Maximum Matchings continued ...

Given a graph $G = (V, E)$, let us orient its edges in an arbitrary way, and let G' denote the obtained directed graph. Let further x_e, $e \in E$ be real variables associated with the edges of G, and let us define a matrix $T_G(x) \in \mathbb{R}^{V \times V}$ as follows:

$$t_{uv}(x) = \begin{cases} x_e & \text{if } e = (u, v) \text{ and } (\overrightarrow{u, v}) \in E(G') \\ -x_e & \text{if } e = (u, v) \text{ and } (\overrightarrow{v, u}) \in E(G') \\ 0 & \text{otherwise,} \end{cases}$$

for all $u, v \in V$. Then, $\det T_G(x)$ is a polynomial of the variables x_e, $e \in E$.

Theorem 1 (Tutte (1947)) *A graph G has a perfect matching if and only if $\det T_G(x)$ is not identically zero.*

Proof. Let us denote by S_V the set of permutations of V, and let $sg(\pi)$ denote the sign of permutation $\pi \in S_V$. Then,

$$\det T_G(x) = \sum_{\pi \in S_V} (-1)^{sg(\pi)} \prod_{v \in V} t_{v,\pi(v)}.$$

Each non-vanishing term in this expansion corresponds to a set of edges G, namely for $\pi \in S_V$ we have $F_\pi = \{(v, \pi(v)) \mid v \in V\}$. Let us first note that if $M = \{(u_i, v_i) \mid i = 1, .., n/2\} \subseteq E$ is a perfect matching, then $\pi(u_i) = v_i$ and $\pi(v_i) = u_i$ is a permutation for which $F_\pi = M$. The corresponding term is $(-1)^{n/2} \prod_{e \in M} x_e^2$, and since x_e appear only twice in M for each $e \in E$, this term cannot be canceled out by any other term of the expansion, hence $\det T_G(x) \not\equiv 0$.

On the other hand, if G does not contain a perfect matching, then each set F_π must contain an odd cycle, oriented naturally by π. Reversing this orientation, we get the same set of edges corresponding to another permutation π', such that the corresponding terms cancel out. Thus, all terms of $\det T_G(x)$ cancel out. $\square$

Theorem 2 (Lovász (1979)) *Let x_e, $e \in E$ be i.i.d. random variables, uniform in $[0, 1]$. Then, $rankT_G(x) = 2\nu(G)$ with probability 1.* $\square$

This result leads to a simple, randomized matching algorithm!!

Edmonds' matching algorithm

Given a graph $G = (V, E)$ and a subset $X \subseteq V$ of its vertices, let $q_G(X)$ denote the *number of odd connected components* in the subgraph of G induced by $V \setminus X$.

Lemma 1 *Let $G = (V, E)$ be a graph, and $X \subseteq V$ be a set of vertices. Then, any matching M in G must leave at least $q_G(X) - |X|$ vertices uncovered.*

Proof. Let $M \subseteq E$ be a maximum matching in G, $|M| = \nu(G)$. Then, in every odd component of the subgraph induced by $V \setminus X$ either we have a vertex uncovered by M, or a vertex covered by an edge of M, the other endpoint of which is in X. Since these other endpoints are pairwise different (M is a matching), the lemma follows. □

Sketch of Edmonds' blossom-shrinking algorithm (1965):

Given a graph $G = (V, E)$ and a matching $M \subseteq E$ in it, we try to find an M-augmenting path, or a proof that such a path does not exists.

We "grow" $|V \setminus V(M)|$ distinct M-alternating trees, i.e., rooted trees in which the root vertices are the ones not covered by M, and in which all unique paths connecting a vertex to the root of its tree are M-alternating. (In particular, all non-root vertices are covered by M.) Initially we start with the vertices $V \setminus V(M)$ as roots of distinct M-alternating trees, and no edges in this M-alternating forest F. We call every second vertex in an M-alternating tree, starting with the root, an *outer* vertex, while all other vertices will be called *inner*.

In a general step we choose an outer vertex x of the forest F, and a neighbor y of it (i.e. $(x, y) \in E$), and check the following cases:

Growing: If y is not a vertex (yet) of the forest F, then we add to the forest the edge (x, y) together with the edge of M covering y.

Augmenting: If y is an outer vertex of F, but belongs to a tree of F different from the one containing x, then the paths connecting x and y to the roots of their respective M-alternating trees, together with (x, y) form an M-augmenting path. (Then we can augment M, and start this procedure from scratch.)

Blossom shrinking: If y is an outer vertex in the same M-alternating tree of F to which x belongs to, then the paths connecting x and y to the root of this tree converge at some vertex r, and the path segments till r together with (x, y) form a blossom, which we shrink into one vertex.

We keep doing the above, as long as there are edges connecting two outer vertices. Finally we may arrive to a situation, when none of the above applies for all the outer vertices, i.e., when all outer vertices have all their neighbors as inner.

Theorem 3 *Either M is augmented in the above procedure, or M is a maximum matching.*

Proof. Proof follows by Lemma 1. Choose X to be the set of inner vertices at the end. By removing X we have every outer vertex in distinct odd components (each outer vertex u represent a set of vertices which from which the repeated blossom shrinking produced u; each of those steps kept the cardinality of the set of vertices corresponding to u odd). Each of the these odd components are connected by one matching edge to X, except the root components, in which exactly one edge is not covered by M. □

Theorem 4 (Gallai-Edmonds structure theorem: Gallai (1964)) *Let $G = (V, E)$ be an arbitrary graph, denote by Y the set of vertices not covered by at least one maximum matching of G, by $X \subseteq V \setminus Y$ the neighbors of Y, and by W the rest of the vertices. Then we have the following:*

(a) Any maximum matching of G contains a perfect matching of $G[W]$, an almost perfect matching of each of the connected components of $G[Y]$, and matches all vertices of X into distinct connected components of $G[Y]$.

(b) The connected components of $G[Y]$ are factor-critical*, i.e., they have a perfect matching after the removal of any of their vertices.*

(c) $2\nu(G) = |V| - q_G(X) + |X|$.

Proof. Choose X to be the set of inner vertices, Y to be the set of vertices shrunk to the outer vertices, and let W be the rest. Then, every maximum matching M misses the same number of $|V \setminus V(M)|$ vertices, which in case of the matching at the end of the procedure is the set of roots of the M-alternating trees of cardinality $q_G(X) - |X|$, proving (c). Since every odd component of the graph induced by $V \setminus X$ must have at least one vertex not covered by a matching edge inside this odd component in any maximum matching, property (a) follows readily. Finally, picking any vertex u in any of the (odd) components of Y, u belongs to one of the trees in the final M-augmenting tree, and in this tree there is an M-augmenting path connecting u to the root. Replacing non-matching edges in this path with matching edges, we get another maximum matching, so that the same trees will form an M-alternating tree representation, but with u as a root vertex. Thus, (b) follows. $\square$

Theorem 5 (Berge (1958)) *Given a graph $G = (V, E)$, we have*

$$2\nu(G) \ + \ \max_{X \subseteq V}(q_G(X) - |X|) \ = \ |V|.$$

Proof. By Lemma 1 we have $|V \setminus V(M)| = |V| - 2\nu(G) \geq q_G(X) - |X|$ for any maximum matching M and any subset $X \subseteq V$.

Choosing X as in the Gallai-Edmonds Structure Theorem, (c) shows that equality can be attained, completing the proof. $\square$

Theorem 6 (Tutte (1947)) *A graph $G = (V, E)$ has a perfect matching if an only if*

$$q_G(X) \ \leq \ |X| \quad \textit{for all subsets} \quad X \subseteq V.$$

Proof. By Lemma 1 we have $|V \setminus V(M)| = |V| - 2\nu(G) \geq q_G(X) - |X|$ for any maximum matching M and any subset $X \subseteq V$ Thus, if M is a perfect matching in G, we get $0 \geq q_G(X) - |X|$ for every subset $X \subseteq V$. For the converse direction, assume that $q_G(X) > |X|$ for a subset X. Then $\max_{X \subseteq V} q_G(X) - |X| > 0$, and thus by Theorem 5 we get $|V| - 2\nu(G) > 0$, proving that G cannot have a perfect matching. $\square$

Matching Polyhedra

Let us associate to a graph $G = (V, E)$ a polyhedron

$$P_G = \left\{ \mathbf{x} \in \mathbb{R}_+^E \;\middle|\; \sum_{u \in N(v)} x_{uv} \leq 1 \quad \text{for all } v \in V \right\}.$$

Clearly, $P_G \cap \mathbb{Z}^E$ contains characteristic vectors of matchings of G. P_G is called the *fractional matching polytope* of G.

Theorem 7 (Hoffman and Kruskal (1956), Heller and Tompkins (1956)) *If G is bipartite, then P_G is integral.*

Proof. For the proof we use the following definition of a vertex (of a convex polyhedron): $\mathbf{v} \in P_G$ is a vertex of P_G if and only if there are no vectors $\mathbf{u}, \mathbf{w} \in P_G$, $\mathbf{u} \neq \mathbf{w}$ such that $\mathbf{v} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$.

Let us now consider an arbitrary vector $\mathbf{x} = (x_e \mid e \in E) \in P_G$, and define $F(\mathbf{x}) = \{e \in E \mid 0 < x_e < 1\}$. We show that if $F \neq \emptyset$, then there are vectors $\mathbf{y}, \mathbf{z} \in P_G$, $\mathbf{y} \neq \mathbf{z}$ such that $\mathbf{x} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}$, which by the above definition of a vertex proves that all vertices of P_G have binary coordinates.

Assume now that $F \neq \emptyset$ and that it contains a cycle $C \subseteq F$. Since G is bipartite, C has an even number of edges. Let us label the edges by integers 1, ..., $|C|$ as we follow these edges along the cycle C, and let $O \subseteq C$ be those edges that have an odd label. Let us choose a tiny $\epsilon > 0$, and define $\mathbf{y} = (y_e \mid e \in E)$ and $\mathbf{z} = (z_e \mid e \in E)$ by

$$y_e = \begin{cases} x_e & \text{if } e \in E \setminus C, \\ x_e + \epsilon & \text{if } e \in O, \\ x_e - \epsilon & \text{if } e \in C \setminus O, \end{cases}$$

and

$$z_e = \begin{cases} x_e & \text{if } e \in E \setminus C, \\ x_e - \epsilon & \text{if } e \in O, \\ x_e + \epsilon & \text{if } e \in C \setminus O. \end{cases}$$

Then we have $\mathbf{y}, \mathbf{z} \in P_G$, $\mathbf{y} \neq \mathbf{z}$ (think it over!), and $\mathbf{x} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}$.

If $F \neq \emptyset$ and it does not contain a cycle, then it must contain a maximal path $P \subseteq F$ between two nodes $a, b \in V$ of the graph. Note that by the maximality of P within F we must have

$$\sum_{u \in N(a)} x_{au} < 1 \text{ and } \sum_{u \in N(b)} x_{bu} < 1.$$

Let us label the edges of P by integers, 1, ..., $|P|$ as we follow these edges along the path P, and define $O \subseteq P$ be those edges of P that have odd label. Let us choose again a tiny $\epsilon > 0$, and define $\mathbf{y} = (y_e \mid e \in E)$ and $\mathbf{z} = (z_e \mid e \in E)$ by

$$y_e = \begin{cases} x_e & \text{if } e \in E \setminus P, \\ x_e + \epsilon & \text{if } e \in O, \\ x_e - \epsilon & \text{if } e \in P \setminus O, \end{cases}$$

and

$$z_e = \begin{cases} x_e & \text{if } e \in E \setminus P, \\ x_e - \epsilon & \text{if } e \in O, \\ x_e + \epsilon & \text{if } e \in P \setminus O. \end{cases}$$

Then we have $\mathbf{y}, \mathbf{z} \in P_G$, $\mathbf{y} \neq \mathbf{z}$ (think it over!), and $\mathbf{x} = \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}$.

This completes the proof of the theorem. □

Theorem 8 (Balinski (1970)) *For any graph G, the vertices of P_G are half-integral.*

Proof. ... try also with the $\pm\epsilon$ technique, as above. □

Let us associate to a graph $G = (V, E)$ another polyhedron, S_G defined by

$$S_G = \{\mathbf{x} \in \mathbb{R}_+^E \mid x_{uv} + x_{uw} \leq 1 \quad \text{for all } u \in V \text{ and for all } v, w \in N(u),\ v \neq w\}.$$

Clearly, $S_G \cap \mathbb{Z}^E$ contains also exactly the characteristic vectors of matchings of G, and clearly $P_G \subset S_G$.

Theorem 9 (Nemhauser and Trotter (1974)) *The vertices of S_G are also half-integral. Furthermore, if $\widehat{\mathbf{x}}$ is an optimal solution to the LP $\max_{\mathbf{x} \in S_G} \mathbf{c}^T\mathbf{x}$, then there exists an optimal solution $\mathbf{x}^*$ to the IP $\max_{\mathbf{x} \in S_G \cap \mathbb{Z}^E} \mathbf{c}^T\mathbf{x}$ for which the following implication holds for all $(u, v) \in E$:*

$$\textit{If } \widehat{x}_{uv} \in \{0, 1\} \textit{ then } x^*_{uv} = \widehat{x}_{uv}.$$

Proof. ... $\pm\epsilon$ technique + ... in any maximum weight matching we can switch to the edges $(u,v) \in E$ for which $\widehat{x}_{uv} = 1$ by the LP optimality of $\widehat{\mathbf{x}}$. $\square$

Given a graph $G = (V,E)$ and vertex $v \in V$, let us denote by $\delta(v) = \{(u,v) \mid u \in V,\ (u,v) \in E\}$ and for a subset $S \subseteq V$ by $\delta(S) = E \cap S \times (V \setminus S)$ the boundary set of edges. For a vector $x \in \mathbb{R}^E$ and subset $F \subseteq E$ we use $x(F) = \sum_{e \in F} x_e$.

Let us further associate to $G = (V,E)$ the following polyhedron

$$Q_G = \left\{ \mathbf{x} \in \mathbb{R}^E \;\middle|\; \begin{array}{ll} 0 \leq x_{uv} \leq 1 & \text{for all } (u,v) \in E \\ x(\delta(v)) = 1 & \text{for all } v \in V \\ x(\delta(S)) \geq 1 & \text{for all } S \subseteq V, |S| \geq 3, |S| \text{ odd} \end{array} \right\}$$

called the *perfect matching polytope* of G.

Theorem 10 (Edmonds (1965)) *For every graph Q_G is integral.*

Proof. See Schrijver's 1983 proof. $\square$

Lemma 2 *Assume that for a vector $x \in Q_G$ and a subset $\emptyset \neq S \subseteq V$ we have $|S|$ even, and $x(\delta(S)) < 1$. Then, for every vertex $u \in S$ we have $x(\delta(S \setminus \{u\})) < 1$.*

Proof. Assume indirectly that $x(\delta(S \setminus \{u\})) \geq 1$. Denote by $F \subseteq \delta(v)$ the set of edges $F = E \cap (S \setminus \{u\} \times \{u\})$. Then we have

$$1 > x(\delta(S)) = x(\delta(S \setminus \{u\})) + x(\delta(v)) - x(F) \geq 2 - x(F)$$

where $x(F) \leq 1$ since $F \subseteq \delta(v)$. Thus we get $2 > 2$, a contradiction proving our claim. $\square$

Lemma 3 *Given $x^* \in \mathbb{R}^E$ satisfying $0 \leq x^*_{uv} \leq 1$ for all $(u,v) \in E$ and $x^*(\delta(v)) = 1$ for all $v \in V$ we can find in polynomial time a nonempty subset $S \subsetneq V$ such that $x^*(\delta(S)) < 1$, or prove that $x^* \in Q_G$.*

Proof. Let as introduce $z \in \{0,1\}^V$ and define

$$g(z) = \sum_{(u,v)\in E} x^*_{uv} \cdot (z_u \bar{z}_v + \bar{z}_u z_v),$$

where $\bar{z}_u = 1 - z_u$. Then for every subset $S \subseteq V$ we have $x^*(\delta(S)) = g(\chi(S))$. Furthermore,

$$\min_{z\in\{0,1\}^V} g(z)$$

is a submodular quadratic minimization problem, solvable in polynomial time by solving an associated min-cut problem (Hammer 1965). Thus, the problem of

$$\min_{\emptyset\neq S\subsetneq V} x^*(\delta(S))$$

is solvable by fixing in all possible ways one of the z variables at 1 and one at 0, and then minimizing $g(z)$ for the remaining variables. If we find that the minimum value of g is always at least 1, then $x^* \in Q_G$. Otherwise we find a nontrivial subset S for which $x^*(\delta(S)) < 1$, and thus by Lemma 3, we also find such an odd set. $\square$