

# Maximum Flows in Networks

Augmenting Path Method – Ford and Fulkerson (1957)

Endre Boros

26:711:653: Discrete Optimization

March 5, 2018

# A travel agency example (Chvatal, 1983)

SF

H

D

A

C

NY

- ▶ A travel agency needs to book as many as possible travelers going from San Francisco (SF) to New York City (NY).



- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

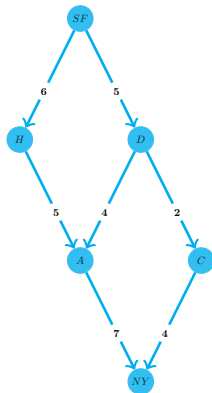
◦ All outgoing flights from  $S$  are fully booked.

◦ No flights with an unoccupied seat are leaving into NY from (from the rest of the network).

◦ **Proof of OPTIMALITY:**

◦ **Assume not optimal** (contradiction)

# A travel agency example (Chvatal, 1983)



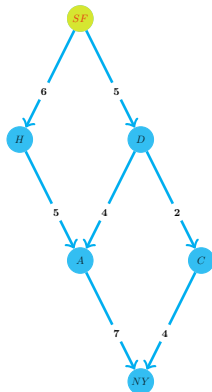
- ▶ A travel agency needs to book as many as possible travelers going from San Francisco (SF) to New York City (NY).
- ▶ The figure shows the available flights and number of seats (where A is Atlanta, C is Chicago, D is Denver and H is Houston.)
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

How many flights from  $S$  to  $NY$  can we book with an unexpired seat to capacity? (Note that the flight from  $D$  to  $C$  is not used.)

ANSWER: OPTIMALITY

MAXIMUM FLOW PROBLEM

# A travel agency example (Chvatal, 1983)



- ▶ A travel agency needs to book as many as possible travelers going from San Francisco (SF) to New York City (NY).

- ▶ We call SF the **source**, and NY the **sink**

- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

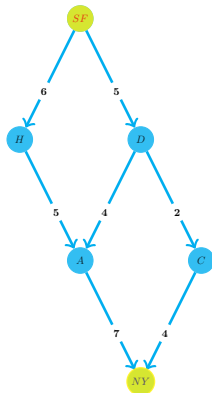
▶ All outgoing flights from  $S$  are fully booked.

▶ No flight with an unbooked seat is leaving the source area (from the rest of the network).

**Proof of OPTIMALITY:**

▶ If there is an unbooked seat from SF to NY, then

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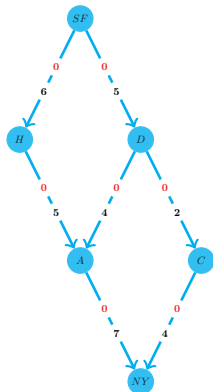
▶ All outgoing flights from  $S$  are fully booked.

▶ No flight with an unbooked seat is leaving the source area (from the rest of the network).

▶ **PROPTIMALITY**

▶ All seats are booked

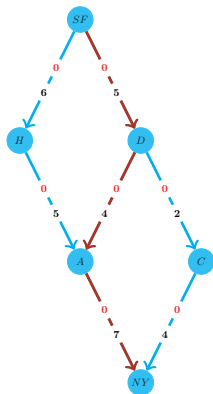
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- ▶ A travel agency needs to book as many as possible travelers going from San Francisco (SF) to New York City (NY).
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  - ▶ **Initialize the flow by 0 on all arcs.**
  - ▶ Find a shortest **augmenting path** from source to sink.
  - ▶ Compute bottleneck capacity:  $\Delta =$
  - ▶ Push through the augmenting path extra  $\Delta$  units of flow.
  - ▶ Compute residual network **precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ All the flights from  $S$  are fully booked.  
▶ The only flight to NY is supplied by routes from  $C$ .  
▶ The maximum flow is 4.

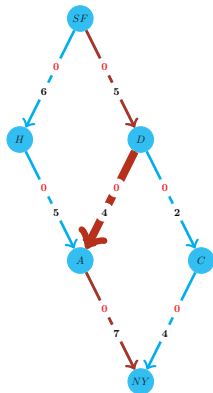
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- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ The flight from A to NY is not selected because it is not a shortest augmenting path from S to NY.

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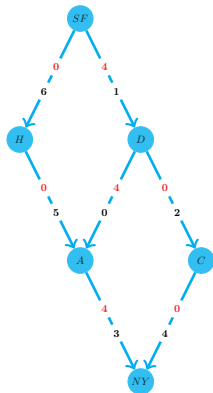


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  - ▶ Find a shortest **augmenting path** from source to sink.
  - ▶ **Compute bottleneck capacity:**  $\Delta = 4$
  - ▶ Push through the augmenting path extra  $\Delta$  units of flow.
  - ▶ Compute residual network **precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ The augmenting path from  $S$  to  $NY$  is selected. The flow on each arc is updated according to the bottleneck capacity of the augmenting path.



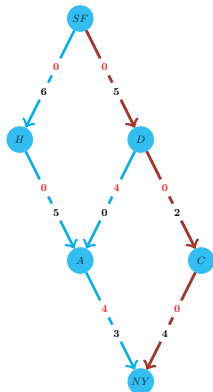
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  - ▶ Compute bottleneck capacity:  $\Delta = 4$
  - ▶ **Push through the augmenting path extra  $\Delta$  units of flow.**
  - ▶ Compute residual network **precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ The augmenting path from  $S$  to  $T$  is fully selected. The augmenting path is  $S \rightarrow D \rightarrow C \rightarrow T$ . The bottleneck capacity is  $\Delta = 2$ .

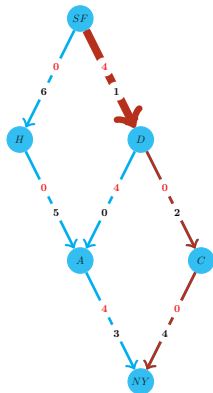
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- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ The augmenting path  $P = \{SF, D, C, NY\}$  is chosen. The flow on each arc is updated accordingly.

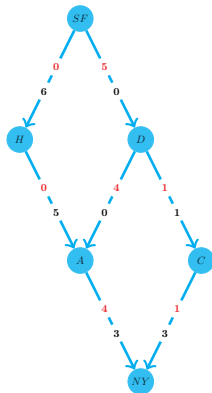
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  - ▶ Initialize the **flow** by 0 on all arcs.
  - ▶ Find a shortest **augmenting path** from source to sink.
  - ▶ **Compute bottleneck capacity:**  $\Delta = 1$
  - ▶ Push through the augmenting path extra  $\Delta$  units of flow.
  - ▶ Compute residual network **precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ The flow on the flight from S to D is fully booked, so the next augmenting path must go through the flight from S to H.

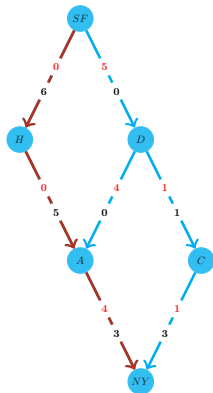
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  - ▶ Compute bottleneck capacity:  $\Delta = 1$
  - ▶ **Push through the augmenting path extra  $\Delta$  units of flow.**
  - ▶ Compute residual network **precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ **Flow** on the flight from S to A fully booked  
▶ **Flow** on the flight from S to D fully booked  
▶ **Flow** on the flight from S to H fully booked

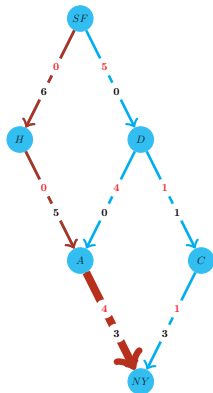
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  - ▶ Initialize the **flow** by 0 on all arcs.
  - ▶ Find a **shortest augmenting path** from source to sink.
  - ▶ Compute bottleneck capacity:  $\Delta = 1$
  - ▶ Push through the augmenting path extra  $\Delta$  units of flow.
  - ▶ Compute residual network **precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ The next flight from S to NY is blocked by the flow on the arc from A to NY. The next flight from S to NY is blocked by the flow on the arc from A to NY.

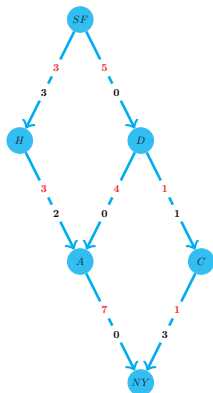
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  - ▶ Initialize the **flow** by 0 on all arcs.
  - ▶ Find a shortest **augmenting path** from source to sink.
  - ▶ **Compute bottleneck capacity:**  $\Delta = 3$
  - ▶ Push through the augmenting path extra  $\Delta$  units of flow.
  - ▶ Compute residual network **precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ The augmenting path  $P = \{SF, A, NY\}$  is chosen. The flow on each arc is updated accordingly.

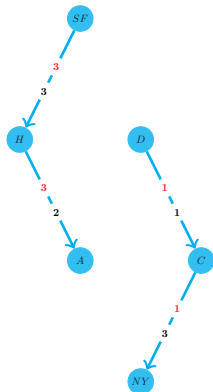
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  - ▶ **Push through the augmenting path extra  $\Delta$  units of flow.**
  - ▶ Compute residual network **precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

▶ The augmenting path  $P = \{SF, D, C, NY\}$  is chosen. The flow on each arc is increased by  $\Delta = 3$  units. The residual network is recomputed.

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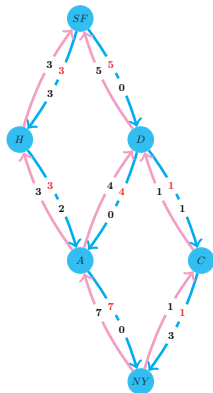


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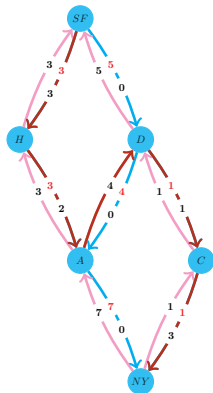
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  - ▶ **Compute residual network precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.

Chvatal, 1983, Fig. 10.10. The flow is 10 units. The residual network is shown in red.

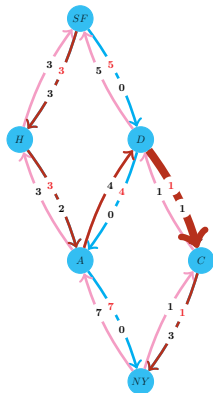
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What is the maximum number of travelers that can be accommodated on flights from San Francisco to New York City?

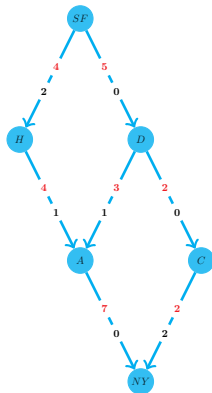
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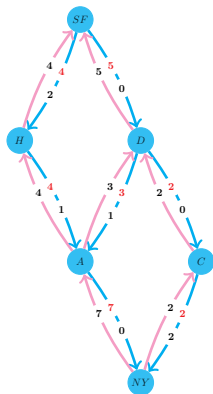
What is the maximum number of travelers that can be accommodated by the agency?

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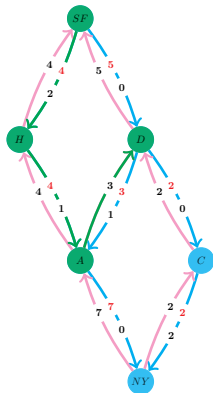
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  - ▶ **Compute residual network precisely!**
- ▶ Consider the source area  $S = \{SF, A, D, H\}$  that is **reachable** from the source.
  - ▶ All outgoing flight from  $S$  are fully booked.

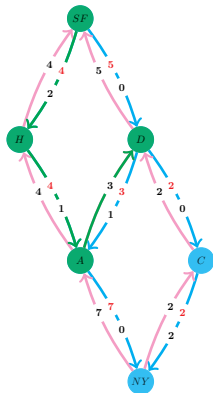
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  - ▶ All outgoing flight from  $S$  are fully booked.
  - ▶ No flight with an occupied seat is coming into this area (from the rest of the network).
  - ▶ Proof of **OPTIMALITY**:

$$MaxFlow = MinCut$$

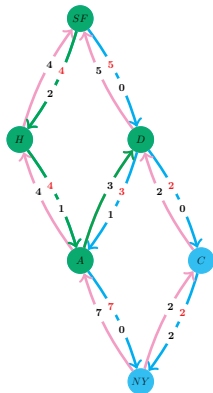
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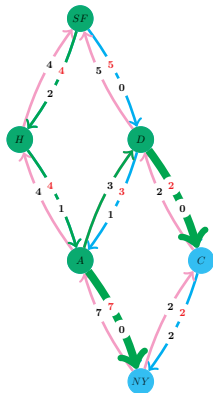


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