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Duality in Generalized Ising Models and Phase Transitions without Local Order Parameters*

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It is shown that any Ising model with positive coupling constants is related to another Ising model by a duality transformation. We define a class of Ising models M_{dn} on d-dimensional lattices characterized by a number $n = 1, 2, \ldots, d$ ($n = 1$ corresponds to the Ising model with two-spin interaction). These models are related by two duality transformations. The models with $1 < n < d$ exhibit a phase transition without local order parameter. A nonanalyticity in the specific heat and a different qualitative behavior of certain spin correlation functions in the low and the high temperature phases indicate the existence of a phase transition. The Hamiltonian of the simple cubic dual model contains products of four Ising spin operators. Applying a star square transformation, one obtains an Ising model with competing interactions exhibiting a singularity in the specific heat but no long-range order of the spins in the low temperature phase.

1. INTRODUCTION

This paper deals with a general concept of duality and with phase transitions without a local order parameter.

Duality[1–5] is an inherent symmetry of the two-dimensional Ising model without crossing interaction bonds. This symmetry relates the partition function and the correlation functions[6–8] of a two-dimensional Ising model at temperature T to the partition function and the correlation functions of its dual Ising model at temperature T^*, where T^* is a decreasing function of T. In this paper the duality transformation is generalized to arbitrary Ising models with positive interaction constants (Sec. 2). This concept of duality is applied to a class of Ising models M_{dn} on d-dimensional lattices (Sec. 3). To obtain the Hamiltonian of the model M_{dn}, one takes the product of all spins located at the two ends of lines ($n = 1$), at the perimeter of surfaces ($n = 2$), and so on. Therefore, $n = 1$ describes the usual Ising model with two-spin interactions. The systems M_{dn} and $M_{d\,d-n}$ on dual lattices without external magnetic field are connected by a duality relation (Sec. 3A). For even dimensions $d = 2n$, one obtains self-dual models (models which are identical with their dual models). If there is only one singularity in the partition function of a self-dual model, then it must occur at $T = T^*$. Self-duality implies a symmetric singularity of the specific heat around the critical temperature (Sec. 3C). If an external magnetic field is present, the systems M_{dn} and $M_{d\,d-n+1}$ on dual lattices are connected by duality relations (Sec. 3A, 3C).

Most known phase transitions can be described by a local order parameter.[9–14] The models M_{dn} with $1 < n < d$ exhibit a phase transition without a local order parameter (Sec. 3B). The existence of a phase transition is indicated by a singularity in the specific heat (at least for $n = d - 1$) and by a qualitatively different asymptotic behavior of certain correlation functions at high and at low temperatures (Sec. 3B). For $n > 1$ the Hamiltonian consists of products of more than two spins. Applying the decoration,[15,16] the star triangle[3–5,17] and/or the star square[18] transformations, one reduces these models to Ising models with two-spin interactions (Sec. 2D). Thus the simple cubic dual model can be transformed to an Ising model with competing two-spin interactions. This model exhibits a singularity in the specific heat, but below the critical temperature there is no long range ordering of the spins (Sec. 4).

2. THE DUALITY TRANSFORMATION

The duality transformation for general Ising models is derived in this section. First (Sec. 2A) the Ising models with general interactions are defined, and some properties, like the degeneracy of the ground state and the spin correlation functions which vanish for all temperatures, are discussed. In Sec. 2B the duality relation for the partition function is stated and proved. The dislocation correlation functions are expressed both in terms of spin correlation functions of the original model and of the dual model in Sec. 2C. We show that a dual model exists for any Ising model (with positive interactions) and that this model can be reduced to an Ising model with only two-spin interactions and an external magnetic field (Sec. 2D).

A. The Model

The most general interaction of a system of N_s Ising spins $S(r) = \pm 1$, located at sites r of a lattice, is

$$H = -\textstyle\sum_b I(b) R(b), \tag{2.1}$$

in which $I(b)$ is the coupling constant of the interaction bond labeled by the index b and

$$R(b) = \Pi_r S(r)^{\theta(r,b)}, \quad \theta(r,b) \in \{0,1\}. \tag{2.2}$$

We express all quantities which may assume two values by the two elements of the set $\{0,1\}$,

$$S(r) = (-1)^{\sigma(r)}, \quad \sigma(r) \in \{0,1\}, \tag{2.3}$$

$$R(b) = (-1)^{\rho(b)}, \quad \rho(b) \in \{0,1\}. \tag{2.4}$$

We define the field operations of addition (modulo 2)

$$0 \oplus 0 = 1 \oplus 1 = 0, \quad 0 \oplus 1 = 1 \oplus 0 = 1 \tag{2.5}$$

and multiplication (modulo 2)

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1 \tag{2.6}$$

for the set $\{0,1\}$.

Then Eq. (2.2) can be written

$$\rho(b) = \oplus_r\ \theta(r,b)\,\sigma(r). \tag{2.7}$$

The operation symbol $\oplus$ with an index denotes summation (2.5) over this index. Let N_θ be the rank of the matrix $\theta(r,b)$. Then there are 2^{N_θ} different configurations[19] $\{\rho(b)\}$. We now restrict ourselves to systems with positive interaction constants, $I(b) > 0$. The ground states of the system (2.1) are defined by $R(b) = 1$ for all b. Therefore, the ground states are determined by the solutions $\{\sigma_0(r)\}$ of the homogeneous equations

$$\oplus_r\,\theta(r,b)\,\sigma_0(r) = 0 \quad \text{for all } b. \tag{2.8}$$

This system of equations has 2^{N_g} solutions with

$$N_g = N_s - N_\theta. \tag{2.9}$$

Therefore, the ground state is 2^{N_g}-fold degenerate. We associate the unitary operators

$$U\{\sigma_0\} = \Pi_r\,S_x(r)^{\sigma_0(r)} \tag{2.10}$$

with all ground states $\{\sigma_0(r)\}$. The operator $S_x(r)$ flips the spin at site r,

$$S_x^2(r) = 1, \qquad S_x(r)\,S(r)\,S_x(r) = -S(r). \tag{2.11}$$

The operators U commute with all operators R:

$$U\{\sigma_0\}\,R(b)\,U\{\sigma_0\}^{-1} = R(b)\,(-1)^{\oplus_r\,\theta(r,b)\sigma_0(r)} = R(b). \tag{2.12}$$

Therefore, all the operators U commute with the Hamiltonian:

$$U\{\sigma_0\}\,HU\{\sigma_0\}^{-1} = H. \tag{2.13}$$

A product of spins $\Pi_r S(r)^{\psi(r)}$, $\psi(r) \in \{0,1\}$, is transformed by $U\{\sigma_0\}$ into

$$U\{\sigma_0\}\,\Pi_r\,S(r)^{\psi(r)}U\{\sigma_0\}^{-1} = \Pi_r\,S(r)^{\psi(r)}(-1)^{\oplus_r\,\psi(r)\sigma_0(r)} \tag{2.14}$$

This product of spins commutes with all operators U if and only if

$$\oplus_r\,\sigma_0(r)\,\psi(r) = 0 \tag{2.15}$$

for all configurations $\{\sigma_0(r)\}$. There are $2^{N_s - N_g} = 2^{N_\theta}$ solutions $\{\psi(r)\}$, since the configurations $\{\sigma_0(r)\}$ form an N_g-dimensional linear manifold.

The product of operators $\Pi_b R(b)^{\phi(b)}$, $\phi(b) \in \{0,1\}$, can be expressed as a product of spin operators,

$$\Pi_b R(b)^{\phi(b)} = \Pi_r\,S(r)^{\psi(r)} \tag{2.16}$$

with

$$\psi(r) = \oplus_b\,\theta(r,b)\,\phi(b). \tag{2.17}$$

Since the rank of the matrix $\theta(r,b)$ is N_θ, the products of $R(b)$ in Eq. (2.16) represent 2^{N_θ} different products of spin operators characterized by the sets $\{\psi(r)\}$ of Eq. (2.17). The products of Eq. (2.16) commute with all operators U. Since there are only 2^{N_θ} different spin products which commute with all U, it follows that a product of spin operators commutes with all operators U if and only if it is a product of operators R. A product of spin operators which does not commute with all operators $U\{\sigma_0\}$ vanishes, since from

$$U\,\Pi_r\,S(r)^{\psi(r)}\,U^{-1} = -\,\Pi_r\,S(r)^{\psi(r)} \tag{2.18}$$

and from Eq. (2.13) it follows that

$$\langle \Pi_r S(r)^{\psi(r)}\rangle = \langle \Pi_r S(r)^{\psi(r)}\,U^{-1}U\rangle = \langle U\,\Pi_r S(r)^{\psi(r)}\,U^{-1}\rangle = -\,\langle \Pi_r S(r)^{\psi(r)}\rangle = 0. \tag{2.19}$$

Therefore, the expectation value of a product of spin operators vanishes if this product cannot be represented by a product of operators R.

It follows from Eqs. (2.16) and (2.17) that those products of operators R which are unity for each spin configuration $\{S(r)\}$ are determined by the $2^{N_b - N_\theta}$ solutions $\{\phi_0(b)\}$ of the system of homogeneous equations

$$\oplus_b\,\theta(r,b)\,\phi_0(b) = 0 \quad \text{for all } r. \tag{2.20}$$

Hereafter we will call any product of operators which is unity for each spin configuration the unit element.

B. The Duality Relation for the Partition Functions

We call two Ising models which are characterized by matrices $\theta(r,b)$, $\theta^*(r^*,b)$ and coupling parameters $K(b) = \beta I(b)$, $K^*(b) = \beta^* I^*(b)$ (β and β^* are the inverse temperatures of these systems) dual to each other if they fulfill these three conditions:

(a) the closure condition

$$\oplus_b\,\theta(r,b)\,\theta^*(r^*,b) = 0 \tag{2.21}$$

for all pairs of r, r^*,

(b) the completeness relation

$$N_\theta + N_\theta^* = N_b, \tag{2.22}$$

in which N_θ and N_θ^* are the ranks of the matrices θ and θ^* and N_b is the number of bonds b, and

(c)

$$\tanh K(b) = e^{-2K^*(b)} \tag{2.23}$$

for all bonds b.

The symmetric partition functions $Y\{K\}$ and $Y^*\{K^*\}$,

$$Y\{K\} = Z\{K\}\,2^{-(N_s+N_g)/2}\,\Pi_b[\cosh 2K(b)]^{-1/2}, \tag{2.24}$$

$$Z\{K\} = \textstyle\sum_{\{S(r)\}} e^{-\beta H\{S\}}, \tag{2.25}$$

(and similarly for $Y^*\{K^*\}$) of two dual Ising models obey

$$Y\{K\} = Y^*\{K^*\}. \tag{2.26}$$

For the particular case of a planar Ising model without crossing bonds, this relation was proved by Wannier.[3] We prove now Eq. (2.26) for the general case, comparing the high-temperature expansion for Z with the low-temperature expansion for Z^*. From

$$Z\{K\} = \sum_{\{S(r)\}} e^{-\beta H\{S\}} = \sum_{\{S(r)\}} \Pi_b e^{K(b)R(b)} \tag{2.27}$$

one obtains

$$Z\{K\} = \Pi_b \cosh K(b) \sum_{\{\phi(b)\}} \Pi_b \tanh K(b)^{\phi(b)} \times \sum_{\{S(r)\}} \Pi_b R(b)^{\phi(b)}, \tag{2.28}$$

since

$$e^{K(b)R(b)} = \cosh K(b)[1 + R(b) \tanh K(b)] = \cosh K(b) \sum_{\{\phi(b)\}} \tanh K(b)^{\phi(b)} R(b)^{\phi(b)} \tag{2.29}$$

follows from $R(b) = \pm 1$.

If the product of the operators R in Eq. (2.28) is the unit element, then the sum over all spin configurations yields 2^{N_s}; otherwise the sum vanishes. The product of the operators R is the unit element for all sets $\{\phi_0(b)\}$ of Eq. (2.20) and only these sets. Therefore, it follows that

$$Z\{K\} = 2^{N_s} \Pi_b \cosh K(b) \sum_{\{\phi_0(b)\}} \Pi_b \tanh K(b)^{\phi_0(b)}. \tag{2.30}$$

The partition function $Z^*\{K^*\}$ can be written

$$Z^*\{K^*\} = \sum_{\{S(r^*)\}} e^{-\beta H^*\{S\}} = \sum_{\{S(r^*)\}} \Pi_b e^{K^*(b)R^*(b)} = \Pi_b e^{K^*(b)} \sum_{\{S(r^*)\}} \Pi_b e^{-2K^*(b)\rho^*(b)}, \tag{2.31}$$

since $R^*(b) = 1 - 2\rho^*(b)$. From the closure condition (2.21), one obtains

$$\oplus_b \theta(r,b)\, \rho^*(b) = \oplus_b \oplus_{r^*} \theta(r,b)\theta^*(r^*,b)\sigma(r^*) = 0. \tag{2.32}$$

Therefore, each set $\{\rho^*(b)\}$ obeys Eq. (2.20) with $\rho^*(b) = \phi_0(b)$. It follows that

$$Z^*\{K^*\} = \Pi_b e^{K^*(b)} \sum_{\{\phi_0(b)\}} N\{\phi_0\}\, \Pi_b e^{-2K^*(b)\phi_0(b)}. \tag{2.33}$$

Here $N\{\phi_0\}$ denotes the number of configurations $\{S(r^*)\}$ which obey

$$\phi_0(b) = \oplus_{r^*} \theta(r^*,b)\sigma(r^*) \quad \text{for all } b. \tag{2.34}$$

If for a given set $\{\phi_0(b)\}$ Eq. (2.34) has no solutions, then $N\{\phi_0\} = 0$; otherwise $N\{\phi_0\} = 2^{N_s*-N_\theta*}$. In particular, for $\beta^* = 0$ it follows that

$$Z^* = 2^{N_s^*} = \sum_{\{\phi_0(b)\}} N\{\phi_0\}. \tag{2.35}$$

There are $2^{N_b-N_\theta} = 2^{N_\theta^*}$ sets $\{\phi_0(b)\}$ [we used the completeness relation (2.22)]. Therefore all N obey $N\{\phi_0\} = 2^{N_s*-N_\theta*}$. From Eq. (2.33) one obtains

$$Z^*\{K^*\} = 2^{N_g^*} \Pi_b e^{K^*(b)} \sum_{\{\phi_0(b)\}} \Pi_b e^{-2K^*(b)\phi_0(b)}. \tag{2.36}$$

From Eqs. (2.30) and (2.36) one obtains $Y\{K\} = Y^*\{K^*\}$, Eq. (2.26), using Eq. (2.23).

If the completeness relation (2.22) is not fulfilled, but

$$N_b - N_\theta - N_\theta^* = N_m > 0, \tag{2.37}$$

and if all $K(b)$ and $K^*(b)$ are positive, then it follows from Eq. (2.33) that

$$Z^*\{K^*\} \le 2^{N_g^*} \Pi_b e^{K^*(b)} \sum_{\{\phi_0(b)\}} \Pi_b e^{-2K^*(b)\phi_0(b)}. \tag{2.38}$$

Using the analogous inequality for $Z\{K\}$, one obtains the inequality

$$2^{-N_m/2}\, Y\{K\} \le Y^*\{K^*\} \le 2^{N_m/2}\, Y\{K\}. \tag{2.39}$$

C. Dislocations

We now consider systems with magnetic dislocations. Let the operator $M(b)$ change the sign of the interaction constant $I(b)$ in the Hamiltonian. Then one obtains

$$\langle \Pi_b M(b)^{\phi^*(b)} \rangle \{K\} = \langle \Pi_b e^{-2\phi^*(b)K(b)R(b)} \rangle = \langle \Pi_b [\cosh 2K(b) - R(b) \sinh 2K(b)]^{\phi^*(b)} \rangle \tag{2.40}$$

and

$$\langle \Pi_b M(b)^{\phi^*(b)} \rangle \{K\} = Z\{(-1)^{\phi^*}K\}/Z\{K\} = Y\{(-1)^{\phi^*}K\}/Y\{K\} \tag{2.41}$$

with $\phi^*(b) \in \{0, 1\}$. From $\tanh K = e^{-2K^*}$, Eq. (2.23), it follows that

$$\tanh(-1)^{\phi^*} K = e^{-2K^* - i\pi\phi^*}. \tag{2.42}$$

Substituting Eq. (2.26) into (2.41) and using (2.42), one obtains

$$\langle \Pi_b M(b)^{\phi^*(b)} \rangle \{K\} = Y^*\{K^* + \tfrac{1}{2} i\pi\phi^*\}/Y^*\{K^*\} = i^{-\Sigma_b \phi^*(b)} \langle \Pi_b e^{i\pi\phi^*(b)R^*(b)/2} \rangle \{K^*\} = \langle \Pi_b R^*(b)^{\phi^*(b)} \rangle \{K^*\}. \tag{2.43}$$

Therefore, the expectation value of a product of dislocation operators equals the expectation value of the corresponding product of operators R^* in the dual lattice. Since R^* is a product of spin operators $S(r^*)$, one may introduce corresponding operators $\mu(r^*)$ in the original model and represent $M(b)$ by

$$M(b) = \Pi_{r^*} \mu(r^*)^{\theta^*(r^*,b)}, \quad \mu^2(r^*) = 1. \tag{2.44}$$

Then one obtains

$$\langle \Pi_{r^*}\mu(r^*)^{\psi(r^*)}\rangle \{K\} = \langle \Pi_{r^*} S(r^*)\rangle^{\psi(r^*)}\rangle \{K^*\}. \tag{2.45}$$

For the particular case of the two-dimensional Ising model without crossing interactions this was derived by Kadanoff and Ceva.[7]

D. Construction of a Dual Ising Model: Reduction to Two-Spin Interactions

A dual Ising model exists for any given Ising model (with positive interactions) of Eq. (2.1). To obtain this dual model, one has to find a complete set of solutions $\{\phi_0(b)\}$ of Eq. (2.20). This set is complete if each solution $\{\phi_0(b)\}$ of Eq. (2.20) is a linear combination of the solutions of that set. Associate with each solution of the set a point $r^*\{\phi_0\}$. Then the lattice which is defined by the matrix

$$\theta^*(r^*\{\phi_0\}, b) = \phi_0\{b\} \tag{2.46}$$

is dual to the original lattice.

The Hamiltonian of the dual lattice may contain products of a large number of spins $S(r^*)$. We list three transformations[20] which reduce these systems with many-spin interactions to Ising models with two-spin interactions and possibly a magnetic field.

The Decoration Transformation[15,16]

The interaction $-IR_1R_2$, in which R_1 and R_2 are products of spins, can be reduced to an interaction $-I_1R_1S - I_2R_2S$, in which S is a new spin or a product of new spins

$$e^{KR_1R_2} = \tfrac{1}{2} f \sum_S e^{K_1R_1S + K_2R_2S} \tag{2.47}$$

with

$$f^2 = [\cosh(K_1 + K_2)\cosh(K_1 - K_2)]^{-1}, \tag{2.48}$$

$$\tanh K = \tanh K_1 \tanh K_2. \tag{2.49}$$

This transformation reduces products of more than three spins in the Hamiltonian to products of three spins.

A Generalized Triangle Transformation[3-5,17]

An interaction $-IS_1S_2S_3$ can be reduced to two-spin interactions and an interaction with a magnetic field by the transformation

$$\exp(KS_1S_2S_3) = \tfrac{1}{2} f \sum_S \exp[K_0S + (K_1S + K_2) \times (S_1 + S_2 + S_3) + K_3(S_1S_2 + S_1S_3 + S_2S_3)] \tag{2.50}$$

with

$$f^8 = f_0^{-1} f_1^{-3} f_2^{-3} f_3^{-1}, \quad e^{8K} = f_0^{-1} f_1^3 f_2^{-3} f_3, \tag{2.51}$$

$$e^{8K_2} = f_0 f_1 f_2^{-1} f_3^{-1}, \quad e^{8K_3} = f_0^{-1} f_1 f_2 f_3^{-1}, \quad f_n = \cosh[K_0 + (2n-3)K_1]. \tag{2.52}$$

For the particular choice $K_1 = -K_0$ the equations simplify to

$$f^8 = [\cosh^4(2K_0)\cosh(4K_0)]^{-1}, \quad e^{8K} = \cosh^4(2K_0)/\cosh(4K_0), \quad e^{8K_2} = e^{-8K_3} = \cosh(4K_0). \tag{2.53}$$

A Star Square Transformation[18]

If the Hamiltonian is invariant under flipping of all spins, then one may prefer to conserve this invariance. Products of more than four spins in the interaction can be reduced to four-spin interactions by the decoration transformation (2.47). The four-spin interactions are reduced to two-spin interactions by a star square transformation

$$\exp(KS_1S_2S_3S_4) = \tfrac{1}{2} f \sum_S \exp[K_0S(S_1 + S_2 + S_3 + S_4) + K_1(S_1S_2 + S_1S_3 + S_1S_4 + S_2S_3 + S_2S_4 + S_3S_4)] \tag{2.54}$$

with

$$e^{8K} = \cosh(4K_0)/\cosh^4(2K_0), \tag{2.55a}$$

$$e^{-8K_1} = \cosh(4K_0), \tag{2.55b}$$

$$f^8 = 1/[\cosh(4K_0)\cosh^4(2K_0)]. \tag{2.55c}$$

For real K_0 the right-hand side of Eq. (2.55a) is less than or equal to 1. Therefore, K must be negative or zero. To obtain negative K's, one may apply the decoration transformation with negative K_1 and K_2, Eq. (2.49).

Therefore, we have shown that there exists a dual Ising model (2.46) to any Ising model and that this can be reduced to an Ising model with only two-spin interactions and possibly an interaction with a magnetic field.

3. THE MODELS M_{dn} AND THEIR PROPERTIES

In this section we consider the models M_{dn}. In Sec. 3A we define the models and derive the duality relations which relate the systems M_{dn} and $M_{d\,d-n+1}$ in an external magnetic field and the duality relation between the systems M_{dn} and $M_{d\,d-n}$ without an external magnetic field. The behavior of the spin correlation functions at high and low temperatures is discussed in Sec. 3B. We prove that there is no local order parameter in the systems with $n > 1$. In Sec. 3C we discuss the thermodynamic properties of the systems.

A. The Models, Duality

We consider a d-dimensional hypervolume divided into C_d hypercells $B^{(d)}$. These are bounded by $(d-1)$-dimensional hypercells $B^{(d-1)}$ (total number C_{d-1}), these again by $(d-2)$-dimensional hypercells

$B^{(d-2)}$ (total number C_{d-2}), and so on, until we arrive at 0-dimensional hypercells which are simply the C_0 corners $B^{(0)}$ of the d-dimensional hypercells. For this original lattice L, one may construct a dual lattice L^* by placing one dual corner $B^{(0)*}$ in each original hypercell $B^{(d)}$, then connecting the dual corners by dual edges $B^{(1)*}$, each of which intersects one hypercell $B^{(d-1)}$, then connecting these dual edges by dual faces $B^{(2)*}$, each of which intersects one hypercell $B^{(d-2)}$, and so on, until we obtain the d-dimensional dual hypercells $B^{(d)*}$, each of which contains one original corner $B^{(0)}$. Denoting the number of the m-dimensional hypercells by C^*_m, we obtain

$$C^*_m = C_{d-m}, \tag{3.1}$$

since by construction there is a one-to-one correspondence of the m-dimensional dual hypercells to the $(d-m)$-dimensional original hypercells. Let us denote the intersection point of $B^{(m)}$ and its dual hypercell $B^{(d-m)*}$ by $r^{(m)} = r^{(d-m)*}$. Then a hypercell $B(r^{(m)})$ and a dual hypercell $B^*(r^{(m)})$ is associated with each point $r^{(m)}$.

Let us consider some examples. A *linear chain* ($d=1$, Fig. 1) of points $r^{(0)} = i$ (black circles) (we denote integers by i, j, k) divides the line into one-dimensional cells (segments). The dual lattice consists of the segments between the points $r^{(1)} = i + \frac{1}{2}$ (open circles). The *square lattice* ($d = 2$, Fig. 2) consists of the squares bounded by the continuous lines; its dual lattice consists of the squares bounded by the broken lines. The corners $r^{(0)} = (i, j)$ of the original lattice are denoted by black circles, the corners $r^{(2)} = (i+\frac{1}{2}, j+\frac{1}{2})$ of the dual lattice are denoted by open circles and the edges of the original lattice intersect the edges of the dual lattice at the points $r^{(1)}$ (triangles). In Fig. 3 a cube of the original *cubic lattice* ($d = 3$) and a cube of the dual lattice are drawn. The corners $r^{(0)} = (i, j, k)$ of the original lattice are denoted by black circles and the corners $r^{(3)} = (i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2})$ by open circles. The edges (continuous lines) of the original lattice and the faces of the dual lattice intersect at points $r^{(1)}$ (open squares), whereas the faces of the original lattice and the edges (broken lines) of the dual lattice intersect at points $r^{(2)}$ (black squares). We have considered only self-dual lattices, that is, lattices which are topologically equivalent to their dual lattice. Not all lattices are self-dual.

Now let us return to the general case and introduce the functions Θ and Θ^*. Let $\Theta(r^{(m-1)}, r^{(m)}) = 1$, if $r^{(m-1)}$ lies on the boundary of $B(r^{(m)})$; otherwise, $\Theta(r^{(m-1)}, r^{(m)}) = 0$. Let $\Theta^*(r^{(m)}, r^{(m-1)}) = 1$ if $r^{(m)}$ lies on the boundary of $B^*(r^{(m-1)})$; otherwise

$$\Theta^*(r^{(m)}, r^{(m-1)}) = \Theta(r^{(m-1)}, r^{(m)}), \tag{3.2}$$

that is, if $r^{(m-1)}$ lies on the boundary of $B(r^{(m)})$, then $r^{(m)}$ lies on the boundary of $B^*(r^{(m-1)})$.

Since the m-dimensional boundaries of $B^{(m+1)}$ form a *closed* m-dimensional hypersurface, two m-dimensional boundaries $B^{(m)}$ of $B^{(m+1)}$ meet in each $(m-1)$-dimensional hypercell at the boundary of $B^{(m+1)}$. Therefore, one obtains

$$\oplus_{r^{(m)}} \Theta(r^{(m-1)}, r^{(m)}) \Theta(r^{(m)}, r^{(m+1)}) = 0. \tag{3.3}$$

The Ising model M_{dn} on the lattice L with n dimensional bonds consists of Ising spins $S(r) = \pm 1$ at all sites $r = r^{(n-1)}$ interacting via

$$-\beta H_{dn} = K \textstyle\sum_{r^{(n)}} \Pi_r S(r)^{\theta(r, r^{(n)})} + h \sum_r S(r). \tag{3.4}$$

The product in the first term of the Hamiltonian runs over all spins $S(r)$ lying on the boundary of the n-dimensional hypercell $B(r^{(n)})$. For $n = 1$, the model (3.4) describes the Ising model with two-spin interactions between spins lying at the two ends of an edge and an external magnetic field

$$B = k_B \mathrm{Th}/\mu_B. \tag{3.5}$$

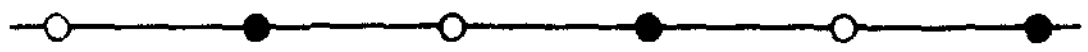

FIG. 1. The linear chain.

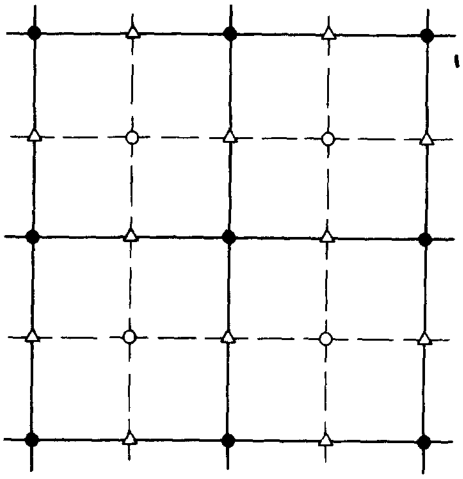

FIG. 2. The square lattice and its dual.

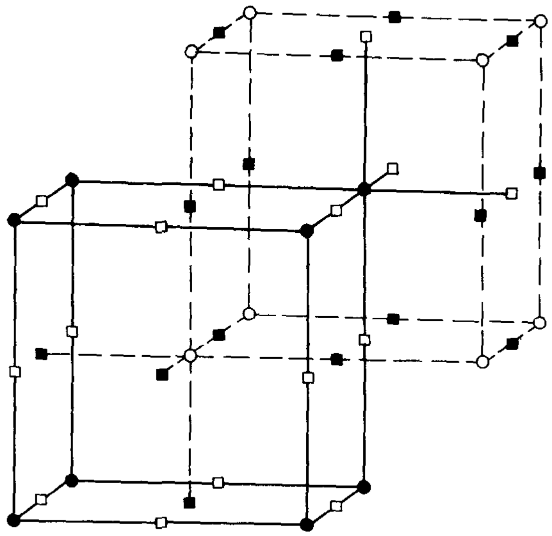

FIG. 3. The simple cubic lattice and its dual.

For the lattices considered above we obtain the Hamiltonians

$$-\beta H_{11}=K\sum_i S(i)S(i+1)+h\sum_i S(i), \tag{3.6}$$

$$-\beta H_{21}=K\sum_{ij}S(i,j)[S(i+1,j)+S(i,j+1)] + h\sum_{ij}S(i,j), \tag{3.7}$$

$$-\beta H_{31}=K\sum_{ijk}S(i,j,k)[S(i+1,j,k)+S(i,j+1,k) + S(i,j,k+1)]+h\sum_{ijk}S(i,j,k). \tag{3.8}$$

For $n=2$ the Hamiltonian contains the products of spins lying at the boundary of the faces $B^{(2)}$,

$$\begin{aligned}-\beta H_{22}=K\sum_{ij}&S(i,j+\tfrac{1}{2})S(i+1,j+\tfrac{1}{2})\\ &\times S(i+\tfrac{1}{2},j)S(i+\tfrac{1}{2},j+1)\\ &+h\sum_{ij}[S(i,j+\tfrac{1}{2})+S(i+\tfrac{1}{2},j)],\end{aligned} \tag{3.9}$$

$$\begin{aligned}-\beta H_{32}=K\sum_{ijk}&[S(i,j+\tfrac{1}{2},k)S(i+1,j+\tfrac{1}{2},k)\\ &\times S(i+\tfrac{1}{2},j,k)S(i+\tfrac{1}{2},j+1,k)\\ &+S(i,j,k+\tfrac{1}{2})S(i+1,j,k+\tfrac{1}{2})S(i+\tfrac{1}{2},j,k)\\ &\times S(i+\tfrac{1}{2},j,k+1)+S(i,j,k+\tfrac{1}{2})\\ &\times S(i,j+1,k+\tfrac{1}{2})S(i,j+\tfrac{1}{2},k)S(i,j+\tfrac{1}{2},k+1)]\\ &+h\sum_{ijk}[S(i,j,k+\tfrac{1}{2})+S(i,j+\tfrac{1}{2},k)\\ &+S(i+\tfrac{1}{2},j,k)].\end{aligned} \tag{3.10}$$

For $n=3$ the Hamiltonian contains the products of spins lying at the boundary of the volumes $B^{(3)}$:

$$\begin{aligned}-\beta H_{33}=K\sum_{ijk}&S(i,j+\tfrac{1}{2},k+\tfrac{1}{2})S(i+1,j+\tfrac{1}{2},k+\tfrac{1}{2})\\ &\times S(i+\tfrac{1}{2},j,k+\tfrac{1}{2})S(i+\tfrac{1}{2},j+1,k+\tfrac{1}{2})\\ &\times S(i+\tfrac{1}{2},j+\tfrac{1}{2},k)S(i+\tfrac{1}{2},j+\tfrac{1}{2},k+1)\\ &+h\sum_{ijk}[S(i,j+\tfrac{1}{2},k+\tfrac{1}{2})\\ &+S(i+\tfrac{1}{2},j,k+\tfrac{1}{2})+S(i+\tfrac{1}{2},j+\tfrac{1}{2},k)].\end{aligned} \tag{3.11}$$

In general, the model M_{dn} on a hypercubic lattice consists of $N_s=\binom{d}{n-1}N$ Ising spins located at the centers of the $(n-1)$-dimensional hypercubes. (N is the number of the d-dimensional hypercubes). The Hamiltonian consists of the sum of the products of the $2n$ spins at the $(n-1)$-dimensional hypersurfaces of the $N_b=\binom{d}{n}N$ hypercubes $B^{(n)}$. Let us denote a subset of n unit vectors e_i along the main axes by E_n; then the model M_{dn} for the d-dimensional hypercubic lattice is defined by

$$-\beta H_{dn}=K\sum_{r^{(0)},E_n}R(r^{(0)},E_n) + h\sum_{r^{(0)},E_{n-1}}S(r^{(0)}+v(E_{n-1})) \tag{3.12}$$

with

$$v(E_n)=\tfrac{1}{2}\sum_{e\in E_n}e, \tag{3.13}$$

$$R(r^{(0)},E_n)=\Pi_{e\in E_n}S(r^{(0)}+v(E_n)-\tfrac{1}{2}e) \times S(r^{(0)}+v(E_n)+\tfrac{1}{2}e). \tag{3.14}$$

Similarly one defines the Ising model M^*_{dn} on the dual lattice L^*. The Ising spins $S(r^*)=\pm 1$ are located at the sites $r^*=r^{(n-1)*}=r^{(d-n+1)}$ and interact via

$$-\beta^* H^*_{dn}=K^*\sum_{r^{(n)*}}\Pi_{r^*}S(r^*)^{\Theta^*(r^*,r^{(n)*})} + h^*\sum_{r^*}S(r^*). \tag{3.15}$$

We now show that the models M_{dn} and $M^*_{d,d-n+1}$ are related by the duality relation

$$Y_{dn}(K,h)=Y^*_{d,d-n+1}(K^*,h^*) \tag{3.16}$$

with

$$\tanh K=e^{-2h^*},\qquad \tanh h=e^{-2K^*}. \tag{3.17}$$

If we label the interaction of the spin $S(r^{(n-1)})$ with the external magnetic field by $b(r^{(n-1)})$ and the interaction of the spins on the boundary of $B(r^{(n)})$ by $b(r^{(n)})$, then we have

$$\theta(r^{(n-1)},b(r^{(n)}))=\Theta(r^{(n-1)},r^{(n)}), \tag{3.18a}$$

$$\theta(r^{(n-1)},b(r^{(n-1)\prime}))=\delta_{r^{(n-1)}r^{(n-1)\prime}}, \tag{3.18b}$$

$$\theta^*(r^{(n)},b(r^{(n)\prime}))=\delta_{r^{(n)}r^{(n)\prime}}, \tag{3.19a}$$

$$\theta^*(r^{(n)},b(r^{(n-1)}))=\Theta^*(r^{(n)},r^{(n-1)}). \tag{3.19b}$$

Substituting Eqs. (3.18) and (3.19) into Eq. (2.21) and using Eq. (3.2), we find that the closure condition is fulfilled. From Eqs. (3.18b) and (3.19a) it follows that $N_\theta=N_s$ and $N^*_\theta=N^*_s$. Since $N_b=N_s+N^*_s$, the completeness relation is fulfilled.

We now compare the models M_{dn} and $M^*_{d\,d-n}$ without external magnetic fields. Then the bonds are connected with the sites $r^{(n)}$ by Eq. (3.18a) and

$$\theta^*(r^{(n+1)},b(r^{(n)}))=\Theta^*(r^{(n+1)},r^{(n)})=\Theta(r^{(n)},r^{(n+1)}). \tag{3.20}$$

From Eq. (3.3) it follows that the closure condition is fulfilled. We now discuss the completeness relation (2.22). In the Appendix we derive relations between the N's, C's, and the topology of the lattice. Here we summarize the results: The exponents N_g and N^*_g of the orders of the degeneracy 2^{N_g} and $2^{N_g^*}$ of the models (3.4) and (3.15) are

$$N_g=t_g+\sum_{m=0}^{n-2}(-1)^{n-m}C_m, \tag{3.21}$$

$$N^*_g=t^*_g+\sum_{m=n+2}^{d}(-1)^{m-n}C_m, \tag{3.22}$$

in which t_g and t^*_g depend only on the boundary conditions and on n. From a generalization of Euler's theorem[21]

$$\sum_{m=0}^{d}(-1)^m C_m=t, \tag{3.23}$$

in which t depends only on the topology (boundary conditions), from

$$N_s = C_{n-1}, \quad N_b = C_n, \quad N_s^* = C_{n+1} \tag{3.24}$$

and from

$$N_m = N_g - N_s + N_b - N_s^* + N_g^*, \tag{3.25}$$

which is derived from Eqs. (2.9) and (2.37), it follows that

$$N_m = t_g + t_g^* + (-)^{d-n}\, t. \tag{3.26}$$

Therefore, N_m depends only on the topology of the system and on n. For a d-dimensional hypersurface wrapped on a $(d+1)$-dimensional hypersphere, one obtains $N_m = 0$ for $1 \le n \le d-1$. Therefore, the duality relation

$$Y_{dn}(K, 0) = Y^*_{d\,d-n}(K^*, 0), \tag{3.27}$$

with

$$\tanh K = e^{-2K^*}, \tag{3.28}$$

holds for this boundary condition. For the two-dimensional Ising model ($d = 2, n = 1$) this was shown in Ref. 3. For systems with periodic boundary conditions one obtains $N_m = \binom{d}{n}$. In the thermodynamic limit the factors $2^{N_m/2}$ in Eq. (2.39) can be neglected, and, using Eqs. (2.24), (2.25), and

$$-\beta F(K) = \ln Z(K), \tag{3.29}$$

we obtain for the free energy

$$\beta^* F^*_{d\,d-n}(K^*) = \beta F_{dn}(K) - \tfrac{1}{2}(N_g^* + N_s^* - N_g - N_s) \times \ln 2 + \tfrac{1}{2} N_b \ln \sinh 2K. \tag{3.30}$$

B. Correlation Functions

In this section we discuss the behavior of the spin correlation functions of the systems M_{dn} without an external magnetic field. We showed in Sec. 2 that an operator

$$U\{\sigma_0\} = \Pi_r\, S_x(r)^{\sigma_0(r)} \tag{2.10}$$

commutes with the Hamiltonian H and all operators R if

$$\oplus_r\, \theta(r, b)\, \sigma_0(r) = 0 \quad \text{for all } b. \tag{2.8}$$

The only solution for $n = 1$ besides the trivial solution $\sigma_0(r) = 0$ is

$$\sigma_0(r) = 1. \tag{3.31}$$

For $n > 1$ we obtain solutions

$$\sigma_0(r) = \Theta(r^{(n-2)}, r), \tag{3.32}$$

which can be verified using Eqs. (3.3) and (3.18a). Therefore, each operator R is invariant under flipping of all spins lying on the $(n-1)$-dimensional hypercells $B(r)$, which meet in the hypercell $B(r^{(n-2)})$. This leads to the high degeneracy 2^{N_g}, where N_g is given by Eq. (3.21). Since, for $r \neq r'$, there exists a neighbor $r^{(n-2)}$ of r with $\Theta(r^{(n-2)}, r) = 1$ and $\Theta(r^{(n-2)}, r') = 0$, we obtain from Eq. (2.19)

$$\langle S(r) S(r') \rangle = \delta_{rr'}. \tag{3.33}$$

Therefore, there is no long-range spin autocorrelation at any temperature. The only products of spins whose expectation values do not vanish can be represented by a product of operators R. These products are the products of all spins lying on the $(n-1)$-dimensional boundary of an n-dimensional hypervolume which consists of n-dimensional hypercells.

We now consider the long-range behavior of $\langle \Pi_r S(r) \rangle$ of the model M_{dn}, Eq. (3.12), where the spins of the product lie on the boundary of an n-dimensional hypercube. From the high temperature expansion it follows that

$$\langle \Pi_r S(r) \rangle = [\tanh K + 2(d-n)(\tanh K)^{1+2n} + \cdots]^v \quad \text{for } n > 1. \tag{3.34}$$

and

$$\langle \Pi_r S(r) \rangle = \tfrac{1}{2}\{\tanh K + [2(d-1)]^{1/2} \times (\tanh K)^2 + \cdots\}^v + \tfrac{1}{2}\{\tanh K - [2(d-1)]^{1/2} (\tanh K)^2 + \cdots\}^v \quad \text{for } n = 1, \tag{3.35}$$

where v is the volume of the hypercube (for $n = 1$, v is the distance between the two spins; for $n = 2$, v is the area of the square spanned by the spins). From the low temperature expansion one obtains

$$\langle \Pi_r S(r) \rangle = (1 - e^{-4(d-n+1)K} + \cdots)^f \quad \text{for } n < d, \tag{3.36}$$

$$\langle \Pi_r S(r) \rangle = (1 - 2e^{-2K} + \cdots)^v \quad \text{for } n = d, \tag{3.37}$$

where f is the hyperarea of the hypercube (for $n = 1$, f is the number of the ends of the line, that is, $f = 2$; for $n = 2$, f is the perimeter of the square). Therefore, we deduce that the behavior of these correlation functions in the limit of large hypercubes is different in the low and the high temperature phases, and we expect

$$\langle \Pi_r S(r) \rangle \propto \begin{cases} \exp[-v/v_0(T)] & \text{for } T > T_c, n < d, \quad (3.38) \\ \exp[-f/f_0(T)] & \text{for } T < T_c, n < d. \quad (3.39) \end{cases}$$

We attribute the qualitatively different assymptotic behavior in both temperature regions to different states of the system above and below a critical temperature T_c.

For $n = d - 1$ the different behavior in both temperature regions becomes more evident if one makes use of the duality relation for dislocations, Eq. (2.43). One obtains

$$\langle \Pi_r S(r) \rangle \{K\} = \langle \Pi_b R(b) \rangle \{K\} = \langle \Pi_b M^*(b) \rangle \{K\}, \tag{3.40}$$

where the product runs over all b's in the $(d-1)$-dimensional hypercube. The expectation value on the right-hand side of Eq. (3.40) is to be taken in the model M^*_{d1}. The logarithm of this expectation value is proportional to the change in free energy due to the dislocations. This free energy is proportional to the $(d-2)$-dimensional hyperarea of the boundary in the disordered state ($T^* > T^*_c$, that is, for $T < T_c$), and it is proportional to the $(d-1)$-dimensional hypervolume in the ordered state of the dual system ($T^* < T^*_c$, that is, for $T > T_c$). This is in agreement with Eqs. (3.38) and (3.39).

We now compare the systems M_{dn} and $M_{d+1,n}$ on a hypercubic lattice. From the theorem of Griffiths generalized by Kelly and Sherman[22] it follows that any expectation value $\langle \Pi_r S(r) \rangle$ in the system M_{dn} is less or equal to the expectation value in the system $M_{d+1,n}$,

$$\langle \Pi_r S(r) \rangle_d \le \langle \Pi_r S(r) \rangle_{d+1}, \tag{3.41}$$

since the $(d+1)$-dimensional system consists of layers of the system M_{dn} plus an additional interaction between the layers. Therefore, if this expectation value shows the long-range behavior, Eq. (3.39), for M_{dn}, then this long-range behavior is also apparent in $M_{d+1,n}$, and we obtain $T_{c,dn} \le T_{c,d+1,n}$, that is,

$$K_{c,d+1,n} \le K_{c,dn}. \tag{3.42}$$

The systems M_{dn} with $n > 1$ exhibit an unusually high ground-state entropy $S_0 \propto N$. Taking

$$S(r^{(0)} + v(E_{n-1})) = 1, \quad \text{if } e_d \in E_{n-1}, \tag{3.43}$$

in the hypercubic models (3.12), we may eliminate all spins with half-valued r_d-component. These systems (we denote them by M'_{dn}) consist of $N_s = N\binom{d-1}{n-1}$ spins and have a much smaller degeneracy,

$$N_g = \frac{N}{N_d}\binom{d-2}{n-2} + \binom{d-2}{n-1},$$

N_d denoting the length of the periodicity in the r_d direction.

For $n = d$ the system disintegrates into linear chains. For $n = 1$ the system is unchanged. For $d = 3, n = 2$ one obtains the Hamiltonian

$$\begin{aligned} -\beta H'_{32} = K\textstyle\sum_{ijk}[&S(i, j+\tfrac{1}{2}, k)\, S(i+1, j+\tfrac{1}{2}, k) \\ &\times S(i+\tfrac{1}{2}, j, k)\, S(i+\tfrac{1}{2}, j+1, k) \\ &+ S(i+\tfrac{1}{2}, j, k)\, S(i+\tfrac{1}{2}, j, k+1) \\ &+ S(i, j+\tfrac{1}{2}, k)\, S(i, j+\tfrac{1}{2}, k+1)]. \end{aligned} \tag{3.44}$$

These systems obey the closure condition (2.21) if one chooses the model $M^*_{d\,d-n}$ on the hypercubic lattice as the dual model. One obtains

$$N_m = \frac{N}{N_d}\binom{d-2}{n-2} + \binom{d-1}{n} + \binom{d-2}{n-1}. \tag{3.45}$$

Therefore, in the thermodynamic limit $N \to \infty$, $N_d \to \infty$, the duality relation (3.30) holds, and the free energies of M_{dn} and M'_{dn} show the same nonanalyticities. In the systems M'_{dn} the spins separated by a vector pointing in the e_d direction are correlated. At high temperatures one obtains

$$\langle S(r) S(r + r_d e_d) \rangle = [\tanh K + 2(d-n) \times (\tanh K)^{1+2n} + \cdots]^{|r_d|} \quad \text{for } n > 1, \tag{3.46}$$

and for low temperatures it follows that

$$\langle S(r) S(r + r_d e_d) \rangle = (1 - e^{-4(d-n+1)K} + \cdots)^{2+2(n-1)|r_d|} \quad \text{for } n < d. \tag{3.47}$$

Therefore, we expect an exponential decay of the correlation function for large r_d at all temperatures if $n > 1$. Here again we find no long range order.

Absence of a Local Order Parameter

A second-order phase transition with local order parameter is characterized as follows: Let us add local operators $\psi_i(r)$ to the Hamiltonian H_0,

$$-\beta H = K H_0 + \textstyle\sum_{ir} h_i \psi_i(r).$$

Then there is a discontinuity of the first-order derivatives with respect to h of the free energy $F(K, \{h\})$ along a ν'-dimensional hypervolume known as a first-order transition line in the ν-dimensional (K, h_i) space. This hypervolume is bounded by a $(\nu' - 1)$-dimensional λ-hypersurface commonly known as λ-point or λ-line, where the second-order phase transition takes place. Any local operator $\psi(r) = \sum h_i \psi_i(r)$ with a discontinuity of $\sum_r \langle \psi(r) \rangle = -\beta \Sigma_i h_i \partial F / \partial h_i$ along the first-order transition can be considered as an order parameter. In the homogeneous phase the limit

$$\lim_{r \to \infty} [\langle \psi(0) \psi(r) \rangle - \langle \psi(0) \rangle \langle \psi(r) \rangle] \tag{3.48}$$

vanishes. If the expectation values in expression (3.48) are averaged over *all* states along a first-order transition, then the limit (3.48) does not vanish. In the Ising model ($n = 1$) with ferromagnetic interactions $\psi(r) = S(r)$ is such a local operator. For $T = 0$ we have $\langle S(0) S(r) \rangle = 1$, whereas $\langle S(r) \rangle = 0$. In the models M_{dn} with $n > 1$ there is no first-order transition for $T < T_{c,dn}$, $h_i = 0$ associated with a local order parameter $\langle\psi(r)\rangle$ if we confine ourselves to operators which are polynomials of spin operators located in a finite region about r. We can see this as follows: Any product of spins $S(r)$ which do not lie on a closed $(n-1)$-dimensional hypersurface of hypercells $B^{(n-1)}$ gives vanishing contributions for a sufficiently large distance r. Therefore, we may confine ourselves to expressions for ψ, which are polynomials of $R(b)$,

$$\psi(r)=P(r;R(b)). \tag{3.49}$$

Applying Eqs. (2.40) and (2.43), one obtains

$$[\langle\psi(0)\psi(r)\rangle-\langle\psi(0)\rangle\langle\psi(r)\rangle](K) = [\langle\psi^*(0)\psi^*(r)\rangle-\langle\psi^*(0)\rangle\langle\psi^*(r)\rangle](K^*) \tag{3.50}$$

with

$$\psi^*(r)=P(r;\cosh 2K^*-R^*(b)\sinh 2K^*). \tag{3.51}$$

Therefore, the correlation of the ψ's in the model M_{dn} below T_c is related to the correlation of the ψ^*'s in the dual model $M^*_{d\,d-n}$ above T_c^*. According to the cluster property of the Ising model, proved rigorously by Ruelle[23] for Ising models with $n=1$, the right-hand side of Eq. (3.49) vanishes for $r\to\infty$. Therefore, there is no first-order transition characterized by a local order parameter in the models M_{dn} with $n>1$ along the K axis.[24]

C. Thermodynamic Properties

In this section we consider the thermodynamic properties of the systems M_{dn}.

The Linear Chain M_{11}

The partition function of the linear chain (3.6) of N Ising spins with nearest neighbor interaction and the periodic boundary condition $S(N+1)=S(1)$ can be calculated[5] explicitly:

$$Z(K,h)=\operatorname{tr}\begin{pmatrix} e^{K+h} & e^{-K}\\ e^{-K} & e^{K-h}\end{pmatrix}^N. \tag{3.52}$$

With Eq. (2.24),

$$Y(K,h)=f^N Z(K,h),\qquad f=(2\cosh 2K\cosh 2h)^{-1/2}, \tag{3.53}$$

one obtains

$$Y(K,h)=\operatorname{tr}\begin{pmatrix} fe^{K+h} & fe^{-K}\\ fe^{-K} & fe^{K-h}\end{pmatrix} = F_N(fe^K\cosh h, f^2(e^{2K}-e^{-2K})). \tag{3.54}$$

The first argument of F_N is half the trace of the 2×2 matrix; the second argument is its determinant. It follows that

$$F_N(t,d)=(t+\sqrt{t^2-d})^N+(t-\sqrt{t^2-d})^N, \tag{3.55}$$

with

$$t=[2(1+e^{-4K})(1+\tanh^2 h]^{-1/2}, \tag{3.56}$$

$$d=\tfrac{1}{2}(1-e^{-4K})(1-\tanh^2 h)(1+e^{-4K})^{-1} \times(1+\tanh^2 h)^{-1}. \tag{3.57}$$

Since the linear chain is a self-dual lattice, one obtains from Eq. (3.16) that

$$Y_{11}(K,h)=Y_{11}(K^*,h^*) \tag{3.58}$$

with Eq. (3.17), which is fulfilled since t and d, Eqs. (3.56) and (3.57), are invariant under this transformation.

The Models M_{dd}

The partition functions $Z_{dd}(K,0)$ of the models M_{dd} without external magnetic field can be calculated from the duality relation (3.16), (3.17):

$$Y_{dd}(K,0)=Y^*_{d1}(\infty,h^*),\qquad \tanh h^*=e^{-2K}. \tag{3.59}$$

Since in the model M^*_{d1} all spins are coupled by a two-spin interaction with infinite K^*, only the two configurations $\{S(r^*)=1\}$ and $\{S(r^*)=-1\}$ contribute

$$Z^*_{d1}(K^*,h^*) \sim \exp(N_bK^*+N^*_sh^*)+\exp(N_bK^*-N^*_sh^*). \tag{3.60}$$

It follows that

$$Z_{dd}(K,0)=2^{N_s}[(\cosh K)^{N_b}+(\sinh K)^{N_b}]. \tag{3.61}$$

The partition functions of the models M_{dd} are analytic in K for all finite K and $h=0$. Since the Ising model $M^*_{d1}, d>1$, shows a phase transition for $h^*=0$ at $K^*=K^*_{c,d1}$, a nonanalyticity is apparent in the partition function $Y_{dd}(K,h)$ for $K\to\infty$ at $h=-\frac{1}{2}\ln[\tanh(K^*_{c,d1})]$.

The Models M_{dn} with $n<d$ without External Magnetic Field

The nonanalyticity which is apparent in the free energy F_{d1} at the critical $K_{c,d1}=\beta_c I$ (β_c is the inverse critical temperature) also occurs in the free energy $F^*_{d\,d-1}$ [Eq. (3.30)]. Since the correlation functions, Eqs. (3.38) and (3.39), show a qualitatively different assymptotic behavior at low and high temperatures, we expect a phase transition for all infinite systems M_{dn} with $1\le n<d$ at some $K=K_{c,dn}$ accompanied by a nonanalyticity of the free energy. The critical K's of the model and its dual model are related by Eq. (3.28), which can be cast in the symmetric form

$$\sinh 2K_{c,dn}\sinh 2K^*_{c,d\,d-n}=1. \tag{3.62}$$

In particular, for self-dual lattices like the hypercubic lattice, one obtains

$$\sinh 2K_{c,dn}\sinh 2K^*_{c,d\,d-n}=1. \tag{3.63}$$

TABLE I. Critical parameters of some three-dimensional Ising models and their dual models.

original lattice	diamond	simple cubic	body-centered cubic	face-centered cubic
K_c	0.3698	0.2217	0.1575	0.1021
K_c^*	0.5195	0.7613	0.9284	1.1426
E_c/E_0	0.432	0.3284	0.270	0.245
E_c^*/E_0^*	0.937	0.9495	0.964	0.971
S_c/S_∞	0.737	0.808	0.845	0.853
$(S_c^* - S_0^*)/(S_\infty^* - S_0^*)$	0.100	0.092	0.072	0.063

For a self-dual model ($d = 2$, self-dual lattice) it follows that

$$K_{c,d,d/2} = \tfrac{1}{2}\ln(\sqrt{2} + 1). \tag{3.64}$$

We derived the inequality $K_{c,d+1,n} \leq K_{c,dn}$ for hypercubic lattices, Eq. (3.42). From Eq. (3.63) one obtains

$$K_{c,d,n-1} \leq K_{c,d+1,n}, \tag{3.65}$$

and from Eq. (3.42) and (3.65) it follows that

$$K_{c,d,n-1} \leq K_{c,d,n}. \tag{3.66}$$

The critical temperature of the hypercubic systems is a decreasing function of n.

Since the duality relation (3.30) relates the free energy F_{dn} at high temperatures to the free energy $F^*_{d\,d-n}$ of its dual model at low temperatures, we deduce that the critical exponent $\alpha_{d\,d-n}$ of the specific heat of the model $M^*_{d\,d-n}$ above T_c^* is given by the critical exponent α'_{dn} of the model M_{dn} below T_c and vice versa:

$$\alpha_{d\,d-n} = \alpha'_{dn}. \tag{3.67}$$

Therefore, any asymetry in the specific heat of the model M_{dn} near T_c is also apparent in the specific heat of the dual model, but the high temperature and the low temperature regions are interchanged. Self-dual systems exhibit a symmetric singularity of the specific heat around the critical temperature.

From the thermodynamic relation

$$E = \frac{\partial(\beta F)}{\partial\beta} = \frac{\partial(KF)}{\partial K}, \tag{3.68}$$

it follows that

$$E^*(K^*)/E_0^* = \cosh(2K) - \sinh(2K)E(K)/E_0, \tag{3.69}$$

in which E_0 denotes the ground state energy $E_0 = -IN_b$. Therefore, using Eqs. (3.30), (3.68), and

$$F = E - TS, \tag{3.70}$$

one is able to calculate the energy E^* and the entropy S^* of the dual model from E and S. From the critical parameters[13] of the Ising model on the diamond, the simple cubic, the body-centered cubic, and the face-centered cubic lattice, we have calculated the critical parameters of their dual models. The results are listed in Table I. The binding energies of the dual models at critical temperature are unusually large [for example, 95% of the ground state energy for the model (3.10)]. This is in agreement with the unusually low critical entropy. For the model (3.10) we obtain $S_c^*/k_BN = 0.82$ which is to be compared with the zero temperature entropy $S_0^*/k_BN = \ln 2 = 0.69$ and the entropy at infinite temperature $S_\infty^*/k_BN = 3\ln 2 = 2.08$.

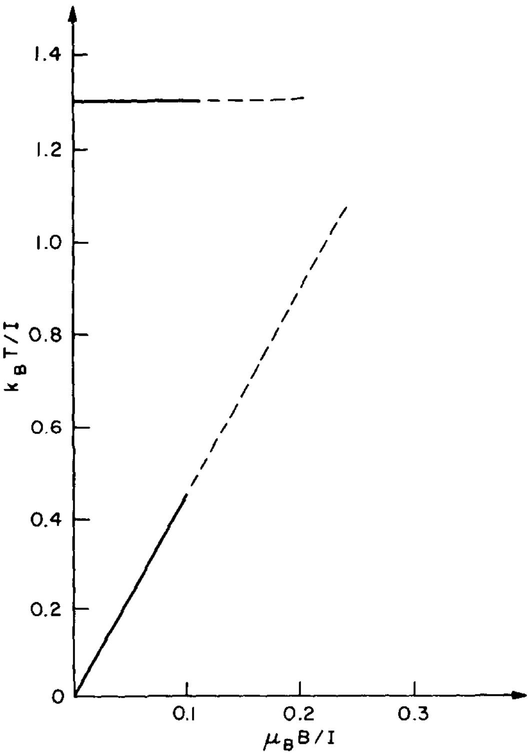

FIG. 4. Phase diagram of the system (3.10).

The Systems M_{dn} with $1 < n < d$ in an External Magnetic Field

Near the critical temperature the Ising models M_{d1} are very sensitive to an external magnetic field, since the spins exhibit a long range correlation. This does not apply to systems M_{dn} with $n > 1$. Therefore, a phase transition line $K = K_c(h)$

is expected. This function $K_c(h)$ can be calculated for small h if one assumes that the nonanalytic part of the free energy depends on $K - K_c(h)$ only[25,26]:

$$F_{\text{sing}}(K,h) = F_{\text{sing}}(K - K_c(h)). \qquad (3.71)$$

From $KF = -I \ln Z$, one obtains Eq. (3.68). The mth derivative of KF with respect to h can be expressed as spin correlation functions involving up to m spins. Only products of spins which can be expressed as products of $R(b)$ yield nonvanishing expectation values. Since $R(b)$ is a product of $2n$ spins for the hypercubic systems, one obtains for $m < 2n$ only constant contributions of the type $\langle S^2(r) S^2(r') \cdots \rangle$. For $m = 2n$ expectation values $\langle R(b) \rangle$ also occur yielding $(2n)!\,E$:

$$\left.\frac{\partial^m (KF)}{\partial h^m}\right|_{h=0} = \begin{cases} \text{const} & \text{for } m < 2n \\ \text{const} + (2n)!\,E & \text{for } m = 2n. \end{cases} \qquad (3.72)$$

From Eq. (3.71) and (3.72) it follows for the hypercubic systems that

$$K_{c,dn}(h) = K_{c,dn}(0) \quad - h^{2n} + \cdots . \qquad (3.73)$$

From the duality relation, Eqs. (3.16), (3.17), one obtains for large K the phase transition line

$$h_{c,dn}(K) = K_{c,d\,d-n}(0) - \sinh 2K_{c,d\,d-n}(0) e^{-4nK} + \cdots . \qquad (3.74)$$

The reduced critical temperature $K_c^{-1} = k_B T_c / I$ is plotted as a function of the reduced magnetic field $h/K_c = \mu_B / I$ in Fig. 4 for the cubic model M_{32}, Eq. (3.10).

In an external magnetic field the systems M_{dn} with $d = 2n - 1$ on self-dual lattices are self-dual [Eq. (3.16)].

4. PHASE TRANSITION IN AN ISING MODEL WITH COMPETING INTERACTIONS

In this section we describe an Ising model with competing two-spin interactions. For special values of temperature and interaction parameters this model is related to the model (3.10) by the star square transformation (2.54). This system shows a singularity in the specific heat, but it shows no long range order below the critical temperature.

As in model (3.10) spins are located at the centers $r^{(1)}$ of the edges of the cubes (open squares in Fig. 3). Moreover, spins are located at the centers $r^{(2)}$ of the faces of these cubes (black squares in Fig. 3). We assume an interaction strength I_1 for nearest neighbor pairs $S(r^{(1)})$ and $S(r^{(2)})$, an interaction strength I_2 for next nearest neighbor pairs of spins $S(r^{(1)})$ and $S(r^{(1)\prime})$, and an interaction strength I_3 for pairs of spins $S(r^{(1)})$ lying opposite a spin $S(r^{(2)})$ (Fig. 5). Denoting the central spin $S(r^{(2)})$ of a face by S_5 and its four nearest neighbor spins by S_1, S_2, S_3, S_4, then the Hamiltonian H' of our model is the sum over all faces

$$\begin{aligned} H' = -\sum [& I_1 S_5 (S_1 + S_2 + S_3 + S_4) \\ & + I_2 (S_1 S_2 + S_2 S_3 + S_3 S_4 + S_4 S_1) \\ & + I_3 (S_1 S_3 + S_2 S_4)]. \end{aligned} \qquad (4.1)$$

We discuss the ground state of this model. The system is invariant under simultaneous reversal of I_1 and S_5. We assume I_1 to be positive. The ground state depends on the ratios I_2/I_1 and I_3/I_1. In Fig. 6 we plot the phase diagram at zero temperature. In region 0 the system is ferromagnetic, that is, all spins $S(r^{(1)})$ point in the same direction. In region 1 one of the four spins $S(r^{(1)})$ of a face points in one direction, all three other spins of the face point in the opposite direction. In region 2 a

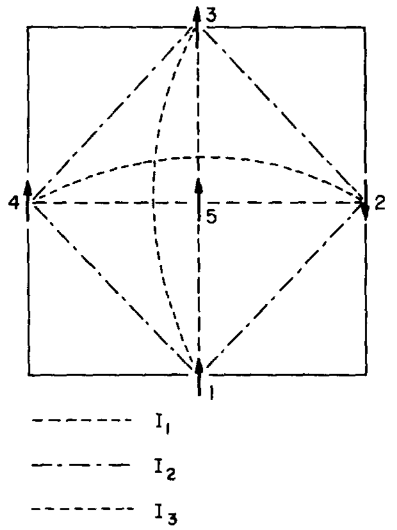

FIG. 5. The interactions in a face of the Ising model (4.1).

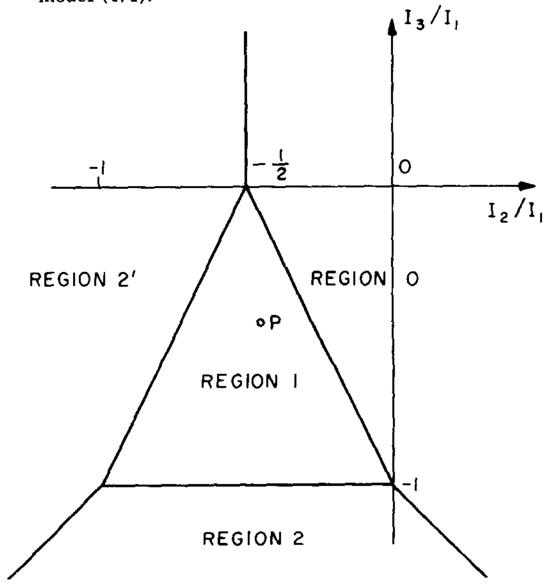

FIG. 6. Phase diagram for the model (4.1) at zero temperature.

pair of neighbored spins $S(r^{(1)})$ points in one direction, the other pair of spins $S(r^{(1)})$ of this face points in the other direction. In region 2′ two opposite spins $S(r^{(1)})$ at a face should point in one direction, the other pair in the other direction, but such an ordering is not possible in three dimensions. Therefore, in this region the ground state cannot be determined by looking merely for the ground state of one face.

Here we are interested in region 1. The ground states of system (4.1) and (3.10) for negative I are the same. The spins S_5 are determined by the surrounding spins S_1, S_2, S_3, S_4. Since the partition function of system (3.10) is invariant under change of sign of I, we obtain the ground state entropy of the system (4.1) in region 1,

$$S_0 = Nk_B \ln 2. \tag{4.2}$$

From the star square transformation (2.54) we find that the partition function Z' of system (4.1) and the partition function $\hat{Z}$ of the Hamiltonian

$$\hat{H} = -\sum [\hat{I}_1 S_1 S_2 S_3 S_4 + \hat{I}_2 (S_1S_2 + S_2S_3 + S_3S_4 + S_4S_1) + \hat{I}_3 (S_1S_3 + S_2S_4)] \tag{4.3}$$

are related by

$$Z'(K_1, K_2, K_3) = (2e^{2\hat{K}-\hat{K}_1})^{3N} \hat{Z}(\hat{K}_1, \hat{K}_2, \hat{K}_3), \tag{4.4}$$

where

$$\hat{K}_2 = K_2 + \hat{K}, \quad \hat{K}_3 = K_3 + \hat{K}, \tag{4.5}$$

$$\cosh 4K_1 = e^{+8\hat{K}}, \cosh 2K_1 = e^{2\hat{K}-2\hat{K}_1}. \tag{4.6}$$

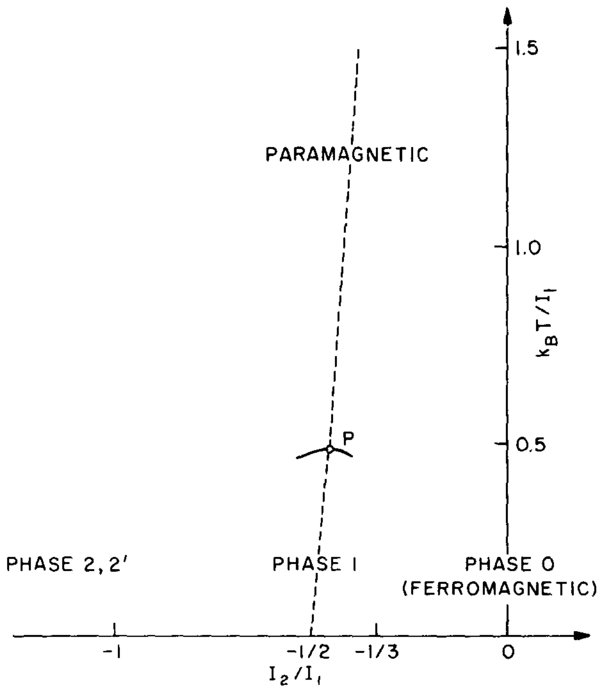

FIG. 7. Phase diagram for the model (4.1) for $I_2 = I_3$. Along the broken line the free energy can be calculated from that of the simple cubic Ising model. The heavy line denotes the phase transition line from Eq. (4.14).

Along the line $\hat{K}_2 = \hat{K}_3 = 0$ (broken line in Fig. 7) the partition function can be expressed in terms of that of the simple cubic Ising model. In particular from the critical singularity of the partition function of the simple cubic Ising model[13] at $I/k_B T_c = 0.2217$ we obtain a singularity of the partition function Z' at $K_{1c} = 2.039, K_{2c} = K_{3c} = -0.9344$, that is, for $I_2/I_1 = I_3/I_1 = -0.4582$, $k_B T_c/I_1 = 0.4904$ (point P of Figs. 6 and 7).

Now let us expand $\hat{Z}(\hat{K}_1, \hat{K}_2, \hat{K}_3)$ into powers of $\hat{K}_2$ and $\hat{K}_3$:

$$\ln \hat{Z}(\hat{K}_1, \hat{K}_2, \hat{K}_3) = \ln Z_{32}(\hat{K}_1) + \sum_{ij} a_{ij} \hat{K}_2^i \hat{K}_3^i. \tag{4.7}$$

The coefficients a_{ij} can be expressed in terms of the spin correlation functions of system (3.10):

$$a_{10} = a_{01} = 0, \tag{4.8}$$

$$\begin{aligned} a_{20} &= \tfrac{1}{2} \partial^2 \ln \hat{Z}/\partial \hat{K}_2^2 \\ &= \tfrac{1}{2} \sum \langle (S_1S_2 + S_2S_3 + S_3S_4 + S_4S_1)^2 \rangle \\ &= 2 \sum (1 + \langle S_1S_2S_3S_4 \rangle) = 6N - 2E/I, \\ a_{02} &= 3N - E/I, \quad a_{11} = 0. \end{aligned} \tag{4.9}$$

In general a $2n$th or $(2n+1)$th derivative of $\ln Z$ can be expressed in terms of cumulants of at most n operators $R(b)$ or products of $R(b)$'s. We expect that such a cumulant shows a singularity at the critical temperature of the form $\epsilon^{2-\alpha-n}$ [with $\epsilon = (T - T_c)/T_c$], since such cumulants occur in the nth derivative of the free energy with respect to the interaction constants in a system with Hamiltonian $-\hat{I}\sum R(b) - \hat{I}_2 \sum RR - \cdots$. The nth derivative with respect to $\hat{I}$ is proportional to the nth temperature derivative of the free energy and is thus proportional to $\epsilon^{2-\alpha-n}$. We assume that the cumulants of products of R's show no stronger singular behavior. Since $\hat{K}_2$ and $\hat{K}_3$ are regular functions of T for fixed $I_2/I_1, I_3/I_1$, we obtain for $I_2/I_1 = I_3/I_1 = -0.4582$

$$\ln Z' = \ln Z_{32}(\hat{K}_1) + \text{regular terms} + O(\epsilon^{3-\alpha}). \tag{4.10}$$

Therefore, we obtain for the singular part of the specific heat $c'_{\text{sing}}(T)$,

$$\begin{aligned} c'_{\text{sing}}(T_c(1+\epsilon)) &= q^{-2} c_{\text{sing},31}(T_{c,31}(1 - q\epsilon)) \\ &\quad \cdot [1 + O(\epsilon)] \\ &= 2.088\, c_{\text{sing},31}\,(T_{c,31}(1 - 0.6924\epsilon)) \\ &\quad \cdot [1 + O(\epsilon)],\ q = (\partial \ln K_1/\partial \ln K)_{Tc}. \end{aligned} \tag{4.11}$$

If we assume that the singular part of the free energy depends only on $T - T_c(I_1, I_2, I_3)$ [compare

Eq. (3.71)], then from Eqs. (4.8) and (4.9) and from

$$\frac{\partial \ln \hat{Z}}{\partial \hat{K}_1} = -\frac{E}{I} \tag{4.12}$$

we obtain

$$\hat{F}_{\text{sing}}(\hat{K}_1, \hat{K}_2, \hat{K}_3) = F_{\text{sing},32}(\hat{K}_1 + 2\hat{K}_2^2 + \hat{K}_3^2 + \cdots). \tag{4.13}$$

Therefore, we may expand the critical temperature in powers of $I_2/I_1 + 0.4582$ and $I_3/I_1 + 0.4582$ (Fig. 7):

$$k_B T_c(I_1, I_2, I_3)/I_1 = 0.4904 - 4.00(I_2/I_1 + 0.4582)^2 - 2.00(I_3/I_1 + 0.4582)^2 - \cdots. \tag{4.14}$$

Since the ground state of this system is the same as for the model (3.10), the two-spin correlations at $T = 0$ vanish and no long range order is expected below T_c.

5. CONCLUSION

In 1966 Mermin and Wagner[27] proved that there is no spontaneous magnetization in the two-dimensional isotropic Heisenberg model. On the other hand, there is evidence from high temperature expansions of the magnetic susceptibility[28] that this system undergoes a phase transition. This raises the question of whether or not it is possible to have a phase transition without a local order parameter. In this paper we have exhibited systems which undergo phase transitions but which do not have a local order parameter. The specific systems were certain classes of Ising models. It would be of some interest to generalize this concept to other types of systems.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor L. P. Kadanoff and Professor H. Wagner for helpful discussions.

APPENDIX

In this appendix we derive the Eqs. (3.21)–(3.23). Any lattice can be created from another lattice with the same boundary conditions by applying one of the following steps (see Fig. 8) as many times as needed.

Step 1: Divide an m-dimensional hypercell into two parts by creating an $(m+1)$-dimensional hypercell.

Step 2: Collapse an $(m+1)$-dimensional hypercell by merging two m-dimensional hypercells with the same boundary together into one.

By applying any of these steps, the left-hand side of

$$\sum_{m=0}^{d} (-1)^{d-m} C_m = t \tag{3.23}$$

remains unchanged. Therefore, t depends only on the boundary conditions. This is a generalization of Euler's theorem.

Next we consider the change of N_g resulting from the application of Step 1. If $m > n$, then the Hamiltonian does not change. For $m = n$ one interaction is effectively duplicated, since one boundary $B^{(n)}$ is duplicated. For $m = n-1$ one spin is replaced by two spins, but for the ground state both must be equal. For $m = n-2$ there is also one additional spin. Taking this spin aligned upwards, one obtains a one-to-one correspondence with the ground state of the original system. But changing the signs of all spins lying on bonds adjacent to one $(n-1)$-dimensional hypercell at the boundary of the new bond, we obtain another ground state. Therefore, the new system has twice the degeneracy of the original system. For $m < n-2$ the Hamiltonian does not change. Therefore, we obtain (Step 2 is just the inverse of Step 1)

$$N_g = t_g + \sum_{m=0}^{n-2} (-1)^{n-m} C_m. \tag{3.21}$$

Since this expression changes only by $+1$ after application of Step 1 and by -1 after application of Step 2 for $m = n-2$, t_g depends only on the boundary conditions and on n. Similarly, we obtain

$$N_g^* = t_g^* + \sum_{m=n+2}^{d} (-1)^{m-n} C_m. \tag{3.22}$$

Therefore, N_m, Eq. (3.16), depends only on the topology of the system and on n.

We consider two topologies: first, a lattice which is topologically equivalent to a d-dimensional hypersurface wrapped on a $(d+1)$-dimensional hypersphere. As a representative we choose the $(d+1)$-dimensional simplex, which is the generalization of the triangle and the tetrahedron. It has

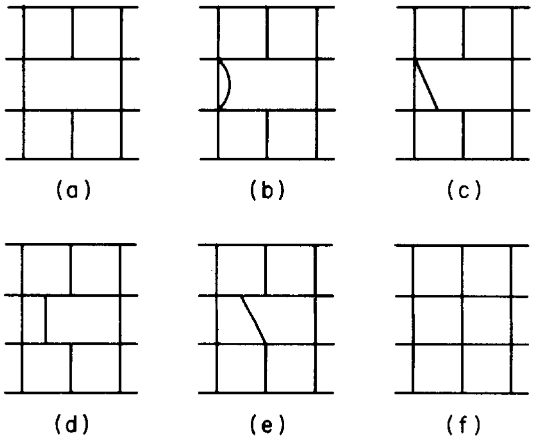

FIG. 8. Example for changing a lattice by applying the Steps 1 and 2. From the lattice (a) the lattice (f) is created by applying once the Step 1 with $m = 1$ (b), twice Step 1 with $m = 0$ (c), (d), twice the Step 2 with $m = 0$ (e), (f). The number $C_2 - C_1 + C_0$ remains unchanged.

$C_0 = d+2$ corners. Any two corners are connected by an edge. Any three edges span a face and so on. It follows that $C_m = \binom{d+2}{m+1}$. Using Eq. (3.13), one obtains

$$t = 1 + (-1)^d. \tag{A1}$$

We may number the spins of the model M_{dn} on this lattice by n indices $1 \le i_1 < i_2 \cdots < i_n \le d+2$. For the ground state all spins with $i_n = d+2$ can be chosen arbitrarily. Then all other spins are given by $S(i_1 \cdots i_n) = S(i_2 \cdots i_n, d+2) \cdot S(i_1 i_3 \cdots i_n, d+2) \cdots S(i_1 \cdots i_{n-1}, d+2)$. Therefore it follows that $N_g = \binom{d+1}{n-1}$. From the Eqs. (3.21), (3.22), (3.26), and (A1) one obtains

$$t_g = (-1)^{n+1}, \quad t_g^* = (-1)^{d+n+1} \quad \text{for } n \le d-1, \tag{A2}$$

$$N_m = 0 \quad \text{for } n \le d-1. \tag{A3}$$

Since $N_m = 0$, the duality relation (3.17) holds for all lattices wrapped on a $(d+1)$-dimensional hypersphere.

Secondly, we consider a lattice with periodic boundary conditions. As a representative we choose the d-dimensional hypercube. Then it follows that $C_m = \binom{d}{m}$, and one obtains from Eq. (3.23)

$$t = 0. \tag{A4}$$

Because of the periodic boundary conditions, all spins occur twice in the products R. Therefore, all spins can be chosen arbitrarily, $N_g = \binom{d}{n-1}$. Then one obtains from the Eqs. (3.21), (3.22), (3.26), and (A1)

$$t_g = \binom{d-1}{n-1}, \qquad t_g^* = \binom{d-1}{n}, \qquad N_m = \binom{d}{n}. \tag{A5}$$

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[24] Assuming that a product of operators R can be expressed by the sum of a constant, an energy density, and operators with a less critical behavior in the sense of the operator algebra due to Kadanoff (Refs. 6 and 7), we find from the assumption $F_{\text{sing}}(K, \{h\}) = F_{\text{sing}}(K - K_c\{h\})$ (Refs. 25 and 26) finite derivatives $\partial K_c\{h\}/\partial h_i$. This gives some evidence that a $(\nu-1)$-dimensional λ-hypersurface $K = K_c\{h\}$ exists which separates the $(K, \{h\})$ space into two phases (in some region around $h_i = 0$) and that this λ-hypersurface is not the boundary of a first-order phase transition.

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