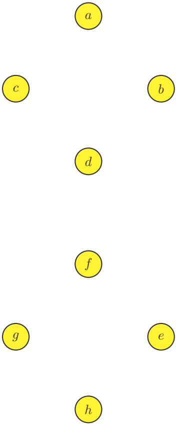
Matchings and Assignments

Endre Boros
26:711:653: Discrete Optimization

February 19, 2018

Matchings and augmenting paths



- A graph $G = (V, E)$ on 8 vertices.
- A matching: $\mathbf{M} = \{(b, d), (c, g), (e, h)\}$
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- An improved matching: $\mathbf{M}' = \mathbf{M} \triangle \mathbf{P}$.
- For any matching M and any M-augmenting path P we have

$$|M \triangle P| = |M| + 1$$

- Berge (1957): A matching M is maximum iff there exists no M-augmenting path.

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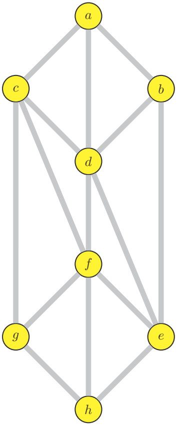

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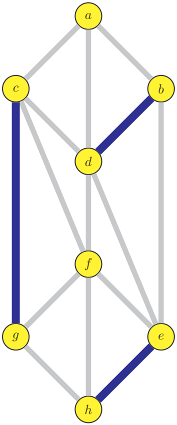

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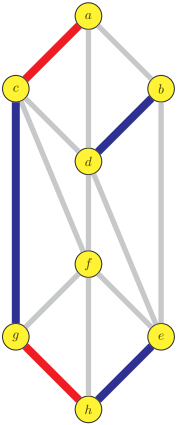

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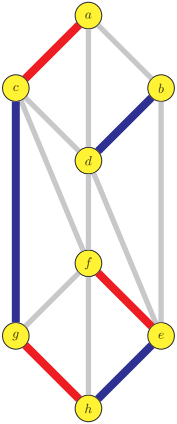

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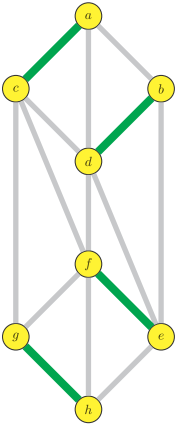

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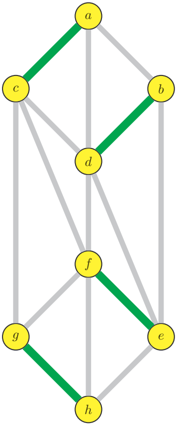

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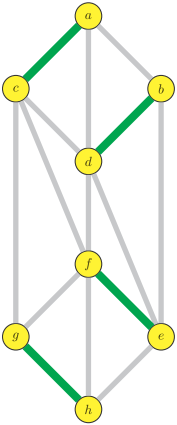

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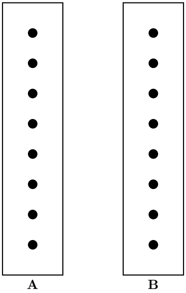

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- with edges $E \subseteq A \times B$.
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- **Then** G does not have edges between $(\mathbf{V} \cup \mathbf{W})$ and $(\mathbf{X} \cup \mathbf{Z})$.
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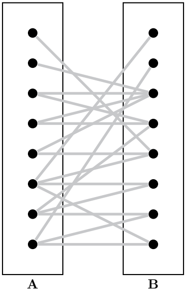

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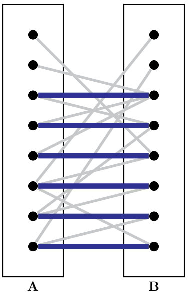

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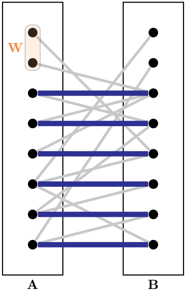

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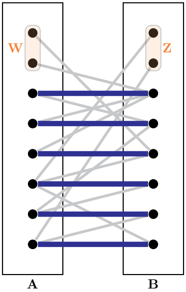

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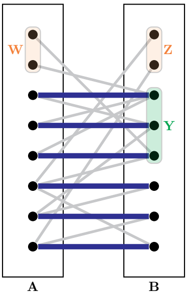

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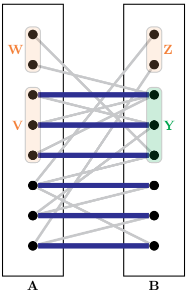

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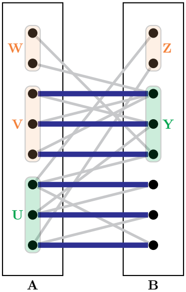

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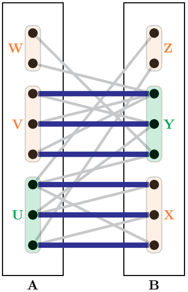

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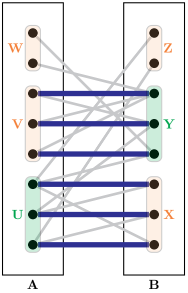

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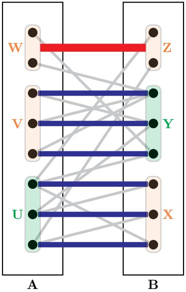

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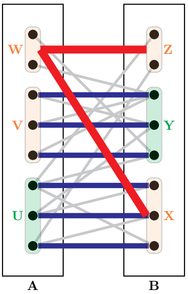

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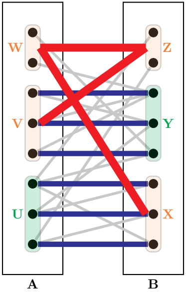

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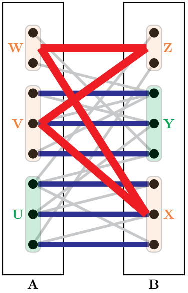

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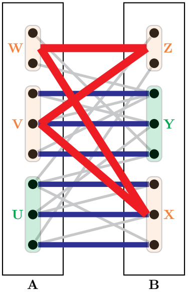

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