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Gaussian Correlation Inequalities for Ferromagnets*

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Various correlation inequalities are derived for families of random variables, $\{X_j\}$, arising in certain ferromagnetic models of statistical mechanics and quantum field theory. It is shown for example that $E(X_{j_1} \dots X_{j_{2m}}) \leqq E(Z_{j_1} \dots Z_{j_{2m}})$ for any $m, j_1, \dots, j_{2m}$, where the Z_j's are jointly Gaussian with zero means and $E(Z_i Z_j) = E(X_i X_j)$. The proof of these inequalities is based on combinatorial methods due to Kelly and Sherman. Some applications to Ising models and φ^4 quantum fields are given.

1. Introduction

In his early work on the law of the iterated logarithm [1], Hinčin used the inequality,

$$E[(\sum a_j X_j)^{2m}] \leqq \frac{(2m)!}{2^m m!} (E[(\sum a_j X_j)^2])^m \qquad (m=1, 2, \dots), \tag{1.1}$$

which he derived for a family, $\{X_j\}$, of independent Bernoulli random variables. In that situation, Hinčin's inequality may be equivalently stated as the fact that

$$E(X_{j_1} \dots X_{j_{2m}}) \leqq E(Z_{j_1} \dots Z_{j_{2m}}) \qquad \forall\, m, j_1, \dots, j_{2m}, \tag{1.2}$$

where the Z_j's are jointly Gaussian mean zero random variables with $E(Z_i Z_j) = E(X_i X_j)$ for all i, j.

Hinčin's inequality has not only found wide application in probability theory but has been used in many areas of analysis as well (e.g., in [2, 3]). Other inequalities, which are similar to (1.2) in that they bound correlations of arbitrary order by some function of the covariance, have lately been seriously applied to various models of statistical mechanics and quantum field theory [4, 5, 6, 7]. The random variables arising in these models are not independent but rather have a particular kind of dependence (describable as "ferromagnetic" and discussed in Sections 2 and 4 below) which is the key to the various correlation inequalities they satisfy.

It has recently been shown that the inequality (1.1) remains valid for these ferromagnetic models, providing that $a_j \geqq 0$ for all j [8, 9, 10]; one major result of *this* paper is that (1.2) remains valid as well. Since the X_j's are now dependent,

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(1.2) does not follow directly from (1.1); furthermore the proof of (1.1) (as given in [9]) was based on entire function techniques involving zeros of moment generating functions, techniques which seem entirely unsuitable for deriving (1.2). The methods used in this paper to obtain (1.2) and other related inequalities are combinatorial in nature and rely heavily on the graphical techniques employed by Kelly and Sherman in [11]. The Kelly-Sherman techniques apply directly only to spin-$\frac{1}{2}$ Ising models, the simplest ferromagnetic systems; however, all inequalities thus obtained then extend to more general systems by fairly simple limit arguments. The more general systems include those random fields arising in φ^4 quantum field models [12].

In Section 2 of this paper we define and discuss spin-$\frac{1}{2}$ Ising models and review some of the correlation inequalities previously known or conjectured for them; we also describe some of the applications of these inequalities in statistical mechanics and quantum field theory. Section 3 contains the statement and proofs of various correlation inequalities (including (1.2)) for spin-$\frac{1}{2}$ Ising models. In Section 4 we define general ferromagnetic families of random variables and show that all the inequalities of Sections 2 and 3 extend to them; we consider several examples, including general Ising models, and discuss the interesting unsolved problem of characterizing the class of ferromagnetic random variables. Section 5 concludes the paper with an application of our inequalities to an analysis of asymptotic independence in general Ising models which strengthens some previously obtained results [5, 7].

We have attempted to make this paper relatively self-contained; however, in the interests of brevity, this was not attempted for the several brief discussions concerning applications to quantum field models. For a general introduction to the probabilistic (Euclidean) approach to constructive quantum field theory, see [13] and many of the articles in [14]; for a more complete discussion of Ising models than we give in the following section, see [15]. We note finally that following the initial announcement of the results of this paper, Sylvester obtained partially alternate proofs of our inequalities which eliminate some of the combinatoric considerations of Section 3 (see the appendix of [16]).

2. Ising Models

We consider (finite) spin-$\frac{1}{2}$ Ising models with pair interactions in an external magnetic field. Such a model consists of a finite family of random variables, $\{X_j\colon j=1,\ldots,N\}$, whose joint probability distribution, ν, on $\mathbb{R}^N$ has the form

$$\nu(x_1,\ldots,x_N)=\frac{1}{Z}\exp(-\beta H(x_1,\ldots,x_N))\prod_{j=1}^{N}\rho_j(x_j), \tag{2.1}$$

with

$$H(x_1,\ldots,x_N)=-\sum_{i<j=1}^{N}J_{ij}x_ix_j-\sum_{j=1}^{N}h_jx_j, \tag{2.2}$$

$$\rho_j(x)=[\delta(x-1)+\delta(x+1)]/2, \tag{2.3}$$

and

$$Z = Z(\{\beta J_{ij}\}, \{\beta h_j\}) = \int_{\mathbb{R}^N} \exp(-\beta H(x_1, \ldots, x_N)) \prod_{j=1}^{N} d\rho_j(x_j). \tag{2.4}$$

The index j should be thought of as labelling atomic sites in some crystal lattice with X_j denoting the spin (i.e. atomic magnetization) of the j'th atom, J_{ij} the interaction strength between X_i and X_j, and h_j the external magnetic field strength at the j-th site; $H(x_1, \ldots, x_N)$ is the energy of the system when $X_j = x_j$ for each j and $\beta \geqq 0$ is proportional to the inverse temperature so that ρ defines the standard Gibbs canonical ensemble and Z is the standard partition function of classical statistical mechanics. When $J_{ij} \geqq 0$ for all i, j, the model is said to ferromagnetic and when $h_j \geqq 0$ for all j (resp. $h_j = 0$ for all j) the model is said to be in a positive (resp. zero) external field. Note that for $J_{ij} \equiv 0$ and $h_j \equiv 0$, the X_j's are simply independent Bernoulli random variables.

Generally the index j labels points in a physical d-dimensional lattice such as $\mathbb{Z}^d$ and J_{ij} is defined geometrically: for example, in the classical nearest neighbor Ising model, $J_{ij} = 0$ unless i and j label nearest neighbor points in $\mathbb{Z}^d$ in which case $J_{ij} = J$ independently of i, j. Ising models are of great importance in rigorous statistical mechanics in that they are the simplest systems which undergo phase transitions. Roughly speaking, phase transitions occur when some thermodynamic quantity $\left(\text{e.g., the magnetization per atom, } \mathrm{E}\left(\sum_{j=1}^{N} X_j\right)/N\right)$ exhibits nonanalyticity (e.g. as a function of h when $h_j = h$ $\forall j$). Such phenomena cannot take place in finite systems and it is consequently necessary to consider infinite Ising models obtained from finite models by means of a "thermodynamic limit" in which $N \to \infty$ (that is, the labelled set of lattice points tends to all of $\mathbb{Z}^d$).

Correlation inequalities have been one of the most useful tools for taking thermodynamic limits, proving the existence of phase transitions, and studying associated phenomena. We proceed to review some of the previously known correlation inequalities and their applications. Although some of the results stated below (in particular, (2.5) and (2.6)) are true for more general models, we will assume throughout that $\{X_j : j = 1, \ldots, N\}$ is a spin-$\frac{1}{2}$ Ising model whose joint distribution is given by (2.1) through (2.4) with $J_{ij} \geqq 0$ and $h_j \geqq 0$ for all i, j. In order to simplify notation we denote an expectation such as the left hand side of (1.2) by the symbol $\langle j_1 \ldots j_{2m} \rangle$.

The Griffiths-Kelly-Sherman inequalities [17, 11] apply for any choice of $m, n, j_1, \ldots, j_{m+n}$:

$$\langle j_1 \ldots j_m \rangle \geqq 0, \tag{2.5}$$

$$\langle j_1 \ldots j_{m+n} \rangle \geqq \langle j_1 \ldots j_m \rangle \langle j_{m+1} \ldots j_{m+n} \rangle. \tag{2.6}$$

These inequalities show that any correlation, $\langle j_1 \ldots j_m \rangle$, is a non-negative non-decreasing function of all the J_{ij}'s and h_j's and this monotonicity is instrumental in obtaining thermodynamic limits [17, 18]. In a translation invariant infinite Ising model, the magnetization per atom equals $E(X_i)$ for any i and consequently the magnetization is also monotonic in the J_{ij}'s. This latter fact has been applied to prove the existence of phase transitions for Ising models (such as nearest neighbor models in dimension greater than two) which are "more ferromagnetic" than models in which a phase transition was already known to occur (such as the nearest neighbor model in two dimensions) [17]. Other correlation inequalities similar to (2.5) and (2.6) have been discovered [19] and have important applications (e.g., [4]).

Many important correlation inequalities involve the Ursell functions (multidimensional cumulants), $U_n(j_1, \ldots, j_n)$. These are symmetric functions of their arguments and may be defined inductively by,

$$\langle j_1, \ldots, j_m \rangle = \sum_{k=1}^{m} \sum^{(k)} U(A_1^k) \ldots U(A_k^k) \tag{2.7}$$

where $\sum^{(k)}$ denotes the sum over partitions of $\{1, \ldots, m\}$ into k nonempty sets $\{A_1^k, \ldots, A_k^k\}$ and $U(\{i_1, \ldots, i_n\}) \equiv U_n(j_{i_1}, \ldots, j_{i_n})$. (2.5) and (2.6) for $m, n=1$ say that $U_1(j) \equiv \langle j \rangle \geqq 0$ and $U_2(j, k) \equiv \langle jk \rangle - \langle j \rangle \langle k \rangle \geqq 0$. The Griffiths-Hurst-Sherman inequality [20] states that

$$U_3(i, j, k) \equiv \langle ijk \rangle - \langle i \rangle \langle jk \rangle - \langle j \rangle \langle ik \rangle - \langle k \rangle \langle ij \rangle + 2 \langle i \rangle \langle j \rangle \langle k \rangle \leqq 0. \tag{2.8}$$

Since $U_3(i, j, k) = \partial^2 \langle i \rangle / \partial h_j \partial h_k$, the GHS inequality implies that when $h_j = h$ for all j, the magnetization is a concave function of h, the external field strength. We also have $U_3(i, j, k) = \partial U_2(i, j)/\partial h_k$ which leads to the monotonicity of various quantities, such as the correlation length in Ising Models [4] and the mass gap in φ^4 field theories [12], as a function of the external field.

The fourth Ursell function does not satisfy an inequality such as (2.8) for arbitrary $h_j \geqq 0$; this can be seen by direct calculation in an Ising model with $N=1$. It has been shown however that

$$\langle ijkl \rangle - \langle ij \rangle \langle kl \rangle - \langle ik \rangle \langle jl \rangle - \langle il \rangle \langle jk \rangle \leqq 0; \tag{2.9}$$

this applies for arbitrary $h_j \geqq 0$, and *when* $h_j \equiv 0$, the left hand side of (2.9) equals $U_4(i, j, k, l)$ since all correlations of odd order then vanish. (2.9) was first stated as such by Lebowitz [21]; it is actually contained in a stronger result used in the original proof of the GHS inequality (2.8) [20], and also follows from (2.8) by applying $\partial/\partial h_l$ at $h_j \equiv 0$. This inequality has already found several applications; for example, in quantum field models, it has been used to show the absence of certain bound states [22, 23] and to derive absolute bounds on physical coupling constants [24].

Following the discovery [25, 26] that in zero external field and for arbitrary non-negative λ_j's,

$$(-1)^{n+1} \sum U_{2n}(j_1, \ldots, j_{2n}) \lambda_{j_1} \ldots \lambda_{j_{2n}} \geqq 0 \tag{2.10}$$

where the sum is over all $j_1, \ldots, j_{2n} \in \{1, \ldots, N\}$, it was conjectured by the author (see also [5]) that in *zero external field* and for any n,

$$(-1)^{n+1} U_{2n}(j_1, \ldots, j_{2n}) \geqq 0. \tag{2.11}$$

(2.11) has recently been proven for $n=3$ [27, 16, 28]. An early application of these conjectured inequalities to the absence of bound states in field theory was given in [5], but these results have since been obtained in [7] through the use of other inequalities [21]; another derivation using the inequalities of this paper is given below in section 5.

Some of the inequalities proved in the following section may be considered as natural extensions of (2.9) (but different from those conjectured in (2.11)). It is proven in particular (see Theorem 3 below) that

$$\langle j_1 \dots j_{2m} \rangle \leqq \sum \langle \Lambda_1 \rangle \dots \langle \Lambda_m \rangle \tag{2.12}$$

where the sum is over all $(2m)!/2^m m!$ distinct partitions of $\{1, \dots, 2m\}$ into m pairs $\{\Lambda_1, \dots, \Lambda_m\}$ and where $\langle \{i, k\} \rangle$ denotes $\langle j_i j_k \rangle$. It is easily seen by direct calculation that for the correlations of a jointly Gaussian family of mean zero random variables, the inequality in (2.12) is replaced by equality (see for example, Nelson's article in [14]). We consequently call (2.12) a Gaussian correlation inequality and note that it is equivalent to the strong version of Hinčin's inequality, (1.2). Note that for $n > 1$, the inequality in (2.11) is also replaced by equality for Gaussian families.

3. Gaussian Correlation Inequalities

In this section we continue to consider only spin-$\frac{1}{2}$ Ising models, $\{X_j\}$, as defined by (2.1) through (2.4) with $J_{ij} \geqq 0$ and $h_j \geqq 0$ for all i, j. We will consider both normalized and unnormalized correlations:

$$\langle j_1 \dots j_m \rangle \equiv E(X_{j_1} \dots X_{j_m}), \tag{3.1}$$

and

$$\begin{aligned} (j_1 \dots j_m) &\equiv Z \langle j_1 \dots j_m \rangle \\ &= (\tfrac{1}{2})^N \sum_{x_1, \dots, x_N = \pm 1} x_{j_1} \dots x_{j_m} \exp\Big(\sum_{i<j} J_{ij} x_i x_j + \sum_j h_j x_j\Big); \end{aligned} \tag{3.2}$$

the β of (2.1) and (2.4) has been absorbed into $\{J_{ij}, h_j\}$ by setting it equal to one. We will also use the notation (Λ) for a set of natural numbers $\Lambda = \{i_1, \dots, i_n\}$ to denote $(j_{i_1} \dots j_{i_n})$ and similarly for $\langle \Lambda \rangle$; in particular $\langle \varnothing \rangle = 1$ and $(\varnothing) = Z$. $|\Lambda|$ will denote the cardinality of Λ.

In the following theorem we consider partitions of a set Λ into disjoint subsets $\{\Lambda_1, \Lambda_2, \dots, \Lambda_k\}$; such a partition will be called a *pair-partition* if, either $|\Lambda_i| = 2$ for all i (when $|\Lambda|$ is even) or $|\Lambda_j| = 1$ for a single j and $|\Lambda_i| = 2$ for all $i \neq j$ (when $|\Lambda|$ is odd).

A family $\mathscr{F}$ of partitions of Λ will be called *admissible* if every pair-partition of Λ is a refinement of some partition in $\mathscr{F}$. Expressions involving correlations are to be considered as functions of the $N(N-1)/2 + N$ variables $\{J_{ij}, h_k: i<j=1, \dots, N;\ k=1, \dots, N\}$; an inequality involving correlations will be said to apply *strongly* if it remains valid (in the region: $J_{ij} \geqq 0, h_k \geqq 0$ for all i, j, k) after the application of any multi-partial-derivative (mixed or unmixed) of the $\{J_{ij}, h_k\}$. For example, (2.5) (after multiplication by Z) and (2.6) (after multiplication by Z^2) apply strongly [11].

Theorem 1. *Let $n=1,2,\ldots$ be arbitrary and let $\mathscr{F}$ be a family of partitions of $\{1,\ldots,n\}$ into disjoint sets $\{A_1,A_2\}$. If $\mathscr{F}$ is admissible, then for any $j_1,\ldots,j_n\in\{1,\ldots,N\}$, the following inequality applies strongly:*

$$Z(j_1,\ldots,j_n)\equiv(\varnothing)(\{1,\ldots,n\})\leqq\sum_{\mathscr{F}}(A_1)(A_2) \tag{3.3}$$

where the sum is over all $\{A_1,A_2\}\in\mathscr{F}$.

Before giving the proof of Theorem 1, we make several remarks and present several theorems which are direct corollaries of Theorem 1.

Remark 1. The condition that $\mathscr{F}$ be admissible is necessary as well as sufficient, in the sense that given a nonadmissible $\mathscr{F}$, one can find an Ising model and a choice of $j_1,\ldots,j_n$ so that (3.3) is not satisfied: Without loss of generality we assume that the pair-partition $\{\{1,2\},\{3,4\},\ldots\}$ is not a refinement of any partition of $\mathscr{F}$ and we then take $j_1=j_2=1$, $j_3=j_4=3,\ldots$. We choose an Ising model in which $J_{ij}\equiv 0$ and either $h_j\equiv 0$ (for n even) or $h_n>0$ and all other h_j's vanish (for n odd); with this choice, the left hand side of (3.3) is positive while the right hand side vanishes.

Theorem 2. *Let $n=1,2,\ldots$ be arbitrary; then for any $j_1,\ldots,j_n$, the following inequality applies strongly:*

$$(\varnothing)(\{1,\ldots,n\})\leqq\sum_{k=2}^{n}(\{1,k\})(\{1,\ldots,n\}\setminus\{1,k\}) \quad \textit{for } n \textit{ even}, \tag{3.4}$$

$$(\varnothing)(\{1,\ldots,n\})\leqq(\{1\})(\{2,\ldots,n\})+\sum_{k=2}^{n}(\{1,k\})(\{1,\ldots,n\}\setminus\{1,k\}) \tag{3.5}$$

for n odd.

Proof. It is easily seen that the family $\mathscr{F}$ of partitions of $\{1,\ldots,n\}$ appropriate to (3.4) or (3.5) is admissible.

Remark 2. (3.4) is a Gaussian correlation inequality in the sense that after division by $(\varnothing)^2$ it would be equality for the correlations of a jointly Gaussian family of mean zero random variables. This either can be seen directly or by noting how (3.4) is used in the proof of Theorem 3 below. When $n=2$, (3.4) is trivial while for $n=4$, it is equivalent to (2.9); when $n=3$, (3.5) is weaker than the GHS inequality (2.8).

Theorem 3. *Let $m=1,2,\ldots$ and $n=2m$ or $2m-1$; then for any $j_1,\ldots,j_n$, the following inequality applies strongly:*

$$(\varnothing)^{m-1}(\{1,\ldots,n\})\leqq\sum(A_1)\ldots(A_m) \tag{3.6}$$

where the summation is over all pair-partitions of $\{1,\ldots,n\}$. Thus for $n=2m$, and any jointly Gaussian family of mean zero random variables, $\{Z_j: j=1,\ldots,N\}$, with $E(Z_iZ_j)=\langle ij\rangle$ for all i,j, we have

$$0\leqq\langle j_1\ldots j_{2m}\rangle\leqq E(Z_{j_1}\ldots Z_{j_{2m}}). \tag{3.7}$$

Proof. For $m=1$, the inequality is trivial; we proceed by induction on m by assuming the result is known for $m=1,\ldots,k$. For $n=2(k+1)$ or $2(k+1)-1$ we apply (3.4) or (3.5) to $(\varnothing)^{k-1}\cdot(\varnothing)(\{1,\dots,n\})$ and then use the induction assumption for $m=k$ to obtain the desired result; we of course use here the fact that all multi-partial-derivatives of $(\varnothing)$ are nonnegative. The final statement of the theorem follows from a standard calculation for Gaussian families after dividing (3.6) by $(\varnothing)^m$ (as in Nelson's article in [14]).

The following theorem is yet another corollary to Theorem 1; it will be used in sections 4 and 5.

Theorem 4. *For arbitrary m,n and $j_1,\dots,j_{m+n}$, the following inequality holds strongly:*

$$\begin{aligned}0&\leqq(\varnothing)(\{1,\dots,m+n\})-(\{1,\dots,m\})(\{m+1,\dots,m+n\})\\&\leqq\sum_{i=1}^{m}\sum_{k=m+1}^{m+n}(\{i,k\})(\{1,\dots,m+n\}\setminus\{i,k\}).\end{aligned}\tag{3.8}$$

It follows that for $m+n$ even,

$$\begin{aligned}0&\leqq\langle\{1,\dots,m+n\}\rangle-\langle\{1,\dots,m\}\rangle\langle\{m+1,\dots,m+n\}\rangle\\&\leqq\min(m,n)\sum_P\langle\Lambda_1\rangle\dots\langle\Lambda_{(m+n)/2}\rangle,\end{aligned}\tag{3.9}$$

where the summation is over all pair-partitions, $P\equiv\{\Lambda_1,\dots,\Lambda_{(m+n)/2}\}$, of $\{1,\dots,m+n\}$ in which at least one of the Λ_i's contains one element each from $\{1,\dots,m\}$ and $\{m+1,\dots,m+n\}$; if m,n are even, then each pair-partition in the sum necessarily contains two such Λ_i's.

Proof. The left hand inequality of (3.8) is just the second GKS inequality (2.6) in its strong form [11] while the right hand inequality is a special case of Theorem 1 in which $\mathcal{F}$ consists of $\{\{1,\dots,m\},\{m+1,\dots,m+n\}\}$ together with all partitions of the form $\{\{i,k\},\{1,\dots,m+n\}\setminus\{i,k\}\}$ for $i\in\{1,\dots,m\}$ and $k\in\{m+1,\dots,m+n\}$; this family $\mathcal{F}$ is easily seen to be admissible. To obtain (3.9) we divide (3.8) by $(\varnothing)^2$ and then apply (3.6) (divided by $(\varnothing)^m$) to each term $\langle\{1,\dots,m+n\}\setminus\{i,k\}\rangle$ appearing on the right hand side. The resulting inequality is

$$\begin{aligned}0&\leqq\langle\{1,\dots,m+n\}\rangle-\langle\{1,\dots,m\}\rangle\langle\{m+1,\dots,m+n\}\rangle\\&\leqq\sum_P M_P\langle\Lambda_1\rangle\dots\langle\Lambda_{(m+n)/2}\rangle\end{aligned}\tag{3.10}$$

with the summation over the same pair partitions, $P\equiv\{\Lambda_1,\dots,\Lambda_{(m+n)/2}\}$, as in the summation of (3.9) and with M_P an integer counting the number of times a particular P appears. It is easily seen that M_P equals the number of Λ_i's in P which have one element each from $\{1,\dots,m\}$ and $\{m+1,\dots,m+n\}$; thus $M_P\leqq\min(m,n)$ which yields (3.9).

Remark 3. None of the right hand inequalities of (3.8), (3.9), and (3.10) (for $m,n>1$) is a Gaussian inequality in the sense discussed above; this is most easily seen by noting that for $m+n$ even, a Gaussian inequality would require elimination of the $\min(m,n)$ factor in (3.9) or equivalently the elimination of all the M_P's in (3.10) (it follows from Theorem 3 that this can be done for $\min(m,n)\leqq2$). We *conjecture* that this strengthened inequality is correct; in fact we make the more general *conjecture* that if $\mathcal{F}$ is a family of partitions $\{1,\dots,n\}$ into k disjoint (but possibly empty) sets $\{\Lambda_1, \ldots, \Lambda_k\}$, then the following inequality applies strongly whenever $\mathscr{F}$ is admissible:

$$(\varnothing)^{k-1} \quad (\{1, \ldots, n\}) \leqq \sum_{\mathscr{F}} (\Lambda_1) \ldots (\Lambda_k). \tag{3.11}$$

Before starting the proof of Theorem 1 proper, we discuss Griffiths' ghost spin method (see [20]) which will allow us to restrict consideration to the case of zero external field. For a positive field Ising model, $\{X_j: j=1, \ldots, N\}$, of the sort we have been discussing with given J_{ij}, h_j, we consider the related zero field Ising model $\{Y_j: j=0, \ldots, N\}$ with the same choice of J_{ij} (for $i \geqq 1$) and $J_{0j}=h_j$ (for $j=1, \ldots, N$). It follows by simple calculation that for any $n, j_1, \ldots, j_n$;

$$E(X_{j_1} \ldots X_{j_n}) = \begin{cases} E(Y_{j_1} \ldots Y_{j_n}) & \text{for } n \text{ even} \\ E(Y_0 Y_{j_1} \ldots Y_{j_n}) & \text{for } n \text{ odd} \end{cases}. \tag{3.12}$$

Our proof of Theorem 1 is based heavily on the techniques of Kelly and Sherman [11] (see also [29]).

Proof of Theorem 1. We first note that by Griffiths' ghost spin method and the definition of admissibility (and of pair-partition for odd cardinality sets), it suffices to consider zero field Ising models with n even. We next note that we may assume without loss of generality that the j_i's are all distinct; this follows from the facts that $(j_1 \ldots j_{n-2} j j) = (j_1 \ldots j_{n-2})$, that $(\Lambda_i) \geqq 0$ strongly for each Λ_i, and that if $\mathscr{F} = \{\{\Lambda_1, \Lambda_2\}\}$ is an admissible family of partitions of $(1, \ldots, n\}$ we obtain an admissible family of partitions of $\{1, \ldots, n-2\}$ by taking $\{\{\tilde{\Lambda}_1, \tilde{\Lambda}_2\}\}$ where $\tilde{\Lambda}_i = \Lambda_i \cap \{1, \ldots, n-2\}$. By relabelling indices, we may now assume that $j_i = i$; it then suffices to show that for any zero field Ising model of the type we have been discussing and any $n = 2m \leqq N$, the inequality

$$(\varnothing)(1 \ldots n) \leqq \sum_{\mathscr{F}} (\Lambda_1)(\Lambda_2) \tag{3.13}$$

applies strongly providing that $\mathscr{F}$ is an admissible family of partitions of $\{1, \ldots, n\}$ into two disjoint sets $\{\Lambda_1, \Lambda_2\}$ with $|\Lambda_1|$ and $|\Lambda_2|$ even.

Since by (3.2), each side of (3.13) is an entire function of the $\{J_{ij}\}$; it suffices to show that each pair of multi-Taylor coefficients satisfies the corresponding inequality in order to conclude that (3.13) applies strongly. We denote multi-indices by $\Gamma = \{m_{ij}\}$ (with $1 \leqq i < j \leqq N$ and each $m_{ij} = 0, 1, 2, \ldots$) and let $J^\Gamma = \prod_{i<j} (J_{ij})^{m_{ij}}$, and $\Gamma! = \prod_{i<j} (m_{ij})!$; we then have for Λ, Λ' subsets of $\{1, \ldots, N\}$:

$$(\Lambda) = \sum_{\Gamma} c_\Gamma(\Lambda) J^\Gamma / \Gamma!, \tag{3.14}$$

$$(\Lambda)(\Lambda') = \sum_{\Gamma} c_\Gamma(\Lambda, \Lambda') J^\Gamma / \Gamma!. \tag{3.15}$$

It remains to prove that for each Γ,

$$c_\Gamma(\varnothing, \{1, \ldots, n\}) \leqq \sum_{\mathscr{F}} c_\Gamma(\Lambda_1, \Lambda_2). \tag{3.16}$$

It is convenient to think of each Γ as an undirected graph with $\{1, \ldots, N\}$ as vertices and m_{ij} edges between vertices i and j; we define $\partial\Gamma$, the boundary of Γ, as the set of vertices with an odd number of incident edges: i.e.,

$$\partial\Gamma = \{k\colon \sum_{j<k} m_{jk} + \sum_{j>k} m_{kj} \text{ is odd}\}.$$

Now an expansion of the exponential in (3.2) (with $h_j \equiv 0$) directly yields that

$$c_\Gamma(A) = \begin{cases} 1 & \text{if } \partial\Gamma = A \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

Since we of course have that

$$c_\Gamma(A_1, A_2) = \sum_{\Gamma_1, \Gamma_2} \frac{\Gamma!}{\Gamma_1!\,\Gamma_2!} c_{\Gamma_1}(A_1)\, c_{\Gamma_2}(A_2) \tag{3.18}$$

where the sum is over all Γ_1, Γ_2 such that $\Gamma_1 + \Gamma_2 = \Gamma$ (i.e. such that $J^{\Gamma_1} J^{\Gamma_2} = J^\Gamma$), it follows that for A_1 and A_2 disjoint, $c_\Gamma(A_1, A_2) = 0$ unless $\partial\Gamma = A_1 \cup A_2$. We may thus assume in (3.16) that $\partial\Gamma = \{1, \ldots, n\}$.

We are now dealing with a purely graphical problem and we first prove (3.16) for simple graphs (i.e., graphs with $\Gamma! = 1$). If Γ has no multiple edges then (with the help of (3.17)) we may simplify (3.18) (for $\{A_1, A_2\}$ a disjoint partition of $\{1, \ldots, n\} = \partial\Gamma$) to

$$c_\Gamma(A_1, A_2) = |\{\Gamma_1 \subseteq \Gamma\colon \partial\Gamma_1 = A_1\}| = |\{\Gamma_2 \subseteq \Gamma\colon \partial\Gamma_2 = A_2\}|, \tag{3.19}$$

where $|A|$ denotes the cardinality of A and where we think of a simple graph Γ topologically as the union of its edges. Since (3.19) also applies with (A_1, A_2) replaced by $(\varnothing, \{1, \ldots, n\})$; we may rewrite (3.16) as

$$|\{\tilde{\Gamma} \subseteq \Gamma\colon \partial\tilde{\Gamma} = \varnothing\}| \leqq \sum_{\mathscr{F}} |\{\Gamma_1 \subseteq \Gamma\colon \partial\Gamma_1 = A_1\}| \tag{3.20}$$

where the sum is over all $\{A_1, A_2\} \in \mathscr{F}$ and where Γ is now any simple graph with $\partial\Gamma = \{1, \ldots, n\}$.

Following [11, 29], we consider the subgraphs of Γ as a group under the operation of symmetric difference: $\Gamma_1 \Delta \Gamma_2 = (\Gamma_1 \setminus \Gamma_2) \cup (\Gamma_2 \setminus \Gamma_1)$. In this group, $G_\varnothing = \{\tilde{\Gamma}\colon \partial\tilde{\Gamma} = \varnothing\}$ is a subgroup and $G_{A_1} = \{\Gamma_1\colon \partial\Gamma_1 = A_1\}$ is a coset of $G_\varnothing$; that is if G_{A_1} is nonempty, then $G_{A_1} = \{\tilde{\Gamma} \Delta \Gamma'\colon \tilde{\Gamma} \in G_\varnothing\}$ for some fixed $\Gamma' \in G_{A_1}$. We thus have the fact [11] that

$$|\{\Gamma_1 \subseteq \Gamma\colon \partial\Gamma_1 = A_1\}| = |\{\tilde{\Gamma} \subseteq \Gamma\colon \partial\tilde{\Gamma} = \varnothing\}| \text{ or } 0. \tag{3.21}$$

(3.20) will be obtained as soon as we see that for at least one $\{A_1, A_2\} \in \mathscr{F}$ there exists a $\Gamma_1 \subseteq \Gamma$ with $\partial\Gamma_1 = A_1$. This follows from the admissibility of $\mathscr{F}$: by choosing paths between boundary vertices in the same connected component, Γ may be partitioned into $n/2$ pieces each containing exactly two boundary vertices; we then choose that $\{A_1, A_2\}$ whose refinement gives the same partitioning of $\partial\Gamma$ and recombine some of the $n/2$ pieces to give a subgraph of Γ whose boundary is A_1.

It only remains to show that (3.16) is true even when Γ is not simple. We do this by showing that there is a related simple graph Γ_s such that $c_{\Gamma_s}(A, A') = c_\Gamma(A, A')$ for any disjoint partition $\{A, A'\}$ of $\partial\Gamma$. Suppose that $m_{12} = M > 1$; then we convert Γ into $\bar{\Gamma}$ by inserting M new vertices into the edges joining vertices 1 and 2 (i.e. we add new vertices $\{N+1, \ldots, N+M\}$ and replace the multiple edges joinging $\{1,2\}$ with simple edges joining $\{1, N+1\}, \ldots, \{1, N+M\}$, $\{2, N+1\}, \ldots$, and $\{2, N+M\}$); a simple calculation then yields that

$$\begin{aligned} c_{\bar{\Gamma}}(\Lambda_1, \Lambda_2) &= \sum_{\bar{\Gamma}_1, \bar{\Gamma}_2} \frac{\bar{\Gamma}!}{\bar{\Gamma}_1! \, \bar{\Gamma}_2!} \\ &= \sum_{\Gamma_1, \Gamma_2} \frac{M!}{M_1! \, M_2!} \frac{\bar{\Gamma}!}{\bar{\Gamma}_1! \, \bar{\Gamma}_2!} = \sum_{\Gamma_1, \Gamma_2} \frac{\Gamma!}{\Gamma_1! \, \Gamma_2!} = c_\Gamma(\Lambda_1, \Lambda_2) \end{aligned} \tag{3.22}$$

where M_i is the number of edges joining $\{1,2\}$ in Γ_i; in the summations, we require $\bar{\Gamma}_1 + \bar{\Gamma}_2 = \bar{\Gamma}$, $\Gamma_1 + \Gamma_2 = \Gamma$ and $\partial \bar{\Gamma}_i = \partial \Gamma_i = \Lambda_i (i=1,2)$. By a finite chain of such alterations, we obtain Γ_s as desired which completes the proof of Theorem 1.

Before ending this section of the paper, we note that the proof of Theorem 1 can be used to obtain strong inequalities different than those in (3.3) which have the form:

$$\sum_{\mathscr{F}'} (\Lambda_1')(\Lambda_2') \leqq \sum_{\mathscr{F}} (\Lambda_1)(\Lambda_2). \tag{3.23}$$

We have in particular the following result.

Theorem 5. *For arbitrary i, j, k, l, the following inequality applies strongly:*

$$(\varnothing)(i\,j\,k\,l) + (i\,j)(k\,l) \geqq (i\,k)(j\,l) + (i\,l)(k\,j); \tag{3.24}$$

thus in a zero field Ising model,

$$0 \leqq -U_4(i,j,k,l) \leqq 2 \min\{\langle i j\rangle \langle k l\rangle, \langle i k\rangle \langle j l\rangle, \langle i l\rangle \langle j k\rangle\}. \tag{3.25}$$

Proof. The proof of Theorem 1 shows that it suffices to consider an arbitrary simple graph Γ with $\partial\Gamma = \{1,2,3,4\}$ and show that the sets $G_\Lambda \equiv \{\tilde{\Gamma} \subseteq \Gamma : \partial\tilde{\Gamma} = \Lambda\}$ satisfy

$$|G_\varnothing| + |G_{\{1,2\}}| \geqq |G_{\{1,3\}}| + |G_{\{1,4\}}|. \tag{3.26}$$

Moreover, as explained in the proof of Theorem 1, $|G_\Lambda| = 0$ or $|G_\varnothing|$ for any Λ; thus we need only show that the number of non vanishing terms on the left hand side of (3.26) is no less than the number on the right hand side. Now $|G_\varnothing| \neq 0$ (since $\varnothing \in G_\varnothing$) so we may assume without loss of generality that neither term on the right hand side of (3.26) vanishes; this is equivalent to assuming that vertices 3 and 4 belong to the same connected component of Γ as does vertex 1 which implies that vertex 2 also belongs to that component (since any component has an even number of boundary vertices) and consequently that $|\{G_{12}\}| \neq 0$ which completes the proof.

Remark 4. The proof of Theorem 5 clearly yields a sufficient condition on two families, $\mathscr{F}'$, $\mathscr{F}$, of partitions of $\{1, \ldots, 2m\}$ into two disjoint even sets which would imply the strong validity of (3.23): given any simple graph Γ with $\partial\Gamma = \{1, \ldots, 2m\}$, we require that $K_{\mathscr{F}'}(\Gamma) \leqq K_{\mathscr{F}}(\Gamma)$ where $K_{\mathscr{F}}(\Gamma)$ is the number of partitions $\{\Lambda_1, \Lambda_2\}$ in $\mathscr{F}$ for which there exists a disjoint partition of Γ, $\Gamma = \Gamma_1 \cup \Gamma_2$, with $\partial\Gamma_i = \Lambda_i$. This sufficient condition seems somewhat unwieldy except for the investigation of very specific correlation inequalities. To obtain more generality, one might consider families of partitions into at most k pieces and inequalities of the form

$$\sum_{\mathscr{F}'} (A'_1) \dots (A'_k) \leqq \sum_{\mathscr{F}} (A_1) \dots (A_k). \tag{3.27}$$

The discovery of necessary and sufficient conditions for such inequalities to be valid would presumably help establish both the conjectures (3.11) and (2.11).

4. General Ferromagnetic Systems

All the inequalities discussed in the last two sections apply to more general systems than the finite spin-$\frac{1}{2}$ Ising models defined in Section 2. Such systems include infinite Ising models, higher spin Ising models, and the random fields associated with certain quantum field models [30, 12]. The extension of spin-$\frac{1}{2}$ results to more general systems is generally based on Griffith's method of "anologue spin-$\frac{1}{2}$ systems" [30] together with simple limit arguments; we incorporate these ideas into a general definition of ferromagnetic systems. Our definition is closely related to the notion of "ferromagnetic limit distributions" discussed in [12].

We will call a finite set $\{Y_1 \dots Y_M\}$ of random variables with finite variances *positive field ferromagnetic* (or simply *ferromagnetic*) if there is a sequence of finite spin-$\frac{1}{2}$ Ising models, $\{X_j(n): j=1, \dots, N(n); n=1,2, \dots\}$ whose joint probability distributions are given by (2.1) through (2.4) with $\beta J_{ij}(n) \geqq 0$, $\beta h_j(n) \geqq 0$ (all i,j), such that for some choice of $\lambda^i_j(n) \geqq 0$ ($j=1, \dots, N(n)$; $i=1, \dots, M$), $\left\{Y_i(n) \equiv \sum_{j=1}^{N(n)} \lambda^i_j(n) X_j(n)\right\}$ converges weakly to $\{Y_i\}$ (i.e. the corresponding joint probability distributions converge weakly) and in addition $E(Y_i(n)\, Y_k(n)) \to E(Y_i\, Y_k)$ for all i,k as $n \to \infty$. We will call $\{Y_i\}$ *mean zero ferromagnetic* if each $\{X_j(n)\}$ can be chosen with $h_j(n) \equiv 0$; Theorem 9 below justifies this choice of terminology. An infinite set of random variables will be called ferromagnetic (resp. mean zero ferromagnetic) if each finite subset is ferromagnetic (resp. mean zero ferromagnetic); a random vector $\vec{Y} = (Y_1, \dots, Y_M)$ will be called ferromagnetic (resp. mean zero ferromagnetic) if $\{Y_1, \dots, Y_M\}$ is ferromagnetic (resp. mean zero ferromagnetic).

The following proposition is modelled after the original quantum field theoretic application of higher order correlation inequalities [6]; it explains why most spin-$\frac{1}{2}$ results extend to general ferromagnetic models.

Theorem 6. *Suppose $\{Y_j(n): j=1, \dots, M\}$ is ferromagnetic (resp. mean zero ferromagnetic) for each $n=1,2, \dots$, and for each j, $E(Y_j(n)^2)$ is bounded uniformly in n. If $\{Y_j(n)\}$ converges weakly to $\{Y_j: j=1, \dots, M\}$ as $n \to \infty$, it follows that $\{Y_j\}$ is ferromagnetic (resp. mean zero ferromagnetic), and that as $n \to \infty$,*

$$E\left(\exp\left(\sum_{j=1}^{M} z_j Y_j(n)\right)\right) \to E\left(\exp\left(\sum_{j=1}^{M} z_j Y_j\right)\right) \tag{4.1}$$

uniformly on compact subsets of $(z_1, \dots, z_M) \in \mathbb{C}^M$, and that

$$E(Y_{j_1}(n) \dots Y_{j_m}(n)) \to E(Y_{j_1} \dots Y_{j_m}) \tag{4.2}$$

for any $m, j_1, \ldots, j_m$. If we do not assume that $\{Y_j(n)\}$ converges weakly, it still follows that some subsequence $\{Y_j(n_k)\}$ converges weakly as $k \to \infty$ and thus that this subsequence satisfies the conclusions of the theorem.

Proof. We suppose first that the $Y_i(n)$'s are positive linear combinations of spins in a spin-$\frac{1}{2}$ Ising model as in the definition of ferromagnetic sets: $Y_i(n) = \sum_j \lambda_j^i(n) X_j(n)$. Now by the multilinearity of (2.8), (i.e. by considering $\sum_{i,j,k} U_3(i,j,k) \lambda_i \lambda_j \lambda_k$ for an appropriate choice of $\lambda_j \geqq 0$), it follows that for $a_i \geqq 0$, $Y_n \equiv \sum a_i Y_i(n)$ satisfies

$$\frac{d^3}{d\lambda^3} \log E(\exp(\lambda Y_n)) \leqq 0 \qquad \text{for } \lambda \geqq 0 \tag{4.3}$$

so that by elementary calculus and the fact that the Taylor coefficients of $\log E(\exp(\lambda Y_n))$ are just the cumulants of Y_n, we have (see [10])

$$\begin{aligned} E(\exp Y_n) &\leqq \exp\{E(Y_n) + \tfrac{1}{2}[E(Y_n^2) - (E(Y_n))^2]\} \\ &\leqq \exp\{(1 + E(Y_n^2))/2\} \end{aligned} \tag{4.4}$$

Now $E(Y_n^2) \leqq C(\sum a_i^2)$, with C independent of n and $\{a_i\}$, because of the uniform bound on $E(Y_j(n)^2)$. In addition, by (2.5),

$$E(Y_{j_1}(n) \ldots Y_{j_m}(n)) \geqq 0 \tag{4.5}$$

so that

$$\begin{aligned} E\{\exp(\textstyle\sum z_j Y_j(n))\} &\leqq E\{\exp(\textstyle\sum (\operatorname{Re} z_j) Y_j(n))\} \\ &\leqq E\{\exp(\textstyle\sum |\operatorname{Re} z_j| Y_j(n))\} \leqq \exp\{(1 + C \textstyle\sum |\operatorname{Re} z_j|^2)/2\} \\ &= O(\exp\{C \textstyle\sum |z_j|^2/2\}). \end{aligned} \tag{4.6}$$

Thus the left hand side of (4.1) is uniformly bounded in n and analytic on compact subsets of $\mathbb{C}^N$ so that all the results follow by standard analytic function arguments (e.g., see [9, 10]). We have also now shown that a general ferromagnetic family $\{Y_j(n)\}$ satisfies (4.4) through (4.6) so that the above argument carries through without the special assumption made at its beginning; this completes the proof.

For a ferromagnetic family $\{Y_j\}$ we use the same notation as in Section 2 and write $\langle j_1 \ldots j_m \rangle$ or $\langle \{1, \ldots, m\} \rangle$ for the right hand side of (4.2). We then have as a corollary to Theorem 6 that all (normalized) correlation inequalities discussed in the previous two sections still apply.

Theorem 7. *If $\{Y_j\}$ is a ferromagnetic family, then all the inequalities,* (2.5), (2.6), (2.8), (2.10), (2.11) *for $n \leqq 3$,* (3.3) *after division by $(\varnothing)^2$,* (3.4) *and* (3.5) *after division by $(\varnothing)^2$,* (3.6) *after division by $(\varnothing)^m$,* (3.7), (3.9), *and* (3.25), *apply.*

Proof. The multilinear nature of all these inequalities shows that they apply to the $\{Y_j(n)\}$ of the spin-$\frac{1}{2}$ approximation to $\{Y_j\}$; consequently by (4.2) they apply in the limit $n \to \infty$.

Remark 5. We also clearly have that $Y \equiv \sum a_j Y_j$ for $a_j \geqq 0$ satisfies an inequality such as (4.4). Besides inequalities, other spin-$\frac{1}{2}$ results extend to ferromagnetic families; in particular, the Lee-Yang theorem implies that for $a_j \geqq 0$, all the zeros of $E(\exp(zY))$ have nonpositive real part and that they are all pure imaginary if $\{Y_j\}$ is mean zero ferromagnetic. The Lee-Yang theorem has many important applications in statistical mechanics and field theory [15, 13]. For a more general context in which the Lee-Yang theorem applies, see [26].

Remark 6. Those inequalities which applied strongly to spin-$\frac{1}{2}$ models also apply strongly to general ferromagnetic models in the sense that if any multi-partial derivative is applied to the spin-$\frac{1}{2}$ inequality and the resulting inequality is divided by the appropriate power of $(\varnothing)$ to convert all (A)'s to $\langle A\rangle$'s, one finally obtains an inequality which applies to general ferromagnetic families. Equivalently, we may take a ferromagnetic family $\{X_j: j=1, \ldots, N\}$ with joint probability distribution v_0 and consider random variables $\{Y_j: j=1, \ldots, N\}$ whose joint distribution is $\tilde{Z}^{-1}\exp(-\beta H(x_1, \ldots, x_N))\, v_0$ where H is given by (2.2) and $\beta>0$ may be required to be small; $\{Y_j\}$ is a ferromagnetic family and will then satisfy the relevant inequalities strongly with $(\varnothing)$ replaced by $\tilde{Z}=E(\exp(-\beta H(X_1, \ldots, X_N)))$. This comment applies in particular to the general Ising models described below (following Theorem 10).

For the purposes of the next theorem, we call a function F (on $\mathbb{R}^N$) strongly positive if its multi-Taylor expansion about the origin is everywhere convergent and has all non-negative coefficients; if F is an nth degree multinomial, it will be called strongly positive of degree n. For simplicity, we restrict the following theorem to the mean zero case; $\operatorname{cov}(X_1, X_2)$ denotes $E(X_1 X_2)-E(X_1)E(X_2)$.

Theorem 8. *Suppose $\vec{Y}$ is a mean zero ferromagnetic random vector and $\vec{Z}$ is mean zero Gaussian with the same covariance as $\vec{Y}$; then for F and G strongly positive (of degrees m and n respectively),*

$$0 \leqq E(F(\vec{Y})) \leqq E(F(\vec{Z})) \tag{4.7}$$

and

$$0 \leqq \operatorname{cov}(F(\vec{Y}), G(\vec{Y})) \leqq K_{m,n} \operatorname{cov}(F(\vec{Z}), G(\vec{Z})) \tag{4.8}$$

where

$$K_{m,n}=\begin{cases} 1 & \textit{if } \min(m,n) \leqq 2, \\ \min(m,n) & \textit{otherwise} \end{cases}. \tag{4.9}$$

Proof. The result follows directly from (3.7), (3.9), and Remark 3.

Remark 7. If (3.11) were correct for all admissible $\mathscr{F}$, it would of course follow that (4.8) would remain valid without the $K_{m,n}$ factor and we conjecture that this is in fact the case. This conjectured inequality (as well as (4.7)) would then extend to nonpolynomial but strongly positive F, G.

The remainder of this section is primarily devoted to examples of ferromagnetic families and to the consideration of some interesting open problems. First we give a theorem which explains our choice of the term "mean zero" ferromagnetic.

Theorem 9. *A ferromagnetic family $\{Y_i\}$ is mean zero ferromagnetic if and only if $E(Y_i)=0$ for all i.*

Proof. Clearly, $E(Y_i)=0$ for all i, is a necessary condition for $\{Y_i\}$ to be mean zero ferromagnetic. Suppose now that $E(Y_i)=0$ for all i; then by Theorems 3 and 7, $E(Y_{j_1} \dots Y_{j_n})=0$ for any *odd* n and any $j_1, \dots, j_n$. We then apply Griffiths' ghost spin method (see (3.12)) to the spin-$\frac{1}{2}$ approximating families $\{Y_j(n)\}$ to obtain zero field approximating families $\{\tilde{Y}_j(n)\}$ all of whose even correlations converge to the even correlations of $\{Y_j\}$; since odd correlations vanish we conclude that all correlations of $\{\tilde{Y}_j(n)\}$ converge to those of $\{Y_j\}$. It then follows from Theorem 6 that $\{\tilde{Y}_j(n)\}$ converges weakly to $\{Y_j\}$ so that $\{Y_j\}$ is mean zero ferromagnetic as desired.

We now consider various examples of ferromagnetic families; our first result concerns multivariate normal distributions.

Theorem 10. *A jointly Gaussian family* $\{Z_j\}$ *is ferromagnetic if and only if* $E(Z_i) \geqq 0$ *and* $\operatorname{cov}(Z_i, Z_j) \geqq 0$ *for all* i, j.

Proof. It clearly follows from the GKS inequalities, (2.5) and (2.6), that these are necessary conditions. To see that the conditions are sufficient we first note that without loss of generality we may assume that the Z_i's have zero means; this is so since it easily follows from our definitions that, $\{Y_i\}$ ferromagnetic and $a_i \geqq 0$ imply that $\{Y_i + a_i\}$ is ferromagnetic. We then consider the $N \times N$ convariance matrix A with $A_{ij} = E(Z_i Z_j)$ and let $\vec{f}_i = A^{\frac{1}{2}} \vec{e}_i$ $(i=1, \dots, N)$ where $A^{\frac{1}{2}}$ is the positive square root of A and $\{\vec{e}_i\}$ is the standard basis for $\mathbb{R}^N$. Since $\vec{f}_i \cdot \vec{f}_j = A_{ij} \geqq 0$, there exists an orthogonal matrix R such that $\{\vec{v}_i = R\vec{f}_i\}$ is contained in the positive orthant of $\mathbb{R}^N$; i.e. $\vec{v}_i = (v_1^i, \dots, v_N^i)$ with $v_k^i \geqq 0$ for all i, k. We now consider the spin-$\frac{1}{2}$ model consisting of *independent* Bernoulli random variables $\{X_k : k=1, \dots, N\}$ and let $Y_i = \sum_{k=1}^{N} v_k^i X_k$; then $E(Y_i Y_j) = \vec{v}_i \cdot \vec{v}_j = \vec{f}_i \cdot \vec{f}_j = A_{ij} = E(Z_i Z_j)$ for $i, j = 1, \dots, N$. A simple application of the (multivariate) central limit theorem then shows that $\{Z_j : j=1, \dots, N\}$ is (mean zero) ferromagnetic. Since this is true for arbitrary N, the proof is complete.

We next consider general Ising models; these are finite families of random variables, $\{X_j : j=1, \dots, N\}$, whose joint probability distribution ν on $\mathbb{R}^N$ is given by (2.1), (2.2), and (2.4) but with general ρ_j (subject of course to the condition that $Z < \infty$). The assumption that $\beta J_{ij} \geqq 0$ and $\beta h_j \geqq 0$ for all i, j, is insufficient to imply that $\{X_j : j=1, \dots, N\}$ is ferromagnetic in our sense (the notion of a "general ferromagnet" used in [13, ch. 8] is quite different) unless each ρ_j is itself the probability distribution of some ferromagnetic random variable. If each ρ_j does satisfy this assumption then $\{X_j\}$ is ferromagnetic (with possibly extra assumptions needed on the smallness of β which would not be necessary if, for example, $\int \exp(bx^2) d\rho_j(x) < \infty$ for all $b > 0$ and each j); this can easily be derived from our definitions (see the second part of Remark 6). Note on the other hand that $\{X_j\}$ may be ferromagnetic even with $\beta J_{ij} < 0$ for some i, j. Such a situation occurs for $\{X_1, X_2, X_3\}$ jointly Gaussian with mean zero and covariance matrix

$$A = (A_{ij}) = (E(X_i X_j)) = \begin{bmatrix} 1 & \varepsilon & 0 \\ \varepsilon & 1 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix} \tag{4.10}$$

with $0 < \varepsilon < 1/\sqrt{2}$. It is easily determined that the joint distribution of $\{X_j\}$ has the form given in (2.1), (2.2), and (2.4) with $J_{1,3} = -(A^{-1})_{1,3} = -\varepsilon^2/(1-2\varepsilon^2) < 0$ even though $\{X_j\}$ is ferromagnetic by Theorem 10.

In order to obtain general Ising models which are ferromagnetic, it is necessary to first obtain ρ_j's which are the distributions of single ferromagnetic variables. We now give some examples of random variables X which are ferromagnetic (it is of course assumed that $E(X) \geqq 0$ in each case):

X assumes $(n+1)$ equally spaced values with equal probabilities, (4.11)

X is uniformly distributed on an interval, (4.12)

X has a probability density proportional to $\exp(-ax^4+bx^3+cx^2+dx)$ with $a>0$, $b \geqq 0$, arbitrary c, and $d \geqq -(b^3+4abc)/8a^2$. (4.13)

The random variables of (4.11) and (4.12), which arise in spin-$n/2$ Ising models and their limits, were shown to be ferromagnetic by Griffiths [30]. In (4.13), $X-b/4a$ has a density similar to X but with $b=0$ and $d \geqq 0$ and is therefore ferromagnetic by the results of Griffiths and Simon [12] (see [31] for an explanation in terms of large deviations in the central limit theorem); such random variables arise in φ^4 quantum field models as we now briefly explain (for a detailed explanation, see [14, 13, 18]).

Euclidean (quantum) field theory is concerned with the construction and detailed analysis of certain (generalized) random fields $\varphi(\vec{r})\,(\vec{r} \in \mathbb{R}^D)$. One method of constructing these random fields is to obtain them as limits of "lattice approximations" much as the Wiener process can be expressed as a limit of random walks. A (finite) lattice approximation consists of a finite collection of random variables $\{X_j\}$ which form a general Ising model as described above; typically, the measures ρ_j in (2.1) will have the form

$$d\rho_j/dx = \exp(-P_j(x)) \tag{4.14}$$

with each P_j a polynomial. For an (even) φ^4 quantum field model each P_j has the form $a_j x^4 - b_j x^2 - e_j$ with $a_j > 0$ so that by example (4.13), $\{X_j\}$ is ferromagnetic; the resulting random field is ferromagnetic in the sense that the collection of random variables $\{\int \varphi(\vec{r}) f(\vec{r})\, d\vec{r} \colon f \geqq 0\}$ is ferromagnetic. This result was proved in [12] and is of considerable importance; combined with Theorem 3 for example, it implies that the correlations (Schwinger functions) of an even φ^4 random field are bounded by the correlations of the related Gaussian (generalized free) field of the same covariance. It is presently not known whether a ferromagnetic random variable can have its probability density given by (4.14) with P_j a polynomial of degree greater than four; correspondingly, no quantum fields other than Gaussian and φ^4 models are known to be ferromagnetic.

The problem of classifying ferromagnetic random variables (even in the mean zero case) is unsolved; as we have seen, it has clear importance for the study of general Ising models and consequently for quantum field theory and may have interesting applications to other areas of probability theory as well. What is needed is a reasonable set of necessary and sufficient conditions (e.g., in terms of its probability distribution or moment generating function) for a random variable to be ferromagnetic. This problem appears analogous to (although presumably more difficult than) the classification of infinitely divisible random variables by the Lévy-Khintchine formula [32, Chap. XVII].

The Lee-Yang Theorem (see [9]) gives a necessary condition for X to be mean zero ferromagnetic; namely, that

$$E(\exp rX) = \exp(br^2) \prod_j (1 + r^2/\alpha_j^2) \tag{4.15}$$

for some $b \geqq 0$ and $\{\alpha_j \in \mathbb{R}\}$ with $\sum (1/\alpha_j^2) < \infty$. This condition is basically equivalent to the requirement that the left hand side of (4.15) be an entire function of r of exponential order two (and finite type) with only pure imaginary zeros. This condition is not sufficient since a random variable X with

$$E(e^{rX}) = (1 + r^2/\alpha^2)\, e^{r^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{rx} \left(1 + \frac{x^2 - 1}{\alpha^2}\right) e^{-x^2/2}\, dx \tag{4.16}$$

($\alpha \geqq 1$), satisfies (4.15) but not the GHS inequality (2.8); in fact the GHS inequality implies that $\{\alpha_j\}$ in (4.15) must be infinite. It is not known whether (4.15) together with the GHS and possibly other inequalities constitute sufficient conditions.

One way of obtaining new ferromagnetic random variables is by adding independent (known) ferromagnetic random variables; thus any X of the form $X = Z + \sum a_i Y_i$, with $\{Y_i\}$ independent Bernoulli random variables, Z a mean zero normal random variable independent of $\{Y_i\}$, and $\sum (a_i)^2 < \infty$, is mean zero ferromagnetic. Such random variables can easily be seen not to exhaust the ferromagnetic class by considering the distribution of α_j's in (4.15). Note first that $\{\pm i\alpha_j\}$ are exactly the zeros of $E(\exp rX)$ for complex r. Now for an X of the above form, $\alpha \in \{\alpha_j\}$ implies that $m\alpha \in \{\alpha_j\}$ for any odd m; however Y, defined as in (4.11) with the equally spaced values $\{k/n+1 \colon k = n, n-2, \ldots, -n\}$, has $\{\alpha_j\} = \{l\pi \colon l \text{ not divisible by } (n+1)\}$ so that for n even, Y cannot be a linear combination of independent Bernoulli and Gaussian random variables.

Before completing this section, we discuss a situation where it would be of considerable interest to show that a particular random variable is ferromagnetic. Let X be a random variable whose probability density is proportional to

$$\sum_{n=1}^{\infty} (4n^4 \pi^2 e^{9x/2} - 6n^2 \pi e^{5x/2}) \exp(-n^2 \pi e^{2x}); \tag{4.17}$$

the moment generating function of X satisfies [33, Chap. 10]

$$E(\exp(zX)) = E(\exp(-zX)) = \xi(z + \tfrac{1}{2})/\xi(\tfrac{1}{2}) \tag{4.18}$$

where

$$\xi(s) = \tfrac{1}{2} s(s-1)\, \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s) \tag{4.19}$$

and ζ is the Riemann zeta function. The conjecture that X is mean zero ferromagnetic is then by the Lee-Yang Theorem (i.e. by (4.15)) stronger than the Riemann hypothesis; this conjecture would also imply that X satisfies the GHS inequality (2.8). In terms of ξ, the GHS inequality would state that

$$\xi''' \xi^2 - 3\xi'' \xi' \xi + 2(\xi')^3 \leqq 0 \quad \text{for } s \geqq \tfrac{1}{2}. \tag{4.20}$$

Numerical calculations indicate that (4.20) is correct [34]; we ask: can this inequality be made rigorous? Are the other inequalities mentioned in Theorem 7 and (4.4) valid (when suitably rewritten in single variable versions) for the zeta function? The connection between the Riemann hypothesis and the Lee-Yang theorem is not a new one; for an interesting discussion, see Kac's comments on a paper of Pólya in [35, p. 424].

5. Asymptotic Independence

In this section we give an application of the correlation inequalities obtained in Section 3 to asymptotic independence estimates; we accordingly consider an infinite collection of random variables $\{X_i\}$ indexed by points in the d-dimensional lattice $\mathbb{Z}^d$. In the case of nearest neighbor Ising models, for example, it is known [36] that for sufficiently high temperature (sufficiently small β in (2.1)), X_i and X_j are exponentially asymptotically independent in the sense that

$$0 \leqq \langle ij\rangle - \langle i\rangle\langle j\rangle \equiv E(X_i X_j) - E(X_i)\,E(X_j) \leqq C\exp(-\|i-j\|/\xi) \tag{5.1}$$

for some choice of C and $\xi<\infty$, where $\|i-j\|$ denotes the Euclidean length of $(i-j)\in\mathbb{R}^d$. The following theorem estimates the rate of asymptotic independence for more general functions of $\{X_j\}$ in terms of the decay of the covariance. We use the notation of Section 3 and denote $\langle j_1 \dots j_n\rangle$ by $\langle\{1,\dots,n\}\rangle$; for Λ_1,Λ_2 finite sets of natural numbers we define $D(\Lambda_1,\Lambda_2)=\min\{\|j_l-j_m\|: j\in\Lambda_1, m\in\Lambda_2\}$.

Theorem 11. *Suppose $\{X_j: j\in\mathbb{Z}^d\}$ is mean zero ferromagnetic with $\langle jk\rangle \leqq F(\|j-k\|)$ for all $j,k\in\mathbb{Z}^d$ where F is a nonincreasing function on $[0,\infty)$; then for Λ_1,Λ_2 disjoint with cardinalities $|\Lambda_1|=m$, $|\Lambda_2|=n$, we have*

$$0\leqq\langle\Lambda_1\cup\Lambda_2\rangle-\langle\Lambda_1\rangle\langle\Lambda_2\rangle\leqq\begin{cases}C_{m,n}F(0)^{(m+n-2)/2}\,F(D(\Lambda_1,\Lambda_2)); & m,n \text{ odd}\\ C_{m,n}F(0)^{(m+n-4)/2}\,[F(D(\Lambda_1,\Lambda_2))]^2; & m,n \text{ even}\end{cases} \tag{5.2}$$

where $C_{m,n}$ is a universal constant depending only on m and n. In particular, if $F(r)=O(\exp(-r/\xi))$, then

$$\langle\Lambda_1\cup\Lambda_2\rangle-\langle\Lambda_1\rangle\langle\Lambda_2\rangle=\begin{cases}O(\exp(-D(\Lambda_1,\Lambda_2)/\xi)); & m,n \text{ odd}\\ O(\exp(-2D(\Lambda_1,\Lambda_2)/\xi)); & m,n \text{ even}.\end{cases} \tag{5.3}$$

Proof. By relabelling, we may assume $\Lambda_1=\{1,\dots,m\}$ and $\Lambda_2=\{m+1,\dots,m+n\}$; the theorem then follows as an immediate corollary to Theorem 4.

Remark 8. It was already known [4] (for considerably more general systems than covered by this theorem) that for arbitrary m and n, $\langle\Lambda_1\cup\Lambda_2\rangle-\langle\Lambda_1\rangle\langle\Lambda_2\rangle= O(\exp(-D(\Lambda_1,\Lambda_2)/\xi))$; we include the odd m and n case for the sake of completeness. The estimate (5.3) for even m and n was also previously known but only for somewhat more special systems: Spencer obtained it under the extra assumptions of translation invariance and a Markoff property for the translations [7]; such an estimate was first suggested by Feldman [5] who showed that it is a consequence of the (as yet) unproved inequalities of (2.11). This result is of considerable significance for φ^4 field theory models in that it yields information about the mass spectrum and the corresponding absence of bound states in the model;

for nearest neighbor Ising models it yields analogous information concerning the eigenvalue spectrum of the Markoff translation operators.

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