Corner Polyhedra

Let us consider an integer programming problem and its set of feasible solutions of the following form

$$P_I \;=\; \text{conv}\,\{x \in \mathbb{Z}^n_+ \mid Ax = b,\ x \geq 0\}$$

where $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$ for some $m \leq n$. We can assume without any loss of generality that A is of full row rank, and write $A = [B, N]$ for a basic subset B of its columns. Then we can write $x = (x_B, x_N)$ by grouping the components of x according to the above partitioning of the columns of A. For such a basic-nonbasic partitioning of the columns, we can write

$$x_B = B^{-1}b - B^{-1}Nx_N.$$

Let us denote by $f = B^{-1}b$, and introduce the notation $[-B^{-1}N] = [r_1, ..., r_{n-m}]$ for the coefficient columns of the nonbasic variables in the above equalities. Furthermore, we simplify our notation by introducing $z = x_B \in \mathbb{R}^m$ and $y = x_N \in \mathbb{R}^{n-m}$.

Following Gomory (1969) we relax now the integrality of the nonbasic variables $y = x_N$ and the nonnegativity of the basic variables $z = x_B$, and define the so called *corner polyhedron* as

$$P_C \;=\; \text{conv}\left\{ \begin{pmatrix} z \\ y \end{pmatrix} \in \mathbb{Z}^m \times \mathbb{R}^{n-m}_+ \;\middle|\; z = f + \sum_{j=1}^{n-m} r_j y_j \right\}$$

Lemma 1 *If $(z, y) \in P_C$, $z \in \mathbb{Z}^m$, then for all $j = 1, ..., n-m$ and for infinitely many unbounded values of $\lambda > 0$ we have*

$$(z + \lambda r_j, y_1, ..., y_{j-1}, y_j + \lambda, y_{j+1}, ..., y_{n-m}) \in P_C.$$

Proof. The vectors r_j are rational for all $j = 1, ..., n-m$, and thus for infinitely many (integer) values of $\lambda > 0$ we have that λr_j is an integer vector for all $j = 1, ..., n-m$. □

Lemma 2 *Assume that $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$ and $P_C \neq \emptyset$. Then tight valid inequalities for P_C that are violated by $(f, 0)$ can be written as*

$$\sum_{j=1}^{n-m} \alpha_j y_j \;\geq\; 1$$

for some $\alpha_j \geq 0$, $j = 1, ..., n-m$.

Proof. An arbitrary valid inequality can be written as

$$\sum_{i=1}^{m} \mu_i z_i + \sum_{j=1}^{n-m} \nu_j y_j \geq \xi.$$

Substituting in $z = f + \sum_{j=1}^{n-m} r_j y_j$ we get

$$\sum_{j=1}^{n-m} \left(\nu_j + \sum_{i=1}^{m} \mu_i r_{ij} \right) y_j \geq \xi - \sum_{i=1}^{m} \mu_i f_i.$$

Since we assume that $(f, 0)$ is violating this inequality we get

$$\xi - \sum_{i=1}^{m} \mu_i f_i > 0.$$

By the previous lemma we get that for infinitely many unbounded nonnegative λ values and for all $j = 1, .., n-m$ we have

$$\left(\nu_j + \sum_{i=1}^{m} \mu_i r_{ij} \right) (y_j + \lambda) \geq \xi - \sum_{i=1}^{m} \mu_i f_i,$$

from which it follows that $\alpha_j := \nu_j + \sum_{i=1}^{m} \mu_i r_{ij} \geq 0$ for all $j = 1, ..., n-m$. □

Intersection Cuts

Let us consider a closed convex set $S \subseteq \mathbb{R}^m$ such that $f \in int(S)$ and that S has no integer point in its interior. To such a *lattice-free* convex set we associate a real valued function $\Phi_S : \mathbb{R}^m \to \mathbb{R}_+$ by defining

$$\Phi_S(r) \;=\; \inf\{t > 0 \mid f + \frac{r}{t} \in S\}.$$

Note that we have $\Phi_S(r) = 0$ if and only if $f + \lambda r \in S$ for all $\lambda \geq 0$.

Lemma 3 (Balas (1971)) *If S is a lattice free closed convex set such that $f \in int(S)$, then*

$$\sum_{j=1}^{n-m} \Phi_S(r_j) y_j \;\geq\; 1$$

is a valid inequality for P_C (that cuts off the corner $(f, 0)$. □

Any lattice free closed convex polyhedron can be written as

$$S \;=\; \{z \in \mathbb{R}^m \mid a_i^T(z-f) \le 1,\ i=1,\dots,\ell\}$$

for some vectors $a_i \in \mathbb{R}^m$, $i=1,\dots,\ell$.

Lemma 4 *For all $r \in \mathbb{R}^m$ we have*

$$\Phi_S(r) \;=\; \max_{i=1,\dots\ell} a_i^T r.$$

$\square$

Lemma 5 *Φ_S is subadditive and positively homogenous:*

$$\Phi_S(r+r') \le \Phi_S(r) + \Phi_S(r') \quad \textit{and} \quad \Phi_S(\lambda r) \;=\; \lambda \Phi_S(r)$$

for all $r, r' \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}_+$. $\square$

Corollary 1 *If S is a lattice free closed and convex polyhedron (of the above form), and $(z,y) \in P_C$, $z \in \mathbb{Z}^m$, then we have*

$$\sum_{j=1}^{n-m} \Phi_S(r_j) y_j = \sum_{j=1}^{n-m} \Phi_S(r_j y_j) \ge \Phi_S(\sum_{j=1}^{n-m} r_j y_j) = \Phi_S(z-f) \ge 1.$$

$\square$

Theorem 1 (Conforti, Cornuejols, Zambelli (2010)) *If $P_C \neq \emptyset$, then every minimal valid inequality for P_C is an intersection cut.* $\square$