11/4/2020

Semidefinite Relaxation of Quadratic Optimization Problem (Zhi-Quan Luo)

A CVX CODE FOR SDR

```
cvx_begin
  variable X(n,n) symmetric
  minimize(trace(C*X));
  subject to
    for i=1:p
      trace(A(:,:,i)*X) >= b(i);
    end
    for i=p+1:m
      trace(A(:,:,i)*X) == b(i);
    end
  X == semidefinite(n);
cvx_end
```

cvx代码如上：算法采用计算 代数运算的次数，对于BQP问题，可以采用内点算法，算法复杂度

$$\mathcal{O}(n^{3.5}\log(1/\epsilon))$$

As an illustration, consider the intuitively appealing idea of applying a rank-one approximation on $X^\star$. Specifically, let $r = \mathrm{rank}(X^\star)$, and let

$$X^\star = \sum_{i=1}^{r} \lambda_i q_i q_i^T$$

考虑如上的eigen-decomposition，所以可以定义

$$\tilde{x} = \sqrt{\lambda_1} q_1$$

对于Boolean quadratic program(BQP)问题

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T C x \\ \text{s.t.} \quad & x_i^2 = 1, \ \ i = 1, \ldots, n. \end{aligned} \qquad (2)$$

可以获得通过如下的方法构造可行解

$$\hat{x} = \mathrm{sgn}(\tilde{x})$$

考虑MIMO detection问题，问题描述如下 N-input M-output model

$$y_C = H_C s_C + v_C. \tag{7}$$

Here, $y_C \in \mathbb{C}^M$ is the received vector, $H_C \in \mathbb{C}^{M \times N}$ is the MIMO channel, $s_C \in \mathbb{C}^N$ is the transmitted symbol vector, and $v_C \in \mathbb{C}^M$ is an additive white Gaussian noise vector.

其中信号满足：

$$s_{C,i} \in \{\pm 1 \pm j\}$$

所以该算法等价于零散的最小二乘法（NP-hard problem)

$$\min_{s_C \in \{\pm 1 \pm j\}^N} \|y_C - H_C s_C\|^2, \tag{8}$$

该问题常用 sphere decoding methods去解决，对于N<21的问题非常有用，但是平均下是 exponential in N.

该问题可以在 $\mathcal{O}(N^{3.5})$ 的算法复杂度解决，如果采用SDR方法，将该问题转化为一个实值的齐次QCQP问题：

$$y = \begin{bmatrix} \Re\{y_C\} \\ \Im\{y_C\} \end{bmatrix}, \quad s = \begin{bmatrix} \Re\{s_C\} \\ \Im\{s_C\} \end{bmatrix}, \quad H = \begin{bmatrix} \Re\{H_C\} & -\Im\{H_C\} \\ \Im\{H_C\} & \Re\{H_C\} \end{bmatrix},$$

===》

$$\min_{s \in \{\pm 1\}^{2N}} \|y - Hs\|^2. \tag{9}$$

===》

$$\begin{aligned} \min_{s \in \mathbb{R}^{2N}, t \in \mathbb{R}} \quad & \|ty - Hs\|^2 \\ \text{s.t.} \quad & t^2 = 1, \; s_i^2 = 1, \; i = 1, \ldots, 2N. \end{aligned} \tag{10}$$

===》

$$\begin{aligned} \min_{s \in \mathbb{R}^{2N}, t \in \mathbb{R}} \quad & \begin{bmatrix} s^T & t \end{bmatrix} \begin{bmatrix} H^T H & -H^T y \\ -y^T H & \|y\|^2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \\ \text{s.t.} \quad & t^2 = 1, \; s_i^2 = 1, \; i = 1, \ldots, 2N. \end{aligned} \tag{11}$$

考虑之前的问题，并且提出randomization的方法

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & x^T C x \\
\text{s.t.} \quad & x^T A_i x \trianglerighteq_i b_i, \quad i = 1, \ldots, m.
\end{aligned} \tag{12}
$$

Now, let $X \in \mathbb{S}^n$ be an arbitrary symmetric positive semidefinite matrix. Consider a random vector $\boldsymbol{\xi} \in \mathbb{R}^n$ drawn according to the Gaussian distribution with zero mean and covariance X, or $\xi \sim \mathcal{N}(0, X)$ for short. The intuition of randomization lies in

优化的思想：考虑如下的stochastic QCQP

considering the following stochastic QCQP:

$$
\begin{aligned}
\min_{X \in \mathbb{S}^n, X \succeq 0} \quad & \mathrm{E}_{\xi \sim \mathcal{N}(0,X)}\{\xi^T C \xi\} \\
\text{s.t.} \quad & \mathrm{E}_{\xi \sim \mathcal{N}(0,X)}\{\xi^T A_i \xi\} \trianglerighteq_i b_i, \quad i = 1, \ldots, m,
\end{aligned} \tag{13}
$$

算法如下：可以提供一个quasi-optimal bit -error-rate performance

GAUSSIAN RANDOMIZATION PROCEDURE FOR BQP

given an SDR solution $X^\star$, and a number of randomizations L.

for $\ell = 1, \ldots, L$

generate $\xi_\ell \sim N(0, X^\star)$, and construct a QCQP-feasible point

$$\tilde{x}_\ell = \mathrm{sgn}(\xi_\ell). \tag{15}$$

end

determine $\ell^\star = \arg\min_{\ell=1,\ldots,L} \tilde{x}_\ell^T C \tilde{x}_\ell$.

output $\hat{x} = \tilde{x}_{\ell^\star}$ as the approximate QCQP solution.

然后可以得到该算法和原问题的解的接近程度(randomization之后所得到的结果)

$$
\begin{aligned}
v_{\mathrm{QP}} = \min_{x \in \mathbb{R}^n} \quad & x^T C x \\
\text{s.t.} \quad & x^T A_i x \geq 1, \ i = 1, \ldots, m
\end{aligned}
$$

$$v_{\mathrm{QP}} \leq v(\hat{x}) \leq \gamma v_{\mathrm{QP}}, \tag{18}$$

where $\gamma = 27m^2/\pi$ is the so-called approximation ratio. Notice

该结果是最坏情况下的结果。

考虑SDR的解的逼近情况：

$$v_{\mathrm{SDR}} \le v_{\mathrm{QP}} \le \gamma v_{\mathrm{SDR}}, \tag{19}$$

where $\gamma = 27m^2/\pi$ is as above.

Inhomogeneous problems

Consider a general inhomogeneous QCQP

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & x^T C x + 2c^T x \\ \text{s.t.} \quad & x^T A_i x + 2a_i^T x \trianglerighteq_i b_i, \quad i = 1, \ldots, m \end{aligned} \tag{22}$$

该问题可以齐次化为（增加一个分量）

$$\begin{aligned} \min_{x\in\mathbb{R}^n, t\in\mathbb{R}} \quad & [x^T \quad t] \begin{bmatrix} C & c \\ c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{s.t.} \quad & t^2 = 1, \\ & [x^T \quad t] \begin{bmatrix} A_i & a_i \\ a_i^T & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \end{aligned}$$

Complex-valued Problems

Consider a general complex-valued homogeneous QCQP

$$\begin{aligned} \min_{x\in\mathbb{C}^n} \quad & x^H C x \\ \text{s.t.} \quad & x^H A_i x \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \end{aligned} \tag{23}$$

where $C, A_1, \ldots, A_m \in \mathbb{H}^n$, with $\mathbb{H}^n$ being the set of all complex $n \times n$ Hermitian matrices. Using the same SDR idea as in the

该问题可以使用SDR变为下列问题：

$$\begin{aligned} \min_{X\in\mathbb{H}^n} \quad & \mathrm{Tr}(CX) \\ \text{s.t.} \quad & \mathrm{Tr}(A_i X) \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \\ & X \succcurlyeq 0, \end{aligned} \tag{24}$$

Complex value problem可以用来解决k-ary quadratic program，如下：

$$\begin{aligned} \min_{x\in\mathbb{C}^n} \quad & x^H C x \\ \text{s.t.} \quad & x_i \in \{1, e^{j2\pi/k}, \ldots, e^{j2\pi(k-1)/k}\}, i = 1, \ldots, n, \end{aligned} \tag{25}$$

上面的k-ary quadratic program可以用下面的问题来近似：（理论和实际效果都还可以）

$$\begin{aligned} \min_{X \in \mathbb{H}^n} \quad & \mathrm{Tr}(CX) \\ \text{s.t.} \quad & X \succeq 0, \; X_{ii} = 1, \; i = 1, \ldots, n. \end{aligned} \tag{26}$$

Separable QCQPs

定义为具有如下形式的QCQP：

Consider a QCQP of the form

$$\begin{aligned} \min_{x_1, \ldots, x_k \in \mathbb{C}^n} \quad & \sum_{i=1}^{k} x_i^H C_i x_i \\ \text{s.t.} \quad & \sum_{l=1}^{k} x_l^H A_{i,l} x_l \trianglerighteq_i b_i, \quad i = 1, \ldots, m. \end{aligned} \tag{27}$$

该问题的SDR版本为：

Let $X_i = x_i x_i^H$ for $i = 1, \ldots, k$. By relaxing the rank constraint on each X_i, we obtain the following SDR of (27):

$$\begin{aligned} \min_{X_1, \ldots, X_k \in \mathbb{H}^n} \quad & \sum_{i=1}^{k} \mathrm{Tr}(C_i X_i) \\ \text{s.t.} \quad & \sum_{l=1}^{k} \mathrm{Tr}(A_{i,l} X_l) \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \\ & X_1 \succeq 0, \ldots, X_k \succeq 0. \end{aligned} \tag{28}$$

Semidefinite Problem中的Rand Reduction,步骤如下：

- formulate the problem as a rank-constrained SDP
- remove the rank constraint to obtain a SDP
- extract a feasible solution of the original rank-constrained SDP

关于SDP问题最优解的rank的一些结果：

（Shapiro-Barvinok-Pataki)对于下列的问题：如果该问题有解

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & \mathrm{Tr}(CX) \\ \text{s.t.} \quad & \mathrm{Tr}(A_i X) \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \\ & X \succeq 0. \end{aligned} \tag{6}$$

则存在最优解满足：, $\mathrm{rank}\,(X^\star) \le \lfloor (\sqrt{8m+1} - 1)/2 \rfloor.$

（Huang and Palomar)对于下列的问题，如果该问题有解

Consider a general complex-valued homogeneous QCQP

$$\begin{aligned} \min_{x \in \mathbb{C}^n} \quad & x^H C x \\ \text{s.t.} \quad & x^H A_i x \trianglerighteq_i b_i, \quad i = 1, \ldots, m, \end{aligned} \tag{23}$$

where $C, A_1, \ldots, A_m \in \mathbb{H}^n$, with $\mathbb{H}^n$ being the set of all complex $n \times n$ Hermitian matrices. Using the same SDR idea as in the

则存在最优解满足： $\operatorname{rank}(X^\star) \leq \sqrt{m}.$

（[27])对于complex-valued separable QCQPs:如果该问题有解

Consider a QCQP of the form

$$\begin{aligned} \min_{x_1, \ldots, x_k \in \mathbb{C}^n} \quad & \sum_{i=1}^{k} x_i^H C_i x_i \\ \text{s.t.} \quad & \sum_{l=1}^{k} x_l^H A_{i,l} x_l \trianglerighteq_i b_i, \quad i = 1, \ldots, m. \end{aligned} \tag{27}$$

则存在最优解，满足：

sible. Then, as shown in [16], there exists a solution $\{X_i^\star\}_{i=1}^k$ to (28) whose ranks satisfy

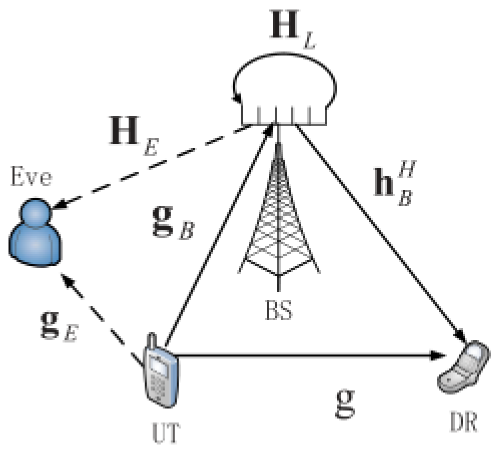
$$\sum_{i=1}^{k} \operatorname{rank}(X_i^\star)^2 \leq m.$$

对于real-valued separable QCQPs:如果该问题有解，则存在解满足：

$$\sum_{i=1}^{k} \frac{\operatorname{rank}(X_i^\star)(\operatorname{rank}(X_i^\star) + 1)}{2} \leq m.$$

Robust Beamforming and Jamming for Enhancing the Physical Layer Security of Full Duplex Radios (Zhengmin Kong)

模型： FD cellular system, 如下图所示：



- BS: base station (Full duplex base station) 作用：downlink transmission
- DR: 单天线 legitimate receiver
- UT : 单天线 legitimate UT 作用：uplink transmission
- Eve: 多天线 Eve

Eve的信息： N_E receiving antennas,

BS的信息： N_T transmitting antennas and N_R receiving antennas t

假设： $N_T \geq N_E + 1,$

- downlink的信息
  - from BS to Eve channel information: $\mathbf{H}_E \in \mathbb{C}^{N_E \times N_T}$
  - from BS to DR channel information: $\mathbf{h}_B^H \in \mathbb{C}^{1 \times N_T}$
  - residual self-interference channel: $\mathbf{H}_L \in \mathbb{C}^{N_R \times N_T}$
  - BS transmits the confidential signal plus AN: $\mathbf{x}_b \triangleq \mathbf{v}s + \mathbf{n}_a,$
  - total available power of the BS: $P_{\text{tot}}.$
  - 功率限制： $$\text{tr}\left(\mathbf{v}\mathbf{v}^H + \Omega\right) \leq P_{\text{tot}}. \quad (2)$$
- uplink的信息
  - UT transmits the confidential signal: $z \sim \mathcal{CN}(0, 1)$
  - UT transmits the signal: power $P_t.$
  - BS, DR, Eve所接收到的信号：

$$y_B = \mathbf{r}^H \left(\sqrt{P_t}\mathbf{g}_B z + \mathbf{H}_L \mathbf{x}_b + \mathbf{n}_B\right), \quad (3)$$

$$y_D = \mathbf{h}_B^H \mathbf{x}_b + \sqrt{P_t} g z + n_D, \quad (4)$$

$$\mathbf{y}_E = \mathbf{H}_E \mathbf{x}_b + \sqrt{P_t}\mathbf{g}_E z + \mathbf{n}_E, \quad (5)$$

其中perfect state information $\mathbf{H}_L$ 对于BS未知,其中 采用maximum ration combining receiver (MRC),而不是minimum mean-square error (MMSE) receiver，也就是采用 $\mathbf{r} = \frac{\mathbf{g}_B}{\|\mathbf{g}_B\|_F},$

Channel State Information Model

假设BS只能获得 imperfect estimate of channel state information:

$$\mathbf{H}_E = \bar{\mathbf{H}}_E + \Delta\mathbf{H}_E, \quad \mathbf{g}_E = \bar{\mathbf{g}}_E + \Delta\mathbf{g}_E, \tag{6}$$

对于CSI估计的不确定性，我们采用deterministic uncertainty model.

$$\left\{\mathbf{H}_L : ||\mathbf{H}_L||_F \le \delta_{\mathbf{H}_L}\right\}, \xi_{\mathbf{H}_E} \triangleq \left\{\Delta\mathbf{H}_E : ||\Delta\mathbf{H}_E||_F \le \delta_{\mathbf{H}_E}\right\}$$

以及：

$$\xi_{\mathbf{g}_E} \triangleq \left\{\Delta\mathbf{g}_E : ||\Delta\mathbf{g}_E||_F \le \delta_{\mathbf{g}_E}\right\}$$

则问题可以转化为优化为：

$$\begin{aligned}
\max_{\mathbf{v},\Omega\succeq\mathbf{0}} \min_{\Delta\mathbf{H}_E\in\xi_{\mathbf{H}_E},\mathbf{H}_L\in\xi_{\mathbf{H}_L},\Delta\mathbf{g}_E\in\xi_{\mathbf{g}_E}} & \ln(1+\eta_1)+\ln(1+\eta_2) \\
& -\ln\det\left(\mathbf{I}_{N_E}+\mathbf{Z}\mathbf{N}^{-1}\right) \\
\text{s.t. } & \operatorname{rank}(\mathbf{Q}) = 1, \\
& \operatorname{tr}(\mathbf{Q}+\Omega) \le P_{\text{tot}},
\end{aligned} \tag{7}$$

where $\eta_1 \triangleq \frac{\mathbf{h}_B^H\mathbf{Q}\mathbf{h}_B}{P_t|g|^2+\mathbf{h}_B^H\Omega\mathbf{h}_B+1}$, $\eta_2 \triangleq \frac{P_t||\mathbf{g}_B||_F^2}{\mathbf{r}^H\mathbf{H}_L(\mathbf{Q}+\Omega)\mathbf{H}_L^H\mathbf{r}+1}$, $\mathbf{Z} \triangleq \mathbf{H}_E\mathbf{Q}\mathbf{H}_E^H + P_t\mathbf{g}_E\mathbf{g}_E^H$, $\mathbf{N} \triangleq \mathbf{H}_E\Omega\mathbf{H}_E^H + \mathbf{I}_{N_E}$.