Maximum Flows and Minimum Cuts

Network Formulation

Given a directed graph $G = (V, A)$, a *source* $s \in V$, a *sink* $t \in V$, and integer capacities $u_{ij} \in \mathbb{Z}_+$ for all arcs $(i, j) \in A$, the so called *maximum flow* problem can be formulated as follows:

$$\max \sum_{(s,j)\in A} x_{sj} - \sum_{(j,s)\in A} x_{js} \tag{1}$$

$$\text{s.t.} \quad \sum_{(i,j)\in A} x_{ij} - \sum_{(j,k)\in A} x_{jk} = 0 \quad \text{for all } j \in V \setminus \{s, t\} \tag{2}$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for all } (i, j) \in A. \tag{3}$$

An assignment $x : A \mapsto \mathbb{R}_+$ is called a *flow*, if it satisfies conditions (2)-(3). Let us introduce the notation

$$F_{G,u}(x) = \sum_{(s,j)\in A} x_{s,j} - \sum_{(j,s)\in A} x_{js} \tag{4}$$

for the objective function value (1), and call it the *flow value* of the flow x.

It is customary to assume that if $(i, j) \in A$ then also $(j, i) \in A$. This is not a restriction of generality, since we can include all arcs with $u_{ij} = 0$, without changing the problem.

It is also customary to assume that

$$\min\{x_{ij}, x_{ji}\} = 0, \tag{5}$$

which is again not a restriction of generality, since switching a flow x to

$$x'_{k\ell} = \begin{cases} x_{ij} - \min\{x_{ij}, x_{ji}\} & \text{if } (k, \ell) = (i, j) \\ x_{ji} - \min\{x_{ij}, x_{ji}\} & \text{if } (k, \ell) = (j, i) \\ x_{k\ell} & \text{if } (k, \ell) \neq (i, j), (j, i) \end{cases} \tag{6}$$

results in another flow with the same the objective function value (1). Operation (6) is called *flow cancellation*. Let us remark that the above property does not hold for an arbitrary flow of problem (1) – (3), and we shall assume (6) only in certain proofs, as a technical condition.

Let us further note that for an arc $a = (i, j) \in A$ we shall use both notations $u(a) = u_{ij}$ and $x(a) = x_{ij}$.

Paths, cycles and decomposition of flows

Given a directed graph $G = (V, A)$ and a subset $F \subseteq A$ of its arcs, let us call an arc $a^* \in F$ a *bottleneck* of F, if $u(a^*) \leq u(a)$ for all $a \in F$, and let us introduce the notation

$$\Delta_{G,u}(F) = u(a^*). \tag{7}$$

Given $G = (V, A)$ and $u : A \mapsto \mathbb{R}_+$, let us consider an $s - t$ path $P = (i_0, a_1, i_1, ..., a_r, i_r)$, i.e., where $a_k = (i_{k-1}, i_k) \in A$ for $k = 1, ..., r$, and where $i_0 = s$ and $i_r = t$. We usually identify P with the set of its arcs, and write $P \subseteq A$, whenever it does not cause confusion. Let us also assume that P is simple, i.e., all vertices $i_0, i_1, ..., i_r$ are distinct. To such a simple path P we associate a binary vector $\chi^P \in \{0, 1\}^A$, the *characteristic* vector of $P \subseteq A$ defined by

$$\chi^P_{ij} = \begin{cases} 1 & \text{if } (i,j) \in P, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

It is easy to see that $x = \alpha\chi^P$ is a flow for every $0 \leq \alpha \leq \Delta_{G,u}(P)$. Such flows are called *path-flows*.

Similarly, if $C = (i_0, a_1, i_1, ..., a_r, i_r)$ $(a_{i_k} = (i_{k-1}, i_k) \in A$ for $k = 1, .., r)$ is a directed cycle $(i_0 = i_r)$, then its characteristic vector is

$$\chi^C_{ij} = \begin{cases} 1 & \text{if } (i,j) \in C, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

and the vector $x = \alpha\chi^C$ is a flow in G for every $0 \leq \alpha \leq \Delta_{G,u}(C)$. Such flows are called *cycle-flows*.

More generally, if Q_k, $k = 1, ..., p$ are all $s - t$ paths or cycles in G, and α_k, $k = 1, ..., p$ are nonnegative reals such that

$$\sum_{k:(i,j) \in Q_k} \alpha_k \leq u_{ij}$$

holds for all $(i,j) \in A$, then $x = \sum_{k=1}^{p} \alpha_k \chi^{Q_k}$ is a flow in the network G with capacities u.

The converse is also true, and in fact the equalities (2) define a convex cone, the generators of which are the characteristic vectors of simple paths and simple cycles in G.

Theorem 1 *Let x be a flow in the network $G = (V, A)$ with source $s \in V$, sink $t \in V$ and arc capacities $u : A \mapsto \mathbb{R}_+$. Let us further assume that $F_{G,u}(x) \geq 0$, and let p be the number of arcs $(i,j) \in A$ of G for which $x_{ij} > 0$. Then, there exist simple $s-t$ paths and cycles Q_k, $k = 1, ..., q$ and corresponding positive reals $\alpha_k > 0$, $k = 1, ..., q$, such that*

$$x = \sum_{k=1}^{q} \alpha_k \chi^{Q_k}$$

and $q \leq p$.

Proof. We shall prove the claim by induction on p. If $p = 0$, i.e., if $x \equiv 0$, then the claim is trivially true. Let us assume next that we already have shown the claim for all cases when the number of arcs with a positive flow on it is less than p, and consider the above case with exactly p such arcs.

Let us then consider the subgraph $G' = (V, A')$ where $A' = \{(i,j) \in A \mid x_{ij} > 0\}$, and assume first that this subgraph contain a directed cycle $C \subseteq A'$. Let us then define

$$x'_{ij} = \begin{cases} x_{ij} & \text{if } (i,j) \in A \setminus C, \\ x_{ij} - \Delta_{G,x}(C) & \text{if } (i,j) \in C. \end{cases}$$

Then, x' is a flow in G, and the number of arcs with a positive flow in x' is at least one less than p.

Let us assume next that there are no directed cycles in (G, A'), and let us choose a maximal path $P \subseteq A'$. Let i be the tail and j be the head of this path. Then, $deg^{out}_{G'}(i) > 0$, and $deg^{in}_{G'}(i) = 0$ (since otherwise P would not be maximal), implying that $i \in \{s, t\}$. Similarly, $deg^{in}_{G'}(j) > 0$ and $deg^{out}_{G'}(j) = 0$ implies that $j \in \{s, t\}$. We claim that $j \neq s$, because otherwise $deg^{in}_{G'}(j) = deg^{in}_{G'}(s) > 0$ and $F_{G,u}(x) \geq 0$ would imply $deg^{out}_{G'}(s) > 0$, contradicting the maximality of P. Thus, $j = t$, and since G' does not contain directed cycles, we must also have $i = s$. Therefore, P is an $s-t$ path, and hence

$$x'_{ij} = \begin{cases} x_{ij} & \text{if } (i,j) \in A \setminus P, \\ x_{ij} - \Delta_{G,x}(P) & \text{if } (i,j) \in P \end{cases}$$

is a flow in G, having $x'_{ij} > 0$ for at most $p-1$ arcs.

Thus, in both cases, x' is a nonnegative combination of at most $p-1$ path and cycle flows, by our hypothesis, i.e.,

$$x' = \sum_{k=1}^{q} \alpha_k \chi^{Q_k}$$

for some $s-t$ paths and cycles Q_k, $k = 1, ..., q$, and for some $q < p$, and consequently, in both cases we can write

$$x = x' + \Delta_{G,x}(Q_{q+1})\chi^{Q_{q+1}} = \sum_{k=1}^{q+1} \alpha_k \chi^{Q_k}$$

where $\alpha_{q+1} = \Delta_{G,x}(Q_{q+1})$ and $Q_{q+1} = C$ in the first case, and $Q_{q+1} = P$ in the second case, and where $q+1 \leq p$. □

Corollary 1 *Given a network $G = (V, A)$, $s, t \in V$ with capacities $u : A \mapsto \mathbb{R}_+$, there exists a maximum flow x^* of the form*

$$x^* = \sum_{k=1}^{p} \alpha_k \chi^{P_k}$$

for some $s-t$ paths P_1, ..., P_p, and nonnegative reals α_1, ..., α_p, where $p \leq m = |A|$.

Proof. Let x be an arbitrary maximum flow in G. Similarly to the above proof, let us obtain flows $x \geq x' \geq x'' \geq ...$ by cancelling cycles, as above. Clearly, cycle cancellation decreases the number of arcs with a positive flow value, thus in at most m steps we arrive to a flow x^* for which there is no cycle with positive flows on all its arcs. It is easy to see that $F_{G,u}(x) = F_{G,u}(x') = F_{G,u}(x'') = \cdots = F_{G,u}(x^*)$. Thus x^* is also a maximum flow, and by the above theorem it can be decomposed into at most m path-flows. □

Minimum cuts

Given a subset $S \subseteq V$, let $\delta^{out}(S) = \{(i,j) \in A \mid i \in S, j \notin S\}$, and similarly let $\delta^{in}(S) = \{(i,j) \in A \mid i \notin S, j \in S\}$. If $s \in S$ and $t \notin S$, then $\delta^{out}(S)$ is called an $s-t$ *cut set*, and the vertex set S is called an $s-t$ *cut*.

Lemma 1 *If $G = (V, A)$ is a directed graph, $s, t \in V$ and $F \subseteq A$ is a minimal subset of the arcs such that $G' = (V, A \setminus F)$ does not contain an $s-t$ path, then $F = \delta^{out}(S)$ for some subset $S \subseteq V$, $s \in S$, $t \notin S$.*

Proof. See in class ... $\square$

The quantity

$$C_{G,u}(S) = \sum_{(i,j)\in\delta^{out}(S)} u_{ij} \tag{10}$$

is called the *cut value* of the cut set $\delta^{out}(S)$ (or simply of the cut S).

Lemma 2 *Given a directed graph $G = (V, A)$, $u : A \mapsto \mathbb{R}_+$, $s, t \in V$ and a flow x in G, for every subset $S \subseteq V$, $s \in S$, $t \notin S$ we have*

$$\sum_{(i,j)\in\delta^{out}(S)} x_{ij} - \sum_{(i,j)\in\delta^{in}(S)} x_{ij} = F_{G,u}(x).$$

Proof. Take the sum of equalities (2) for all $j \in S$. $\square$

Lemma 3 *Given a directed graph $G = (V, A)$, $u : A \mapsto \mathbb{R}_+$, and $s, t \in V$, for every subset $S \subseteq V$, $s \in S$, $t \notin S$ and every flow x we have*

$$C_{G,u}(S) \geq F_{G,u}(x).$$

Proof. Since x is a flow, we have $0 \leq x_{ij} \leq u_{ij}$ for all arcs $(i,j) \in A$. Thus,

$$C_{G,u}(S) = \sum_{(i,j)\in\delta^{out}(S)} u_{ij}$$

$u_{ij} \geq x_{ij}$ for all arcs $(i,j) \in A$ implying

$$\geq \sum_{(i,j)\in\delta^{out}(S)} x_{ij}$$

$x_{ij} \geq 0$ for all arcs $(i,j) \in A$ implying

$$\geq \sum_{(i,j)\in\delta^{out}(S)} x_{ij} - \sum_{(i,j)\in\delta^{in}(S)} x_{ij}$$

which is by Lemma 2 equals to

$$= F_{G,u}(x).$$

□

Let us call a subset $S^* \subseteq V$ a *minimum cut* if

$$C_{G,u}(S^*) = \min_{\substack{S\subseteq V \\ s\in S \\ t\notin S}} C_{G,u}(S).$$

Lemma 4 *If S and S' are minimum cuts in $G = (V, A)$ with arc capacities $u : A \mapsto \mathbb{R}_+$, then so is $S \cap S'$ and $S \cup S'$.*

Proof. We have

$$C_{G,u}(S \cup S') + C_{G,u}(S \cap S') \leq C_{G,u}(S) + C_{G,u}(S').$$

(In other words, $C_{G,u}(\cdot)$ is a *sub-modular* function.) □

Corollary 2 *For every network $G = (V, A)$, with source $s \in V$, sink $t \in V$, and with arc capacities $u : A \mapsto \mathbb{R}_+$ there exists a unique minimal minimum cut $S^* \subseteq V$ ($s \in S^*$, $t \notin S^*$).*

Proof. Take the intersection of all minimum cuts, and apply Lemma 4. □

Augmenting paths

The results on flow decomposition may suggest to try to construct a maximum flow by successively adding path-flows. This however may fail, if we do it in a straightforward way. [See example in class.] We may succeed however, if we realize that flow *push-back* is a feasible operation [see demonstration in class]. We can formalize the idea in the following way.

Given a simple flow x (satisfying (5)) in a directed network $G = (V, A)$ with source $s \in V$, sink $t \in V$ and arc capacities $u : A \mapsto \mathbb{R}_+$, let us associate to it the so called *residual capacities* $\tilde{u} = \tilde{u}(G, u, x)$ defined by

$$\tilde{u}_{ij} = u_{ij} - x_{ij} + x_{ji}$$

for all arcs $(i, j) \in A$ (remember, we assume that if $(i, j) \in A$, then so is $(j, i) \in A$). The pair $G, \tilde{u}$ is called the *residual network* of G, u corresponding to the flow x.

We shall say that a vertex $k \in V$ is *x-reachable* from s in the residual network if there is an $s - k$ path P for which $\tilde{u}_{i,j} > 0$ for all arcs $(i, j) \in P$. If t is x-reachable from s in the residual network $G, \tilde{u}$, then an $s - t$ path P for which $\tilde{u}_{ij} > 0$ for all arcs $(i, j) \in P$ is called an *x-augmenting path.*

Theorem 2 (Ford and Fulkerson (1956)) *Given a network $G = (V, A)$ with source $s \in V$, sink $t \in V$ and arc capacities $u : A \mapsto \mathbb{R}_+$, a flow x is a maximum flow iff there exists no x-augmenting path.*

Proof. Clearly, if there is an x-augmenting path P, then by increasing the flow along P by $\Delta_{G,\tilde{u}}(P)$ we can obtain another flow x' for which $F_{G,u}(x') = F_{G,u}(x) + \Delta_{G,\tilde{u}}(P) > F_{G,u}(x)$, contradicting that x was chosen as a maximum flow.

For the other direction, let us assume that there exists no x-augmenting path, and let us denote by R the set of vertices which are x-reachable from s, i.e., $t \notin R$. We claim that for all arcs $(i, j) \in A$, $i \in R$, $j \notin R$ we have $x_{ij} = u_{ij}$ and $x_{ji} = 0$, because otherwise we would have $\tilde{u}_{ij} > 0$, implying that j is also x-reachable from s. Thus, we have

$$\sum_{(i,j)\in\delta^{in}(R)} x_{ij} = 0 \text{ and } \sum_{(i,j)\in\delta^{out}(R)} x_{ij} = \sum_{(i,j)\in\delta^{out}(R)} u_{ij}.$$

Therefore, by Lemmas 2 we get

$$\begin{aligned} F_{G,u}(x) &= \sum_{(i,j)\in\delta^{out}(R)} x_{ij} - \sum_{(i,j)\in\delta^{in}(R)} x_{ij} \\ &= \sum_{(i,j)\in\delta^{out}(R)} x_{ij} \\ &= \sum_{(i,j)\in\delta^{out}(R)} u_{ij} \\ &= C_{G,u}(R). \end{aligned}$$

Thus, Lemma 3 implies that x is a maximum flow and R is a minimum cut. $\square$

An immediate corollary of the above is the following:

Theorem 3 (Max-Flow-Min-Cut) *Given a network $G = (V, A)$ with source $s \in V$, sink $t \in V$ and arc capacities $u : A \mapsto \mathbb{R}_+$*

$$\max_{x \text{ is an } s-t \text{ flow}} F_{G,u}(x) \;=\; \min_{S \text{ is an } s-t \text{ cut}} C_{G,u}(S).$$

$\square$

Theorem 4 *Given a network $G = (V, A)$ with source $s \in V$, sink $t \in V$ and arc capacities $u : A \mapsto \mathbb{R}_+$, let x be an arbitrary maximum flow, and let R be the set of vertices x-reachable from s. Then $R = S^*$ is the unique minimal minimum cut.*

Proof. See HW. $\square$

The augmenting path algorithm

This method was proposed by Ford and Fulkerson (1956) to solve the maximum flow problem.

AUGMENTING PATH ALGORITHM

Input: Directed graph $G = (V, A)$, source $s \in V$, sink $t \in V$ and arc capacities $u : A \mapsto \mathbb{R}_+$.

Initialization: Set $k = 0$, $x^0 = 0$.

Step 1: Construct the residual network $G, \tilde{u}$ with respect to x^k.

Step 2: Find an x^k-augmenting $s-t$ path P_k. If there is no such path, **STOP** and output x^k.

Step 3: Set $x^{k+1} = x^k + \Delta_{G,\tilde{u}}(P_k)\chi^{P_k}$, set $k = k + 1$, and return to **Step 1**.

Proof of correctness: Follows by Theorem 2.

Analysis of complexity: Assume that u is integral. Then in every iteration the flow value is increased by at least 1. On the other hand, the maximum flow value is not more than the minimum cut value by Theorem 3, which is not more than the cut value of the cut $S = \{s\}$, which is bounded from above by nU, where U is an upper bound on the arc capacities in G. Thus, we need at most $O(nU)$ iterations before termination. Each time Steps 1 and 2 takes $O(m)$ time, while Step 3 takes $O(n)$ time. Thus, in total we need at most $O(mnU)$ time.

Let us note that if the capacities are irrational numbers the above algorithm may take infinitely many steps (see Ford and Fulkerson (1962), or Zwick (1995)).

Theorem 5 *If a network $G = (V, A)$ with source $s \in V$, sink $t \in V$ has integral arc capacities $u : A \mapsto \mathbb{Z}_+$, then it has an integral maximum flow.*

Proof. Run the AUGMENTING PATH ALGORITHM. It terminates in a finite number of steps, according to the above analysis, and in each step it increases an integral current flow by an integral value. □

Scaling

The performance of the augmenting path approach can easily be improved by using capacity scaling, introduced by Edmonds and Karp (1970,1972) and Dinitz (1973).

A WEAKLY POLYNOMIAL AUGMENTING PATH ALGORITHM

Input: Directed graph $G = (V, A)$, source $s \in V$, sink $t \in V$ and arc capacities $u : A \mapsto \mathbb{R}_+$.

Initialization: Set $U = \max_{(i,j)\in A} u_{ij}$, $L = \lceil \log U \rceil$, $k = 0$, and $x^0_{ij} = 0$ for all $(i, j) \in A$.

Main Loop: For $k = 1, ..., L$, let $u^k = \lfloor \frac{u}{2^{L-k}} \rfloor$ and compute a maximum flow x^k of the network (G, u^k) starting from the flow $2x^{k-1}$, by using the AUGMENTING PATH ALGORITHM.

Output: Flow x^L.

Proof of correctness: Let us observe that $2x^{k-1}$ is a feasible flow in (G, u^k), furthermore that $u^L = u$.

Analysis of complexity: Let us note that the maximum flow value of (G, u^k) is more than that of (G, u^{k-1}) by at most m. This is because if S is a minimum cut in (G, u^{k-1}), then for every arc $(i, j) \in \delta^{out}(W S)$ we have

$$2u^{k-1}_{ij} = 2x^{k-1}_{ij} \le u^k_{ij} \le 2x^{k-1}_{ij} + 1.$$

Thus, x^k can be constructed, starting from the feasible flow $2x^{k-1}$, by adding at most m augmenting paths. Since each augmenting path computation can be accomplished in $O(m)$ time, each iteration of the **Main Loop** can be finished in $O(m^2)$ time. Thus, the algorithm will terminate in $O(m^2 L) = O(m^2 \log U)$ time. □

A strongly polynomial augmenting path method

Given a directed graph $G = (V, A)$, source and sink vertices $s, t \in V$, and a subset $F \subseteq A$, let us denote by $\sigma(F)$ the length of a shortest $s - t$ path in (V, F), and let $\Sigma(F) \subseteq F$ denote the set of arcs belonging to such a shortest $s - t$ path in (V, F). For a subset $F \subseteq A$ let F^{-1} denote the set of reverse arcs, i.e., $F^{-1} = \{(j, i) \mid (i, j) \in F\}$; for $a = (i, j) \in A$ we write $a^{-1} = (j, i)$.

Let us also recall that if x is a feasible flow, then A_x denotes the set of arcs in the residual network (having a positive residual capacity). Furthermore, for $G = (V, A)$ we use $n = |V|$ and $m = |A|$.

Let us then start with a simple statement, claimed originally by Dinitz (1970).

Lemma 5 *Let $G = (V, A)$ be a directed graph, $s, t \in V$, and let $A' = A \cup \Sigma(A)^{-1})$. Then we have $\sigma(A) = \sigma(A')$ and $\Sigma(A) = \Sigma(A')$.*

Proof. Let us observe that if P is an $s - t$ path in (V, A) containing an arc $a = (i, j) \in \Sigma(A)$, and P' is an $s - t$ path in $A \cup \Sigma(A)^{-1}$ containing $a^{-1} = (j, i)$ such that $|P' \cap \Sigma(A)^{-1}| = k$, then there exists an $s - t$ path in $(V, A \cup \Sigma(A)^{-1})$ which has length at most $(|P| + |P'| - 2)/2$, and contains at most $k - 1$ arcs from $\Sigma(A)^{-1}$. $\square$

Theorem 6 (Dinitz (1970), Edmonds and Karp (1972)) *If in the* Augmenting Path Algorithm *a shortest $s - t$ path is chosen for flow augmentation in every iteration, then the algorithm terminates with a maximum flow in at most $|V||A|$ iterations, i.e., it can be implemented to run in $O(nm^2)$ total time.*

Proof. Let us denote by x a feasible flow obtained in the course of the algorithm, and let A_x denote the set of arcs of the residual network (having positive residual capacity). Let us then choose a shortest $s - t$ path for flow augmentation, and let x' denote the obtained augmented flow. Then we have $A_{x'} \subseteq A_x \cup \Sigma(A_x)$, thus $\sigma(A_{x'}) \geq \sigma(A_x \cup \Sigma(A_x)^{-1}) = \sigma(A_x)$, by the definitions, and by Lemma 5. Furthermore, if $\alpha(A_{x'}) = \sigma(A_x)$, then we also have $\Sigma(A_{x'}) \subseteq \Sigma(A_x \cup \Sigma(A_x)^{-1}) = \Sigma(A_x)$, again by the definition and by lemma 5. Moreover, since at least one arc in the chosen shortest $s - t$ path will have a zero residual capacity, we have $\Sigma(A_{x'}) \subsetneq \Sigma(A_x)$. Thus, we can have at most $O(m)$ iterations before $\Sigma(A_{x'})$ becomes empty, i.e., before $\sigma(A_{x'}) > \sigma(A_x)$. Since the length of the shortest path can increase at most n times, we get that the number of iterations is not exceeding $O(mn)$, from which the claim follows. $\square$

Other strongly polynomial algorithms for maximum flows

We recall here a few improved algorithms for network flow computations. We leave out the proofs, but encourage the reader to consult the original publications or mor erecent textbooks.

Theorem 7 (Dinitz (1970)) *A maximum flow can be computed in $O(n^2m)$ time by successively constructing so called* blocking flows.

Karzanov (1974) gave a faster algorithm to compute a blocking flow in an acyclic digraph.

Theorem 8 (Karzanov (1974)) *Given an acyclic directed graph $G = (V, A)$ with source $s \in V$, sink $t \in V$, and capacities $u : A \mapsto \mathbb{Z}_+$, a blocking $s-t$-flow can be found in $O(n^2)$ time, and thus a maximum flow can be computed in $O(n^3)$ time.*

Goldberg (1985, 1987) and Goldberg and Tarjan (1986,1988) introduced a new maximum flow algorithm based on *preflows* and on a so called *push-relabel* procedure.

Theorem 9 (Goldberg (1985)) *The* PREFLOW PUSH - RELABEL ALGORITHM *finds a maximum flow in time $O(n^3)$.*