Disjunctive Programming/Lift and Project Methods for IP

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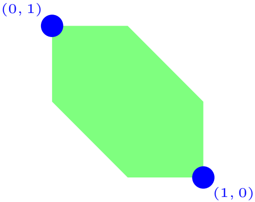
26:711:653: Discrete Optimization

Spring 2019

Disjunctive Programming (Balas, 1975)



- Consider the polyhedron $\mathbf{P}$ defined by the inequalities

$$\mathbf{P} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{rrl} x_1 & +x_2 & \geq \frac{1}{2} \\ x_1 & +x_2 & \leq \frac{3}{2} \\ x_1 & & \leq 1 \\ & x_2 & \leq 1 \\ x_1, & x_2 & \geq 0 \end{array} \right\}$$

- Consider

$$\mathbf{Q}_0 = \{(x_1, x_2) \mid x_1 = 0\} \quad \text{and} \quad \mathbf{Q}_1 = \{(x_1, x_2) \mid x_2 = 1\}$$

- Then we have

$$\mathbf{P_I} \subseteq \mathbf{P}^{(x_1)} = \operatorname{conv}((\mathbf{Q}_0 \cap \mathbf{P}) \cup (\mathbf{Q}_1 \cap \mathbf{P})) \subseteq \mathbf{P}$$

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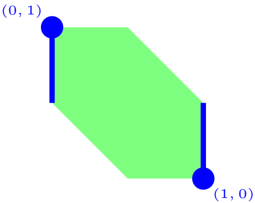
Disjunctive Programming (Balas, 1975)



- Consider the polyhedron $\mathbf{P}$ defined by the inequalities

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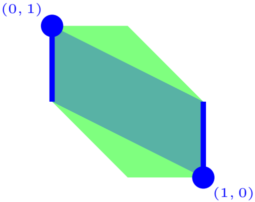

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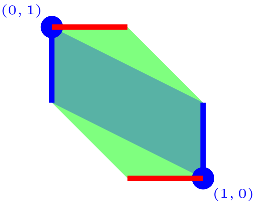

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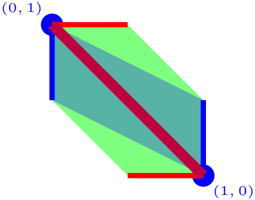

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Lift-and-Project (Balas and Ceria, 1991)

- Consider the polyhedron defined by

$$\begin{array}{rrcl}
x_1 & +x_2 & \geq & \frac{1}{2} \\
x_1 & +x_2 & \leq & \frac{3}{2} \\
x_1 & & \leq & 1 \\
 & x_2 & \leq & 1 \\
x_1 & & \geq & 0 \\
 & x_2 & \geq & 0
\end{array}$$

- Let us multiply all inequalities by $t_1(x) = x_1$ and $t_2(x) = 1 - x_1$ ...

$$\begin{array}{rrcl}
x_1^2 & +x_1x_2 & \geq & \frac{1}{2} \cdot x_1 \\
x_1^2 & +x_1x_2 & \leq & \frac{3}{2} \cdot x_1 \\
x_1^2 & & \leq & x_1 \\
 & x_1x_2 & \leq & x_1 \\
x_1^2 & & \geq & 0 \\
 & x_1x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{rrcl}
x_1(1-x_1) & +x_2(1-x_1) & \geq & \frac{1}{2}(1-x_1) \\
x_1(1-x_1) & +x_2(1-x_1) & \leq & \frac{3}{2}(1-x_1) \\
x_1(1-x_1) & & \leq & (1-x_1) \\
 & x_2(1-x_1) & \leq & (1-x_1) \\
x_1(1-x_1) & & \geq & 0 \\
 & x_2(1-x_1) & \geq & 0
\end{array}$$

$$\begin{array}{rrl}
x_1 & +(1-x_1) & = 1 \\
x_1^2 & +x_1(1-x_1) & = x_1 \\
x_1x_2 & +x_2(1-x_1) & = x_2
\end{array}$$

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- Let us multiply all inequalities by $t_1(x) = x_1$ and $t_2(x) = 1 - x_1$ ...

$$\begin{array}{rrl} x_1^2 & +x_1x_2 & \geq \frac{1}{2} \cdot x_1 \\ x_1^2 & +x_1x_2 & \leq \frac{3}{2} \cdot x_1 \\ x_1^2 & & \leq x_1 \\ & x_1x_2 & \leq x_1 \\ x_1^2 & & \geq 0 \\ & x_1x_2 & \geq 0 \end{array} \qquad \begin{array}{rrl} x_1(1-x_1) & +x_2(1-x_1) & \geq \frac{1}{2}(1-x_1) \\ x_1(1-x_1) & +x_2(1-x_1) & \leq \frac{3}{2}(1-x_1) \\ x_1(1-x_1) & & \leq (1-x_1) \\ & x_2(1-x_1) & \leq (1-x_1) \\ x_1(1-x_1) & & \geq 0 \\ & x_2(1-x_1) & \geq 0 \end{array}$$

$$\begin{array}{rrl} x_1 & +(1-x_1) & = 1 \\ x_1^2 & +x_1(1-x_1) & = x_1 \\ x_1x_2 & +x_2(1-x_1) & = x_2 \end{array}$$

Lift-and-Project (Balas and Ceria, 1991)

- Let us multiply all inequalities by $t_1(x) = x_1$ and $t_2(x) = 1 - x_1$ ...

$$
\begin{array}{rrl}
x_1^2 & +x_1 x_2 & \geq \frac{1}{2} \cdot x_1 \\
x_1^2 & +x_1 x_2 & \leq \frac{3}{2} \cdot x_1 \\
x_1^2 & & \leq x_1 \\
 & x_1 x_2 & \leq x_1 \\
x_1^2 & & \geq 0 \\
 & x_1 x_2 & \geq 0
\end{array}
\qquad
\begin{array}{rrl}
x_1(1-x_1) & +x_2(1-x_1) & \geq \frac{1}{2}(1-x_1) \\
x_1(1-x_1) & +x_2(1-x_1) & \leq \frac{3}{2}(1-x_1) \\
x_1(1-x_1) & & \leq (1-x_1) \\
 & x_2(1-x_1) & \leq (1-x_1) \\
x_1(1-x_1) & & \geq 0 \\
 & x_2(1-x_1) & \geq 0
\end{array}
$$

$$
\begin{array}{rrl}
x_1 & +(1-x_1) & = 1 \\
x_1^2 & +x_1(1-x_1) & = x_1 \\
x_1 x_2 & +x_2(1-x_1) & = x_2
\end{array}
$$

- Substitute $x_1^2 = x_1$, $x_1(1-x_1) = 0$ ...

$$
\begin{array}{rrl}
\frac{1}{2}x_1 & +x_1 x_2 & \geq 0 \\
-\frac{1}{2}x_1 & +x_1 x_2 & \leq 0 \\
x_1 & & \leq x_1 \\
-x_1 & +x_1 x_2 & \leq 0 \\
x_1 & & \geq 0 \\
 & x_1 x_2 & \geq 0
\end{array}
\qquad
\begin{array}{rrl}
\frac{1}{2}x_1 & +x_2(1-x_1) & \geq \frac{1}{2} \\
\frac{3}{2}x_1 & +x_2(1-x_1) & \leq \frac{3}{2} \\
x_1 & & \leq 1 \\
x_1 & +x_2(1-x_1) & \leq 1 \\
0 & & \geq 0 \\
 & x_2(1-x_1) & \geq 0
\end{array}
$$

$$
\begin{array}{rrl}
x_1 & +(1-x_1) & = 1 \\
x_1 & & = x_1 \\
x_1 x_2 & +x_2(1-x_1) & = x_2
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Lift-and-Project (Balas and Ceria, 1991)

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$$\begin{array}{rrl}
x_1^2 & +x_1x_2 & \geq \frac{1}{2}\cdot x_1 \\
x_1^2 & +x_1x_2 & \leq \frac{3}{2}\cdot x_1 \\
x_1^2 & & \leq x_1 \\
 & x_1x_2 & \leq x_1 \\
x_1^2 & & \geq 0 \\
 & x_1x_2 & \geq 0
\end{array}
\qquad
\begin{array}{rrl}
x_1(1-x_1) & +x_2(1-x_1) & \geq \frac{1}{2}(1-x_1) \\
x_1(1-x_1) & +x_2(1-x_1) & \leq \frac{3}{2}(1-x_1) \\
x_1(1-x_1) & & \leq (1-x_1) \\
 & x_2(1-x_1) & \leq (1-x_1) \\
x_1(1-x_1) & & \geq 0 \\
 & x_2(1-x_1) & \geq 0
\end{array}$$

$$\begin{array}{rrl}
x_1 & +(1-x_1) & = 1 \\
x_1^2 & +x_1(1-x_1) & = x_1 \\
x_1x_2 & +x_2(1-x_1) & = x_2
\end{array}$$

- Substitute $x_1^2 = x_1$, $x_1(1-x_1) = 0$ ...

$$\begin{array}{rrl}
\frac{1}{2}x_1 & +x_1x_2 & \geq 0 \\
-\frac{1}{2}x_1 & +x_1x_2 & \leq 0 \\
x_1 & & \leq x_1 \\
-x_1 & +x_1x_2 & \leq 0 \\
x_1 & & \geq 0 \\
 & x_1x_2 & \geq 0
\end{array}
\qquad
\begin{array}{rrl}
\frac{1}{2}x_1 & +x_2(1-x_1) & \geq \frac{1}{2} \\
\frac{3}{2}x_1 & +x_2(1-x_1) & \leq \frac{3}{2} \\
x_1 & & \leq 1 \\
x_1 & +x_2(1-x_1) & \leq 1 \\
0 & & \geq 0 \\
 & x_2(1-x_1) & \geq 0
\end{array}$$

$$\begin{array}{rrl}
x_1 & +(1-x_1) & = 1 \\
x_1 & & = x_1 \\
x_1x_2 & +x_2(1-x_1) & = x_2
\end{array}$$

Lift-and-Project (Balas and Ceria, 1991)

- Substitute $x_1^2 = x_1$, $x_1(1-x_1) = 0$ ...

$$
\begin{array}{rrl}
\frac{1}{2}x_1 & +x_1x_2 & \geq 0 \\
-\frac{1}{2}x_1 & +x_1x_2 & \leq 0 \\
x_1 & & \leq x_1 \\
-x_1 & +x_1x_2 & \leq 0 \\
x_1 & & \geq 0 \\
 & x_1x_2 & \geq 0
\end{array}
\qquad
\begin{array}{rrl}
\frac{1}{2}x_1 & +x_2(1-x_1) & \geq \frac{1}{2} \\
\frac{3}{2}x_1 & +x_2(1-x_1) & \leq \frac{3}{2} \\
x_1 & & \leq 1 \\
x_1 & +x_2(1-x_1) & \leq 1 \\
0 & & \geq 0 \\
 & x_2(1-x_1) & \geq 0
\end{array}
$$

$$
\begin{array}{rrl}
x_1 & +(1-x_1) & = 1 \\
x_1 & & = x_1 \\
x_1x_2 & +x_2(1-x_1) & = x_2
\end{array}
$$

- Substitute $y_{21} = x_2 t_1(x) = x_1 x_2$, and $y_{22} = x_2 t_2(x) = x_2(1-x_1)$, and eliminate redundant inequalities ...

$$
\begin{array}{rrl}
\frac{1}{2}x_1 & +y_{21} & \geq 0 \\
-\frac{1}{2}x_1 & +y_{21} & \leq 0 \\
0 & & \leq 0 \\
-x_1 & +y_{21} & \leq 0 \\
x_1 & & \geq 0 \\
 & y_{21} & \geq 0
\end{array}
\qquad
\begin{array}{rrl}
\frac{1}{2}x_1 & +y_{22} & \geq \frac{1}{2} \\
\frac{3}{2}x_1 & +y_{22} & \leq \frac{3}{2} \\
x_1 & & \leq 1 \\
x_1 & +y_{22} & \leq 1 \\
0 & & \geq 0 \\
 & y_{22} & \geq 0
\end{array}
$$

$$
\begin{array}{rrl}
0 & & = 0 \\
0 & & = 0 \\
y_{21} & +y_{22} & = x_2
\end{array}
$$

Lift-and-Project (Balas and Ceria, 1991)

- Substitute $x_1^2 = x_1$, $x_1(1 - x_1) = 0$ ...

$$
\begin{array}{rrlrrl}
\frac{1}{2}x_1 & +x_1x_2 & \geq 0 & \frac{1}{2}x_1 & +x_2(1-x_1) & \geq \frac{1}{2} \\
-\frac{1}{2}x_1 & +x_1x_2 & \leq 0 & \frac{3}{2}x_1 & +x_2(1-x_1) & \leq \frac{3}{2} \\
x_1 & & \leq x_1 & x_1 & & \leq 1 \\
-x_1 & +x_1x_2 & \leq 0 & x_1 & +x_2(1-x_1) & \leq 1 \\
x_1 & & \geq 0 & 0 & & \geq 0 \\
 & x_1x_2 & \geq 0 & & x_2(1-x_1) & \geq 0
\end{array}
$$

$$
\begin{array}{rrl}
x_1 & +(1-x_1) & = 1 \\
x_1 & & = x_1 \\
x_1x_2 & +x_2(1-x_1) & = x_2
\end{array}
$$

- Substitute $y_{21} = x_2 t_1(x) = x_1x_2$, and $y_{22} = x_2 t_2(x) = x_2(1 - x_1)$, and eliminate redundant inequalities ...

$$
\begin{array}{rrlrrl}
\frac{1}{2}x_1 & +y_{21} & \geq 0 & \frac{1}{2}x_1 & +y_{22} & \geq \frac{1}{2} \\
-\frac{1}{2}x_1 & +y_{21} & \leq 0 & \frac{3}{2}x_1 & +y_{22} & \leq \frac{3}{2} \\
0 & & \leq 0 & x_1 & & \leq 1 \\
-x_1 & +y_{21} & \leq 0 & x_1 & +y_{22} & \leq 1 \\
x_1 & & \geq 0 & 0 & & \geq 0 \\
 & y_{21} & \geq 0 & & y_{22} & \geq 0
\end{array}
$$

$$
\begin{array}{rrl}
0 & & = 0 \\
0 & & = 0 \\
y_{21} & +y_{22} & = x_2
\end{array}
$$

Lift-and-Project (Balas and Ceria, 1991)

- Substitute $y_{21} = x_2 t_1(x) = x_1 x_2$, and $y_{22} = x_2 t_2(x) = x_2(1 - x_1)$, and eliminate redundant inequalities ...

$$
\begin{array}{rrrl}
\frac{1}{2}x_1 & +y_{21} & \geq & 0 \\
-\frac{1}{2}x_1 & +y_{21} & \leq & 0 \\
-x_1 & +y_{21} & \leq & 0 \\
x_1 & & \geq & 0 \\
 & y_{21} & \geq & 0
\end{array}
\qquad
\begin{array}{rrrl}
\frac{1}{2}x_1 & +y_{22} & \geq & \frac{1}{2} \\
\frac{3}{2}x_1 & +y_{22} & \leq & \frac{3}{2} \\
x_1 & & \leq & 1 \\
x_1 & +y_{22} & \leq & 1 \\
 & y_{22} & \geq & 0
\end{array}
$$

$$y_{21} + y_{22} = x_2$$

- $$x_1 + 2x_2 \geq 1$$
- $$x_1 + 2x_2 \leq 2$$

Lift-and-Project (Balas and Ceria, 1991)

- Substitute $y_{21} = x_2 t_1(x) = x_1 x_2$, and $y_{22} = x_2 t_2(x) = x_2(1 - x_1)$, and eliminate redundant inequalities ...

$$
\begin{array}{rrll}
\frac{1}{2}x_1 & +y_{21} & \geq & 0 \\
-\frac{1}{2}x_1 & +y_{21} & \leq & 0 \\
-x_1 & +y_{21} & \leq & 0 \\
x_1 & & \geq & 0 \\
 & y_{21} & \geq & 0
\end{array}
\qquad
\begin{array}{rrll}
\frac{1}{2}x_1 & +y_{22} & \geq & \frac{1}{2} \\
\frac{3}{2}x_1 & +y_{22} & \leq & \frac{3}{2} \\
x_1 & & \leq & 1 \\
x_1 & +y_{22} & \leq & 1 \\
 & y_{22} & \geq & 0
\end{array}
$$

$$y_{21} + y_{22} = x_2$$

-
$$x_1 + 2x_2 \geq 1$$

-
$$x_1 + 2x_2 \leq 2$$

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$$
\begin{array}{rrll}
\frac{1}{2}x_1 & +y_{21} & \geq & 0 \\
-\frac{1}{2}x_1 & +y_{21} & \leq & 0 \\
-x_1 & +y_{21} & \leq & 0 \\
x_1 & & \geq & 0 \\
 & y_{21} & \geq & 0
\end{array}
\qquad
\begin{array}{rrll}
\frac{1}{2}x_1 & +y_{22} & \geq & \frac{1}{2} \\
\frac{3}{2}x_1 & +y_{22} & \leq & \frac{3}{2} \\
x_1 & & \leq & 1 \\
x_1 & +y_{22} & \leq & 1 \\
 & y_{22} & \geq & 0
\end{array}
$$

$$y_{21} + y_{22} = x_2$$

- $$x_1 + 2x_2 \geq 1$$

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