Encoding an Oscillator into Many Oscillators

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An outstanding challenge for quantum information processing using bosonic systems is Gaussian errors such as excitation loss and added thermal noise errors. Thus, bosonic quantum error correction is essential. Most bosonic quantum error correction schemes encode a finite-dimensional logical qubit or qudit into noisy bosonic oscillator modes. In this case, however, the infinite-dimensional bosonic nature of the physical system is lost at the error-corrected logical level. On the other hand, there are several proposals for encoding an oscillator mode into many noisy oscillator modes. However, these oscillator-into-oscillators encoding schemes are in the class of Gaussian quantum error correction. Therefore, these codes cannot correct practically relevant Gaussian errors due to the established no-go theorems that state that Gaussian errors cannot be corrected by using only Gaussian resources. Here, we circumvent these no-go results and show that it is possible to correct Gaussian errors by using Gottesman-Kitaev-Preskill (GKP) states as non-Gaussian resources. In particular, we propose a non-Gaussian oscillator-into-oscillators code, namely the GKP two-mode squeezing code, and demonstrate that it can quadratically suppress additive Gaussian noise errors in both the position and momentum quadratures except for a small sublogarithmic correction. Furthermore, we demonstrate that our GKP two-mode squeezing code is near optimal in the weak noise limit by proving via quantum information theoretic tools that quadratic noise suppression is optimal when we use two physical oscillator modes. Lastly, we show that our non-Gaussian oscillator encoding scheme can also be used to correct excitation loss and thermal noise errors, which are dominant error sources in many realistic bosonic systems.

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Introduction.—Continuous variable (CV) bosonic quantum systems are ubiquitous in various quantum computing and communication architectures [1,2] and provide unique advantages to, e.g., quantum simulation of bosonic systems such as boson sampling [3–6] and simulation of vibrational quantum dynamics of molecules [7–10]. However, realistic imperfections such as excitation losses and added thermal noise are major challenges for realizing large-scale and fault-tolerant CV quantum information processing.

Quantum error correction (QEC) is essential for scalable and fault-tolerant quantum information processing [11]. Recently, there have been significant advances in using bosonic systems to realize QEC in a more hardware-efficient manner [12]. In many bosonic QEC schemes proposed so far, a finite-dimensional quantum system (e.g., a qubit) is encoded into an oscillator [13–18] or into many oscillators [19–28]. For example, the four-component cat code [15] encodes a logical qubit into an oscillator using cat states with an even number of excitations, i.e., $|0_L\rangle \propto |\alpha\rangle + |-\alpha\rangle$ and $|1_L\rangle \propto |i\alpha\rangle + |-i\alpha\rangle$. Thanks to the inherent hardware efficiency, various qubit-into-an-oscillator bosonic QEC schemes have been realized experimentally [29–37]. However, such schemes are not tailored to CV information processing tasks since the error-corrected systems are described by discrete variables such as Pauli operators.

On the other hand, an error correction scheme where an infinite-dimensional oscillator mode is encoded into many noisy oscillators can still be tailored to various CV quantum information processing tasks. So far, there have been several proposals for encoding an oscillator into many oscillators [38–45]. For example, in the case of the three-mode Gaussian-repetition code [38,39], an infinite-dimensional oscillator mode is encoded into three oscillators by repeatedly appending the position eigenstates: $|\hat{q}_L = q\rangle \equiv |\hat{q}_1 = q\rangle|\hat{q}_2 = q\rangle|\hat{q}_3 = q\rangle$. Note that in this case, the logical Hilbert space is infinite dimensional because q can be any real number.

However, none of the previously proposed oscillator-into-oscillators codes can correct Gaussian errors. This is because they are *Gaussian quantum error correction* schemes and the established no-go theorems state that Gaussian errors cannot be corrected by using only Gaussian operations [26,46,47]. Since Gaussian errors include excitation losses, thermal noise, and additive Gaussian noise errors that are ubiquitous in many realistic bosonic systems, these no-go results set a hard limit on the practical utility of the proposed Gaussian oscillator encoding schemes.

Moreover, since the quantum capacity [48] of Gaussian channels is finite [49–55], one may be tempted to deduce that useful infinite-dimensional oscillator encoding against Gaussian errors is not possible.

In this Letter, we demonstrate that useful oscillator encoding is nevertheless possible despite the Gaussian no-go results and the finite quantum capacity. That is, we circumvent the established Gaussian no-go results by using the Gottesman-Kitaev-Preskill (GKP) states [14] as a non-Gaussian resource. We also show that nontrivial QEC gain (larger than unity) with oscillator encoding is compatible with the finiteness of Gaussian quantum channel capacity.

GKP states as non-Gaussian resources.—Examples of non-Gaussian resources [56,57] include the single-photon Fock state and photon-number-resolving measurements [3,58], Kerr nonlinearities [59], cubic phase state and gate [14], SNAP gate [60], Schrödinger cat states [13], and GKP states [14,61]. Among these non-Gaussian resources, we focus on the GKP states.

The canonical GKP state (or the grid state) [14,62] is defined as the unique (up to an overall phase) simultaneous eigenstate of the two commuting displacement operators $\hat{S}_q \equiv \exp[i\sqrt{2\pi}\hat{q}]$ and $\hat{S}_p \equiv \exp[-i\sqrt{2\pi}\hat{p}]$ with unit eigenvalues. Explicitly, the canonical GKP state is given by

$$|\mathrm{GKP}\rangle \propto \sum_{n\in\mathbb{Z}} |\hat{q} = \sqrt{2\pi}n\rangle \propto \sum_{n\in\mathbb{Z}} |\hat{p} = \sqrt{2\pi}n\rangle \qquad (1)$$

and thus has definite values of *both* the position and the momentum operators modulo $\sqrt{2\pi}$, i.e., $\hat{q} = \hat{p} = 0$ mod $\sqrt{2\pi}$. That is, we circumvent the Heisenberg uncertainty principle (i.e., $\hat{q}$ and $\hat{p}$ cannot be measured simultaneously) by measuring the quadrature operators modulo $\sqrt{2\pi}$. Note that the effects of such compromise are negligible if the quantities of interest are much smaller than $\sqrt{2\pi}$, which will be the case below for our purposes [see Eq. (11)].

In Fig. 1(a), we plot the Wigner function of the canonical GKP state with an average photon number $\bar{n} = 5$ [63]. Note that negative peaks in the Wigner function indicate that the GKP state is a non-Gaussian state. We also remark that the ability to prepare the canonical GKP state (combined with Gaussian operations) allows us to measure the quadrature operators modulo $\sqrt{2\pi}$ [see Ref. [14] and also Fig. 1(b)], which is needed in our oscillator encoding scheme introduced below [see Fig. 2(b)].

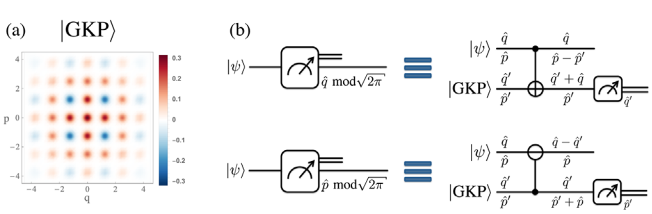

FIG. 1. (a) An approximate GKP state with an average photon number $\bar{n} = 5$. (b) Measurement of the position or momentum operator modulo $\sqrt{2\pi}$. The controlled $\oplus$ and $\ominus$ symbols, respectively, represent the SUM and the inverse-SUM gates where $\mathrm{SUM}_{j\to k} \equiv \exp[-i\hat{q}_j\hat{p}_k]$. The final measurement in the bottom mode is the homodyne measurement of the position or the momentum quadrature operator.

GKP two-mode squeezing code.—Let $|\psi\rangle = \int dq\psi(q)|\hat{q}_1 = q\rangle$ be an arbitrary bosonic state (or data) that we want to encode into two bosonic modes. We define the encoded state of the GKP two-mode squeezing code as follows:

$$|\psi_L\rangle = \mathrm{TS}_{1,2}(G)|\psi\rangle \otimes |\mathrm{GKP}\rangle. \qquad (2)$$

Here, $|\mathrm{GKP}\rangle$ is the canonical GKP state in the second mode and $\mathrm{TS}_{1,2}(G) = \exp[g(\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2)]$ is the two-mode squeezing operation acting on modes 1 and 2 with a gain $G = \cosh^2(g) \geq 1$ [hence the name of the code; see Fig. 2(a)]. We refer to the first mode as the data mode and the second mode as the ancilla mode. In the Heisenberg picture, the two-mode squeezing operation $\mathrm{TS}_{1,2}(G)$ transforms the quadrature operator $\hat{\boldsymbol{x}} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2)^T$ into $\hat{\boldsymbol{x}}' = (\hat{q}_1', \hat{p}_1', \hat{q}_2', \hat{p}_2')^T = \boldsymbol{S}_{\mathrm{TS}}(G)\hat{\boldsymbol{x}}$, where the 4×4 symplectic matrix $\boldsymbol{S}_{\mathrm{TS}}(G)$ associated with $\mathrm{TS}_{1,2}(G)$ is given by (see, e.g., Ref. [2])

$$\boldsymbol{S}_{\mathrm{TS}}(G) = \begin{bmatrix} \sqrt{G}\boldsymbol{I} & \sqrt{G-1}\boldsymbol{Z} \\ \sqrt{G-1}\boldsymbol{Z} & \sqrt{G}\boldsymbol{I} \end{bmatrix}. \qquad (3)$$

Here, $\boldsymbol{I} = \mathrm{diag}(1,1)$ and $\boldsymbol{Z} = \mathrm{diag}(1,-1)$. Note that the gain G can be chosen at will to optimize the performance of the error correction scheme.

For the noise model, we consider the independent and identically distributed additive Gaussian noise errors, i.e., $\mathcal{N}^{(1)}[\sigma] \otimes \mathcal{N}^{(2)}[\sigma]$, mainly due to their experimental relevance (see also [69]). In the Heisenberg picture, $\mathcal{N}^{(k)}[\sigma]$ adds Gaussian random noise $\xi_q^{(k)}$ and $\xi_p^{(k)}$ to the quadrature operators of the kth mode. Thus, the quadrature operator $\hat{\boldsymbol{x}}'$ is further transformed via the additive Gaussian noise error into $\hat{\boldsymbol{x}}'' = \hat{\boldsymbol{x}}' + \boldsymbol{\xi}$, where $\boldsymbol{\xi} = [\xi_q^{(1)}, \xi_p^{(1)}, \xi_q^{(2)}, \xi_p^{(2)}]^T$ is the quadrature noise vector following $\boldsymbol{\xi} \sim_{\mathrm{iid}} \mathcal{N}(0,\sigma^2)$ [see Fig. 2(a)].

The decoding procedure [shown in Fig. 2(b)] starts with an application of the inverse of the encoding circuit $[\mathrm{TS}_{1,2}(G)]^\dagger$. Then, the quadrature operator is transformed into $\hat{\boldsymbol{x}}''' = [\boldsymbol{S}_{\mathrm{TS}}(G)]^{-1}\hat{\boldsymbol{x}}'' = \hat{\boldsymbol{x}} + \boldsymbol{z}$, where $\boldsymbol{z} \equiv [\boldsymbol{S}_{\mathrm{TS}}(G)]^{-1}\boldsymbol{\xi}$ is the reshaped quadrature noise vector, which is given by

$$\boldsymbol{z} = \begin{bmatrix} \sqrt{G}\xi_q^{(1)} - \sqrt{G-1}\xi_q^{(2)} \\ \sqrt{G}\xi_p^{(1)} + \sqrt{G-1}\xi_p^{(2)} \\ \sqrt{G}\xi_q^{(2)} - \sqrt{G-1}\xi_q^{(1)} \\ \sqrt{G}\xi_p^{(2)} + \sqrt{G-1}\xi_p^{(1)} \end{bmatrix} \equiv \begin{bmatrix} z_q^{(1)} \\ z_p^{(1)} \\ z_q^{(2)} \\ z_p^{(2)} \end{bmatrix}. \qquad (4)$$

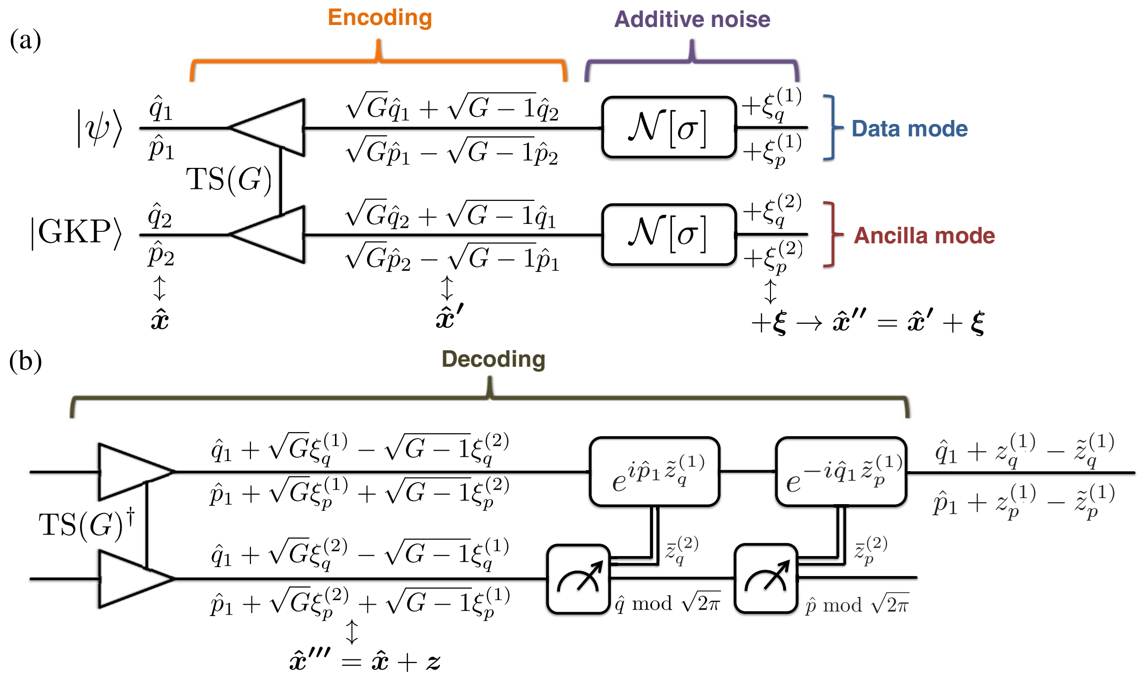

FIG. 2. (a) Encoding circuit of the GKP two-mode squeezing code subject to independent and identically distributed additive Gaussian noise errors. (b) Decoding circuit of the GKP two-mode squeezing code. Note that the circuits for the measurements of the position and the momentum operators modulo $\sqrt{2\pi}$ (at the end of the decoding) are defined in Fig. 1(b).

The role of the two-mode squeezing operations in the encoding and the decoding circuits is clear by now: they transform uncorrelated additive noise $\boldsymbol{\xi}$ into *correlated* additive noise $\boldsymbol{z}$. This means that after noise reshaping, we can extract useful information about the data noise $z_q^{(1)}$ and $z_p^{(1)}$ by measuring only the ancilla noise $z_q^{(2)}$ and $z_p^{(2)}$. Importantly, the encoded logical information in the data mode is not revealed through this process because the data mode need not be measured. Note also that we need to measure *both* the position and momentum noise in the ancilla mode. This is precisely the reason why the simultaneous quadrature measurement (modulo $\sqrt{2\pi}$) in Fig. 1(b) is needed at the end of the decoding circuit [71].

Based on the outcomes of the simultaneous measurement of the ancilla quadrature noise modulo $\sqrt{2\pi}$, we estimate that $z_q^{(2)}$ and $z_p^{(2)}$ are given by the smallest displacements that are compatible with the modular measurement outcomes, i.e.,

$$\bar{z}_q^{(2)} = R_{\sqrt{2\pi}}[z_q^{(2)}] \quad \text{and} \quad \bar{z}_p^{(2)} = R_{\sqrt{2\pi}}[z_p^{(2)}], \tag{5}$$

where $R_s(z) \equiv z - n^\star(z)s$ and $n^\star(z) \equiv \mathrm{argmin}_{n\in\mathbb{Z}}|z - ns|$ [72]. We then further estimate that the data quadrature noise $z_q^{(1)}$ and $z_p^{(1)}$ are

$$\bar{z}_q^{(1)} = -\frac{2\sqrt{G(G-1)}}{2G-1}\bar{z}_q^{(2)} \quad \text{and}$$
$$\bar{z}_p^{(1)} = \frac{2\sqrt{G(G-1)}}{2G-1}\bar{z}_p^{(2)}, \tag{6}$$

which are obtained by taking into account the correlation in the reshaped noise $\boldsymbol{z}$ and using a maximum likelihood estimation method as detailed in the Supplemental Material [64].

The decoding operation is simply to remove the estimated noise in the data mode by applying the counterdisplacement operations $\exp[i\hat{p}_1\bar{z}_q^{(1)}]$ and $\exp[-i\hat{q}_1\bar{z}_p^{(1)}]$ to the data mode [see Fig. 2(b)]. As a result, we are left with the following logical position and momentum quadrature noise:

$$\xi_q^{(\mathrm{out})} \equiv z_q^{(1)} - \bar{z}_q^{(1)} \quad \text{and} \quad \xi_p^{(\mathrm{out})} \equiv z_p^{(1)} - \bar{z}_p^{(1)}, \tag{7}$$

whose variances $(\sigma_q)^2 = (\sigma_p)^2 = (\sigma_L)^2$ are given by

$$(\sigma_L)^2 = \frac{\sigma^2}{2G-1} + \sum_{n\in\mathbb{Z}} \frac{4G(G-1)}{(2G-1)^2} 2\pi n^2 \times q_n(\sigma). \tag{8}$$

Here, $q_n(\sigma) \equiv \int_{(n-1/2)\sqrt{2\pi}}^{(n+1/2)\sqrt{2\pi}} dz p[\sqrt{2G-1}\sigma](z)$, where $p[\sigma](z) \equiv (1/\sqrt{2\pi\sigma^2})\exp[-(z^2/2\sigma^2)]$ is the probability density function of the Gaussian normal distribution $\mathcal{N}(0,\sigma^2)$. See the Supplemental Material [64] for the derivation of Eq. (8).

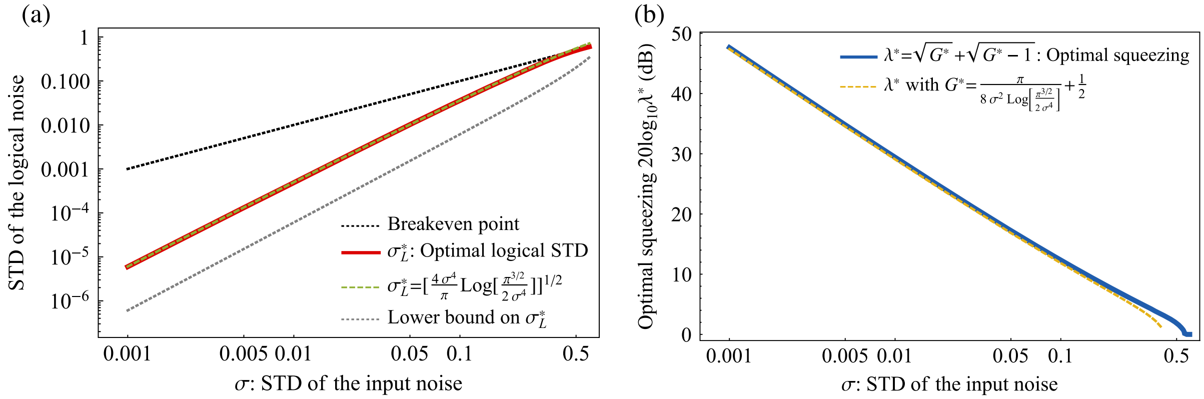

FIG. 3. (a) The minimum standard deviations of the output logical quadrature noise $\sigma_q = \sigma_p = \sigma_L^\star$ as a function of the input standard deviation σ for the GKP two-mode squeezing code and (b) the optimal two-mode squeezing gain $G^\star$ that achieves $\sigma_L^\star$, translated to the required single-mode squeezing in the unit of decibel $20\log_{10}\lambda^\star$, where $\lambda^\star \equiv \sqrt{G^\star} + \sqrt{G^\star - 1}$. The black dotted line, the green dashed line, and the grey dotted line in (a) represent the breakeven point $\sigma_L = \sigma$, the asymptotic expression $\sigma_L^\star = (2\sigma^2/\sqrt{\pi})\sqrt{\log_e[\pi^{3/2}/2\sigma^4]}$, and the lower bound on σ_L in Eq. (12), respectively. Also, the yellow dashed line in (b) represents $G^\star = (\pi/8\sigma^2)(\log_e[\pi^{3/2}/2\sigma^4])^{-1} + \frac{1}{2}$.

Recall that we can freely choose the gain G to optimize the performance of the GKP two-mode squeezing code. Here, we choose G such that the standard deviation of the output logical quadrature noise σ_L is minimized. In Fig. 3, we plot the minimum standard deviation of the output logical quadrature noise $\sigma_L^\star$ and the optimal gain $G^\star$ that achieves $\sigma_L^\star$ as a function of the input noise standard deviation σ. These optimal values are obtained via a brute-force numerical optimization.

From the numerical optimization, we find that for $\sigma \geq \sigma_c \equiv 0.558$, the optimal gain $G^\star$ is trivially given by $G^\star = 1$ and thus the GKP two-mode squeezing code cannot reduce the noise standard deviation: $\sigma_L^\star = \sigma$. On the other hand, if the input noise is small enough, i.e., $\sigma < \sigma_c = 0.558$, the optimal gain $G^\star$ is strictly larger than 1, and the minimum standard deviation of the output quadrature noise σ_L can be made smaller than the input noise standard deviation σ: $\sigma_L^\star < \sigma$. Moreover in the $\sigma \ll 1$ regime, we analytically find that the optimal gain $G^\star$ is asymptotically given by

$$G^\star \overset{\sigma\ll 1}{\to} \frac{\pi}{8\sigma^2}\left(\log_e\left[\frac{\pi^{3/2}}{2\sigma^4}\right]\right)^{-1} + \frac{1}{2}, \tag{9}$$

and the optimal output standard deviation $\sigma_L^\star$ is given by

$$\sigma_L^\star \overset{\sigma\ll 1}{\to} \frac{2\sigma^2}{\sqrt{\pi}}\sqrt{\log_e\left[\frac{\pi^{3/2}}{2\sigma^4}\right]}. \tag{10}$$

See the Supplemental Material [64] for a detailed derivation. As shown in Fig. 3, these asymptotic expressions agree well with the exact numerical results in the small σ regime. Note that the standard deviations (STDs) of the ancilla noise are given by

$$\mathrm{STD}[z_q^{(2)}] = \mathrm{STD}[z_q^{(2)}] = \sqrt{(2G-1)}\sigma = o(1) \tag{11}$$

and thus are much smaller than the spacing of the GKP state $\sqrt{2\pi}$ at the optimal gain $G^\star$. Moreover, Eq. (10) implies that the output standard deviation $\sigma_L^\star$ decreases quadratically in σ (i.e., $\sigma_L^\star \propto \sigma^2$) except for a small sublogarithmic correction. We remark that the quadratic suppression is the best one can hope for when using two physical oscillator modes. That is, σ_L is fundamentally lower bounded by

$$\sigma_L \geq \frac{\sigma^2}{\sqrt{e}(1-\sigma^2)} > \frac{\sigma^2}{\sqrt{e}} \tag{12}$$

due to the finite quantum capacity of $\mathcal{N}[\sigma]$ (see the Supplemental Material [64] for the proof). Thus, our GKP two-mode squeezing code is asymptotically near optimal in the weak noise limit up to an overall constant factor [see Fig. 3(a) for an illustration].

Excitation loss and thermal noise errors.—Excitation loss errors with external thermal noise are described by Gaussian thermal-loss channels that can be converted via a quantum-limited amplification to an additive Gaussian noise channel [12,54]. For instance, the bosonic pure-loss channel with loss probability $\gamma \in [0,1]$ can be converted to an additive Gaussian noise channel $\mathcal{N}[\sigma]$ with $\sigma = \sqrt{\gamma}$. Hence, the GKP two-mode squeezing code can also handle the excitation loss errors because we can simply convert the loss errors into the additive noise errors and then apply the same decoding scheme presented above [73].

Assuming the amplification decoding, the critical value of the standard deviation $\sigma_c = 0.558$ corresponds to the critical loss probability $\gamma_c = (\sigma_c)^2 = 0.311$ in the case of pure excitation loss. Thus, the GKP two-mode squeezing code helps when the loss probability is below 31.1%. For example, for 1% loss probability (i.e., $\gamma = 0.01$ and

$\sigma = \sqrt{\gamma} = 0.1$), the optimal two-mode squeezing gain is given by $G^\star = 4.806$, which requires $20\log_{10}\lambda^\star = 12.35$ dB single-mode squeezing operations. In this case, the output noise standard deviation is given by $\sigma_L^\star = 0.036$, equivalent to the loss probability 0.13%. This corresponds to a QEC "gain" for the protocol of $1/0.13 \simeq 7.7$ in terms of the loss probability and $0.1/0.036 \simeq 2.8$ in terms of displacement noise.

Experimental realizations and potential applications.— Our oscillator encoding scheme requires Gaussian operations and GKP states. While Gaussian resources are readily available in many realistic bosonic systems, preparing a canonical GKP state is not strictly possible because it would require infinite squeezing. Recently, however, there have been many proposals for preparing an approximate GKP state in various experimental platforms [14,74–86]. Notably, approximate GKP states have been realized in both trapped ion [33–35] and circuit QED [37] systems. Thus, our GKP two-mode squeezing code can in principle be realized in the state-of-the-art quantum computing platforms.

Imperfections in realistic GKP states such as finite squeezing will add additional quadrature noise to the system. Indeed, we show in the Supplemental Material [64] that a nontrivial QEC gain $\sigma^2/(\sigma_L^\star)^2 > 1$ can be achieved only when the supplied GKP states have a squeezing larger than the critical value 11.0 dB. On the other hand, the squeezing of the experimentally realized GKP states ranges from 5.5 to 9.5 dB [34,37]. In this regard, we stress that our oscillator encoding scheme is compatible with *nondeterministic* GKP state preparation schemes since the GKP states can be prepared off-line and then supplied to the error correction circuit in the middle of the decoding procedure (in a similar fashion to the standard magic state injection protocol [87]). Thus, in near-term experiments, it will be more advantageous to use post-selection to achieve sufficiently high GKP squeezing at the expense of success probability [88].

Our work paves the way toward robust CV quantum information processing via oscillator encoding schemes that can correct experimentally relevant Gaussian errors. As an example, we discuss in the longer version of the paper [89] how to perform logical beam splitter operations, which are needed, e.g., for error-corrected boson sampling. Also, a recent follow-up work [90] has theoretically demonstrated that the GKP two-mode squeezing code proposed in this Letter can be used to enhance the robustness of CV distributed sensing protocols. Moreover, we also expect that our oscillator encoding scheme can be useful for overcoming loss errors in transduction protocols [91,92].

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[64] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.080503, which includes Refs. [65–68] and contains detailed discussions of approximate GKP states and maximum likelihood estimation method for deriving Eq. (6). It also includes derivations of Eqs (8), (10), (12) and analysis of our GKP two-mode squeezing code with noisy ancilla GKP states.

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[72] More explicitly, $R_s(z)$ is a periodic function with period s and is given by $R_s(z)=z$ if $z\in[-s/2,s/2]$.

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