Quantum convolutional data-syndrome codes

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Abstract

We consider performance of a simple quantum convolutional code in a fault-tolerant regime using several syndrome measurement/decoding strategies and three different error models, including the circuit model. Unlike the quantum stabilizer codes which are designed to be robust agiant qubit errors, our goal is to make the code robust against both qubit error and measurement error. The conventional approach of repeated measurement protocal is used as a benchmark.

Objective: Design Data-Syndrome Codes with Optimal Decoders

QCC → DS-QCC: repeated measurement encoded by repetition code; redundant measurement encoded by convolutional code

Viterbi decoder → Hybrid Viterbi decoder

QCC: quantum convolutional code

DS-QCC: data-syndrome quantum convolutional code

Data-Syndrome(DS) codes

In Data-Syndrome(DS) codes, we have a qubit error vector $\mathbf{e}\in\mathbb{F}_2^n$, and a syndrome measurement error $\boldsymbol{\epsilon}\in\mathbb{F}_2^r$; the extracted syndrome vector is given by $\mathbf{s}^T=\mathbf{G}*\mathbf{e}^T+\boldsymbol{\epsilon}^T$. The error can be characterized by $(\mathbf{e}\,|\,\boldsymbol{\epsilon})$, a pair of a quaternary and a binary vectors, with the inner product

$$(\mathbf{e}_1,\boldsymbol{\epsilon}_1)\star(\mathbf{e}_2,\boldsymbol{\epsilon}_2)^T\equiv\mathbf{e}_1*\mathbf{e}_2^T+\boldsymbol{\epsilon}_1\boldsymbol{\epsilon}_2^T. \tag{1}$$

By analogy with stabilizer codes, we define an additive code $\mathcal{C}_{\rm DS}\subseteq\mathbb{F}_4^n\oplus\mathbb{F}_2^r$ with the generator matrix

$$\mathbf{G}_{\rm DS}=\left(\,\mathbf{G}\,|\,\mathbf{I}\,\right) \tag{2}$$

Convolutional DS codes

Now, given an $[[n,k]]$ quantum code $\mathcal{Q}$ with the (full-row-rank) generating matrix $\mathbf{G}(\mathcal{Q})$ of size $(n-k)\times n$, we introduce redundant measurements by adding some linearly dependent rows. Without limiting generality, a set of r' additional rows $\mathbf{F}=\mathbf{A}\mathbf{G}(\mathcal{Q})$ can be obtained by multiplying the original generating matrix by an $r'\times(n-k)$ binary matrix $\mathbf{A}$, so that the generating matrix $\mathbf{G}_{\rm DS}$ in (3) of the resulting DS code has the form

$$\mathbf{G}_{\rm DS}=\left(\begin{array}{c|cc}\mathbf{G}(\mathcal{Q}) & \mathbf{I}_{n-k} & \\ \mathbf{A}\mathbf{G}(\mathcal{Q}) & & \mathbf{I}_{r'}\end{array}\right)\longrightarrow\mathbf{G}'_{\rm DS}=\left(\begin{array}{c|cc}\mathbf{G}(\mathcal{Q}) & \mathbf{I}_{n-k} & \\ & \mathbf{A} & \mathbf{I}_{r'}\end{array}\right) \tag{3}$$

$\mathbf{G}_{\rm DS}$ can be rewritten in the row-equivalent form $\mathbf{G}'_{\rm DS}$.

When choosing $\mathbf{A}^T$ to be the generator matrix of a classical convolutional code, it is called a *convolutional DS code*. This set up allows a low-complexity decoder, as explained in the next section.

Decoding of Convolutional DS Codes

Big advantage of classical convolutional codes is that one can use the maximum-likelihood Viterbi decoding using a code trellis. The "stripe" form of a generator matrix of a convolutional code (with small band width) ensures that its code trellis has relatively small number of states, which means that the Viterbi decoding has relatively small complexity.

In our case, we modify $\mathbf{G}_{\rm DS}$ into the "stripe" form and design a hybrid Viterbi decoder to decode qubits and syndrome bits simulataneously. We show an example as following, where $\mathbf{G}(\mathcal{Q})$ and $\mathbf{A}$ are generated by vectors $(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)$ and $(\mathbf{u}_1|\mathbf{u}_2|\mathbf{u}_3)$, respectively, and assume that $\mathbf{v}_i$ and $\mathbf{u}_i$ have lengths n and n'. Then, in a particular case, the DS code generator $\mathbf{G}'_{\rm DS}$ in (3) has the form

$$\mathbf{G}'_{\rm DS}=\begin{pmatrix}\mathbf{v}_1&\mathbf{v}_2&\mathbf{v}_3&&&1&&&&&&&\\&\mathbf{v}_1&\mathbf{v}_2&\mathbf{v}_3&&&1&&&&&&\\&&\mathbf{v}_1&\mathbf{v}_2&\mathbf{v}_3&&&1&&&&&\\&&&&&\mathbf{u}_1^T&&&\mathbf{I}&&&&\\&&&&&\mathbf{u}_2^T&\mathbf{u}_1^T&&&\mathbf{I}&&&\\&&&&&\mathbf{u}_3^T&\mathbf{u}_2^T&\mathbf{u}_1^T&&&\mathbf{I}&&\\&&&&&&\mathbf{u}_3^T&\mathbf{u}_2^T&&&&\mathbf{I}&\\&&&&&&&\mathbf{u}_3^T&&&&&\mathbf{I}\end{pmatrix},$$

where $\mathbf{I}$ is the $n'\times n'$ identity matrix. With an appropriate permutation of columns and rows, we can transform the above matrix into the form

$$\mathbf{G}''_{\rm DS}=\begin{pmatrix}\mathbf{v}_1&\mathbf{v}_2&&&\mathbf{v}_3&1&&&&&&&\\&&\mathbf{I}&&&\mathbf{u}_1^T&&&&&&&\\&\mathbf{v}_1&&&\mathbf{v}_2&&&\mathbf{v}_3&1&&&&\\&&&\mathbf{I}&&\mathbf{u}_2^T&&&\mathbf{u}_1^T&&&&\\&&&&\mathbf{v}_1&&&\mathbf{v}_2&&&\mathbf{v}_3&1&\\&&&&&\mathbf{u}_3^T&\mathbf{I}&&\mathbf{u}_2^T&&&\mathbf{u}_1^T&\\&&&&&&&&\mathbf{u}_3^T&\mathbf{I}&&\mathbf{u}_2^T&\\&&&&&&&&&&&\mathbf{u}_3^T&\mathbf{I}\end{pmatrix},$$

where we marked the small matrix block that defines the repeating section of the syndrome trellis. Now, the method in Ref. [2] gives the syndrome trellis, a particular form of the code trellis.

Numerical results

For typical Noisy Intermediate-Scale Quantum (NISQ) devices like superconducting transmon qubits, measurement/read-out error has 10 times larger probability compared with gate error. This ratio was considered in our simulation parameters. We start with a simple quantum convolutional code $\mathbf{G}(\mathcal{Q}_6)$ with the parameters $[[24,6,3]]$ and syndrome generators of weight 6 [3]. Explicitly we show $\mathbf{G}(\mathcal{Q}_1)$ in Eq (4)

$$\mathbf{G}(\mathcal{Q}_1)=\left(\begin{array}{ccc|ccc|ccc}1&\omega&\overline{\omega}&&&&&&\\\overline{\omega}&\omega&1&&&&&&\\1&1&1&1&\omega&\overline{\omega}&&&\\\omega&\omega&\omega&\omega&\overline{\omega}&1&&&\\&&&1&1&1&1&\omega&\overline{\omega}\\&&&\omega&\omega&\omega&\omega&\overline{\omega}&1\\&&&&&&\overline{\omega}&\omega&1\\&&&&&&1&1&1\end{array}\right). \tag{4}$$

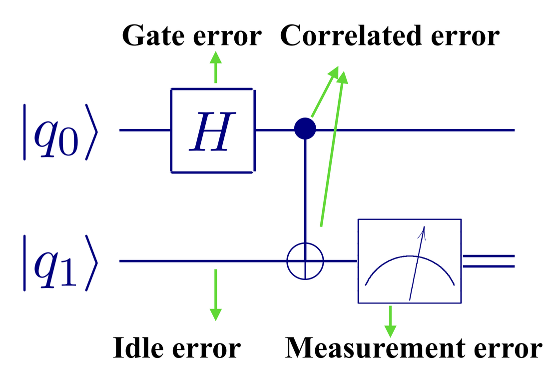

Figure 1: Circuit-based error model (C)

Three Error Models (A) (B) & (C)

Phenomenological error model (A) is a depolarizing channel with qubit error probability p and independent syndrome bit error probability q.

Phenomenological error model (B) includes several rounds of syndrome measurement, and qubit errors that happen before each round (depolarizing noise with probability p; these errors accumulate between measurement rounds), and independent syndrome measurement errors with probability q.

Model (C) is a circuit error model.

For simulations one includes an additional round with perfect syndrome measurement.

Four measurement protocols

(i) Code "GA", a quantum DS CC (3) with the 16×18 matrix $\mathbf{A}^T$ chosen as the generating matrix of the binary convolutional code (CC) with the generator row $\mathbf{g}=(11|01|11)$. Explicitly,

$$\mathbf{A}_{\rm GA}^T=\begin{pmatrix}1\,1\,0\,1\,1\,1&&\\&1\,1\,0\,1\,1\,1&\\&&\ldots\ldots\ldots\end{pmatrix}. \tag{5}$$

Rows of matrix $\mathbf{F}=\mathbf{A}_{\rm GA}\mathbf{G}(\mathcal{Q}_k)$ have weights ${\rm wt}(\mathbf{f}_j)\in\{6,9\}$.

(ii) Code "GR" (here R stands for "repetition") is constructed similarly, except the matrix $\mathbf{A}^T$ is formed by a trivial CC code with $\mathbf{g}=(11)$. Explicitly, it has the form

$$\mathbf{A}_{\rm GR}^T=\begin{pmatrix}1\,1&&\\&1\,1&\\&&\ldots\end{pmatrix}. \tag{6}$$

It is easy to see that such a matrix results from three-times repeated measurement of the original set of generators in the 18 rows of $\mathbf{G}(\mathcal{Q}_6)$. Respectively, only the original stabilizer generators of weights 3 and 6 need to be measured here.

(iii) Code "GI" is a trivial DS code with $\mathbf{A}_{\rm GI}=\mathbf{0}$. The name is due to the structure of the matrix (3): in this case it has the form $\mathbf{G}_{\rm DS}=(\mathbf{G}(\mathcal{Q}_6)\,|\,\mathbf{I}_{18})$. With phenomenological error model (A) and three-times repeated measurement, we use this code as a simpler alternative to code "GR". Namely, we first perform majority vote on every bit of the syndrome, then use the DS code GI for actual decoding.

(iv) Finally, the code "G" stands for yet another simple DS decoding protocol for three-fold repeated measurements. Again, the syndrome bits are obtained using majority vote, but the resulting syndrome is considered as error-free, and the decoding is done directly using the QCC $\mathcal{Q}_6$. Main difference with the previous case is that here a single-bit syndrome error after majority vote necessarily results in a decoding fault.

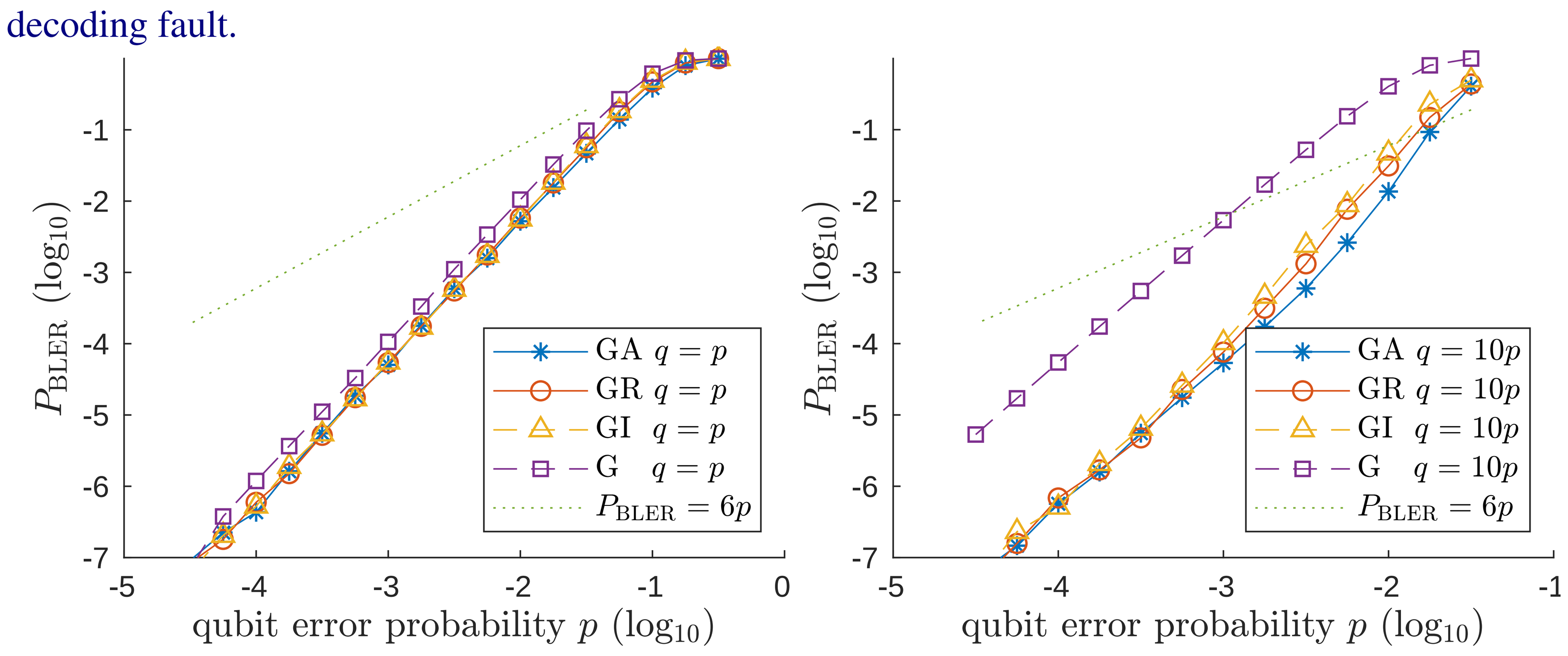

Figure 2: Phenomenological noise model (A) with depolarizing errors (probability p) and syndrome bit measurement error probability $q=p$ (left) and $q=10p$ (right). Symbols show the block error probability $P_{\rm BLER}$ for four decoders as indicated, see text for details. Dotted lines give the nominal single-qubit break-even threshold, $P_{\rm BLER}=p$. Results indicate that (with the exception of the simplest decoder G) all decoders are able to correct most syndrome measurement errors with $q=p$, and also for $q=10p$ in the interval $p\lesssim 10^{-3}$. With larger error rates, code GA works best, consistent with its larger distance for syndrome errors.

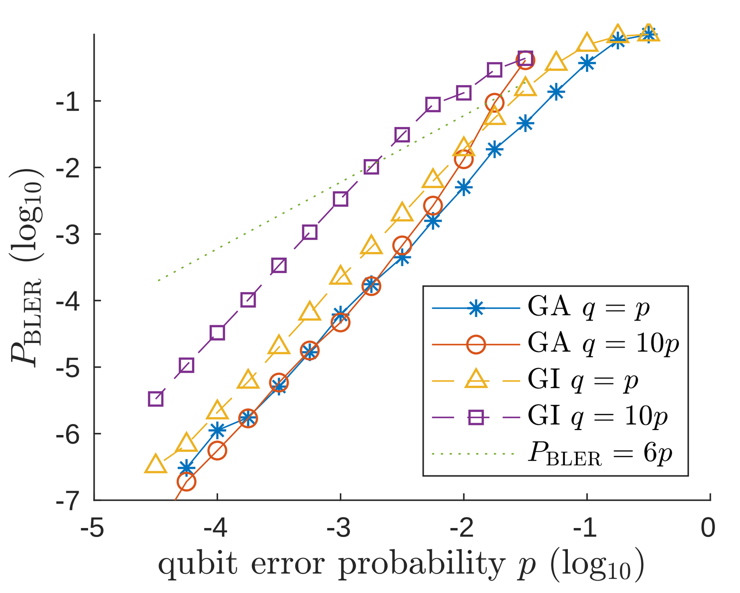

Figure 3: Effective block error rate $P_{\rm BLER}$ with phenomenological error model B. Only results for GA and single-interval GI decoders are shown. The results are largely consistent with those for phenomenological error model (A).

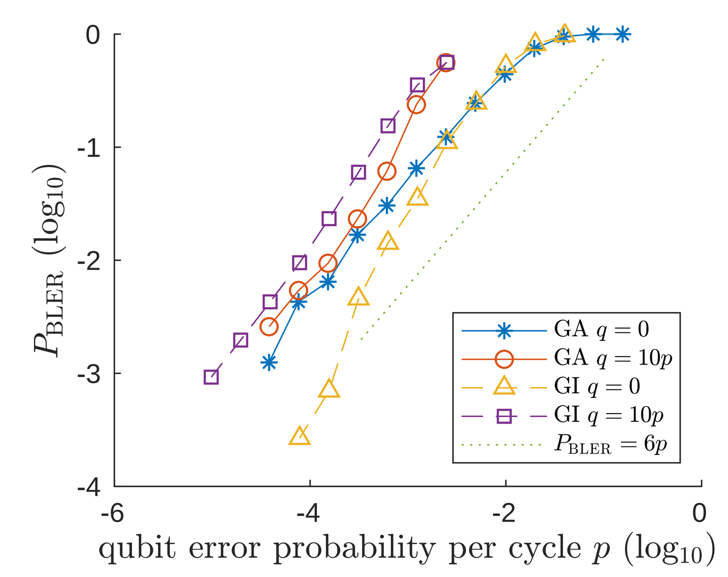

Figure 4: Effective block error rate (per gate) for phenomenological error model C as a function of gate error probability p_1 scaled by cycle duration. One striking difference with phenomenological error models A and B is that the calculated curves no longer have quadratic dependence on BER, as would be expected for a code with distance $d=3$.

Conclusion

1. We introduced quantum convolutional data-syndrome codes.
2. We constructed an efficient decoder for this class of codes.
3. We analyzed numerically the performance of a family of DS codes based on a $[[24,6,3]]$ QCC using three distinct error models.
4. Before this work [1], DS code has not been simulated with the circuit error model.
5. Our simulation results indicate that a DS code with large-distance classical syndrome code may show competitive performance in the regime where measurement errors are significant, even if code generators have larger weights.
6. This regime is experimentally relevant to NISQ devices.

Future work

1. The QCCs we used have relatively high weights of stabilizer generators. It is an open question whether degenerate QCCs exist, with small-weight generators, large distances, and trellises with reasonably small memory sizes. For the purpose of constructing convolutional DS codes, one would further like to have a QCC with a redundant set of minimum-weight stabilizer generators. For such codes, degenerate Viterbi decoding algorithm would be particularly useful.
2. It is an open question whether similarly constructed non-convolutional DS codes could be useful in this regime, e.g., for optimizing the performance of surface codes in NISQ devices.
3. One promising way for improving the practical performance of DS codes is by using fault-tolerant gadgets for generator measurements, since such circuit prevent error propagation.

Reference

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[2] V. Sidorenko and V. Zyablov, "Decoding of convolutional codes using a syndrome trellis," *IEEE Trans. Inf. Th.*, vol. 40, pp. 1663–1666, 1994.

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