1 Formalism and Methodology

The first work on stellar oscillation of compact stars in General relativity were a series of papers [1, 2, 3, 4, 5] published by Thorne and his fellows between 1967 and 1970. Later Lee Lindblom and Steven L. Detweiler improved the differential equations and make them suitable for numerical calculation [6, 7]. Chandrasekhar and Ferrari derived the equations for compact stellar oscillation from another approach similar to the one for black hole, and an useful algorithm was suggested. Here we make a review of the formalism and numerical computation algorithm. We adopt the notation in [6, 7]. The perturbed metric tensor for a non-rotating neutron star is written as,

$$ds^2 = -e^{\nu}(1+r^l H_0 Y_m^l e^{i\omega t})dt^2 - 2i\omega r^{l+1} H_1 Y_m^l e^{i\omega t} dt dr + e^{\lambda}(1-r^l H_0 Y_m^l e^{i\omega t})dr^2 + r^2(1-r^l K Y_m^l e^{i\omega t})(d\theta^2 + \sin^2\theta d\phi^2) \ . \quad (1)$$

The perturbation of the fluid elements is described by the polar components of Lagrangian displacement,

$$\begin{aligned} \xi^r &= r^{l-1} e^{-\lambda/2} W Y_m^l e^{i\omega t} \ , \\ \xi^\theta &= -r^{l-2} V \partial_\theta Y_m^l e^{i\omega t} \ , \\ \xi^\phi &= -r^l (r\sin\theta)^{-2} V \partial_\phi Y_m^l e^{i\omega t} \ . \end{aligned} \quad (2)$$

In the above equations, H_0, H_1, K, W, and V are perturbation functions. The spherical symmetric metric tensor of unperturbed stars is described by its elements e^ν and e^λ. The variables p and ρ are the unperturbed pressure and density inside the neutron stars. Unperturbed stellar model is described by the solutions of TOV equations. Y_m^l are the conventional spherical harmonics. The above five perturbation variables are not all independent and they have the following relation,

$$\begin{aligned} H_0 = [3M + \tfrac{1}{2}(l+2)(l-1)r + 4\pi r^3 p]^{-1} \Big\{ & 8\pi r^3 e^{-\nu/2} X - [\tfrac{1}{2}l(l+1)(M+4\pi r^3 p) - \omega^2 r^3 e^{-(\lambda+\nu)}]H_1 \\ & + [\tfrac{1}{2}(l+2)(l-1)r - \omega^2 r^3 e^{-\nu} - r^{-1}e^{\lambda}(M+4\pi r^3 p)(3M - r + 4\pi r^3 p)]K \Big\} \ , \end{aligned} \quad (3)$$

where M is defined as $\int_0^r 4\pi\rho r^2 dr$. After we work out the Einstein tensor and stress-energy tensor in terms of the above perturbation metric and fluid functions, we can substitute them into Einstein's equation and continuity equation,

$$\delta G_{\mu\nu} = 8\pi \delta T_{\mu\nu} \ , \qquad \delta\left(T^{\mu\nu}{}_{;\mu}\right) = 0 \ , \quad (4)$$

and obtain the following $4th$ order coupling linear differential equations,

$$H_1' = -r^{-1}[l + 1 + 2Me^{\lambda} r^{-1} + 4\pi r^2 e^{\lambda}(p-\rho)]H_1 + r^{-1}e^{\lambda}[H_0 + K - 16\pi(\rho+p)V] \ , \quad (5)$$

$$K' = r^{-1}H_0 + \tfrac{1}{2}l(l+1)r^{-1}H_1 - [(l+1)r^{-1} - \tfrac{1}{2}\nu']K - 8\pi(\rho+p)e^{\lambda/2}r^{-1}W \ , \quad (6)$$

$$W' = -(l+1)r^{-1}W + re^{\lambda/2}[\gamma^{-1}p^{-1}e^{-\nu/2}X - l(l+1)r^{-2}V + \tfrac{1}{2}H_0 + K] \ , \quad (7)$$

$$\begin{aligned} X' = & -lr^{-1}X + (\rho+p)e^{\nu/2}\{\tfrac{1}{2}(r^{-1} - \tfrac{1}{2}\nu')H_0 + \tfrac{1}{2}[r\omega^2 e^{-\nu} + \tfrac{1}{2}l(l+1)r^{-1}]H_1 \\ & + \tfrac{1}{2}(\tfrac{3}{2}\nu' - r^{-1})K - \tfrac{1}{2}l(l+1)\nu' r^{-2}V - r^{-1}[4\pi(\rho+p)e^{\lambda/2} + \omega^2 e^{\lambda/2-\nu} - \tfrac{1}{2}r^2(r^{-2}e^{-\lambda/2}\nu')']W\} \ . \end{aligned} \quad (8)$$

The function X is used instead of V, and X is defined as,

$$V = \omega^{-2}(\rho+p)^{-1}e^{\nu}[e^{-\nu/2}X + r^{-1}p'e^{-\lambda/2}W - \tfrac{1}{2}(\rho+p)H_0] \ . \quad (9)$$

The derivative is taken with respect to r. γ is the adiabatic index, and $\gamma = \frac{\rho+p}{p}\left(\partial p/\partial\rho\right)_s$. Let's denote the four functions H_1, K, W and X as a vector field $\mathbf{Y}(r) = \{H_1, K, W, X\}$ and (5)-(8) as,

$$\mathbf{Y}'(r,l,w) = \mathbf{Q}(r,l,w)\cdot\mathbf{Y}(r,l,w) \ . \quad (10)$$

We begin our numerical calculation by solving the TOV equations for unperturbed non-rotating stellar model. After that we can integrate (5)-(8) from the center of the star to the stellar surface. Since (10) are 4th order differential equations, there are four independent solutions, and the general solution would be a composition of them. The regular condition at the center reduce the number of independent solutions from four to two. If we take the limit $r \to 0$ on the right hand side of (10) and let the left hand side equals 0, we get,

$$X(0) = (\rho_0 + p_0)e^{\nu_0/2}\left\{\left[\frac{4\pi}{3}(\rho_0 + 3p_0) - \omega^2 e^{-\nu_0/l}\right]W(0) + \frac{1}{2}K(0)\right\} \ , \quad (11)$$

$$H_1(0) = \{2lK(0) + 16\pi(\rho_0 + p_0)W(0)\}/l(l+1) \ . \quad (12)$$

We also have $H_0(0) = K(0)$ from (3) by taking the limit $r \to 0$. The two independent solutions can be found by integrating (5)-(8) with $W(0) = 1$ and $K(0) = \pm(\rho_0 + p_0)$. And the general solution is expressed as a combination of those two independent solutions. The unique solution with a specific complex frequency ω is determined by the condition X vanishes at the surface, $X(R) = 0$, which can be obtained from (9) in which ρ, p and p' vanish at the stellar surface.

Outside the star, there would be no fluid motion anymore. The perturbation equation reduces to a second-order differential equation for a Zerilli function Z which is defined in terms of the metric perturbation functions $H_0(r)$ and $K(r)$ as following,

$$\begin{pmatrix} Z(r^*) \\ dZ(r^*)/dr^* \end{pmatrix} = \frac{1}{gk-h} \begin{pmatrix} -a & k-b \\ ga\ . & gb-h \end{pmatrix} \begin{pmatrix} H_0(r) \\ K(r) \end{pmatrix} \tag{13}$$

All the above functions $a(r)$, b, $g(r)$, $k(r)$, $h(r)$ are defined as,

$$a(r) = -(nr+3M)/[\omega^2 r^2 - (n+1)M/r]\ , \tag{14}$$

$$b(r) = \frac{nr(r-2M) - \omega^2 r^4 + M(r-3M)}{(r-2M)[\omega^2 r^2 - (n+1)M/r]}\ , \tag{15}$$

$$g(r) = \frac{n(n+1)r^2 + 3nMr + 6M^2}{r^2(nr+3M)}\ , \tag{16}$$

$$h(r) = \frac{-nr^2 + 3nMr + 3M^2}{(r-2M)(nr+3M)}\ , \tag{17}$$

$$k(r) = -r^2/(r-2M)\ , \tag{18}$$

and,

$$n = \tfrac{1}{2}(l-1)(l+2)\ , \tag{19}$$

$$r^* = r + 2M\log(r/2M - 1) \quad (or, \quad dr^* = \frac{dr}{1-2M/r})\ . \tag{20}$$

The boundary value of Z at the stellar surface is determined by $H_0(R)$, $K(R)$ and (13). And the wave equation of the Zerilli function Z reads,

$$\frac{d^2}{dr^{*2}} Z = [V_z(r^*) - \omega^2] Z\ , \tag{21}$$

where $V_z(r^*)$ is an effective potential and is expressed as,

$$V_z(r^*) = \frac{(1-2M/r)}{r^3(nr+3M)^2}[2n^2(n+1)r^3 + 6n^2Mr^2 + 18nM^2 r + 18M^3]\ . \tag{22}$$

Now we integrate (22) to a large enough r, for example 20 times the stellar radius. The zerilli function has a asymptotic behavior at infinity, which is a composition of an outgoing wave and an incoming wave,

$$Z \sim A_{out} e^{-i\omega r^*} \sum_{j=0}^{\infty} \alpha_j r^{-j} + A_{in} e^{i\omega r^*} \sum_{j=0}^{\infty} \bar{\alpha}_j r^{-j}\ , \tag{23}$$

where A_{out} and A_{in} are the magnitude of the outgoing and incoming gravitational waves, and α_j are constants given by,

$$\alpha_0 = 1\ , \tag{24}$$

$$\alpha_1 = -i(n+1)\omega^{-1}\ , \tag{25}$$

$$\alpha_2 = -\tfrac{1}{2}\omega^{-2}[n(n+1) - \tfrac{3}{2} i M\omega(1+2/n)]\ . \tag{26}$$

A quasi normal mode only allows pure outgoing wave at infinity, that is $A_{in} = 0$. We adopted two algorithms to locate the mode frequencies. One is a predictor-corrector-modifier integration algorithm from [6], and the other is a slightly different algorithm from [8] and [9]. Our code can reproduce the results in [6] and [10]. For f-mode and p-mode stellar oscillations, the imaginary part of the complex eigenfrequency is much smaller than the real part. For this reason, we can approach the eigenfrequency by using a real frequency to perform all our calculation. By doing so, the situation would become significantly simpler since all the perturbation functions would be real. The coefficients of the incoming and outgoing waves are generally complex but conjugate to each other in this case, that is $A_{in} = \bar{A}_{out}$. And the gravitational wave becomes a standing wave at infinity. The predictor-corrector-modifier integration algorithm initially uses two real trial frequencies (ω_1 and ω_2) to calculate the magnitudes of the incoming wave ($A_{in,1}$ and $A_{in,2}$). These two trial frequencies ω_1 and ω_2 should be both close to the real part of the mode frequency. If we assume that the function $A_{in}(\omega)$ is analytic, we can always approach it by a linear function[1] near the complex mode frequency, that is,

$$A_{in}(\omega) \approx A_0 + A_1 \cdot \omega \ . \tag{27}$$

With the two points $(\omega_1, A_{in,1})$ and $(\omega_2, A_{in,2})$ we can calculate out A_0 and A_1 in (27). By letting (27) equal zero we approximately get the mode frequency,

$$\omega = -\frac{A_0}{A_1} \qquad or, \begin{cases} \omega_r = Re\left(-\frac{A_0}{A_1}\right) \\ \omega_i = Im\left(-\frac{A_0}{A_1}\right) \end{cases} . \tag{28}$$

Then we substitute with ω_r into the trial frequency ω_i that is further away from ω_r, and do all the above again. We iterate several times until ω_r is close enough to the previous one. For another algorithm discussed in [8] and [9], it is proven that the amplitude of the standing gravitational wave $(\alpha^2 + \beta^2)^{1/2}$ at infinity would be a minimum if the frequency ω is one of the mode frequencies. Here α and β are the real and the imaginary parts of A_{in}. The predictor-corrector-modifier integration algorithm is convenient to find the f-mode frequencies. But since it requires the two trial frequencies to be very close to the p-mode frequencies, we use the later algorithm to search for p-modes.

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