Chvátal-Gomory cuts

Given $A = [a_1, ..., a_n] \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, let as before

$$P = \{x \mid Ax \leq b\}$$

and

$$P_I = \operatorname{conv} P \cap \mathbb{Z}^n.$$

Let us recall that $\lfloor * \rfloor$ and linear functions are superadditive, that composition of superadditive functions is again superadditive and consequently

Fact 1 *For every $u \in \mathbb{R}^m_+$ the inequality*

$$\sum_{j=1}^{n} \lfloor u^T a_j \rfloor x_j \leq \lfloor u^T b \rfloor \tag{1}$$

is valid for P_I.

Proof. Consider the function $F : \mathbb{R}^M \to \mathbb{R}$ defined by

$$F(b) = \lfloor u^T b \rfloor.$$

As we observed above, it is a composition of superadditive functions, and hence itself is superadditive, and thus the result follows by the main lemma of superadditive functions (Lemma 8 in the previous handout). $\square$

In particular, if $u \in \mathbb{R}^m_+$ such that $u^T A \in \mathbb{Z}^n$, then (1) is called a *Chvátal cut* (honoring the paper by Chvátal (1973)).

Let us note furthermore that if $Q = \{x \mid \widehat{A}x = \widehat{b}\}$, then we can also write $Q = \{x \mid Ax \leq b\}$, where

$$A = \begin{bmatrix} \widehat{A} \\ -\widehat{A} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \widehat{b} \\ -\widehat{b} \end{bmatrix}.$$

Thus, when Fact 1 is applied in this way to a system of equations, inequality (1) is also called a *Gomory cut* (honoring the paper by Gomory (1960)).

Chvátal closure

Let us call inequality (1) a *C-G cut of the polytope* P (of rank 1), whenever $u^T A \in \mathbb{Z}^n$ for some $u \in \mathbb{R}^m_+$. Let us further define

$$P' = \left\{ x \in \mathbb{R}^n \;\middle|\; u^T Ax \leq \lfloor u^T b \rfloor \text{ for all } u \in \mathbb{R}^m_+ \text{ for which } u^T A \in \mathbb{Z}^n \right\}$$

the so called *Chvátal closure of* P.

Clearly, Fact 1 implies that

$$P_I = P'_I \subseteq P' \subseteq P.$$

Applying the same to P', we can obtain $P'' \subseteq P'$ such that $P''_I = P_I$. The C-G cuts for P' are also valid inequalities for P_I, since we have $P_I = P'_I$. They are called C-G cuts of P of rank 2, etc. Defining $P^{(0)} = P$, and $P^{(t+1)} = \left(P^{(t)}\right)'$ be the Cvhátal closure of $P^{(t)}$ for $t = 0, 1, 2, ...$, we have

$$P_I \subseteq \cdots \subseteq P^{(t+1)} \subseteq P^{(t)} \subseteq \cdots \subseteq P^{(1)} \subseteq P^{(0)} = P.$$

The defining inequalities of $P^{(t+1)}$ (the C-G cuts of rank 1 of $P^{(t)}$) are called *C-G cuts of rank* t *of the polytope* P.

Theorem 1 (Chvátal (1973)) *For every rational bounded polyhedron* P *there exists a finite threshold* $t^* \in \mathbb{Z}$ *such that* $P_I = P^{(t)}$ *for all* $t \geq t^*$.

Proof. We shall prove the statement for polyhedra within the unit cube, i.e., for

$$P = \{x \mid Ax \leq b,\; 1 \geq x \geq 0\}$$

where $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, and the 1 and 0 in the definitons of P denote the full one and full zero vectors of dimension n, respectively. The proof we recall here is from the book by Nemhauser and Woolsey (1988).

We shall show that if $\pi^T x \leq \pi_0$ is a rational valid inequality for P_I with $\pi \in \mathbb{Z}^n$ and $\pi_0 \in \mathbb{Z}$, then it is (or it is dominated by) a C-G inequality of finite rank. This clearly will imply the statement, since P_I is defined by a finite number of facets, each being a rational valid inequality for P_I.

Clearly, if $P = P_I$, then there is not much to prove, since

Claim 1 *If* $\pi^T x \leq \pi_0$ *is a rational valid inequality for* P*, then it is (or it is dominated by) a C-G inequality of rank* 1.

Proof. Any valid inequality for P can be obtained from P by taking a non-negative linear combination of the defining inequalities. Since π is rational, this can be achieved by a rational linear combination. Multiplying then both sides of the obtained rational inequality by a large integer, we can get an inequality with integral coefficients, i.e., we can obtain a dominating inequality with integral coefficients for some $u \in \mathbb{Q}^m_+$. $\square$

Let us then assume that $P \neq P_I$, i.e. that P has some non-integral vertices, and denote by $V(P)$ the set of vertices of P.

Claim 2 *If $\pi^T x \leq \pi_0$ is valid for P_I, and $I \cup J$ is a partition of the index set $[n] = \{1, 2, ..., n\}$, then there exists a real $w \geq 1$ such that*

$$\pi^T x - w \left(\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \right) \leq \pi_0 \tag{2}$$

is a valid inequality for P.

Proof. Let us note that for all $v \in V(P)$ we have

$$\sum_{i \in I} v_i + \sum_{j \in J} (1 - v_j) \geq 0$$

and for $v \in V(P) \setminus \mathbb{Z}^n$ we must have strict inequality.

Thus, for all $v \in V(P) \cap \mathbb{Z}^n$ inequality (2) must hold, since $V(P) \cap \mathbb{Z}^n \subseteq P_I$ and $\pi^T x \leq \pi_0$ is valid for P_I.

Furthermore, we also have

$$\alpha = \min_{v \in V(P) \setminus \mathbb{Z}^n} \sum_{i \in I} v_i + \sum_{j \in J} (1 - v_j) \quad > \quad 0. \tag{3}$$

Let us choose

$$\gamma = \max_{x \in P} \left(\pi^T x - \pi_0 \right) \tag{4}$$

and set $w = \max\{1, \frac{\gamma}{\alpha}\}$.

Thus, for all $v \in V(P) \setminus \mathbb{Z}^n$ we have

$$\pi^T v - w \left(\sum_{i \in I} v_i + \sum_{j \in J} (1 - v_j) \right) \leq \gamma + \pi_0 - w\alpha \leq \pi_0$$

where the first inequality follows by (3) and (4), while the last one is implied by our selection of w. Consequently, (2) must also hold for all $v \in V(P) \setminus \mathbb{Z}^n$, completing the proof of the claim. $\square$

Claim 3 *If $\pi^T x \leq \pi_0$ is valid for P_I, $\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$, and $\pi^T x \leq \pi_0 + 1$ is a C-G cut, then for all partitions $I \cup J = [n]$ the inequality*

$$\pi^T x - \sum_{i \in I} x_i - \sum_{j \in J} (1 - x_j) \leq \pi_0 \tag{5}$$

is also a C-G cut.

Proof. By applying Claim 1 to the inequality (2) of Claim 2, we get that (2) is a C-G inequality for some $w \geq 1$. Furthermore

$$\pi^T x \leq \pi_0 + 1 \tag{6}$$

is also a C-G cut by our assumption. Then taking the convex combination of (2) and (6) by coefficients $\frac{1}{w}$ and $\frac{w-1}{w}$ we get that

$$\pi^T x - \sum_{i \in I} x_i - \sum_{j \in J} (1 - x_j) \leq \pi_0 + \frac{w-1}{w}$$

is a valid inequality for P_I, with integer coefficients on the left hand side, and thus by rounding down the right hand side we get (5), proving the claim. $\square$

Claim 4 *If $\pi^T x \leq \pi_0$ is valid for P_I, $\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$, and $\pi^T x \leq \pi_0 + 1$ is a C-G cut, then for all subsets $I, J \subseteq [n]$, $I \cap J = \emptyset$ the inequality*

$$\pi^T x - \sum_{i \in I} x_i - \sum_{j \in J} (1 - x_j) \leq \pi_0 \tag{7}$$

is also a C-G cut.

Proof. Let us now prove the claim by induction on $|[n] \setminus (I \cup J)|$. We already proved this for the case $0 = |[n] \setminus (I \cup J)|$ in Claim 3.

Assume $\ell \in [n] \setminus (I \cup J)$. Then, applying our inductive hypothesis for the pairs of sets $I \cup \{\ell\}, J$ and $I, J \cup \{\ell\}$, we know that the inequalities

$$\begin{array}{rcl} \pi_T x - \sum_{i \in I \cup \{\ell\}} x_i - \sum_{j \in J} (1 - x_j) & \leq & \pi_0 \\ \pi_T x - \sum_{i \in I} x_i - \sum_{j \in J \cup \{\ell\}} (1 - x_j) & \leq & \pi_0 \end{array}$$

are both C-G inequalities. Then, by taking their convex combination with weights $\frac{1}{2}$ - $\frac{1}{2}$, we get the valid inequality

$$\pi_T x - \sum_{i \in I} x_i - \sum_{j \in J} (1 - x_j) \quad \leq \quad \pi_0 + \frac{1}{2},$$

from which by rounding we obtain (7). $\square$

Finally, the theorem follows from Claim 4 when applying it to $I = J = \emptyset$. $\square$

The threshold t^* in Theorem 1 is called the Chvátal rank of P. From the above proof we can derive an upper bound on t^*, which may be exponential in i, in the worst case. The truth is that the Chvátal rank of a polytope may be exponential.

Gomory's fractional cutting plane algorithm

For practical purpose, we may not want to generate all Chvátal cuts of a certain rank – since it may happen that already P' has exponentially many defining inequalities, in terms of the size of P. Gomory (1960) proposed a sequential procedure in which such cuts are generated one-by-one, until an integral solution is found.

Consider the integer programming problem

$$\max\{c^T x \mid Ax = b, \ x \geq 0, \ integral\} \tag{8}$$

and assume that we solve the linear programming relaxation by a dual-simplex type procedure. At the optimum we have

$$x_B + B^{-1} A_N x_N = B^{-1} b$$

for the optimal basis B. If $B^{-1}b$ here is an integral vector, then $x_B = B^{-1}b$, $x_N = 0$ is an integer optimum of (8). Otherwise, we have an index i in the basis for which

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i \notin \mathbb{Z}$$

where $\bar{a}_{ij}$, $j \in N$ denotes the ith row of $B^{-1}A_N$ and $\bar{b}_i$ denotes the ith component of $B^{-1}b$. Thus, the inequality

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor$$

is a valid C-G cut for the set of feasible points of (8). Hence adding the equality

$$\sum_{j \in N} \{\bar{a}_{ij}\} x_j - x_{n+1} = \{\bar{b}_i\}$$

to the set of equations in (8), where $\{x\} = x - \lfloor x \rfloor$, defines a new system of equations, satisfied by all integral solution of the original problem, but violated by the current LP optimum. Extending B by $\{n+1\}$, we get a basis of the new system, which is dual feasible, but not primal feasible. Hence we can continue with the dual-simplex algorithm. In fact the first pivot will kick out x_{n+1} from the basis.

Gomory (1960) proved that in a finite number of steps the above procedure terminates with an integral basic solution, which is an optimum of (8).