

## ON TENSOR PRODUCTS OF CSS CODES

BENJAMIN AUDOUX AND ALAIN COUVREUR

**ABSTRACT.** CSS codes are in one-to-one correspondance with length 3 chain complexes. The latter are naturally endowed with a tensor product  $\otimes$  which induces a similar operation on the former. We investigate this operation, and in particular its behavior with regard to minimum distances. Given a CSS code  $C$ , we give a criterion which provides a lower bound on the minimum distance of  $C \otimes D$  for every CSS code  $D$ . From this criterion arises a generic bound for the minimum distance which is twice larger than the single bound previously known in the literature. We apply these results to study the behaviour of iterated tensor powers of codes. Such sequences of codes are logarithmically LDPC and we prove in particular that their minimum distances tend generically to infinity. More precisely, their minimum distance increases as  $O(n^\alpha)$  for some  $\alpha > 0$ , where  $n$  is the codes length, while the row weight of their parity-check matrices grows as  $O(\log(n))$ . This entails a rather surprising fact: even if a CSS code does not have quantum degeneracy, for a large enough  $\ell$ , its  $\ell$ -th iterated tensor power does. Different known results are also reinterpreted in terms of tensor products and three new families of LDPC CSS codes are studied.

## INTRODUCTION

In the last century, error-correcting codes were developed to overcome the emergence of anomalies in data while transmitting or storing them. In the quantum setting, such correction systems are all the more important as quantum decoherence eventually produces such errors. At the fall of the  $XX^{\text{th}}$ , several constructions were given for quantum error-correcting codes; among them, CSS codes, developed by A.R Calderbank, P. Shor and A. Steane [CS96, Ste96], are constructed from two classical codes orthogonal to each other. Because of their strong relation with classical codes, they have been subject to intense study. CSS codes can alternatively be related to the topological notion of chain complexes. Not only this approach provides a way to construct CSS codes, but parameters such as length, dimension and minimum distance can also be read through the chain complex and its (co)homology. In this perspective, the non detectable error patterns correspond to (co)cycles with nonzero representatives in the (co)homology of the chain complex. The dimension of the quantum code is nothing but the dimension of the homology group, and the quantum minimum distance nothing but the minimum weight of a (co)homologically non trivial cycle. This point of view was pioneered by M. Freedman, D. Meyer [FM01] and A. Kitaev [Kit03].

Swiftness in error-correction is crucial since error correction should occur faster than errors arise. In the classical setting, LDPC (Low Density Parity Check) codes, that is codes with sparse parity-check matrices [Gal62] are known to have very efficient decoding algorithms. These so-called *iterative decoding algorithms* have a very low complexity and can be applied to some LDPC codes with error correction performance very close to the Shannon limit; see for instance [RSU01]. The notion of LDPC code can be transposed to the quantum setting and efficient classical iterative decoding algorithms exist. However, while good classical LDPC codes can easily be obtained by random generation, there is no manner to generate randomly quantum LDPC codes. Hence, to date, the construction of a quantum LDPC codes rests on methods of algebraic topology [Kit03, BMD07, Zém09, Aud14] or combinatorics [TZ14, CDZ13, Del13]. The list of references is far from being exhaustive. Classical and quantum LDPC codes differ in yet another important point. While a generic sequence of classical LDPC codes has a minimum distance which is linear in the code length, the best known families of quantum LDPC codes have a minimum distance in  $O(\sqrt{n \log n})$  [FML02]. The question whether this square root barrier is fortuitous or not remains open. It is worth noting that by “LDPC” we mean that the code has parity check matrices with row weight in  $O(1)$  or  $O(\log n)$  where  $n$  denotes the code length. Indeed, a recent result of Bravyi and Hastings [BH14] proves the existence of MDPC (Moderate Density Parity Check) quantum codes, that is CSS codes described by matrices whose row weight is in  $O(\sqrt{n})$ , with dimensions and minimum distances linear in the code length. Bravyi and

Hastings' construction is performed by choosing two CSS codes of the same length and by computing their so-called *homological product*, that we shall denote here by  $\boxtimes$ . It produces a CSS code whose minimum distance is linear in its length with a nonzero probability.

In the present paper, we deepen the interplay between chain complexes and CSS codes by transposing to the latter the standard notion of tensor product  $\otimes$  defined for the former. We also introduce a reduced notion  $\otimes_r$  of tensor product which, compared with the standard one, improves the relative parameters since it decreases the length but preserves the dimension and the minimum distance. Though distinct, Bravyi and Hastings' homological product and (reduced) tensor products are closely related. Relationship between them are discussed in Section 3.4.

We study families of codes obtained by iterated tensor powers of a CSS code. This operation does not improve the relative parameters but can reasonably preserve them while providing codes with sparser parity check matrices. Actually, for  $C_1$  and  $C_2$  two CSS codes, the length and the dimension of the product  $C_1 \otimes C_2$  enjoy closed formulae more or less close to the rough product of the corresponding parameters of  $C_1$  and  $C_2$ . The minimum distance  $d_{C_1 \otimes C_2}$  of the product is more difficult to evaluate. Our main result is a criterion that provides a lower bound, using as large as possible sets of (co)homologically non trivial elements with as small as possible *overlaps* (see Definition 2.6):

**Theorem 2.8 and Theorem 3.10.** *Let  $C$  be a CSS code defined as a pair of classical codes  $C_2 \subseteq C_1^\perp$  given by full rank parity-check matrices. Let  $g_1, \dots, g_k \in C_1^\perp$  and  $g_1^*, \dots, g_k^* \in C_2^\perp$  be such that*

$$C_1^\perp = C_2 \oplus \text{Span}(g_1, \dots, g_k), \quad C_2^\perp = C_1 \oplus \text{Span}(g_1^*, \dots, g_k^*) \quad \text{and} \quad \forall i, j, \langle g_i^*, g_j \rangle = \delta_{ij}.$$

*If, for any  $j_0 \in \{1, \dots, k\}$ , there exists  $\Omega_{j_0} \subseteq g_{j_0}^* + C_1$  and  $\Omega'_{j_0} \subseteq g_{j_0} + C_2$ , with  $|\Omega_{j_0}|, |\Omega'_{j_0}| \geq N$  and  $\text{overlap}(\Omega_{j_0}), \text{overlap}(\Omega'_{j_0}) \leq K$ . Then, for all CSS code  $D$ :*

$$d_{C \otimes D}, d_{C \boxtimes D} \geq \left\lceil \frac{N}{K} d_D \right\rceil.$$

As a simple application of our criterion, we obtain then

**Corollary 2.18 and Corollary 3.11.** *If  $C$  and  $D$  are two CSS codes described by matrices which have no columns of zeros, then*

$$2 \max(d_C, d_D) \leq d_{C \otimes D}, d_{C \boxtimes D}.$$

This lower bound is twice better than the previously known lower bound [BH14, Lemma 2]. It follows that the iterated tensor powers of any CSS code described by matrices with no zero column is an LDPC family whose minimum distances have a non trivial growth tending to infinity:

**Corollary 2.23.** *If  $C = (\mathbf{H}_X, \mathbf{H}_Z)$  is any CSS code such that none of  $\mathbf{H}_X$  or  $\mathbf{H}_Z$  has a zero column, then the family  $(C^{\otimes \ell})_{\ell \in \mathbb{N}}$  is logarithmically LDPC with  $d_{C^{\otimes \ell}} \geq 2^\ell$  for every  $\ell \in \mathbb{N}^*$ .*

In particular, the minimum distance grows exponentially fast compared to the row weight of the parity check matrices. So, even if a CSS code has no *quantum degeneracy*<sup>1</sup>, for a large enough  $\ell$ , its  $\ell$ -th iterated power does.

Our criterion for estimating the minimum distance turns out to be quite efficient when applied with construction involving classical codes with a large group of automorphism. We give three such examples:

- binary codes from finite geometry, on which acts  $\mathbf{PGL}(3, \mathbb{F}_q)$ , lead to a CSS code QFG( $s$ ) for each  $s \in \mathbb{N}^*$ .
- binary cyclic codes of length  $n$  on which acts  $\mathbb{Z}/n\mathbb{Z}$ , lead to a CSS code QCC( $4^s, 2^s$ ) for each  $s \in \mathbb{N}^*$ .
- binary Reed–Muller codes RM( $r, s$ ) on which acts the affine group  $\mathbf{Aff}(r, \mathbb{F}_2)$ , lead to a CSS code QRM( $s$ ) for each  $s \in \mathbb{N}^*$ .

For these three examples, the sequence of iterated  $\ell$ -th tensor powers have length  $N_\ell$  tending to infinity and minimum distance which can be larger than  $N_\ell^\alpha$  for any  $\alpha < \frac{1}{2}$ . Moreover, these codes are *logarithmically LDPC*, i.e. they have parity check matrices with row weight in  $O(\log N_\ell)$  and the number of stabilizers acting nontrivially on a qubit (i.e. the column weight) is in  $O(\log N_\ell)$  too. The first two examples provide

<sup>1</sup>i.e. its quantum minimum distance is not larger than the minimum of the distances of the two classical codes defining it.

sequences of CSS codes with constant dimension, while the third one has a dimension sequence tending to infinity. Moreover, by diagonal extraction, the latter leads to a family which is almost LDPC, in the sense that the weights grow slower than  $N_\ell^\varepsilon$  for any  $\varepsilon > 0$ , and whose dimensions and minimum distances are respectively larger than  $N_\ell^{\frac{\alpha}{2}}$  and  $N_\ell^\alpha$  for any  $\alpha < 1$ .

*Remark.* One can note that, for all the LDPC families provided in this paper, the lower bound on the minimum distance culminates at, but does not exceed, the “square root of the length” barrier. Unfortunately, this is no coincidence, since a simple remark (Remark 2.13) shows that if the above criterion is sharp for a given code  $C$ , then the minimum distance of  $C$  is at most the square root of the length. Without saying anything on the square root barrier conjecture in general (even for iterated tensor powers of codes), the examples given above are hence somehow optimal as corollaries of Theorem 2.8.

**Organization.** Section 1 contains a brief review of the needed definitions from homological algebra (Section 1.1), classical codes (Section 1.2) and CSS codes (Section 1.3). In particular, we recall there the deep connection between CSS codes and chain complexes.

In Section 2, we use the latter connection to transport the notion of tensor product for chain complexes to CSS codes (Section 2.1). We provide then the main theorem, which gives a lower bound for the minimum distance of the product of two CSS codes (Section 2.2), and state a number of direct consequences on rougher, but general, lower bounds and on parameters of iterated tensorial powers (Section 2.3).

As examples of applications, we provide in Section 3 some elementary interpretations, in term of tensor products, of known results such as the hypergraph product codes given by J.-P. Tillich and G. Zemor in [TZ14] (Section 3.1) or Khovanov codes given by the first author in [Aud14] (Section 3.2). We also discuss the product of Steane codes, already discussed by S. Bravyi and M. Hastings in [BH13, BH14] (Section 3.3) and we relate tensor products to their homological products (Section 3.4). Note that the relationship of hypergraph and homological products with tensor products was already notified in [FH14].

Finally, section 4 is devoted to the description of three new families of LDPC CSS codes, based on finite geometry (Section 4.1), cyclic codes (Section 4.2) and Reed–Muller codes (Section 4.3).

The paper ends with two technical appendices with the details of the computation of lengths for iterated tensor powers (Appendix A) and iterated reduced tensor powers (Appendix B).

**Notation.** We shall consider  $\mathbb{F}_2$ -spaces, which are finite-dimensional vector spaces over the field  $\mathbb{F}_2$ . All the theoretical material present in this paper can actually be adapted to work for any field but, in order to simplify notation, and since it is sufficient for all the applications we consider here, we restrict this presentation to the  $\mathbb{F}_2$  case.

For any  $\mathbb{F}_2$ -space  $C$ , we denote by  $C^* := \text{Hom}(C, \mathbb{F}_2)$  the dual space of  $C$ . Every map  $f : A \rightarrow B$  induces a dual map  $f^* : B^* \rightarrow A^*$  defined by  $f^*(\varphi) = \varphi \circ f$  for every  $\varphi \in B^*$ . For every  $X \subseteq C$ , we denote its *orthogonal space* by  $X^\perp := \{\varphi \in C^* \mid \varphi|_X \equiv 0\}$ .

If  $C$  is given with a basis  $\mathcal{B}$ , then the bijection  $(A \subset \mathcal{B} \mapsto \sum_{b \in A} b \in C)$  identifies the elements of  $C$  with the subsets of  $\mathcal{B}$ . We shall use freely this identification, denoting subsets  $\{a_1, \dots, a_s\} \subset \mathcal{B}$ , and hence the related elements of  $C$ , by concatenations  $a_1 a_2 \dots a_s$ .<sup>2</sup> Associated to  $\mathcal{B}$ , there is a natural dual basis  $\mathcal{B}^* := \{b^* \mid b \in \mathcal{B}\}$  for  $C^*$ , where  $b^*$  is defined by  $b^*(b') = \delta_{bb'}$  for all  $b' \in \mathcal{B}$ . Here,  $\delta$  stands for the Kronecker delta. Using the subset identification mentioned above, we shall denote by  $b \in x$ , where  $x \in C$  and  $b \in \mathcal{B}$ , the fact that  $b^*(x) \neq 0$ , which means that  $b$  appears in the decomposition of  $x$ ; and by  $|x|$  the Hamming weight of  $x \in C$ , that is the number of  $b \in \mathcal{B}$  such that  $b \in x$ . We shall also denote with brackets the usual scalar product defined on  $C$  by  $\langle b_1, b_2 \rangle := \delta_{b_1 b_2}$  for all  $b_1, b_2 \in \mathcal{B}$ . The following map:

$$\begin{aligned} C &\longrightarrow C^* \\ x &\longmapsto y \mapsto \langle x, y \rangle \end{aligned}$$

is then an isomorphism sending  $\mathcal{B}$  on  $\mathcal{B}^*$ . For every  $X \subset C$ , it induces an isomorphism between  $X^\perp$  and  $\{x \in E \mid \forall y \in C, \langle x, y \rangle = 0\}$ . In order to reduce the amount of notation, we shall use freely this identification without necessarily mentioning it. The dual of a map  $f : A \rightarrow B$  would hence be seen as  $f^* : B \rightarrow A$ .

<sup>2</sup>note that the order of the  $a_i$ 's in this notation is irrelevant

By convention and unless otherwise specified,  $\mathbb{F}_2$ -spaces shall be denoted using roman capital letters, with an index  $i$  when it corresponds to the degree  $i$  part of a graded<sup>3</sup> space; chain complexes<sup>3</sup> using cursive capital letters; maps of chain complexes by  $\partial$ , possibly with a distinctive index or exponent; quantum codes<sup>3</sup> using calligraphic capital letters; and classical codes<sup>3</sup> using a slightly modified type of calligraphic capital letters. A same letter shall be used for associated objects: typically  $C$  shall be the CSS code<sup>3</sup> associated to the chain complex  $\mathcal{C}$  defined as the 2-nilpotent<sup>3</sup> map  $\partial$  (or  $\partial_{\mathcal{C}}$ ) defined on  $C := \bigoplus_{i \in \mathbb{Z}} C_i$ . The map  $\partial_i$  shall be then the restricted map  $\partial|_{C_i}$ . If a classical code is involved in the story, then it should be  $\mathcal{C}$ .

**Acknowledgements.** The present paper is the fruit of several meetings organized by the TOCQ team and funded by the CNRS (for the PEPS ICQ 2013 grant TOCQ), the Labex Archimède and the INRIA. The authors are hence willing to thank the other members of the team for all the discussions and all their comments on the paper: Anthony Leverrier, Jean-Pierre Tillich and Gilles Zémor and especially Nicolas Delfosse for his relevant observations. Finally, the authors express their gratitude to Matthew Hastings for for his feedback on a first version of the present article and for interesting discussions.

## 1. SOME BACKGROUND

### 1.1. Chain complexes.

1.1.1. *Definitions.* For the sake of self-containedness, we begin by a review of standard notions of homological algebra (see e.g. [Wei94] for further details).

In the litterature, chain complexes are often defined as a sequence of  $\mathbb{F}_2$ -spaces  $(C_i)_{i \in \mathbb{Z}}$  which are all zero but a finite number of them, together with a collection of maps either all of the form  $\partial_i : C_i \rightarrow C_{i+1}$  or all of the form  $\partial_i : C_i \rightarrow C_{i-1}$ . Another way to describe them is to consider the direct sum  $C := \bigoplus_{i \in \mathbb{Z}} C_i$  and regard the collection of maps  $(\partial_i)_{i \in \mathbb{Z}}$  as a graded endomorphism of  $C$ . In the present paper, we shall adopt the latter approach.

**Definition 1.1.** A linear map  $\partial \in \text{End}(C)$ , for some  $\mathbb{F}_2$ -space  $C$ , is **2-nilpotent** if it satisfies  $\partial^2 = 0$ . An  $\varepsilon$ -chain complex  $\mathcal{C}$ , for  $\varepsilon = \pm 1$ , is a 2-nilpotent map  $\partial \in \text{End}(C)$  such that

- $C$  is  $\mathbb{Z}$ -graded, that is decomposes into  $C := \bigoplus_{i \in \mathbb{Z}} C_i$ ;
- $\partial$  **increases the degree by exactly  $\varepsilon$** , that is  $\text{Im}(\partial|_{C_i}) \subset C_{i+\varepsilon}$  for every  $i \in \mathbb{Z}$ .

If omitted and unless otherwise specified,  $\varepsilon$  shall be assumed to be equal to 1.

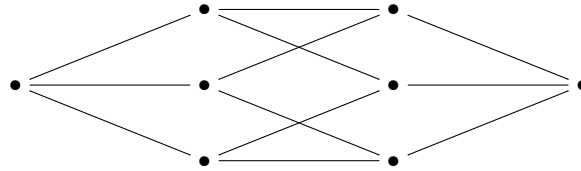
Since  $C$  is **finite-dimensional**, there is only a finite number of degrees  $i$  such that  $C_i \neq \{0\}$ . The **support** of a chain complex is the smallest interval  $[a, b]$  of integers such that  $C_i = \{0\}$  for  $i < a$  or  $i > b$  and the value  $b - a + 1$  is called the **length of the chain complex**.

A **basis  $\mathcal{B}$**  for  $\mathcal{C}$  is the data of a basis for each non zero space  $C_i$ , that is an identification of  $C_i$  with a power of  $\mathbb{F}_2$ .

**Notation 1.2.** Chain complexes shall be represented as

$$\cdots \xrightarrow{\partial_{i-2}} C_{i-1} \xrightarrow{\partial_{i-1}} C_i \xrightarrow{\partial_i} C_{i+1} \xrightarrow{\partial_{i+1}} C_{i+2} \xrightarrow{\partial_{i+2}} \cdots$$

In explicit cases given with a basis,  $C_i$  shall be represented by dots, one for each generator, and  $\partial_i$  shall be represented by edges joining a generator  $x$  to the elements of  $\partial_i(x)$ . For instance, the following picture:



represents the complex  $\text{Span}(w_1) \xrightarrow{\partial_0} \text{Span}(x_1, x_2, x_3) \xrightarrow{\partial_1} \text{Span}(y_1, y_2, y_3) \xrightarrow{\partial_2} \text{Span}(z_1)$ , where **Span** denotes the vector space spanned by the given generators and where  $\partial_0(w_1) = x_1 + x_2 + x_3$ ,  $\partial_1(x_1, x_2, x_3) = (y_1 + y_2, y_1 + y_3, y_2 + y_3)$  and  $\partial_2(y_1) = \partial_2(y_2) = \partial_2(y_3) = z_1$ .

<sup>3</sup>see next section for definitions

**Definition 1.3.** For any  $\varepsilon$ -chain complex  $\mathcal{C}$ , we define its **dual  $\mathcal{C}^*$**  as the  $(-\varepsilon)$ -chain complex  $\partial^* \in \text{End}(\mathcal{C}^*)$  defined by  $C^* := \bigoplus_{i \in \mathbb{Z}} \text{Hom}(C_i, \mathbb{F}_2)$  and  $\partial^*(\varphi) = \varphi \circ \partial$  for every  $\varphi \in C^*$ .

We say that a chain complex is **symmetric** if it is isomorphic, as a chain complex, to its dual.

**Proposition 1.4.** If  $\mathcal{B}$  is a basis for a  $\varepsilon$ -chain complex  $\mathcal{C}$ , then  $\text{Mat}_{\mathcal{B}^*}(\partial^*) = {}^t \text{Mat}_{\mathcal{B}}(\partial)$ .

*Remark 1.5.* If  $\mathcal{C}$  is given with a basis, then the maps  $\partial_i$  can be given by their matrices. Using the identification between an  $\mathbb{F}_2$ -space and its dual mentioned in the Notation section,  $\mathcal{C}^*$  can be seen as the chain complex obtained by reversing all the arrows and transposing all the matrices.

Furthermore, over  $\mathbb{F}_2$ , Proposition 1.4 is proven by noting that, for every generators  $x$  and  $y$ ,

$$y \in \partial(x) \iff y^*(\partial(x)) \neq 0 \iff \partial^*(y^*)(x) \neq 0 \iff x^* \in \partial^*(y^*).$$

So, if  $\mathcal{C}$  is given using Notation 1.2, then  $\mathcal{C}^*$  is obtained by reading the graph from right to left.

**Definition 1.6.** For  $\mathcal{C}$  any  $\varepsilon$ -chain complex, we define its **homology groups**  $H_\bullet(\mathcal{C}) := \bigoplus_{i \in \mathbb{Z}} H_i(\mathcal{C})$  by  $H_i(\mathcal{C}) := \text{Ker}(\partial_i) / \text{Im}(\partial_{i-\varepsilon})$ . For any  $x \in \text{Ker}(\partial)$ , we denote by  $[x]$  its image in  $H_\bullet(\mathcal{C})$ .

**Definition 1.7.** If  $\mathcal{C}$  is a  $\varepsilon$ -chain complex given with a basis  $\mathcal{B}$ , then, for each  $i \in \mathbb{Z}$  we denote by

- $n_i(\mathcal{C}) := \dim(C_i)$  and define the *length* of  $\mathcal{C}$  as  $n_{\mathcal{C}} := n_0(\mathcal{C})$ ;
- $k_i(\mathcal{C}) := \dim(H_i(\mathcal{C}))$  and define the *dimension* of  $\mathcal{C}$  as  $k_{\mathcal{C}} := k_0(\mathcal{C})$ ;
- $d_i(\mathcal{C}) := \min \{|x| \mid [x] \in H_i(\mathcal{C}) \setminus \{0\}\}$  and define the *minimum distance* of  $\mathcal{C}$  as  $d_{\mathcal{C}} := d_0(\mathcal{C})$ ;
- $w_i(\mathcal{C}) := \max \{|x| \mid x \text{ row of } \text{Mat}_{\mathcal{B}}(\partial_i)\}$  and define the *weight* of  $\mathcal{C}$  as  $w_{\mathcal{C}} := w_0(\mathcal{C})$ .

*Remark 1.8.* The above parameters have only a relative dependency with regard to the basis. Indeed,  $w_{\mathcal{C}}$  depends on the entire  $\mathcal{B}$ ,  $d_{\mathcal{C}}$  depends only on its restriction  $\mathcal{B}_{|C_0}$ ,  $n_{\mathcal{C}}$  and  $k_{\mathcal{C}}$  are independent of  $\mathcal{B}$ .

#### 1.1.2. Operations on chain complexes.

**Definition 1.9.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two  $\varepsilon$ -chain complexes. We define their **direct sum**  $\mathcal{C} \oplus \mathcal{D}$  as the  $\varepsilon$ -chain complex  $\partial_{\mathcal{C} \oplus \mathcal{D}} \in \text{End}\left(\bigoplus_{i \in \mathbb{Z}} (C_i \oplus D_i)\right)$ .

**Proposition 1.10.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two  $\varepsilon$ -chain complexes given with basis. Then

- $(\mathcal{C} \oplus \mathcal{D})^* \cong \mathcal{C}^* \oplus \mathcal{D}^*$ ;

and for each  $i \in \mathbb{Z}$ ,

- $H_i(\mathcal{C} \oplus \mathcal{D}) \cong H_i(\mathcal{C}) \oplus H_i(\mathcal{D})$ ;
- $n_i(\mathcal{C} \oplus \mathcal{D}) = n_i(\mathcal{C}) + n_i(\mathcal{D})$ ,  $k_i(\mathcal{C} \oplus \mathcal{D}) = k_i(\mathcal{C}) + k_i(\mathcal{D})$ ,  $d_i(\mathcal{C} \oplus \mathcal{D}) = \min(d_i(\mathcal{C}), d_i(\mathcal{D}))$  and
- $w_i(\mathcal{C} \oplus \mathcal{D}) = \max(w_i(\mathcal{C}), w_i(\mathcal{D}))$ .

*Proof.* All the statements, except the one on minimum distances and the one on weights, are classical results of homological algebra.

For the statement on minimum distances, let  $x \in C_i$  and  $y \in D_i$  be homologically non trivial elements of minimum weight. Then  $(x, 0)$  and  $(0, y) \in C_i \oplus D_i$  are homologically non trivial, so  $d_i(\mathcal{C} \oplus \mathcal{D}) \leq \min(d_i(\mathcal{C}), d_i(\mathcal{D}))$ . Conversely, every homologically non trivial element  $(a, b) \in C_i \oplus D_i$  is such that either  $a$  or  $b$  is homologically non trivial and its weight is hence larger than either the weight of  $(x, 0)$  or of  $(0, y)$ .

For the statement on weights, consider the maps  $\partial_{\mathcal{C},i} : C_i \rightarrow C_{i+1}$  and  $\partial_{\mathcal{D},i} : D_i \rightarrow D_{i+1}$ . Let  $M_{\mathcal{C},i}$  and  $M_{\mathcal{D},i}$  be their matrix representations. Then, the map  $\partial_{\mathcal{C} \oplus \mathcal{D},i} : C_i \oplus D_i \rightarrow C_{i+1} \oplus D_{i+1}$  is represented by the matrix:

$$\begin{pmatrix} M_{\mathcal{C},i} & 0 \\ 0 & M_{\mathcal{D},i} \end{pmatrix}$$

which yields the result on the maximum weight of the rows.  $\square$

In particular, we emphasize the fact that adding a direct summand given with its own basis and which has null homology does not affect the parameters except the length which is increased consequently. But conversely, detecting and removing a direct summand may alter the minimum distance if the basis does not respect the direct sum decomposition.

**Definition 1.11.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two  $\varepsilon$ -chain complexes. We define the **tensor product**  $\mathcal{C} \otimes \mathcal{D}$  as the  $\varepsilon$ -chain complex  $\text{Id}_{\mathcal{C}} \otimes \partial_{\mathcal{D}} + \partial_{\mathcal{C}} \otimes \text{Id}_{\mathcal{D}} \in \text{End}\left(\bigoplus_{i \in \mathbb{Z}} \left(\bigoplus_{r \in \mathbb{Z}} (C_r \otimes D_{i-r})\right)\right)$ .

**Proposition 1.12.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two  $\varepsilon$ -chain complexes. Then

- $(\mathcal{C} \otimes \mathcal{D})^* \cong \mathcal{C}^* \otimes \mathcal{D}^*$ ;

and for each  $i \in \mathbb{Z}$ ,

- $H_i(\mathcal{C} \otimes \mathcal{D}) \cong \bigoplus_{r \in \mathbb{Z}} (H_r(\mathcal{C}) \otimes H_{i-r}(\mathcal{D}))$  (Künneth formula);
- $n_i(\mathcal{C} \otimes \mathcal{D}) = \sum_{r \in \mathbb{Z}} n_r(\mathcal{C}) \cdot n_{i-r}(\mathcal{D})$ ,  $k_i(\mathcal{C} \otimes \mathcal{D}) = \sum_{r \in \mathbb{Z}} k_r(\mathcal{C}) \cdot k_{i-r}(\mathcal{D})$ ;
- if  $\mathcal{C}$  and  $\mathcal{D}$  were given with bases  $\mathcal{B}_{\mathcal{C}}$  and  $\mathcal{B}_{\mathcal{D}}$ , then  $\mathcal{B}_{\mathcal{C}} \otimes \mathcal{B}_{\mathcal{D}}$  provides a basis for  $\mathcal{C} \otimes \mathcal{D}$  such that  $w_i(\mathcal{C} \otimes \mathcal{D}) = \max \{w_j(\mathcal{C}) + w_k(\mathcal{D}) \mid j + k = i\}$ .

*Proof.* The statements on duals, homologies and lengths are classical results of homological algebra. The statement on weights follows from  $\text{Mat}_{\mathcal{B}_{\mathcal{C}} \otimes \mathcal{B}_{\mathcal{D}}}(\partial_{\mathcal{C} \otimes \mathcal{D}})$  being obtained as a sum of Kronecker products of the form  $\text{Mat}_{\mathcal{B}_{\mathcal{C}}}(\partial_{\mathcal{C}}) \otimes \text{Id}$  and  $\text{Id} \otimes \text{Mat}_{\mathcal{B}_{\mathcal{D}}}(\partial_{\mathcal{D}})$ .  $\square$

The isomorphism of Künneth formula, proven for instance in [Wei94, Thm 3.6.3]<sup>4</sup>, is actually induced from maps defined at the chain complex level. It induces hence the following proposition which shall be needed further in the proof of Lemma 2.7.

**Proposition 1.13.** If, for each  $r \in \mathbb{Z}$ ,  $x_1^r, \dots, x_{j_r}^r \in C_r$  and  $y_1^r, \dots, y_{j_r}^r \in D_r$  induce basis for, respectively,  $H_r(\mathcal{C})$  and  $H_r(\mathcal{D})$ , then, for every  $i \in \mathbb{Z}$ , the elements of the form  $x_j^r \otimes y_{j-r}^{i-r}$  induce a basis for  $H_i(\mathcal{C} \otimes \mathcal{D})$ .

*Remark 1.14.* Evaluating  $d_i(\mathcal{C} \otimes \mathcal{D})$  is less straightforward and it shall be the aim of Section 2.2.

1.1.3. *Short complexes and reduction.* In this paper, we shall be mostly interested in length 3 portions of chain complex centered around degree zero. This motivates the following definitions.

**Definition 1.15.** A chain complex  $\mathcal{C}$  is said to be a **short complex** if it has a support contained in  $\{-1, 0, 1\}$ .

It is hence of the following form:  $C_{-1} \xrightarrow{\partial_{-1}} C_0 \xrightarrow{\partial_0} C_1$ . The chain complex is said to be *balanced* if it has non trivial homology only in degree zero. A balanced short complex is said to be *reduced*. For a short complex, being reduced is equivalent to require  $\partial_{-1}$  to be injective and  $\partial_0$  to be surjective. So the chain complex is of the form  $C_{-1} \xrightarrow{\hookrightarrow} C_0 \xrightarrow{\twoheadrightarrow} C_1$ . Equivalently, it consists in requiring that  $\dim(H_0(\mathcal{C})) = \dim(C_0) - \dim(C_{-1}) - \dim(C_1)$ .

Note that a short complex  $C_{-1} \xrightarrow{\partial_{-1}} C_0 \xrightarrow{\partial_0} C_1$  is symmetric if and only if  $C_{-1} \simeq C_1^*$  and  $\partial_0 = \partial_{-1}^*$ .

Any chain complex  $\mathcal{C}$  can be turned into a short one by truncating the degrees higher than 1 and lower than  $-1$ . More precisely, by shifting beforehand the degree, one can extract any length 3 portion of  $\mathcal{C}$ . However, the result is generally not balanced, even if  $\mathcal{C}$  was. There is nonetheless a reduction process to turn a short complex into a reduced one (almost) without altering its parameters. Indeed, if

$\mathcal{C} := C_{-1} \xrightarrow{\partial_{-1}} C_0 \xrightarrow{\partial_0} C_1$  is given with a basis  $\mathcal{B}$ , and if  $\mathcal{D}$  is obtained from  $\mathcal{C}$  by removing all redundant rows of  $\text{Mat}_{\mathcal{B}}(\partial_{-1})$  and/or  $\text{Mat}_{\mathcal{B}}(\partial_0)$  and by modifying  $C_{-1}$  and  $C_1$  consequently, then  $\mathcal{D}$  is reduced and it is mostly a consequence of Remark 1.8 that  $n_{\mathcal{D}} = n_{\mathcal{C}}$ ,  $k_{\mathcal{D}} = k_{\mathcal{C}}$ ,  $d_{\mathcal{D}} = d_{\mathcal{C}}$  and  $w_{\mathcal{D}} \leq w_{\mathcal{C}}$ . From a linear algebraic point of view, it consists in replacing  $C_{-1}$  by a complement space for  $\text{Ker}(\partial_{-1})$  spanned by vectors of  $\mathcal{B}$ , and replacing  $C_1$  by its quotient under a complement space of  $\text{Im}(\partial_0)$  spanned by vectors of  $\mathcal{B}$ . This process is however non canonical since it requires the choice of complement spaces, or equivalently, the choice of the redundant rows to be removed.

As a conclusion, any length 3 portion of a chain complex can be grading-shifted so it is centred in degree zero, and then the above (non canonical) reduction process can turn it into a reduced complex without altering the parameters, except the weight which may even be decreased.

*Remark 1.16.* Adding  $\text{Ker}(\partial_{-1})$  in degree  $-2$  and  $\text{Coker}(\partial_0)$  in degree  $2$  is another way to turn a short complex into a balanced one. The chain complex is then of length five. This provides a balancing process

<sup>4</sup>noting that, since we are working over a field, any module is flat and hence Tor is zero

which is canonical and which preserves all the parameters. However, for length reasons, we shall consider in this paper, only the (non canonical) reduction process, and not the (canonical) balancing one.

**1.2. Classical codes.** As they shall play a keyrole in several constructions, we set here a few notation on classical codes. A classical code  $C$  is a subspace of an  $\mathbb{F}_2$ -space  $E$  given with a basis  $\mathcal{B}_E$ . It can be described by either a generating map  $g_C : A \longrightarrow E$  such that  $\text{Im}(g_C) = C$  or a parity-check map  $p_C : E \longrightarrow B$  such that  $\text{Ker}(p_C) = C$ . For any such code, we define:

- its length  $n_C$  as the dimension of  $E$ ;
- its dimension  $k_C$  as the dimension of  $C$ ;
- its minimum distance  $d_C$  as the minimum weight for a non trivial element of  $C$ , using the basis  $\mathcal{B}_E$ ;
- its weight as the maximal weight of a row of  $\text{Mat}_{\mathcal{B}_E, \mathcal{B}_B}(p_C)$ , where  $\mathcal{B}_B$  is a given basis for  $B$ .

We define the *dual* of  $C$  as the code  $C^\perp$  defined by  $C^\perp \subset E^*$ , which can also be seen, using the identification mentionned in the Notation section, as  $\{x \in E \mid \forall y \in C, \langle x, y \rangle = 0\} \subset E$ . It is easily checked that  $p_C^*$  and  $g_C^*$  are, respectively, a generating map and a parity-check map for  $C^\perp$  so that, up to transpose,  $C$  and  $C^\perp$  exchange their generating and parity-check matrices; and that  $n_{C^\perp} = n_C$  and  $k_{C^\perp} = n_C - k_C$ .

**1.3. CSS codes.** The CSS codes were developed in [CS96, Ste96]. They are a special case of stabilizer quantum error correcting codes associated to pairs of orthogonal classical codes, that is codes  $C_1$  and  $C_2$  such that  $C_2 \subseteq C_1^\perp$ ; or equivalently to matrices  $\mathbf{H}_X$  and  $\mathbf{H}_Z$  such that  $\mathbf{H}_X^t \mathbf{H}_Z = 0$ . A quick review can be found in section 1.1 of [Aud14] but for a more comprehensive treatment, we refer the reader to [NC10, Pre, Del12]. A CSS code is said to be *symmetric* if  $C_1 = C_2$  or, equivalently, if  $\mathbf{H}_X = \mathbf{H}_Z$ .

For a CSS code  $C$ , some relevant parameters are

- $n_C$  the length of  $C$ , that is the common length of the codes  $C_1$  and  $C_2$ ;
- $k_C$  the dimension of  $C$ , that is the dimension of  $C_1^\perp / C_2$
- $d_C$  the minimum distance of  $C$ , that is the minimum weight of an element of  $(C_1^\perp \setminus C_2) \cup (C_2^\perp \setminus C_1)$ ;
- $w_C$  the weight of  $C$ , that is the highest weight realized by a row of  $\mathbf{H}_X$  or  $\mathbf{H}_Z$ .

They shall be gathered in the notation  $\llbracket n_C; k_C; d_C; w_C \rrbracket$ .

Chain complexes turn out to be efficient for constructing such CSS codes. Indeed, once equipped with a basis, they not only naturally provide matrices whose product is zero, but parameters can also be read from them, their duals and the associated homologies. The following classical statement reformulates the usual matrix-based description of CSS codes in terms of chain complexes, and this allows a more intrinsic description of these objects. Similar statements appear, for instance, in [Aud14, Prop. 1.7] or [Del12].

**Proposition 1.17.** *To a short complex  $\mathcal{C} := C_{-1} \xrightarrow{\partial_{-1}} C_0 \xrightarrow{\partial_0} C_1$  given with a basis  $\mathcal{B}$ , there is an associated CSS code  $C := (\text{Mat}_{\mathcal{B}}(\partial_0), {}^t\text{Mat}_{\mathcal{B}}(\partial_{-1}))$  with parameters  $n_C = n_{\mathcal{C}}$ ,  $k_C = k_{\mathcal{C}}$ ,  $d_C = \min(d_{\mathcal{C}}, d_{\mathcal{C}^*})$  and  $w_C = \max(w_{\mathcal{C}}, w_{\mathcal{C}^*})$ .*

*Remark 1.18.* If the chain complex is symmetric, then the CSS code is symmetric.

Compared to the definition of CSS codes given above, the codes  $C_1$  and  $C_2$  correspond to  $C_1 = \text{Ker}(\partial_0)^\perp$  and  $C_2 = \text{Im}(\partial_{-1})$ , or equivalently to  $C_1 = \text{Im}(\partial_0^*)$  and  $C_2 = \text{Ker}(\partial_{-1}^*)^\perp$ . Conversely, two matrices  $\mathbf{H}_X$  and  $\mathbf{H}_Z$  such that  $\mathbf{H}_X^t \mathbf{H}_Z = 0$  provide a chain complex

$$0 \longrightarrow \mathbb{F}_2^{k_1} \xrightarrow{{}^t\mathbf{H}_Z} \mathbb{F}_2^{n_C} \xrightarrow{\mathbf{H}_X} \mathbb{F}_2^{k_2} \longrightarrow 0$$

where  $k_1$  and  $k_2$  are, respectively, the numbers of rows in  $\mathbf{H}_Z$  and  $\mathbf{H}_X$ . **There is hence a one-to-one correspondence between CSS codes and, up to isomorphisms, short complexes given with a basis.** However, as a consequence of the discussion on reduction given in Section 1.1.3, removing redundant rows in  $\mathbf{H}_Z$  and  $\mathbf{H}_X$  does not affect the parameters, except the weight which may even be decreased. It is hence natural to focus on CSS codes associated to reduced complexes.

*Remark 1.19.* The data of a 2-nilpotent map  $\partial \in \text{End}(C)$  is actually sufficient to construct a CSS code as  $C_1 = \text{Ker}(\partial)^\perp$  and  $C_2 = \text{Im}(\partial)$ . This code is actually the code associated to the short complex  $C \xrightarrow{\partial} C \xrightarrow{\partial} C$ . Note that from every pair of classical codes  $C_1, C_2$  such that  $C_2 \subseteq C_1^\perp$ , one can always construct a 2-nilpotent

map  $\partial : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$  whose image is  $C_2$  and kernel is  $C_1^\perp$ . This means that every quantum code can be represented by a 2-nilpotent chain complex. This description of quantum CSS code from 2-nilpotent map is used in [BH14] and shall be discussed in Section 3.4.

## 2. TENSOR PRODUCTS OF CSS CODES

**2.1. Definitions.** As mentionned in the previous section, CSS codes are in one-to-one correspondence with short complexes. As such, they inherit the notions of direct sum and tensor product. The former is well defined since it sends short (respectively reduced) complexes to short (respectively reduced) complexes. The latter requires some more attention. Indeed, **the tensor product of two short complexes is, in general, not anymore short but of length 5**. One way to correct this shortcoming is to roughly truncate.

**Definition 2.1.** For  $(C_i)_{i \in I}$  a finite family of CSS codes, we define  $\otimes_{i \in I} C_i$  as the CSS code associated to the degrees  $\{-1, 0, 1\}$ -truncation of  $\otimes_{i \in I} \mathcal{C}_i$ , where, for each  $i \in I$ ,  $\mathcal{C}_i$  is the short complex associated to  $C_i$ . When the family is made of  $\ell$  copies of the same code  $C$ , we also denote it by  $C^{\otimes \ell}$ .

However, the truncation produces two major drawbacks:

- the operation is not associative in the sense that, in general,  $(\otimes_{i \in I} C_i) \otimes (\otimes_{j \in J} C_j) \neq (\otimes_{i \in I \sqcup J} C_i)$ ;
- the result is, in general, not reduced even if all the factors are.

To remedy these issues, one can use the reduction process described in Section 1.1.3.

**Definition 2.2.** For  $(C_i)_{i \in \{1, \dots, k\}}$  a finite family of CSS codes, we define recursively  $\otimes_{i \in \{1, \dots, k\}} C_i$  as the CSS code associated to the reduction of the degrees  $\{-1, 0, 1\}$ -truncation of  $\mathcal{C}_{1 \dots (k-1)} \otimes \mathcal{C}_k$ , where  $\mathcal{C}_{1 \dots (k-1)}$  is the reduced complex associated to  $\otimes_{i \in \{1, \dots, k-1\}} C_i$  and  $\mathcal{C}_k$  the short complex associated to  $C_k$ . When the family is made of  $\ell$  copies of the same code  $C$ , we also denote it by  $C^{\otimes_r \ell}$ .

*Warning 2.3.* The notation  $\otimes_r$  is an abuse of notation since it is not canonically defined and requires, at each step, the choice of redundant rows to be removed.

*Remark 2.4.* In order to get a canonical notion of somehow reduced tensor product, one can use the balancing process mentionned in Remark 1.16. It however ends with slightly longer codes.

**2.2. A minimum distance result.** Let  $\mathcal{C}$  be a chain complex given with bases  $(a_1, \dots, a_{n-1})$  for  $C_{-1}$ ,  $(b_1, \dots, b_{n_0})$  for  $C_0$  and  $(c_1, \dots, c_n)$  for  $\bigoplus_{i \in \mathbb{Z} \setminus \{-1, 0\}} C_i$ . We denote the matrix associated to  $\partial_{-1}$  by

$$\begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n_0} \\ \vdots & & \vdots \\ \lambda_{n_1 1} & \cdots & \lambda_{n_1 n_0} \end{pmatrix} = \begin{pmatrix} \Lambda_1, \dots, \Lambda_{n_0} \end{pmatrix}.$$

We fix  $g_1, \dots, g_r$  elements in  $\text{Ker}(\partial_0)$  which generate a basis of  $H_0(\mathcal{C})$ , and set  $g_j := \sum_{i=1}^{n_0} \gamma_i^j b_i$  for all  $j \in \{1, \dots, r\}$ . Then we complete  $g_1, \dots, g_r$  into a basis of  $C_0$  and, for all  $j_0 \in \{1, \dots, r\}$ , we define

$$\text{Ker}_{j_0}^\oplus := g_{j_0}^* + \text{Ker}(\partial_0)^\perp$$

which, roughly speaking, corresponds to the elements in  $C_0$  which are orthogonal to all generators of  $\text{Ker}(\partial_0)$  except  $g_{j_0}$ .<sup>5</sup>

*Remark 2.5.* Elements of  $\text{Ker}_{j_0}^\oplus$  are in  $\text{Im}(\partial_{-1})^\perp = \text{Ker}(\partial_{-1}^*)$ , but not in  $\text{Ker}(\partial_0)^\perp = \text{Im}(\partial_0^*)$ , they are hence cohomologically non trivial elements.

**Definition 2.6.** For every subset  $\Omega \subset C_0$ , we define  $\text{overlap}(\Omega) := \max_{i \in \{1, \dots, n_0\}} \left| \{p \in \Omega \mid b_i \in p\} \right|$ . If stacking, as rows of a matrix, the elements of  $\Omega$ , it corresponds to the maximum weight of a column.

<sup>5</sup>note that, since we are working over  $\mathbb{F}_2$ ,  $\text{Ker}_{j_0}^\oplus$  is an affine subspace, but over any other field, it should be the union of affine subspaces  $\bigcup_{\lambda \in \mathbb{K}^*} \lambda g_{j_0}^* + \text{Ker}(\partial_0)^\perp$



**Lemma 2.7.** *Let  $N, K \in \mathbb{N}^*$ . If, for any  $j_0 \in \{1, \dots, r\}$ , there exists  $\Omega_{j_0} \subset \text{Ker}_{j_0}^\oplus$  such that  $|\Omega_{j_0}| \geq N$  and  $\text{overlap}(\Omega_{j_0}) \leq K$ , then, for every chain complex  $\mathcal{D}$  such that either  $\mathcal{C}$  or  $\mathcal{D}$  is balanced,*

$$d_{\mathcal{C} \otimes \mathcal{D}} \geq \left\lceil \frac{N}{K} d_{\mathcal{D}} \right\rceil.$$

*Proof.* We consider  $x_0 = \sum_{i=1}^{n-1} a_i \otimes a_i + \sum_{i=1}^{n_0} b_i \otimes b_i + \sum_{i=1}^n c_i \otimes c_i$ , with  $a_i, b_i, c_i \in \mathcal{D}$ , a minimally weighted representative of a non trivial class in  $H_0(\mathcal{C} \otimes \mathcal{D})$ . As a consequence of **Künneth formula**, and since either  $\mathcal{C}$  or  $\mathcal{D}$  is balanced,  $H_0(\mathcal{C} \otimes \mathcal{D}) \cong H_0(\mathcal{C}) \otimes H_0(\mathcal{D})$  and there exist  $g_1, \dots, g_r \in \text{Ker}(\partial_{\mathcal{D}})$  and elements  $x_j, y_i, z_j \in \mathcal{D}$ , such that  $[g_{j_0}]$  is non trivial in  $H_0(\mathcal{D})$  for at least one given  $j_0 \in \{1, \dots, r\}$  and  $x_j, y_i, z_j \in \mathcal{D}$ , such that

$$\begin{aligned} x_0 &= \sum_{j=1}^r g_j \otimes g_j + \partial_{\mathcal{C} \otimes \mathcal{D}} \left( \sum_{i=1}^{n-1} a_i \otimes x_i + \sum_{i=1}^{n_0} b_i \otimes y_i + \sum_{i=1}^k c_i \otimes z_j \right) \\ &= \sum_{j=1}^r \sum_{i=1}^{n_0} \gamma_i^j b_i \otimes g_j + \sum_{j=1}^{n-1} \sum_{i=1}^{n_0} \lambda_{ij} b_i \otimes x_j + \sum_{i=1}^{n_0} b_i \otimes \partial_{\mathcal{D}}(y_i) + \text{non relevant terms}, \end{aligned}$$

where the non relevant terms are of the form  $a_i \otimes \cdot$  or  $c_i \otimes \cdot$  for some  $i$ .

For every  $i \in \{1, \dots, n_0\}$ , we project on terms of the form  $b_i \otimes \cdot$  and obtain then

$$b_i = \sum_{j=1}^r \gamma_i^j g_j + \sum_{j=1}^{n-1} \lambda_{ij} x_j + \partial_{\mathcal{D}}(y_i).$$

For all  $p = (p_1, \dots, p_{k_1}) \in \mathbb{F}_2^{n_0}$ , we have then

$$\sum_{i=1}^{n_0} p_i b_i = \sum_{j=1}^r \sum_{i=1}^{n_0} p_i \gamma_i^j g_j + \sum_{j=1}^{n-1} \sum_{i=1}^{n_0} p_i \lambda_{ij} x_j + \partial_{\mathcal{D}}(\text{something}),$$

which can be reformulated as

$$\sum_{i \in p} b_i = \sum_{j=1}^r \langle p, g_j \rangle g_j + \sum_{j=1}^{n_0} \langle p, \Lambda_j \rangle x_j + \partial_{\mathcal{D}}(\text{something}).$$

If  $p \in \text{Ker}_{j_0}^\oplus$ , we obtain  $\sum_{i \in p} b_i = g_{j_0} + \partial_{\mathcal{D}}(\text{something})$  which is non trivial in  $H_0(\mathcal{D})$ , and hence  $\sum_{i \in p} |b_i| \geq \left| \sum_{i \in p} b_i \right| \geq d_{\mathcal{D}}$ .

Now we consider  $\Omega_{j_0}$  as in the statement of the lemma. We obtain then

$$K d_{\mathcal{C} \otimes \mathcal{D}} = K |x_0| = K \left( \sum_{i=1}^{n-1} |a_i| + \sum_{i=1}^{n_0} |b_i| + \sum_{i=1}^n |c_i| \right) \geq K \sum_{i=1}^{n_0} |b_i| \geq \sum_{p \in \Omega} \sum_{i \in p} |b_i| \geq N d_{\mathcal{D}},$$

and since  $d_{\mathcal{C} \otimes \mathcal{D}}$  is an integer, it is bounded below by  $\left\lceil \frac{N}{K} d_{\mathcal{D}} \right\rceil$ .  $\square$

Using the correspondence set in Proposition 1.17, this leads to the following statement, formulated in terms of classical codes and matrices.

**Theorem 2.8.** *Let  $C$  be a CSS code defined as a pair of classical codes  $C_2 \subseteq C_1^\perp$  given by full rank parity-check matrices. Let  $g_1, \dots, g_k \in C_1^\perp$  and  $g_1^*, \dots, g_k^* \in C_2^\perp$  be such that*

$$C_1^\perp = C_2 \oplus \text{Span}(g_1, \dots, g_k), \quad C_2^\perp = C_1 \oplus \text{Span}(g_1^*, \dots, g_k^*) \quad \text{and} \quad \forall i, j, \langle g_i^*, g_j \rangle = \delta_{ij}.$$

*If, for any  $j_0 \in \{1, \dots, k\}$ , there exists  $\Omega_{j_0} \subseteq g_{j_0}^* + C_1$  and  $\Omega'_{j_0} \subseteq g_{j_0} + C_2$ , with  $|\Omega_{j_0}|, |\Omega'_{j_0}| \geq N$  and  $\text{overlap}(\Omega_{j_0}), \text{overlap}(\Omega'_{j_0}) \leq K$ . Then, for all CSS code  $\mathcal{D}$ , the minimum distance of  $C \otimes \mathcal{D}$  satisfies*

$$d_{C \otimes \mathcal{D}} \geq \left\lceil \frac{N}{K} d_{\mathcal{D}} \right\rceil.$$

Lemma 2.7 implies actually a stronger result since

- (1) the matrices  $\mathbf{H}_X$  and  $\mathbf{H}_Z$  may not have full rank as soon as the matrices of  $\mathcal{D}$  do;
- (2) one may use distinct bases to define the sets  $\Omega$ 's and  $\Omega'$ 's.

However, our applications shall use only the statement of Theorem 2.8, and mostly in its one dimensional case, which is even simpler to state.

**Corollary 2.9.** *For a CSS code  $C$  of dimension 1 associated to a pair of classical codes  $C_2 \subseteq C_1^\perp$  given by full rank parity-check matrices. Then, if there exists a subset  $\Omega \subseteq C_1^\perp \setminus C_2$  and a subset  $\Omega' \subseteq C_2^\perp \setminus C_1$  with  $|\Omega|, |\Omega'| \geq N$  and  $\text{overlap}(\Omega), \text{overlap}(\Omega') \leq K$ . Then for all CSS code  $\mathcal{D}$ , the minimum distance of  $C \otimes \mathcal{D}$  satisfies*

$$d_{C \otimes \mathcal{D}} \geq \left\lceil \frac{N}{K} d_{\mathcal{D}} \right\rceil.$$

*Remark 2.10.* If dealing with a symmetric code described by a symmetric chain complexe, then it is sufficient to consider the sets  $\Omega$  in Theorem 2.8 or Corollary 2.9, since they can be used again as sets  $\Omega'$ .

**2.3. Some direct consequences.** In this section, we state some direct consequences of Lemma 2.7.

**2.3.1. Rougher lower bounds for the minimal distance of tensor products.**

**Corollary 2.11.** *If  $\mathcal{C}$  is a chain complex satisfying the hypothesis of Lemma 2.7, then  $d_{\mathcal{C}} \geq \left\lceil \frac{N}{K} \right\rceil$ .*

*Proof.* Apply Lemma 2.7 with the reduced complex  $\mathcal{D} := 0 \longrightarrow \mathbb{F}_2 \longrightarrow 0$ .  $\square$

*Remark 2.12.* Applied to the case  $\text{Im}(\partial_{-1}) = 0$ , Corollary 2.11 provides also a lower bound for the minimum distance of a classical code. In this context, the statement essentially follows from the remark that, if an element  $c$  of a code has a non trivial scalar product with a vector  $\omega$ , then  $c$  has at least one non trivial entry shared with  $\omega$ . Now, with the notation of Section 2.2 and for  $j_0$  such that  $g_{j_0} \in c$ ,  $c$  has at least one non trivial entry in common with each of the  $N$  elements of  $\Omega_{j_0}$ . But each of these entries can appear at most  $K$  times. It follows that  $c$  has at least  $\frac{N}{K}$  non trivial entries.

*Remark 2.13.* One may cherish the hope to provide an LDPC family of CSS codes with minimum distances growing faster than the square root of the lengths by applying Corollary 2.11 to evaluate the minimum distance of a CSS code  $C$  as some  $\frac{N}{K}$ . This is however doomed to fail. Indeed, if stacking the elements of  $\Omega$  into a matrix and counting, column by column, the number  $V$  of non trivial entries, we obtain  $V \leq n_C K$ ; but on the other hand, counting  $V$  row by row, we obtain  $V \geq N d_{\mathcal{C}^*}$  since, as noticed in Remark 2.5, elements of  $\Omega$  are non trivial cohomology classes. It follows that  $\frac{N}{K} d_{\mathcal{C}^*} \leq n_C$ . In particular, if  $\frac{N}{K}$  is close enough to  $d_{\mathcal{C}}$ , we obtain  $d_{\mathcal{C}} d_{\mathcal{C}^*} \lesssim n_C$  and hence  $d_C = \min(d_{\mathcal{C}}, d_{\mathcal{C}^*}) \lesssim \sqrt{n_C}$ .

**Corollary 2.14.** *If  $\mathcal{C}$  and  $\mathcal{D}$  are two chain complexes such that one of them is balanced, then*

$$\max(d_{\mathcal{C}}, d_{\mathcal{D}}) \leq d_{\mathcal{C} \otimes \mathcal{D}} \leq d_{\mathcal{C}} d_{\mathcal{D}}.$$

*Proof.* The right hand side is a direct consequence of Proposition 1.13. The left hand one is obtained by applying Lemma 2.7 with  $N = K = 1$ ; this always holds when  $\Omega_{j_0}$  is a singleton.  $\square$

As we shall see in several examples, the upper bound is sharp. On the contrary, the lower bound is not, except in some trivial cases.

**Proposition 2.15.** *If  $\mathcal{C}$  and  $\mathcal{D}$  are two chain complexes such that  $\mathcal{C} : C^{-1} \xrightarrow{\partial_{-1}} C_0 \xrightarrow{\partial_0} C_1$  is balanced and  $\partial_0$  is non zero on every elements of the basis of  $C_0$ , then  $d_{\mathcal{C} \otimes \mathcal{D}} \geq 2d_{\mathcal{D}}$ .*

*Proof.* With the notation of Section 2.2, we consider, for each  $j_0 \in \{1, \dots, r\}$ ,  $\Omega_{j_0} := \text{Ker}_{j_0}^\oplus$ . It can actually be described as  $\omega_{j_0} + \text{Ker}(\partial_0)^\perp$  with  $\omega_{j_0}$  any element in  $\text{Ker}_{j_0}^\oplus$ . Now, for any  $i \in \{1, \dots, n_0\}$ , there is at least one element  $f_i \in \text{Ker}(\partial_0)^\perp$  such that  $\langle f_i, b_i \rangle = 1$ , otherwise  $b_i$  would be contained in  $(\text{Ker}(\partial_0)^\perp)^\perp \cong \text{Ker}(\partial_0)$  and  $\partial_0(b_i)$  would be zero. But, as a generator,  $b_i \in f_i$ , and the map  $(x \mapsto x + f_i)$  induces a bijection between the elements in  $\Omega_{j_0}$  which contain  $b_i$  and those who do not. It follows that  $\text{overlap}(\Omega_{j_0}) = \frac{1}{2} |\Omega_{j_0}|$  and the statement is proved using Lemma 2.7.  $\square$

*Remark 2.16.* Similar results can be obtained for  $q$ -ary CSS codes. In this case we would get  $d_{\mathcal{C} \otimes \mathcal{D}} \geq \frac{q}{q-1} \max(d_{\mathcal{C}}, d_{\mathcal{D}})$ .

*Remark 2.17.* The lower bound in Corollary 2.14 can hence be sharp only if  $\partial_0$  vanishes on some generator. If this generator is not in the image of  $\partial_{-1}$ , then  $d_{\mathcal{C}} = 1$  and  $d_{\mathcal{C}}d_{\mathcal{D}} = \max(d_{\mathcal{C}}, d_{\mathcal{D}})$ . If it does, then  $\mathcal{C}$  is the direct sum of a chain complex with a (useless) summand of the form  $\text{Span}(b'_i) \xrightarrow{\quad} \text{Span}(b_i) \longrightarrow 0$ , where  $b'_i$  is the unique preimage of  $b_i$ ; and this summand can be removed without altering the minimum distance. It follows that the lower bound of Corollary 2.14 is sharp if and only if it is equal to the upper bound.

Corollary 2.14 and Proposition 2.15 have the following consequence for the (reduced) tensor product of CSS codes.

**Corollary 2.18.** *If  $\mathcal{C}$  and  $\mathcal{D}$  are two CSS codes described by matrices which have no columns of zeros, then*

$$2 \max(d_{\mathcal{C}}, d_{\mathcal{D}}) \leq d_{\mathcal{C} \otimes \mathcal{D}}.$$

In Section 3.3 we shall see an example where  $d_{\mathcal{C} \otimes \mathcal{D}} = (2 + \frac{1}{3}) \max(d_{\mathcal{C}}, d_{\mathcal{D}}) < d_{\mathcal{C}}d_{\mathcal{D}}$ .

*Remark 2.19.* For chain complexes, the minimum distance of a tensor product is bounded above by the product of the minimum distances of the summands but, because of the interplay with dual chain complexes, this does not hold anymore for CSS codes. An example is given by Tillich–Zémor construction, presented in Section 3.1.

### 2.3.2. Parameters of iterated tensor powers.

**Corollary 2.20.** *Let  $\mathcal{C}$  be the CSS code associated to  $\mathcal{C} := \mathbb{F}_2^{b_1} \xrightarrow{\quad} \mathbb{F}_2^a \xrightarrow{\quad} \mathbb{F}_2^{b_2}$ , where  $a, b_1, b_2 \in \mathbb{N}$ . If  $\mathcal{C}$  and  $\mathcal{C}^*$  satisfy both the hypothesis of Lemma 2.7 for the same integers  $N, K \in \mathbb{N}^*$ , then  $(\mathcal{C}^{\otimes \ell})_{\ell \in \mathbb{N}}$  is a family of CSS codes with parameters*

$$\left\| \sim \frac{1}{2} \sqrt{\frac{a+2\sqrt{b_1b_2}}{\pi \ell \sqrt{b_1b_2}}} \left( a+2\sqrt{b_1b_2} \right)^\ell ; (a-b_1-b_2)^\ell ; \geq \left( \frac{N}{K} \right)^\ell ; \leq a\ell \right\|.$$

*In particular, the family is logarithmically LDPC and the minimum distance grows strictly faster than the  $\frac{\log(a+2\sqrt{b_1b_2})}{\log N - \log K}$ -th power of the length.*

*Proof.* The statement on

- the length follows from an adaptation of the proofs of Prop. 4.1, A.1 and A.2 in [Aud14], details can be found in Appendix A;
- the dimension is a direct consequence of Künneth formula;
- the bound on the minimum distance follows from an inductive use of Lemma 2.7;
- the weight follows from an inductive use of Proposition 1.12 and the fact that  $b_1, b_2 \leq a$ , which is a consequence of the reducedness of  $\mathcal{C}$ .  $\square$

Corollary 2.20 can be improved by using the reduced notion of tensor powers defined in Section 2.1: parameters  $k$  and  $d$  are kept untouched, parameter  $w$  is possibly reduced and parameter  $n$  is significantly reduced. To avoid making the text cumbersome, we only state here the case  $b = b_1 = b_2$ , but the general statement is given in Appendix B.

**Corollary 2.21.** *Let  $\mathcal{C}$  be the CSS code associated to  $\mathcal{C} := \mathbb{F}_2^{b_1} \xrightarrow{\quad} \mathbb{F}_2^a \xrightarrow{\quad} \mathbb{F}_2^{b_2}$ , where  $a, b \in \mathbb{N}$ . If  $\mathcal{C}$  and  $\mathcal{C}^*$  satisfy both the hypothesis of Lemma 2.7 for the same integers  $N, K \in \mathbb{N}^*$ , then for every  $\ell \in \mathbb{N}$ ,  $\mathcal{C}^{\otimes \ell}$  is a CSS code with parameters*

$$\left\| \frac{2(a+b)^\ell + (a-2b)^\ell}{3} ; (a-2b)^\ell ; \geq \left( \frac{N}{K} \right)^\ell ; \leq a\ell \right\|.$$

*This provides a family logarithmically LDPC with a minimum distance which grows at least as the  $\frac{\log(a+b)}{\log N - \log K}$ -th power of the length.*

*Proof.* Compared to Corollary 2.20, only the statement on lengths needs a further proof. We set  $\mathcal{C}_1 := \mathcal{C}$  and define recursively  $\mathcal{C}_\ell$  as the tensor product of  $\mathcal{C}$  with the reduction of  $\mathcal{C}_{\ell-1}$ . We define the sequences of integers  $(a_\ell)_{\ell \in \mathbb{N}^*}$  and  $(b_\ell)_{\ell \in \mathbb{N}^*}$  by  $\mathcal{C}_\ell := \mathbb{F}_2^{b_\ell} \hookrightarrow \mathbb{F}_2^{a_\ell} \twoheadrightarrow \mathbb{F}_2^{b_\ell}$ . Developping  $\mathcal{C}_\ell \otimes \mathcal{C}$  and using Künneth formula to say that the homology is trivial except in degree zero, we obtain the following — for simplicity, we have written only the dimensions of the different spaces — where the second line corresponds to the result after reduction:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & bb_\ell & \longrightarrow & ab_\ell + ba_\ell & \longrightarrow & aa_\ell + 2bb_\ell & \longrightarrow & ab_\ell + ba_\ell & \longrightarrow & bb_\ell & \longrightarrow & 0 \\ & & & & \nearrow & & \searrow & & & & & & \\ & & & & ab_\ell + ba_\ell - bb_\ell & & ab_\ell + ba_\ell - bb_\ell & & & & & & \end{array} .$$

It follows that  $\begin{pmatrix} a_{\ell+1} \\ b_{\ell+1} \end{pmatrix} = A \begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix}$  with  $A = \begin{pmatrix} a & 2b \\ b & a-b \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a-2b & 0 \\ 0 & a+b \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ , and hence that  $\begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} = A^{\ell-1} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2(a+b)^\ell + (a-2b)^\ell \\ (a+b)^\ell - (a-2b)^\ell \end{pmatrix}$ .  $\square$

*Remark 2.22.* The value  $\left(\frac{N}{K}\right)^\ell$  given as a lower bound for  $d_\ell$  in Corollaries 2.20 and 2.21 can actually be sharpened into  $\left\lceil \dots \left\lceil \left\lceil \frac{N}{K} \right\rceil \frac{N}{K} \right\rceil \frac{N}{K} \dots \right\rceil$ . It provides, in general, a slightly better constant.

Proposition 2.15 together with either Corollary 2.20 or 2.21 imply that, by considering its tensor powers, any CSS code, even the poorest one as soon as it is not unreasonable, provides a logarithmically LDPC family with a minimum distance tending to infinity:

**Corollary 2.23.** *If  $C = (\mathbf{H}_X, \mathbf{H}_Z)$  is a CSS code such that none of  $\mathbf{H}_X$  or  $\mathbf{H}_Z$  has a zero column, then the families  $(C^{\otimes \ell})_{\ell \in \mathbb{N}}$  and  $(C^{\otimes_r \ell})_{\ell \in \mathbb{N}}$  are logarithmically LDPC with  $d_{C^{\otimes \ell}}, d_{C^{\otimes_r \ell}} \geq 2^\ell$  for every  $\ell \in \mathbb{N}^*$ .*

In particular, and even if a CSS code has no quantum degeneracy, i.e. its quantum minimum distance is not larger than the minimum of the distances of the two classical codes defining it, for a large enough  $\ell$ , its  $\ell$ -th iterated power does.

*Remark 2.24.* The previous statement assert that the row weight of the  $\ell$ -th iterated tensor power  $C^{\otimes \ell}$  (or  $C^{\otimes_r \ell}$ ) has a row weight which is linear in  $\ell$  and hence logarithmic in the code length. Very similar arguments would show that the column weights, which are related to the number of stabilizers acting non trivially on a given qubit, are also in  $O(\ell)$  and hence logarithmic in the code length.

### 3. REINTERPRETATION OF KNOWN RESULTS

**3.1. Tillich–Zemor codes.** In [TZ14], J.-P. Tillich and G. Zémor give a construction of a CSS code from any two classical codes. Their construction is based on a graph point of view. In this section, we give an alternative approach of their construction based on tensor products.

Any linear map between two  $\mathbb{F}_2$ -spaces can be seen as a chain complex of length 2 and, by adding a null space on the right or on the left, as a short complex. If interested only in reduced complexes, one can apply the reduction process described in Section 1.1.3 or, equivalently, consider only injective maps (with a null space on their right) and surjective map (with a null space on their left). The following two propositions can be straightforwardly verified.

**Proposition 3.1.** *If  $\mathcal{C} := C_{-1} \xrightarrow{g} C_0 \twoheadrightarrow 0$  is given with a basis  $\mathcal{B}$ , then*

- $n_{\mathcal{C}} = \dim(C_0)$ ;
- $H_0(\mathcal{C}) = \text{Coker}(g)$  so  $k_{\mathcal{C}} = \dim(C_0) - \dim(C_{-1})$  and, if  $\text{Coker}(g) \neq 0$ ,  $d_{\mathcal{C}} = 1$ ;
- $w_{\mathcal{C}} = 0$ .

**Proposition 3.2.** *If  $\mathcal{C} := 0 \hookrightarrow C_0 \xrightarrow{p} C_1$  is given with a basis  $\mathcal{B}$ , then*

- $n_{\mathcal{C}} = \dim(C_0)$ ;
- $H_0(\mathcal{C}) = \text{Ker}(p)$  so  $k_{\mathcal{C}} = \dim(C_0) - \dim(C_1)$  and, if  $\text{Ker}(p) \neq 0$ ,  $d_{\mathcal{C}} = \min_{x \in \text{Ker}(p) \setminus \{0\}} |x|$ ;
- $w_{\mathcal{C}}$  is the maximal weight of a row in  $\text{Mat}_{\mathcal{B}}(p)$ .

Now, noting that

$$\left( C_{-1} \xrightarrow{g} C_0 \twoheadrightarrow 0 \right)^* = C_{-1} \xleftarrow{g^*} C_0 \longleftarrow 0 \text{ and } \left( 0 \longrightarrow C_0 \xrightarrow{p} C_1 \right)^* = 0 \longleftarrow C_0 \xleftarrow{p^*} C_1,$$

and recalling from Section 1.2 that a classical code  $C$  can be given either by an injective generating map  $g_C$  or by a surjective parity-check map  $p_C$ , and that, up to transpose,  $C$  and  $C^\perp$  exchange their generating and parity-check maps, we obtain as a corollary of Proposition 1.17:

**Proposition 3.3.** *If  $C$  is a classical code, then the CSS code  $C_p$  associated to  $0 \longrightarrow C_0 \xrightarrow{p_C} C_1$  has parameters  $n_{C_p} = n_C$ ,  $k_{C_p} = k_C$ ,  $d_{C_p} = 1$  and  $w_{C_p} = w_C$ ; and the CSS code  $C_g$  associated to  $C_{-1} \xrightarrow{g_C} C_0 \twoheadrightarrow 0$  has parameters  $n_{C_g} = n_{C^\perp}$ ,  $k_{C_g} = k_{C^\perp}$ ,  $d_{C_g} = 1$  and  $w_{C_g} = w_{C^\perp}$ .*

It can hence be noted that CSS codes associated to classical codes have very poor minimum distances. However, combining Proposition 1.17 with Proposition 1.12 and Corollary 2.14, we obtain Tillich–Zemor result which can be stated as:

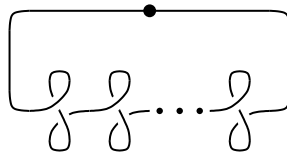
**Theorem 3.4.** [TZ14] *If  $C$  and  $\mathcal{D}$  are two classical codes given, respectively, by a parity-check map  $p_C$  and a generating map  $g_{\mathcal{D}}$ , then the CSS code  $C \otimes \mathcal{D}$  associated to  $\left( C_0 \xrightarrow{p_C} C_1 \right) \otimes \left( D_{-1} \xrightarrow{g_{\mathcal{D}}} D_0 \right)$  has parameters  $n_{C \otimes \mathcal{D}} = n_C n_{\mathcal{D}^\perp} + k_C k_{\mathcal{D}^\perp}$ ,  $k_{C \otimes \mathcal{D}} = k_C k_{\mathcal{D}^\perp}$ ,  $d_{C \otimes \mathcal{D}} = \min(d_C, d_{\mathcal{D}^\perp})$  and  $w_{C \otimes \mathcal{D}} = \max(w_C + w_{\mathcal{D}^\perp}, w_{C^\perp} + w_{\mathcal{D}})$ .*

**Remark 3.5.** The fact that one has to combine two classical codes described, respectively, by a parity-check and a generating matrix should be compared to the necessity, in Tillich–Zemor construction, to deal with a classical code and the dual of another one. This is pictured by the butterfly crossed polygon in the right-hand side of [TZ14, Figure 5].

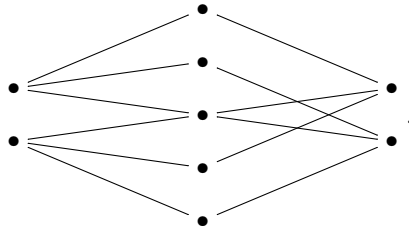
**3.2. Khovanov codes.** Chain complexes arise naturally in the context of topology, and in particular in the framework of knots and links theory. Khovanov homology is an example of link invariant which is defined as the homology of a chain complex  $\text{Ch}(D)$  associated to any link diagram  $D$ . In [Aud14], the first author used it to define CSS codes associated to link diagrams. Khovanov homology is related to tensor products via the following proposition:

**Proposition 3.6.** *For any pointed link diagrams  $D_1$  and  $D_2$ ,  $\text{Ch}(D_1 \# D_2) = \text{Ch}(D_1) \otimes \text{Ch}(D_2)$  where  $\#$  denotes the pointed connected sum.*

**3.2.1. Unknot codes.** The diagrams used in [Aud14] to define the unknot codes are not iterated connected sums of a given diagram. However, their Khovanov chain complexes is isomorphic to that of the following diagrams



which are iterated connected sums. It follows that the chain complexes underlying unknot codes are the  $\ell$ -th tensor power of



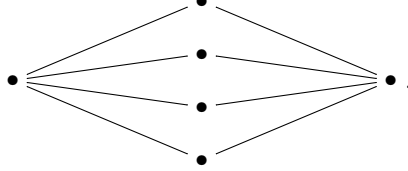
We denote the generators in the middle degree, from top to bottom, by positive integers from 1 to 5. Using the subset notation described in the Notation section, the homology is generated by 14. Since the chain complex

is symmetric, it is sufficient to use Corollary 2.20 with  $\Omega_{14} := \{12, 45\}$  to obtain back the parameters  $\llbracket \frac{3^{2\ell+1}}{\sqrt{8\pi\ell}}, 1, 2^\ell, 3\ell \rrbracket$ . Using Corollary 2.21, it can be improved into a family with asymptotical parameters  $\llbracket \frac{2.7^\ell}{3}, 1, 2^\ell, 3\ell \rrbracket$ .

**3.2.2. Unlink codes.** The diagrams considered in [Aud14] to define unlink codes are iterated connected sums of the following diagram

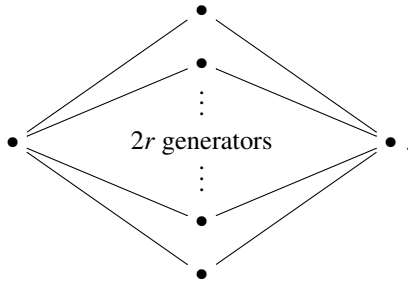


so the associated chain complexes are iterated tensor powers of



We denote the generators in the middle degree, from top to bottom, by positive integers from 1 to 4. Using the subset notation, the homology is generated by 12 and 13, and using Corollary 2.20 with  $\Omega_{12} := \{24, 13\}$  and  $\Omega_{13} := \{34, 12\}$ , we obtain back the parameters  $\llbracket \sqrt{\frac{3}{2\pi\ell}} 6^\ell, 2^\ell, 2^\ell, 4\ell \rrbracket$ . Using Corollary 2.21, it can be improved into a family with asymptotical parameters  $\llbracket \frac{2}{3} 5^\ell, 2^\ell, 2^\ell, 4\ell \rrbracket$ .

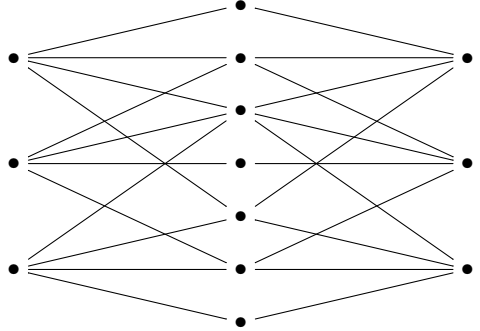
*Remark 3.7.* Forgetting its Khovanov origin, the above family can be extended to a two-parameters family defined as the  $\ell$ -th tensor power of



From a coding theoretic point of view, the corresponding CSS code is symmetric and associated to the code  $\mathcal{C} \subseteq \mathcal{C}^\perp$  where  $\mathcal{C}$  is the repetition code and  $\mathcal{C}^\perp$  is the parity code. The homology is generated by  $\{1i \mid i \in \{2, \dots, 2r-1\}\}$  and, using Corollary 2.21 with  $\Omega_{1i} := \left\{ i(2r), \prod_{\substack{j=1 \\ j \neq i}}^{2r-1} j \right\}$ , we obtain codes with asymptotical parameters  $\llbracket \frac{2}{3}(2r+1)^\ell, (2r-2)^\ell, 2^\ell, 2r\ell \rrbracket$  when  $r$  is fixed and  $\ell$  tends to infinity.

**3.3. Product of Steane  $\llbracket 7; 1; 3 \rrbracket$  codes.** In [BH13, Section V.A], which is the extended version of [BH14], Bravyi and Hastings study in details the Steane code with parameter  $\llbracket 7; 1; 3 \rrbracket$ . In its principal symmetric

form, it can be described as the CSS code  $\mathcal{S}_{7;1;3}$  associated to



We denote the generators in the middle degree, from top to bottom, by positive integers from 1 to 7. Using the subset notation, it is easily computed that  $\text{Ker}(\partial_0) = \mathbb{F}_2\langle 1235, 2346, 3567, 124 \rangle$  and  $\text{Im}(\partial_{-1}) = \mathbb{F}_2\langle 1235, 2346, 3567 \rangle$ . The homology is hence generated by 124.

Bravyi and Hastings computed that  $d_{\mathcal{S}_{7;1;3}^{\otimes 2}} = 7$ , and indeed, using Theorem 2.7 with

$$\Omega_{124} := \{124, 136, 157, 237, 256, 345, 467\},$$

we obtain that  $d_{\mathcal{S}_{7;1;3}^{\otimes 2}} \geq \frac{7}{3}d_{\mathcal{S}_{7;1;3}} = 7$ . This is an example where Theorem 2.7 gives a sharp lower bound whereas  $K \neq 1$  and  $\frac{N}{K} \notin \mathbb{N}$ .

We shall see in Section 4.1 a generalization of  $\mathcal{S}_{7;1;3}$ .

**3.4. Bravyi–Hastings homological product.** In [BH14], Bravyi and Hastings present a notion of homological product for CSS codes which are described by 2-nilpotent maps. This product is closely related to the tensor product of codes.

**Definition 3.8.** Let  $C, D$  be two  $\mathbb{F}_2$ -spaces, and  $\partial_C \in \text{End}(C), \partial_D \in \text{End}(D)$  be two 2-nilpotent maps<sup>6</sup>. As recalled in Remark 1.19, these data provide two CSS codes  $C, D$  and the *homological product*  $C \boxtimes D$  is defined as the CSS code associated to the 2-nilpotent map

$$\partial_C \boxtimes \partial_D := \partial_C \otimes \text{Id}_D + \text{Id}_C \otimes \partial_D \in \text{End}(C \otimes D).$$

**Proposition 3.9** ([BH14]). *If  $C$  and  $D$  are two CSS codes described by 2-nilpotent maps, then  $k_{C \boxtimes D} = k_C k_D$  and  $\max(d_C, d_D) \leq d_{C \boxtimes D} \leq d_C d_D$ .*

Bravyi and Hastings show moreover that for a random CSS code  $C$  of length  $n$ , the minimum distance of  $C^{\boxtimes 2}$  is larger than  $cn^2$  for some positive constant  $c$  with a probability tending to 1 when  $n$  tends to infinity.

In the coming section, we explain how Bravyi and Hastings' homological product can be understood as extracted from the tensor product. It shall follow that our criterion for a lower bound of the minimal distance, as well as all its corollaries, apply the same to homological products. From this perspective, homological products appear as an improvement of tensor products since they reduce the length of the outputs while preserving the dimension. Minimum distances are however more difficult to compare, even if they share a same lower bound.

Conversely, we show in a second time that the situation is inverted when starting from chain complexes: tensor products can be understood as extracted from the homological products of the associated ungraded 2-nilpotent maps. In this situation, minimum distances for tensor and homological products are equal, so the tensor product has globally better relative parameters.

**3.4.1. From 2-nilpotent maps to chain complexes.** Let  $C$  be a CSS code described by a 2-nilpotent map  $\partial_C \in \text{End}(C)$ . From the chain complex point of view,  $C$  is also the CSS code associated to

$$\mathcal{C} := C \xrightarrow{\partial_C} C \xrightarrow{\partial_C} C,$$

which can be reduced into

$$C_- \xrightarrow{\partial_C} C \xrightarrow{\partial_C} C_+,$$

<sup>6</sup>Recall from Definition 1.1 that a 2-nilpotent map is an endomorphism  $\partial$  satisfying  $\partial^2 = 0$ .

where  $C =: C_- \oplus \text{Ker}(\partial_C)$  and  $C_+ := C/C'_+$  with  $C := C'_+ \oplus \text{Im}(\partial_C)$ . We set  $\pi_C : C \rightarrow C_-$  the projection along  $\text{Ker}(\partial_C)$ ,  $p_C : C \rightarrow C_+$  the canonical projection and  $\partial_C^{-1}$  the inverse map of  $\partial_C : C_- \rightarrow C_+$ . We set similar notation for  $\mathcal{D}$ , another CSS code described by a 2-nilpotent map.

The length 3 middle part of  $\mathcal{C} \otimes_r \mathcal{D}$  is equal to

$$\begin{array}{ccccc}
 & & C_- \otimes D_+ & \xrightarrow{\partial_C \otimes \text{Id}_D} & C \otimes D_+ \\
 C_- \otimes D & \xrightarrow{\text{Id}_C \otimes \partial_D} & \oplus & \xrightarrow{\text{Id}_C \otimes \partial_D} & \\
 & \searrow \partial_C \otimes \text{Id}_D & & \searrow \partial_C \otimes \text{Id}_D & \\
 \oplus & \xrightarrow{\text{Id}_C \otimes \partial_D} & C \otimes D & \xrightarrow{\partial_C \otimes \text{Id}_D} & \oplus \\
 C \otimes D_- & \xrightarrow{\partial_C \otimes \text{Id}_D} & \oplus & \xrightarrow{\text{Id}_C \otimes \partial_D} & C_+ \otimes D \\
 & \searrow \partial_C \otimes \text{Id}_D & & \searrow \partial_C \otimes \text{Id}_D & \\
 & & C_+ \otimes D_- & & 
 \end{array}$$

It can be decomposed as the direct sum  $\mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3$ , where

- $\mathcal{P}_1$  is the chain subcomplex defined as  $\text{Span} \left( \begin{array}{ccccc} & & w \otimes x & & \\ & \nearrow & \oplus & \searrow & \\ u \otimes v & & & & 0 \\ & \searrow & \partial_C(w) \otimes \partial_D^{-1}(x) & \nearrow & \\ \oplus & & & & \oplus \\ 0 & \nearrow & \oplus & \searrow & \\ & \searrow & 0 & \nearrow & \\ & & 0 & & 0 \end{array} \right),$

that is, as the direct summand  $C_- \otimes D$  in degree  $-1$ ; as the space spanned by elements of the form  $(w \otimes x) \oplus (\partial_C(w) \otimes \partial_D^{-1}(x)) \oplus 0 \in (C_- \otimes D_+) \oplus (C \otimes D) \oplus (C_+ \otimes D_-)$  for some  $w \in C_-$  and  $x \in D_+$ , in degree 0; and as zero in degree 1;

- $\mathcal{P}_2$  is the chain subcomplex defined as  $\text{Span} \left( \begin{array}{ccccc} & & \pi_C(w) \otimes p_D(x) & & \\ & \nearrow & \oplus & \searrow & \\ \pi_C(u) \otimes v & & & & y \otimes p_D(z) \\ & \searrow & w \otimes x & \nearrow & \oplus \\ \oplus & & & & \\ u \otimes \pi_D(v) & \nearrow & \oplus & \searrow & p_C(y) \otimes z \\ & \searrow & p_C(w) \otimes \pi_D(x) & \nearrow & \end{array} \right);$

- $\mathcal{P}_3$  is the chain subcomplex defined as  $\text{Span} \left( \begin{array}{ccccc} & & 0 & & \\ & \nearrow & \oplus & \searrow & \\ 0 & & & & 0 \\ & \searrow & 0 & \nearrow & \oplus \\ \oplus & & & & \\ 0 & \nearrow & \oplus & \searrow & y \otimes z \\ & \searrow & w \otimes x & \nearrow & \end{array} \right).$

It is easily checked that  $H_0(\mathcal{P}_1) \cong H_0(\mathcal{P}_3) \cong \{0\}$  and that  $\mathcal{P}_2$  is isomorphic, as a chain complex, to

$$C \otimes D_- + C_- \otimes D \xrightarrow{\text{Id}_C \otimes \partial_D + \partial_C \otimes \text{Id}_D} C \otimes D \xrightarrow{\text{Id}_C \otimes \partial_D + \partial_C \otimes \text{Id}_D} C \otimes D / C'_+ \otimes D'_+,$$

which is a partially reduced form of the chain complex associated to  $\partial_C \boxtimes \partial_D$ . In degree 0, the isomorphism is nothing but the projection  $\psi$  onto the central summand  $C \otimes D$ . As a consequence, we obtain that  $H(\partial_C \boxtimes \partial_D) \cong H_0(\mathcal{C} \otimes_r \mathcal{D}) \cong H_0(\mathcal{C} \otimes \mathcal{D}) \cong H_0(\mathcal{C}) \otimes H_0(\mathcal{D}) \cong H(\partial_C) \otimes H(\partial_D)$ . The homological product can hence be seen as a subcomplex of the tensor product that contains all the homology. This provides a substantial reduction of the length, but the variation of the minimum distance is, again, more difficult to estimate. However, the criterion for a lower bound given in Theorem 2.8 still holds.



**Theorem 3.10.** *Let  $C$  be a CSS code defined by a 2-nilpotent map  $\partial_C$ , and let  $g_1, \dots, g_k \in \text{Ker}(\partial)$  and  $g_1^*, \dots, g_k^* \in \text{Im}(\partial)^\perp$  be such that*

$$\text{Ker}(\partial) = \text{Im}(\partial) \oplus \text{Span}(g_1, \dots, g_k), \quad \text{Im}(\partial)^\perp = \text{Ker}(\partial)^\perp \oplus \text{Span}(g_1^*, \dots, g_k^*) \quad \text{and} \quad \forall i, j, \langle g_i^*, g_j \rangle = \delta_{ij}.$$

*If, for any  $j_0 \in \{1, \dots, k\}$ , there exist  $\Omega_{j_0} \subseteq g_{j_0}^* + \text{Ker}(\partial)^\perp$  and  $\Omega'_{j_0} \subseteq g_{j_0} + \text{Im}(\partial)$ , with  $|\Omega_{j_0}|, |\Omega'_{j_0}| \geq N$  and  $\text{overlap}(\Omega_{j_0}), \text{overlap}(\Omega'_{j_0}) \leq K$ . Then, for any CSS code  $\mathcal{D}$  defined by a 2-nilpotent map, we have*

$$d_{C \otimes \mathcal{D}} \geq \left\lceil \frac{N}{K} d_{\mathcal{D}} \right\rceil.$$

*Proof.* Let  $\mathcal{C}$  and  $\mathcal{D}$  denote the chain complexes underlying  $C$  and  $\mathcal{D}$ . Lemma 2.7 gives a lower bound for the weight of homologically non trivial elements in the kernel of  $\partial_{\mathcal{C} \otimes \mathcal{D}}$ . In particular, it holds for elements in  $\mathcal{P}_2$ , and  $\psi$  provides a one-to-one correspondence between them and homologically non trivial elements in the kernel of  $\partial_{\mathcal{C} \otimes \mathcal{D}}$ . However, the map  $\psi$  does not preserve the weight. Nonetheless, using notation from Section 2.2,  $\psi(x_0)$  is actually equal to  $\sum_{j=1}^{n_0} b_j \otimes b_j$  so its weight is  $\sum_{j=1}^{n_0} |b_j|$ , and this is precisely the part of  $|x_0|$  which is bounded below in the proof of Lemma 2.7.  $\square$

**Corollary 3.11.** *If  $C$  and  $\mathcal{D}$  are CSS codes described by 2-nilpotent matrices which have no columns of zeros, then*

$$2 \max(d_C, d_{\mathcal{D}}) \leq d_{C \otimes \mathcal{D}}.$$

**3.4.2. From chain complexes to 2-nilpotent maps.** Forgetting the grading provides a canonical way to produce a 2-nilpotent map from any chain complex. We explain now how the tensor product of two chain complexes can be seen as extracted from the homological product of the associated 2-nilpotent maps. This actually corresponds to the case of 2-nilpotent maps given with a basis such that their matrices are block-subdiagonal with two subdiagonal blocks.

Given a CSS code  $C$  associated to a chain complex

$$\mathcal{C} = \dots \longrightarrow C_i \xrightarrow{\partial_i} C_{i+1} \xrightarrow{\partial_{i+1}} \dots$$

where the  $C_i$ 's are all  $\{0\}$  but finitely many of them, we can define  $C := \bigoplus_{i \in \mathbb{Z}} C_i$  and  $\partial_C := \bigoplus_{i \in \mathbb{Z}} \partial_i$ . The map  $\partial_C$  is 2-nilpotent and it is easily checked that

$$\text{Ker}(\partial_C) / \text{Im}(\partial_C) = H_\bullet(\mathcal{C}) = \bigoplus_{i \in \mathbb{Z}} H_i(\mathcal{C}).$$

In particular,  $\text{ker}(\partial_C) / \text{Im}(\partial_C) \cong H_0(\mathcal{C})$  whenever  $\mathcal{C}$  is balanced.

If  $\mathcal{C}$  and  $\mathcal{D}$  are two reduced complexes, then  $C \otimes D \xrightarrow{\partial_C \otimes \partial_D} C \otimes D \xrightarrow{\partial_C \otimes \partial_D} C \otimes D$  decomposes into the direct sum  $\bigoplus_{i \in \mathbb{Z}} \{\mathcal{C} \otimes \mathcal{D}\}_i$ , where  $\{\mathcal{C} \otimes \mathcal{D}\}_i$  is the length three truncature of  $\mathcal{C} \otimes \mathcal{D}$  centered in degree  $i$ . They all have null homology except for the summand  $i = 0$  which actually corresponds to the central part of  $\mathcal{C} \otimes \mathcal{D}$ . Moreover, any basis induced from basis on  $\mathcal{C}$  and  $\mathcal{D}$  respects this direct sum decomposition. It follows that  $k_{\mathcal{C} \otimes \mathcal{D}} = k_{\mathcal{C} \otimes \mathcal{D}}$  and  $d_{\mathcal{C} \otimes \mathcal{D}} = d_{\mathcal{C} \otimes \mathcal{D}}$ . Besides, it is easily checked that  $n_{\mathcal{C} \otimes \mathcal{D}} = n_{\mathcal{C}} n_{\mathcal{D}}$ . Consequently, for  $\mathcal{C} := \mathbb{F}_2^{b_1} \hookrightarrow \mathbb{F}_2^a \twoheadrightarrow \mathbb{F}_2^{b_2}$  a reduced complex defining a CSS code  $C$ , the iterated powers  $C^{\otimes \ell}$ ,  $C^{\otimes \ell}$  and  $C^{\otimes, \ell}$  have same dimensions and minimum distances but different lengths, which are respectively  $(a + b_1 + b_2)^\ell$ ,  $O\left(\frac{(a+2\sqrt{b_1 b_2})^\ell}{\sqrt{\ell}}\right)$  and  $O\left((a + \sqrt{b_1 b_2})^\ell\right)$ .

**3.4.3. Comparison between tensor and homological powers.** There are two natural notions of product for CSS codes, namely tensor and homological ones, and we have observed how to switch from one to the other. They both generate LDPC families when used iteratively. It is natural to question whether a construction is better than the other. The answer is actually negative, and the qualities of the family of codes obtained by iterated tensor or homological powers depend on the initial descriptive type of the input codes:

- if the input code is described by a 2-nilpotent map, then one can see it as coming from a chain complex with repeated space and map. In this situation, the homological powers of the original 2-nilpotent map provide shorter codes with same dimensions than the tensorial powers. Moreover, the control of the minimum distances provided by the present paper is equal for both.

- if the input code is described by a general complex, then one can consider the underlying 2-nilpotent map by forgetting the grading. In this situation, the tensor powers of the original chain complex provide shorter codes with same dimensions and minimum distances, hence better relative parameters, than the homological powers.

A good philosophy should hence be to stick to the original nature of the inputs and use homological products when dealing with 2-nilpotent maps and tensor products when dealing with chain complexes.

#### 4. NEW FAMILIES OF CODES

In this section we present new families of CSS codes defined as iterated tensor powers of some given CSS code. They all share a logarithmic LDPC structure and, for a length  $N_\ell$  which tends to infinity, their minimum distance can be “as closed as possible to  $\sqrt{N_\ell}$ ” in the sense that, for all  $\alpha < \frac{1}{2}$ , there is such a family whose minimum distance is larger than  $N_\ell^\alpha$ .

To control minimum distances, we use Theorem 2.8 which requires the construction of large sets of cohomologically non trivial vectors  $\Omega$  with small overlap. For this sake, it is natural to seek among codes with many automorphisms. This feature is indeed shared by our three examples, namely:

- codes from finite geometry, endowed with a natural action of  $\mathbf{PGL}(3, \mathbb{F}_q)$ ;
- cyclic codes, i.e. codes of length  $n$  with a natural action of the cyclic group of order  $n$ ;
- Reed Muller codes, endowed with a natural action of the affine group.

**4.1. Quantum finite geometry codes.** In this section, we set  $q = 2^s$  for some positive integer  $s$ . The idea relies on using points/lines incidence structures of affine and projective spaces over finite fields to construct LDPC CSS codes. It has already been used to construct classical LDPC codes in [KLF01] and moderate density parity check quantum codes in [Far12].

Here, we shall consider two incidence structures:

- the point/line incidence structure;
- the point/affine charts incidence structure.

**4.1.1. The projective plane.** The projective plane  $\mathbb{P}^2(\mathbb{F}_q)$  is defined as the set of lines of  $\mathbb{F}_q^3$  passing through the origin. Let us remind classical facts of this finite geometry:

**Proposition 4.1.**

- (i) The plane contains  $q^2 + q + 1$  point and  $q^2 + q + 1$  lines.
- (ii) Every line contains  $q + 1$  points and every point is contained in  $q + 1$  lines.
- (iii) Every two distinct points are contained in a unique line and every two distinct lines meet at a unique point.

Note that each of the above statements express the principle of duality in projective planes, which swaps point and lines and reverses inclusions.

*Example 4.2.* For  $q = 2$ , the projective plane is also called *Fano plane*. It contains 7 points and 7 lines and the point/line incidence structure is usually represented by the picture given in Figure 1 in which the 6 lines and the circle represent the 7 lines of  $\mathbb{P}^2(\mathbb{F}_2)$ .

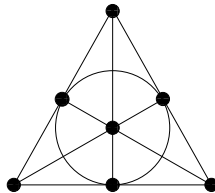


FIGURE 1. The projective plane  $\mathbb{P}^2(\mathbb{F}_2)$ .

Additionally we consider the *affine charts* of the projective plane.

**Definition 4.3.** An affine chart of  $\mathbb{P}^2(\mathbb{F}_q)$  is the complement of a line.

Let us list some properties of affine charts.

**Proposition 4.4.**

- (i) An affine chart is isomorphic to the affine plane over  $\mathbb{F}_q$ ; in particular it contains  $q^2$  elements.
- (ii) Let  $L$  be a line in  $\mathbb{P}^2(\mathbb{F}_q)$  and  $U$  an affine chart. Then,
  - either  $L$  is the complement of  $U$  and hence  $L \cap U = \emptyset$ ;
  - or  $L \cap U$  is an affine line and hence has  $q$  elements.

In particular, since  $q$  is even, the number of points of  $L \cap U$  is always even.

- (iii) The number of affine charts of  $\mathbb{P}^2(\mathbb{F}_q)$  equals the number of lines and hence equals  $q^2 + q + 1$ .

4.1.2. *Classical codes associated to projective planes in characteristic 2.* We construct two binary codes associated to the projective space  $\mathbb{P}^2(\mathbb{F}_q)$ , with length  $|\mathbb{P}^2(\mathbb{F}_q)| = q^2 + q + 1$ . Vectors of  $\mathbb{F}_2^{q^2+q+1}$  can be regarded as subsets of  $\mathbb{P}^2(\mathbb{F}_q)$  and we shall freely speak of either vectors or subsets of  $\mathbb{P}^2(\mathbb{F}_q)$ . From this point of view, the canonical inner product on  $\mathbb{F}_2^{q^2+q+1}$  can be given a geometric interpretation since, for  $S, S' \subseteq \mathbb{P}^2(\mathbb{F}_q)$ :

$$\langle S, S' \rangle = |S \cap S'| \pmod{2}.$$

We introduce the codes

- $C_{\text{lines}}(s)$ , spanned by lines of  $\mathbb{P}^2(\mathbb{F}_q)$ ;
- $C_{\text{planes}}(s)$ , spanned by the affine charts of  $\mathbb{P}^2(\mathbb{F}_q)$ .

**Warning 4.5.** We want to stress the fact that, even though the projective spaces are defined over  $\mathbb{F}_q$ , the associated classical codes, and hence the quantum codes to follow, are defined over  $\mathbb{F}_2$ .

The dimension of  $C_{\text{lines}}(s)$  is well-known.

**Proposition 4.6** ([Smi69]). *For all  $s > 0$ , we have  $\dim_{\mathbb{F}_2}(C_{\text{lines}}(s)) = 3^s + 1$ .*

**Proposition 4.7.** *For all  $s \geq 1$ ,*

- (i)  $C_{\text{planes}}(s) \subseteq C_{\text{planes}}(s)^\perp$ ;
- (ii)  $C_{\text{planes}}(s) \subseteq C_{\text{lines}}(s)$ ;
- (iii)  $C_{\text{planes}}(s) \subseteq C_{\text{lines}}(s)^\perp$ ;
- (iv)  $C_{\text{lines}}(s) = C_{\text{planes}}(s) \oplus \text{Span}(L)$  for every line  $L \subseteq \mathbb{P}^2(\mathbb{F}_q)$ ;
- (v)  $\dim(C_{\text{lines}}(s)) - \dim(C_{\text{planes}}(s)) = 1$ .

*Proof.* To prove (i), note first that an affine chart has an even number of points and hence is orthogonal to itself. Let  $A_1, A_2$  be two distinct affine charts. Then, there exist two distinct lines  $L_1, L_2$  such that if we denote by  ${}^cX$  the complement of a subset  $X$  of  $\mathbb{P}^2(\mathbb{F}_q)$ , then

$$A_1 = {}^cL_1 \quad \text{and} \quad A_2 = {}^cL_2.$$

Thus,

$$A_1 \cap A_2 = {}^c(L_1 \cup L_2)$$

Next, since  $L_1, L_2$  are distinct to each other,  $|L_1 \cup L_2| = 2q + 1$  and hence

$$\langle A_1, A_2 \rangle \equiv |A_1 \cap A_2| \equiv |\mathbb{P}^2(\mathbb{F}_q)| - |L_1 \cup L_2| \equiv q^2 - q \equiv 0 \pmod{2}.$$

To prove (ii), consider an affine chart  $A$  and let  $L$  be the line such that  $A = {}^cL$ . Let  $P \in L$  be a point and  $L_1, \dots, L_q$  be all the lines containing  $P$  but  $L$ . Then

$$A = L_1 + \dots + L_q.$$

Indeed, every point  $Q \in A$  is in exactly one of the  $L_i$ 's. Moreover, the  $L_i$ 's all meet at  $P$  which is the only point in  ${}^cA$  contained in the union of  $L_i$ 's. Since the number of the  $L_i$ 's is  $q$  and hence is even, then  $P \notin L_1 + \dots + L_q$ . This proves that every affine chart is a sum of lines.

Point (iii) is a direct consequence of Proposition 4.4(ii).

To prove (iv), denote by  $\mathbf{1}$  the all-one vector  $(1, \dots, 1)$ . Then for all affine chart  $A$ , there is a line  $L$  such that  ${}^cA = L$ . In terms of vectors, we get  $A = L + \mathbf{1}$ . Then, let  $L, L'$  be two lines of  $\mathbb{P}^2(\mathbb{F}_q)$  and  $A, A'$  be respectively the affine charts  ${}^cL$  and  ${}^cL'$ , then

$$L + L' = L + L' + \mathbf{1} + \mathbf{1} = A + A'.$$

Thus,

$$L = A + A' + L'.$$

So far, we prove that every line  $L'$  of  $\mathbb{P}^2(\mathbb{F}_q)$  is a sum of  $L$  and an element of  $C_{\text{planes}}(s)$ . This proves that

$$C_{\text{lines}}(s) = C_{\text{planes}}(s) + \text{Span}(L).$$

But  $\langle L, L \rangle \equiv |L| \equiv 1 \pmod{2}$ , so  $L \notin C_{\text{lines}}^\perp$  and it follows hence, from (i), that  $L \notin C_{\text{planes}}(s)$ .

Finally, (iv) is a direct consequence of (v).  $\square$

*Remark 4.8.* Actually,  $C_{\text{planes}}(s)$  is nothing but the *even subcode* of  $C_{\text{lines}}(s)$  i.e. the subcodes of vectors of even weight.

#### 4.1.3. Quantum CSS codes from the projective plane in characteristic 2.

**Definition 4.9.** We define  $\text{QFG}(s)$  as the quantum code of length  $q^2 + q + 1$  associated to  $C_{\text{planes}}(s) \subseteq C_{\text{lines}}(s)$ .

After reduction, the corresponding chain complex is

$$\mathcal{C}_{\text{FG}}(s) := \mathbb{F}_2^{3^s} \hookrightarrow \mathbb{F}_2^{2^{2s}+2^s+1} \twoheadrightarrow \mathbb{F}_2^{2^{2s}+2^s-3^s}.$$

Indeed, Proposition 4.7(v) together with Proposition 4.6 assert that  $\dim_{\mathbb{F}_2}(C_{\text{planes}}(s)) = 3^s$ .

*Remark 4.10.* The code  $\text{QFG}(1)$  is nothing but the  $[[7, 1, 3]]$  Steane code. This fact is actually well-known, since the Steane code is known to be constructed from the Hamming code and its dual while the Hamming code is already known to be the code  $C_{\text{lines}}(1)$  spanned by the lines of  $\mathbb{P}^2(\mathbb{F}_2)$ .

**Lemma 4.11.** Let  $\Omega$  be the set of lines of  $\mathbb{P}^2(\mathbb{F}_q)$ . We have

$$\Omega \subset C_{\text{lines}}(s) \setminus C_{\text{planes}}(s) \quad \text{and} \quad \Omega \subset C_{\text{planes}}(s)^\perp \setminus C_{\text{lines}}(s)^\perp.$$

*Proof.* The first inclusion is a direct consequence of Proposition 4.7(iv). From Proposition 4.4(ii) every line of  $\mathbb{P}^2(\mathbb{F}_q)$  is in  $C_{\text{planes}}(s)^\perp$ . But a line  $L$  is not in  $C_{\text{lines}}(s)^\perp$ . Indeed, let  $L'$  be a line distinct from  $L$ , then  $\langle L, L' \rangle \equiv |L \cap L'| \equiv 1 \pmod{2}$ .  $\square$

**Lemma 4.12.**  $|\Omega| = q^2 + q + 1$  and  $\text{overlap}(\Omega) = q + 1$ .

*Proof.* It is a direct consequence of Proposition 4.1.  $\square$

**Proposition 4.13.** For every  $s \geq 1$ , the family of iterated tensor powers  $\text{QFG}(s)^{\otimes \ell}$  has parameters

$$\left\| \sim \frac{K_s}{\sqrt{\ell}} \left( (2^{2s} + 2^s + 1) + 2 \left( 2\sqrt{3} \right)^s \sqrt{1 + \left( \frac{1}{2} \right)^s - \left( \frac{3}{4} \right)^s} \right)^\ell, 1, \geq \left( 2^s + \frac{1}{2^s + 1} \right)^\ell, \leq (2^{2s} + 2^s + 1)\ell \right\|$$

for some constant  $K_s$  depending only on  $s$  and the family of iterated reduced tensor powers  $\text{QFG}(s)^{\otimes_r \ell}$  has parameters

$$\left\| \sim K'_s \left( (2^{2s} + 2^s + 1) + \left( 2\sqrt{3} \right)^s \sqrt{1 + \left( \frac{1}{2} \right)^s - \left( \frac{3}{4} \right)^s} \right)^\ell, 1, \geq \left( 2^s + \frac{1}{2^s + 1} \right)^\ell, \leq (2^{2s} + 2^s + 1)\ell \right\|$$

for some constant  $K'_s$  depending only on  $s$ .

*Proof.* The minimum distance is a consequence of Lemma 4.12 and Corollary 2.9. The other parameters are obtained using Corollary 2.20 and Proposition B.1.  $\square$

**4.2. Quantum cyclic codes.** The following example is based on classical cyclic codes.

4.2.1. *Cyclic codes.* Here we recall very classical facts about cyclic codes. For further details we refer the reader to [MS77, Chapter 7].

A binary *cyclic code*  $C \subseteq \mathbb{F}_2^n$  is a code which is stable under the action of the automorphism

$$\sigma: \begin{cases} \mathbb{F}_2^n & \longrightarrow \mathbb{F}_2^n \\ (x_0, \dots, x_{n-1}) & \longmapsto (x_{n-1}, x_0, \dots, x_{n-2}). \end{cases}$$

In what follows we identify  $\mathbb{F}_2^n$  and  $\mathbb{F}_2[X]/(X^n - 1)$  using the  $\mathbb{F}_2$ -linear isomorphism

$$\begin{cases} \mathbb{F}_2[X]/(X^n - 1) & \xrightarrow{\sim} \mathbb{F}_2^n \\ f = f_0 + f_1X + \dots + f_{n-1}X^{n-1} & \longmapsto (f_0, \dots, f_{n-1}) \end{cases}$$

and we define the weight of a polynomial as the number of its nonzero coefficients. Using this identification, the automorphism  $\sigma$  corresponds in  $\mathbb{F}_2[X]/(X^n - 1)$  to the multiplication by  $X$ . A code  $C \subset \mathbb{F}_2[X]/(X^n - 1)$  is hence cyclic if it is stable under the multiplication by  $X$ , that is if it is an ideal. Since  $\mathbb{F}_2[X]$  is a principal ideal ring, the ideals of  $\mathbb{F}_2[X]/(X^n - 1)$  are in one-to-one correspondence with the divisors of  $X^n - 1$ .

Given  $h \in \mathbb{F}_2[X]$  such that  $h \mid X^n - 1$ , the code  $Cyc(h)$  is defined as the code corresponding to the ideal generated by  $h$ . It is well-known that this code has dimension  $n - \deg(h)$ . The polynomial  $h$  is referred to as a *generating polynomial* of the code. It is unique up to multiplication by an invertible element of  $\mathbb{F}_2[X]/(X^n - 1)$ . Note that if  $h_1 \mid h_2 \mid X^n - 1$ , then  $Cyc(h_2) \subseteq Cyc(h_1)$ .

The dual of a cyclic code is cyclic and its generating polynomial can be obtained as follows. Given a polynomial  $f \in \mathbb{F}_2[X]/(X^n - 1)$  we define

$$\bar{f} := X^{\deg f} f \left( \frac{1}{X} \right),$$

and referred to as the *reciprocal polynomial* of  $f$ . Over  $\mathbb{F}_2$ ,  $X^n - 1$  is equal to its reciprocal polynomial so, if  $f \mid X^n - 1$ , then  $\bar{f} \mid X^n - 1$ . Let  $h$  be the polynomial such that  $\bar{f}h = X^n - 1$ , then

$$C(f)^\perp = C(h).$$

4.2.2. *A construction of CSS codes.* The case  $n = 2^s$  is actually never considered in the study of classical cyclic codes since, in that case, the polynomial  $X^n - 1$  is completely inseparable and all the constructions based on choosing divisors of  $X^n - 1$  having a prescribed set of roots, such as BCH codes (see for instance [MS77, Chapter 9]), are not possible. But oddly enough, this is precisely the case which shall lead to interesting families of CSS codes defined by iterated tensor powers.

In this situation  $X^{2^s} - 1 = (X - 1)^{2^s}$  and hence the divisors of  $X^n - 1$  are of the form  $(X - 1)^r$  for all  $r \in \{0, \dots, 2^s\}$ . The corresponding cyclic codes are thus  $Cyc((X - 1)^r)$ . Such a code has dimension  $n - r$  and

$$Cyc((X - 1)^r)^\perp = Cyc((X - 1)^{n-r}).$$

**Definition 4.14.** For any  $r < n$ , we define  $QCC(n, r)$  as the CSS code of dimension 1 associated to the pair of codes  $Cyc((X - 1)^r) \subseteq Cyc((X - 1)^{n-r})$ . If  $g_r : \mathbb{F}_2^{n-r} \rightarrow \mathbb{F}_2^n$  and  $g_{n-r+1} : \mathbb{F}_2^{r-1} \rightarrow \mathbb{F}_2^n$  are, respectively, generating maps for  $Cyc((X - 1)^r)$  and  $Cyc((X - 1)^{n-r})^\perp = Cyc((X - 1)^{r+1})$ , then  $QCC(n, r)$  is also defined as the CSS code associated to

$$\mathcal{C}_{Cyc} := \mathbb{F}_2^{n-r} \xrightarrow{g_r} \mathbb{F}_2^n \xrightarrow{g_{n-r+1}^*} \mathbb{F}_2^{r-1}.$$

**Lemma 4.15.** Let  $r \leq n$  be a non negative integer. Then, the weight of  $(X - 1)^r \in \mathbb{F}_2[X]$  equals  $2^{w_2(r)}$ , where  $w_2(r)$  denotes the binary weight of  $r$ , i.e. the weight of its decomposition in base 2.

*Proof.* We prove it by induction on  $w_2(r)$ . If  $w_2(r) = 1$ , then  $r = 2^a$  for some non negative integer  $a$  and  $(X - 1)^{2^a} = X^{2^a} - 1$  has weight 2. For  $w_2(r) > 1$  then let  $a := \lceil \log_2(r) \rceil$ . Then  $r = 2^a + r'$  where  $r' < 2^a$  and  $w_2(r') = w_2(r) - 1$ . By induction, the weight of  $(X - 1)^{r'}$  equals  $2^{w_2(r')}$ . Hence

$$(X - 1)^r = (X - 1)^{2^a} (X - 1)^{r'} = X^{2^a} (X - 1)^{r'} + (X - 1)^{r'}.$$

Since  $r' < 2^a$ , the polynomials  $X^{2^a} (X - 1)^{r'}$  and  $(X - 1)^{r'}$  have no common monomials and hence, the weight of  $(X - 1)^r$  is twice that of  $(X - 1)^{r'}$ . This concludes the proof.  $\square$

**Proposition 4.16.** *For every  $\ell \in \mathbb{N}$ , the  $\ell$ -th tensor power of  $QCC(n, r)$  has dimension 1 and minimum distance at least  $2^{\ell(s-w)}$ , where  $w = \max(w_2(r-1), w_2(n-r))$ .*

*Proof.* To prove the statement, we apply Corollary 2.9 with

$$\begin{aligned}\Omega &:= \{X^i(X-1)^{r-1} \bmod (X-1)^n \mid i \in \{0, \dots, n-1\}\} \\ \Omega' &:= \{X^i(X-1)^{n-r} \bmod (X-1)^n \mid i \in \{0, \dots, n-1\}\}.\end{aligned}$$

Both sets have cardinality  $n = 2^s$ , and their respective overlaps are

$$\text{overlap}(\Omega) = w_2(r-1) \quad \text{and} \quad \text{overlap}(\Omega') = w_2(n-r).$$

Indeed, stacking the elements of  $\Omega$ , we obtain a circulant matrix whose column weight equals the row weight. But the latter is given by Lemma 4.15. This concludes the proof.  $\square$

**Corollary 4.17.** *For  $n = 2^s$  and  $r = 2^{\frac{s}{2}} = \sqrt{n}$ , where  $s$  is an even integer, the family of iterated tensor powers  $QCC(n, r)^{\otimes \ell}$  has parameters*

$$\left\| \sim \frac{K_n}{\sqrt{\ell}} \left( n + 2n^{\frac{3}{4}} \left( 1 - \frac{1}{\sqrt{n}} \right) \right)^\ell, 1, \geq \sqrt{n}^\ell, \leq \sqrt{n} \ell \right\|$$

for some constant  $K_n$  depending only on  $n$  and the family of iterated reduced tensor powers  $QCC(n, r)^{\otimes \ell}$  has parameters

$$\left\| \sim K'_n \left( n + n^{\frac{3}{4}} \left( 1 - \frac{1}{\sqrt{n}} \right) \right)^\ell, 1, \geq \sqrt{n}^\ell, \leq \sqrt{n} \ell \right\|$$

for some constant  $K'_n$  depending only on  $n$ .

*Proof.* We have  $r-1 = 1 + 2 + \dots + 2^{\frac{s}{2}-1}$  and  $n-r = 2^s - 2^{\frac{s}{2}} = 2^{\frac{s}{2}}(2^{\frac{s}{2}} - 1) = 2^{\frac{s}{2}}(1 + 2 + \dots + 2^{\frac{s}{2}-1})$ . It follows that  $w_2(r-1) = w_2(n-r) = \frac{s}{2}$ . The minimum distance is thus a consequence of Corollary 2.9 and the length a consequence of Corollary 2.20 and Proposition B.1.  $\square$

**4.3. Quantum Reed–Muller codes.** In this section, we define a two-parameters family of CSS codes based on classical Reed–Muller codes. Such CSS codes have been studied in [ZF97]. Another construction of stabilizer codes (which are not CSS) based on Reed–Muller codes was also proposed by Steane in [Ste99].

To this end, we define, for every  $r \in \mathbb{N}^*$  and  $s \in \{0, \dots, r\}$ :

- $\text{Pol}_r := \mathbb{F}_2[X_1, \dots, X_r] / (X_1^2 - X_1, \dots, X_r^2 - X_r)$  given with the basis  $\{X_I := \prod_{i \in I} X_i \mid I \subset \{1, \dots, r\}\}$ ;
- $\text{Pol}_{r,s}$  the restriction of  $\text{Pol}_r$  to elements of degree<sup>7</sup> at most  $s$ ;
- $\phi_r : \text{Pol}_r \hookrightarrow \mathbb{F}_2^{2^r}$  the map which sends a polynomial  $P$  to  $(P(x))_{x \in \mathbb{F}_2^r}$ ;
- $\phi_{r,s}$  the restriction of  $\phi_r$  to  $\text{Pol}_{r,s}$ .

**Definition 4.18.** The Reed–Muller code  $\text{RM}(r, s)$  is the classical code with generating map  $\phi_{r,s}$ .

**Proposition 4.19** ([MS77, Theorem 13.4]). *For every  $r \in \mathbb{N}^*$  and  $s \in \{0, \dots, r\}$ ,  $\text{RM}(r, s)^\perp = \text{RM}(r, r-s-1)$ .*

**Definition 4.20.** For every  $r \in \mathbb{N}^*$ , we define the quantum Reed–Muller code  $\text{QRM}(r)$  as the CSS code associated to

$$\mathcal{C}_{\text{RM}}(r) := \text{Pol}_{2r, r-1} \xhookrightarrow{\phi_{2r, r-1}} \mathbb{F}_2^{4r} \xrightarrow{\phi_{2r, r-1}^*} \text{Pol}_{2r, r-1}.$$

**Proposition 4.21.** *For every  $r \in \mathbb{N}^*$ , the family of iterated tensor powers  $\text{QRM}(r)^{\otimes \ell}$  has parameters*

$$\left\| \sim \sqrt{\frac{2^{2r+1} - \binom{2r}{r}}{2\pi\ell(4^r - \binom{2r}{r})}} \left( 2^{2r+1} - \binom{2r}{r} \right)^\ell, \binom{2r}{r}^\ell, 2^{r\ell}, \leq 4^r \ell \right\|$$

and the family of iterated reduced tensor powers  $\text{QRM}(r)^{\otimes \ell}$  has parameters

$$\left\| \frac{2 \left( \frac{3 \cdot 4^r - \binom{2r}{r}}{2} \right)^\ell + \binom{2r}{r}^\ell}{3}, \binom{2r}{r}^\ell, 2^{r\ell}, \leq 4^r \ell \right\|.$$

<sup>7</sup>defined by  $\deg(X_I) = |I|$ , with the convention that 0 has degree  $-\infty$

*Proof.* It follows from Proposition 4.19 that  $\text{Ker}(\phi_{2r,r-1}^*) = \text{RM}(2r, r)$ . But, on the other hand,  $\text{Im}(\phi_{2r,r-1}) = \text{RM}(2r, r-1)$ ; the homology of  $\mathcal{C}_{\text{RM}}(r)$  is hence generated by the images through  $\phi_{2r}$  of the elements of  $\text{Pol}_{2r}$  which are of degree exactly  $r$ .

Now, let us consider such a generator  $\phi_{2r}(X_{I_0})$ , with  $I_0 \subset \{1, \dots, 2r\}$  of cardinality  $r$ ; and set  $I_0^c := \{1, \dots, 2r\} \setminus I_0$ . For every  $I, J \subset \{1, \dots, 2r\}$ ,  $\langle \phi_{2r}(X_I), \phi_{2r}(X_J) \rangle = \sum_{x \in \mathbb{F}_2^{4r}} X_I X_J(x) = 1$  if and only if  $I \cup J = \{1, \dots, 2r\}$ . Using notation from Section 2.2, it follows then that

$$\text{Ker}_{\phi_{2r}(X_{I_0})}^{\oplus} = \phi_{2r}(X_{I_0^c}) + \text{RM}(2r, r)^{\perp} = \phi_{2r}(X_{I_0^c} + \text{Pol}_{2r,r-1}).$$

Since  $\text{QRM}(r)$  is symmetric, it is now sufficient to apply Corollary 2.20 or 2.21 with

$$\Omega_{\phi_{2r}(X_{I_0})} := \left\{ \phi_{2r} \left( \prod_{i \in I_0^c} (X_i + \varepsilon_i) \right) \mid \forall i \in I_0^c, \varepsilon_i \in \mathbb{F}_2 \right\}.$$

The elements of  $\Omega_{\phi_{2r}(X_{I_0})}$  have indeed disjoint support: the  $x$ -th coordinate of  $\phi_{2r} \left( \prod_{i \in I_0^c} (X_i + \varepsilon_i) \right)$ , where  $x = (x_i)_{i \in \{1, \dots, 2r\}} \in \mathbb{F}_2^{2r}$ , is 1 if and only if  $x_i = 1 - \varepsilon_i$  for every  $i \in I_0^c$ .

On the other hand,  $\phi_{2r}(X_{\{1, \dots, r\}})$  is an element of weight  $2^r$  which survives in homology.  $\square$

By extracting the diagonal subfamily  $\ell = r$ , it follows from Stirling series that we obtain an  $r$ -indexed family with parameters

$$\left\| \sim \frac{\left(\frac{3}{2}\right)^{r-1} 4^{r^2}}{e^{\frac{1}{9\pi}} e^{\frac{1}{3} \sqrt{\frac{r}{\pi}}}}, \sim \frac{4^{r^2}}{e^{\frac{1}{3} \sqrt{\pi r}}}, 2^{r^2}, \leq 4^r r \right\|.$$

The family is not, *stricto sensu*, logarithmically LDPC, but the weight grows slower than any positive power of the length, the dimension faster than any “ $< 1$ ”-power of the length, and the minimum distance faster than any “ $< \frac{1}{2}$ ”-power of the length.

## REFERENCES

- [Aud14] Benjamin Audoux, *An application of Khovanov homology to quantum codes.*, Ann. Inst. Henri Poincaré D, Comb. Phys. Interact. **1** (2014), no. 2, 185–223.
- [BH13] Sergey Bravyi and Matthew B. Hastings, *Homological product codes*, arXiv:1311.0885, 2013.
- [BH14] ———, *Homological product codes*, Proceedings of the 46th annual ACM symposium on theory of computing, STOC ’14. New York, NY, USA, May 31– June 04, 2014, ACM, 2014, pp. 273–282.
- [BMD07] Hector Bombin and Miguel Martin-Delgado, *Homological error correction: classical and quantum codes*, J. Math. Phys. **48** (2007), no. 5, 052105.
- [CDZ13] Alain Couvreur, Nicolas Delfosse, and Gilles Zémor, *A construction of quantum LDPC codes from Cayley graphs*, IEEE Trans. Inform. Theory **59** (2013), no. 9, 6087–6098.
- [CS96] Robert Calderbank and Peters W. Shor, *Good quantum error-correcting codes exist*, Phys. Rev. A **54:1098** (1996), 1098–1105.
- [Del12] Nicolas Delfosse, *Constructions et performances de codes LDPC quantiques*, Ph.D. thesis, University of Bordeaux, 2012.
- [Del13] ———, *Tradeoffs for reliable quantum information storage in surface codes and color codes*, Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on, 2013, pp. 917–921.
- [Far12] Jacob Farinholt, *Quantum LDPC codes constructed from point-line subsets of the finite projective plane*, ArXiv:1207.0732, 2012.
- [FH14] Michael H. Freedman and Matthew B. Hastings, *Quantum systems on non-k-hyperfinite complexes: A generalization of classical statistical mechanics on expander graphs*, Quantum Inf. Comput. **14** (2014), no. 1–2, 144–180.
- [FM01] Michael H. Freedman and David A. Meyer, *Projective plane and planar quantum codes*, Found. Comput. Math. **1** (2001), no. 3, 325–332.
- [FML02] Michael H. Freedman, David A. Meyer, and Feng Luo, *Z<sub>2</sub>-systolic freedom and quantum codes*, Mathematics of quantum computation, Comput. Math. Ser., Chapman & Hall/CRC, Boca Raton, FL, 2002, pp. 287–320.
- [Gal62] Robert G. Gallager, *Low density parity check codes*, Ph.D. thesis, Massachusetts Institute of Technology, 1962.
- [Kit03] Alexey Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Physics **303** (2003), no. 1, 2–30.
- [KLF01] Yu Kou, Shu Lin, and Marc P. C. Fossorier, *Low-density parity-check codes based on finite geometries: a rediscovery and new results*, IEEE Trans. Inform. Theory **47** (2001), no. 7, 2711–2736.
- [MS77] Florence J. MacWilliams and Neil J. A. Sloane, *The theory of error-correcting codes. I*, North-Holland Publishing Co., Amsterdam, 1977, North-Holland Mathematical Library, Vol. 16.
- [NC10] Michael A. Nielsen and Isaac L. Chuang, *Quantum computation and quantum information*, 10th anniversary editions ed., Cambridge University Press. xxxi, Cambridge, 2010.
- [Pre] John Preskill, *Quantum information and computation*, <http://www.theory.caltech.edu/people/preskill/ph229/>.

- [RSU01] Thomas J. Richardson, M. Amin Shokrollahi, and Rüdiger L. Urbanke, *Design of capacity-approaching irregular low-density parity-check codes*, IEEE Trans. Inform. Theory **47** (2001), 619–637.
- [Smi69] K. J. C. Smith, *On the  $p$ -rank of the incidence matrix of points and hyperplanes in a finite projective geometry*, J. Combin. Theory **7** (1969), no. 2, 122–129.
- [Ste96] Andrew Steane, *Multiple particle interference and quantum error correction*, Proc. Roy. Soc. Lond. A **452**:2551 (1996), 2551–2577.
- [Ste99] ———, *Quantum reed-muller codes*, IEEE Trans. Inform. Theory **45** (1999), no. 5, 1701–1703.
- [TZ14] Jean-Pierre Tillich and Gilles Zémor, *Quantum LDPC codes with positive rate and minimum distance proportional to the square root of the blocklength*, IEEE Trans. Inform. Theory **60** (2014), no. 2, 1193–1202.
- [Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.
- [Zém09] Gilles Zémor, *On Cayley graphs, surface codes, and the limits of homological coding for quantum error correction*, Coding and cryptography, Lecture Notes in Comput. Sci., vol. 5557, Springer, Berlin, 2009, pp. 259–273.
- [ZF97] Lin Zhang and Ian Fuss, *Quantum reed-muller codes*, ArXiv:quant-ph/9703045, 1997.

#### APPENDIX A. LENGTH OF TENSOR POWERS

In this appendix, we prove the length part of Corollary 2.20. Using inductively Proposition 1.12, it is easily seen that the length of  $C^{\otimes \ell}$  is equal to the constant term in the Laurent polynomial  $(bt^{-1} + a + b't)^{\ell}$ . The statement is hence a consequence of Proposition A.5 given below. But before proving it, we set some technical lemmata.

**Definition A.1.** For every  $x > 0$  and every  $\varepsilon = \pm 1$ , we define<sup>8</sup>  $y_{\varepsilon} = \frac{1+2x+\varepsilon\sqrt{1+4x}}{2x}$ .

Moreover, for every  $\ell \in \mathbb{N}$ , we define  $T_{\ell}(x) := \sum_{r=0}^{\ell} \binom{\ell}{r} \binom{2r}{r} x^r$  and  $P_{\ell}^{\varepsilon}(x) := x^{\ell} y_{\varepsilon}^{\ell} \sum_{r=0}^{\ell} \binom{\ell}{r}^2 y_{\varepsilon}^{-2r}$ .

It is directly checked that  $y_+ + y_- = \frac{1+2x}{x}$ ,  $y_+ - y_- = \frac{\sqrt{1+4x}}{x}$  and  $y_+ y_- = 1$ . It follows from the latter equality that  $P_{\ell}^{+}(x) = P_{\ell}^{-}(x) = x^{\ell} y_+^{\ell} \sum_{r=0}^{\ell} \binom{\ell}{r}^2 y_-^{2r}$ . Moreover, it can be straightforwardly computed that:

**Lemma A.2.** For every  $\ell, r \in \mathbb{Z}$ ,

- $(2\ell + 1) \binom{\ell}{r} \binom{2r}{r} - \ell \binom{\ell-1}{r} \binom{2r}{r} + 2(2\ell + 1) \binom{\ell}{r-1} \binom{2r-2}{r-1} - 4\ell \binom{\ell-1}{r-1} \binom{2r-2}{r-1} = (\ell + 1) \binom{\ell+1}{r} \binom{2r}{r};$
- $(2\ell + 1) \left( \binom{\ell}{r}^2 + \binom{\ell}{r-1}^2 \right) - \ell \left( \binom{\ell-1}{r}^2 - 2\binom{\ell-1}{r-1}^2 + \binom{\ell-1}{r-2}^2 \right) = (\ell + 1) \binom{\ell+1}{r}^2;$

with the convention that  $\binom{\ell}{r} = 0$  whenever  $\ell < 0$  or  $r \notin \{0, \dots, \ell\}$ .

**Lemma A.3.** For every  $\ell \in \mathbb{N}$  and  $x \geq 0$ ,  $T_{\ell}(x) = P_{\ell}^{+}(x) = P_{\ell}^{-}(x)$ .

*Proof.* For every  $\ell \in \mathbb{N}^*$  and every  $x > 0$ ,

$$(\ell + 1)T_{\ell+1}(x) - (2\ell + 1)(1 + 2x)T_{\ell}(x) + \ell(1 + 4x)T_{\ell-1}(x) = 0.$$

Indeed,

$$\begin{aligned} (2\ell + 1)(1 + 2x)T_{\ell}(x) &= (2\ell + 1) \sum_{r \in \mathbb{Z}} \left( \binom{\ell}{r} \binom{2r}{r} x^r + 2 \binom{\ell}{r} \binom{2r}{r} x^{r+1} \right); \\ &= \sum_{r \in \mathbb{Z}} \left( (2\ell + 1) \binom{\ell}{r} \binom{2r}{r} + 2(2\ell + 1) \binom{\ell}{r-1} \binom{2r-2}{r-1} \right) x^r; \end{aligned}$$

and

$$\begin{aligned} \ell(1 + 4x)T_{\ell-1}(x) &= \ell \sum_{r \in \mathbb{Z}} \left( \binom{\ell-1}{r} \binom{2r}{r} x^r + 4 \binom{\ell-1}{r} \binom{2r}{r} x^{r+1} \right) \\ &= \sum_{r \in \mathbb{Z}} \left( \ell \binom{\ell-1}{r} \binom{2r}{r} x^r + 4\ell \binom{\ell-1}{r-1} \binom{2r-2}{r-1} \right) x^r \end{aligned}$$

<sup>8</sup>in order to avoid heavy notation, the dependence on  $x$  shall be omitted



so, using Lemma A.2,

$$(2\ell + 1)(1 + 2x)T_\ell(x) - \ell(1 + 4x)T_{\ell-1}(x) = (\ell + 1) \sum_{r \in \mathbb{Z}} \binom{\ell + 1}{r} \binom{2r}{r} x^r = (\ell + 1)T_{\ell+1}(x).$$

But in the other hand,

$$(\ell + 1)P_{\ell+1}^+(x) - (2\ell + 1)(1 + 2x)P_\ell^+(x) + \ell(1 + 4x)P_{\ell-1}^+(x) = 0.$$

Indeed,

$$\begin{aligned} (2\ell + 1)(1 + 2x)P_\ell^+(x) &= (2\ell + 1)x(y_+ + y_-)x^\ell y_+^\ell \sum_{r \in \mathbb{Z}} \binom{\ell}{r}^2 y_-^{2r} \\ &= x^{\ell+1} y_+^{\ell+1} \sum_{r \in \mathbb{Z}} (2\ell + 1) \binom{\ell}{r}^2 (y_-^{2r} + y_-^{2r+2}) \\ &= x^{\ell+1} y_+^{\ell+1} \sum_{r \in \mathbb{Z}} (2\ell + 1) \left( \binom{\ell}{r}^2 + \binom{\ell}{r-1}^2 \right) y_-^{2r}, \end{aligned}$$

and

$$\begin{aligned} \ell(1 + 4x)P_{\ell-1}^+(x) &= \ell x^2 (y_+ - y_-)^2 x^{\ell-1} y_+^{\ell-1} \sum_{r \in \mathbb{Z}} \binom{\ell-1}{r}^2 y_-^{2r} \\ &= x^{\ell+1} y_+^{\ell+1} \sum_{r \in \mathbb{Z}} \ell \binom{\ell-1}{r}^2 (y_-^{2r} - 2y_-^{2r+2} + y_-^{2r+4}) \\ &= x^{\ell+1} y_+^{\ell+1} \sum_{r \in \mathbb{Z}} \ell \left( \binom{\ell-1}{r}^2 - 2\binom{\ell-1}{r-1}^2 + \binom{\ell-1}{r-2}^2 \right) y_-^{2r}, \end{aligned}$$

so, using Lemma A.2,

$$(2\ell + 1)(1 + 2x)P_\ell^+(x) - \ell(1 + 4x)P_{\ell-1}^+(x) = (\ell + 1)x^{\ell+1} y_+^{\ell+1} \sum_{r \in \mathbb{Z}} \binom{\ell+1}{r}^2 y_-^{2r} = (\ell + 1)P_{\ell+1}^+(x).$$

Finally,  $T_0(x) = 1 = P_0^+(x)$  and  $T_1(x) = 1 + 2x = x(y_+ + y_-) = P_1^+(x)$ , so  $T_\ell(x) = P_\ell^+(x)$  for every  $\ell \in \mathbb{N}$ .  $\square$

**Corollary A.4.** *For every  $x \geq 0$ , when  $\ell$  tends to infinity,*

$$T_\ell(x) \sim \frac{1}{2} \sqrt{\frac{1+4x}{\pi \ell x}} (1+4x)^\ell.$$

*Proof.* Applying Proposition A.1 in [Aud14] to Lemma A.3 (for  $\varepsilon = 1$ ), we get

$$T_\ell(x) \sim x^\ell y_+^\ell \frac{\left(1 + \frac{1}{y_+}\right)^{2\ell+1}}{2 \sqrt{\frac{\pi \ell}{y_+}}} = \frac{x^\ell}{2 \sqrt{\pi \ell}} \left( \sqrt{y_+} + \frac{1}{\sqrt{y_+}} \right)^{2\ell+1}$$

But, since  $y_+^{-1} = y_-$ ,  $\left( \sqrt{y_+} + \frac{1}{\sqrt{y_+}} \right)^2 = y_+ + 2 + y_- = \frac{1+4x}{x}$ , and the result follows.  $\square$

**Proposition A.5.** *Let  $a, b, b' > 0$ . For every  $\ell \in \mathbb{N}$ , we denote by  $c_\ell$  the constant term in  $(bt^{-1} + a + b't)^\ell$ . Then, when  $\ell$  tends to infinity,*

$$c_\ell \sim \frac{1}{2} \sqrt{\frac{a + 2\sqrt{bb'}}{\pi \ell \sqrt{bb'}}} (a + 2\sqrt{bb'})^\ell.$$

*Proof.* Following closely the arguments given in the proof of Proposition 4.1 in [Aud14], we begin by

$$\begin{aligned}
(bt^{-1} + a + b't)^\ell &= \left( (\sqrt{b}t^{-\frac{1}{2}} + \sqrt{b'}t^{\frac{1}{2}})^2 + a - 2\sqrt{bb'} \right)^\ell \\
&= \sum_{r=0}^{\ell} \binom{\ell}{r} (\sqrt{b}t^{-\frac{1}{2}} + \sqrt{b'}t^{\frac{1}{2}})^{2r} (a - 2\sqrt{bb'})^{\ell-r} \\
&= (a - 2\sqrt{bb'})^\ell \sum_{r=0}^{\ell} \binom{\ell}{r} \left( \frac{1}{a - 2\sqrt{bb'}} \right)^r \sum_{s=0}^{2r} \binom{2r}{s} \sqrt{b}^s \sqrt{b'}^{2r-s} t^{s-r} \\
&= (a - 2\sqrt{bb'})^\ell \sum_{r=0}^{\ell} \sum_{s=0}^{2r} \binom{\ell}{r} \binom{2r}{s} \left( \frac{b'}{a - 2\sqrt{bb'}} \right)^r \sqrt{\frac{b}{b'}}^s t^{s-r}.
\end{aligned}$$

In this sum, only the terms with  $s = r$  contribute to the constant term. We obtain hence that

$$c_\ell = (a - 2\sqrt{bb'})^\ell \sum_{r=0}^{\ell} \binom{\ell}{r} \binom{2r}{r} \left( \frac{\sqrt{bb'}}{a - 2\sqrt{bb'}} \right)^r = (a - 2\sqrt{bb'})^\ell T_\ell \left( \frac{\sqrt{bb'}}{a - 2\sqrt{bb'}} \right).$$

Using Lemma A.4, we obtain then

$$c_\ell \sim \frac{1}{2} \sqrt{\frac{1 + \frac{4\sqrt{bb'}}{a-2\sqrt{bb'}}}{\frac{\pi\ell\sqrt{bb'}}{a-2\sqrt{bb'}}}} \left( (a - 2\sqrt{bb'}) \left( 1 + \frac{4\sqrt{bb'}}{a - 2\sqrt{bb'}} \right) \right)^\ell = \frac{1}{2} \sqrt{\frac{a + 2\sqrt{bb'}}{\pi\ell\sqrt{bb'}}} (a + 2\sqrt{bb'})^\ell.$$

□

## APPENDIX B. LENGTH OF REDUCED TENSOR POWERS

Using the same technic than in its proof, Corollary 2.21 can be generalized to the case  $b \neq b'$ :

**Proposition B.1.** *Let  $C$  be the CSS code associated to  $\mathcal{C} := \mathbb{F}_2^{b\ell} \hookrightarrow \mathbb{F}_2^a \twoheadrightarrow \mathbb{F}_2^{b'\ell}$ , where  $a, b, b' \in \mathbb{N}$ . If  $\mathcal{C}$  and  $\mathcal{C}^*$  satisfy both the hypothesis of Lemma 2.7 for the same integers  $N, K \in \mathbb{N}^*$ , then for every  $\ell \in \mathbb{N}$ ,  $C^{\otimes \ell}$  is a CSS code with parameters  $\llbracket n_\ell, (a - b - b')^\ell, d_\ell, w_\ell \rrbracket$ , where  $\left(\frac{N}{K}\right)^\ell \leq d_\ell \leq d_1^\ell$ ,  $w_\ell \leq a\ell$  and*

$$n_\ell = \frac{2bb'(a - b - b')^\ell + (b^2 + b'^2 + (b + b')\sqrt{bb'}) (a + \sqrt{bb'})^\ell + (b^2 + b'^2 - (b + b')\sqrt{bb'}) (a - \sqrt{bb'})^\ell}{2(b^2 + bb' + b'^2)}.$$

*This provides a family logarithmically LDPC with a minimum distance which grows at least as the  $\frac{\log(a + \sqrt{bb'})}{\log N - \log K}$ -th power of the length.*

*Proof.* As for Corollary 2.21, only the statement on the length need some attention.

We define the sequences of integers  $(a_\ell)_{\ell \in \mathbb{N}^*}$ ,  $(b_\ell)_{\ell \in \mathbb{N}^*}$  and  $(b'_\ell)_{\ell \in \mathbb{N}^*}$  by  $\mathcal{C}_\ell =: \mathbb{F}_2^{b_\ell\ell} \hookrightarrow \mathbb{F}_2^{a_\ell\ell} \twoheadrightarrow \mathbb{F}_2^{b'_\ell\ell}$ . Developing  $\mathcal{C}_\ell \otimes \mathcal{C}$  as in the proof of Corollary 2.21, we obtain

$$\begin{pmatrix} b_{\ell+1} \\ a_{\ell+1} \\ b'_{\ell+1} \end{pmatrix} = \begin{pmatrix} a - b & b & 0 \\ b' & a & b \\ 0 & b' & a - b' \end{pmatrix} \begin{pmatrix} b_\ell \\ a_\ell \\ b'_\ell \end{pmatrix}.$$

It can be computed that

$$\begin{pmatrix} a - b & b & 0 \\ b' & a & b \\ 0 & b' & a - b' \end{pmatrix} = P \begin{pmatrix} a - \sqrt{bb'} & 0 & 0 \\ 0 & a - b - b' & 0 \\ 0 & 0 & a + \sqrt{bb'} \end{pmatrix} P^{-1}$$

with

$$P = \begin{pmatrix} -\sqrt{b} & b^2 & \sqrt{b} \\ \sqrt{b'} - \sqrt{b} & -bb' & \sqrt{b'} + \sqrt{b} \\ \sqrt{b'} & b'^2 & \sqrt{b'} \end{pmatrix}.$$

The result follows by considering the  $\ell$ -th power. □

AIX MARSEILLE UNIVERSITÉ, I2M, UMR 7373, 13453 MARSEILLE, FRANCE  
*E-mail address:* benjamin.audoux@univ-amu.fr

INRIA & LIX, UMR 7161, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU, FRANCE  
*E-mail address:* alain.couvreur@lix.polytechnique.fr