Hermite Normal Form

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26:711:653: Discrete Optimization

Spring, 2019

Is there an integer solution to the system of equations?

$$36x_1 + 30x_2 + 21x_3 = 10$$

- **If and only if** $gcd(36, 30, 21)$ is a divisor of 10!
- $gcd(36, 30, 21)$ ... by Euclid's algorithm ($\approx$ 300 BC)
- ... by computing the **Hermite normal form** of $[36, 30, 21]$:
- $$\begin{aligned}&[36, 30, \mathbf{21}]\\ &\quad\to [15, \mathbf{9}, 21] \to [21, 15, \mathbf{9}]\\ &\qquad\to [\mathbf{3}, 6, 9] \to [9, 6, \mathbf{3}]\\ &\qquad\quad\to [0, 0, 3] \to [3, 0, 0]\end{aligned}$$
- Answer is **NO**!

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$$\begin{bmatrix} 15 & -6 & 9 & 6 \\ -1 & 0 & -7 & 10 \\ 4 & -1 & 2 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 15 \\ 3 \\ 7 \end{pmatrix}$$

Row 1: switch the sign ... and sort ...

$$\begin{bmatrix} 15 & \color{red}{\mathbf{-6}} & 9 & 6 \\ -1 & 0 & -7 & 10 \\ 4 & -1 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 15 & 6 & 9 & 6 \\ -1 & 0 & -7 & 10 \\ 4 & 1 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 15 & 9 & 6 & 6 \\ -1 & -7 & 0 & 10 \\ 4 & 2 & 1 & 8 \end{bmatrix}$$

Row 1: take away the smallest positive ... and resort ...

$$\begin{bmatrix} 15 & 9 & 6 & \color{red}{\mathbf{6}} \\ -1 & -7 & 0 & 10 \\ 4 & 2 & 1 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 3 & 0 & \color{red}{\mathbf{6}} \\ -21 & -17 & -10 & 10 \\ -12 & -6 & -7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 3 & 3 & 0 \\ 10 & -21 & -17 & -10 \\ 8 & -12 & -6 & -7 \end{bmatrix}$$

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$$\begin{bmatrix} 6 & 3 & \color{red}{\mathbf{3}} & 0 \\ 10 & -21 & -17 & -10 \\ 8 & -12 & -6 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & \color{red}{\mathbf{3}} & 0 \\ 44 & -4 & -17 & -10 \\ 20 & -6 & -6 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 0 & 0 \\ -17 & 44 & -4 & -10 \\ -6 & 20 & -6 & -7 \end{bmatrix}$$

Row 2: switch the sign ... and sort ...

$$\left[\begin{array}{r|rrr} 3 & 0 & 0 & 0 \\ \hline -17 & 44 & \color{red}{\mathbf{-4}} & \color{red}{\mathbf{-10}} \\ -6 & 20 & -6 & -7 \end{array}\right] \rightarrow \left[\begin{array}{r|rrr} 3 & 0 & 0 & 0 \\ \hline -17 & 44 & 4 & 10 \\ -6 & 20 & 6 & 7 \end{array}\right] \rightarrow \left[\begin{array}{r|rrr} 3 & 0 & 0 & 0 \\ \hline -17 & 44 & 10 & 4 \\ -6 & 20 & 7 & 6 \end{array}\right]$$

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$$\left[\begin{array}{r|rrr} 3 & 0 & 0 & 0 \\ \hline -17 & 4 & \mathbf{2} & 0 \\ -6 & 6 & -5 & -46 \end{array}\right] \to \left[\begin{array}{r|rrr} 3 & 0 & 0 & 0 \\ \hline -17 & 0 & 2 & 0 \\ -6 & 16 & -5 & -46 \end{array}\right] \to \left[\begin{array}{r|rrr} 3 & 0 & 0 & 0 \\ \hline -17 & 2 & 0 & 0 \\ -6 & -5 & 16 & -46 \end{array}\right]$$

Row 3: switch the sign ... and sort ...

$$\left[\begin{array}{rr|rr} 3 & 0 & 0 & 0 \\ -17 & 2 & 0 & 0 \\ \hline -6 & -5 & 16 & \mathbf{-46} \end{array}\right] \to \left[\begin{array}{rr|rr} 3 & 0 & 0 & 0 \\ -17 & 2 & 0 & 0 \\ \hline -6 & -5 & 16 & 46 \end{array}\right] \to \left[\begin{array}{rr|rr} 3 & 0 & 0 & 0 \\ -17 & 2 & 0 & 0 \\ \hline -6 & -5 & 46 & 16 \end{array}\right]$$

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Use the columns to make the matrix diagonally dominated ...

$$\left[\begin{array}{rrr|r} 3 & 0 & 0 & 0 \\ -17 & \mathbf{2} & 0 & 0 \\ -6 & -5 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -51 & -5 & \mathbf{2} & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$$

- $$B = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \qquad B \cdot y = \begin{pmatrix} 15 \\ 3 \\ 7 \end{pmatrix}$$
- $y_1 = 5$
- $y_2 = -1$
- $y_3 = \frac{3}{2}$
- The answer is **NO**!

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