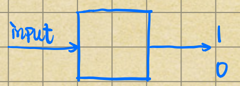
— NP-Complete Problem:

- If there is polynomial-time algorithm for a problem that problem is <u>tractable</u>.
- $O(2^n)$; $O((1+\alpha)^n)$ $\alpha > 0$

1. Decision Problem



(a). Given a graph is it connected?

(b). Given a integer is it prime?

(c). Given a boolean expression is it satsification? (SAT)

$X \wedge (y \vee (\neg z \wedge X))$ $X=1$, $y=1$ $z=0$.

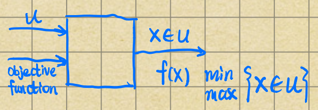
2. Search Problem:

(a). Given a graph, find a path from vertex a to vertex b.

(b). Given a set, find an item in that set.

(c). Sort, determinent

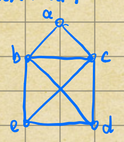
3. Optimization Problem:



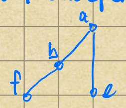
(a). Shortest Path / $\substack{\text{min} \\ \text{max}}$ spanning tree

(b). Maximum Clique Problem → Decision Version of the Max-CLiQue Problem:

(1). Given a graph $G=(V,E)$ and an integer k, does G have a clique of size k or larger?



(c). Maximum Independent Set Problem:



— Search Problem $\Longleftrightarrow$ Set / language membership

An example: Satisfiability problem:

- Alphabet $\{x, \ X, \ \wedge, \vee, \neg, (,), T, F\}$

• Alphabet: $\{x_1, \ldots, x_n, \wedge, \vee, \neg, (,), T, F\}$

U^n_{sat} = set of legal (well-formed) boolean expression : $\begin{cases} x_1 (\wedge x_2 (\vee \rightsquigarrow \text{illegal} \\ x_1 \wedge x_2 \vee (x_3 \vee \neg (x_4 \wedge \neg x_5)) \in U \rightsquigarrow \text{legal} \end{cases}$

$\Sigma^n_{SAT} \subseteq U^n_{SAT} = \{ e \in U \mid$ there is a assignment of T, F to variable in e that makes the expression true $\}$.

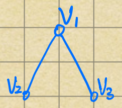
$x_1 \wedge (x_2 \vee \neg x_3) \in \Sigma^n_{SAT}$

$x_1 = T \quad x_2 = T \quad x_3 = F$

$x_1 \wedge (\neg x_1) \wedge x_2 \notin \Sigma^n_{SAT}$

— Graph Connectivity: $G(V, E)$.

• Alphabet = $\{v_1, \ldots, v_n, \text{“}\{\text{”}, \text{“}\}\text{”}\}$



$v_1, v_2, v_3 \{v_1, v_2\}, \{v_2, v_3\}$.

$U = \{$all sequence representing graphs on n-vertices$\}$

Σ^n_{SAT} = The set of connect graphs :

$\Sigma^n \subseteq U^n$; $\Sigma^n = \{$There exists a path from any vertex to any other vertex$\}$.

1. Certificate (polynomial-time)

Σ^n

• Given $w \in U$, does $w \overset{?}{\in} \Sigma$

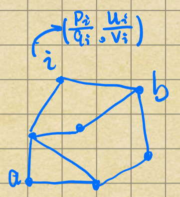
— For SAT: Certificate is a particular assignment of T, F to each Var x_i.

For CLIQUE: Certificate is a subset of vertices $v_1, \ldots, v_0$

Non-deterministic Polynomial time algorithm:

• NP = $\{$The set of decision Problem for which there is a non-deterministic poly time algorithm$\}$

EX.



• Does there exist a path of larger $\alpha = \frac{z}{w}$ or shorter?

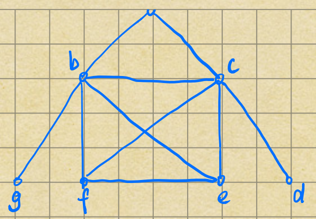
$$\sqrt{\left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right)^2 + \left(\frac{u_1}{v_1} - \frac{u_2}{v_2}\right)^2} + \sqrt{\cdots} + \sqrt{\quad} \leq \alpha = \frac{z}{w}$$

Polynomial-time Reduction:

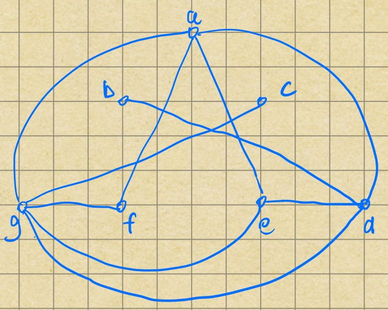
A decision Problem Σ_1 is polynomial-time reducible to another decision problem Σ_2

— Max independent set problem is reducible to MAX CLIQUE Problem:

α

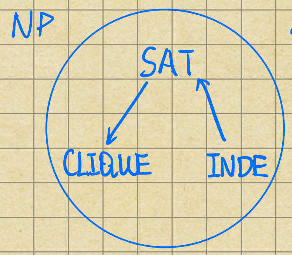


Independent: $\{c, g, f, d\}$



— Σ_2 if there is a polynomial-time algorithm that is allowed to call the decision Problem Σ_2 a polynomial number of time and solve Σ_1

— NP



- NP-Complete Problem is a problem / language where every after language in NP is reducible to it.

— CNF: Conjuctive normal form

$(X_1 \vee \bar{X}_4 \vee \cdots) \wedge (\qquad) \wedge \cdots (\qquad)$

$(X_1 \vee \bar{X}_2)(X_1 \vee X_3 \vee \bar{X}_5)(X_2 \vee X_3)$

$X_1 + X_2\overline{(X_3 + \bar{X}_4)} + (\overline{X_3\bar{X}_4} + X_4)$

- De Morgan's laws:

$\overline{(X_1+X_2)} = \bar{X}_1 \cdot \bar{X}_2$ $\qquad$ $\overline{X_1X_2} = \bar{X}_1 + \bar{X}_2$

$\overline{(X_1+\cdots X_n)} = \bar{X}_1 \cdots \bar{X}_n$ $\qquad$ $\overline{X_1 \cdots X_n} = \bar{X}_1 + \cdots + \bar{X}_n$

- CNF-SAT is NP-Complete

— SAT is NP-complete (Cook's Theorem)

CNF-SAT is NP-Complete, SAT $\xrightarrow{\text{reduce}}$ CNF-SAT

3-SAT 〃 〃 〃, CNF $\xrightarrow{\text{reduce}}$ 3-SAT

Max CLIQUE 〃 〃, 3AT $\longrightarrow$ Max CLIQUE

Max INDEP 〃 〃, Max CLIQUE $\longrightarrow$ Max INDEP

– 3-SAT:

$(\cdot + \cdot + \cdot)(\cdot + \cdot + \cdot) \cdots\cdots (\cdot + \cdot + \cdot)$

- CNF-SAT $\xrightarrow{\text{reduce}}$ 3-SAT:

$$(X_1 + X_2) = (X_1 + X_2 + y)(X_1 + X_2 + \bar{y})$$

$$\begin{cases} X \to (x+y)(x+\bar{y}) \\ \quad \to (x+y+z)(x+y+\bar{z})(x+\bar{y}+u)(x+\bar{y}+\bar{u}) \end{cases}$$

$(X+y+Z+W) = (X+y+U)(U \equiv Z+W).$ $\qquad a \equiv b \Leftrightarrow (a+\bar{b})(\bar{a}+b)$

$\to (U + \overline{Z+W})(\bar{U} + Z + W)$

$(U + \bar{Z}\,\bar{W})(\bar{U} + Z + W)$

$(U+\bar{Z})(U+\bar{W})(\bar{U}+Z+W)$

$(U+\bar{Z}+\alpha)(U+\bar{Z}+\bar{\alpha})(U+\bar{W}+\beta)(U+\bar{W}+\bar{\beta})(\bar{U}+Z+W)$

– MAX-CLIQUE $\longleftarrow$ 3-SAT

$$(\bar{X}_1 + X_2 + X_3)(X_1 + \bar{X}_3 + X_5)$$

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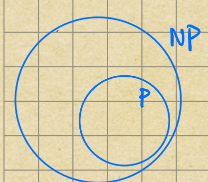
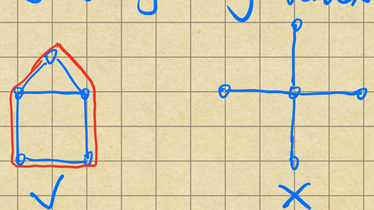


$P \subseteq NP$

$P \stackrel{?}{=} NP$

– Hamiltonian Cycle:

go through every vertex at once



- TSP Problem

– Integer Programming (IP)

$$\text{Max/Min } w_1 x_1 + \cdots w_n x_n \quad \text{s.t.} \begin{cases} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \le b_1 \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n \le b_n \end{cases}$$

$X_i \in \{0, 1, 2, \ldots, n\}.$