Notes on Disjunctive Programming and Lift-and-Project Approaches

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1 Introduction

Let us consider a linear binary programming problem

$$\max\{c^T x \mid Ax \leq b,\ x \in \mathbb{B}^n\} \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $\mathbb{B} = \{0,1\}$, and denote by $P = \{x \mid Ax \leq b,\ 0 \leq x \leq 1\}$ the set of feasible points of its standard linear programming relaxation, and by $P_I = \text{conv}\ (P \cap \mathbb{B}^n)$ the so called *integer core of* P, i.e., the convex hull of the feasible binary solutions of (1).

Cutting plane methods in general aim at finding tighter descriptions of P_I in terms of linear inequalities (see e.g., Gomory [1963], Chvátal [1973]), and in particular the so called *lift-and-project* type algorithms (see e.g., Sherali and Adams [1990], Lovász and Schrijver [1991], Balas et al. [1993], Lasserre [2001], Bienstock and Zuckerberg [2005]) were introduced for providing both efficient computational methods to derive deep cuts, as well as complete hierarchies of tighter and tighter reformulations of the integer core of polytopes.

Lift-and-project type methods generally consists of three steps: first we introduce a nonlinear description for P, which in the next step is linearized, by possibly introducing many new variables, and finally the obtained description is projected back into the space of the original variables. While during the introduction of nonlinearities in step 1 we do not change the set of feasible solutions, the linearization step involves the application of identities $x^2 = x$, which are valid only for binary values, and hence this step generally restricts the set of feasible assignments to the original variables, without eliminating any of the integer assignments. Thus, after the projection, we obtain a generally tighter reformulation of P_I. Furthermore, the higher dimensional representation allows us to generate some of the facets of the projection by simple linear programming techniques, providing in this way efficiently computable cutting plane families for P.

In this set of short notes we introduce a common framework for methods of this type, inspired primarily by ideas from disjunctive programming (see e.g., Balas [1975, 1985, 1998]) and by the algebraic hierarchy for quadratic binary optimization, introduced in Boros et al. [1990]. We provide elementary proofs for correctness and show that this framework includes, as special cases, several of the approaches cited above.

2 Disjunctive Closures

To describe the proposed lift-and-project approach, it is best to recall first a few basic facts about the geometry of convex polyhedra, convex hulls, disjunctive programming, and to introduce some definitions and notations to simplify our presentation. For simplicity, we consider binary integer programming problems.

One of the motivating ideas is the so called disjunctive programming, introduced by Balas [1975]. One of the main observations we can restate as follows:

Theorem 1 (Balas [1975]) *If $F_i = \{x \in \mathbb{R}^n \mid A^i x \geq b^i,\ x \geq 0\}$ are given polyhedra for $i = 1, ..., q$, then*

$$\operatorname{conv}\left(\bigcup_{i=1}^{q} F_i\right) = \left\{ \sum_{i=1}^{q} \xi^i \;\middle|\; \begin{array}{c} A^i\xi^i - b^i\xi_0^i \geq 0 \quad i = 1, ..., q \\ \displaystyle\sum_{i=1}^{q} \xi_0^i \;=\; 1 \\ (\xi^i, \xi_0^i) \geq 0 \quad \textit{for all} \quad i = 1, ..., q \end{array} \right\} \subseteq \mathbb{R}^n.$$

□

To see a potential application of the above result, let us call a family $\mathcal{Q} = \{Q_1, Q_2, ..., Q_q\}$ of n dimensional convex sets *covering* if $\mathbb{B}^n \subseteq \bigcup_{k=1}^{q} Q_k$, and for a given polyhedron $P \subseteq \mathbb{R}^n$ let us denote by

$$P^{\mathcal{Q}} = \operatorname{conv}\left(\bigcup_{k=1}^{q} (P \cap Q_k)\right) \tag{2}$$

the convex hull of the intersections of P with the convex sets Q_k, $k = 1, ..., q$. It is immediate to see that for every covering family $\mathcal{Q}$ of convex sets we have

$$P_I \subseteq P^{\mathcal{Q}} \subseteq P, \tag{3}$$

where $P_I = \operatorname{conv} P \cap \mathbb{B}^n$. As we shall see, for appropriately selected covering family $\mathcal{Q}$ the polyhedron $P^{\mathcal{Q}}$ approximates P_I much better than P.

Assuming that the polyhedra $F_k = P \cap Q_k$ are defined by m_k linear inequalities for $k = 1, ..., q$, we can derive from Theorem 1 that $P^{\mathcal{Q}}$ is a polyhedral region defined in terms of $M = 1 + \sum_{k=1}^{q} m_k$ linear inequalities in $N = q(n+1)$ nonnegative variables.

Thus, if q and $\sum_{k=1}^{q} m_k$ are not too large, we can use Theorem 1 to optimize over $P^{\mathcal{Q}}$ instead of P, obtaining better bounds for (1). Furthermore, the inequalities defining $P^{\mathcal{Q}}$ may serve as valuable cutting planes for P_I.

3 Properties of Disjunctive Closures

In what follows, we show that essentially the same description as in Theorem 1 can also be obtained by an algebraic approach, which provide a common scheme for the techniques proposed by Sherali and Adams [1990], Lovász and Schrijver [1991] and Balas et al. [1993].

Let us note first that the mapping $P \longmapsto P^{\mathcal{Q}}$ can be viewed as an operator for every covering family $\mathcal{Q}$, which maps P into a *tighter* polytope $P^{\mathcal{Q}}$, while preserving the feasibility of all integral vectors in P.

More specifically, let us call the covering family $\mathcal{Q} = \{Q_1, ..., Q_q\}$ a *subcube covering* if all sets in $\mathcal{Q}$ are subcubes of $[0,1]^n$, i.e., if $Q_k = B(P_k, N_k)$ for some subsets $P_k, N_k \subseteq [n]$ with $P_k \cap N_k = \emptyset$ for $k = 1, ..., q$, and where

$$B(P_k, N_k) = \{\, x \in [0,1]^n \;|\; x_j = 1 \text{ for } j \in P_k, \quad \text{and } x_j = 0 \text{ for } j \in N_k \} \,. \tag{4}$$

Given two covering families $\mathcal{Q}$ and $\mathcal{Q}'$ let us define their *common refinement* by

$$\mathcal{Q} \odot \mathcal{Q}' = \{Q \cap Q' \mid Q \in \mathcal{Q},\ Q' \in \mathcal{Q}'\},$$

and note that $\mathcal{Q} \odot \mathcal{Q}' = \mathcal{Q}' \odot \mathcal{Q}$. It is easy see that for two covering families $\mathcal{Q}$ and $\mathcal{Q}'$ we have

$$P_I \subseteq P^{\mathcal{Q}\odot\mathcal{Q}'} \subseteq \left(P^{\mathcal{Q}}\right)^{\mathcal{Q}'} \subseteq P.$$

A slightly stronger statement can be claimed for subcube coverings.

Fact 1 *If $\mathcal{Q}$ and $\mathcal{Q}'$ are both subcube coverings, then*

$$P^{\mathcal{Q}\odot\mathcal{Q}'} = \left(P^{\mathcal{Q}}\right)^{\mathcal{Q}'}.$$

Proof. It is enough to show that for every $Q' \in \mathcal{Q}'$ we have $Q' \cap P^{\mathcal{Q}} \subseteq P^{\mathcal{Q}\odot\mathcal{Q}'}$. Consider a point $x^* \in Q' \cap P^{\mathcal{Q}}$. Then, by the definition of $P^{\mathcal{Q}}$ there must exists vectors $x^Q \in P \cap Q$ for some $Q \in \mathcal{Q}$ and reals $\lambda_Q \geq 0$, $\sum_{Q\in\mathcal{Q}} \lambda_Q = 1$ such that $x^* = \sum_{Q\in\mathcal{Q}} \lambda_Q x^Q$. We claim that since Q' is a subcube, we must have $x^Q \in Q'$ whenever $\lambda_Q > 0$, which will imply $x^Q \in P \cap (Q \cap Q')$ for $\lambda_Q > 0$, i.e., that $x \in P^{\mathcal{Q}\odot\mathcal{Q}'}$. To see this claim, assume that $Q' = B(P, N)$ for some subsets $P, N \subseteq [n]$, $P \cap N = \emptyset$, and let us define $\ell(x) = \sum_{j\in N} x_j + \sum_{j\in P}(1 - x_j)$ for an arbitrary vector $x \in \mathbb{R}^n$. Note that $\ell(x) = 0$ if $x \in Q'$, and $\ell(x) > 0$ for all $[0,1]^n \setminus Q'$. Since $\ell(x)$ is a linear function of x, we get $\ell(x^*) = \sum_{Q\in\mathcal{Q}} \lambda_Q \ell(x^Q)$. Thus we have $\ell(x^*) = 0$, since $x^* \in Q'$, implying by the nonnegativity of λ_Q and $\ell(x^Q)$, $q \in \mathcal{Q}$ that $\ell(x^Q) = 0$ must hold whenever $\lambda_Q > 0$, proving the claim, and completing the proof. $\square$

Let us finally call a subcube covering $\mathcal{Q}$ *balanced*, if there are positive reals $\omega_Q > 0$, $Q \in \mathcal{Q}$ for which

$$1 = \sum_{Q \in \mathcal{Q}:\ Q \ni x} \omega_Q \tag{5}$$

holds for all binary vectors $x \in \mathbb{B}^n$.

Fact 2 *If both $\mathcal{Q}$ and $\mathcal{Q}'$ are balanced subcube coverings, then so is $\mathcal{Q} \odot \mathcal{Q}'$.*

Proof. Assuming $1 = \sum_{Q \in \mathcal{Q}:\ Q \ni x} \omega_Q$ and $1 = \sum_{Q' \in \mathcal{Q}':\ Q' \ni x} \omega_{Q'}$ hold for all binary vectors $x \in \mathbb{B}^n$, let us define

$$\omega_{Q \cap Q'} = \begin{cases} \omega_Q \omega_{Q'} & \text{if } \mathbb{B}^n \cap Q \cap Q' \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to verify that $1 = \sum_{X \in \mathcal{Q} \odot \mathcal{Q}':\ X \ni x} \omega_X$ also holds for all binary vectors $x \in \mathbb{B}^n$. $\square$

Let us consider some examples.
We define first the following set of subcubes

$$\begin{aligned} Q_1 &= \{x \in [0,1]^3 \mid x_1 = 1\}, \\ Q_2 &= \{x \in [0,1]^3 \mid x_1 = 0,\ x_2 = 1\}, \\ Q_3 &= \{x \in [0,1]^3 \mid x_1 = 0,\ x_2 = 0\}. \end{aligned}$$

Then $\mathcal{Q} = \{Q_1, Q_2, Q_3\}$ is a balanced subcube covering since we have

$$x_1 + \bar{x}_1 x_2 + \bar{x}_1 \bar{x}_2 \ = \ 1 \quad \text{for all } \ (x_1, x_2, x_3) \in \{0,1\}^3.$$

Similarly, if we define subcubes as

$$\begin{aligned} Q'_1 &= \{x \in [0,1]^3 \mid x_1 = 0\}, \\ Q'_2 &= \{x \in [0,1]^3 \mid x_1 = 1,\ x_3 = 1\}, \\ Q'_3 &= \{x \in [0,1]^3 \mid x_1 = 1,\ x_3 = 0\}. \end{aligned}$$

Then $\mathcal{Q}' = \{Q'_1, Q'_2, Q'_3\}$ is also a balanced subcube covering since we have

$$\bar{x}_1 + x_1 x_3 + x_1 \bar{x}_3 \ = \ 1 \quad \text{for all } \ (x_1, x_2, x_3) \in \{0,1\}^3.$$

Consequently, $\mathcal{R} = \mathcal{Q} \odot \mathcal{Q}'$ is also a balanced subcube covering, as shown by the following identity over $\{0,1\}^3$

$$\begin{aligned} 1 &= (x_1 + \bar{x}_1 x_2 + \bar{x}_1 \bar{x}_2)(\bar{x}_1 + x_1 x_3 + x_1 \bar{x}_3) \\ &= x_1 x_3 + x_1 \bar{x}_3 + \bar{x}_1 x_2 + \bar{x}_1 \bar{x}_2 \end{aligned}$$

implying that $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$, where

$$\begin{aligned} R_1 &= \{x \in [0,1]^3 \mid x_1 = 1,\ x_3 = 1\}, \\ R_2 &= \{x \in [0,1]^3 \mid x_1 = 1,\ x_3 = 0\}, \\ R_3 &= \{x \in [0,1]^3 \mid x_1 = 0,\ x_2 = 1\}, \\ R_4 &= \{x \in [0,1]^3 \mid x_1 = 0,\ x_2 = 0\}. \end{aligned}$$

The following fact follows directly from the definitions:

Fact 3 *Given a polytope P, if $Q \cap P$ is a subcube for all sets Q of a subcube covering $\mathcal{Q}$, then we have $P_I = P^{\mathcal{Q}}$.* □

To a subcube covering $\mathcal{Q} = \{B(P_k, N_k) \mid k = 1, ..., q\}$, where $P_k, N_k \subseteq [n]$ and $P_k \cap N_k = \emptyset$ for $k = 1, ..., q$, let us associate multilinear monomials

$$t_k(x) = \left(\prod_{j \in P_k} x_j\right)\left(\prod_{j \in N_k} \bar{x}_j\right), \tag{6}$$

for $k = 1, ..., q$, where $\bar{x} = 1 - x$. Note that by the definitions, we have

$$B(P_k, N_k) = \{x \in [0,1]^n \mid t_k(x) = 1\}. \tag{7}$$

Let us also note that if $\omega_k > 0$, $k = 1, ..., q$ denote some real coefficients satisfying (5), and hence showing that $\mathcal{Q}$ is balanced, then we also have

$$\sum_{k=1}^{q} \omega_k t_k(x) \ \equiv\ 1 \tag{8}$$

hold identically for every binary vector $x \in \mathbb{B}^n$. In fact the reverse is also true.

Fact 4 *For some sets $P_k, N_k \subseteq [n]$ with $P_k \cap N_k = \emptyset$ for $k = 1, ..., q$ the family $\mathcal{Q} = \{B(P_k, N_k) \mid k = 1, ..., q\}$ is a balanced subcube covering if and only if for some positive reals $\omega_k > 0$, $k = 1, ..., q$ the equality* (8) *holds for every binary vector $x \in \mathbb{B}^n$.* □

Let us remark that (8) holds for all binary vectors if and only if it holds for all $x \in \mathbb{R}^n$, due to the well-known fact (see Hammer and Rudeanu [1968]) that the left hand side has a unique multilinear polynomial expression in terms of $x_1, ..., x_n$. Thus, there is a one-to-one correspondence between balanced subcube coverings and multilinear polynomial identities (8).

Fact 5 *If* $\sum_{k=1}^{q} \omega_k t_k(x) \equiv 1$ *and* $\sum_{k=1}^{q'} \omega'_k t'_k(x) \equiv 1$ *are two multilinear polynomial identities over* $\mathbb{B}^n$*, and* $\mathcal{Q} = \{B(P_k, N_k) \mid k = 1, ..., q\}$ *and* $\mathcal{Q}' = \{B(P'_k, N'_k) \mid k = 1, ..., q'\}$ *denote respectively the corresponding balanced subcube coverings, then we have*

$$\sum_{\substack{k\in[q],\ell\in[q'] \\ (P_k\cap N'_\ell)\cup(N_k\cap P'_\ell)=\emptyset}} \omega_k\omega'_\ell \left(\prod_{j\in P_k\cup P'_\ell} x_j\right)\left(\prod_{j\in N_k\cup N'_\ell} \bar{x}_j\right) \equiv 1$$

also as an identity over $\mathbb{B}^n$*, corresponding to the balanced subcube covering* $\mathcal{Q} \odot \mathcal{Q}'$. □

Let us note that if we consider the identity $x_j + \bar{x}_j \equiv 1$ and denote by $\mathcal{Q}$ the corresponding balanced subcube covering, then we have that $P^{\mathcal{Q}}$ is the same reformulation of P as $P^{(j)}$ introduced in Balas et al. [1993]. In what follows we extend the method of Balas et al. [1993] and provide a simple algebraic derivation for $P^{\mathcal{Q}}$ for an arbitrary balanced subcube covering $\mathcal{Q}$, generalizing in this way (and changing slightly the standard linearizations, used typically by) the techniques of Sherali and Adams [1990], Lovász and Schrijver [1991], Balas et al. [1993]. By extending slightly further the derived algebraic description, we can arrive as a special case to the $N(K)$ operator introduced in Lovász and Schrijver [1991]. Furthermore, using the algebraic description, we can also introduce several hierarchies of ever tightening reformulations, including as a special case the hierarchy introduced by Sherali and Adams [1990].

4 A Lift-and-Project Approach

In this section we describe a simple lift-and-project type algebraic procedure, which for every balanced subcube covering $\mathcal{Q}$ and polytope P yield an $O(n|\mathcal{Q}|)$ dimensional linear system $S^{\mathcal{Q}}$, the projection of which in $\mathbb{R}^n$ is exactly $P^{\mathcal{Q}}$. Furthermore, using $S^{\mathcal{Q}}$, we can derive for every $x^* \notin P^{\mathcal{Q}}$ in polynomial time in the size of P and $|\mathcal{Q}|$ a linear inequality $\alpha^T x \geq \beta$ which is valid for $P^{\mathcal{Q}}$ and for which $\alpha^T x^* < \beta$.

To be able to describe our approach, let us view the polytope P corresponding to problem (1) as the set of feasible solutions to the inequalities

$$\begin{array}{rcll} \sum_{j=1}^n a_{ij}x_j & \leq & b_i & \text{for } i = 1, ..., m, \\ x_j & \leq & 1 & \text{for } j = 1, ..., n, \\ -x_j & \leq & 0 & \text{for } j = 1, ..., n. \end{array} \tag{9}$$

Let us further consider an arbitrary balanced subcube covering

$$\mathcal{Q} = \{B(P_k, N_k) \mid k = 1, ..., q\} \tag{10}$$

corresponding to the multilinear polynomial identity (8), i.e., where $t_k(x)$ are defined by (6).

We are now ready to describe a lift-and-project algorithm, which will be shown to produce $P^{\mathcal{Q}}$.

Introducing nonlinearity: For a given index $k \in \{1, ..., q\}$, let us multiply each of the $m+2n$ inequalities in (9) by $t_k(x)$, and let us denote by $M(k)$ the set of obtained nonlinear inequalities:

$$\begin{array}{rcll} \sum_{j=1}^{n} a_{ij} x_j t_k(x) & \leq & b_i t_k(x) & \text{for } i = 1, ..., m, \\ x_j t_k(x) & \leq & t_k(x) & \text{for } j = 1, ..., n, \\ -x_j t_k(x) & \leq & 0 & \text{for } j = 1, ..., n. \end{array} \tag{$M(k)$}$$

Let us write these inequalities for $k = 1, ..., q$, and let us also write down equality (8), plus the n additional equations we can obtain from it by multiplying with the individual variables x_j, for $j = 1, ..., n$:

$$\begin{array}{rrcll} & \sum_{k=1}^{q} \omega_k t_k(x) & = & 1 & \\ x_j & -\sum_{k=1}^{q} \omega_k x_j t_k(x) & = & 0 & \text{for } j = 1, ..., n. \end{array} \tag{11}$$

Linearization: The nonlinear products $x_j t_k(x)$ are not necessarily multilinear, since they may contain x_j^2 or $x_j \bar{x}_j$. Let us substitute in all of the above inequalities x_j^2 by x_j, and hence $x_j \bar{x}_j$ by 0, for all $j = 1, ..., n$. Furthermore, introduce new variables $z_k = t_k(x)$ for $k = 1, ..., q$, and

$$y_{jk} = x_j t_k(x) \text{ for all pairs } (j,k) \in [n] \times [q] \text{ for which } j \notin P_k \cup N_k, \tag{12}$$

and substitute everywhere $x_j t_k(x)$ by y_{jk}, and $t_k(x)$ by z_k. Let us further introduce $R_k = [n] \setminus (P_k \cup N_k)$ for $k = 1, ..., q$, and note that after the above substitutions all inequalities become linear in the

$$N_{\mathcal{Q}} = n + q + \sum_{k=1}^{q} |R_k| = O(nq) \tag{13}$$

variables $x = (x_1, ..., x_n)$, $y = (y_{jk} \mid k \in [q],\ j \in R_k)$, and $z = (z_1, ..., z_q)$.

The system of inequalities obtained in this way from ($M(k)$) can be rewritten, after rearrangements, and the deletion of trivial ones, as

$$\begin{array}{rrcll} \displaystyle\sum_{j \in R_k} a_{ij} y_{jk} & + \left(-b_i + \displaystyle\sum_{j \in P_k} a_{ij}\right) z_k & \leq & 0 & \text{for } i = 1, ..., m, \\ y_{jk} & -z_k & \leq & 0 & \text{for } j \in R_k, \\ -y_{jk} & & \leq & 0 & \text{for } j \in R_k, \\ & -z_k & \leq & 0, & \end{array} \tag{$S(k)$}$$

where the last inequality comes from the second group of inequalities of $(M(k))$ for $j \in N_k$, and also from the third group of inequalities of $(M(k))$ for $j \in P_k$.

Inequalities (11) can be rewritten as

$$\begin{aligned} \sum_{k=1}^{q} \omega_k z_k &= 1, \\ \sum_{k:j \in R_k} \omega_k y_{jk} + \left(\sum_{k:j \in P_k} \omega_k \right) z_k &= x_j \quad \text{for } j = 1, ..., n. \end{aligned} \tag{14}$$

Note that $(S(k))$, for $k = 1, ..., q$, and (14) together form a system of linear inequalities consisting of

$$M_{\mathcal{Q}} = n + 1 + q(m+1) + 2 \sum_{k=1}^{q} |R_k| = O(q(n+m)) \tag{15}$$

inequalities in $N_{\mathcal{Q}} = O(nq)$ variables. Let us denote by $S^{\mathcal{Q}} \subseteq \mathbb{R}^{N_{\mathcal{Q}}}$ the set of feasible (x, y, z) solutions to this linear system.

Projection: Project the convex set $S^{\mathcal{Q}}$ back into the space of the original n variables $x = (x_1, ..., x_n)$, and denote the resulting convex polyhedron by $R^{\mathcal{Q}} \subseteq \mathbb{R}^n$. In other words,

$$R^{\mathcal{Q}} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \exists z \in \mathbb{R}^q \text{ and } y \in \mathbb{R}^{\sum_{k=1}^{q} |R_k|} \\ \text{such that } (x, y, z) \in S^{\mathcal{Q}} \end{array} \right\}. \tag{16}$$

Theorem 2 *For an arbitrary polytope P described by a set of inequalities as in* (9) *and an arbitrary balanced subcube covering $\mathcal{Q}$ given as in* (10)*, we have*

$$R^{\mathcal{Q}} = P^{\mathcal{Q}}.$$

Proof. To prove that $P^{\mathcal{Q}} \subseteq R^{\mathcal{Q}}$, let us consider an arbitrary vector $x \in P^{\mathcal{Q}}$. Then it can be represented as a convex combination of vectors from $P \cap Q_k$, $k = 1, ..., q$, i.e.,

$$x = \sum_{k=1}^{q} \lambda_k x^k$$

for some reals $\lambda_k \geq 0$, $\sum_{k=1}^{q} \lambda_k = 1$ and vectors $x^k \in P \cap B(P_k, N_k)$, $k = 1, ..., q$. Let us further consider a set of real coefficients $\omega_k > 0$, $k = 1, ..., q$ satisfying (8), which prove that $\mathcal{Q}$ is a balanced covering, set

$$z_k = \frac{\lambda_k}{\omega_k} \quad \text{for} \quad k = 1, ..., q,$$

and define

$$y_{jk} = x_j^k z_k \quad \text{for} \quad k = 1, ..., q \text{ and } j \in R_k.$$

Note that since $x^k \in P$ it satisfies all inequalities in (9), and since $x^k \in B(P_k, N_k)$ we have $x_j^k = 1$ for all $j \in P_k$ and $x_j^k = 0$ for all $j \in N_k$. Thus, for every index k the above defined quantities z_k and y_{jk} for $j \in R_k$ satisfy all inequalities in $(S(k))$. Furthermore, the equalities in (14) are also satisfied, since we have $\sum_{k=1}^q \omega_k z_k = \sum_{k=1}^q \lambda_k = 1$ and the additional n equalities of (14) follow by the equation $x = \sum_{k=1}^q \lambda_k x^k$. These imply that $(x, y, z) \in S^{\mathcal{Q}}$, and $x \in R^{\mathcal{Q}}$ follows.

For the reverse direction $R^{\mathcal{Q}} \subseteq P^{\mathcal{Q}}$, assume that $x \in R^{\mathcal{Q}}$, i.e., that $(x, y, z) \in S^{\mathcal{Q}}$ for some real vectors $y \in \mathbb{R}^{\sum_{k=1}^q |R_k|}$ and $z \in \mathbb{R}^q$. Let us then define

$$\lambda_k = \omega_k z_k \quad \text{for } k = 1, ..., q,$$

implying that $\lambda_k \geq 0$, $k = 1, ..., k$ and that $\sum_{k=1}^q \lambda_k = 1$. We claim that we can define vectors $x^k \in P \cap B(P_k, N_k)$ for all indices k for which $\lambda_k > 0$, such that $x = \sum_{k=1}^q \lambda_k x^k$ holds (for arbitrary vectors $x^k \in P \cap B(P_k, N_k)$ for all other indices with $\lambda_k = 0$), which then proves that $x \in P^{\mathcal{Q}}$, implying $R^{\mathcal{Q}} \subseteq P^{\mathcal{Q}}$, and completing the proof of the theorem. To see this last claim, consider an index k for which $\lambda_k > 0$, i.e., for which $z_k > 0$ since $\omega_k > 0$ by definition, and set

$$x_j^k = \begin{cases} \frac{y_{jk}}{z_k} & \text{for all } j \in R_k, \\ 1 & \text{for all } j \in P_k, \\ 0 & \text{for all } j \in N_k. \end{cases}$$

It is simple to check that since z_k and y_{jk}, $j \in R_k$ satisfy $(S(k))$, the above defined vector x^k satisfies all inequalities in (9), and thus it belongs to both P and $B(P_k, N_k)$. □

This theorem has several consequences. In the following statements we assume that a polytope P is given as in (9), and that balanced subcube coverings are given as in (10). We also assume that, as will be the case in all special cases we mention later, the coefficients ω_Q, $Q \in \mathcal{Q}$ can be represented in $poly(q)$ space, where $q = |\mathcal{Q}|$, as before.

Corollary 1 *The optimization problem* $\max\{c^T x \mid x \in P^{\mathcal{Q}}\}$ *can be solved in polynomial time in the input size of* P *and* $q = |\mathcal{Q}|$.

Proof. By Theorem 2 and by the definitions we have

$$\max\{c^T x \mid x \in P^{\mathcal{Q}}\} \;=\; \max\{c^T x \mid (x, y, z) \in S^{\mathcal{Q}}\}.$$

Since $S^{\mathcal{Q}}$ is described by the inequalities $(S(k))$ for $k = 1, ..., q$ and (14), the size of this second linear programming problem is polynomial in the size of (9) and q. □

Valid inequalities for $P^{\mathcal{Q}}$, which are also valid for P_I by (3), may serve as effective cutting planes in solving the integer programming problem (1). In fact such cutting planes can be generated efficiently:

Corollary 2 *Given a vector $x^* \notin P^{\mathcal{Q}}$, a hyperplane*

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \leq \beta \tag{17}$$

separating x^* from $P^{\mathcal{Q}}$ (and hence from P_I) can be generated in polynomial time in the size of (9) *and q.*

Proof. If $x^* \notin P^{\mathcal{Q}}$, then by Theorem 2 the system of inequalities $(S(k))$ for $k = 1, ..., q$ and (14), with x_j^*, $j = 1, ..., n$ on its right hand side, is inconsistent. Thus, by Farkas' lemma there exists real multipliers $\beta, -\alpha_1, ... - \alpha_n$ for the equalities of (14) and nonnegative multipliers μ_{ik}, $i = 1, ..., m$, ζ_{jk} and ξ_{jk} for $j \in R_k$ and ν_k for the inequalities of $(S(k))$ for $k = 1, ..., q$ such that taking the linear combination with these coefficients we obtain identically zero left hand side, and a negative right hand side. Thus, we have $\beta - \alpha_1 x_1^* - \cdots - \alpha_n x_n^* < 0$ on the right hand side, proving that x^* does not satisfy (17). On the other hand, for all feasible $x \in P^{\mathcal{Q}}$ (i.e., for all $(x, y, z) \in S^{\mathcal{Q}}$) we have (17) hold, since it is obtained as valid linear combination of valid equalities and inequalities. This proves that the obtained hyperplane indeed separates x^* from $P^{\mathcal{Q}}$. Clearly, the coefficients of such a linear combination can be found by solving the dual linear program, in which we maximize $\alpha_1 x_1^* + \cdots + \alpha_n x_n^*$. Since this LP has $M_{\mathcal{Q}}$ variables and $N_{\mathcal{Q}}$ constraints, the claim follows by (13) and (15). $\square$

Let us remark that the formulation $S^{\mathcal{Q}}$ is highly structured, the blocks $(S(k))$ do not share variables for different values of k, and only the $n+1$ equalities in (14) connect these blocks. As a consequence, Lagrangean type methods could be employed, speeding up the solution of the optimization problem over $S^{\mathcal{Q}}$, as well as the generation of separating hyperplanes for it.

Let us also remark that if $\mathcal{Q}_i$, $i = 1, ..., \ell$ are balanced subcube coverings, then $P^* = \bigcap_{i=1}^{\ell} P^{\mathcal{Q}_i}$ provides even tighter description for P_I, and the above corollaries imply that both optimization and separation for P^* can be solved in polynomial time in the input size of P and in $\sum_{i=1}^{\ell} |\mathcal{Q}_i|$.

5 Some Special Cases

Let us note that an arbitrary multilinear identity of the form (8) provides us with a balanced subcube covering, and hence there is a great versatility in the ways the above can be applied.

As a first example, let us consider the identity

$$x_j + \bar{x}_j = 1$$

for some index $j \in [n]$, and let $\mathcal{Q}^{(j)}$ denote the corresponding balanced subcube covering (i.e., $\mathcal{Q}^{(j)}$ consists of two subcubes of dimension $n-1$). Then $P^{\mathcal{Q}^{(j)}}$ is exactly the polytope introduced in Balas et al. [1993]. Moreover, the same hierarchy can be defined by introducing $\mathcal{Q}^{(I)} = \bigodot_{i \in I} \mathcal{Q}^{(i)}$, i.e., by considering the identity

$$\sum_{S \subseteq I} \left(\prod_{j \in S} x_j \prod_{j \in I \setminus S} \bar{x}_j \right) \; = \; 1.$$

We also get that optimization over $\mathcal{Q}^{(I)}$ can be done efficiently if $|I|$ is a constant, moreover by Facts 1 and 3 that $P_I = \mathcal{Q}^{([n])}$.

As another example, let us consider the identity

$$\sum_{I \subseteq [n], |I|=k} \sum_{S \subseteq I} \frac{1}{\binom{n}{k}} \left(\prod_{j \in S} x_j \prod_{j \in I \setminus S} \bar{x}_j \right) \; = \; 1,$$

for some $k \in [n]$, and let $\mathcal{Q}_k$ denote the corresponding balanced subcube covering. Then, we have

$$P_I = P^{\mathcal{Q}_n} \subseteq \cdots \subseteq P^{\mathcal{Q}_1} \subseteq P,$$

providing an analogous hierarchy as in Sherali and Adams [1990].

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