Second Homework Set

Take home exam. Submit solutions via email, preferably in pdf format.

Scanned images are OK, as long as they are readable.

Solutions are due by midnight on Monday, May 13, 2019 (SHARP!)

Problem 1: Consider the following real functions: $F_1(x) = \min\{\lfloor 5x-2 \rfloor; \lfloor 7x-12 \rfloor\}$ and $F_2(x) = \min\{\lfloor 5x-4 \rfloor; \lfloor x+12 \rfloor\}$. Which one is superadditive and why?

Problem 2: Assume that $a_i \in \mathbb{R}^n$ for $i = 1, \ldots, m$, and define

$$\Phi(x) \;=\; \max_{1 \le i \le m} a_i^T x.$$

Prove that $\Phi(x)$ is a subadditive positive homogeneous function. In other words prove that it satisfies the following relations:

$$\begin{array}{rll} \Phi(x+y) & \le \Phi(x) + \Phi(y) & \text{for all } x, y \in \mathbb{R}^n, \\ \Phi(\lambda x) & = \lambda \Phi(x) & \text{for all } x \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R}_+. \end{array}$$

Problem 3: Consider the matrix A given below:

$$A \;=\; \begin{pmatrix} 5 & -4 & 2 & 0 & -8 \\ -1 & 3 & 3 & -5 & 7 \\ 0 & 1 & 0 & 2 & 3 \end{pmatrix}$$

- What is its Hermite normal form?
- Does the vector $\mathbf{b} = (1, 2, 0)$ belong to the generated lattice $\langle A \rangle$?

Problem 4: Consider the following integer programming problem:

$$\begin{array}{rrl} \min x & & \\ s.t. & & \\ -2x & +6y & \le 15 \\ 2x & -10y & \le -25 \\ x, & y & \text{are integers} \end{array}$$

What is the optimum $(x^*, y^*) \in \mathbb{R}^2$ of the linear programming relaxation? You can use AMPL to compute it. Find a CG-cut of rank one that cuts off (x^*, y^*).

Problem 5: Consider the firehouse location problem and the small numerical example presented in class (see also the AMPL files). Assume that each of the firehouses that could be built in A or B can handle only up to 5000 fires each, while those that could be built in C, D, or E have the capacity of 3000 fires, each. Update the mathematical model to accommodate such capacity limits. What is the new optimal solution in our small numerical example? (Use AMPL, if needed.)

Problem 6: Consider the cone generated by vectors $(2,3)$ and $(2,11)$ in $\mathbb{R}^2$. What is its minimal Hilbert basis? Is that unique?

Problem 7: Consider the planar graph in Figure 1, and recall the algorithm to compute a maximum cut in planar graphs. What is a maximum cut in this example?

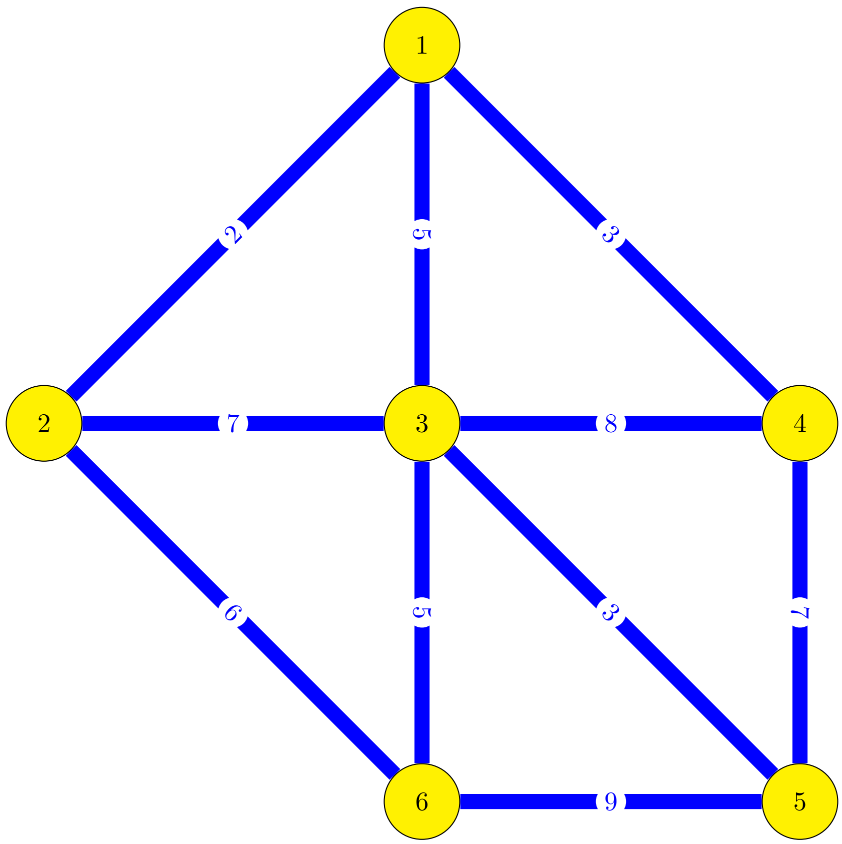

Figure 1: Planar graph with weighted edges

Problem 8: Can you find an integer solution to the following system of equations?

$$\begin{pmatrix} 3 & 7 & 15 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Problem 9: Consider the directed network on vertices $V = \{s, 1, 2, 3, 4, 5, t\}$ given by the following list. An element of this list has the form (i, j, c) where

(i, j) is an arc of the network, and c is the capacity of arc (i, j):

$$(s, 1, 17),\ (s, 2, 11),\ (1, 2, 9),\ (1, 3, 12),\ (1, 4, 5),\ (2, 1, 6),\ (2, 3, 4),$$
$$(3, 4, 2),\ (3, t, 15),\ (3, 5, 11),\ (3, t, 12),\ (4, 5, 9),\ (4, t, 15),\ (5, t, 7)$$

- What is the maximum $s \to t$ flow in this network? (You may use AMPL to compute this.)
- What is a minimum cut?
- **EXTRA CREDIT; mandatory for PhD students:** Recall the definition of the *cut value* $C(S)$ for subsets $S \subseteq V$, $s \in S$, $t \notin S$. Prove that C is a submodular function, that is for $S_i \subseteq V$, $s \in S_i$, $t \notin S_i$, $i = 1$, we have
$$C(S_1 \cap S_2) + C(S_1 \cup S_2) \le C(S_1) + C(S_2).$$

Problem 10 [**ONLY for PhD students**]
Hermitia is formed as the peaceful economic union of Parodia and Utopia, neither of which however wanted to give up their own currencies, the BING and the BANG. To simplify matters they agreed to use common coins in three nominations. On each of the coins the common value on one side is printed in BANGs, the currency of Parodia, and on the other side it is printed in BINGs, the currency of Utopia. The three coins are the following: the **filler** worth 3 bangs and 1 bing, the **kuna** worth 7 bangs and 2 bings, and the **taller** worth 15 bangs and 4 bings. (Don't even ask ..., the negotiations took over three years.)

Stores are required to post the prices of items as pairs of nonnegative integers. For instance, a pencil maybe sold for (11,3), meaning its bang price is 11 and its bing price is 3. A customer may buy it by giving two kunas to the cashier who returns one filler to the customer: 2(7,2) - (3,1) = (11,3).

- Assuming that the store has unlimited number of coins in each of the three nominations, can you buy a large eraser, the price of which is 3 bangs in Parodia, and 5 bings in Utopia? Why and how?
- In Hilberton, a smaller town in Hermitia, the municipality forbids the store keepers to return any money (in order to save the cashiers form bringing in and keeping large quantities of coins for opening; in other words, you can buy there something only if you can pay the exact price, in both currencies). To make this law feasible, the municipality also orders all stores to post price combinations *x bang/y bing* only for integers satisfying the inequalities $3y \le x$ and $4x \le 15y$. Is it indeed possible to pay for any item, the price of which satisfies the municipality's rule, with an appropriate combination of the above coins? Why? What other nomination(s) would make the municipality's rule feasible?

[Hint: Be careful and do not jump to conclusions too fast! Take some of the names, as suggestions ...]