Integrality of Polyhedra

Let us first recall a few basic notions of polyhedra.

Valid inequalities, faces, facets, and vertices

Let us denote by $[m] = \{1, 2, ..., m\}$ the set of indices, and let

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^T \mathbf{x} \leq b_i,\ i \in [m]\} \tag{1}$$

be a *polyhedron*, where $\mathbf{a}_i \in \mathbb{R}^n$, $i \in [m]$ are given vectors, and $b_i \in \mathbb{R}$, $i \in [m]$ are given reals. We can assume w.l.o.g. that $rank\{\mathbf{a}_i \mid i \in [m]\} = n$, i.e., in particular that $n \leq m$.

We say that a linear inequality $\mathbf{a}^T\mathbf{x} \leq b$ is *valid* for P if every vector $\mathbf{x} \in P$ satisfies it. We say that $L = L(\mathbf{a}, b) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = b\}$ is a supporting hyperplane, if $\mathbf{a}^T\mathbf{x} \leq b$ is a valid inequality for P, and $F = L \cap P \neq \emptyset$. The intersection F is called a *face* of P. A maximal face of P is called a *facet* of P, while a minimal face F of it is called a *vertex* if $F = \{\mathbf{v}\}$ for some $\mathbf{v} \in P$. A polyhedron P is called *pointed*, if it has at least one vertex. A polyhedron is called a *polytope*, if it is bounded.

Fact 1 *A nonempty set $F \subseteq P$ is a face of P (given by* (1)*) if and only if there exists a subset $I \subseteq [m]$ such that*

$$F = \{\mathbf{x} \in P \mid \mathbf{a}_i^T\mathbf{x} = b_i \textit{ for } i \in I\} \ = \ P \cap \bigcap_{i \in I} L(\mathbf{a}_i, b_i).$$

Proof. Assume first that F is a face of P, let $F = P \cap L$, where $L = L(\mathbf{a}, b)$, consider the linear programming problem $\min\{\sum_{i\in[m]} y_i b_i \mid \sum_{i\in[m]} y_i \mathbf{a}_i = \mathbf{a},\ y_i \geq 0,\ i \in [m]\}$, let $y^* = (y_1^*, ..., y_m^*)$ be an optimal solution to it, and let $I = \{i \in [m] \mid y_i^* > 0\}$. Then, we have $F = \{\mathbf{x} \in P \mid \mathbf{a}_i^T\mathbf{x} = b_i \text{ for } i \in I\}$ by complementary slackness.

Conversely, if $F = \{\mathbf{x} \in P \mid \mathbf{a}_i^T\mathbf{x} = b_i \text{ for } i \in I\}$ for some $I \subseteq [m]$, then let $\mathbf{a} = \sum_{i\in I} \mathbf{a}_i$, $b = \sum_{i \in I} b_i$, and set $L = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = b\}$. Then, we have $F = P \cap L$, and for every $\mathbf{x} \in P \setminus F$ we have $\mathbf{a}^t\mathbf{x} < b$. $\square$

Clearly, the larger the index set I defining a face F the smaller the face is.

Fact 2 *If $F = \{\mathbf{x} \in P \mid \mathbf{a}_i^T\mathbf{x} = b_i \text{ for } i \in I\}$ is a minimal nonempty face of P, where $I \subseteq [m]$, then*

$$rank\{\mathbf{a}_i \mid i \in I\} \;=\; rank\{\mathbf{a}_i \mid i \in [m]\}.$$

$\square$

It follows that a vertex is the unique solution to a subsystem, which has the same rank as the entire system, i.e., P has a vertex if and only if $rank\{\mathbf{a}_i \mid i \in [m]\} = n$.

Fact 3 *If a polyhedron is pointed, then every minimal nonempty face of it is a vertex.* $\square$

Fact 4 *A polytope P has a finite set V of vertices, and $P = \mathrm{conv}\,(V)$.* $\square$

Fact 5 *Let $\mathbf{v} \in P$ be an element of the polyhedron P. Then, $\mathbf{v}$ is a vertex if and only if it cannot be written as $\mathbf{v} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$ for some vectors $\mathbf{u}, \mathbf{w} \in P \setminus \{\mathbf{v}\}$.*

Proof. Suppose $\mathbf{v}$ is a vertex of $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^T\mathbf{x} \le b_i,\; i \in [m]\}$, i.e., $\{\mathbf{v}\} = \{\mathbf{x} \in P \mid \mathbf{a}_i^{\mathbf{x}} = b_i,\; i \in I\}$ for some subset $I \subseteq [m]$, and assume indirectly that $\mathbf{v} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$ for some vectors $\mathbf{u}, \mathbf{w} \in P \setminus \{\mathbf{v}\}$. Then $\mathbf{a}_i^T\mathbf{u} = b_i$ follows for all $i \in I$, contradicting the facts that $\mathbf{v}$ is the unique solution to that system of equalities, and that $\mathbf{v} \ne \mathbf{u}$.

Conversely, assume that $\mathbf{v}$ cannot be expressed as the middle point between two vectors from $P \setminus \{\mathbf{v}\}$, let $I = \{i \in [m] \mid \mathbf{a}_i^T\mathbf{v} = b_i\}$, and let $S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^T x = \mathbf{b}_i,\; i \in I\}$. Assume first that $S \ne \{\mathbf{v}\}$, and let $\mathbf{z} \in S \setminus \{\mathbf{v}\}$ and $R = \{\mathbf{v} + \lambda(\mathbf{z} - \mathbf{v}) \mid \lambda \in \mathbb{R}\} \subseteq S$. Since we have $\mathbf{a}_i^T\mathbf{v} < b_i$ for all $i \in [m] \setminus I$, for a sufficiently small $\epsilon > 0$ we have both $\mathbf{u} = \mathbf{v} - \epsilon(\mathbf{z} - \mathbf{v}) \in P$ and $\mathbf{w} = \mathbf{v} + \epsilon(\mathbf{z} - \mathbf{v}) \in P$. Since $\mathbf{v} = \frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$ and $\{\mathbf{u}, \mathbf{w}\} \subseteq P \setminus \{\mathbf{v}\}$, this contradicts our assumption on $\mathbf{v}$, and thus proves that $S = \{\mathbf{v}\}$, i.e., that $\mathbf{v}$ is a vertex of P. $\square$

Let us next considered polytopes and polyhedra given by a system of equalities:

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = b, \quad \mathbf{x} \ge 0\}, \tag{2}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Fact 6 *A feasible point* $\mathbf{v} \in P$ *is a vertex of* P *if and only if the columns of* A *corresponding to the nonzero components of* $\mathbf{v}$ *are linearly independent.*

Proof. See the handout on Convexity. $\square$

Given a full row rank matrix $A \in \mathbb{R}^{m \times n}$, an $m \times m$ submatrix B of it is called a *basis* of A if $det(B) \neq 0$. A vectors $\mathbf{x}$ satisfying $A\mathbf{x} = b$ is called a *basic solution*, if the nonzeros of $\mathbf{x}$ correspond to linearly independent columns of A. A vector $\mathbf{x}$ satisfying $A\mathbf{x} = b$ is called a *feasible basic solution* if it is basic and $\mathbf{x} \geq 0$.

Fact 7 *Given* $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, *consider the polyhedron*

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0\}.$$

Then, $\mathbf{v} \in P$ *is a vertex of* P *if and only if the vector* $\mathbf{z} = (\mathbf{x}, \mathbf{b} - A\mathbf{x})$ *is a feasible basic solution of the system of equations* $[A \mid I_m]\mathbf{z} = \mathbf{b}$.

Proof. Follows analogously to the above proof from Fact 2. $\square$

In the theory of linear programming the variables in $\mathbf{z}$ corresponding to $\mathbf{b} - A\mathbf{x}$ are called *slack* variables.

Integral Polytopes and Polyhedra

In the sequel we shall consider *rational polyhedra* $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^T\mathbf{x} \leq b_i,\ i \in [m]\}$, i.e. where $\mathbf{a}_i \in \mathbb{Q}^n$ and $b_i \in \mathbb{Q}$ for all $i \in [m]$. The *integral core* P_I of P is defined as

$$P_I = conv(P \cap \mathbb{Z}^n).$$

We say that P is *integral*, if $P = P_I$. Note that minimal faces of P are either vertices, or extremal directions, and hence the above definition is equivalent with saying that P is integral if all minimal faces of P contain integral vectors. Note also that in the bounded case, i.e., for polytopes, this is again equivalent with requiring that all vertices are integral vectors.

We shall denote by A the $m \times n$ matrix the rows of which are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_m$ (remember, the column vectors of A are denoted by $\mathbf{a}^1$, ..., $\mathbf{a}^n$).

The first important result characterizing integrality of polytopes is the following theorem:

Theorem 1 (Hoffman (1974)) *A rational polytope P is integral if and only if for all integral vectors $\mathbf{c} \in \mathbb{Z}^n$ the optimum value of the linear programming problem* $\max\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P\}$ *is an integer.*

Proof. It is immediate that if P is integral, then $\max\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P\}$ is an integer for all $\mathbf{c} \in \mathbb{Z}^n$, since the optimum value of a linear programming problem over a bounded polyhedron is attained at a vertex.

To see the converse direction, let $\mathbf{v} = (v_1, ..., v_n)$ be a vertex of P, and choose $\mathbf{c} \in \mathbb{Z}^n$ such that $\mathbf{v}$ is the unique optimal solution[1] to $\max\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P\}$. By multiplying $\mathbf{c}$ with a large integer, if necessary, we can assure[2] that $\mathbf{c}^T\mathbf{v} - \mathbf{c}^T\mathbf{u} > u_1 - v_1$ for all vertices $\mathbf{u}$ of P other than $\mathbf{v}$, implying that $\mathbf{c}^T\mathbf{v} + v_1 > \mathbf{c}^T\mathbf{u} + u_1$ for all vertices $\mathbf{u} \neq \mathbf{v}$ of P, i.e. that $\mathbf{v}$ is also a (unique) optimal solution to the linear programming problem $\max\{\bar{\mathbf{c}}^T\mathbf{x} \mid \mathbf{x} \in P\}$, where $\bar{\mathbf{c}} = (c_1 + 1, c_2 ..., c_n) \in \mathbb{Z}^n$. Since we assume that both optimum values $\mathbf{c}^T\mathbf{v}$ and $\bar{\mathbf{c}}^T\mathbf{v}$ are integers, $v_1 = \bar{\mathbf{c}}^T\mathbf{v} - \mathbf{c}^T\mathbf{v} \in \mathbb{Z}$ follows. Repeating this argument for v_2, ..., v_n, we get that $\mathbf{v}$ is an integral vector, and since we picked $\mathbf{v}$ as an arbitrary vertex of P, the theorem follows. $\square$

We recall here stronger extensions of this theorem for (not necessarily bounded) polyhedra:

[1] WHY is there such a $\mathbf{c}$?

[2] WHY?

Theorem 2 (Edmonds and Giles (1977)) *A rational polyhedron P is integral if and only if for each integral vector $\mathbf{c} \in \mathbb{Z}^n$ the optimum value of the linear programming problem $\max\{\mathbf{c}^T\mathbf{x} \mid \mathbf{x} \in P\}$ is an integer, whenever it is finite.*

Theorem 3 (Edmonds and Giles (1977)) *Let $A \in \mathbb{Q}^{m\times n}$, $\mathbf{b} \in \mathbb{Q}^m$, and consider the rational polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$. Then, P is integral if and only if each rational supporting hyperplane of P contains integral vectors.*

Proof. The necessity of the condition is trivial by the definitions. For the sufficiency, let us assume that each rational supporting hyperplane of P contains and integral vertex, and that P has a minimal face F which does not contain integral vertices. By Fact 1 it follows that there exists a subset $I \subseteq [m]$ of the rows such that $F = \{\mathbf{x} \in \mathbb{R}^n \mid A[I;]\mathbf{x} = \mathbf{b}[I]\}$. If this system of equalities does not have any integral solution, then by a result of Kronecker (1884) there exists $\mathbf{y} \in \mathbb{Q}^I$ such that $\mathbf{c}^T = \mathbf{y}^T A[I;]$ is integral and $\beta = \mathbf{y}^T\mathbf{b}[I]$ is not. We can assume that $\mathbf{y} \geq 0$, since adding a common multiple of the denominators of the rational entries in $A[I;]$ and $\mathbf{b}[I]$ does not change the above facts. Then the rational hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^T\mathbf{x} = \beta\}$ is supporting P (since $H \supseteq F$, and since by the nonnegativity of $\mathbf{y}$ we have $\mathbf{y}^T A[I;]\mathbf{x} \leq \mathbf{y}^T\mathbf{b}[I]$ for all $\mathbf{x} \in P$). However, for every integral vector $\mathbf{x} \in \mathbb{Z}^n$ we have $\mathbf{c}^T\mathbf{x} \in \mathbb{Z}$, and hence $\mathbf{c}^T\mathbf{x} \neq \beta$! This contradicts our assumption that H contains an integral vector, completing the proof. $\square$

Let us note that in the above proof we have $\beta = \max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$, and thus Theorem 2 follows similarly.

Lemma 1 *Let $A \in \mathbb{Z}^{m\times m}$ be a nonsingular integral square matrix. Then $A^{-1}\mathbf{b}$ is an integral vector for all integral vectors $\mathbf{b} \in \mathbb{Z}^m$ if and only if $det(A) = \pm 1$.*

Proof. Assume first that $det(A^{-1}) = \pm 1$. Then, by Cramer's rule, the matrix A^{-1} is also integral, and thus $A^{-1}b$ is integral for all integral vector $\mathbf{b} \in \mathbb{Z}^m$. For the converse direction, let us note that the integrality of $A^{-1}\mathbf{e}_i$ for $i = 1, ..., m$, where $\mathbf{e}_i$ denotes the ith unit vector, implies that A^{-1} itself is integral, and thus by $1 = det(I) = det(AA^{-1}) = det(A)det(A^{-1})$ we get that $det(A) = \pm 1$. $\square$

Let us call a matrix $A \in \mathbb{R}^{m\times n}$ *totally unimodular* if all square submatrices of it have a determinant equal to 0, 1, or -1. Note that this implies that all elements of A belong to $\{-1, 0, 1\}$, and thus in particular that A is integral.

Lemma 2 *A matrix $A \in \mathbb{R}^{m\times n}$ is totally unimodular if and only if $A' = [A \mid I_m]$ is totally unimodular.*

Proof. Let us denote by $A[I; J]$ the submatrix of A formed by the elements in the intersections of rows $i \in I$ and columns $j \in J$. Let us now consider an arbitrary square submatrix B of A, i.e., $B = A[I; J]$ for some $|I| = |J|$. Let $L = J \cup \{n+i \mid i \notin I\}$, and let $B' = A'[; L]$. Then B' is an $m \times m$ submatrix A', and $|det(B)| = |det(B')|$. □

Lemma 3 *A matrix $A \in \mathbb{R}^{m\times n}$ is totally unimodular of and only if*

$$A' = \begin{pmatrix} A \\ -A \\ I_n \end{pmatrix}$$

is totally unimodular. □

The following result characterizes the integrality of a large class of polyhedra, in terms of total unimodularity of the correspond coefficient matrix.

Theorem 4 (Hoffman and Kruskal (1956)) *Let $A \in \mathbb{Z}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and consider the polyhedra*

$$P_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0\}.$$

Then $P_{\mathbf{b}}$ is integral for every integral vector $\mathbf{b} \in \mathbb{Z}^m$ if and only if A is totally unimodular.

Proof. Assume first that A is totally unimodular. Then $[A \mid I_m]$ is unimodular by Lemma 2, and hence every basis B of it has determinant equal to ± 1, and thus $B^{-1}\mathbf{b}$ is integral for every integral vector $\mathbf{b} \in \mathbb{Z}^m$, implying the claim by Fact 7.

For the converse direction, let B be a basis of $[A \mid I_m]$. By Lemmas 1 and 2 it is enough to show that $B^{-1}\mathbf{v}$ is integral for all integral vectors $\mathbf{v} \in \mathbb{Z}^m$. For this, let us now fix an arbitrary vector $\mathbf{v} \in \mathbb{Z}^m$, choose $\mathbf{u} \in \mathbb{Z}^m$ such that $\mathbf{u} + B^{-1}\mathbf{v} \geq 0$, and set $\mathbf{b} = B(\mathbf{u} + B^{-1}\mathbf{v})$. Note that by extending $\mathbf{u}+B^{-1}\mathbf{v}$ by $n-m$ zeros, we can get a vector $\mathbf{z} \in \mathbb{R}^n$ for which $[A \mid I_m]\mathbf{z} = \mathbf{b}$ holds. Note also that $\mathbf{b} = B\mathbf{u} + \mathbf{v}$ is integral, since B is a submatrix of the integral matrix $[A \mid I_m]$, and u and v were chosen as integral vectors. Since the nonzeros of z correspond to the columns of B, and since B is a basis of $[A \mid I_m]$, the columns of B are linearly independent, implying by Fact 7 that $\mathbf{z}$ is a vertex of P_b. Since b is integral, this implies by our assumption that $\mathbf{z}$ is integral, i.e. that $\mathbf{u} + B^{-1}\mathbf{v}$ is integral, i.e. that $B^{-1}\mathbf{v}$ is integral. □

Corollary 1 *An integral matrix $A \in \mathbb{Z}^{m\times n}$ is totally unimodular if and only if the polyhedron*

$$P_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} \leq Ax \leq \mathbf{b},\ \mathbf{c} \leq \mathbf{x} \leq \mathbf{d}\}$$

is integral for all integral vectors $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$, and $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^n$.

Proof. Note that we have

$$P_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} = \mathbf{c} \ + \ \left\{ \mathbf{x}' \in \mathbb{R}^n \ \middle| \ \begin{array}{rcl} A\mathbf{x}' & \leq & \mathbf{b} - A\mathbf{c} \\ -A\mathbf{x}' & \leq & -\mathbf{a} + A\mathbf{c} \\ \mathbf{x}' & \leq & \mathbf{d} - \mathbf{c} \\ \mathbf{x}' & \geq & 0 \end{array} \right\}.$$

Thus the claim follows by Theorem 4 and Lemma 3, since we can realize an arbitrary integral vector $\mathbf{b}' \in \mathbb{Z}^{n+2m}$ in the form

$$\mathbf{b}' = \begin{pmatrix} \mathbf{b} - A\mathbf{c} \\ -\mathbf{a} + A\mathbf{c} \\ \mathbf{d} - \mathbf{c} \end{pmatrix}$$

with some integral $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$ and $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^n$. □

Corollary 2 *Let $A \in \mathbb{Z}^{m\times n}$ be a totally unimodular matrix, and $\mathbf{b} \in \mathbb{Z}^m$. Then, the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ is integral.*

Proof. Due to Fact 7, $\mathbf{v}$ is a vertex of P if and only if $\mathbf{v} = \mathbf{u} - \mathbf{w}$ for a vertex $(\mathbf{u}, \mathbf{w})$ of $Q = \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{2n} \mid [A \mid -A](\mathbf{y}, \mathbf{z}) \leq \mathbf{b},\ (\mathbf{y}, \mathbf{z}) \geq 0\}$. Furthermore, $[A \mid -A]$ is totally unimodular, if and only if A is. Therefore, Q is integral by Theorem 4, whenever $\mathbf{b}$ is an integral vector, and thus by the above correspondence, P is also integral. $\square$

Corollary 3 *Let A be an $m \times n$ integral matrix, $\mathbf{b} \in \mathbb{Z}^m$, $\mathbf{c} \in \mathbb{R}^n$, and suppose that the linear programming problem*

$$\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0\} \tag{3}$$

has an optimal solution $\mathbf{x}^*$ such that $\bar{A} = A[;J]$ is totally unimodular, where $J = \{j \mid \mathbf{x}^*_j \neq 0\}$. Then, (3) has an integral optimum.

Proof. Since $\mathbf{x}^*$ is an optimal solution to (3), we have

$$\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0\} \ = \ \max\{\bar{\mathbf{c}}^T\bar{\mathbf{x}} \mid \bar{A}\bar{\mathbf{x}} \leq \mathbf{b},\ \bar{\mathbf{x}} \geq 0\} \tag{4}$$

where $\bar{\mathbf{c}} = \mathbf{c}[J]$. Since the coefficient matrix of the right hand side problem is totally unimodular by our assumption, there exists an integral optimal solution to it by Corollary 2 (and by the fundamental theorem of linear programming). Extending this integral vector by zeros, we obtain an integral optimal solution to (3) by the equality (4). $\square$

Totally Unimodular Matrices

Example 1 (Poincaré (1900)) *Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix in which every column contains at most one $+1$ and at most one -1 entry. Then A is totally unimodular.*

Proof. We need to show that every square submatrix B of A has a determinant equal to 0 or ± 1. We shall show this by induction on the order of B. Clearly, for 1×1 submatrices this follows from the fact that every element of A is either 0 or ± 1. Assume that we have shown the claim for up to $k \times k$ submatrices of A and consider a submatrix B of order $k+1$. Note that if every column of B contains exactly 0 or 2 nonzero entries, then the sum of the rows of B must be equal to 0, and thus $det(B) = 0$ follows. Otherwise, B has a column with exactly one nonzero entry. Computing its determinant by expansion along this column, we get that $|det(B)|$ is the same as the absolute value of a $k \times k$ submatrix of A, since the only nonzero entry in this column is ± 1. Hence, the claim follows by our inductive hypothesis. $\square$

As a special case, we get the well-known fact from network theory:

Example 2 *The vertex-arc incidence matrix of a directed graph is totally unimodular.* $\square$

Another well-known case also can be derived from a special case of Example 1:

Example 3 *The vertex-edge incidence matrix A of an undirected graph $G = (V, E)$ is totally unimodular of and only if G is bipartite.*

Proof. Note first that if G is not bipartite, it contains a chordless odd cycle $W = \{w_1, w_2, ..., w_k\} \subseteq V$ with edge set $F = \{f_1, ..., f_k\} \subseteq E$, where $f_i = (v_i, v_{i+1})$ for $i = 1, ..., k-1$ and with $f_k = (v_k, v_1)$. Then the $k \times k$ submatrix $B = A[W; F]$ has $det(B) = \pm 2$.

On the other hand, if G is bipartite, let $V = A \cup B$ denote the partition of its vertices into two classes, such that $E \subseteq A \times B$, and obtain A' from A by changing the sign of its rows corresponding to $v \in B$. Note that A is totally unimodular if and only if A' is.[3] Since A' has the property of Example 1, the claim now follows. $\square$

[3] WHY?

A useful characterization of totally unimodular matrices is the following:

Theorem 5 (Ghouila-Houri (1962)) *An integral matrix $A \in \mathbb{Z}^{m\times n}$ is totally unimodular if and only if for every subset $C \subseteq [n]$ of the columns there is a partition $C = C_1 \cup C_2$ such that*

$$-\mathbf{e} \leq \sum_{j\in C_1} \mathbf{a}^j - \sum_{j\in C_2} \mathbf{a}^j \leq \mathbf{e}$$

where $\mathbf{e} = (1, 1, ..., 1) \in \mathbb{R}^m$.

Proof. Assume first that A is totally unimodular, and let $C \subseteq [n]$ be an arbitrary subset of the column indices. Let us denote by $\mathbf{d} \in \{0,1\}^n$ the characteristic vector of C, and consider the polyhedron

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid 0 \leq \mathbf{x} \leq \mathbf{d},\ \ \lfloor \frac{1}{2} A\mathbf{d} \rfloor \leq A\mathbf{x} \leq \lceil \frac{1}{2} A\mathbf{d} \rceil\}.$$

Since P is bounded and nonempty ($\frac{1}{2}\mathbf{d} \in P$), it has at least one vertex $\mathbf{v} \in P$ which is integral by Corollary 1. Then the vector $\mathbf{y} = \mathbf{d} - 2\mathbf{v} \in \{-1, 0, +1\}^n$ and $y_j \neq 0$ if and only if $j \in C$. Setting $C_1 = \{j \mid y_j = +1\}$ and $C_2 = \{j \mid y_j = -1\}$, we get

$$\sum_{j\in C_1} \mathbf{a}^j - \sum_{j\in C_2} \mathbf{a}^j = A(\mathbf{d} - 2\mathbf{v}).$$

Now, by $\mathbf{v} \in P$ we get

$$-\mathbf{e} + A\mathbf{d} \leq 2\lfloor \frac{1}{2} A\mathbf{d} \rfloor \leq 2A\mathbf{v} \leq 2\lceil \frac{1}{2} A\mathbf{d} \rceil \leq \mathbf{e} + A\mathbf{d}$$

from which

$$-\mathbf{e} \leq A(\mathbf{d} - 2\mathbf{v}) \leq \mathbf{e}$$

follows, as required.

For the converse direction we prove by induction on k that every $k \times k$ submatrix of A has a determinant equal to 0 or ± 1.

Applying the condition of the theorem for subsets $C \subseteq [n]$ of size $|C| = 1$, we get that every element of A is either 0 or ± 1, from which the claim follows for $k = 1$.

Assume now that we have already shown the claim for all $k \times k$ submatrices, and consider a submatrix $B = A[I; J]$ of order $|I| = |J| = k + 1$. If $det(B) = 0$ then we are done, otherwise by Cramer's rule, every entry in

$B^* = (det(B))B^{-1}$ is a determinant of a $k \times k$ submatrix of B, and whence of A, and thus by our induction hypothesis, all these entries are either 0 or ± 1, i.e., $B^* \in \{-1, 0, +1\}^{(k+1)\times(k+1)}$. Let $\mathbf{b}^*$ be the first column of B^*. Thus, we have $\mathbf{b}^* \in \{-1, 0, +1\}^{k+1}$, and $B\mathbf{b}^* = (det(B))\mathbf{e}_1$, where $\mathbf{e}_1$ denotes the 1st unite vector. Let us then define a subset $C \subseteq J \subseteq [n]$ of the columns of A by $C = \{j \mid b^*_j \neq 0\}$. By our assumption, there exists a partition of C into two subsets $C = C_1 \cup C_2$ such that

$$-\mathbf{e} \leq B\mathbf{y} \leq \mathbf{e} \tag{5}$$

where $\mathbf{e} = (1, 1, ..., 1) \in \mathbb{R}^{k+1}$, and where $\mathbf{y} \in \{-1, 0, +1\}^J$ is defined by

$$y_j = \begin{cases} 1 \text{ if } j \in C_1 \\ -1 \text{ if } j \in C_2 \\ 0 \text{ if } j \in J \setminus C. \end{cases}$$

Note that we have both b^*_j and y_j to belong to $\{-1, +1\}$ for $j \in C$ and equal to 0 for $j \in J \setminus C$. This implies that $\mathbf{b}^* - \mathbf{y}$ has only even coordinates, and thus by $B\mathbf{b}^* = (det(B))\mathbf{e}_1$ we get from (5) that

$$-\mathbf{e}_1 \leq B\mathbf{y} \leq \mathbf{e}_1.$$

Note that $\mathbf{b}^* \neq 0$, and thus $\mathbf{y} \neq 0$ by definitons, and therefore by the nonsingularity of B we have $B\mathbf{y} \neq 0$. Since $B\mathbf{y}$ is an integral vector, we can conclude from the above that either $B\mathbf{y} = \mathbf{e}_1$ or $B\mathbf{y} = -\mathbf{e}_1$, from which $(det(B))\mathbf{y} = \pm(det(B))B^{-1}\mathbf{e}_1 = \pm\mathbf{b}^*$ follows. This implies finally $det(B) = \pm 1$, because both $\mathbf{y}$ and $\mathbf{b}^*$ are vectors of -1, 0 or 1 entries. □

It is immediate by the definitions that the role of rows and columns can be interchanged in the above characterization. Thus, an immediate consequence of the above theorem is the following claim, used in scheduling problems:

Example 4 *Let A be a binary matrix, such that every column of A contains a contiguous sequence of nonzeros. Then, A is totaly unimodular.*

Proof. For an arbitrary subset R of the rows, assign the elements of R alternating into R_1 and R_2. This partition will satisfy the conditions of Theorem 5. □

There are several results in the literature showing various interesting properties of totally unimodular matrices. Let us recall here the following nice characterization:

Theorem 6 (Baum and Trotter (1977)) *Given an integral matrix $A \in \mathbb{Z}^{m\times n}$, let us denote by $P_{k,\mathbf{b}}$ the polyhedron*

$$P_{k,\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq k\mathbf{b},\ \mathbf{x} \geq 0\},$$

where $\mathbf{b} \in \mathbb{R}^m$ and $k \in \mathbb{Z}_+$. Then, A is totally unimodular if and only if for each integral vector $\mathbf{b} \in \mathbb{Z}^m$, integer $k \geq 1$, and integral vector $\mathbf{y} \in P_{k,\mathbf{b}} \cap \mathbb{Z}^n$ there exist integral vectors $\mathbf{x}_j \in P_{1,\mathbf{b}} \cap \mathbb{Z}^n$, $j = 1, ..., k$ such that $\mathbf{y} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$. $\square$

Seymour (1980) proved that totally unimodular matrices arise in a certain way from network matrices and from the following two special matrices[4]

$$M_5 = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad N_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}. \qquad (6)$$

We do not describe in detail this decomposition result, but recall an important consequence of it:

Corollary 4 (Seymour (1980)) *The problem of testing if a given matrix is totally unimodular has a good characterization.* $\square$

In fact Schrijver (1986) constructed a polynomial time algorithm which recognizes if a given matrix is totally unimodular, or not.

[4]Prove that both M_5 and N_5 are totally unimodular!

Unimodular Matrices

We shall call a full row rank matrix A *unimodular*, if $det(B) = \pm 1$ for all bases B of A.

The following result characterizes unimodular matrices, in terms of the integrality of a related class of polyhedra (cf. Veinott and Dantzig (1968)):

Theorem 7 (Hoffmand and Kruskal (1956)) *Let $A \in \mathbb{Z}^{m\times n}$ be a full row rank integral matrix, and consider the following polyhedra for given vectors $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$:*

$$P_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\} \quad \text{and} \quad Q_{\mathbf{c}} = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y}^T A \geq \mathbf{c}^T\}.$$

Then, the following properties are equivalent:

(i) A is unimodular;

(ii) for each integral vector $\mathbf{b} \in \mathbb{Z}^m$ the polyhedron $P_{\mathbf{b}}$ is integral;

(iii) for each integral vector $\mathbf{c} \in \mathbb{Z}^n$ the polyhedron $Q_{\mathbf{c}}$ is integral.

Proof. *(i)⇒(ii)*: Every basis B of A has determinant equal to ± 1, and thus $B^{-1}\mathbf{b}$ is integral for every integral vector $\mathbf{b} \in \mathbb{Z}^m$, implying the claim by Fact 6.
(i)⇒(iii): Let $\mathbf{c} \in \mathbb{Z}^n$, and consider a minimal face F of $Q_{\mathbf{c}}$. Then by Facts 1 and 2 we have that $F = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^T A[;J] = \mathbf{c}^T[J]\}$ for a subset J of the columns consisting of linearly independent columns with $rank(A[;J]) = rank(A) = m$. Then, by the unimodularity of A it follows that $det(A[;I]) = \pm 1$, and thus F contains an integral $\mathbf{y} \in F$.
(ii)⇒(i): Let B be a basis of A. By Lemma 1 it is enough to show that $B^{-1}\mathbf{v}$ is integral for all integral vectors $\mathbf{v} \in \mathbb{Z}^m$. For this, let us now fix an arbitrary vector $\mathbf{v} \in \mathbb{Z}^m$, choose $\mathbf{u} \in \mathbb{Z}^m$ such that $\mathbf{u} + B^{-1}\mathbf{v} \geq 0$, and set $\mathbf{b} = B(\mathbf{u} + B^{-1}\mathbf{v})$. Note that by extending $\mathbf{u} + B^{-1}\mathbf{v}$ by $n - m$ zeros, we can get a vector $\mathbf{z} \in \mathbb{R}^n$ for which $A\mathbf{z} = \mathbf{b}$ holds. Note also that $\mathbf{b} = B\mathbf{u} + \mathbf{v}$ is integral, since B is a submatrix of the integral matrix A, and $\mathbf{u}$ and $\mathbf{v}$ were chosen as integral vectors. Since the nonzeros of $\mathbf{z}$ correspond to the columns of B, and since B is a basis of A, the columns of B are linearly independent, implying by Fact 6 that $\mathbf{z}$ is a vertex of $P_{\mathbf{b}}$. Since $\mathbf{b}$ is integral, this implies by our assumption that $\mathbf{z}$ is integral, i.e. that $\mathbf{u} + B^{-1}\mathbf{v}$ is integral, i.e. that $B^{-1}\mathbf{v}$ is integral.

(iii)$\Rightarrow$*(i)*: Follows analogously to the above, since by Facts 1 and 2 the vertices of $Q_{\mathbf{c}}$ are also in a one-to-one correspondence with the bases of A.$\square$

An immediate corollary of the above is the following claim:

Corollary 5 *Let A be a full column rank integral matrix. Then A^T is unimodular if and only if both sides of the linear programming duality equality*

$$\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\} \;=\; \min\{\mathbf{y}^T\mathbf{b} \mid \mathbf{y}^T A = \mathbf{c}^T,\; \mathbf{y} \geq 0\}$$

are attained by integral vectors $\mathbf{x}$ and $\mathbf{y}$, for all integral vectors $\mathbf{b}$ and $\mathbf{c}$. $\square$

Several other extensions of totally unimodular matrices were considered in the literature, including *balanced matrices* (see e.g., Berge and Las Vergnas (1970), Fulkerson, Hoffman and Oppenheim (1974), and Conforti, Cornuejols and Rao (1999)), *perfect matrices* (see e.g., Conforti, Cornuejols and de Francesco (1997), Guenin (1998), and Tamari (2000)), *ideal matrices* (see e.g., Nobili and Sassano (1995, 1998), and Cornuejols and Guenin (2002)), etc.