Semiparametric Estimation of Oaxaca-Blinder Decompositions with Continuous Groups

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September 22, 2018

Preliminary, please do not cite.

Abstract

This paper considers Oaxaca-Blinder type decompositions with continuous groups. In particular, we decompose the differences between outcomes at a series of values and at a particular value (with at two particular values as a special case) of the continuous group variable into (i) composition effects and (ii) structural effects. The composition effects are due to differences in observed characteristics (e.g. race or education) for individuals at different values of the continuous group variable. The structural effects are due to differences in the "return" to characteristics at different values of the continuous group variable. We also consider detailed decompositions of both the composition and structural effects. Our procedure is based on semiparametric smooth coefficient models. This approach is distinct from previous work on decompositions with continuous groups. We develop the limiting distribution of our estimators. We apply our method to decompose earnings differentials for individuals across their parents' income (i.e. parents' income is the continuous group).

Keywords: Local Linear Regression, Oaxaca-Blinder Decomposition, Continuous Decomposition, Intergenerational Income Mobility

JEL Codes: C14, J62

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1 Introduction

Since the seminal work of Oaxaca (1973) and Blinder (1973) in the early 1970s, there are many studies on decomposition of differences in the mean and other distributional statistics between two groups or its changes over time into various explanatory variables using a variety of methodologies. For instance, Juhn, Murphy, and Pierce (1993) decompose wage inequality for males. Firpo, Fortin, and Lemieux (2009) consider the direct effect of union status on male log wage using unconditional quantile regression. Chernozhukov, Fernández-Val, and Melly (2013) decompose quantile differences using distribution regression. [1] Fortin, Lemieux, and Firpo (2011) offers a comprehensive overview of decomposition methods for differences between two groups.

Almost universally, although Blinder–Oaxaca decomposition is used to compare the outcomes between two discrete groups, e.g., male and female, in many applications groups are not readily divided into discrete bins. Suppose the group variable is continuous (e.g., years of work experience and parents' income) instead of binary or categorical (e.g., union status or races), one idea would be to divide the population by the group variable into a small number of groups and perform the Oaxaca-Blinder decomposition (see, e.g., Cameron and Heckman (2001), Frenette et al. (2007), Du, Renyu, He, and Zhang (2014) and Richey and Rosburg (2018)). Using this approach, one needs to choose the cutoffs for the group in a some arbitrary way.

There are a few papers that work on decomposition for a continuous group variable. Ñopo (2008) is the first one which extends the Blinder–Oaxaca decomposition from two groups to a continuum of groups. Ulrick (2012) proposes a technique which allows the structure (unexplained) component to vary across values of the group variable. Both papers above focus on aggregate effects (the explained and unexplained components) and do not study detailed decomposition effects. To extend literature on decomposition for a continuous group variable, this paper develops a semiparametric decomposition technique based on a semiparametric smooth coefficient model (Li, Huang, Li, and Fu (2002)) and Oaxaca-Blinder decomposition.See more research related to semiparametric smooth coefficient model, e.g., Hastie and Tibshirani (1993), Cai, Fan, and Li (2000), Fan, Zhang, et al. (1999), Zhang, Lee, and Song (2002), and Li and Racine (2010) and

[1]Freeman (1980) looks at the difference in the variance of wages between the union and non-union sector. DiNardo, Fortin, and Lemieux (1996) analyze changes in the U.S. distribution of wages by applying kernel density methods. See also Donald, Green, and Paarsch (2000), Machado and Mata (2005), Altonji, Bharadwaj, and Lange (2012), Gelbach (2016), Elder, Goddeeris, and Haider (2015) among others.

among others.

Ñopo (2008) and Ulrick (2012) assume that the conditional expectation follows some particular parametric model, but it seems challenging in practice to specify the right functional form. Ulrick (2012) allows group variable to interact with the covariates in the functional form of conditional expectation of outcome but it is difficult to include the interactions appropriately.[2] Richey and Rosburg (2016) propose a decomposition method that extends the traditional Oaxaca-Blinder decomposition to a continuous group membership setting that can be applied to any distributional measure of interest.

Differently, our approach is quite flexible, in which we allow the coefficients on the covariates – the effects of covariates on the outcome – to depend on the values of group variable. For instance, in our application the effects of parents' education on child's income can vary with parents' income. Also, our method does not suffer from the curse of dimensionality and is feasible even with the moderate amount of data typically available in applications. We estimate the conditional expectation of covariate using local linear regression.[3]

Specifically, our approach decomposes mean differences between outcomes at a series values of and at a particular value of the continuous group variable by combining the local constant estimation of smooth coefficient model and standard Oaxaca-Blinder decomposition. The method does not suffer from choosing the cutoffs for the group variable arbitrarily. It allows the differences between outcomes to vary across values of the group variable and is able to perform detailed decompositions in addition to aggregate decompositions. The decomposition results do not depend on the order of decomposition, that is, the results are path independent. We also establish the joint asymptotic normality of the decomposition components at a series values of the group variable.

We apply our method to decompose earnings differentials for children across their parents' income (i.e. parents' income is the continuous group) to study the changing roles of education, race and other observable characteristics in the contribution to the differences between the child's incomes conditional on parents' incomes. More specifically, we decompose differences between child's incomes conditional at a series of values of parents' income and at the mean of parents'

[2] See Ulrick, Mongeon, and Giannetto (2018) and Nieto and Ramos (2015) for applications related to Ñopo (2008) and Ulrick (2012).

[3] Ñopo (2008) notes the conditional expectations would require some non-parametric regressions and Ulrick (2012) estimate the conditional expectation using a regression without specifying the type of regression.

income in our dataset.[4] We find that the differences are more contributed by the differences in the "return" to covariates (structure effects) than by the differences in the distribution of covariates (composition effects) across different values of parents' income, which is more obvious when the parents' income is at the higher quantiles. We also find that the composition effects are mostly contributed by the effects associated to parents' education attainments and race when the parents' income is lower but the effects are mostly explained by the effects associated to parents' education attainments when the parents' income is higher. It is interesting to find that the "return" to parents' eduction is higher and significant when the parents' income is lower. Also it is surprising to find that the "return" to race is non-significant across different values of parents' income.

We organize the rest of the paper as follows. Section 2 describes the parameters of interest. In Section 3, we present our new approach. The asymptotic theory of the decomposition components is developed in Section 4. Section 5 applies the method to decompose earnings differentials for children across their parents' income. We conclude in Section 6.

2 Parameters of Interest

Let Y denotes outcome variable, T a continuous group variable and X a $(1+\mathbb{K}) \times 1$ vector of covariates including a constant. Our goal is to decompose differences in average between outcomes conditional on two particular values of a continuous group variable: $E(Y|T=t) - E(Y|T=t_0)$. The difference can be written as:

$$
\begin{aligned}
\Delta_O(t) &\equiv E(Y|T=t) - E(Y|T=t_0) \\
&= E\big(E(Y|X,t)|T=t\big) - E\big(E(Y|X,t_0|T=t_0)\big) && (2.1)\\
&= E(X'|T=t)\beta(t) - E(X'|T=t_0)\beta(t_0) && (2.2)\\
&= \underbrace{\big(E(X'|T=t) - E(X'|T=t_0)\big)\,\beta(t)}_{\Delta_C(t)} + \underbrace{E(X'|T=t_0)\,\big(\beta(t)-\beta(t_0)\big)}_{\Delta_S(t)} && (2.3)\\
&= \sum_{k=1}^{\mathbb{K}} \underbrace{\big(E(X_k|T=t) - E(X_k|T=t_0)\big)\,\beta_k(t)}_{\Delta_{C_k}(t)} + \sum_{k=1}^{\mathbb{K}} \underbrace{E(X_k|T=t_0)\,\big(\beta_k(t)-\beta_k(t_0)\big)}_{\Delta_{S_k}(t)} && (2.4)
\end{aligned}
$$

[4]One can choose the particular value of parents' income to be poverty line or average family income of the United States.

where $\Delta_C(t)$ and $\Delta_S(t)$ denote aggregate composition and structure effects, respectively. $\Delta_{C_k}(t)$ and $\Delta_{S_k}(t)$ represent detailed composition and structure effects associated to the kth covariate, respectively.[5]

In the paper, we estimate $E(Y|T)$ based on local constant estimation of smooth coefficient models.

Remark 1. *Our approach allows the heterogeneous response of the outcome to changes in distribution of covariates conditional on continuous group variable. That is, $\beta(t)$ indexed by t is a function of t allowing changes across the values of group variable. For instance, t is parents' income, a X is father's education, and Y is the children's income. Our model allows father's education to have different impacts on the children's income conditional on various parents' income. This encompasses the case in which t is binary.*

Remark 2. *The first parameters that we consider are $\Delta_C(t)$ and $\Delta_S(t)$ both indexed by group variable t, that is, both are functions of t and straightforward measures to plot. Our approach is useful because it allows one to examine the role that the variation of distribution of covariates plays across t's in term of difference between outcomes conditional on t's. More specifically, we can test whether $\left(\Delta_C(t_1), \cdots, \Delta_C(t_q)\right)$ is jointly statistically different from zero or other values which includes testing the significance of $a\Delta_C(t)$ being different from zero as a special case. The estimate $\Delta_S(t)$ is also a function of t and straightforward to plot. This allows one to examine the role that the differences in the returns to covariates across t's play in term of differences between outcomes conditional on t's.*

Remark 3. *To examine the role that the differences in covariates plays in terms of difference between outcomes conditional on t's, we consider the parameters $\Delta_{C_k}(t)$ which are straightforward to plot, as a function of t. For instance, we can examine the role that the distribution of head's education plays in term of differences between child's incomes conditional on parents' income. We can plot the differences between the child's incomes which are contributed by the difference in the distribution of head's education against parents' incomes. To examine the role that the returns to covariates plays in terms of differences between outcomes conditional on t's, we also consider the parameters $\Delta_{S_k}(t)$ which, as a function of t, are also straightforward to plot. For instance, we can*

[5]Note that the detailed composition effect linked to a constant is zero.

examine the role that the return to head's education plays in term of differences between child's incomes conditional on parents' incomes. We can plot the differences between child's incomes contributed by the difference in the return to head's education against parents' incomes.

In the paper, we decompose differences between outcomes at a series of values of and a fixed particular value of continuous group variable, which including decomposing differences between outcomes at two particular values of continuous group variable as a special case.[6] In next section, we first show how to use our approach to estimate the decomposition components of difference between outcomes at a value of and a fixed particular value of continuous group variable. Then, decomposing differences between outcomes at a series of t's and a fixed particular t_0 is simply to repeat the decomposition for t's.

3 Estimation

In this section we first describes semiparametric smooth coefficient models. Then, we use plug-in approach to compute the decomposition components shown in Equations 2.3 and 2.4. Lastly, compute the decomposition components repeatedly for differences between outcomes at a series of values of and a fixed particular value of continuous group variable.

3.1 Semiparametric Smooth Coefficient Models

Semiparametric smooth coefficient models (Li, Huang, Li, and Fu (2002)) is

$$Y_i = X_i'\beta(T_i) + \epsilon_i \tag{3.1}$$

The local constant estimator (Li and Racine (2007), p.302) of the model above is given by the minimizer to the following optimization

$$\hat{\beta}(t) = \arg\min_{\beta(t)} \sum_{i=1}^{n} K_{h_t}(T_i - t)(Y_i - X_i'\beta(t))^2. \tag{3.2}$$

[6] In this paper, we decompose differences between outcomes conditional two values of group variable because it allows the flexibility of selecting values which are of interest to researchers. For instance, one might be interested in what drives the differences between the incomes of children from low income family and the ones of children from higher income family. In this case, one could choose the poverty line as the particular value of group variable and decompose the differences across family incomes.

We rewrite this optimization as

$$\hat{\beta}(t) = \arg\min_{\beta(t)} \left\| W_{h_t}^{1/2} \big(Y - X\beta(t) \big) \right\|_2^2. \tag{3.3}$$

where Y is a $n \times 1$ vector, X is an $n \times (1 + \mathbb{K})$ matrix, $W_{h_t} = Diag(K(\frac{T_1 - t}{h_t}), \ldots, K(\frac{T_n - t}{h_t})$ and $\|\cdot\|_2$ denotes L-2 norm.

Thus, $\hat{\beta}(t)$ can be computed by

$$\hat{\beta}(t) = \big(X' W_{h_t} X \big)^{-1} X' W_{h_t} Y \tag{3.4}$$

In practice, we select a bandwidth which makes the bias terms in the decomposition components converge to zero.

3.2 Decomposition

In this paper, we use local linear regression to estimate $E(X|T)$. Then, combined with estimates $\hat{\beta}(\cdot)$, the components of decomposition for a difference are straightforward to compute by plugging $\hat{E}(X|T)$ and $\hat{\beta}(\cdot)$ into Equations 2.3 and 2.4. Then, one can repeatedly compute the decomposition components for differences between outcomes at a series of t's and a fixed particular t_0 using the same approach.

4 Asymptotic Theory

This section shows joint asymptotic normality of decomposition components in differences between outcomes at a series of values of and at a particular value of continuous variable, which including a difference between outcomes at two particular values of group variable as a special case. We establish theorems which allow one to conduct inferences on a series of decomposition components simultaneously, although the theorems are not for uniform inference.

For simplicity, we assume all bandwidths converge at the same rate as h. That is, $h_t = C_t h$ and $h_k = C_k h, k = 1, \cdots, \mathbb{K}$ where h_k is the bandwidth used in estimating $E(X_k|T = \cdot)$ based on local linear regression, and C_t and C_k are both some constants.

Assumption 1. *Assume all bandwidths converge at the same rate as* h*:* $h_t = C_t h$ *and* $h_k = C_k h$*, where* C_t *and* C_k *are some constants.*

Let $X = g(T) + \epsilon$. We also assume the regularity conditions for local linear regression model (see, for example, Li and Racine (2007)[Theorem 2.7]) and local constant estimator of semiparametric smooth coefficient model (Li, Huang, Li, and Fu (2002)).

4.1 Joint Asymptotic Normality of Decomposition Components

Four theorems below establish respectively the asymptotic normality of the estimated decomposition components. κ_2 and $g''(t)$ shown in the following theorems are defines as $\kappa_2 = \int k(v)v^2 dv$, $g''(t) = \partial^2 g(t)/\partial t^2$.

4.1.1 Aggregate Components

Theorem 1 establishes the joint asymptotic normality of the aggregate composition affects across a series of values of the group variable from t_1 to t_q.

Theorem 1. *Under regularity conditions stated in Lemma 1 and 2 and Assumption 1, the joint asymptotic normality of the aggregate composition effect* $\hat{\Delta}_C(t)$*'s across a series of values of the group variable from* t_1 *to* t_q *is as follows*

$$\sqrt{nh}\left(\hat{\Delta}_C(t_1) - \Delta_C(t_1) - Bias_{\Delta_C}(t_1), \quad \cdots, \quad \hat{\Delta}_C(t_q) - \Delta_C(t_q) - Bias_{\Delta_C}(t_q)\right)' \xrightarrow{d} N(0, \mathbb{V}_C)$$

where $\mathbb{V}_C = R_{\Delta_C}\Sigma_{\Delta_C}R'_{\Delta_C}$ *where* R_{Δ_C}*,* Σ_{Δ_C} *and* $Bias_{\Delta_C}(\cdot)$ *are given in section A.1 of the appendix.*

Theorem 1 shows the joint asymptotic normality of the aggregate composition effects of differences between outcomes conditional on a series of values and a particular reference value of a group variable which includes as a special case one aggregate composition effect of difference between outcomes conditional on one value and a reference value of group variable. Although Theorem 1 does not enable uniform inference, it is useful because it allows one to consider inference on a list of aggregate composition effects and will be important for testing whether the aggregate composition effects are jointly significantly different from zero.

Simultaneous consistency of $\Delta_C(t_1)$ through $\Delta_C(t_q)$ requires that $nh \to 0$ and $h^2 \to 0$ in order for the bias and the variances to both converge to zero.

Theorem 2 establishes the joint asymptotic normality of the aggregate structure effects of differences between outcomes conditional on a series of values and a particular reference value of a group variable which includes as a special case one aggregate composition effect of difference between outcomes conditional on one value and a reference value of group variable.

Theorem 2. *Under regularity conditions stated in Lemma 1 and 2 and Assumption 1, the joint asymptotic normality of the aggregate composition effect $\hat{\Delta}_S(t)$'s across a series of values of the group variable from t_1 to t_q is as follows*

$$\sqrt{nh}\left(\hat{\Delta}_S(t_1) - \Delta_S(t_1) - Bias_{\Delta_S}(t_1), \quad \cdots, \quad \hat{\Delta}_S(t_q) - \Delta_S(t_q) - Bias_{\Delta_S}(t_q)\right)' \xrightarrow{d} N(0, \mathbb{V}_S)$$

where $\mathbb{V}_S = R_{\Delta_S}\Sigma_{\Delta_S}R'_{\Delta_S}$ where R_{Δ_S}, Σ_{Δ_S} and $Bias_{\Delta_S}(\cdot)$ are given in section A.2 of the appendix.

Although Theorem 2 does not allow uniform inference, it is useful because it allows one to consider inference on a list of aggregate composition effects thus will be important for testing whether the aggregate structure effects are jointly significantly different from zero. Simultaneous consistency of $\Delta_S(t_1)$ through $\Delta_S(t_q)$ requires that $nh \to 0$ and $h^2 \to 0$ in order for the bias and the variances to both converge to zero.

4.1.2 Detailed Components

The two following theorems establish respectively the joint asymptotic normality of the detailed composition and structure effects of . Theorem 3 shows the joint asymptotic normality of the detailed composition effects of differences between outcomes conditional on a series of values and a particular reference value of a group variable which includes as a special case one detailed composition effect.

Theorem 3. *Under regularity conditions stated in Lemma 1 and 2 and Assumption 1, the joint asymptotic normality of the detailed composition effect $\hat{\Delta}_{C_k}(t)$'s across a series of values of the group variable from t_1 to t_q is as follows*

$$\sqrt{nh}\left(\hat{\Delta}_{C_k}(t_1) - \Delta_{C_k}(t_1) - Bias_{\Delta_{C_k}}(t_1), \quad \cdots, \quad \hat{\Delta}_{C_k}(t_q) - \Delta_{C_k}(t_q) - Bias_{\Delta_{C_k}}(t_q)\right)'$$

$$\xrightarrow{d} N(0, \mathbb{V}_{C_k})$$

where $\mathbb{V}_{C_k} = R_{\Delta_{C_k}} \Sigma_{\Delta_{C_k}} R'_{\Delta_{C_k}}$ *where* $R_{\Delta_{C_k}}$, $\Sigma_{\Delta_{C_k}}$ *and* $\Delta_{C_k}(\cdot)$ *are given in section A.3 of the appendix.*

Theorem 3 allows one to jointly test whether the difference in distribution of a covariate conditional on the group variable contribute significantly to the differences between outcomes conditional a series of values and a particular value of group variable. Simultaneous consistency of $\Delta_{C_k}(t_1)$ through $\Delta_{C_k}(t_q)$ requires that $nh \to 0$ and $h^2 \to 0$ in order for the bias and the variances to both converge to zero.

Theorem 4 establishes the joint asymptotic normality of the detailed structure effects. It is useful because it allows one to jointly test whether a return to a covariate significantly contributes to the differences between outcomes conditional a series of values and a particular value of group variable.

Theorem 4. *Under regularity conditions stated in Lemma 1 and 2 and Assumption 1, the joint asymptotic normality of the detailed structure effect $\hat{\Delta}_{S_k}(t)$'s across a series of values of the group variable from t_1 to t_q is as follows*

$$\sqrt{nh}\left(\hat{\Delta}_{S_k}(t_1) - \Delta_{S_k}(t_1) - Bias_{\Delta_{S_k}}(t_1), \quad \cdots, \quad \hat{\Delta}_{S_k}(t_q) - \Delta_{S_k}(t_q) - Bias_{\Delta_{S_k}}(t_q)\right)'$$
$$\xrightarrow{d} N(0, \mathbb{V}_{S_k})$$

where $\mathbb{V}_{S_k} = R_{\Delta_{S_k}} \Sigma_{\Delta_{S_k}} R'_{\Delta_{S_k}}$ *where* $R_{\Delta_{S_k}}$, $\Sigma_{\Delta_{S_k}}$ *and* $\Delta_{S_k}(\cdot)$ *are given in section A.4 of the appendix.*

Simultaneous consistency of $\Delta_{S_k}(t_1)$ through $\Delta_{S_k}(t_q)$ requires that $nh \to 0$ and $h^2 \to 0$ in order for the bias and the variances to both converge to zero.

In practice, we carry out inference using bootstrap.

5 An Application

It is well known that parents' income is correlated with the income of their children (see Solon (1992) and Solon (1999), among many others). However, there are many other "covariates" that are also correlated with both parents' income and child's income– for example, education and race. For instance, Callaway and Huang (2018) find that adjusting for differences in background characteristics such as head's education and race tends to reduce (though not eliminate) the overall effect of parents' income on child's income as well as reduce differences in the local intergenerational income elasticity and variance of child's income across parents' income levels. Richey and Rosburg (2017) investigate the changing roles of ability and education in the transmission of economic status across generations.

We apply new decomposition approaches stated above to decompose differences between child's incomes at a series of values of and the mean of parents' incomes into other observable characteristics. The decomposition allows us to examine the roles that these characteristics play in explaining the differences at various parents' income levels. We suspect the roles that family head's education and race contribute to the differences would vary across parents' incomes because the distribution of and/or the "returns" to these two characteristics conditional on parents' income are likely various with parents' income. That is, low-income families are likely having less educated family head and high-income families are likely to have family head with higher education. Also, the returns to the same education level could be not the same among families with different family incomes. Although both family heads have the same education levels, one family head, who has higher income, could affect the child – and therefore the child's income – differently from the other family heads, who have lower income.

For any decomposition dealing with dummy variables, we face the omitted variable problem. In this paper, we choose to omit the following variables: white, female child, non-veteran head and head with high school education.[7] [8] Also, the group chosen as reference could also affect the decomposition results. We show the main results using as reference the coefficients from smooth coefficient model in which the child's income at a series of values of parents' income is the dependent variable. We show the results as robustness checks by switching the reference. The advantage of our approach is that the order of decomposition does not matter, that is, the decomposition results are path independent.

[7]There exists the "omitted group" problem in the decomposition and there is no general solution to this problem yet. See Jones (1983), Oaxaca and Ransom (1999) and Gelbach (2016).

[8]For the sake of comparison with the literature, we choose to omit high school graduates. See (Fortin, Lemieux, and Firpo (2011) p. 42), " ...the common practice of using high school graduates as the omitted category allows the comparison of detailed decomposition results when this omitted category is comparable across studies."

To perform the decomposition, the first step is to choose the optimal bandwidth for smooth coefficient model. The bandwidth chosen needs to make the bias terms converge to zero. In this paper, we require $h = c \cdot n^{-1/5}$ in which c is selected by by leave-one-out cross-validation. The optimal bandwidth used for smooth coefficient model is 0.3072. We use the same bandwidth for local linear regression of X on T.[9]

5.1 Data Description

The dataset used in this paper is from Callaway and Huang (2018). It has been a long history for researchers using Panel Study of Income Dynamics (PSID) to study intergenerational income mobility. In the paper, the outcome variable is children's family income and the continuous group variable is parents' family income. The covariates include child's gender and year of birth, family head's race, educational attainment, and veteran status.

5.2 Results

Figure 1 shows the decomposition results. For the aggregate decomposition, we show that the aggregate components change across the parents' incomes. On average, the ratio of the composition effect to the overall estimated difference is higher when the parents' income is in the lower percentiles; the ratio is lower when the parents' income is in the higher percentiles. For example, at the 10th percentile of parents' incomes, differing characteristics between children from different families explain 40% of the estimated difference, with differing returns for these characteristics explaining the remaining difference. However, at the 90th percentile of parents' incomes, differing characteristics between children from different families explain 24.5% of the estimated difference, with differing returns for these characteristics explaining the remaining gap. The average ratio of the composition effect to the overall estimated difference is about 36.7% when the parents' income is within 20th and 40th percentiles. However, that average ratio decreases to 24.9% when the parents' income is within 60th and 90th percentiles.

Panel b of Figure 1 shows that the differing education explains most of the composition effect. Another interesting finding is that the differences in the distribution of race plays a significant role

[9]In this paper, the optimal bandwidth becomes smaller as fewer variables are included.

when the parents' income is in the lower percentiles; however, the differences in race distribution does not seem to contribute to the composition effect when the parents' income is in the higher percentiles.

Panel c of Figure 1 shows that the structure effect is most contributed by the returns for the other characteristics and the difference between the constant terms. We note an important finding that is the return to the education positively and significantly contributes to the structure effect when the parents' income is in the lower percentiles. This means that the return to education for the children from lower income families is higher than the one for the children from families with average income. This finding has important policy implications. It is also surprising to find that the return to race does not change across parents' incomes.

Figure 2 to 3 plots the overall differences and the effects across parents' incomes with %95 confidence bands obtained by bootstrapping. Figure 5 to 8 in the Appendix B show the decomposition results in which we change the reference group used to compute the effects.

6 Conclusion

The paper has developed a semiparametric method to decompose differences between outcomes conditional on different values of a continuous group variable, which can be useful in many interesting applications. We apply this method to study the effects of characteristics of family background like race and parents' education attainments on the differences between child's income conditional on parents' income. We find that conditional on parents' income the differences between child's income are mostly from the differences in the "returns" to family characteristics. We also find that the "return" to parents' education attainments is higher for children from low income family and the "return" to race is not significant different across different values of parents' income. Also, we find that the composition effects across parents' income are mostly contributed by the effects associated to parents' education, especially from children from low income family.

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A Proofs

Before we show the proof of Theorem 1 to 4, we first prove the following three Lemmas. Lemma 1 show some results about the estimate from semiparametric smooth coefficient model; Lemma 2 shows results from local linear regression. Lemma 3 shows results on the components in $\hat{\Delta}_C(t)$.

We assume regularity conditions (Li, Huang, Li, and Fu (2002)) and other assumptions required for Theorem 9.3 (Li and Racine (2007)).

Lemma 1. *Under some regular conditions (see Li, Huang, Li, and Fu (2002)), one can have*

$$\begin{aligned}\hat{\beta}(t) - \beta(t) - Bias_t &= (nh_t)^{-1}\sum_{i=1}^{n} H_t^{-1} X_i \epsilon_i K(\frac{T_i - t}{h_t}) + (s.o.)\\ &:= (nh_t)^{-1}\sum_{i=1}^{n}\psi_1(t) + (s.o.)\end{aligned}$$

where

$$\begin{aligned}Bias_t &= \kappa_2 h_t^2 E[XX'\{\beta'(t)f_t'(X,t)/f(X|t) + \frac{1}{2}\beta''(t)f(t)\}|t]\\ \psi_1(t) &= H_t^{-1} X_i \epsilon_i K(\frac{T_i - t}{h_t})\\ H_t &= f(t)E(XX'|T = t)\end{aligned}$$

Proof. See the proof of Theorem 9.3 in Li and Racine (2007). □

Lemma 2. *Let $X = g(T) + \epsilon$ and under standard regularity conditions for local linear estimators (see, for example, Li and Racine (2007)[Theorem 2.7]), one can show that*

$$\begin{aligned}\hat{E}(X|T = t) - E(X|T = t) - Bias_X(t) &= (nh_X)^{-1}\sum_{i=1}^{n}\frac{1}{f(t)}K(\frac{T_i - t}{h_X})U_i + (s.o.)\\ &:= (nh_X)^{-1}\sum_{i=1}^{n}\psi_2(t) + (s.o.)\end{aligned}$$

where $Bias_X(t) = \frac{\kappa_2}{2} f(t) g_X''(t) h_X^2$ and $\psi_2(t) = \frac{1}{f(t)}K(\frac{T_i - t}{h_X})U_i$.

Proof. The proof follows from Li and Racine (2007) Theorem 2.7. □

A.1 Proof of Theorem 1

To show the joint asymptotic normality of the aggregate composition effects stated in Theorem 1, we first introduce the next lemma which is a result about $\hat{\Delta}_C - \Delta_C$.

Lemma 3. *Under regularity conditions stated in Lemma 1 and 2,*

$$\hat{\Delta}_C(t) - \Delta_C(t) = Bias_{\Delta_C}(t) + R_{\Delta_C}(t)W_{\Delta_C}(t) + (s.o.)$$

where

$$\begin{aligned}
Bias_{\Delta_C}(t) &= \left(Bias_X(t) - Bias_X(t_0)\right)\beta(t) + \left(E(X'|T=t) - E(X'|T=t_0)\right) Bias_t \\
R_{\Delta_C}(t) &= \Big(\beta(t)', \quad -\beta(t)', \quad \left(E(X'|T=t) - E(X'|T=t_0)\right)\Big), \\
W_{\Delta_C}(t) &= \left(\left((nh_X)^{-1}\sum_{i=1}^{n}\psi_2(t)\right)', \left((nh_X)^{-1}\sum_{i=1}^{n}\psi_2(t_0)\right)', \left((nh_t)^{-1}\sum_{i=1}^{n}\psi_1(t)\right)'\right)'
\end{aligned}$$

.

Proof. The proof of Lemma 3 mainly follows three steps.

(i) Subtract $\hat{E}(X'|T=t)$ with $E(X'|T=t)$, $\hat{E}(X'|T=t_0)$ with $E(X'|T=t_0)$, and $\hat{\beta}(t)$ with $\beta(t)$ while adding or subtracting related terms to keep the equality hold.

(ii) Rearrange to have the terms stated in Lemma 1 and 2.

(iii) Apply the Lemma 1 and 2 to complete the proof.

$$\begin{aligned}
&\hat{\Delta}_C(t) - \Delta_C(t) \\
&= \left(\hat{E}(X'|T=t) - \hat{E}(X'|T=t_0)\right)\hat{\beta}(t) - \left(E(X'|T=t) - E(X'|T=t_0)\right)\beta(t) \\
&\qquad\qquad\qquad \text{Follow step (i)} \\
&= \underbrace{\left(\left(\hat{E}(X'|T=t) - E(X'|T=t)\right) - \left(\hat{E}(X'|T=t_0) - E(X'|T=t_0)\right)\right)\left(\hat{\beta}(t) - \beta(t)\right)}_{A} \\
&\quad + \hat{E}(X'|T=t)\beta(t) + E(X'|T=t)\left(\hat{\beta}(t) - \beta(t)\right) - \hat{E}(X'|T=t_0)\beta(t) - E(X'|T=t_0)\left(\hat{\beta}(t) - \beta(t)\right) \\
&\quad - \left(E(X'|T=t) - E(X'|T=t_0)\right)\beta(t) \\
&\qquad\qquad\qquad \text{Follow step (ii)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\hat{E}(X'|T=t) - E(X'|T=t)\right)\beta(t) - \left(\hat{E}(X'|T=t_0) - E(X'|T=t_0)\right)\beta(t) \\
&\quad + \left(E(X'|T=t) - E(X'|T=t_0)\right)\left(\hat{\beta}(t) - \beta(t)\right) \\
&\qquad\qquad\qquad \text{Follow step (iii)} \\
&= \left(Bias_X(t) + (nh_X)^{-1}\sum_{i=1}^{n}\psi_2(t)\right)\beta(t) - \left(Bias_X(t_0) + (nh_X)^{-1}\sum_{i=1}^{n}\psi_2(t_0)\right)\beta(t) \\
&\quad + \left(E(X'|T=t) - E(X'|T=t_0)\right)\left(Bias_t + (nh_t)^{-1}\sum_{i=1}^{n}\psi_1(t)\right) + (s.o.) \\
&:= Bias_{\Delta_C}(t) + R_{\Delta_C}(t)W_{\Delta_C}(t) + (s.o.)
\end{aligned}
$$

where it is straightforward to show the 3th equality holds using standard arguments that Term A is asymptotically negligible.

□

Proof of Theorem 1.

Following Lemma 3, it is straightforward to prove Theorem 1 by Liapunov Central Limit Theorem.

$$
\begin{aligned}
&\left(\hat{\Delta}_C(t_1) - \Delta_C(t_1) - Bias_{\Delta_C}(t_1), \quad \cdots, \quad \hat{\Delta}_C(t_q) - \Delta_C(t_q) - Bias_{\Delta_C}(t_q)\right)' \\
&= \left(R_{\Delta_C}(t_1)W_{\Delta_C}(t_1), \quad \cdots, \quad R_{\Delta_C}(t_q)W_{\Delta_C}(t_q)\right)' + (s.o.) \\
&= R_{\Delta_C}W_{\Delta_C} + (s.o.)
\end{aligned}
$$

where

$$
\begin{aligned}
R_{\Delta_C} &= Diag\left(R_{\Delta_C}(t_1), \quad \cdots, \quad R_{\Delta_C}(t_q)\right), \\
W_{\Delta_C} &= \left(W'_{\Delta_C}(t_1), \quad \cdots, \quad W'_{\Delta_C}(t_q)\right)'
\end{aligned}
$$

and the first equality holds by Lemma 3.

Under Assumption 1, the proof is completed by Liapunov Central Limit Theorem.

$$
\sqrt{nh}\left(\hat{\Delta}_C(t_1) - \Delta_C(t_1) - Bias_{\Delta_C}(t_1), \quad \cdots, \quad \hat{\Delta}_C(t_q) - \Delta_C(t_q) - Bias_{\Delta_C}(t_q)\right)' \xrightarrow{d} N(0, \mathbb{V}_C)
$$

where

$$
\begin{aligned}
\mathbb{V}_C &= R_{\Delta_C}\Sigma_{\Delta_C}R'_{\Delta_C},\\
\Sigma_{\Delta_C} &= E\left(nhW_{\Delta_C}W'_{\Delta_C}\right)\\
&= nhE\begin{pmatrix} W_{\Delta_C}(t_1)W'_{\Delta_C}(t_1) & W_{\Delta_C}(t_1)W'_{\Delta_C}(t_2) & \cdots & W_{\Delta_C}(t_1)W'_{\Delta_C}(t_q)\\ W_{\Delta_C}(t_2)W'_{\Delta_C}(t_1) & W_{\Delta_C}(t_2)W'_{\Delta_C}(t_2) & \cdots & W_{\Delta_C}(t_2)W'_{\Delta_C}(t_q)\\ \vdots & \vdots & \ddots & \vdots\\ W_{\Delta_C}(t_q)W'_{\Delta_C}(t_1) & W_{\Delta_C}(t_q)W'_{\Delta_C}(t_2) & \cdots & W_{\Delta_C}(t_q)W'_{\Delta_C}(t_q)\end{pmatrix}\\
&= \begin{pmatrix} \Sigma^{11}_{\Delta_C} & \Sigma^{12}_{\Delta_C} & \cdots & \Sigma^{1q}_{\Delta_C}\\ \Sigma^{21}_{\Delta_C} & \Sigma^{22}_{\Delta_C} & \cdots & \Sigma^{2q}_{\Delta_C}\\ \vdots & \vdots & \ddots & \vdots\\ \Sigma^{q1}_{\Delta_C} & \Sigma^{q2}_{\Delta_C} & \cdots & \Sigma^{qq}_{\Delta_C}\end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma^{\iota\iota}_{\Delta_C} &= nhE\begin{pmatrix} (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_\iota)\psi'_2(t_\iota), & (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_\iota)\psi'_2(t_0), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_2(t_\iota)\psi'_1(t_\iota)\\ (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_0)\psi'_2(t_\iota), & (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_0)\psi'_2(t_0), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_2(t_0)\psi'_1(t_\iota)\\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_1(t_\iota)\psi'_2(t_\iota), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_1(t_\iota)\psi'_2(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_\iota)\psi'_1(t_\iota)\end{pmatrix}\\
&= \begin{pmatrix} C_X^{-1}\frac{1}{f(t_\iota)}\kappa E(U^2|t_\iota) & 0 & 0\\ 0 & C_X^{-1}\frac{1}{f(t_0)}\kappa E(U^2|t_0) & 0\\ 0 & 0 & C_t^{-1}f(t_\iota)\kappa H_{t_\iota}^{-1}E(XX'\epsilon^2|t_\iota)H_{t_\iota}^{-1}\end{pmatrix}\\
\Sigma^{\iota\gamma}_{\Delta_C} &= nhE\begin{pmatrix} (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_\iota)\psi'_2(t_\gamma), & (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_\iota)\psi'_2(t_0), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_2(t_\iota)\psi'_1(t_\gamma)\\ (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_0)\psi'_2(t_\gamma), & (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_0)\psi'_2(t_0), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_2(t_0)\psi'_1(t_\gamma)\\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_1(t_\iota)\psi'_2(t_\gamma), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_1(t_\iota)\psi'_2(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_\iota)\psi'_1(t_\gamma)\end{pmatrix}\\
&= \begin{pmatrix} 0 & 0 & 0\\ 0 & C_X^{-1}\frac{1}{f(t_0)}\kappa E(U^2|t_0) & 0\\ 0 & 0 & 0\end{pmatrix}
\end{aligned}
$$

where $\kappa = \int k^2(v)dv$,

$$E[\sum_{i=1}^n \psi_1(t_\iota)\psi'_2(t_\gamma)] = nh_Xf(t_\iota)f(t_\gamma)^{-1}H_{t_\iota}^{-1}E(X\epsilon U|t_\iota)\int k(v)k(\frac{h_tv+t_\iota-t_\gamma}{h_X})dv = 0$$

$$E[\sum_{i=1}^n \psi_1(t_\iota)\psi'_1(t_\gamma)]_{\iota\neq\gamma} = nh_tf(t_\iota)H_{t_\iota}^{-1}\int X_iX'_i\epsilon_{i,t_\iota}\epsilon_{i,t_\gamma}f(X,\epsilon|t_\iota)dxd\epsilon H_{t_\gamma}^{-1}\int k(v)k(\frac{h_tv+t_\iota-t_\gamma}{h_t})dv = 0$$

$$E[\sum_{i=1}^n \psi_2(t_\iota)\psi'_2(t_\gamma)]_{\iota\neq\gamma} = nh_Xf^{-1}(t_\gamma)\int U_{i,t_\iota}U_{i,t_\gamma}du\int k(v)k(\frac{h_Xv+t_\iota-t_\gamma}{h_X})dv = 0$$

□

A.2 Proof of Theorem 2

The proof of Theorem 2 is quite similar to the one of Theorem 1. Before proving the main result, we first consider the following result.

Lemma 4. *Under regularity conditions stated in Lemma 1 and 2,*

$$\hat{\Delta}_S(t) - \Delta_S(t) = Bias_{\Delta_S}(t) + R_{\Delta_S}(t) W_{\Delta_S}(t) + (s.o.)$$

where

$$\begin{aligned}
Bias_{\Delta_S}(t) &= Bias_X(t_0)\left(\beta(t) - \beta(t_0)\right) + E(X'|T = t_0)\left(Bias_t - Bias_{t_0}\right) \\
R_{\Delta_S}(t) &= \Big(\left(\beta(t) - \beta(t_0)\right)', \quad E(X'|T = t_0), \quad -E(X'|T = t_0)\Big), \\
W_{\Delta_S}(t) &= \Big(\big((nh_X)^{-1}\sum_{i=1}^{n}\psi_2(t_0)\big)', \big((nh_t)^{-1}\sum_{i=1}^{n}\psi_1(t)\big)', \big((nh_t)^{-1}\sum_{i=1}^{n}\psi_1(t_0)\big)'\Big)'.
\end{aligned}$$

Proof. The proof of this result follows using essentially the same arguments as in Lemma 3. □

Proof of Theorem 2. Using the result of Lemma 4, the proof follows essentially the same arguments as in Theorem 1.

$$\sqrt{nh}\Big(\hat{\Delta}_S(t_1) - \Delta_S(t_1) - Bias_{\Delta_S}(t_1), \quad \cdots, \quad \hat{\Delta}_S(t_q) - \Delta_S(t_q) - Bias_{\Delta_S}(t_q)\Big)' \xrightarrow{d} N(0, \mathbb{V}_S)$$

where

$$\begin{aligned}
\mathbb{V}_S &= R_{\Delta_S}\Sigma_{\Delta_S}R'_{\Delta_S}, \\
R_{\Delta_S} &= Diag\Big(R_{\Delta_S}(t_1), \quad \cdots, \quad R_{\Delta_S}(t_q)\Big), \\
W_{\Delta_S} &= \Big(W'_{\Delta_S}(t_1), \quad \cdots, \quad W'_{\Delta_S}(t_q)\Big)', \\
\Sigma_{\Delta_S} &= E\left(nhW_{\Delta_S}W'_{\Delta_S}\right)
\end{aligned}$$

$$
= nhE \begin{pmatrix} W_{\Delta_S}(t_1)W'_{\Delta_S}(t_1) & W_{\Delta_S}(t_1)W'_{\Delta_S}(t_2) & \cdots & W_{\Delta_S}(t_1)W'_{\Delta_S}(t_q) \\ W_{\Delta_S}(t_2)W'_{\Delta_S}(t_1) & W_{\Delta_S}(t_2)W'_{\Delta_S}(t_2) & \cdots & W_{\Delta_S}(t_2)W'_{\Delta_S}(t_q) \\ \vdots & \vdots & \ddots & \vdots \\ W_{\Delta_S}(t_q)W'_{\Delta_S}(t_1) & W_{\Delta_S}(t_q)W'_{\Delta_S}(t_2) & \cdots & W_{\Delta_S}(t_q)W'_{\Delta_S}(t_q) \end{pmatrix}
$$

$$
= \begin{pmatrix} \Sigma^{11}_{\Delta_S} & \Sigma^{12}_{\Delta_S} & \cdots & \Sigma^{1q}_{\Delta_S} \\ \Sigma^{21}_{\Delta_S} & \Sigma^{22}_{\Delta_S} & \cdots & \Sigma^{2q}_{\Delta_S} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma^{q1}_{\Delta_S} & \Sigma^{q2}_{\Delta_S} & \cdots & \Sigma^{qq}_{\Delta_S} \end{pmatrix}
$$

where

$$
\Sigma^{\iota\iota}_{\Delta_S} = nhE \begin{pmatrix} (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_0)\psi_2'(t_0), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_2(t_0)\psi_1'(t_\iota), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_2(t_0)\psi_1'(t_0) \\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_1(t_\iota)\psi_2'(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_\iota)\psi_1'(t_\iota), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_\iota)\psi_1'(t_0) \\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_1(t_0)\psi_2'(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_0)\psi_1'(t_\iota), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_0)\psi_1'(t_0) \end{pmatrix}
$$

$$
= \begin{pmatrix} C_X^{-1}\frac{1}{f(t_0)}\kappa E(U^2|t_0) & 0 & 0 \\ 0 & C_t^{-1}f(t_\iota)\kappa H_{t_\iota}^{-1}E(XX'\epsilon^2|t_\iota)H_{t_\iota}^{-1} & 0 \\ 0 & 0 & C_t^{-1}f(t_0)\kappa H_{t_0}^{-1}E(XX'\epsilon^2|t_0)H_{t_0}^{-1} \end{pmatrix}
$$

$$
\Sigma^{\iota\gamma}_{\Delta_S} = nhE \begin{pmatrix} (nh_X)^{-2}\sum_{i=1}^n \psi_2(t_0)\psi_2'(t_0), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_2(t_0)\psi_1'(t_\gamma), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_2(t_0)\psi_1'(t_0) \\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_1(t_\iota)\psi_2'(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_\iota)\psi_1'(t_\gamma), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_\iota)\psi_1'(t_0) \\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_1(t_0)\psi_2'(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_0)\psi_1'(t_\gamma), & (nh_t)^{-2}\sum_{i=1}^n \psi_1(t_0)\psi_1'(t_0) \end{pmatrix}
$$

$$
= \begin{pmatrix} C_X^{-1}\frac{1}{f(t_0)}\kappa E(U^2|t_0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_t^{-1}f(t_0)\kappa H_{t_0}^{-1}E(XX'\epsilon^2|t_0)H_{t_0}^{-1} \end{pmatrix}
$$

□

A.3 Proof of Theorem 3

Lemma 5. *Under regularity conditions stated in Lemma 1 and 2,*

$$
\hat{\Delta}_{C_k}(t) - \Delta_{C_k}(t) = Bias_{\Delta_{C_k}}(t) + R_{\Delta_{C_k}}(t)W_{\Delta_{C_k}}(t) + (s.o.)
$$

where

$$\begin{aligned}
Bias_{\Delta_{C_k}}(t) &= \left(Bias_{X_k}(t) - Bias_{X_k}(t_0)\right)\beta(t) + \left(E(X_k|T=t) - E(X_k|T=t_0)\right) Bias_{t,k} \\
R_{\Delta_{C_k}}(t) &= \left(\beta_k(t)', \quad -\beta_k(t)', \quad \left(E(X_k|T=t) - E(X_k|T=t_0)\right)\right), \\
W_{\Delta_{C_k}}(t) &= \left(\left((nh_{X_k})^{-1}\sum_{i=1}^{n}\psi_{2,k}(t)\right)', \left((nh_{X_k})^{-1}\sum_{i=1}^{n}\psi_{2,k}(t_0)\right)', \left((nh_t)^{-1}\sum_{i=1}^{n}\psi_{1,k}(t)\right)'\right)'
\end{aligned}$$

where k denotes the kth element.

Proof. The proof of this result follows using essentially the same arguments as in Lemma 3. □

Proof of Theorem 3. Using the result of Lemma 5, the proof follows essentially the same arguments as in Theorem 1.

$$\sqrt{nh}\left(\hat{\Delta}_{C_k}(t_1) - \Delta_{C_k}(t_1) - Bias_{\Delta_{C_k}}(t_1), \quad \cdots, \quad \hat{\Delta}_{C_k}(t_q) - \Delta_{C_k}(t_q) - Bias_{\Delta_{C_k}}(t_q)\right)' \xrightarrow{d} N(0, \mathbb{V}_{C_k})$$

where

$$\begin{aligned}
\mathbb{V}_{C_k} &= R_{\Delta_{C_k}}\Sigma_{\Delta_{C_k}}R'_{\Delta_{C_k}}, \\
R_{\Delta_{C_k}} &= Diag\left(R_{\Delta_{C_k}}(t_1), \quad \cdots, \quad R_{\Delta_{C_k}}(t_q)\right), \\
W_{\Delta_{C_k}} &= \left(W'_{\Delta_{C_k}}(t_1), \quad \cdots, \quad W'_{\Delta_{C_k}}(t_q)\right)' \\
\Sigma_{\Delta_{C_k}} &= E\left(nhW_{\Delta_{C_k}}W'_{\Delta_{C_k}}\right) \\
&= nhE\begin{pmatrix}
W_{\Delta_{C_k}}(t_1)W'_{\Delta_{C_k}}(t_1) & W_{\Delta_{C_k}}(t_1)W'_{\Delta_{C_k}}(t_2) & \cdots & W_{\Delta_{C_k}}(t_1)W'_{\Delta_{C_k}}(t_q) \\
W_{\Delta_{C_k}}(t_2)W'_{\Delta_{C_k}}(t_1) & W_{\Delta_{C_k}}(t_2)W'_{\Delta_{C_k}}(t_2) & \cdots & W_{\Delta_{C_k}}(t_2)W'_{\Delta_{C_k}}(t_q) \\
\vdots & \vdots & \ddots & \vdots \\
W_{\Delta_{C_k}}(t_q)W'_{\Delta_{C_k}}(t_1) & W_{\Delta_{C_k}}(t_q)W'_{\Delta_{C_k}}(t_2) & \cdots & W_{\Delta_{C_k}}(t_q)W'_{\Delta_{C_k}}(t_q)
\end{pmatrix} \\
&= \begin{pmatrix}
\Sigma^{11}_{\Delta_{C_k}} & \Sigma^{12}_{\Delta_{C_k}} & \cdots & \Sigma^{1p}_{\Delta_{C_k}} \\
\Sigma^{21}_{\Delta_{C_k}} & \Sigma^{22}_{\Delta_{C_k}} & \cdots & \Sigma^{2p}_{\Delta_{C_k}} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma^{p1}_{\Delta_{C_k}} & \Sigma^{p2}_{\Delta_{C_k}} & \cdots & \Sigma^{pp}_{\Delta_{C_k}}
\end{pmatrix}
\end{aligned}$$

where

$$\Sigma^{\iota\iota}_{\Delta_{C_k}} = nhE\begin{pmatrix} (nh_{X_k})^{-2}\sum_{i=1}^n \psi_{2,k}(t_\iota)\psi'_{2,k}(t_\iota), & (nh_{X_k})^{-2}\sum_{i=1}^n \psi_{2,k}(t_\iota)\psi'_{2,k}(t_0), & n^{-2}h_{X_k}^{-1}h_t^{-1}\sum_{i=1}^n \psi_{2,k}(t_\iota)\psi'_{1,k}(t_\iota) \\ (nh_{X_k})^{-2}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{2,k}(t_\iota), & (nh_{X_k})^{-2}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{2,k}(t_0), & n^{-2}h_{X_k}^{-1}h_t^{-1}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{1,k}(t_\iota) \\ n^{-2}h_{X_k}^{-1}h_t^{-1}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{2,k}(t_\iota), & n^{-2}h_{X_k}^{-1}h_t^{-1}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{2,k}(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{1,k}(t_\iota) \end{pmatrix}$$

$$= \begin{pmatrix} C_{X_k}^{-1}\frac{1}{f(t_\iota)}\kappa E(U_k^2|t_\iota) & 0 & 0 \\ 0 & C_{X_k}^{-1}\frac{1}{f(t_0)}\kappa E(U_k^2|t_0) & 0 \\ 0 & 0 & C_t^{-1}f(t_\iota)\kappa H_{t_\iota,k}^{-1}E(X_k^2\epsilon_k^2|t_\iota)H_{t_\iota,k}^{-1} \end{pmatrix}$$

$$\Sigma^{\iota\gamma}_{\Delta_{C_k}} = nhE\begin{pmatrix} (nh_{X_k})^{-2}\sum_{i=1}^n \psi_{2,k}(t_\iota)\psi'_{2,k}(t_\gamma), & (nh_{X_k})^{-2}\sum_{i=1}^n \psi_{2,k}(t_\iota)\psi'_{2,k}(t_0), & n^{-2}h_{X_k}^{-1}h_t^{-1}\sum_{i=1}^n \psi_{2,k}(t_\iota)\psi'_{1,k}(t_\gamma) \\ (nh_{X_k})^{-2}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{2,k}(t_\gamma), & (nh_{X_k})^{-2}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{2,k}(t_0), & n^{-2}h_{X_k}^{-1}h_t^{-1}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{1,k}(t_\gamma) \\ n^{-2}h_{X_k}^{-1}h_t^{-1}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{2,k}(t_\gamma), & n^{-2}h_{X_k}^{-1}h_t^{-1}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{2,k}(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{1,k}(t_\gamma) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{X_k}^{-1}\frac{1}{f(t_0)}\kappa E(U_k^2|t_0) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where k denotes the kth element of a vector or the kth row of a matrix.

□

A.4 Proof of Theorem 4

As a first step, we prove the following lemma.

Lemma 6. *Under regularity conditions stated in Lemma 1 and 2,*

$$\hat{\Delta}_{S_k}(t) - \Delta_{S_k}(t) = Bias_{\Delta_{S_k}}(t) + R_{\Delta_{S_k}}(t)W_{\Delta_{S_k}}(t) + (s.o.)$$

where

$$\begin{aligned} Bias_{\Delta_{S_k}}(t) &= Bias_{X_k}(t_0)\big(\beta_k(t) - \beta_k(t_0)\big) + E(X_k|T=t_0)\big(Bias_{t,k} - Bias_{t_0,k}\big) \\ R_{\Delta_{S_k}}(t) &= \left(\big(\beta_k(t)-\beta_k(t_0)\big)', \quad E(X_k|T=t_0), \quad -E(X_k|T=t_0)\right), \\ W_{\Delta_{S_k}}(t) &= \left(\big((nh_{X_k})^{-1}\sum_{i=1}^n \psi_{2,k}(t_0)\big)', \big((nh_t)^{-1}\sum_{i=1}^n \psi_{1,k}(t)\big)', \big((nh_{t_0})^{-1}\sum_{i=1}^n \psi_{1,k}(t_0)\big)'\right)'. \end{aligned}$$

Proof. The proof follows essentially the same argument as in Lemma 5. □

The proof of Theorem 4. Using the result of Lemma 6, the proof follows essentially the same arguments as in Theorem 3.

$$\sqrt{nh}\left(\hat{\Delta}_{S_k}(t_1) - \Delta_{S_k}(t_1) - Bias_{\Delta_{S_k}}(t_1), \quad \cdots, \quad \hat{\Delta}_{S_k}(t_q) - \Delta_{S_k}(t_q) - Bias_{\Delta_{S_k}}(t_q)\right)' \xrightarrow{d} N(0, \mathbb{V}_{S_k})$$

where

$$
\begin{aligned}
\mathbb{V}_{S_k} &= R_{\Delta_{S_k}} \Sigma_{\Delta_{S_k}} R'_{\Delta_{S_k}}, \\
R_{\Delta_{S_k}} &= Diag\left(R_{\Delta_{S_k}}(t_1), \quad \cdots, \quad R_{\Delta_{S_k}}(t_q) \right), \\
W_{\Delta_{S_k}} &= \left(W'_{\Delta_{S_k}}(t_1), \quad \cdots, \quad W'_{\Delta_{S_k}}(t_q) \right)', \\
\Sigma_{\Delta_{S_k}} &= E\left(nhW_{\Delta_{S_k}} W'_{\Delta_{S_k}} \right) \\
&= nhE \begin{pmatrix} W_{\Delta_{S_k}}(t_1)W'_{\Delta_{S_k}}(t_1) & W_{\Delta_{S_k}}(t_1)W'_{\Delta_{S_k}}(t_2) & \cdots & W_{\Delta_{S_k}}(t_1)W'_{\Delta_{S_k}}(t_q) \\ W_{\Delta_{S_k}}(t_2)W'_{\Delta_{S_k}}(t_1) & W_{\Delta_{S_k}}(t_2)W'_{\Delta_{S_k}}(t_2) & \cdots & W_{\Delta_{S_k}}(t_2)W'_{\Delta_{S_k}}(t_q) \\ \vdots & \vdots & \ddots & \vdots \\ W_{\Delta_{S_k}}(t_q)W'_{\Delta_{S_k}}(t_1) & W_{\Delta_{S_k}}(t_q)W'_{\Delta_{S_k}}(t_2) & \cdots & W_{\Delta_{S_k}}(t_q)W'_{\Delta_{S_k}}(t_q) \end{pmatrix} \\
&= \begin{pmatrix} \Sigma^{11}_{\Delta_{S_k}} & \Sigma^{12}_{\Delta_{S_k}} & \cdots & \Sigma^{1p}_{\Delta_{S_k}} \\ \Sigma^{21}_{\Delta_{S_k}} & \Sigma^{22}_{\Delta_{S_k}} & \cdots & \Sigma^{2p}_{\Delta_{S_k}} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma^{p1}_{\Delta_{S_k}} & \Sigma^{p2}_{\Delta_{S_k}} & \cdots & \Sigma^{pp}_{\Delta_{S_k}} \end{pmatrix}
\end{aligned}
$$

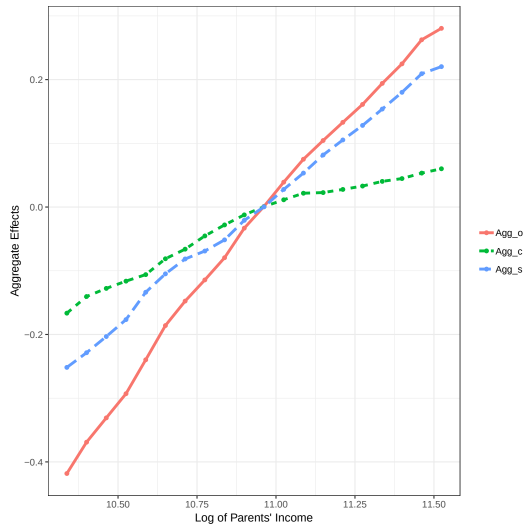
where

$$
\begin{aligned}
\Sigma^{\iota\iota}_{\Delta_{S_k}} &= nhE \begin{pmatrix} (nh_X)^{-2}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{2,k}(t_0), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{1,k}(t_\iota), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{1,k}(t_0) \\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{2,k}(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{1,k}(t_\iota), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{1,k}(t_0) \\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_{1,k}(t_0)\psi'_{2,k}(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_0)\psi'_{1,k}(t_\iota), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_0)\psi'_{1,k}(t_0) \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}_{X_k}\frac{1}{f(t_0)}\kappa E(U_k^2|t_0) & 0 & 0 \\ 0 & C_t^{-1}f(t_\iota)\kappa H^{-1}_{t_\iota,k}E(X_k^2\epsilon_k^2|t_\iota)H^{-1}_{t_\iota,k} & 0 \\ 0 & 0 & C_t^{-1}f(t_0)\kappa H^{-1}_{t_0,k}E(X_k^2\epsilon_k^2|t_0)H^{-1}_{t_0,k} \end{pmatrix} \\
\Sigma^{\iota\gamma}_{\Delta_{S_k}} &= nhE \begin{pmatrix} (nh_X)^{-2}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{2,k}(t_0), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{1,k}(t_\gamma), & n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_{2,k}(t_0)\psi'_{1,k}(t_0) \\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{2,k}(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{1,k}(t_\gamma), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_\iota)\psi'_{1,k}(t_0) \\ n^{-2}h_X^{-1}h_t^{-1}\sum_{i=1}^n \psi_{1,k}(t_0)\psi'_{2,k}(t_0), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_0)\psi'_{1,k}(t_\gamma), & (nh_t)^{-2}\sum_{i=1}^n \psi_{1,k}(t_0)\psi'_{1,k}(t_0) \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}_{X}\frac{1}{f(t_0)}\kappa E(U_k^2|t_0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_t^{-1}f(t_0)\kappa H^{-1}_{t_0,k}E(X_k^2\epsilon_k^2|t_0)H^{-1}_{t_0,k} \end{pmatrix}
\end{aligned}
$$

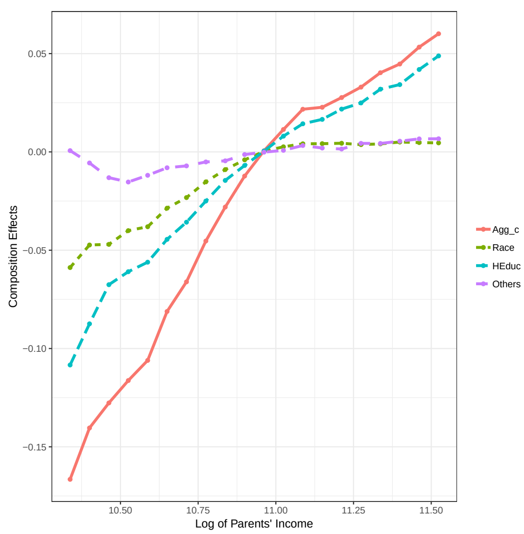
□

B Appendix 1 - Main Tables and Figures

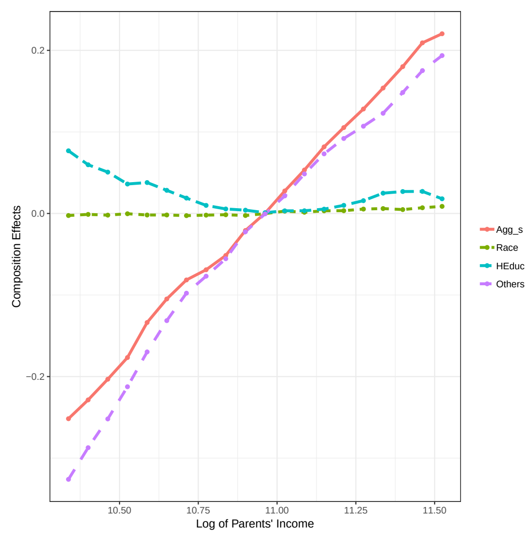
Figure 1: Decomposition across Parents' Incomes



(a) Aggregate effects



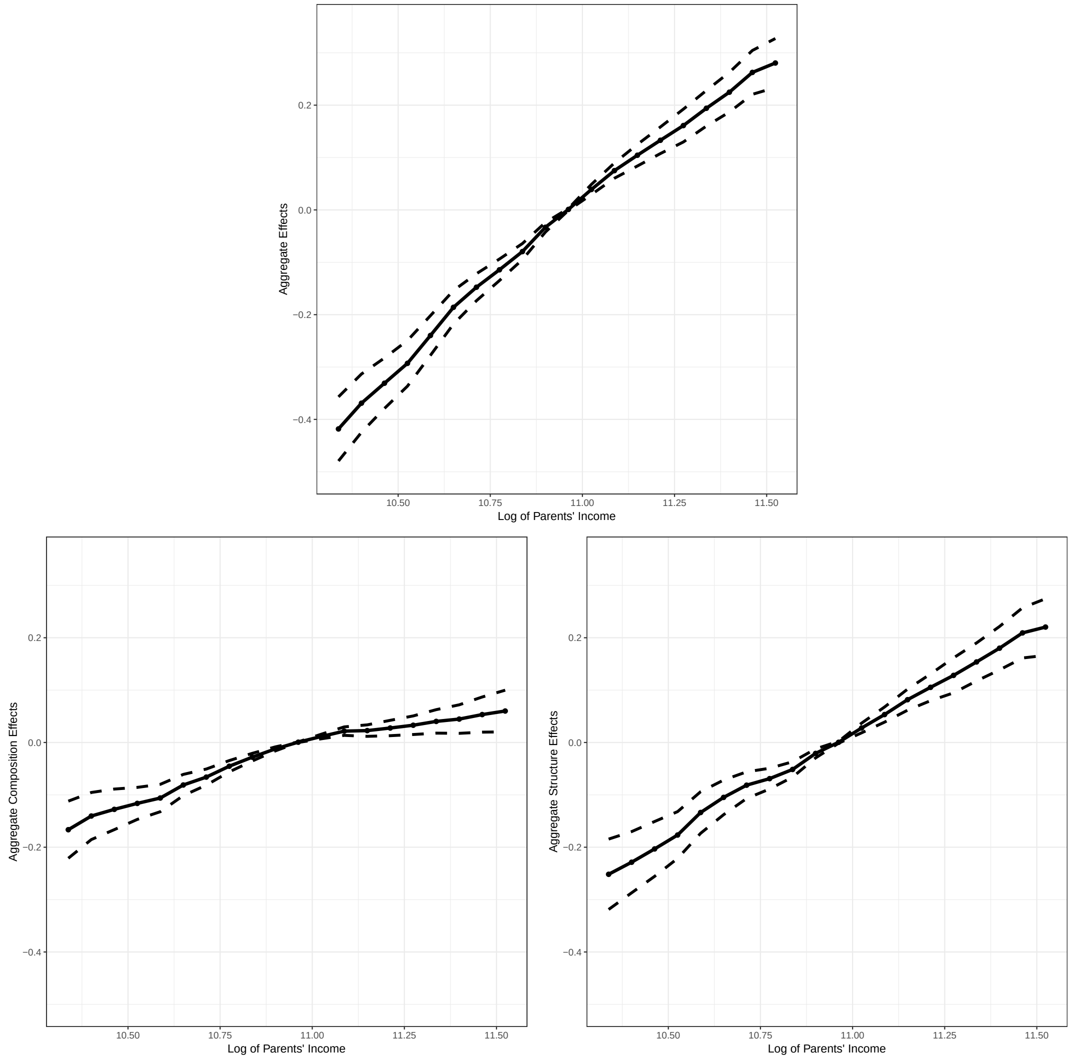
(b) Composition effects



(c) Structure effects

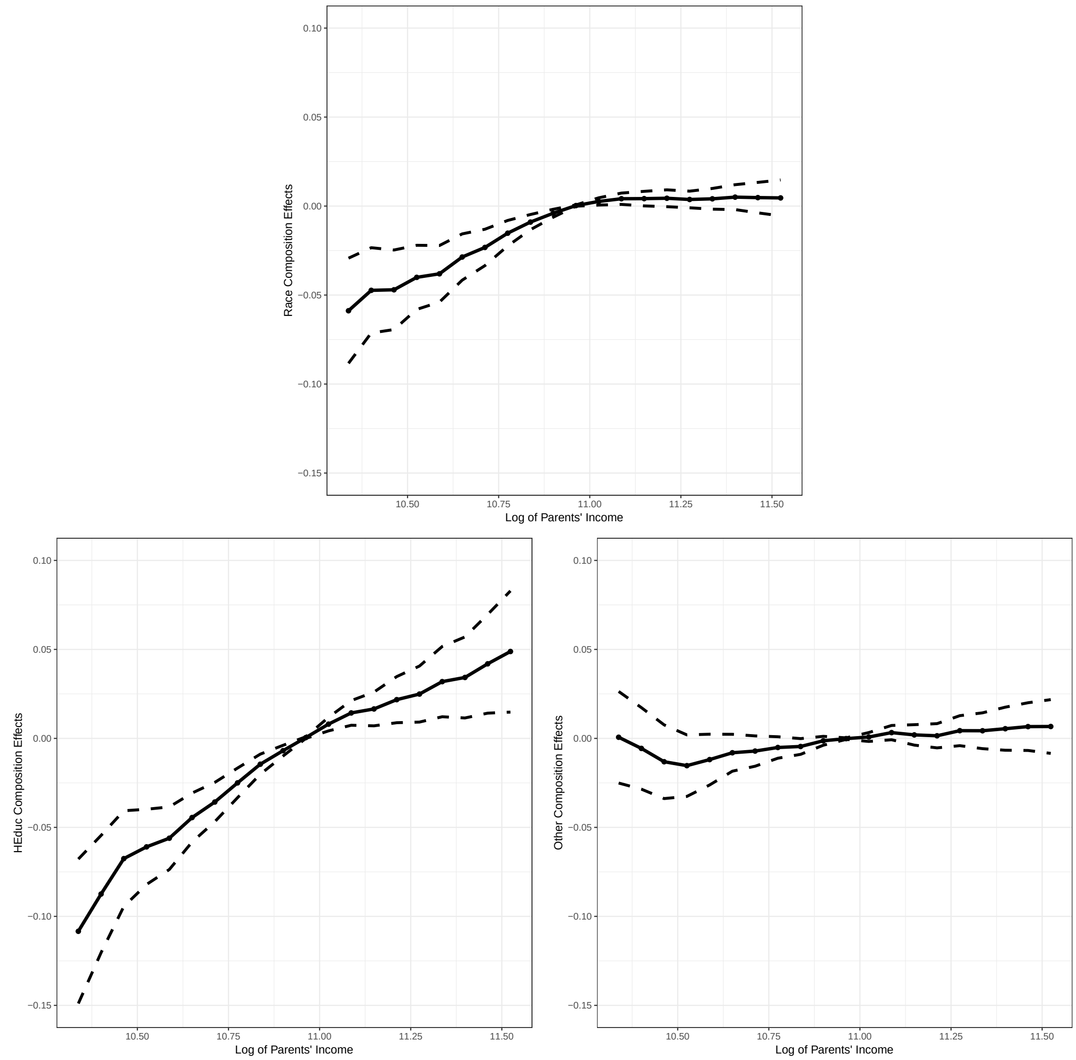
Notes: The top panel plots the aggregate aggregates across parents' incomes. The bottom left papel plots the detailed composition effects across parents' incomes. The bottom right panel plots the detailed structure effects across parents' incomes.

Figure 2: Decomposition across Parents' Incomes



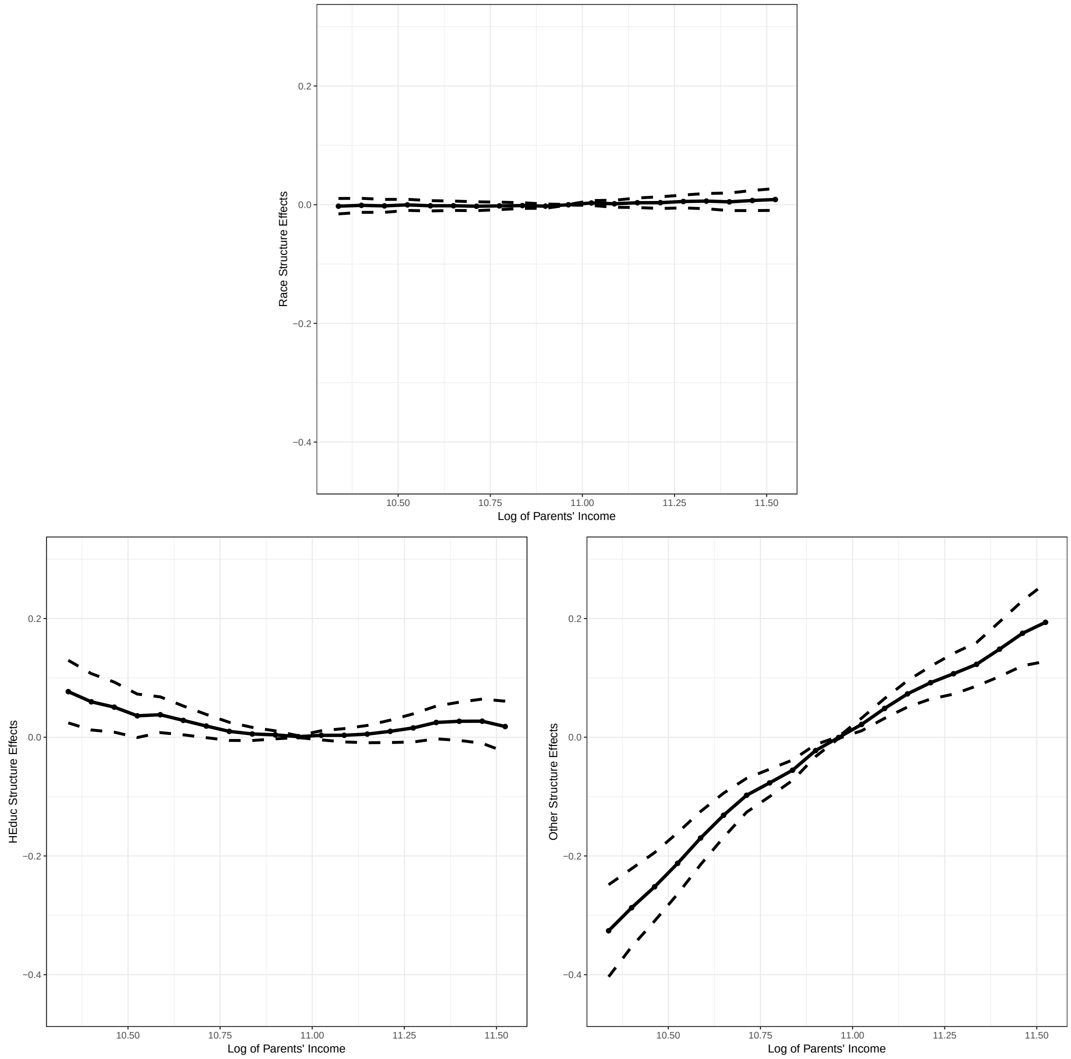
Notes: The top panel plots the overall differences across parents' incomes. The bottom left papel plots the aggregate composition effects across parents' incomes. The bottom right panel plots the aggregate structure effects across parents' incomes.

Figure 3: Decomposition across Parents' Incomes



Notes: The top panel plots the composition effects associated to race across parents' incomes. The bottom left papel plots the composition effects associated to head's education attainment across parents' incomes. The bottom right panel plots the sum of the composition effects associated to other characteristics across parents' incomes.

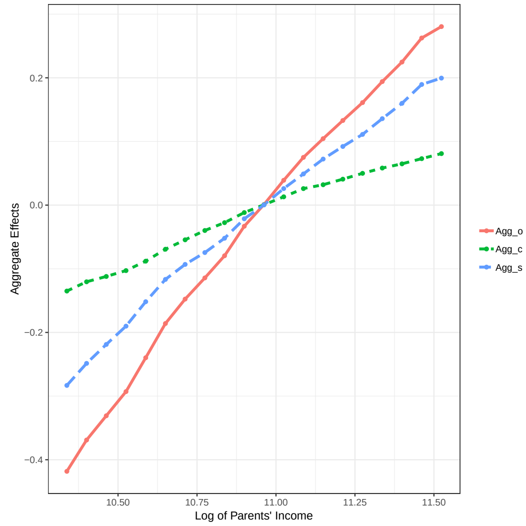
Figure 4: Decomposition across Parents' Incomes



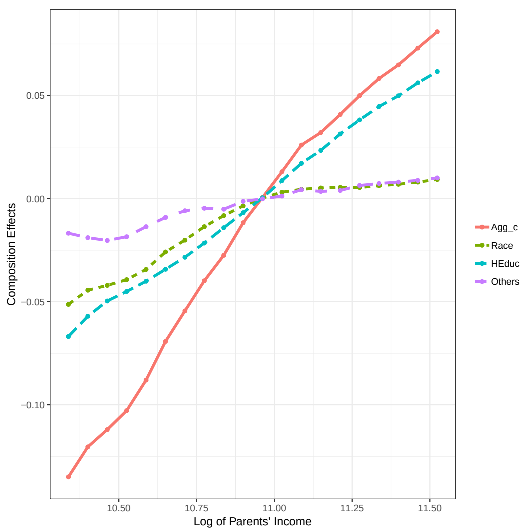
Notes: The top panel plots the structure effects associated to race across parents' incomes. The bottom left papel plots the structure effects associated to head's education attainment across parents' incomes. The bottom right panel plots the sum of the structure effects associated to other characteristics across parents' incomes.

Figure 5 to 8 plot the decomposition results as robustness check in which we switch the reference. More specifically, we use the coefficients from smooth coefficient models in which the we estimate the conditional expectation of child's income conditional on t instead of t_0.

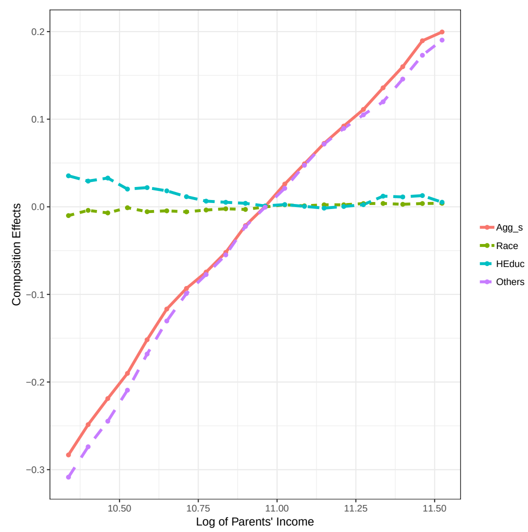
Figure 5: Decomposition across Parents' Incomes



(a) Aggregate effects



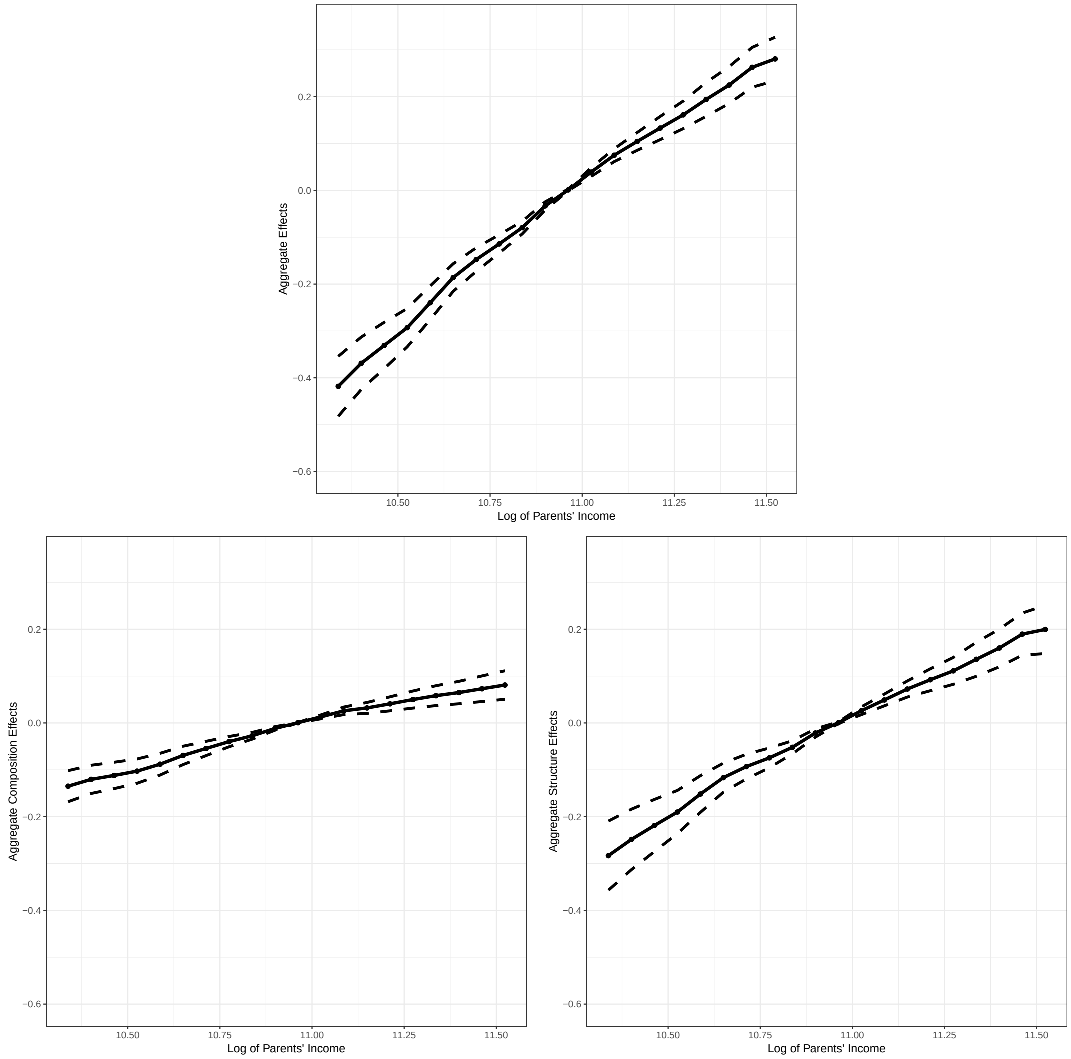
(b) Composition effects



(c) Structure effects

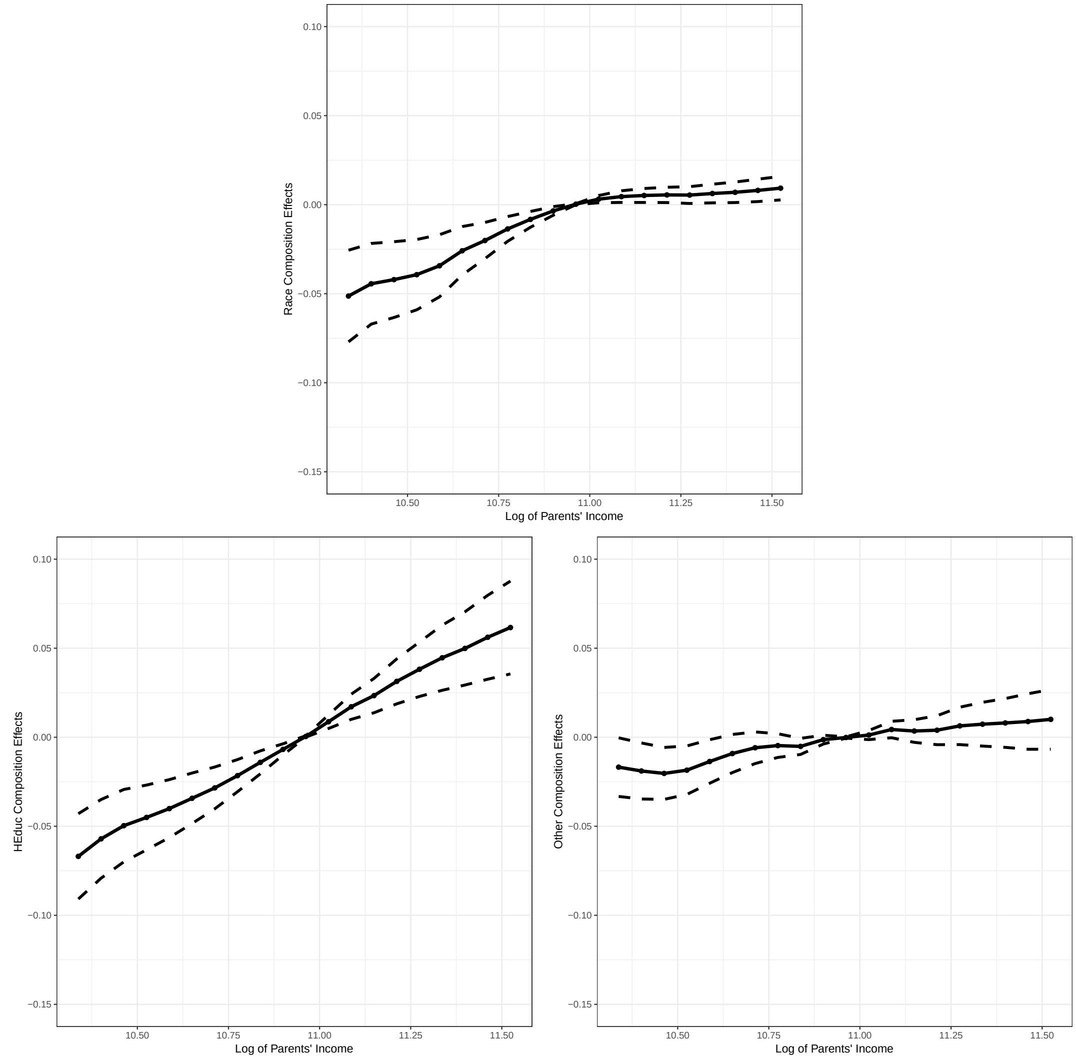
Notes: The top panel plots the aggregate aggregates across parents' incomes. The bottom left papel plots the detailed composition effects across parents' incomes. The bottom right panel plots the detailed structure effects across parents' incomes.

Figure 6: Decomposition across Parents' Incomes



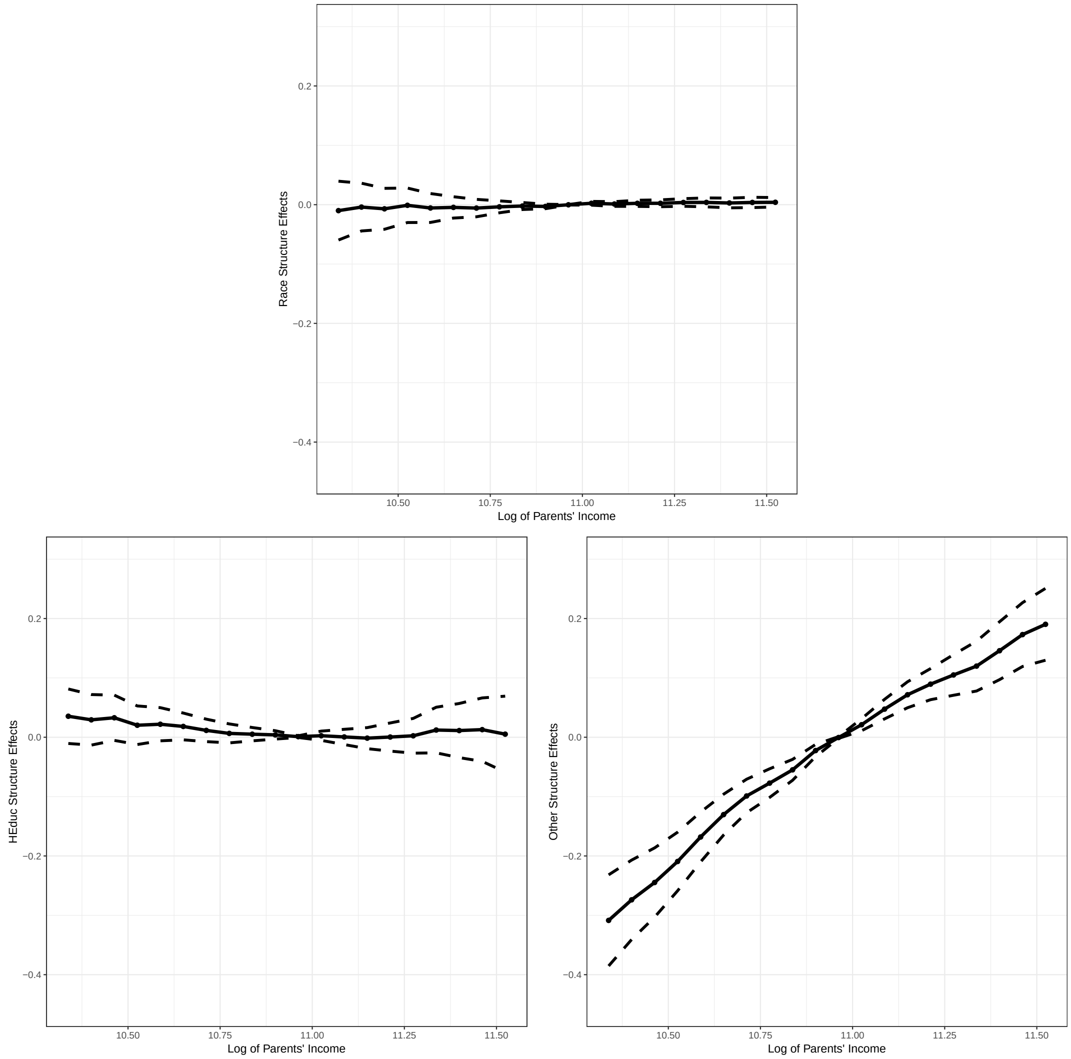
Notes: The top panel plots the overall differences across parents' incomes. The bottom left papel plots the aggregate composition effects across parents' incomes. The bottom right panel plots the aggregate structure effects across parents' incomes.

Figure 7: Decomposition across Parents' Incomes



Notes: The top panel plots the composition effects associated to race across parents' incomes. The bottom left papel plots the composition effects associated to head's education attainment across parents' incomes. The bottom right panel plots the sum of the composition effects associated to other characteristics across parents' incomes.

Figure 8: Decomposition across Parents' Incomes



Notes: The top panel plots the structure effects associated to race across parents' incomes. The bottom left papel plots the structure effects associated to head's education attainment across parents' incomes. The bottom right panel plots the sum of the structure effects associated to other characteristics across parents' incomes.