Lattice Gauge Theories and Spin Models

Manu Mathur* and T. P. Sreeraj†
S. N. Bose National Centre for Basic Sciences, Salt Lake, JD Block, Sector 3, Kolkata 700098, India

The Wegner Z_2 gauge theory-Z_2 Ising spin model duality in $(2+1)$ dimensions is revisited and derived through a series of canonical transformations. These Z_2 results are directly generalized to SU(N) lattice gauge theory in $(2+1)$ dimensions to obtain a dual SU(N) spin model in terms of the SU(N) magnetic fields and electric scalar potentials. The gauge-spin duality naturally leads to a new gauge invariant disorder operator for SU(N) lattice gauge theory. A variational ground state of the dual SU(2) spin model with only nearest neighbour interactions is constructed to analyze SU(2) lattice gauge theory.

I. INTRODUCTION

In 1971 Franz Wegner, using duality transformations, showed that in two space dimensions Z_2 lattice gauge theory can be exactly mapped into a Z_2 Ising model describing spin half magnets [1]. This is the earliest and the simplest example of the intriguing gauge-spin duality. Wegner's work, in turn, was strongly motivated by the self-duality of planar Ising spin model discovered by Kramers and Wannier 30 years earlier [2]. Such alternative dual descriptions have been extensively discussed in the past [3–10] as they are useful to understand theories and their phases at a deeper level . In particular, in the context of QCD, such duality transformations may connect the important and relevant degrees of freedoms at high, low energies providing a better picture of non-perturbative issues like color confinement and vacuum structure. In this work, the Wegner Z_2 gauge-spin duality ideas are revisited and obtained through a series of canonical transformations. The techniques are directly generalized to non-abelian lattice gauge theories to obtain the corresponding spin models. We also analyze SU(N) lattice gauge theories in terms of the resulting dual SU(N) spin operators.

In the simple context of Z_2 lattice gauge theory, the two essential or key features of the Wegner duality [1] are

- It eliminates all unphysical gauge degrees of freedom in Z_2 lattice gauge theory mapping it into Z_2 spin model with a Z_2 global symmetry. There are no Z_2 Gauss law constraints in the dual Z_2 spin model.

- It maps the interacting (non-interacting) terms in the Z_2 lattice gauge theory Hamiltonian into non-interacting (interacting) terms in the Z_2 spin model Hamiltonian resulting in the inversion of the coupling constant.

It is important to note that this gauge-spin duality is through the loop description of Z_2 lattice gauge theory. In fact, the dual Z_2 spin degrees of freedom are the original Z_2 plaquette-loop degrees of freedom which are gauge invariant, mutually independent as well as complete. As a consequence, the redundant or unphysical gauge degrees of freedom do not appear in the dual spin model. The dual spin model, in turn, has a global Z_2 symmetry which is physical and completely independent of the initial Z_2 gauge group. This discrete symmetry is spontaneously broken leading to a phase transition separating ferromagnetic or ordered phase from the paramagnetic or disordered phase. In fact, the initial motivation to study Z_2 gauge-spin duality was to get better understanding of the phase transition in Z_2 lattice gauge theory in terms of the order-disorder phase transitions in the dual spin model [1]. The Z_2 duality transformations show that the confinement and free phases of Z_2 lattice gauge theory correspond to the ordered and disordered phases of the spin model (see section II A 5). The dual formulation also leads to construction of a (non-local) disorder operator for Z_2 lattice gauge theory.

In this work, we show that the Z_2 gauge-spin duality as well as the Kramers-Wannier self-duality of planar Ising model can be constructively obtained though a series of canonical transformations. More importantly, these Z_2 gauge-spin duality transformations are directly generalized to SU(N) group to obtain a dual formulation of SU(N) lattice gauge theory in $(2+1)$ dimensions in terms of the SU(N) spins. Like in Z_2 lattice gauge theory, the SU(N) spins are the dual SU(N) electric potentials. In fact, in $(2+1)$ dimensions the dual SU(N) spin model is also a minimal loop description of SU(N) lattice gauge theory [22, 23] where the loops are chosen over every lattice plaquette. This is again exactly analogous to the Wegner Z_2 duality mentioned before because the Z_2 flux loops over the plaquettes are the Z_2 spin degrees of freedom. However, unlike Z_2 case, the global SU(N) invariance of the dual SU(N) spin model is not independent and corresponds to the global part of SU(N) gauge invariance. This residual global part of the gauge symmetry has to be fixed by hand (see section II B 3). We define a new disorder operator for the SU(N) lattice gauge theory in terms of the dual SU(N) spin operators. This is again a natural generalization of the Z_2 disorder operator to SU(N) lattice gauge theory (see section II A 5 and section II B 6). In fact, through-

*Electronic address: manu@bose.res.in
†Electronic address: sreerajtp@bose.res.in

out this work the SU(N) gauge theory duality results are discussed in the background of the corresponding (much simpler) Z_2 results. Many similar features between Z_2 and SU(N) dualities are emphasized and the differences are also identified and analyzed. This transition from Z_2 to SU(N) lattice gauge theory in the next section helps us understand the present approach to non-abelian duality better. It also makes the presentation simple and transparent.

We obtain Z_2 and SU(N) duality transformations through a series of canonical transformations. These canonical transformations in $(2+1)$ dimensions, also discussed in [22], lead to a dual description of gauge theories as they (a) solve the Gauss law constraints, (b) define the magnetic fields and its conjugate electric scalar potentials [3, 5] as the new fundamental conjugate pairs. In the Z_2 case, the canonical transformations convert all Z_2 conjugate pair operators (Z_2 electric fields, conjugate magnetic vector potentials) on links into the following two distinct and mutually independent classes of operators:

1. *Z_2 spin or plaquette loop operators*: representing the Z_2 magnetic fields and its conjugate electric scalar potentials over plaquettes (see Figure 4-a),

2. *Z_2 string operators*: representing the Z_2 electric fields and the Z_2 flux operators of the unphysical string degrees of freedom. These strings isolate all Z_2 gauge degrees of freedom (see Figure 4-b).

The first set, containing Z_2 plaquette loop operators, are all possible and mutually independent physical (gauge invariant) degrees of freedom of the Z_2 lattice gauge theory. The corresponding physical Hilbert space is denoted by $\mathcal{H}^p$. The second complimentary set, containing Z_2 string operators, represents all possible unphysical gauge degrees of freedom. As expected, all strings are frozen due to the Z_2 Gauss law constraints at lattice sites. This decoupling of strings leads to the Wegner Z_2 gauge-Z_2 spin duality within $\mathcal{H}^p$. *Note that no gauge fixing is needed at any stage.* The dual Z_2 electric scalar potential defines the disorder operator of the Z_2 lattice gauge theory (see section II A 5).

All the features of the canonical transformations mentioned above remain valid when the gauge group Z_2 is replaced by SU(N). In section II B, we show that this generalization is straightforward. As in Z_2 case, the SU(N) disorder operator is most naturally defined using dual SU(N) electric potential. Recently, these canonical transformations were used to solve Mandelstam constraints in SU(N) lattice gauge theories [22, 23]. In [22], we described SU(N) loop dynamics without any loop redundancy. In $(2+1)$ dimension these set of fundamental plaquette operators also define the dual spin operators as there is no local gauge invariance left in the dual description. The dual SU(N) spin Hamiltonian (unlike the dual Ising model in the Z_2 case) is non-local [22] and is briefly discussed in section II B 5. In this work, we further analyze this non-local SU(N) spin Hamiltonian. We show that for the low energy disordered states the non-local interactions can be ignored as a first approximation. This is because they have vanishing expectation values within these disordered states. A simple 'single spin' variational ground state of the dual SU(N) spin model is constructed. The Wilson loop in this ground state is shown to have area law behaviour.

The plan of the paper is as follows. The section II discusses gauge-spin duality through canonical transformations. In section II A, we discuss Z_2 gauge theory-Z_2 Ising model duality in 2+1 dimensions [1]. We obtain the old and well established results with canonical transformations as the new ingredients. The Kramers-Wannier self-duality in $(1+1)$ dimension is shown to be a simple consequence of these canonical transformations. To the best of our knowledge, a canonical transformation approach has not been used to establish these old and well known dualities. In section II B, these Z_2 canonical transformations are extended to SU(N) lattice gauge theories leading to a SU(N) spin model. To keep the presentation simple, the SU(N) discussions in section II B are exactly along the lines of Z_2 discussions in section II A. A comparative summary all Z_2 gauge-spin and SU(N) gauge-spin operators is given in Table 1. We then proceed to construct a new SU(N) disorder operator in terms of the dual scalar electric potential and the magnetic field operators. The section III is devoted to variational analyses of the truncated SU(N) spin model. In Appendix A, we give the steps involved in the iterative Z_2 canonical transformations. In Appendix B, we show that the 'single spin' variational state satisfies Wilson's area law. In Appendix C, the expectation value of the truncated dual spin Hamiltonian is computed in the variational ground state. It is also shown that the expectation value of the non-local part of the spin Hamiltonian in the above variational ground state vanishes.

Throughout this paper we use Hamiltonian formulation of lattice gauge theories [19] with open boundary conditions. We work in two space dimensions on a finite lattice Λ with $\mathcal{N}\,(=(\mathsf{N}+1)\times(\mathsf{N}+1))$ sites, $\mathcal{L}\,(=2\mathsf{N}(\mathsf{N}+1))$ links and $\mathcal{P}\,(=\mathsf{N}^2)$ plaquettes satisfying: $\mathcal{L}=\mathcal{P}+(\mathcal{N}-1)$. Sometimes, a lattice site is denoted by $(\vec{n})\equiv(m,n)$ with $m,n=0,1,\cdots,\mathsf{N}$, the links are denoted by (l) or by $(m,n;\ \hat{i})$ with $i=1,2$, the plaquettes are denoted by p,p' etc.. Any conjugate pair operator P,X will be denoted by $\{P;X\}$.

II. GAUGE-SPIN DUALITY AND CANONICAL TRANSFORMATIONS

A. Z_2 lattice gauge theory and Z_2 spin model

Z_2 (Z_N) lattice gauge theories and the corresponding spin models are the simplest theories having rich features. Due to their enormous simplicity compared to SU(N) lattice gauge theories and the presence of a confining phase, the Z_2 (Z_N) lattice gauge theories have been used as a

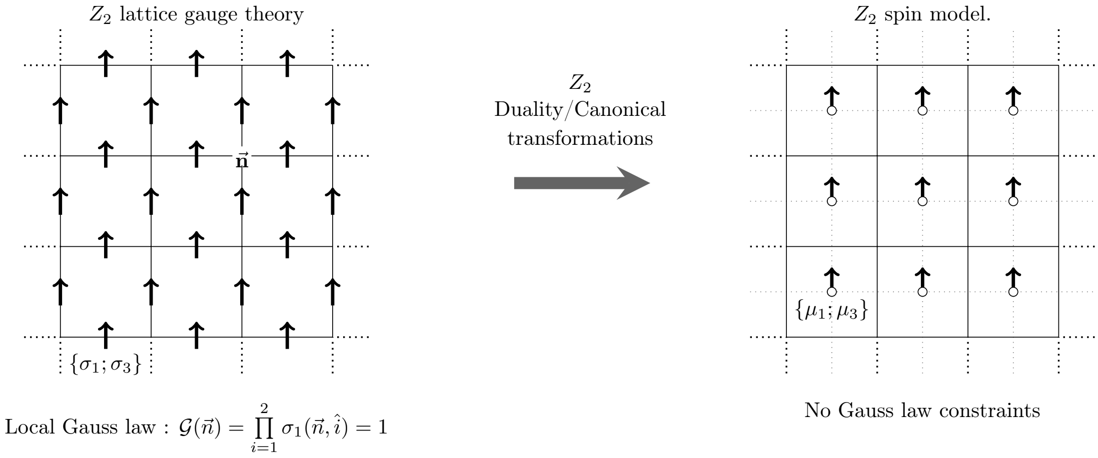

FIG. 1: Duality between Z_2 lattice gauge theory and Z_2 (Ising) spin model. The initial and the final conjugate pairs $\{\sigma_1;\sigma_3\}$ and $\{\mu_1;\mu_3\}$, are defined on the links and the plaquettes or dual sites respectively. The corresponding SU(N) duality is illustrated in Figure 6.

simple theoretical laboratory to test various confinement ideas in lattice gauge theories [10]. They also provide an explicit realization of the Wilson-'t Hooft's algebra of order and disorder operators characterizing different possible phases of the SU(N) gauge theories [4, 10]. In 1964, Schultz, Mattis and Lieb showed that the two-dimensional Z_2 Ising model is equivalent to a system of locally coupled Fermi- ons [11]. This result was later extended to Z_2 lattice gauge theory which also allows an equivalent description in terms of locally interacting fermions [12]. These are old and well known results. In the recent past, on the other hand, Z_2 (Z_N) lattice gauge theories have been useful to understand quantum spin models [13], quantum computations [14], tensor network or matrix product states [15] and their topological properties [16], cold atom simulations [17] and entanglement entropy [18]. In view of the above wide applications, the Z_2 (Z_N) lattice gauge theories and their duality studies are important in their own right. The Z_2 techniques developed in this section can also be easily generalized to Z_N gauge group.

The Z_2 lattice gauge theory involves Z_2 conjugate spin operators $\{\sigma_1(l);\sigma_3(l)\}$ on the link $l \in \Lambda$. The anti-commutation relations amongst these conjugate pairs on every link l are

$$\sigma_1(l)\ \sigma_3(l) + \sigma_3(l)\ \sigma_1(l) = 0. \tag{1}$$

They further satisfy: $\sigma_3(l)^2 = \sigma_1(l)^2 = 1$. In order to maintain a 1-1 correspondence with SU(N) lattice gauge theory (discussed in the next section), it is convenient to identify the conjugate pairs $\{\sigma_1(l);\sigma_3(l)\}$ with Z_2 electric field, $E(l)$ and Z_2 vector potential, $A(l)$ as:

$$\sigma_1(l) = e^{i\pi E(l)}, \qquad \sigma_3(l) = e^{iA(l)}. \tag{2}$$

Above $E(l) \equiv \{0,1\}$ and $A(l) \equiv \{0,\pi\}$. A basis of the two dimensional Hilbert space on each link l is chosen to be the eigenstates $|\pm,l\rangle$ of $\sigma_3(l)$ with eigenvalue ±1 with $\sigma_1(l)$ acting as a spin flip operator:

$$\sigma_3(l)\,|\pm,l\rangle = \pm\,|\pm,l\rangle\,, \qquad \sigma_1(l)|\pm,l\rangle = |\mp,l\rangle. \tag{3}$$

The Z_2 lattice gauge theory Hamiltonian is given by

$$H = -\sum_{l\in\Lambda}\sigma_1(l) - \lambda\sum_{p\in\Lambda}\sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4) \equiv H_E + \lambda H_B. \tag{4}$$

In (4), $\sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4)$ represents the product of σ_3 operators along the four links of a plaquette and the sum over p in the second term in (4) is over all plaquettes. The parameter λ is the Z_2 gauge theory coupling constant. The first term H_E and the second term H_B in (4) represent the Z_2 electric and magnetic field operators respectively. The electric field operator $\sigma_1(l)$ is fundamental while the latter is a composite of the Z_2 magnetic vector potential operators $\sigma_3(l)$ around a plaquette. After a series of canonical transformations, the above characterization of electric, magnetic field will be reversed. More explicitly, the dynamics will be described by the Hamiltonian (4) rewritten in terms of the fundamental magnetic field (the second term) and the electric field operator (the first term) will be composite of the dual electric scalar potentials (see (27) and (28)). The same feature will be repeated in the SU(N) case discussed in the next section.

The Hamiltonian (4) remains invariant if all 4 spins attached to the 4 links emanating from a site n are flipped simultaneously. This symmetry operation is implimented by the Gauss law operator $\mathcal{G}$:

$$\mathcal{G}(n) \equiv \prod_{l_n}\sigma_1(l_n) \tag{5}$$

at lattice site $n \in \Lambda$. In (5), $\prod_{l_n}$ represents the product over 4 links (denoted by l_n) which share the lattice site n

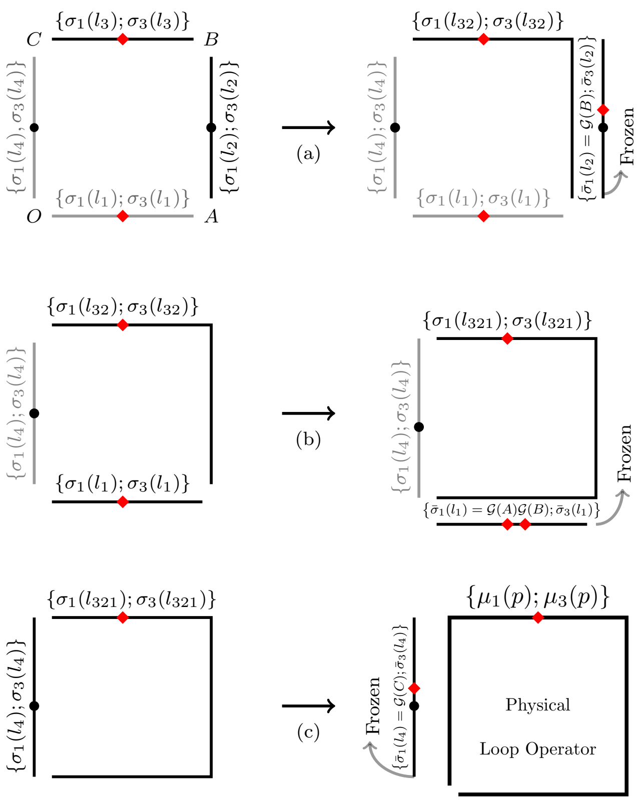

FIG. 2: The Z_2 canonical transformations (11), (13), (14a) and (14b) are pictorially illustrated in (a), (b) and (c) respectively. The ♦ and • represent the electric fields of the initial horizontal and vertical links respectively.

in two space dimensions. The Z_2 gauge transformations are

$$\begin{aligned}
\sigma_1(l) &\to \mathcal{G}^{-1}(n)\sigma_1(l)\mathcal{G}(n)=\sigma_1(l), \qquad \forall\, l\in\Lambda,\\
\sigma_3(l_n) &\to \mathcal{G}^{-1}(n)\sigma_3(l_n)\mathcal{G}(n)=-\sigma_3(l_n), \\
H &\to \mathcal{G}^{-1}(n)\ H\ \mathcal{G}(n)=H.
\end{aligned} \tag{6}$$

Thus, under a gauge transformation at site n, the 4 link flux operator $\sigma_3(l_n)$ on the 4 links l_n sharing the lattice site n change sign. All other $\sigma_3(l)$ remain invariant. The physical Hilbert space $\mathcal{H}^p$ consists of the states $|phys\rangle$ satisfying the Gauss law constraints:

$$\mathcal{G}(n)\,|phys\rangle = |phys\rangle \quad \text{or} \quad \mathcal{G}(n)\approx 1 \quad \forall n\in\Lambda. \tag{7}$$

In other words, $\mathcal{G}(n)$ are unit operators within the physical Hilbert space $\mathcal{H}^p$. All operator identities valid only within $\mathcal{H}^p$ are expressed by $\approx$ sign. We now canonically transform this simplest Z_2 gauge theory with constraints (5) at every lattice site into Z_2 spin model without any constraints as shown in Figure 1. To keep the discussion simple, we start with a single plaquette OABC shown in Fig. 2-a before dealing with the entire lattice. As the canonical transformations are iterative in nature, this simple example contains all the essential ingredients required to understand the finite lattice case . The four links OA, AB, BC, CO will be denoted by l_1,l_2,l_3,l_4 respectively. In this simplest case there are four Z_2 gauge transformation or equivalently Gauss law operators (5) at each of the four corners O, A, B and C:

$$\begin{aligned}
\mathcal{G}(O) &= \mathcal{G}(0,0)=\sigma_1(l_4)\sigma_1(l_1)\approx 1,\\
\mathcal{G}(A) &= \mathcal{G}(1,0)=\sigma_1(l_1)\sigma_1(l_2)\approx 1,\\
\mathcal{G}(B) &= \mathcal{G}(1,1)=\sigma_1(l_2)\sigma_1(l_3)\approx 1,\\
\mathcal{G}(C) &= \mathcal{G}(0,1)=\sigma_1(l_3)\sigma_1(l_4)\approx 1.
\end{aligned} \tag{8}$$

Note that these Gauss law operators satisfy a trivial operator identity:

$$\mathcal{G}(O)\ \mathcal{G}(A)\ \mathcal{G}(B)\ \mathcal{G}(C)\ \equiv\ 1. \tag{9}$$

The above identity states the obvious result that a simultaneous flippings at all 4 sites has no effect. This is because of the abelian nature of the gauge group. We now start with the four initial conjugate pairs on links l_1,l_2,l_3 and l_4:

$$\begin{aligned}
&\{\sigma_1(l_1);\sigma_3(l_1)\}, &&\{\sigma_1(l_2);\sigma_3(l_2)\},\\
&\{\sigma_1(l_3);\sigma_3(l_3)\}, &&\{\sigma_1(l_4);\sigma_3(l_4)\}.
\end{aligned} \tag{10}$$

Using canonical transformations we define four new but equivalent conjugate pairs. The first three string conjugate pairs:

$$\{\bar\sigma_1(l_1);\bar\sigma_3(l_1)\},\quad \{\bar\sigma_1(l_2);\bar\sigma_3(l_2)\},\quad \{\bar\sigma_1(l_4);\bar\sigma_3(l_4)\}$$

describe the collective excitations on the links OA, AB, BC and shown in Figures 2-b,a,c respectively. The remaining collective excitations over the plaquette or the loop $p\equiv OABC$ are described by

$$\{\mu_1(p);\mu_3(p)\}$$

and shown in Figure 2-c. As a consequence of the three mutually independent Gauss law constraints $\mathcal{G}(A),\mathcal{G}(B)$ and $\mathcal{G}(C)$, string electric fields are frozen to the value $+1$. Therefore, there is no dynamics associated with the three strings. In other words, string degrees of freedom completely decouple from $\mathcal{H}^p$. We are thus left with the final physical Z_2 spin operators $\{\mu_1(p);\mu_3(p)\}$ which are explicitly Z_2 gauge invariant. These duality transformations from gauge variant link operators to gauge invariant spin or loop operators are shown in Figure 1. To demonstrate the above results, we start with the initial link operators $\{\sigma_1(l_3);\sigma_3(l_3)\}$ and $\{\sigma_1(l_2);\sigma_3(l_2)\}$ as shown in Fig. (2)-a. We glue them canonically as follows:

$$\begin{aligned}
\bar\sigma_3(l_2) &\equiv \quad \sigma_3(l_2), &\sigma_3(l_{32}) &\equiv \sigma_3(l_3)\sigma_3(l_2)\\
\bar\sigma_1(l_2) &= \sigma_1(l_3)\sigma_1(l_2)\equiv \mathcal{G}(B), &\sigma_1(l_{32}) &= \sigma_1(l_3).
\end{aligned} \tag{11}$$

The canonical transformations (11) are illustrated in Fig. 2-a. After the transformations, the two new but equivalent canonical sets $\{\bar\sigma_1(l_2)=\mathcal{G}(B);\bar\sigma_3(l_2)\}$, $\{\sigma_1(l_{32});\sigma_3(l_{32})\}$ are attached to the links l_2 and $l_{32}\equiv$

$l_3 l_2$ respectively. They satisfy the same commutation relations as the original operators (1):

$$\bar{\sigma}_1(l_2)\bar{\sigma}_3(l_2) + \bar{\sigma}_3(l_2)\bar{\sigma}_1(l_2) = 0, \tag{12}$$
$$\sigma_1(l_{32})\sigma_3(l_{32}) + \sigma_3(l_{32})\sigma_1(l_{32}) = 0.$$

One can easily check: $\bar{\sigma}_1^2(l_2) = 1,\ \bar{\sigma}_3^2(l_2) = 1, \sigma_1(l_{32})^2 = 1, \sigma_3(l_{32})^2 = 1$. Further, note that the two conjugate pairs $\{\bar{\sigma}_1(l_2);\bar{\sigma}_3(l_2)\}$ and $\{\sigma_1(l_3l_2);\sigma_3(l_3l_2)\}$ are also mutually independent as they commute with each other. As an example, $[\bar{\sigma}_1(l_2), \sigma_3(l_3l_2)] \equiv [\sigma_1(l_3)\sigma_1(l_2), \sigma_3(l_3)\sigma_3(l_2)] = 0$. The new conjugate pair $\{\bar{\sigma}_1(l_2);\bar{\sigma}_3(l_2)\}$ is frozen due to the Gauss law at B: $\bar{\sigma}_1(l_2) = \mathcal{G}(B) \approx 1$ in $\mathcal{H}^p$. We now repeat (11) with l_2, l_3 replaced by l_1, l_{32} respectively to define new conjugate operators $\{\bar{\sigma}_1(l_1);\bar{\sigma}_3(l_1)\}$ and $\{\sigma_1(l_{321});\sigma_3(l_{321})\}$ attached to the links l_1 and $l_{321}(\equiv l_3l_2l_1)$ respectively:

$$\bar{\sigma}_3(l_1) \equiv \sigma_3(l_1), \qquad \sigma_3(l_{321}) \equiv \sigma_3(l_{32})\sigma_3(l_1) \tag{13}$$
$$\bar{\sigma}_1(l_1) = \sigma_1(l_{32})\sigma_1(l_1) = \mathcal{G}(A)\mathcal{G}(B), \quad \sigma_1(l_{321}) = \sigma_1(l_3).$$

As before, the new conjugate pair $\{\bar{\sigma}_1(l_1);\bar{\sigma}_3(l_2)\}$ becomes unphysical as $\bar{\sigma}_1(l_1) = \mathcal{G}(A)\mathcal{G}(B) \approx 1$ in $\mathcal{H}^p$. The last canonical transformations involve gluing the conjugate pairs $\{\sigma_1(l_{321});\sigma_3(l_{321})\}$ with $\{\sigma_1(l_4);\sigma_3(l_4)\}$ to define the dual and gauge invariant plaquette variables $\{\mu_1(p);\mu_3(p)\}$, with $p \equiv l_1l_2l_3l_4$:

$$\mu_1(p) \equiv \sigma_3(l_{321})\sigma_3(l_4) \equiv \sigma_3(l_3)\sigma_3(l_2)\sigma_3(l_1)\sigma_3(l_4).$$
$$\mu_3(p) \equiv \sigma_1(l_{321}) \ = \sigma_1(l_3), \tag{14a}$$

$$\bar{\sigma}_3(l_4) \equiv \sigma_3(l_4), \tag{14b}$$
$$\bar{\sigma}_1(l_4) = \sigma_1(l_{321})\sigma_1(l_4) = \sigma_1(l_3)\sigma_1(l_4) \equiv \mathcal{G}(C).$$

To summarize, the three canonical transformations (11), (13), (14a) and (14b) transform the initial four conjugate sets $\{\sigma_1(l_1);\sigma_3(l_1)\}$,$\{\sigma_1(l_2);\sigma_3(l_2)\}$,$\{\sigma_1(l_3);\sigma_3(l_3)\}$, $\{\sigma_1(l_4);\sigma_3(l_4)\}$ attached to the links l_1, l_2, l_3, l_4 to four new and equivalent canonical sets $\{\bar{\sigma}_1(l_2);\bar{\sigma}_3(l_2)\}$, $\{\bar{\sigma}_1(l_1);\bar{\sigma}_3(l_1)\}$, $\{\bar{\sigma}_1(l_4);\bar{\sigma}_3(l_4)\}$ and $\{\mu_1(p);\mu_3(p)\}$ attached to the links l_2, l_1, l_4 and the plaquette p respectively. The advantage of the new sets is that all the three independent Gauss law constraints at A, B and C are automatically solved. They freeze the three strings leaving us only with the physical spin or plaquette loop conjugate operators $\{\mu_1(p);\mu_3(p)\}$. The defining canonical relations (11), (13), (14a) and (14b) can also be inverted. The inverse transformations from the new spin flux operators to Z_2 link flux operators are

$$\sigma_3(l_1) = \bar{\sigma}_3(l_1), \quad \sigma_3(l_2) = \bar{\sigma}_3(l_2), \tag{15}$$
$$\sigma_3(l_3) = \mu_1(p)\bar{\sigma}_3(l_4)\bar{\sigma}_3(l_1)\bar{\sigma}_3(l_2), \quad \sigma_3(l_4) = \bar{\sigma}_3(l_4).$$

Similarly, the initial conjugate Z_2 electric field operators on the links are

$$\sigma_1(l_1) = \mu_3(p)\ \bar{\sigma}_1(l_1) = \mu_3(p)\ \mathcal{G}(A)\mathcal{G}(B) \approx \mu_3(p),$$
$$\sigma_1(l_2) = \mu_3(p)\ \bar{\sigma}_1(l_2) = \mu_3(p)\ \mathcal{G}(B) \approx \mu_3(p) \tag{16}$$
$$\sigma_1(l_3) = \mu_3(p),$$
$$\sigma_1(l_4) = \mu_3(p)\ \bar{\sigma}_1(l_4) = \mu_3(p)\ \mathcal{G}(C) \approx \mu_3(p).$$

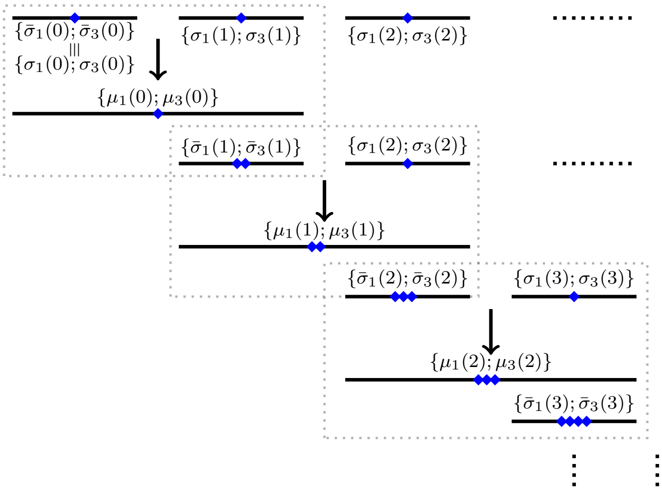

FIG. 3: Kramers-Wannier duality through canonical transformations. The first three steps of duality or canonical transformations (19) are illustrated.

Thus the complete set of gauge-spin duality relations over a plaquette and their inverses are given in (11), (13), (14a), (14b) and (15), (16) respectively. Note that the Gauss law constraint at the origin does not play any role as $\mathcal{G}(O) \approx \mathcal{G}(A)\ \mathcal{G}(B)\ \mathcal{G}(C)$. The total number of degrees of freedom also match. The initial Z_2 gauge theory had 4 spins with 3 Gauss law constraints. In the final dual spin model the 3 gauge non-invariant strings take care of the 3 Gauss law constraints leaving us with the single gauge invariant spin described by $\{\mu_1(p), \mu_3(p)\}$ on the plaquette p. The single plaquette Z_2 lattice gauge theory Hamiltonian (4) can now be rewritten in terms of the new gauge invariant spins as:

$$H \approx -4\ \mu_3(p) - \lambda\ \mu_1(p) = -\begin{pmatrix} \lambda & 4 \\ 4 & -\lambda \end{pmatrix}. \tag{17}$$

Note that the equivalence of the gauge and spin Hamiltonians (4) and (17) respectively is valid only within the physical Hilbert space $\mathcal{H}^p$. The two energy eigenvalues of H are $\epsilon_\pm = \pm 4\sqrt{\left(1 + \left(\frac{\lambda}{4}\right)^2\right)}$.

Before generalizing the above single plaquette results to the entire lattice, we apply canonical transformations to $(1+1)$ dimensional Ising model to get the Kramers-Wannier duality. The 1-d Ising model Hamiltonian is

$$H \ = \ \sum_{m=0}^{\infty}\Big[\sigma_1(m) - \lambda\ \sigma_3(m)\sigma_3(m+1)\Big]. \tag{18}$$

The Kramers-Wannier duality is obtained by the following iterative canonical transformations along a line:

$$\mu_1(m) \equiv \bar{\sigma}_3(m)\sigma_3(m+1),$$
$$\mu_3(m) = \bar{\sigma}_1(m) \tag{19}$$
$$\bar{\sigma}_3(m+1) = \sigma_3(m+1),$$
$$\bar{\sigma}_1(m+1) = \bar{\sigma}_1(m)\sigma_1(m+1) = \mu_3(m)\sigma_1(m+1).$$

In (19), the starting spin is defined as $\bar{\sigma}_3(m=0) \equiv \sigma_3(m=0)$. The above canonical transformations iteratively replace the conjugate pair $\{\sigma_1(m);\sigma_3(m)\}$ or equivalently $\{\bar{\sigma}_1(m);\bar{\sigma}_3(m)\}$ by a new conjugate pair $\{\mu_1(m);\mu_3(m)\}$. Unlike gauge theories, there are no spurious (string) degrees of freedom. This process is graphically illustrated in Figure 3. The above relations lead to $\mu_3(m) = \prod_{s=0}^{m}\sigma_1(s)$. Therefore, $\mu_3(m)\mu_3(m-1) = \sigma_1(m)$ with the convention $\mu_3(m=-1) \equiv 1$. The Ising model Hamiltonian can now be rewritten in its self-dual form in terms of the dual conjugate pairs $\{\mu_1(m);\mu_3(m)\}$:

$$H = \sum_{m=0}^{\infty}\Big[\mu_3(m)\mu_3(m+1) - \lambda\,\mu_1(m)\Big]. \qquad (20)$$

Having discussed the essential ideas, we now directly write down the general Z_2 gauge-spin duality or canonical relations over the entire lattice. The details of these iterative canonical transformations (analogous to (11), (13), (14a) and (14b)) are given in Appendix A. Note that there are $\mathcal{L}$ initial spins (one on every link) with $\mathcal{N}$ Gauss law constraints (one at every site) satisfying the identity:

$$\prod_{(m,n)\in\Lambda}\mathcal{G}(m,n) \equiv 1. \qquad (21)$$

The above identity again states that simultaneous flipping of all spins around every lattice site is an identity operator because each spin is flipped twice. As mentioned earlier, it is a property of all abelian gauge theories which reduces the number of Gauss law constraints from $\mathcal{N}$ to $(\mathcal{N}-1)$. In the non-abelian SU(N) case, discussed in the next section, there is no such reduction. The global SU(N) gauge transformations, corresponding to the extra Gauss law constraints at the origin $\mathcal{G}^a(0,0)=1$, need to be fixed by hand to get the correct number of physical degrees of freedom (see section II B 3). After canonical transformations in Z_2 lattice gauge theory, there are (a) $\mathcal{P}$ physical plaquette spins (analogous to $\{\mu_1(p);\mu_3(p)\}$ in the single plaquette case) shown in Figure 4-a and (b) $(\mathcal{N}-1)$ stringy spins (analogous to $\{\bar{\sigma}_1(l_1);\bar{\sigma}_3(l_1)\}$; $\{\bar{\sigma}_1(l_2);\bar{\sigma}_3(l_2)\}$ and $\{\bar{\sigma}_1(l_4);\bar{\sigma}_3(l_4)\}$ in the single plaquette case) as every lattice site away from the origin can be attached to a unique string. This is shown in Figure 4-b. The degrees of freedom before and after the canonical transformations match as $\mathcal{L} = \mathcal{P} + (\mathcal{N}-1)$. All $(\mathcal{N}-1)$ strings decouple because of the $(\mathcal{N}-1)$ Gauss law constraints. The algebraic details of these transformations leading to freezing of all strings are worked out in detail in Appendix A.

From now onward the $\mathcal{P}$ physical plaquette spin/loop operators are labelled by the top right corners of the corresponding plaquettes as shown in Figure 4-a). The vertical (horizontal) stringy spin operators are labelled by the top (right) end points of the corresponding links as shown in Figure 4-b. The same notation will be used to label the dual SU(N) operators in section II B.

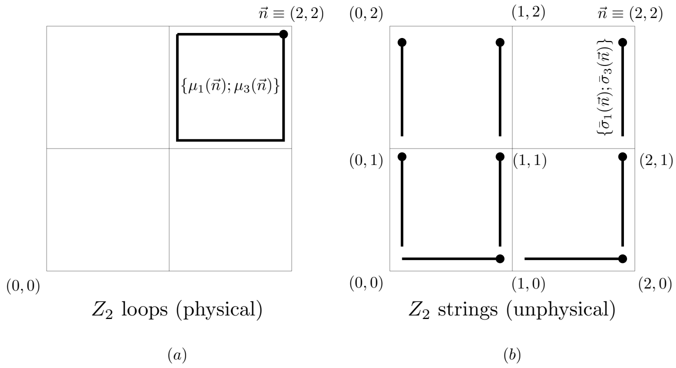

FIG. 4: The physical Z_2 spin conjugate pairs $\{\mu_1(\vec{n});\mu_3(\vec{n})\}$ and the unphysical string conjugate pairs $\{\bar{\sigma}_1(\vec{n});\bar{\sigma}_3(\vec{n})\}$ dual to Z_2 lattice gauge theory are shown in (a) and (b) respectively. We label the spin operators by their top right corners and the horizontal (vertical) string operators by their right (top) endpoint. The strings decouple from the physical Hilbert space as $\bar{\sigma}_1(\vec{n}) = \mathcal{G}(\vec{n}) \approx 1$ by Gauss law constraint at $\vec{n}$. The corresponding dual SU(N) spin and SU(N) string operators are shown in Figure 7-a,b respectively.

1. Physical sector and Z_2 dual potentials

The final duality relations between the initial conjugate sets $\{\sigma_1(m,n;\hat{i});\sigma_3(m,n;\hat{i})\}$ on every lattice link $(m,n;\hat{i})$ and the final physical conjugate loop operators $\{\mu_1(m,n);\,\mu_3(m,n)\}$ are (see Appendix A)

$$\begin{aligned}\mu_1(m,n) = \sigma_3(m-1,n-1;\hat{1})\;\sigma_3(m-1,n-1;\hat{2})\\ \sigma_3(m,n;-\hat{2})\sigma_3(m,n,-\hat{1})\end{aligned} \qquad (22a)$$

$$\mu_3(m,n) = \prod_{n'=n}^{N}\sigma_1(m-1,n',\hat{1}). \qquad (22b)$$

In (22a) we have defined $\sigma_1(m,n;-\hat{1}) \equiv \sigma_1(m-1,n;\hat{1})$ and $\sigma_1(m,n;-\hat{2}) \equiv \sigma_1(m,n-1;\hat{2})$. The relations (22a) and (22b) are the extension of the single plaquette relations (14a) to the entire lattice. They are illustrated in Figure 5-a. The canonical commutation relations are

$$\mu_1(m,n)\mu_3(m,n) + \mu_3(m,n)\mu_1(m,n) = 0. \qquad (23)$$

Further, $\mu_3(m,n)^2 = 1, \mu_1(m,n)^2 = 1$. *The canonical transformations (22a) are important as they define the magnetic field operators $\mu_1(m,n)$ as a new fundamental operator in the dual spin description.* Note that originally the electric field $\sigma_1(m,n)$ was fundamental and the magnetic field was written in terms of its conjugate magnetic vector potentials as $\sigma_3(l_1)(\sigma_3(l_2)(\sigma_3(l_3)(\sigma_3(l_4)$. After duality, the magnetic fields $\mu_1(m,n)$ are fundamental and their canonical conjugate, called electric scalar potential $\mu_3(m,n)$, define the electric field (see (27)).

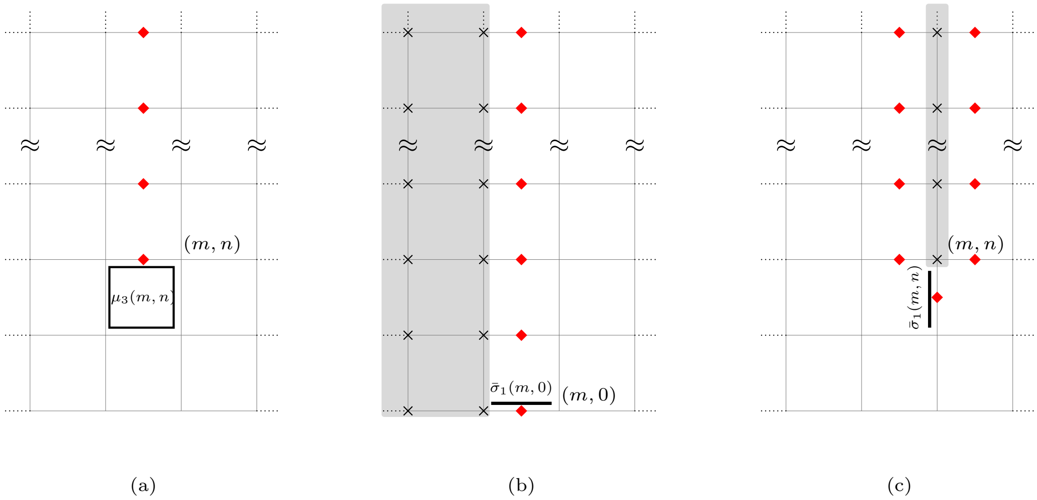

FIG. 5: The non-local relations in the Z_2 gauge-spin duality transformations: (a) We show the relations (22b) expressing $\mu_3(m,n)$ as the product of σ_1 operators which are denoted by ♦. In (b) and (c), we show the relations (24a) expressing $\bar\sigma_1(m,0)$ and $\bar\sigma_1(m,n)$; $n \neq 0$ respectively as the product of σ_1 operators denoted by ♦. These ($\bar\sigma_1(m,n)$) equal the product of Gauss law operators at sites marked by x in the shaded region. For the corresponding SU(N) relations, see Figures 8-a,b.

2. Unphysical sector and Z_2 string operators

The unphysical string conjugate pair operators are (see Appendix A)

$$
\begin{aligned}
\bar\sigma_3(m,0) &= \sigma_3(m-1,0,\hat{1}), \\
\bar\sigma_3(m,n) &= \sigma_3(m,n-1,\hat{2}); \quad n \neq 0
\end{aligned} \tag{24a}
$$

$$
\begin{aligned}
\bar\sigma_1(m,0) &= \prod_{m'=0}^{m-1}\prod_{n'=0}^{N} \mathcal{G}(m',n') \approx 1, \\
\bar\sigma_1(m,n) &= \prod_{n'=n}^{N} \mathcal{G}(m,n') \approx 1; \quad n \neq 0.
\end{aligned} \tag{24b}
$$

These Z_2 string operators are analogous to the three Z_2 string operators in (11), (13) and (14b) in the single plaquette case. The relations (24a) and (24b) are illustrated in Figure 5-b and Figure 5-c respectively. It is easy to see that in the full gauge theory Hilbert space $\bar\sigma_1(m,n)\bar\sigma_3(m,n) + \bar\sigma_3(m,n)\bar\sigma_1(m,n) = 0$ and different string operators located at different lattice sites commute with each others. As expected, their conjugate electric fields $\bar\sigma_1(m,n)$ are entirely in terms of the Z_2 Gauss law operators. Hence these string degrees of freedom are frozen and not dynamical. Further, one can check that all strings and plaquette operators are mutually independent and commute with each other:

$$
\begin{aligned}
&[\mu_3(m,n), \bar\sigma_1(m',n')] = 0, [\mu_3(m,n), \bar\sigma_3(m',n')] = 0, \\
&[\mu_1(m,n), \bar\sigma_1(m',n')] = 0, [\mu_1(m,n), \bar\sigma_3(m',n')] = 0.
\end{aligned} \tag{25}
$$

3. Inverse relations

The inverse relations for the flux operators over the entire lattice are

$$
\begin{aligned}
\sigma_3(m,0;\hat{1}) &= \bar\sigma_3(m+1,0), \\
\sigma_3(m,n;\hat{2}) &= \bar\sigma_3(m,n+1) \\
\sigma_3(m,n;\hat{1}) &= \left(\prod_{l=1}^{n}\bar\sigma_3(m,l)\right)\left(\prod_{q=1}^{n}\bar\sigma_3(m+1,q)\right) \\
&\qquad \left(\prod_{p=1}^{n}\mu_1(m+1,p)\right); \quad n \neq 0
\end{aligned} \tag{26}
$$

On the other hand, the conjugate electric field operators are

$$
\begin{aligned}
\sigma_1(m,n;\hat{1}) &= \mu_3(m,n)\mu_3(m,n+1), \\
\sigma_1(m,n,\hat{2}) &= \mu_3(m,n+1)\mu_3(m+1,n+1).
\end{aligned} \tag{27}
$$

In the second relation in (27), we have used Gauss laws at (m,l) ; $l = n+1, n+2, \cdots$. The above relations are analogous to the inverse relations (15) and (16) in the single plaquette case.

4. Z_2 dual dynamics

Therefore, within $\mathcal{H}^p$ where $\mathcal{G}(m,n) \approx 1$, the Z_2 lattice gauge theory Hamiltonian (4) in terms of the physical loop spin operators takes the simple nearest neighbour interaction form:

$$H = -\sum_{<p,p'>} \mu_3(p)\mu_3(p') - \lambda \sum_p \mu_1(p) \equiv H_E + \lambda H_B,$$
$$= \lambda\Big[-\sum_p \mu_1(p) - \frac{1}{\lambda} \sum_{<p,p'>} \mu_3(p)\mu_3(p')\Big] \qquad (28)$$

In (28) $\sum_{<p,p'>}$ denotes the sum over the nearest neighbour plaquettes. As expected, after the duality transformations the electric and the magnetic field descriptions in terms of potentials have interchanged. The original fundamental Z_2 electric field operator is now in terms of the (dual) electric scalar potential denoted by $\mu_3(p)$ and the Z_2 magnetic field has now acquired an independent status. Thus the Z_2 gauge theory initially written in terms of electric field and magnetic vector potential operators $\{\sigma_1(l); \sigma_3(l)\}$ in (4) is now written in terms of the magnetic field, electric scalar potential operators $\{\mu_1(p); \mu_3(p)\}$ in (28). Therefore, the duality or canonical transformations maps $(2+1)$ dimensional Z_2 lattice gauge theory at coupling λ to a $(2+1)$ dimensional Z_2 spin model at coupling $(1/\lambda)$, i.e,

$$H^{Z_2}_{gauge}(\lambda) \sim \lambda \; H^{Z_2}_{spin}(1/\lambda).$$

We have used $\sim$ above to emphasizes that this equivalence is only within the physical Hilbert space $\mathcal{H}^p$.

5. Z_2 order and disorder

The resulting Z_2 spin model (28) on an infinite lattice is invariant under the global Z_2 transformation:

$$\mu_1(p) \to \mu_1(p), \qquad \mu_3(p) \to -\mu_3(p), \qquad \forall p \in \Lambda. \qquad (29)$$

Its generator $G_\Lambda \equiv \prod_{p\in\Lambda} \mu_1(p)$ leaves the Hamiltonian (28) invariant: $G_\Lambda H G_\Lambda^{-1} = H$. Unlike the initial Z_2 gauge symmetry of Z_2 gauge theory, the global Z_2 symmetry of the dual spin model (28) is the symmetry of the spectrum. Being independent of gauge invariance, it allows the Ising spin model (28) to be magnetized through spontaneous symmetry breaking for $\lambda << 1$. As a consequence of canonical transformations or duality:

$$\Big\langle \mu_1(p) \Big\rangle_{H^{z_2}_{spin}(1/\lambda)} = \Big\langle \sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4) \Big\rangle_{H^{z_2}_{gauge}(\lambda)}$$
$$\Big\langle \mu_3(m,n) \Big\rangle_{H^{z_2}_{spin}(1/\lambda)} = \Big\langle \prod_{n'=n}^{N} \sigma_1(m,n') \Big\rangle_{H^{z_2}_{gauge}(\lambda)} \qquad (30)$$

The above two equations describe the relationship between order and disorder in the gauge and the dual spin system. Note that we always measure order or disorder with respect to the potentials. The first relation above states that at low temperature or large coupling $\lambda >> 1$, the gauge system is in ordered phase. This is because all magnetic vector potentials ($\sigma_3(l)$) are aligned (close to unity) leading to $\sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4) \approx 1$. This is the free phase of Z_2 gauge theory mentioned in the introduction. However, the dual spin system is now at high temperature. It is in the disordered phase as the dual electric scalar potential or the spin values $\mu_3(p) = \pm 1$ are equally probable. On the other hand, at small coupling ($\lambda << 1$), the spin system is ordered with all electric scalar potentials aligned to the value $\mu_3(p) = +1$ or -1. The gauge system is now disordered as the two values of the magnetic vector potentials $\sigma_3(l) = \pm 1$ are equally probable. This is the confining phase with the Z_2 Wilson loop $W_{[\mathcal{C}]}$ around a closed curve $\mathcal{C}$ following the area law:

$$W_{[\mathcal{C}]} \equiv \prod_{l\in\mathcal{C}} \sigma_3(l) \approx exp\left\{-k(\lambda)\; Area(\mathcal{C})\right\}. \qquad (31)$$

The disorder in the gauge system is the order in the dual spin system which is measured by the expectation value of electric scalar potential $\mu_3(p)$. It is a (non-local) product of the original link electric fields which flip the magnetic vector potentials $\sigma_3(l)$ along an infinite path. This is shown in Figure 5-a. For latter convenience, the Z_2 disorder operator is written as:

$$\Sigma(m,n) \equiv \mu_3(m,n) = \prod_{n'=n}^{N} \sigma_1(m,n'). \qquad (32)$$

We further define $\mu_3(m,n) \equiv e^{i\pi\mathcal{E}(m,n)}$. Using (2), we get:

$$\Sigma(m,n) = exp\; i\Big(\pi\, \mathcal{E}(m,n)\Big)$$
$$= exp\; i\Big(\pi \sum_{n'=n}^{N} E(m,n';\hat{1})\Big). \qquad (33)$$

We will generalize (32) or equivalently (33) to SU(N) lattice gauge theory to define a SU(N) disorder operator in section II B 6 after making SU(N) duality transformations in the next section.

B. SU(N) lattice gauge theory and SU(N) spin model

In this section, we construct SU(N) spin model which is dual to SU(N) lattice gauge theory. In fact, dual approaches to abelian and non-abelian lattice gauge theories have been extensively discussed in the past [3–10, 20, 21]. Many of these studies are in the path integral approach and involve abelian gauge groups. The purpose and the motivation is to make the compactness of the abelian and non-abelian gauge groups manifest in the form of topological (magnetic monopoles) degrees of freedom. However, our aim here is to show that SU(N) lattice gauge theory can also be systematically dualized

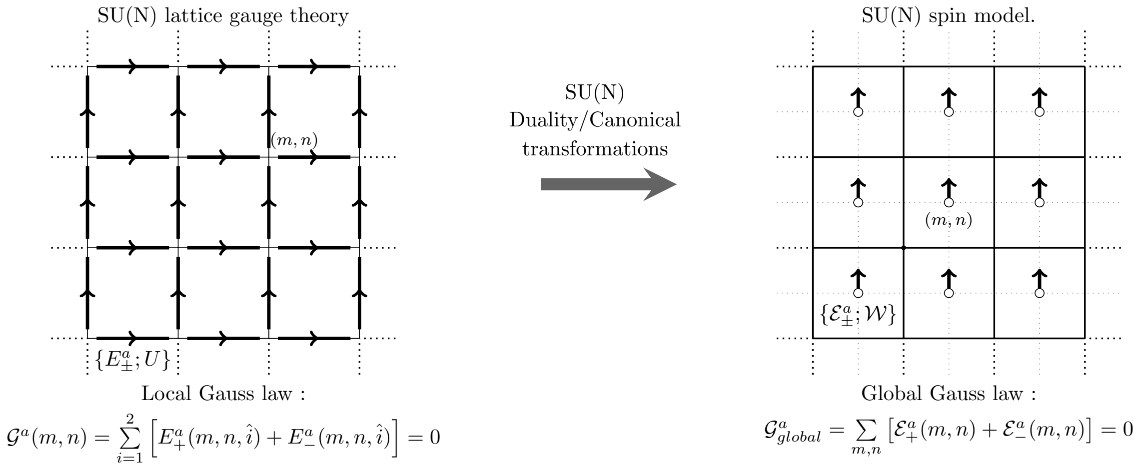

FIG. 6: Duality between SU(N) lattice gauge theory and an SU(N) spin model. Unlike the corresponding Z_2 duality in Figure 1-a,b, global SU(N) Gauss law constraints at the origin remain unsolved.

using Hamiltonian approach like Z_2 lattice gauge theory in the previous section. The associated topological degrees of freedom and their properties are subjects of future studies. The SU(N) dual (spin) operators also lead to a new SU(N) disorder operator discussed in section II B 6.

The Z_2 spin conjugate pairs $\{\mu_1(m,n); \mu_3(m,n)\}$ on the plaquettes are now SU(N) plaquette spin pairs $\{\mathcal{E}^a_\mp(m,n); \mathcal{W}_{\alpha\beta}(m,n)\}$ representing the dual SU(N) electric scalar potential and the scalar SU(N) magnetic fields on the plaquettes respectively. These dual operators are shown in Figure 6-b. Similarly in the unphysical sector, the Z_2 string conjugate pairs $\{\bar{\sigma}_1(m,n); \bar{\sigma}_3(m,n)\}$ are now SU(N) string conjugate pairs $\{\mathsf{E}^a_\mp(m,n); \mathsf{T}_{\alpha\beta}(m,n)\}$. The algebraic details of SU(N) canonical transformations can be found in [22]. In this section, we directly motivate the SU(N) results through the Z_2 lattice gauge theory duality discussed in the previous section. For the sake of comparison and convenience, all the initial and the final (dual) operators involved in Z_2 and $SU(N)$ lattice gauge theories are shown in Table-1.

We start with a very brief review of Hamiltonian formulation of SU(N) lattice gauge theory [19] to have completeness and consistent notations. The basic operators involved in the Kogut-Susskind Hamiltonian formulation of SU(N) lattice gauge theories are $SU(N)$ flux operators $U(\vec{n};\hat{i})$ and the corresponding left, right electric fields $E^a_+(\vec{n};\hat{i})$ and $E^a_-(\vec{n}+\hat{i};\hat{i})$ and on every link $(\vec{n};\hat{i})$. They satisfy the following canonical commutation relations:

$$\left[E^a_+(\vec{n};\hat{i}), U_{\alpha\beta}(\vec{n};\hat{i})\right] = -\left(\frac{\lambda^a}{2}U(\vec{n};\hat{i})\right)_{\alpha\beta},$$
$$\left[E^a_-(\vec{n}+\hat{i};\hat{i}), U_{\alpha\beta}(\vec{n};\hat{i})\right] = \left(U(\vec{n};\hat{i})\frac{\lambda^a}{2}\right)_{\alpha\beta} \tag{34a}$$

$$\left[E^a_+(\vec{n};\hat{i}), E^b_+(\vec{n};\hat{i})\right] = if^{abc}E^c_+(\vec{n};\hat{i}),$$
$$\left[E^a_-(\vec{n};\hat{i}), E^b_-(\vec{n};\hat{i})\right] = if^{abc}E^c_-(\vec{n};\hat{i}). \tag{34b}$$

In (34a) and (34b), $\lambda^a (a = 1, 2, \cdots, N^2-1)$ are the representation matrices in the fundamental representation of SU(N) satisfying $Tr\left(\lambda^a\lambda^b\right) = \frac{1}{2}\delta^{ab}$ and f^{abc} are the SU(N) structure constants. $E^a_-(\vec{n}+\hat{i};\hat{i})$ and $E^a_+(\vec{n};\hat{i})$ are the generators of right and left gauge transformations on the link flux operator $U(\vec{n};\hat{i})$. The left and the right electric fields are not independent and are related by:

$$E^a_-(\vec{n}+\hat{i};\hat{i}) = -R_{ab}(U^\dagger(\vec{n};\hat{i}))E^b_+(\vec{n};\hat{i});$$
$$R_{ab}(U(\vec{n};\hat{i})) \equiv \frac{1}{2}Tr\left(\lambda^a U(\vec{n};\hat{i})\lambda^b U^\dagger(\vec{n};\hat{i})\right). \tag{35}$$

The rotation operator R satisfies $R^TR = RR^T = 1$. The local SU(N) gauge transformations rotate the link operators and the electric fields as:

$$E_\pm(\vec{n};\hat{i}) \to \Lambda(\vec{n})\ E_\pm(\vec{n};\hat{i})\ \Lambda^\dagger(\vec{n}),$$
$$U(\vec{n};\hat{i}) \to \Lambda(\vec{n})\ U(\vec{n};\hat{i})\ \Lambda^\dagger(\vec{n}+\hat{i}) \tag{36}$$

the generators of SU(2) gauge transformations at any lattice site n are:

$$\mathcal{G}^a(\vec{n}) = \sum_{i=1}^{d=2}\Big(E^a_-(\vec{n};\hat{i}) + E^a_+(\vec{n};\hat{i})\Big), \qquad \forall\ \vec{n},\ a. \tag{37}$$

Therefore, there is a Gauss law constraint $\mathcal{G}^a(\vec{n})|\psi\rangle_{phys} = 0$ at each lattice site $\vec{n}$, where $|\psi\rangle_{phys}$ is any physical state. The Hamiltonian is given by

$$H = \frac{g^2}{2}\sum_l E^a(l)E^a(l) + \frac{1}{2g^2}\sum_p\Big(2N - (Tr\ U_p + h.c)\Big)$$
$$\equiv \left(g^2H_E + \frac{1}{g^2}H_B\right). \tag{38}$$

Above, l and p refer to links and plaquettes on the lattice. U_p is the product of link operators corresponding to the links along a plaquette. g^2 is the coupling constant. Like in Z_2 gauge theory Hamiltonian (4), all interactions are contained in the magnetic part H_B of the Hamiltonian. The electric part H_E, with no interactions, can be easily diagonalized leading to gauge invariant strong coupling expansion in terms of loop states [19, 21]. After duality in the next section, like Z_2 lattice gauge theory in section II A 4, their roles will be reversed.

1. Physical sector and SU(N) dual potentials

We now define the dual SU(N) spin and SU(N) string operators analogous to the Z_2 spins and strings in (22a), (22b) and (24a), (24b) respectively. As expected, due to the non-abelian nature of the electric field and the flux operators, the SU(N) duality relations have additional non-abelian structures. To begin with, the $\mathcal{N}$ SU(N) Gauss law constraints at $\mathcal{N}$ different lattice sites are all mutually independent. In other words, identities like (21) do not exist. As a result, there is a global SU(N) invariance in the SU(N) spin model which corresponds to gauge transformations at the origin. All dual operators transform covariantly under this global SU(N). Further, the duality transformations involve parallel transports from the origin to the site of the dual operators. It is convenient to define a SU(N) parallel transport or string flux operator $\mathsf{T}(m,n)$ which connects operators at the origin to the operators at a lattice site (m,n) through the path $(0,0) \rightarrow (m,0) \rightarrow (m,n)$:

$$\mathsf{T}(m,n) = \left(\prod_{m'=0}^{m} U(m',0;\hat{1}) \prod_{n'=0}^{n} U(m,n';\hat{2})\right), \quad (39a)$$

$$\mathsf{E}_{+}^{a}(m,n) = \mathcal{G}^{a}(m,n) \approx 0, \quad (39b)$$

The strings and their electric fields $\mathsf{E}_{+}^{a}(m,n)$ are shown in Figure 7-b and Figure 8-b respectively. The relations (39a) and (39b) are the SU(N) analogues of the Z_2 string relations (24a) and (24b) respectively. The dual SU(N) spin and the SU(N) scalar electric potential operators in terms of the original Kogut Susskind operators are defined [22] as

$$\mathcal{W}(m,n) = \mathsf{T}(m{-}1,n{-}1)\; U_p(m,n)\; \mathsf{T}^{\dagger}(m{-}1,n{-}1), \quad (40a)$$

$$\mathcal{E}_{+}^{a}(m,n) = \sum_{n'=n}^{\mathsf{N}} R_{ab}(S(m,n,n'))\; E_{-}^{b}(m,n';\hat{1}). \quad (40b)$$

The two SU(N) dual operators defined in (40a) and (40b) are the non-abelian extensions of the two Z_2 dual operators defined in (22a) and (22b) repectively. In (40a), (40b), the Kogut-Susskind plaquette operators and the parallel transport $S(m,n,n')$ are

$$\begin{aligned} U_p(m,n) &= U(m-1,n-1;\hat{1})\; U(m,n-1;\hat{2}) \\ &\quad U^{\dagger}(m-1,n;\hat{1})\; U^{\dagger}(m-1,n-1;\hat{2}), \quad (41) \\ S(m,n,n') &\equiv \mathsf{T}(m{-}1,n)\; U(m{-}1,n;\hat{1}) \prod_{q=n}^{n'} U(m,q;\hat{2}). \end{aligned}$$

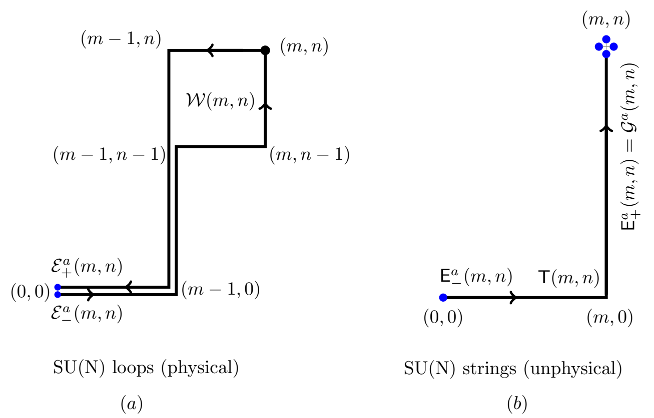

FIG. 7: The physical SU(N) spin conjugate pairs $\{\mathcal{E}_{\pm};\mathcal{W}_{\alpha\beta}\}$ and the unphysical SU(N) string conjugate pairs $\{\mathsf{E}_{\pm}(n);\mathsf{T}(n)\}$ dual to SU(N) lattice gauge theory are shown in (a) and (b) respectively. Like in Z_2 case in Figure 4, we label the SU(N) spin operators by their top right corners and the SU(N) string operators by their endpoints. The strings decouple from the physical Hilbert space as $\mathsf{E}_{+}^{a}(m,n) \approx 0$ by the Gauss law constraints in $\mathcal{H}^p$.

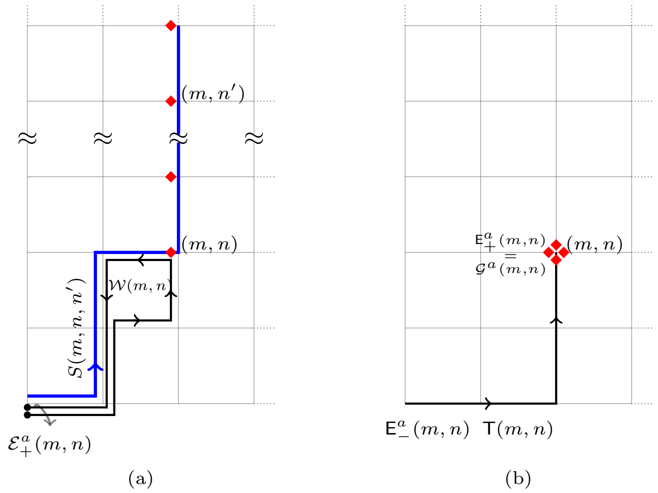

FIG. 8: The non-local relations in SU(N) duality transformations and the Gauss law constraints. (a) We show the relations (40b) expressing $\mathcal{E}_{+}^{a}(m,n)$ as the sum of $E_{-}^{b}(m,n';\hat{1})$. The Kogut-Susskind electric fields and and the plaquette loop electric fields are denoted by ♦ and • respectively. In (b) we show the SU(N) Gauss law constraints (46). The corresponding Z_2 illustrations are in Figure 5-a,b,c.

Z_2 **lattice gauge theory**		**SU(N) lattice gauge theory**	
Gauge Operators	Dual/Spin Operators	Gauge Operators	Dual/Spin Operators
	$\{\mu_1(m,n);\mu_3(m,n)\}$ (Z_2 Loops/Z_2 Ising spins)		$\{\mathcal{E}^a_\pm(m,n);\ \mathcal{W}_{\alpha\beta}(m,n)\}$ (SU(N) Loops/SU(N) spins)
$\left\{\sigma_1(m,n;\hat{i});\sigma_3(m,n;\hat{i})\right\}$		$\left\{E^a_\pm(m,n;\hat{i});U_{\alpha\beta}(m,n;\hat{i})\right\}$	
	$\{\bar{\sigma}_1(m,n);\bar{\sigma}_3(m,n)\}$ (Frozen Z_2 Strings)		$\{\mathsf{E}^a_\pm(m,n);\ \mathsf{T}_{\alpha\beta}(m,n)\}$ (Frozen $SU(N)$ Strings)

TABLE I: The basic conjugate operators of the original and the dual Z_2, SU(N) gauge theories in $(2+1)$ dimensions.

The relation (40a) defines the SU(N) magnetic field operator as a fundamental operator. The second relation in (40b) defines the (dual) SU(N) electric scalar potential $\mathcal{E}^a(m,n)$. The appearance of the $\mathsf{T}(m,n)$ and $S(m,n,n')$ in (41) in the construction of the above dual operators is due to the non-abelian nature of the operators. These parallel transports from the origin are required to have consistent global transformation properties of the SU(N) magnetic fields and the SU(N) electric scalar potentials (see (47)). These dual or loop operators satisfy the expected non-abelian duality or quantization rules:

$$\begin{aligned}\left[\mathcal{E}^a_-(m,n),\mathcal{W}_{\alpha\beta}(m,n)\right] &= -\left(\frac{\lambda^a}{2}\mathcal{W}(m,n)\right)_{\alpha\beta},\\ \left[\mathcal{E}^a_+(m,n),\mathcal{W}_{\alpha\beta}(m,n)\right] &= \left(\mathcal{W}(m,n)\frac{\lambda^a}{2}\right)_{\alpha\beta},\end{aligned} \tag{42a}$$

$$\begin{aligned}\left[\mathcal{E}^a_-(m,n),\mathcal{E}^b_-(m,n)\right] &= if^{abc}\mathcal{E}^c_-(m,n),\\ \left[\mathcal{E}^a_+(m,n),\mathcal{E}^b_+(m,n)\right] &= if^{abc}\mathcal{E}^c_+(m,n).\end{aligned} \tag{42b}$$

Further, the two electric fields are related through parallel transport and commute:

$$\begin{aligned}\mathcal{E}^a_-(m,n) &\equiv -R_{ab}(\mathcal{W}^\dagger(m,n))\mathcal{E}^b_+(m,n)\\ &=> \left[\mathcal{E}^a_-(m,n),\mathcal{E}^b_+(m,n)\right] = 0.\end{aligned} \tag{43}$$

The quantization relations (42a), (42b) and (43) are exactly similar to the original quantization rules (34a) and (34b) respectively. Thus the electric field operator $E^a(m,n;\hat{i})$ and the magnetic vector potential operator $U_{\alpha\beta}(m,n;i)$ have been replaced by their dual electric scalar potential $\mathcal{E}^a(m,n)$ and the dual magnetic field operator $\mathcal{W}_{\alpha\beta}(m,n)$. This is similar to Z_2 lattice gauge theory duality where $\{\sigma_1(m,n);\sigma_3(m,n)\}$ get replaced by $\{\mu_3(m,n);\mu_1(m,n)\}$. We again emphasize that $\mathcal{E}^a(m,n)$ is the dual electric scalar potential as it is conjugate to the fundamental magnetic flux operator $\mathcal{W}_{\alpha\beta}(m,n)$.

2. Unphysical sector and SU(N) string operators

The unphysical sector consists of the string flux operators $\mathsf{T}(m,n)$ and their conjugate electric fields $\mathsf{E}^a(m,n)$ satisfying the canonical quantization relations:

$$\begin{aligned}\left[\mathsf{E}^a_-(m,n),\mathsf{T}_{\alpha\beta}(m,n)\right] &= -\left(\frac{\lambda^a}{2}\mathsf{T}(m,n)\right)_{\alpha\beta},\\ \left[\mathsf{E}^a_+(m,n),\mathsf{T}_{\alpha\beta}(m,n)\right] &= \left(\mathsf{T}(m,n)\frac{\lambda^a}{2}\right)_{\alpha\beta}\end{aligned} \tag{44a}$$

$$\begin{aligned}\left[\mathsf{E}^a_-(m,n),\mathsf{E}^b_-(m,n)\right] &= if^{abc}\mathsf{E}^c_-(m,n),\\ \left[\mathsf{E}^a_+(m,n),\mathsf{E}^b_+(m,n)\right] &= if^{abc}\mathsf{E}^c_+(m,n).\end{aligned} \tag{44b}$$

Again, the operators T^a_+ and T^b_- are related through parallel transport and commute amongst themselves:

$$\begin{aligned}\mathsf{E}^a_-(m,n) &\equiv -R_{ab}(\mathsf{T}^\dagger(m,n))\mathsf{E}^b_+(m,n)\\ &=> \quad \left[\mathsf{E}^a_-(m,n),\mathsf{E}^b_+(m,n)\right] = 0.\end{aligned} \tag{45}$$

The right string electric fields are

$$\begin{aligned}\mathsf{E}^a_+(m,n) &= \textstyle\sum_{i=1}^2 \left[E^a_-(m,n;\hat{i}) + E^a_+(m,n;\hat{i})\right]\\ &= \mathcal{G}^a(m,n) \approx 0, \quad \forall (m,n) \neq (0,0).\end{aligned} \tag{46}$$

Thus as in Z_2 lattice gauge theory, the SU(N) Gauss law constraints freeze all SU(N) string degrees of freedom as shown in Figure 7-b. This, as expected, leads to complete decoupling of the string degrees of freedom from SU(N) spin dynamics (see (51)).

3. The Residual Gauss law

The decoupling of the strings implements all local Gauss law constraints except the one at the origin. Unlike Z_2 lattice gauge theory, the SU(N) Gauss law at the origin is independent of the SU(N) Gauss laws at other sites. In other words, the abelian identity (21) has no non-abelian analogue. Under this residual global gauge invariance at the origin $\Lambda \equiv \Lambda(0,0)$, all loop operators transform like adjoint matter fields:

$$\mathcal{E}_\pm(p) \to \Lambda\ \mathcal{E}_\pm(p)\ \Lambda^\dagger, \quad \mathcal{W}(p) \to \Lambda\ \mathcal{W}(p)\ \Lambda^\dagger. \tag{47}$$

Above, $\mathcal{E}_\pm(p) \equiv \mathcal{E}_\pm(m,n), \mathcal{W}(p) \equiv \mathcal{W}(m,n)$ and $\Lambda \equiv \Lambda(0,0)$ is the gauge transformation at the origin. This global invariance at the origin is fixed by the (N^2-1) global SU(N) Gauss laws:

$$\mathcal{G}^a \equiv \mathcal{G}^a(0,0) = E^a_+(0,0,\hat{1}) + E^a_-(0,0,\hat{2})$$
$$= \sum_{m=1}^{\mathsf{N}}\sum_{n=1}^{\mathsf{N}}\Big[\underbrace{\mathsf{E}^a_-(m,n)}_{=0} + \underbrace{\mathcal{E}^a_+(m,n) + \mathcal{E}^a_-(m,n)}_{\equiv \mathbb{L}^a(m,n)}\Big]$$
$$\equiv \sum_{m=1}^{\mathsf{N}}\sum_{n=1}^{\mathsf{N}} \mathbb{L}^a(m,n) = 0. \qquad (48)$$

In (48), the total left and right electric flux operators on a plaquette located at $p=(m,n)$ are donated by $\mathbb{L}^a(m,n)$ and equations (45), (46) are used to get $\mathsf{E}^a_-(m,n)=0$.

4. Inverse relations

The inverse flux operator relations, analogous to the Z_2 relations (26), are

$$U(m,n;\hat{1}) = \mathsf{T}^\dagger(m,n)\ \mathcal{W}(m+1,n)\ \mathcal{W}(m+1,n-1)\cdot \cdots \mathcal{W}(m+1,1)\ \mathsf{T}(n+1,y),$$
$$U(m,n;\hat{2}) = \ T(m,n+1)\ T^\dagger(m,n). \qquad (49)$$

The inverse electric field relations, analogous to the Z_2 electric field relations (27), are

$$E^a_+(m,n;\hat{1}) = R_{ab}(\mathsf{T}(m,n))\Big\{\mathcal{E}^b_-(m+1,n+1) + \mathcal{E}^b_+(m+1,n) + \delta_{n,0}\sum_{\bar{m}=m+2}^{L}\sum_{\bar{n}=1}^{\mathsf{N}}\mathbb{L}^b(\bar{m},\bar{n})\Big\},$$
$$E^a_+(m,n;\hat{2}) = R_{ab}(\mathsf{T}(m,n))\Big\{\mathcal{E}^b_+(m+1,n+1) + R_{bc}(W(m,n))\mathcal{E}^c_-(m,n+1) + \sum_{\bar{n}=n+2}^{\mathsf{N}}\mathbb{L}^b(m+1,\bar{n})\Big\}(50)$$

In the last step in (50) we have defined,

$$R_{bc}(W(m,n)) \equiv R_{bc}\big(\mathcal{W}(m,n)\mathcal{W}(m,n-1)\cdots\mathcal{W}(m,1)\big)$$

and $\mathbb{L}^a(m,n) \equiv \big(\mathcal{E}^a_-(m,n) + \mathcal{E}^a_+(m,n)\big)$.

5. SU(N) dual dynamics

The Hamiltonian of pure SU(N) gauge theory in terms of the dual operators is

$$H = \sum_{m,n\in\Lambda}\frac{g^2}{2}\Big\{\Big[\vec{\mathcal{E}}_-(m+1,n+1) + \vec{\mathcal{E}}_+(m+1,n) + \Delta_{XY}(m,n)\Big]^2 + \Big[\vec{\mathcal{E}}_+(m+1,n+1) + R_{bc}(W(m,n))\vec{\mathcal{E}}^c_-(m,n+1) + \Delta_Y(m,n)\Big]^2\Big\} + \frac{1}{2g^2}\Big(2N - (Tr\ \mathcal{W}(m,n) + h.c)\Big) \equiv g^2\tilde{H}_E + \frac{1}{g^2}\tilde{H}_B. \qquad (51)$$

We have defined $\Delta^a_{XY}(m,n) \equiv \delta_{m,0}\sum_{\bar{m}=m+2}^{\mathsf{N}}\sum_{\bar{n}=1}^{\mathsf{N}}\mathbb{L}^a(\bar{m},\bar{n})$ and $\Delta^a_Y(m,n) \equiv \sum_{\bar{n}=n+2}^{\mathsf{N}}\mathbb{L}^a(m,\bar{n})$, where $\mathbb{L}(m,n)$ is defined in the equation (48).

The Hamiltonian (51) is the SU(N) spin Hamiltonian dual to the SU(N) Kogut-Susskind Hamiltonian (38). All the salient features of duality, mentioned in the context of Z_2 lattice gauge theory, are satisfied. It is clear that the SU(N) spin Hamiltonian, unlike the Z_2 lattice gauge theory case, is non-local and the non-localities comes from the terms $\mathcal{R}(\mathcal{W}), \Delta^a_{XY}(x,y)$ and $\Delta^a_Y(x,y)$. But, since $\mathcal{R}(\mathcal{W}) = 1 + o(g) + o(g^2) + \cdots$, $\mathbb{L}$ is of the order of g which implies that $\Delta^a_{XY}(x,y)$ and $\Delta^a_Y(x,y)$ are both at least of the order of g. Therefore, we expect that in the $g^2 \to 0$ continuum limit , these non-local parts can be ignored to the lowest order at low energies. This leads to a simplified local effective Hamiltonian H_{spin} which may describe pure SU(N) gauge theory at low energies, sufficiently well.

$$H_{spin} = \frac{g^2}{2}\Big\{\sum_{p=1}^{\mathcal{P}}4\vec{\mathcal{E}}^{\,2}(p) + \sum_{<p,p'>}\vec{\mathcal{E}}_-(p)\cdot\vec{\mathcal{E}}_+(p')\Big\} + \frac{1}{2g^2}\Big\{2N - (Tr\mathcal{W}(p) + h.c)\Big\} \equiv \frac{g^2}{2}\tilde{H}'_E + \frac{1}{2g^2}\tilde{H}_B. \ (52)$$

In (52), $<p,p'>$ is used to show the nearest plaquettes. The above simplified SU(N) spin Hamiltonian H_{spin} describes nearest neighbouring SU(N) spins interacting through their left and right electric fields. All the interactions are now contained in the 'electric part' of the Hamiltonian $\tilde{H}'_E$ unlike the standard link formulation. Note that the coupling constant of the dual model is the inverse of that of the original Kogut Susskind model:

$$H^{SU(N)}_{gauge}\big(\tfrac{1}{g^2}\big) \sim H^{SU(N)}_{spin}\big(g^2\big).$$

We have used $\sim$ above to state that this equivalence is only within the physical Hilbert space $\mathcal{H}^p$.The above relation is SU(N) analogue of the Z_2 result $H^{Z_2}_{gauge}(\lambda) \sim H^{Z_2}_{spin}(\lambda^{-1})$ discussed earlier. Note that the global SU(N) invariance of the dual SU(N) spin model is not physical and needs to be fixed by demanding the Gauss law (48). In fact, the degrees of freedom exactly match only after implementing the residual Gauss law constraints at the origin. We have converted the initial $3\mathcal{L}$ link operators into $3\mathcal{P}$ plaquette loop operators and $3(\mathcal{N}-1)$ string operators (see Table 1) and $\mathcal{L} = \mathcal{P} + (\mathcal{N}-1)$. There are $\mathcal{N}$ mutually independent Gauss laws in SU(N) (but $(\mathcal{N}-1)$ in Z_2 case) lattice gauge theory. Out of these, $(\mathcal{N}-1)$ freeze the $(\mathcal{N}-1)$ strings. We are thus left with a single Gauss law constraint (48) in SU(N) spin model after duality and none in Z_2 case.

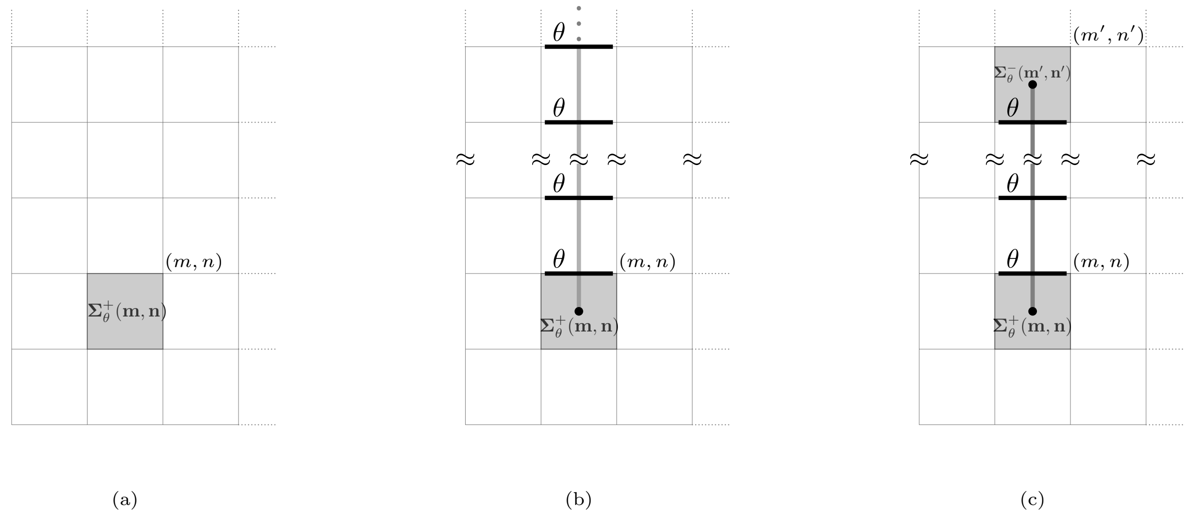

FIG. 9: Graphical illustration of the disorder operator $\Sigma^+_\theta(m,n)$ creating a plaquette vortex (monopole) in terms of (a) the dual operators, (b) the original Kogut-Susskind link operators but now with infinitely long Dirac string, (c) a vortex-antivortex (monopole-anti-monopole) pair connected through a finite length Dirac string. The dark heavy horizontal links across the Dirac strings in (b) and (c) represent rotations of the Kogut-Susskind link flux operators $U(m-1,n';\hat{1}),\ n' \geq n$ by θ.

6. Magnetic disorder operator

Exploiting duality transformations, we now construct a SU(N) gauge invariant operator which measure the magnetic disorder in the gauge system [4]. Such disorder operators and their correlations in the context of $2-d$ Ising model and Kramers-Wannier duality have been discussed in [24].

We work with dual SU(N) spin model with global SU(N) gauge invariance (47). In this section, we focus on a single plaquette $p = (m,n)$ in Λ. We write the SU(2) magnetic plaquette flux operator in the magnetic field eigenbasis as:

$$\mathcal{W}(m,n) \equiv cos\left(\frac{\omega_{(m,n)}}{2}\right)\sigma_0 + i\left(\hat{w}_{(m,n)}\cdot\vec{\sigma}\right)\sin\left(\frac{\omega_{(m,n)}}{2}\right)$$
$$\hat{w}_{(m,n)}\cdot\hat{w}_{(m,n)} = 1, \quad \forall\ (m,n). \tag{53}$$

In (53), $\omega(m,n)$ are gauge invariant angles, $\hat{w}(m,n)$ are the unit vectors in the group manifold S^3 and $\sigma_0, \vec{\sigma}(\equiv \sigma_1,\sigma_2,\sigma_3)$ are the unit, Pauli matrices. Under global gauge transformation $\Lambda \equiv \Lambda(0,0)$ in (47), $(\omega,\hat{w})$ transform as:

$$\omega(m,n)\rightarrow\omega(m,n), \quad \hat{w}^a(m,n) \rightarrow R_{ab}(\Lambda)\ \hat{w}^b(m,n). \tag{54}$$

Above, $R_{ab}(\Lambda)$ are defined in (35). We define two unitary operators:

$$\Sigma^\pm_\theta(m,n) \equiv exp\ i\Big(\hat{w}(m,n)\cdot\mathcal{E}_\pm(m,n)\ \frac{\theta}{2}\Big), \tag{55}$$

which are located on a plaquette $p \equiv (m,n)$ as shown in the Figure 9-a. They both are gauge invariant because $\mathcal{E}^a_\pm(m,n)$ and $\hat{w}(m,n)$ gauge transform like vectors as shown in (47) and (54). In other words, $\left[\mathcal{G}^a, \Sigma^\pm_\theta(m,n)\right] = 0$, where $\mathcal{G}^a$ is defined in (48). As the left and right SU(N) electric scalar potentials are related through (43), $\Sigma^\pm_\theta(m,n)$ are not independent and satisfy:

$$\Sigma^+_\theta(m,n)\ \Sigma^-_\theta(m,n) = \Sigma^-_\theta(m,n)\ \Sigma^+_\theta(m,n) = I. \tag{56}$$

Above I denotes the unit operator in the physical Hilbert space $\mathcal{H}^p$ and $\Sigma^-_\theta = \Sigma^+_{-\theta}$. The physical meaning of the operators $\Sigma^\pm_\theta(m,n)$ is simple. The non-abelian electric scalar potentials $\mathcal{E}^a_\pm(m,n)$ are conjugate to the magnetic flux operators $\mathcal{W}_{\alpha\beta}(m,n)$. They satisfy the canonical commutation relations (42a). Therefore, the gauge invariant operator $\Sigma^\pm_\theta(m,n)$ locally and continuously changes the magnetic flux on the plaquette $p = (m,n)$ as a function of θ. To see this explicitly, we consider common eigenstates $|\omega(m,n),\hat{w}(m,n)\rangle$ of $\mathcal{W}_{\alpha\beta}(m,n)$ on a single plaquette. These states are explicitly constructed in (B1) in Appendix B. They satisfy:

$$Tr\mathcal{W}\ \ |\omega,\hat{w}\rangle = 2\cos\left(\frac{\omega}{2}\right)\ \ |\omega,\hat{w}\rangle\,. \tag{57}$$

We have ignored the irrelevant plaquette index $p \equiv (m,n)$ in (57) as we are dealing with a single plaquette. It is easy to check:

$$|\omega,\hat{w}\rangle_{\pm\theta} \equiv \Sigma^\pm_\theta\ |\omega,\hat{w}\rangle = |\omega\pm\theta,\hat{w}\rangle\,, \tag{58}$$

implying,

$$Tr\ \mathcal{W}\ |\omega,\hat{w}\rangle_{\pm\theta} = 2\ cos\left(\frac{\omega\pm\theta}{2}\right)\ |\omega,\hat{w}\rangle_{\pm\theta}. \tag{59}$$

We further define:

$$\Sigma_{2\pi} \equiv \Sigma^+_{\theta=2\pi} = \Sigma^-_{\theta=2\pi}, \qquad \left(\Sigma_{2\pi}\right)^2 = I. \tag{60}$$

The equations (57) and (60) state:

$$\Sigma_{2\pi}\,(Tr\mathcal{W}) = -\ (Tr\mathcal{W})\ \Sigma_{2\pi}. \tag{61}$$

We thus recover the standard Wilson-'t Hooft loop Z_2 algebra [4] for SU(2) at $\theta = 2\pi$. The operator $\Sigma_{2\pi}$ is the SU(2) 't Hooft operator. The plaquette magnetic flux operators $\Sigma^{\pm}_{\theta}(m,n)$ in (55) can also be written as a non-local sum of Kogut-Susskind link electric fields along a line and the corresponding parallel transports using (40b). The magnetic charge on the plaquette $p = (m,n)$ thus develops an infinite Dirac string in the original (standard) $\{E^a(l); U(l)\}$ description. This is similar to the discrete Z_2 disorder operator $\Sigma(m,n)$ written in terms of the original electric field operators $\sigma_1(m,n')$ in (32) and (33). The singular Dirac string is shown in Figure 9-b. Note that the disorder operators in the strong coupling vacuum satisfy:

$$\langle 0|\Sigma^{\pm}_{\theta}(m,n)|0\rangle = 1,$$

showing that the strong coupling ground state $|0\rangle$ is maximally disordered with respect to the original magnetic vector potentials. The Wilson loops, on the other hand, are zero: $\langle 0|W_C(U)|0\rangle = 0$. It will be interesting to study its behaviour for the finite values of the coupling along with the vacuum correlation functions of $\langle \Sigma^{\pm}_{\theta}(p)\Sigma^{\mp}_{\theta}(p')\rangle$, shown in Figure 9-c, as $|p-p'| \to \infty$. The work in this direction is in progress and will be reported elsewhere.

III. A VARIATIONAL GROUND STATE OF SU(N) SPIN MODEL

In this section, we study the ground state of the dual spin model with nearest neighbour interactions. We then compare the results with those obtained from the variational analysis of the standard Kogut Susskind formulation [25–27]. Note that after canonical transformations each plaquette loop is a fundamental degree of freedom. Therefore, gauge invariant computations in the dual spin model become much simpler. For simplicity we consider $N = 2$. For the ground state of SU(2) gauge theory, the magnetic fluctuations in a region are independent of fluctuation in another region sufficiently far away [30, 31]. So, the largest contributions to the vacuum state comes from states with little magnetic correlations. Therefore, we use the following separable state without any spin-spin correlations as our variational ansatz:

$$|\psi_0\rangle = e^{S/2}|0\rangle; \qquad S = \alpha\sum_p Tr\mathcal{W}(p).$$
$$= \prod_p\ |\psi_0\rangle_p. \tag{62}$$

Above, $|0\rangle$ is the strong coupling vacuum state defined by $\mathcal{E}^a_{\pm}(m,n)|0\rangle = 0$ and α is the variational parameter. Since this state doesn't have long distance correlations, it satisfies Wilson's area law criterion. We consider a

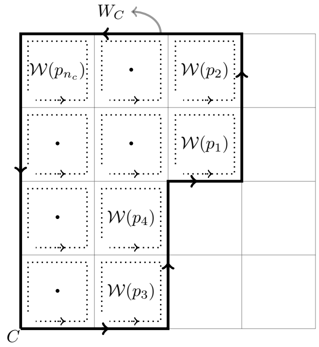

FIG. 10: A Wilson loop W_C can be written as the product of fundamental plaquette loop operators $\mathcal{W}(p)$. $W_C = \mathcal{W}(p_1)\ \mathcal{W}(p_2)\ \mathcal{W}(p_3)\cdots\cdots\mathcal{W}(p_{n_c})$. The tails of the fundamental plaquette loop operators connecting them to the origin (see Figure 7-a) are not shown for clarity.

Wilson loop $Tr\ (W_C)$ along a large space loop C on the lattice and compute its ground state expectation value: $<\psi_0|TrW_C|\psi_0>\ /\ \langle\psi_0|\psi_0\rangle$. In the dual spin model any Wilson loop W_C can be written in terms of the $\mathcal{P}$ fundamental loops $\mathcal{W}_{\alpha\beta}$ as shown in Figure 10:

$$W_C = \mathcal{W}(p_1)\ \mathcal{W}(p_2)\ \mathcal{W}(p_3)\cdots\cdots\mathcal{W}(p_{n_c}). \tag{63}$$

Here p_1 is the plaquette operator in the bottom right corner of C and p_{n_c} is the plaquette operator at the left top corner of C. As shown in the Appendix B (see (B3)):

$$\frac{\langle\psi_0|TrW_C|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = 2\left(\frac{I_2(2\alpha)}{I_1(2\alpha)}\right)^{n_c} = 2e^{-n_c\ln\left(\frac{I_1(2\alpha)}{I_2(2\alpha)}\right)} \tag{64}$$

Here, n_c is the number of plaquettes in the loop C and I_l is the l-th order modified Bessel function of the first kind. The string tension is given by $\sigma_T(\alpha) = \ln\left(\frac{I_1(2\alpha)}{I_2(2\alpha)}\right)$.

We now calculate α by minimizing

$$\langle H_{spin}\rangle = \frac{\langle\psi_0|H_{spin}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}.$$

In order to calculate $\langle H_{spin}\rangle$, we first find the expectation value of $\mathcal{E}_-(p)\cdot\mathcal{E}_+(p')$ and $\mathcal{E}(p)\cdot\mathcal{E}(p) \equiv \mathcal{E}_+(p)\cdot\mathcal{E}_+(p) \equiv \mathcal{E}_-(p)\cdot\mathcal{E}_-(p)$ in (52). This calculation is done in Appendix C. The expectation values are (see (C5))

$$\langle\mathcal{E}_-(p)\cdot\mathcal{E}_+(p')\rangle = 0, \qquad \langle\mathcal{E}(p)\cdot\mathcal{E}(p)\rangle = \frac{3\alpha}{16}\langle Tr\mathcal{W}(p)\rangle.$$

Putting $n_c = 1$ in equation (64), we get $\langle Tr\mathcal{W}(p)\rangle = \frac{2I_2(2\alpha)}{I_1(2\alpha)}$. Therefore, the expectation value of the effective Hamiltonian H_{spin} is

$$\frac{\langle\psi_0|H_{spin}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = 2\mathcal{P}\left\{\left(\frac{3\alpha}{4}g^2 - \frac{1}{g^2}\right)\frac{I_2(2\alpha)}{I_1(2\alpha)} + \frac{1}{g^2}\right\}. \tag{65}$$

Above, $\mathcal{P}$ is the number of plaquettes in the lattice. $\frac{I_2(2\alpha)}{I_1(2\alpha)}$ is a monotonously increasing bounded function of

α. It takes values between $+1$ and -1 with $+1$ at $\alpha \to \infty$ and -1 at $\alpha \to -\infty$. In the weak coupling limit, $g^2 \to 0$, $\frac{I_2(2\alpha)}{I_1(2\alpha)}$ should be maximum for the expectation value of H_{spin} to be minimum and therefore, $\alpha \to \infty$. But, using the asymptotic form of the modified Bessel function of the first kind $I_l(2\alpha)$,

$$I_l(2\alpha) \xrightarrow{\alpha\to\infty} \frac{e^{2\alpha}}{\sqrt{2\pi(2\alpha)}}\left(1+\frac{(1-2l)(1+2l)}{16\alpha}+\cdots\right)$$

In the weak coupling limit , $\frac{I_2(2\alpha)}{I_1(2\alpha)} \approx 1 - \frac{3}{4\alpha}$. Hence,

$$\frac{\langle\psi_0|H_{spin}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = \sum_p 2\left\{\left(\frac{3\alpha}{4}g^2 - \frac{1}{g^2}\right)\left(1-\frac{3}{4\alpha}\right)+\frac{1}{g^2}\right\} \tag{66}$$

Minimizing the expectation value in the weak coupling limit, $\alpha = \frac{1}{g^2}$. The string tension is given by $\sigma_T(\frac{1}{g^2}) = ln\left(I_1(\frac{1}{g^2})/I_2(\frac{1}{g^2})\right)$. This is exactly the result obtained in [25, 26] using variational calculation with the fully disordered ground state and Kogut-Susskind Hamiltonian (38) which is dual to the full non-local spin Hamiltonian. As shown in Appendix C, the expectation value of the non-local part of the Hamiltonian in the variational ground state $|\psi_0\rangle$ vanishes. So, the simplified Hamiltonian with nearest neighbour interactions gives the same variational ground state to the lowest order as the full Hamiltonian. The disorder operator expectation value in this variational ground state is

$$\frac{\langle\psi_0|\Sigma^{\pm}_{\theta=2\pi}(m,n)|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = \frac{16\pi^2}{{}_p\langle\psi_0|\psi_0\rangle_p} = \frac{\pi\alpha}{2I_1(2\alpha)}. \tag{67}$$

Above, we have defined $p \equiv (m,n)$ and written the separable state $|\psi_0\rangle$ as the direct product of the state vectors corresponding to each plaquette i.e, $|\psi_0\rangle = \prod_p |\psi_0\rangle_p$.

IV. SUMMARY AND DISCUSSION

In this work, we have shown that the canonical transformations provide a method to generalize Wegner duality between Z_2 lattice gauge theory and quantum Ising model to SU(N) lattice gauge theories. The SU(N) dual formulation leads to a new gauge invariant disorder operator. The disorder operator can be measured in Monte-Carlo simulations. It will be interesting to see its behaviour across the deconfinement transition. In the weak coupling continuum limit, the Hamiltonian of the dual model reduces to an effective SU(N) spin Hamiltonian with nearest neighbouring interactions. We use a variational analysis of the spin model with a completely disordered ground state ansatz. The effective spin Hamiltonian leads to the same results as the standard Kogut Susskind Hamiltonian. Further analysis of the SU(N) spin model and its spectrum is required. This is the subject of our future investigations. It will also be interesting to generalize these transformations to $(3+1)$ dimensions to define dual electric vector potentials with a dual gauge group. The work in this direction is also in progress.

Acknowledgments

Acknowledgments: We thank Ramesh Anishetty for many discussions throughout this work and for reading the manuscript. M.M would like to thank Michael Grady for correspondence. T.P.S thanks Council of Scientific and Industrial Research (CSIR), India for financial support.

Appendix A: Wegner gauge-spin duality through canonical transformations

In this Appendix, we describe the canonical transformation involved in the construction of the duality relation between the basic operators of Z_2 gauge theory and Ising model in $2+1$ dimensions. The net effect of the canonical transformation involved in the construction of the spin operators on a single plaquette, described in section II A, can be summarized as follows:

- It replaces the top link l_3 on the plaquette by a plaquette spin operator with the same 'electric field' as the top link.

$$\mu_1(p) = \sigma_3(l_1)\sigma_3(l_2)\sigma_3(l_3)\sigma_3(l_4), \quad \mu_3(p) = \sigma_1(l_3) \tag{A1}$$

- The 'electric field' of the top link l_3 that vanishes gets absorbed into the electric fields of other links l_1, l_2, l_4.

$$\begin{aligned}
\bar\sigma_3(l_1) &= \sigma_3(l_1), & \bar\sigma_1(l_1) &= \sigma_1(l_1)\sigma_1(l_3)\\
\bar\sigma_3(l_2) &= \sigma_3(l_2), & \bar\sigma_1(l_2) &= \sigma_1(l_2)\sigma_1(l_3)\\
\bar\sigma_3(l_4) &= \sigma_3(l_4), & \bar\sigma_1(l_4) &= \sigma_1(l_4)\sigma_1(l_3)
\end{aligned} \tag{A2}$$

It is convenient to call the above net canonical transformation a 'plaquette canonical transformation (C.T)'. We now generalize the duality transformation relation to a finite lattice by iterating the plaquette C.T all over the two dimensional lattice starting from the top left plaquette of the lattice and systematically repeating it from top to bottom and left to right. We will illustrate this procedure on a 2×2 lattice which contains all the essential features of the construction on any finite lattice. The sites of the lattice are labelled as $O \equiv (0,0), A \equiv (0,1), B \equiv (0,2), C \equiv (1,0), D \equiv (1,1), E \equiv (1,2), F \equiv (2,0), G \equiv (2,1), H \equiv (2,2)$ and the plaquettes are numbered from top to bottom and left to right (see Figure 11) for convenience. The dual spin operators are constructed on a 2×2 lattice in 4 steps.

1. We begin by performing the plaquette canonical transformation (A1),(A2) on plaquette 1. The spin conju-

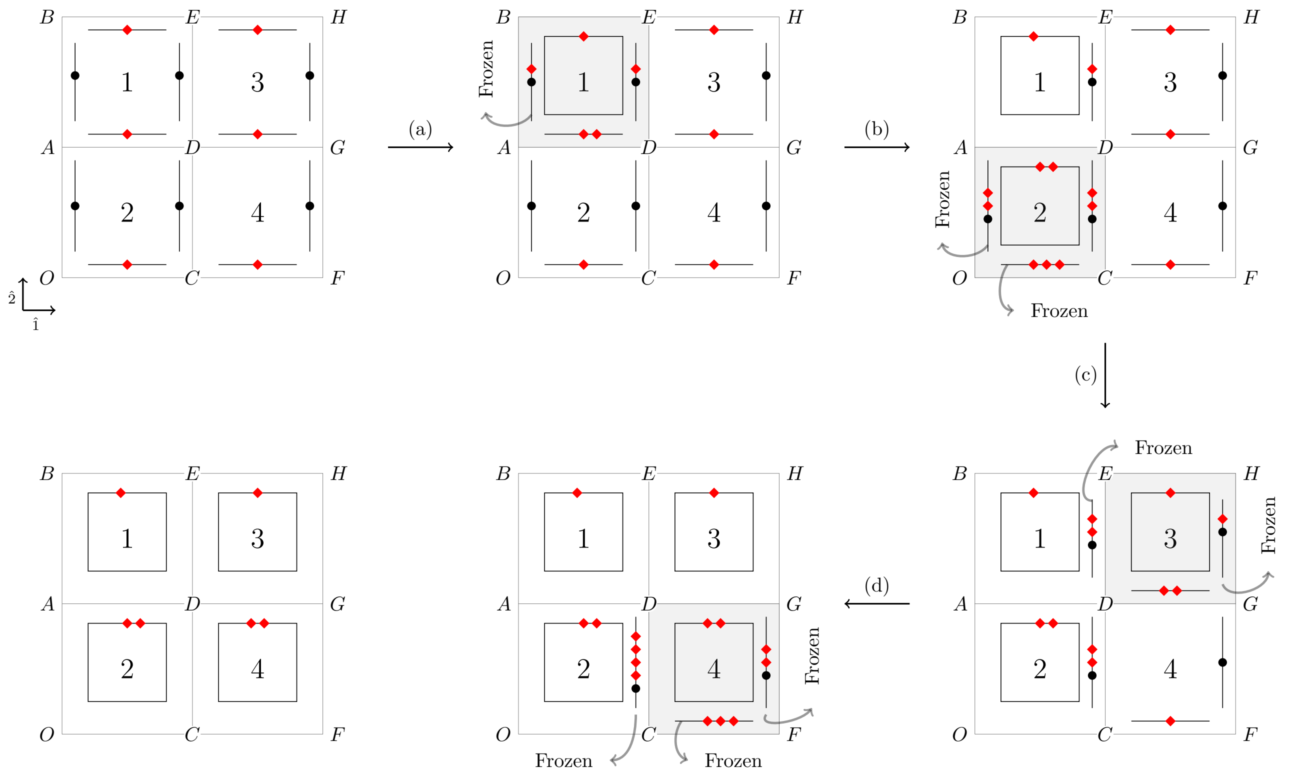

FIG. 11: The 'plaquette' canonical transformations involved in the construction of the duality transformation between Z_2 lattice gauge theory and Z_2 spin model on a 2×2 lattice. The steps (a), (b), (c) and (d) are plaquette CTs on plaquettes 1, 2, 3 and 4 respectively. The electric field $\sigma_1(l)$ corresponding to the vertical and horizontal links are denoted by $\bullet$ and $\blacklozenge$ respectively.

gate operators $\{\mu_1(1); \mu_3(1)\}$ on plaquette 1 are

$$\mu_1(1) \equiv \mu_1(E) = \sigma_3(A,\hat{1})\sigma_3(D,\hat{2})\sigma_3(B,\hat{1})\sigma_3(A,\hat{2}),$$
$$\mu_3(1) \equiv \mu_3(E) = \sigma_1(B,\hat{1}). \tag{A3}$$

The redefined link and string operators around plaquette 1 are

$$\sigma_{3[x]}(D,\hat{2}) = \sigma_3(D,\hat{2}), \quad \sigma_{1[x]}(D,\hat{2}) = \sigma_1(D,\hat{2})\sigma_1(B,\hat{1})$$
$$\sigma_{3[x]}(A,\hat{2}) = \sigma_3(A,\hat{2}), \quad \sigma_{1[x]}(A,\hat{2}) = \sigma_1(A,\hat{2})\sigma_1(B,\hat{1})$$
$$\sigma_{3[x]}(A,\hat{1}) = \sigma_3(A,\hat{1}), \quad \sigma_{1[x]}(A,\hat{1}) = \sigma_1(A,\hat{1})\sigma_1(B,\hat{1}).$$

Our notation is such that $\sigma_3(A,\hat{1})$ denotes the σ_3 variable of the link which starts at site A and is in the $\hat{1}$ direction. The subscript $[x]$ on $\sigma_{3[x]}(A,\hat{1})$ indicates that the electric field $\sigma_1(A,\hat{1})$ absorbs the electric field of the vanishing horizontal link $(B,\hat{1})$ to become $\sigma_{1[x]}(A,\hat{1})$ during the plaquette C.T. Note that by our convention, the plaquette or spin operators are labelled by the top right corner of the plaquette. This plaquette C.T is illustrated in Figure 11 (a). As a result of Gauss law at B:

$$\sigma_{1[x]}(A,\hat{2}) \equiv \mathcal{G}(B) \approx 1.$$

Therefore, $\{\sigma_{1[x]}(A,\hat{2}); \sigma_{3[x]}(A,\hat{2})\} \equiv \{\bar{\sigma}_1(B); \bar{\sigma}_3(B)\}$ are frozen and hence decouple from the physical Hilbert space. Again, as in the main text, the string operators are labelled by their right/top endpoints. We are now left with the conjugate spin operators $\{\mu_1(1); \mu_3(1)\}$ and the two link conjugate pair operators $\{\sigma_{1[x]}(D,\hat{2}); \sigma_{3[x]}(D,\hat{2})\}$, $\{\sigma_{1[x]}(A,\hat{1}); \sigma_{3[x]}(A,\hat{1})\}$. These link operators undergo further canonical transformations.

2. We now iterate the plaquette C.T. on plaquette 2 to construct the spin or plaquette conjugate operators $\{\mu_1(2); \mu_3(2)\}$ and the link conjugate operators $\{\sigma_{1[x]}(C,\hat{2}); \sigma_{3[x]}(C,\hat{2})\}$, $\{\sigma_{1[x]}(O,\hat{1}); \sigma_{3[x]}(O,\hat{1})\}$, $\{\sigma_{1[x]}(O,\hat{2}); \sigma_{3[x]}(O,\hat{2})\}$ as illustrated in Figure 11-b. The spin operators are

$$\mu_1(2) \equiv \mu_1(D) = \sigma_3(A,\hat{1})\sigma_3(O,\hat{2})\sigma_3(O,\hat{1})\sigma_3(C,\hat{2}),$$
$$\mu_3(2) \equiv \mu_3(D) = \sigma_{1[x]}(A,\hat{1}) = \sigma_1(A,\hat{1})\sigma_1(B,\hat{1}) \tag{A4}$$

The redefined link and new string operators around plaquette 2 are

$$\sigma_{3[x]}(C,\hat{2}) = \sigma_3(C,\hat{2}),$$
$$\sigma_{1[x]}(C,\hat{2}) = \sigma_1(C,\hat{2})\sigma_{1[x]}(A,\hat{1}) = \sigma_1(C,\hat{2})\sigma_1(A,\hat{1})\sigma_1(B,\hat{1})$$
$$\sigma_{3[x]}(O,\hat{1}) = \sigma_3(O,\hat{1}),$$

$$\sigma_{1[x]}(O,\hat{1}) = \sigma_1(O,\hat{1})\sigma_{1[x]}(A,\hat{1}) = \mathcal{G}(O)\mathcal{G}(A)\mathcal{G}(B) \approx 1$$
$$\sigma_{3[x]}(O,\hat{2}) = \sigma_3(O,\hat{2}), \quad \text{(A5)}$$
$$\sigma_{1[x]}(O,\hat{2}) = \sigma_1(O,\hat{2})\sigma_{1[x]}(A,\hat{1}) = \mathcal{G}(A)\mathcal{G}(B) \approx 1.$$

Thus the string conjugate pairs $\{\sigma_{1[x]}(O,\hat{1}); \sigma_{3[x]}(O,\hat{1})\} \equiv \{\bar{\sigma}_1(C); \bar{\sigma}_3(C)\}$ and $\{\sigma_{1[x]}(O,\hat{2}); \sigma_{3[x]}(O,\hat{2})\} \equiv \{\bar{\sigma}_1(A); \bar{\sigma}_3(A)\}$ are frozen due to Gauss law at O, A and B.

3. The third step involves iterating the plaquette C.T. on plaquette 3 as shown in Figure 11(c). This leads to decoupling of $\{\sigma_{1[x]}(G,\hat{2}); \sigma_{3[x]}(G,\hat{2})\} \equiv \{\bar{\sigma}_1(H); \bar{\sigma}_3(H)\}$, $\{\sigma_{1[xx]}(D,\hat{2}); \sigma_{3[xx]}(D,\hat{2})\} \equiv \{\bar{\sigma}_1(E); \bar{\sigma}_3(E)\}$ due to the Z_2 Gauss laws at E and H. The canonical transformations on plaquette 3 defining the spins are

$$\mu_1(3) \equiv \mu_1(H) = \sigma_3(E,\hat{1})\sigma_3(G,\hat{2})\sigma_3(D,\hat{1})\sigma_3(D,\hat{2}),$$
$$\mu_3(3) \equiv \mu_3(H) = \sigma_3(E,\hat{1}). \quad \text{(A6)}$$

The redefined links and strings around plaquette 3 are

$$\sigma_{3[xx]}(D,\hat{2}) = \sigma_{3[x]}(D,\hat{2}) = \sigma_3(D,\hat{2}),$$
$$\sigma_{1[xx]}(D,\hat{2}) = \sigma_{1[x]}(D,\hat{2})\sigma_1(E,\hat{1}) = \mathcal{G}(E) \approx 1$$
$$\sigma_{3[x]}(G,\hat{2}) = \sigma_3(G,\hat{2}),$$
$$\sigma_{1[x]}(G,\hat{2}) = \sigma_1(G,\hat{2})\sigma_1(E,\hat{1}) = \mathcal{G}(H) \approx 1$$
$$\sigma_{3[x]}(D,\hat{1}) = \sigma_3(D,\hat{1}),$$
$$\sigma_{1[x]}(D,\hat{1}) = \sigma_1(D,\hat{1})\sigma_1(E,\hat{1}) \quad \text{(A7)}$$

4. Finally, we iterate the plaquette C.T. on plaquette 4 which are shown in Figure 11(d). The conjugate spin operators $\{\mu_1(4); \mu_3(4)\}$ on plaquette 4 are

$$\mu_1(4) \equiv \mu_1(G) = \sigma_3(D,\hat{1})\sigma_3(F,\hat{2})\sigma_3(C,\hat{1})\sigma_3(C,\hat{2}),$$
$$\mu_3(4) \equiv \mu_3(G) = \sigma_{1[x]}(D,\hat{1}) = \sigma_1(D,\hat{1})\sigma_1(E,\hat{1}) \quad \text{(A8)}$$

The remaining string operators are

$$\sigma_{3[xx]}(C,\hat{2}) = \sigma_{3[x]}(C,\hat{2}) = \sigma_3(C,\hat{2}),$$
$$\sigma_{1[xx]}(C,\hat{2}) = \sigma_{1[x]}(C,\hat{2})\sigma_{1[x]}(D,\hat{1}) = \mathcal{G}(D)\mathcal{G}(E) \approx 1$$
$$\sigma_{3[x]}(C,\hat{1}) = \sigma_3(C,\hat{1})$$
$$\sigma_{1[x]}(C,\hat{1}) = \sigma_1(C,\hat{1})\sigma_{1[x]}(D,\hat{1}) = \mathcal{G}(C)\mathcal{G}(O)\mathcal{G}(A)\mathcal{G}(D)\mathcal{G}(B)\mathcal{G}(E) \approx 1$$
$$\sigma_{3[x]}(F,\hat{2}) = \sigma_3(F,\hat{2}), \quad \text{(A9)}$$
$$\sigma_{1[x]}(F,\hat{2}) = \sigma_1(F,\hat{2})\sigma_{1[x]}(D,\hat{1}) = \mathcal{G}(G)\mathcal{G}(H) \approx 1.$$

Gauss laws at O, A, B, C, D, E, G and H implies that the remaining string operators $\{\sigma_{1[xx]}(C,\hat{2}); \sigma_{3[xx]}(C,\hat{2})\} \equiv \{\bar{\sigma}_1(D); \bar{\sigma}_3(D)\}$, $\{\sigma_{1[x]}(C,\hat{1}); \sigma_{3[x]}(C,\hat{1})\} \equiv \{\bar{\sigma}_1(F); \bar{\sigma}_3(F)\}$ and $\{\sigma_{1[x]}(F,\hat{2}); \sigma_{3[x]}(F,\hat{2})\} \equiv \{\bar{\sigma}_1(G); \bar{\sigma}_3(G)\}$ are frozen. As a result, after the series of 4 plaquette C.T.s, all the Gauss law constraints are solved. Only the plaquette/spin variables $\{\mu_1(1); \mu_3(1)\}, \{\mu_1(2); \mu_3(2)\}, \{\mu_1(3); \mu_3(3)\}$ and $\{\mu_1(4); \mu_3(4)\}$ remains in the physical Hilbert space. This leads to a dual Z_2 spin model. These results can be directly generalized to any finite lattice without any new issues, to give the duality relations (22a),(22b), (24a)-(24b).

Appendix B: The ground state & area law

In this Appendix, we calculate the expectation value of a large Wilson loop W_C in the variational ground state $|\psi_0\rangle$ and show that it satisfies Wilson's Area law criterion. Any Wilson loop can be written as the product of fundamental plaquette loop operators, $W_C = \prod_{p_i} \mathcal{W}(p_i)$. Here, p_i denotes the plaquettes inside the loop C in the order bottom right to top left (See Figure 10). It is convenient to define a complete basis $\prod_p |\omega_p, \hat{w}_p\rangle$,which diagonalises all Wilson loops. Above, $\prod_p$ is over all the plaquettes in the lattice and

$$|\omega_p, \hat{w}_p\rangle = \sum_{jm_-m_+} \sqrt{2j+1}\; D^{\,j}_{m_-m_+}(\omega_p, \hat{w}_p)\, |jm_-m_+\rangle_p\,. \quad \text{(B1)}$$

In (B1), $D^{\,j}_{m_-m_+}(\omega_p, \hat{w}_p)$ is a Wigner D matrix and $|jm_-m_+\rangle_p$ is the eigenbasis of $\left(\vec{\mathcal{E}}_+(p)\right)^2 = \left(\vec{\mathcal{E}}_-(p)\right)^2 \equiv \left(\vec{\mathcal{E}}(p)\right)^2$, $\vec{\mathcal{E}}^{a=3}_-(p)$ and $\vec{\mathcal{E}}^{a=3}_+(p)$. Also, $(\omega_p, \hat{w}_p)$ is the angle axis parameterization of the $SU(2)$ group element associated with the plaquette loop operator at p, as defined in section II B 6. The plaquette loop operator $\mathcal{W}_{\alpha\beta}(p)$ is diagonal in this basis,

$$\mathcal{W}_{\alpha\beta}(p)|\omega_p, \hat{w}_p\rangle = z_{\alpha\beta}(p)|\omega_p, \hat{w}_p\rangle.$$

with

$$z_{\alpha\beta} = \begin{bmatrix} \left(\cos\frac{\omega}{2} - i\sin\frac{\omega}{2}\cos\theta\right) & i\sin\frac{\omega}{2}\sin\theta e^{-i\phi} \\ -i\sin\frac{\omega}{2}\sin\theta e^{-i\phi} & \left(\cos\frac{\omega}{2} + i\sin\frac{\omega}{2}\cos\theta\right) \end{bmatrix}_{\alpha\beta}$$

where θ and ϕ are the angles characterizing $\hat{w}_p$. In particular,

$$Tr\mathcal{W}(p)|\omega_p, \hat{w}_p\rangle = 2\cos(\omega_p/2)|\omega_p, \hat{w}_p\rangle.$$

The expectation value of TrW_C in $|\psi_0\rangle$ is given by

$$\begin{aligned}\langle TrW_C\rangle &\equiv \frac{\langle\psi_0|TrW_C|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} \\ &= \frac{1}{\langle\psi_0|\psi_0\rangle}\prod_{p\in p_i}\int d\mu(\omega_p, \hat{w}_p)\langle 0|e^S Trz(C)|\omega_p, \hat{w}_p\rangle\langle\omega_p, \hat{w}_p|0\rangle \\ &= \frac{\prod_p \int d\mu(\omega_p, \hat{w}_p)\; e^{2\alpha\cos\omega_p/2}\; 2\cos\left(\omega(C)/2\right)}{\prod_p \int d\mu(\omega_p, \hat{w}_p)\; e^{2\alpha\cos(\frac{\omega_p}{2})}} \end{aligned} \quad \text{(B2)}$$

In (B2), $\int d\mu(\omega_p, \hat{\omega}_p) \equiv \int_0^{2\pi} 4\sin^2\frac{\omega}{2}d\omega \int_0^{\pi}\sin\theta d\theta \int_0^{2\pi} d\phi$. We have also used the completeness relation of the $|\omega, \hat{w}\rangle$

basis. $z(C)$ is the eigenvalue of W_C corresponding to the eigenstate $\prod_p |\omega_p, \hat{w}_p\rangle$. Since $W_C = \prod_{p_i} \mathcal{W}(p_i)$, $z(C) = \prod_{p_i} z(p_i)$ and $Trz(C) = 2\cos(\omega(C)/2)$. Here, $\omega(C)$ is the gauge invariant angle characterizing the $SU(2)$ matrix $z(C)$ in its angle axis representation. Using the expression for the product of 2 SU(2) matrices [33] repeatedly, it is easy to show that $\cos(\omega(C)/2) = \prod_i \cos(\omega_{p_i}/2)$ + terms which vanish on θ integration[34] Therefore,

$$\langle TrW_C\rangle = 2\left(\frac{I_2(2\alpha)}{I_1(2\alpha)}\right)^{n_c} = 2e^{-n_c\ \ln\left(\frac{I_1(2\alpha)}{I_2(2\alpha)}\right)} \tag{B3}$$

In (B3), n_c is the number of plaquettes in the loop C and $I_l(2\alpha)$ is the l-th order modified Bessel function of the first kind. We have used the relation

$$I_l(2\alpha) = \frac{1}{\pi}\int_0^{\pi} e^{2\alpha\cos\omega}\cos l\omega\ d\omega \tag{B4}$$

and the recurrence relation [32]

$$I_{l-1}(2\alpha) - I_{l+1}(2\alpha) = \frac{2l}{2\alpha} I_l(2\alpha) \tag{B5}$$

to arrive at (B3).

The expectation value of the disorder operator in the variational ground state $|\psi_0\rangle$ is

$$\frac{\langle\psi_0|\Sigma^{\pm}_{\theta=2\pi}(P)|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = \frac{\prod\limits_{\bar{p}\neq P} {}_{\bar{p}}\langle\psi_0|\psi_0\rangle_{\bar{p}}\ {}_P\langle\psi_0|\Sigma^{\pm}_{2\pi}(P)|\psi_0\rangle_P}{\prod\limits_{\bar{p}\neq P} {}_{\bar{p}}\langle\psi_0|\psi_0\rangle_{\bar{p}}\ {}_P\langle\psi_0|\psi_0\rangle_P}$$

$$= \frac{\int d\mu(\omega_p,\hat{\omega}_p) e^{2\alpha[\cos(\frac{\omega_P+2\pi}{2})+\cos\frac{\omega_P}{2}]}}{\int d\mu(\omega_p,\hat{\omega}_p)e^{2\alpha\cos\omega_P/2}} = \frac{\pi\alpha}{2I_1(2\alpha)} \tag{B6}$$

Above, we have again used relations (B4) and (B5) to get the last equality.

Appendix C: Calculation of $\frac{\langle\psi_0|H_{spin}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}$.

The local effective SU(2) spin model Hamiltonian is

$$H_{spin} = \sum_{p=1}^{\mathcal{P}}\left\{4g^2\vec{\mathcal{E}}^{\ 2}(p) + \frac{1}{g^2}\left[2-(Tr\mathcal{W}(p))\right]\right\} + g^2\sum_{\langle p,p'\rangle}\left\{\vec{\mathcal{E}}_-(p)\cdot\vec{\mathcal{E}}_+(p')\right\} \tag{C1}$$

First, lets calculate $\langle\psi_0|\mathcal{E}^a_-(p)\mathcal{E}^a_+(P)|\psi_0\rangle$. Here, P is any plaquette.

$$\begin{aligned}\langle\psi_0|&\mathcal{E}^a_-(p)\mathcal{E}^a_+(P)|\psi_0\rangle\\ &= \left\langle 0\right|\left(e^{S/2}\mathcal{E}^a_-(p)e^{-S/2}\right)e^S\left(e^{-S/2}\mathcal{E}^a_+(P)e^{S/2}\right)\left|0\right\rangle\\ &= \frac{-1}{4}\langle\psi_0|\left[\mathcal{E}^a_-(p),S\right]\left[\mathcal{E}^a_+(P),S\right]|\psi_0\rangle\end{aligned} \tag{C2}$$

In (C2), we have used the fact that $\mathcal{E}_\pm|0\rangle = 0$. Evaluating $\langle\psi_0|\mathcal{E}^a_-(p)\mathcal{E}^a_+(P)|\psi_0\rangle$ in a different way,

$$\begin{aligned}\langle\psi_0|&\mathcal{E}^a_-(p)\mathcal{E}^a_+(P)|\psi_0\rangle\\ &= \left\langle 0\right|e^{S/2}\mathcal{E}^a_-(p)e^{S/2}\left(e^{-S/2}\mathcal{E}^a_+(P)e^{S/2}\right)\left|0\right\rangle\\ &= \frac{1}{2}\langle\psi_0|\Big[\mathcal{E}^a_-(p),\left[\mathcal{E}^a_+(P),S\right]\Big]|\psi_0\rangle\\ &\quad + \frac{1}{4}\langle\psi_0|\left[\mathcal{E}^a_+(P),S\right]\left[\mathcal{E}^a_-(p),S\right]|\psi_0\rangle\end{aligned} \tag{C3}$$

The equations (C2) and (C3) implies:

$$\langle\psi_0|\mathcal{E}_-(p)\cdot\mathcal{E}_+(P)|\psi_0\rangle = \frac{1}{4}\langle\psi_0|\Big[\mathcal{E}^a_-(p),\left[\mathcal{E}^a_+(P),S\right]\Big]|\psi_0\rangle \tag{C4}$$

The expression in (C4) vanishes when $P\neq p$. In particular,

$$\begin{aligned}\langle\psi_0|\mathcal{E}_-(p)\cdot\mathcal{E}_+(p')|\psi_0\rangle &= 0,\\ \langle\psi_0|\mathcal{E}_-(p)\cdot\mathcal{E}_-(p)|\psi_0\rangle &= \tfrac{3\alpha}{16}\langle\psi_0|Tr\mathcal{W}(p)|\psi_0\rangle.\end{aligned} \tag{C5}$$

Above p, p' are nearest neighbours. Putting $n_c = 1$ in equation (B3), $\langle Tr\mathcal{W}(p)\rangle = \frac{2I_2(2\alpha)}{I_1(2\alpha)}$. Using the above relations, the expectation value of H_{spin} is

$$\frac{\langle\psi_0|H_{spin}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = \sum_p\left\{\left(\frac{3\alpha}{4}g^2 - \frac{1}{g^2}\right)\frac{2I_2(2\alpha)}{I_1(2\alpha)} + \frac{2}{g^2}\right\} \tag{C6}$$

The general non-local Hamiltonian H differs from the above effective local spin Hamiltonian H_{spin} by terms of the form $R_{ab}(W)\ \mathcal{E}^a_-(p)\mathcal{E}^b_+(\bar{p})$, where p and $\bar{p}$ are any 2 plaquettes on the lattice which are at least 2 lattice spacing away from each other. Above, W is in general the product of many plaquette loop operators. The expectation value of the full Hamiltonian in the variational ground state $|\psi_0\rangle$ reduces to $\langle\psi_0|H_{spin}|\psi_0\rangle$ as the expectation value of the non-local terms in $|\psi_0\rangle$ vanishes.

[1] F.J. Wegner, J. Math. Phys. **12** (1971) 2259, arXiv:1411.5815.

[2] H.A. Kramers and G.H. Wannier, Phys. Rev. **60** (1941) 252, 263; G.H. Wannier, Rev. Mod. Phys. **17** (1945) 50.

[3] S. Mandelstam, Phys. Rev. **D 19** (1979) 2391.

[4] G. t' Hooft, Nucl. Phys. **B 138** (1978) 1, Nucl. Phys. B 153 (1979) 141; G. t' Hooft in 'Recent Progress in Gauge Theories', G. 't Hooft, et al., Plenum Press, 117, (1980), G. Mack, ibidem, 217, M. E. Peskin, Annals of Phys. 113 (1978) 122.

[5] T. Banks, R. Myerson, J. Kogut, Nucl. Phys. **B 129** (1977) 493.

[6] E. Fradkin, L. Susskind, Phys. Rev. **D 17** (1978) 2637.
[7] J.B. Kogut, Rev. Mod. Phys. **51** (1979) 659.
[8] E. Fradkin, S. Raby, Phys. Rev. **D 20** (1979) 2566.
[9] R. Savit, Rev. Mod. Phys. **52**(1980) 453.
[10] D. Horn, M. Weinstein, S. Yankielowicz, Phys. Rev. **D 19** (1979) 3715, T. Yoneya, Nucl. Phys. **B 144** (1978) 195, A. Ukawa and P. Windey, A. H. Guth, Phys. Rev. **D 21** (1980) 1013, C.P. Korthal Altes, Nucl. Phys. **B 142** (1978) 315, G. Mack, V. B. Petkova, 123 (1979) 442.
[11] T. Schultz, D. Mattis, and E. Lieb, Rev. Mod. Phys. 36 (1964) 856.
[12] E. Fradkin, M. Srednicki, L. Susskind, Phys. Rev. **D 21** (1980) 2885; M. Srednicki, Phys. Rev. **D 21** (1980) 2878.
[13] E. Fradkin, "Field Theories of Condensed Matter Physics", Second Edition, Cambridge University Press, Cambridge, UK, 2013.
[14] A.Y. Kitaev, Ann. Phys., 303 (2003) 2.
[15] T. Sugihara, JHEP 0507 (2005) 022.
[16] B. Swingle, X. G. Wen, arXiv:1001.4517; S. Aoki, T. Iritani, M. Nozaki, T. Numasawa, N. Shiba, H. Tasaki, JHEP 1506 (2015) 187.
[17] E. Zohar, J. I. Cirac, B. Reznik, phys. Rev. **A 88** (2013) 023617.
[18] P.V. Buividovich, M.I. Polikarpov, Phys. Lett. **B 670** (2008) 141.
[19] J. Kogut, L Susskind, Phys. Rev. **D 11** (1975) 395.
[20] Ramesh Anishetty and H. S. Sharatchandra, Phys. Rev. Letts., 65 (1990) 813; R. Anishetty, P. Majumdar, H.S. Sharatchandra, Phys. Lett. B 478 (2000) 373; R. Oeckl, H. Pfeiffer Nucl.Phys. **B 598** (2001) 400.
[21] M. Mathur, Nuclear Physics **B 779** (2007) 32-62; M. Mathur, Phys. Lett. **B 640** (2006) 292.
[22] Manu Mathur, T. P. Sreeraj, Phys. Rev. **D 92** (2015) 125018.
[23] Manu Mathur, T. P. Sreeraj, Phys. Lett. **B 749** (2015) 137.
[24] L. P. Kadanoff, H. Ceva, Phys. Rev. **B 11** (1971) 3918.
[25] P. Suranyi , Nuclear Physics **B 210** (1982) 519.
[26] H.Arisue, M.Kato and T.Fujiwara, Prog. Theor. Phys 70 (1983) 229 ; H. Arisue , Prog. Theor. Phys, 84 (1990) 951.
[27] D. W. Heys and D. R. Stump, Phys. Rev. **D 29** (1984) 1791. and references therein.
[28] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, *Quantum Theory of Angular Momentum*, World Scientific (1988).
[29] Daniel R. Stump, Journ. of Math. Phys. 39 (1998) 3661.
[30] R. P. Feynman, Nucl. Phys. **B 188** (1981) 479.
[31] J.P, Greensite, Nucl. Phys. **B 158** (1979) 469; Nucl. Phys. **B 166** (1980) 113; T. Hofsäs and R. Horsley, Phys. Lett. **B 123** (1983) 65.
[32] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of standards, Applied mathematical series 55 (1964).
[33] Product of 2 SU(2) matrices characterized by $(\omega_1, \hat{w}_1)$ and $(\omega_2, \hat{w}_2)$ gives an SU(2) matrix characterized by $(\omega, \hat{w})$ with

$$\begin{aligned} \cos\frac{\omega}{2} &= \cos\frac{\omega_1}{2}\cos\frac{\omega_2}{2} - (\hat{w}_1\cdot\hat{w}_2)\sin\frac{\omega_1}{2}\sin\frac{\omega_2}{2}; \\ \hat{w}\sin\frac{\omega}{2} &= \hat{w}_1\sin\frac{\omega_1}{2}\cos\frac{\omega_2}{2} + \hat{w}_2\sin\frac{\omega_2}{2}\cos\frac{\omega_1}{2} \\ &\quad - [\hat{w}_1\times\hat{w}_2]\sin\frac{\omega_1}{2}\sin\frac{\omega_2}{2}. \end{aligned} \tag{C7}$$

[34] The integrand under θ integration contains either $\sin 2\theta$ or a $\cos\theta$, both vanish on θ integration from 0 to π.