Greedy Algorithms, Independence Systems, and Matroids.

Endre Boros

26:711:653: Discrete Optimization

Independence systems

- A hypergraph $(E, \mathcal{F})$, $\mathcal{F} \subseteq 2^E$ is called an **independence system** if it satisfies the following two axioms:

  (M0) $\emptyset \in \mathcal{F}$

  (M1) If $X \subseteq Y \in \mathcal{F}$ then $X \in \mathcal{F}$.

- Sets in $\mathcal{F}$ are called **independent**, while those in $2^E \setminus \mathcal{F}$ are called **dependent**.

- Maximal independent sets are called **bases**. The set of bases of $\mathcal{F}$ is denoted by $\mathcal{B} \subseteq \mathcal{F}$.

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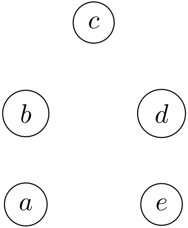

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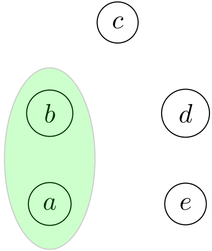

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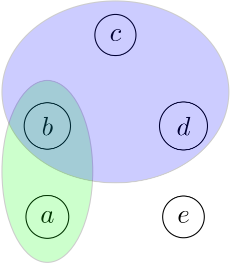

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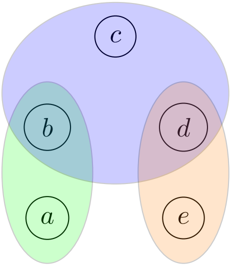

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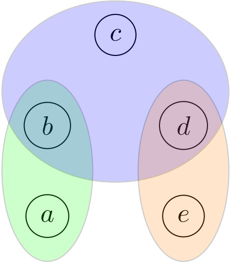

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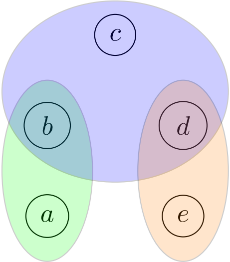

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Maximum Weight Independent Set Problem

- Given an independence system $(E, \mathcal{F})$ $(\mathcal{F} \subseteq 2^E)$ and nonnegative real weights $c : \mathcal{F} \mapsto \mathbb{R}_+$, find an independent set (a hyperedge) $F \in \mathcal{F}$ such that
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is as large as possible.
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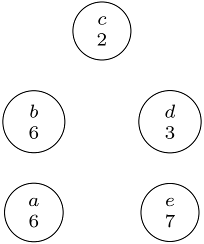

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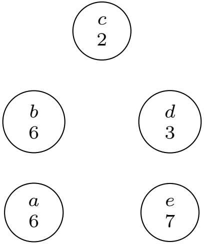

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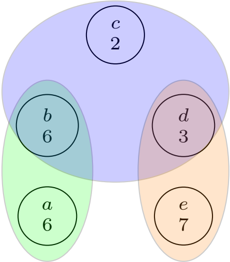

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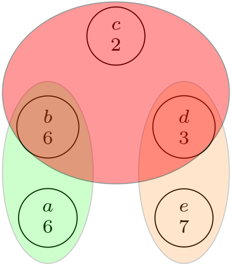

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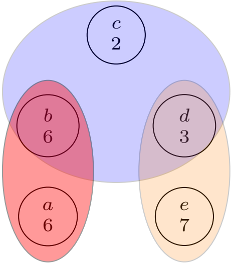

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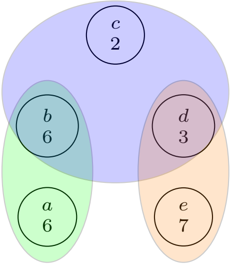

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is as large as possible.
- **MWISP**:
$$\max_{F \in \mathcal{F}} c(F)$$
- Independence systems are typically not given explicitly, but via an independence **oracle**: for a given $S \subseteq E$ such an oracle tells if $S \in \mathcal{F}$ or not.

Examples for MWISP: maximum weight stable set

- Given a hypergraph $\mathcal{H} \subseteq 2^V$, a subset $S \subseteq V$ is called **independent** (stable) if $H \not\subseteq S$ for all $H \in \mathcal{H}$.
- **The family $\mathcal{F}$ of all stable sets form an independence system. (WHY?)**
- The hypergraph $(V, \mathcal{H})$ serves as an independence oracle for $\mathcal{F}$:
  - Given $S \subseteq V$, one need to check for all $H \in \mathcal{H}$ if $H \subseteq S$ or not.
- Consider $V = \{1, 2, 3, 4, 5\}$ and $\mathcal{H} = \{12, 234, 45\} \subseteq 2^V$
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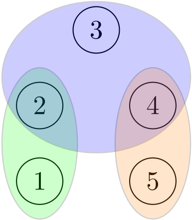

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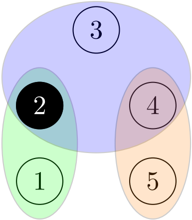

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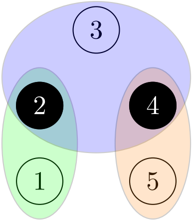

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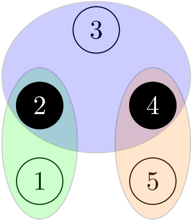

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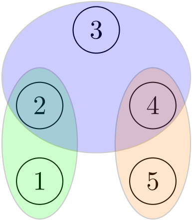

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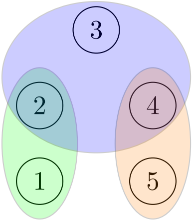

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- Maximum weight matching
- Minimum weight perfect matching
- Maximum weight spanning tree
- Shortest path
- Generalized knapsack
- Set covering
- **Some of the above are easy (polynomial), while others are hard (NP-hard).**
- The simple and fast HEURISTICS (BEST-in-GREEDY) will provide a feasible solution for all (with some guarantees).

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Best-in-Greedy

Input: An independence system $(E, \mathcal{F})$, and weights $c : E \longrightarrow \mathbb{R}$.

Initialize: **Set** $F^G = \emptyset$, and **sort** $E = \{e_1, e_2, ..., e_m\}$ such that

$$c(e_1) \geq c(e_2) \geq \cdots \geq c(e_m).$$

Main Loop: **For** $k = 1, ..., m$ **do**:

If $F^G \cup \{e_k\} \in \mathcal{F}$ **then** $F^G = F^G \cup \{e_k\}$.

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Correctness and Running time of Best-in-Greedy

Complexity: Sorting in **Initialization** takes $O(m \log m)$ time, while **Main Loop** can be executed in $O(m)$ time with $O(m)$ calls to the independence oracle. Thus, the total time needed to run (Best-In) Greedy is $O(m \log m + mI)$, where I is the worst case time we need to run the independence oracle.

Correctness: It is easy to verify that we have $F^G \in \mathcal{F}$ upon termination, since in the **Main Loop** F^G is updated only if it remains independent.

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Analysis of Best-in-Greedy

- To a subset $X \subseteq E$ we associate the **induced sub-hypergraph** $\mathcal{F}_X = \{X \cap F \mid F \in \mathcal{F}\}$, and let us denote by $\mathcal{B}_X$ the set of bases of $\mathcal{F}_X$.
- Let us define the **rank** of a subset $X \subseteq E$ by

$$r(X) \;=\; \max_{F \in \mathcal{F}} |F \cap X| \;=\; \max_{B \in \mathcal{B}_X} |B|.$$

- Let the **lower rank** of X be defined by

$$\rho(X) \;=\; \min_{B \in \mathcal{B}_X} |B|.$$

- Let the **rank quotient** of $(E, \mathcal{F})$ be defined as

$$q(E, \mathcal{F}) \;=\; \min_{X \subseteq E} \frac{\rho(X)}{r(X)}.$$

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Analysis of Best-in-Greedy cont'd

Lemma 1

We have the following relations hold for all independent sets $F \in \mathcal{F}$*, subsets* $X \subseteq E$*, and bases* $B \in \mathcal{B}_X$ *of the induced independence system* $\mathcal{F}_X$*:*

$$|B| \geq \rho(X) = \min_{A \in \mathcal{B}_X} |A|. \tag{1a}$$

$$\frac{\rho(X)}{r(X)} \geq q(E, \mathcal{F}) = \min_{Y \subseteq E} \frac{\rho(Y)}{r(Y)} \tag{1b}$$

$$|F \cap X| \leq r(X) = \max_{F \in \mathcal{F}} |F \cap X| \tag{1c}$$

□

Analysis of Best-in-Greedy cont'd

Theorem 2 (Jenkins (1976), Korte and Hausmann (1978))

Given an independence system $(E, \mathcal{F})$ *and weights* $c : E \mapsto \mathbb{R}_+$*, let* F^G *denote the solution obtained by the best-in-greedy procedure, and let* F^{OPT} *denote the optimal solution. Then we have*

$$q(E, \mathcal{F}) \leq \frac{c(F^G)}{c(F^{OPT})} \leq 1.$$

Furthermore, for every independence system there exist weights such that the lower bound is attained in the above statement.

- Guarantee $q(E, \mathcal{F})$ does not depend on the weights $c : E \mapsto \mathbb{R}_+$!!!

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