SUPPLEMENTARY MATERIAL

Appendix: Convergence analysis

In this section, we prove all the theorems in this paper.

1. proof of Theorem 1

Proof From Eq. (9), we know that

$$vec(\nabla_{\mathcal{G}} H(\bar{\mathcal{G}}, A_{(N)}) - \nabla_{\mathcal{G}} H(\mathcal{G}, A_{(N)})) = vec((\bar{\mathcal{G}} - \mathcal{G})(\otimes_{i=N}^{1}((A_i)^T A_i))).$$

Since $\|\mathcal{G}\|_F = \|\mathrm{vec}(\mathcal{G})\|_F$, thus we infer

$$\begin{aligned}&\|\nabla_{\mathcal{G}} H(\bar{\mathcal{G}}, A_{(N)}) - \nabla_{\mathcal{G}} H(\mathcal{G}, A_{(N)})\|_F \\ &= \|vec((\bar{\mathcal{G}} - \mathcal{G})(\otimes_{i=N}^{1}((A_i)^T A_i)))\|_F \\ &\leq \| \otimes_{i=N}^{1} ((A_i)^T A_i)\|_F \|vec(\bar{\mathcal{G}} - \mathcal{G})\|_F \\ &= L_{\nabla_{\mathcal{G}} H} \|\bar{\mathcal{G}} - \mathcal{G}\|_F,\end{aligned}$$

and from Eq. (9), we also have

$$\begin{aligned}&\|\nabla_{A_n} H(\mathcal{G}, \{A_j\}_{j=1}^{n-1}, \bar{A}, \{A_j\}_{j=n+1}^{N}) - \nabla_{A_n} H(\mathcal{G}, A_{(N)})\|_F \\ &\leq L_{\nabla_{A_n} H} \|\bar{A} - A_n\|_F,\end{aligned}$$

where $B_n = G^{(n)}(\otimes_{i=N, i\neq n}^{1} A_i)^T$ and

$$L_{\nabla_{A_n} H} = \begin{cases} \|B_n(B_n)^T\|_F + \alpha\|L\|_F, & \text{if } n = N, \\ \|B_n(B_n)^T\|_F, & \text{else.} \end{cases}$$

Thus, Theorem 1 is true.

2. proof of Theorem 2

Proof (i) If step 12 of Algorithm 2 is not true, then the Theorem 2(i) is true, if step 12 of Algorithm 2 is true, since Theorem 1 and Proposition 4, we know that

$$\begin{aligned}H(\mathcal{G}^{k+1}, A_{(N)}^k) \leq &\langle \nabla_{\mathcal{G}} H(\mathcal{G}^k, A_{(N)}^k), \mathcal{G}^{k+1} - \mathcal{G}^k\rangle \\ &+ H(\mathcal{G}^k, A_{(N)}^k) \\ &+ \frac{L_{\nabla_{\mathcal{G}^k} H}}{2}\|\mathcal{G}^{k+1} - \mathcal{G}^k\|_F^2. \end{aligned} \tag{15}$$

From Eq. (10), we obtain

$$\begin{aligned}F_G(\mathcal{G}^k) \geq &F_G(\mathcal{G}^{k+1}) + \frac{1}{2\sigma_G^k}\|\mathcal{G}^{k+1} - \mathcal{G}^k\|_F^2 \\ &+ \langle \nabla_{\mathcal{G}} H(\mathcal{G}^k, A_{(N)}^k), \mathcal{G}^{k+1} - \mathcal{G}^k\rangle. \end{aligned} \tag{16}$$

Then sum of the Eq. (15) and Eq. (16), we have

$$\begin{aligned}H(\mathcal{G}^{k+1}, A_{(N)}^k) \leq &H(\mathcal{G}^k, A_{(N)}^k) + F_G(\mathcal{G}^k) - F_G(\mathcal{G}^{k+1}) \\ &- (\frac{1}{2\sigma_G^k} - \frac{L_{\nabla_{\mathcal{G}^k} H}}{2})\|\mathcal{G}^{k+1} - \mathcal{G}^k\|_F^2. \end{aligned} \tag{17}$$

Similarly, since Theorem 1 and Proposition 4, we know that

$$\begin{aligned}H(\mathcal{G}^{k+1}, A_1^{k+1}, \{A_i^k\}_{i=2}^N) \leq &\langle \nabla_{A_1} H(\mathcal{G}^{k+1}, A_{(N)}^k), A_1^{k+1} - A_1^k\rangle \\ &+ H(\mathcal{G}^{k+1}, A_{(N)}^k) \\ &+ \frac{L_{\nabla_{A_1^k} H}}{2}\|A_1^{k+1} - A_1^k\|_F^2. \end{aligned} \tag{18}$$

From Eq. (10), we can also obtain

$$\begin{aligned}F_1(A_1^k) \geq &F_1(A_1^{k+1}) + \frac{1}{2\sigma_1^k}\|A_1^{k+1} - A_1^k\|_F^2 \\ &+ \langle \nabla_{A_1} H(\mathcal{G}^{k+1}, A_{(N)}^k), A_1^{k+1} - A_1^k\rangle. \end{aligned} \tag{19}$$

Then sum of the Eq. (18), Eq. (19) and Eq. (17), we have

$$\begin{aligned}H(\mathcal{G}^{k+1}, A_1^{k+1}, \{A_i^k\}_{i=2}^N) \leq &H(\mathcal{G}^k, A_{(N)}^k) + F_1(A_1^k) + F_G(\mathcal{G}^k) \\ &- F_1(A_1^{k+1}) - F_G(\mathcal{G}^{k+1}) \\ &- \rho\|A_1^{k+1} - A_1^k\|_F^2 \\ &- \rho\|\mathcal{G}^{k+1} - \mathcal{G}^k\|_F^2, \end{aligned} \tag{20}$$

where $\rho = \min(\frac{1}{2\sigma_G^k} - \frac{L_{\nabla_{\mathcal{G}^k} H}}{2}, \frac{1}{2\sigma_1^k} - \frac{L_{\nabla_{A_1^k} H}}{2})$. Assuming Theorem 2(i) holds when $n = N - 1$, i.e.,

$$\begin{aligned}H(\mathcal{G}^{k+1}, A_{(N-1)}^{k+1}, A_N^k) \leq &H(\mathcal{G}^k, A_{(N)}^k) + F_G(\mathcal{G}^k) + \sum_{i=1}^{N-1} F_i(A_i^k) \\ &- \sum_{i=1}^{N-1} F_i(A_i^{k+1}) - F_G(\mathcal{G}^{k+1}) \\ &- \rho\|A_{(N-1)}^{k+1} - A_{(N-1)}^k\|_F^2 \\ &- \rho\|\mathcal{G}^{k+1} - \mathcal{G}^k\|_F^2, \end{aligned} \tag{21}$$

where $\rho = \min(\frac{1}{2\sigma_G^k} - \frac{L_{\nabla_{\mathcal{G}^k} H}}{2}, \{\frac{1}{2\sigma_i^k} - \frac{L_{\nabla_{A_i^k} H}}{2}\}_{i=1}^{N-1})$. Similarly, from Eq. (10), we infer

$$\begin{aligned}F_N(A_N^k) \geq &F_N(A_N^{k+1}) + \frac{1}{2\sigma_N^k}\|A_N^{k+1} - A_N^k\|_F^2 \\ &+ \langle \nabla_{A_N} H(\mathcal{G}^{k+1}, A_{(N-1)}^{k+1}, A_N^k), A_N^{k+1} - A_N^k\rangle. \end{aligned} \tag{22}$$

From Theorem 1 and Proposition 4, we also have

$$\begin{aligned}H(\mathcal{G}^{k+1}, A_{(N)}^{k+1}) \leq &H(\mathcal{G}^{k+1}, A_{(N-1)}^{k+1}, A_N^k) \\ &+ \langle \nabla_{A_N} H(\mathcal{G}^{k+1}, A_{(N-1)}^{k+1}, A_N^k), A_N^{k+1} - A_N^k\rangle \\ &+ \frac{L_{\nabla_{A_N^k} H}}{2}\|A_N^{k+1} - A_N^k\|_F^2. \end{aligned} \tag{23}$$

Then sum of the Eq. (21), Eq. (22) and Eq. (23), we have

$$J(\mathcal{G}^{k+1}, A_{(N)}^{k+1}) \leq J(\mathcal{G}^k, A_{(N)}^k) - \rho\|z^{k+1} - c^k\|_F^2, \tag{24}$$

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where $\rho=\min(\frac{1}{2\sigma_G^k}-\frac{L_{\nabla_{\mathcal{G}^k}H}}{2},\{\frac{1}{2\sigma_i^k}-\frac{L_{\nabla_{A_i^k}H}}{2}\}_{i=1}^N)$. This shows the Theorem 2(i) is true.

(ii) We know that $\frac{\alpha}{2}\operatorname{Tr}((A_N)^TLA_N)$ in Eq. (6) is nonnegative. Hence, Eq. (6) is nonnegative. From Eq. (24), we have

$$\rho\|z^{k+1}-c^k\|_F^2\leq J(\mathcal{G}^k,A_{(N)}^k)-J(\mathcal{G}^{k+1},A_{(N)}^{k+1}).$$

Sum of both side, since Eq. (6) is nonnegative, thus we have

$$\begin{aligned}\rho\sum_{k=1}^{\infty}\|z^{k+1}-c^k\|_F^2&\leq\sum_{k=1}^{\infty}(J(\mathcal{G}^k,A_{(N)}^k)-J(\mathcal{G}^{k+1},A_{(N)}^{k+1}))\\&=J(\mathcal{G}^1,A_{(N)}^1)-\inf J\\&<\infty.\end{aligned}$$

It follows that

$$\lim_{k\to\infty}\|z^{k+1}-c^k\|_F^2=0.$$

3. proof of Lemma 1

Proof By Proposition 2 and Proposition 3, Eq. (10) follows that

$$0\in\partial_{\mathcal{G}}F_G(\mathcal{G}^{k+1})-\frac{1}{\sigma_G^k}(\mathcal{Y}^k-\mathcal{G}^{k+1})+\nabla_{\mathcal{G}}H(\mathcal{Y}^k,A_{(N)}^k).\tag{25}$$

Thus we infer

$$\begin{aligned}f_G^{k+1}&=\nabla_{\mathcal{G}}H(\mathcal{G}^{k+1},A_{(N)}^k)-\nabla_{\mathcal{G}}H(\mathcal{Y}^k,A_{(N)}^k)\\&\quad+\frac{1}{\sigma_G^k}(\mathcal{Y}^k-\mathcal{G}^{k+1})\\&\in\nabla_{\mathcal{G}}H(\mathcal{G}^{k+1},A_{(N)}^k)+\partial_{\mathcal{G}}(F_G(\mathcal{G}^{k+1})+\sum_{i=1}^N F_i(A_i^k))\\&=\partial_{\mathcal{G}}J(\mathcal{G}^{k+1},A_{(N)}^k).\end{aligned}\tag{26}$$

Similarly, by Proposition 2 and Proposition 3, Eq. (10) follows that

$$\begin{aligned}0&\in\partial_{A_i}F_i(A_i^{k+1})-\frac{1}{\sigma_i^k}(Y_i^k-A_i^{k+1})\\&\quad+\nabla_{A_i}H(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i-1},Y_i^k,\{A_j^k\}_{j=i+1}^N).\end{aligned}\tag{27}$$

Thus we also have

$$\begin{aligned}f_i^{k+1}&=\nabla_{A_i}H(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i},\{A_j^k\}_{j=i+1}^N)\\&\quad-\frac{1}{\sigma_i^k}(A_i^{k+1}-Y_i^k)\\&\quad-\nabla_{A_i}H(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i-1},Y_i^k,\{A_j^k\}_{j=i+1}^N)\\&\in\nabla_{A_i}H(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i-1},A_i^{k+1},\{A_j^k\}_{j=i+1}^N)\\&\quad+\partial_{A_i}(F_G(\mathcal{G}^{k+1})+\sum_{j=1}^{i}F_j(A_j^{k+1})+\sum_{j=i+1}^{N}F_j(A_j^k))\\&=\partial_{A_i}J(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i-1},A_i^{k+1},\{A_j^k\}_{j=i+1}^N).\end{aligned}\tag{28}$$

Since Theorem 1, we have

$$\begin{aligned}\|f_G^{k+1}\|_F&=\|\nabla_{\mathcal{G}}H(\mathcal{G}^{k+1},A_{(N)}^k)+\frac{1}{\sigma_G^k}(\mathcal{Y}^k-\mathcal{G}^{k+1})\\&\quad-\nabla_{\mathcal{G}}H(\mathcal{Y}^k,A_{(N)}^k)\|_F,\\&\leq\|\nabla_{\mathcal{G}}H(\mathcal{G}^{k+1},A_{(N)}^k)-\nabla_{\mathcal{G}}H(\mathcal{Y}^k,A_{(N)}^k)\|_F\\&\quad+\|\frac{1}{\sigma_G^k}(\mathcal{Y}^k-\mathcal{G}^{k+1})\|_F\\&\leq(L_{\nabla_{\mathcal{G}^k}H}+\frac{1}{\sigma_G^k})\|\mathcal{G}^{k+1}-\mathcal{Y}^k\|_F.\end{aligned}\tag{29}$$

Similarly, we also have

$$\begin{aligned}\|f_i^{k+1}\|_F&=\|\nabla_{A_i}H(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i},\{A_j^k\}_{j=i+1}^N)\\&\quad+\frac{1}{\sigma_i^k}(Y_i^k-A_i^{k+1})\|\\&\quad-\nabla_{A_i}H(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i-1},Y_i^k,\{A_j^k\}_{j=i+1}^N)\|_F,\\&\leq\|\nabla_{A_i}H(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i},\{A_j^k\}_{j=i+1}^N)\\&\quad-\nabla_{A_i}H(\mathcal{G}^{k+1},\{A_j^{k+1}\}_{j=1}^{i-1},Y_i^k,\{A_j^k\}_{j=i+1}^N)\|_F\\&\quad+\|\frac{1}{\sigma_i^k}(A_i^{k+1}-Y_i^k)\|_F\\&\leq(L_{\nabla_{A_i^k}H}+\frac{1}{\sigma_i^k})\|A_i^{k+1}-Y_i^k\|_F.\end{aligned}\tag{30}$$

Thus we infer

$$\|f_G^{k+1}\|_F+\|\{f_i^{k+1}\}_{i=1}^N\|_F\leq\rho_b\|z^{k+1}-c^k\|_F,\tag{31}$$

where $\rho_b=\max(\frac{1}{\sigma_G^k}+L_{\nabla_{\mathcal{G}^k}H},\{\frac{1}{\sigma_i^k}+L_{\nabla_{A_i^k}H}\}_{i=1}^N)$.

4. proof of Theorem 3

Proof (i) Observe that z' can be viewed as an intersection of compact sets

$$z'=\cap_{q\in\mathbb{N}}\overline{\cup_{k\geq q}\{z^k\}}.$$

From proposition 1, z' is compact.

(ii) $\forall\overline{z}\in z'$, there exists a subsequence z^{k_j} such that

$$\lim_{j\to\infty}z^{k_j}=\overline{z}.$$

Let

$$F(z^{k_j})=\sum_{i=1}^N F_i(A_i^{k_j})+F_G(\mathcal{G}^{k_j}).\tag{32}$$

Since F_i and F_G are lower semicontinuous [1], from Definition 3 and Eq. (32), we obtain that

$$\lim_{j\to\infty}\inf F(z^{k_j})\geq F(\overline{z}).\tag{33}$$

Choosing $k = k_j - 1$, from Eq. (10), we infer

$$\begin{aligned}\lim_{j\to\infty}\sup F_G(\mathcal{G}^{k_j}) &\le \lim_{j\to\infty}\sup(F_G(\overline{\mathcal{G}}) + \frac{1}{2\sigma_G^k}\|\overline{\mathcal{G}} - \mathcal{Y}^k\|_F^2 \\ &\quad + \langle\nabla_{\mathcal{G}}H(\mathcal{Y}^k, A_{(N)}^k), \overline{\mathcal{G}} - \mathcal{Y}^k\rangle), \\ \lim_{j\to\infty}\sup F_i(A_i^{k_j}) &\le \lim_{j\to\infty}\sup(F_i(\overline{A_i}) + \frac{1}{2\sigma_i^k}\|\overline{A_i} - Y_i^k\|_F^2 \\ &\quad + \langle\nabla_{A_i}h_i(Y_i^k), \overline{A_i} - Y_i^k\rangle),\end{aligned} \tag{34}$$

where $h_i(Y_i^k) = H(\mathcal{G}^{k+1}, \{A_n^{k+1}\}_{n=1}^{i-1}, Y_i^k, \{A_n^k\}_{n=i+1}^N)$, $\lim_{j\to\infty}\mathcal{G}^{k_j} = \overline{\mathcal{G}}$, $\lim_{j\to\infty} A_i^{k_j} = \overline{A_i}$. Since $\lim_{j\to\infty} z^{k_j} = \overline{z}$ and $\lim_{j\to\infty}\|z^{k_j} - c^{k_j-1}\|_F = 0$ (Theorem 2(ii)), we can get that

$$\begin{aligned}\lim_{j\to\infty}\|\overline{A_i} - Y_i^k\|_F &\le \lim_{j\to\infty}\|\overline{A_i} - A_i^{k_j}\|_F \\ &\quad + \lim_{j\to\infty}\|A_i^{k_j} - Y_i^{k_j-1}\|_F \\ &= 0, \\ \lim_{j\to\infty}\|\overline{\mathcal{G}} - \mathcal{Y}^k\|_F &\le \lim_{j\to\infty}\|\overline{\mathcal{G}} - \mathcal{G}^{k_j}\|_F \\ &\quad + \lim_{j\to\infty}\|\mathcal{G}^{k_j} - \mathcal{Y}^{k_j-1}\|_F \\ &= 0.\end{aligned} \tag{35}$$

From Eq. (34) and Eq. (35), we infer

$$\begin{aligned}\lim_{j\to\infty}\sup F_i(A_i^{k_j}) &\le \lim_{j\to\infty}\sup F_i(\overline{A_i}), \\ \lim_{j\to\infty}\sup F_G(\mathcal{G}^{k_j}) &\le \lim_{j\to\infty}\sup F_G(\overline{\mathcal{G}}),\end{aligned} \tag{36}$$

where $\lim_{j\to\infty}\mathcal{G}^{k_j} = \overline{\mathcal{G}}$, $\lim_{j\to\infty} A_i^{k_j} = \overline{A_i}$. Therefore, from Eq. (32) and Eq. (36), we have

$$\lim_{j\to\infty}\sup F(z^{k_j}) \le \lim_{j\to\infty}\sup F(\overline{z}). \tag{37}$$

From Eq. (32), Eq. (37) and Eq. (33), we infer

$$\lim_{j\to\infty} F(z^{k_j}) = F(\overline{z}). \tag{38}$$

Since Theorem 2, we have

$$\lim_{j\to\infty} H(z^{k_j}) = H(\overline{z}). \tag{39}$$

Thus from Eq. (38) and Eq. (39), we infer

$$\begin{aligned}\lim_{j\to\infty} H(z^{k_j}) + \lim_{j\to\infty} F(z^{k_j}) &= \lim_{j\to\infty}(H(z^{k_j}) + F(z^{k_j})) \\ &= \lim_{j\to\infty} J(z^{k_j}) \\ &= J(\overline{z}).\end{aligned}$$

This means J is constant on z'.

(iii) From Eq. (26) and Eq. (28), we know that

$$\begin{aligned}&f_G^{k+1} \in \partial_{\mathcal{G}} J(\mathcal{G}^{k+1}, A_{(N)}^k), \\ &f_i^{k+1} \in \partial_{A_i} J(\mathcal{G}^{k+1}, \{A_j^{k+1}\}_{j=1}^i, \{A_j^k\}_{j=i+1}^N),\end{aligned} \tag{40}$$

where the definitions of f_i^{k+1} and f_G^{k+1} are the same as in Eq. (12). From Theorem 2(ii) and Lemma 1, we infer

$$\begin{aligned}\lim_{k\to\infty}(\|\{f_i^{k+1}\}_{i=1}^N\|_F + \|f_G^{k+1}\|_F) &\le \lim_{k\to\infty}\rho_b\|z^{k+1} - c^k\|_F \\ &= 0.\end{aligned} \tag{41}$$

Let

$$z_1^k = \mathcal{G}^k, z_{i+1}^k = A_i^k. \tag{42}$$

From Theorem 2(i), we know that

$$\begin{aligned}\lim_{k\to\infty} J(\{z_i^{k+1}\}_{i=1}^{j-1}, z_j^{k+1}, \{z_i^k\}_{i=j+1}^{N+1}) &= \lim_{k\to\infty} J(z^k) \\ &= J(z^*).\end{aligned} \tag{43}$$

Therefore, from Eq. (40), Eq. (41), Eq. (42) and Eq. (43), we have

$$\begin{aligned}0 &\in \partial_{z_j}\lim_{k\to\infty} J(\{z_i^{k+1}\}_{i=1}^{j-1}, z_j^{k+1}, \{z_i^k\}_{i=j+1}^{N+1}) \\ &= \partial_{z_j} J(z^*).\end{aligned}$$

This means $0 \in \partial J(z^*)$.

5. proof of Theorem 4

Proof From Theorem 1, we know that $\frac{\alpha}{2}\operatorname{Tr}((A_N)^T L A_N)$ is an infinitely differentiable function, and the norm of its derivatives of any order is also continuous, so $\frac{\alpha}{2}\operatorname{Tr}((A_N)^T L A_N)$ is a real analytic function (real analytic functions are all KŁ functions [2]). Since the Frobenius norm, ℓ_p-norm and Eq. (7) are also all KŁ functions [1], it follows that Eq. (6) is a KŁ function. Therefore, from Definition 6, there exists a concave function ϕ so that

$$\phi'(J(z^k) - J(\overline{z}))dist(0, \partial J(z^k)) \ge 1. \tag{44}$$

From ϕ is the convex function, we have

$$\begin{aligned}\phi(J(z^{k+1}) - J(\overline{z})) &\le \phi(J(z^k) - J(\overline{z})) \\ &\quad + \phi'(J(z^k) - J(\overline{z}))(J(z^{k+1}) - J(z^k)).\end{aligned} \tag{45}$$

From Lemma 1 and Theorem 3, we infer

$$dist(0, \partial J(z^k)) \le \rho_b\|z^k - c^{k-1}\|_F. \tag{46}$$

Since the Eq. (44) and Eq. (46), we infer

$$\begin{aligned}\phi'(J(z^k) - J(\overline{z})) &\ge \frac{1}{dist(0, \partial J(z^k))} \\ &\ge \frac{1}{\rho_b\|z^k - c^{k-1}\|_F}.\end{aligned} \tag{47}$$

Let $J(k) = J(z^k) - J(\overline{z})$, from Eq. (45), Eq. (46), Eq. (47) and Theorem 2(i), we have

$$\begin{aligned}\phi(J(k)) - \phi(J(k+1)) &\geq \phi^{'}(J(k))(J(k) - J(k+1)) \\ &\geq \frac{J(k) - J(k+1)}{\rho_b \|z^k - c^{k-1}\|_F} \\ &\geq \frac{\rho \|z^{k+1} - c^k\|_F^2}{\rho_b \|z^k - c^{k-1}\|_F}.\end{aligned}$$

Define C=$\frac{\rho}{\rho_b}$, C is a constant, so we infer

$\|z^{k+1} - c^k\|_F^2 \leq C(\phi(J(k)) - \phi(J(k+1)))\|z^k - c^{k-1}\|_F$.

Using the fact that $2ab \leq a^2 + b^2$

$2\|z^{k+1} - c^k\|_F \leq C(\phi(J(k)) - \phi(J(k+1))) + \|z^k - c^{k-1}\|_F$.

Sum both sides

$$\begin{aligned}2\sum_{k=l+1}^{K} \|z^{k+1} - c^k\|_F \leq& \sum_{k=l+1}^{K} \|z^k - c^{k-1}\|_F \\ &+ C(\phi(J(l+1)) - \phi(J(K+1))) \\ &= C(\phi(J(l+1)) - \phi(J(K+1))) \\ &+ \|z^{l+1} - c^l\|_F + \sum_{k=l+1}^{K} \|z^{k+1} - c^k\|_F.\end{aligned} \tag{48}$$

From Eq. (48), we can get that

$$\begin{aligned}\lim_{K\to\infty} \sum_{k=l+1}^{K} \|z^{k+1} - c^k\|_F \leq& \|z^{l+1} - c^l\|_F + C\phi(J(l+1)) \\ &- \lim_{K\to\infty} C\phi(J(K+1)) \\ &< \infty.\end{aligned}$$

Thus we have

$$\sum_{k=l+1}^{\infty} \|z^{k+1} - c^k\|_F < \infty. \tag{49}$$

From Algorithm 2, no matter $c^k = z^k + \beta_k(z^k - z^{k-1})$ or $c^k = z^k$, we always have

$$\|z^{k+1} - z^k\|_F - \beta_k\|z^k - z^{k-1}\|_F \leq \|z^{k+1} - c^k\|_F. \tag{50}$$

From Eq. (49) and Eq. (50), we know that

$$\begin{aligned}\sum_{k=l+1}^{\infty} (\|z^{k+1} - z^k\|_F - \beta_k\|z^k - z^{k-1}\|_F) \leq& \sum_{k=l+1}^{\infty} \|z^{k+1} - c^k\|_F \\ &< \infty.\end{aligned}$$

Since

$$\begin{aligned}&\|z^{k+1} - z^k\|_F - \beta_{max}\|z^k - z^{k-1}\|_F \leq \\ &\|z^{k+1} - z^k\|_F - \beta_k\|z^k - z^{k-1}\|_F,\end{aligned}$$

we also have

$$\begin{aligned}&\sum_{k=l+1}^{\infty} (\|z^{k+1} - z^k\|_F - \beta_{max}\|z^k - z^{k-1}\|_F) \\ &\leq \sum_{k=l+1}^{\infty} (\|z^{k+1} - z^k\|_F - \beta_k\|z^k - z^{k-1}\|_F) \\ &< \infty,\end{aligned} \tag{51}$$

and

$$\begin{aligned}&\sum_{k=l+1}^{\infty} \|z^{k+1} - z^k\|_F - \sum_{k=l+1}^{\infty} \beta_{max}\|z^k - z^{k-1}\|_F \\ &= \sum_{k=l+1}^{\infty} (\|z^{k+1} - z^k\|_F - \beta_{max}\|z^{k+1} - z^k\|_F) \\ &+ \beta_{max}\|z^{l+1} - z^l\|_F \\ &= \sum_{k=l+1}^{\infty} (1 - \beta_{max})\|z^{k+1} - z^k\|_F + \beta_{max}\|z^{l+1} - z^l\|_F.\end{aligned} \tag{52}$$

Let $s^k = \|z^{k+1} - z^k\|_F$, from Eq. (51) and Eq. (52), we have

$$\begin{aligned}\sum_{k=l+1}^{\infty} (1 - \beta_{max})s^k - \beta_{max}s^l =& \sum_{k=l}^{\infty}(s^{k+1} - \beta_{max}s^k) \\ &\leq \sum_{k=l+1}^{\infty} \|z^{k+1} - c^k\|_F \\ &< \infty.\end{aligned}$$

Thus we infer

$$\sum_{k=l+1}^{\infty} (1 - \beta_{max})s^k < \infty. \tag{53}$$

From $(1 - \beta_{max})$ is a positive constant and Eq. (53), we infer

$$\sum_{k=l+1}^{\infty} \|z^{k+1} - z^k\|_F = \sum_{k=l+1}^{\infty} s^k < \infty.$$

Thus we have

$$\sum_{k=0}^{\infty} \|z^{k+1} - z^k\|_F = \sum_{k=0}^{\infty} s^k < \infty.$$

This shows that

$$\begin{aligned}\lim_{K\to\infty}\|z^{K+p}-z^K\|_F &= \lim_{K\to\infty}\|\sum_{k=K+1}^{K+p}(z^{k+1}-z^k)\|_F \\ &\le \lim_{K\to\infty}\sum_{k=K+1}^{K+p}\|z^{k+1}-z^k\|_F \\ &= \lim_{K\to\infty}\sum_{k=K+1}^{\infty}s^k \\ &= 0. \end{aligned} \tag{54}$$

This means the Theorem 4 is true.

6. proof of Theorem 5

Proof From Appendix B of [3] and Section 3.5 of [4], we know that the computational cost of $\nabla_{A_i}H(\mathcal{G},A_{(N)})$ is

$$\mathcal{O}(\sum_{j=1}^{N}(\sum_{i=1}^{j}I_i)(\sum_{i=j}^{N}r_i)). \tag{55}$$

Similarly, the computational cost of $\nabla_{\mathcal{G}}H(\mathcal{G},A_{(N)})$ is

$$\mathcal{O}(\sum_{j=1}^{N}(\sum_{i=1}^{j}r_i)(\sum_{i=j}^{N}I_i)). \tag{56}$$

The cost of computing the Lipschitz constant, projection to nonnegative, and tensor unfolding is negligible compared to the cost of computing partial gradients $\nabla_{\mathcal{G}}H(\mathcal{G},A_{(N)})$ and $\nabla_{A_j}H(\mathcal{G},A_{(N)})$. Similarly, the cost of the sparsity projection is also negligible compared to the cost of computing these partial gradients, as the time complexity of finding the largest s elements in an array of p elements is $\mathcal{O}(p+slog_2(s))$. Therefore, from Eq. (55) and Eq. (56), the time complexity of the Algorithm 1 and Algorithm 2 in each iteration is approximately estimated as follows.

$$\mathcal{O}(\sum_{j=1}^{N}(\sum_{i=1}^{j}r_i)(\sum_{i=j}^{N}I_i)+N(\sum_{j=1}^{N}(\sum_{i=1}^{j}I_i)(\sum_{i=j}^{N}r_i))).$$

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Chinese Journal of Electronics