Generalized Concatenated Quantum Codes

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We introduce the concept of generalized concatenated quantum codes. This generalized concatenation method provides a systematical way for constructing good quantum codes, both stabilizer codes and nonadditive codes. Using this method, we construct families of new single-error-correcting nonadditive quantum codes, in both binary and nonbinary cases, which not only outperform any stabilizer codes for finite block length, but also asymptotically achieve the quantum Hamming bound for large block length.

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Quantum error-correcting codes (QECCs) play a vital role in reliable quantum information transmission as well as fault-tolerant quantum computation (FTQC). So far, most good quantum codes constructed are stabilizer codes, which correspond to classical additive codes. There is a rich theory of stabilizer codes, and a thorough understanding of their properties [1, 2]. However, these codes are suboptimal in certain cases—there exist nonadditive codes which encode a larger logical space than any stabilizer code of the same length that is capable of tolerating the same number of errors [3, 4, 5].

The recently introduced codeword stabilized (CWS) quantum codes [6, 7, 8] framework, followed by the idea of union of stabilizer codes construction [9, 10], provides a unifying way of constructing a large class of quantum codes, both stabilizer codes and nonadditive codes. The CWS framework naturally allows to search for good quantum codes, and some good nonadditive codes that outperform any stabilizer codes have been found. However, this search algorithm is very inefficient [7], which prevents us from searching for good quantum codes of length $n \geq 10$ in the binary case and even smaller lengths in the nonbinary case.

This letter introduces the concept of generalized concatenated quantum codes (GCQCs), which is a systematical way of constructing good QECCs, both stabilizer codes and nonadditive codes. Compared to the usual concatenated quantum code construction, the role of the basis vectors of the inner quantum code is taken on by subspaces of the inner code. The idea of concatenated codes, originally described by Forney in a seminal book in 1966 [11], was introduced to quantum computation community three decades later [1, 12, 13, 14, 15]. These concatenated quantum codes play a central role in FTQC, as well as the study of constructing good degenerate QECCs.

The classical counterpart of GCQCs, i.e., generalized concatenated codes, was introduced by Blokh and Zyablov [16], followed by Zinoviev [17]. These codes improve the parameters of conventional concatenated codes for short block lengths [17] as well as their asymptotic performance [18]. Many good codes, linear and nonlinear, can be constructed from this method. One may expect that moving to the quantum scenario, the GCQC method should be also a powerful one in making good codes, which we show is the case.

We demonstrate the power of this new GCQC method by showing that some good stabilizer quantum codes, such as some quantum Hamming codes, can be constructed this way. We then further construct families of nonadditive single-error-correcting CWS quantum codes, in both binary and nonbinary cases, which outperform any stabilizer codes. This is the first known systematical construction of these good nonadditive codes, while previous codes were found by exhaustive or random numerical search with no structure to generalize to other cases. We also show that these families of nonadditive codes asymptotically achieve the quantum Hamming bound.

Basic Principle A general quantum code Q of n q-dimensional systems, encoding K levels, is a K-dimensional subspace of the Hilbert space $\mathcal{H}_q^{\otimes n}$. We say Q is of distance d if all $d-1$ errors (i.e., operators acting nontrivially on less than d individual $\mathcal{H}_q$s) can be detected or have no effect on Q, and we denote the parameters of Q by $((n,K,d))_q$.

Recall that concatenated quantum codes are constructed from two quantum codes, an *outer* code A and an *inner* code B. If B is an $((n,K,d))_q$ code with basis vectors $\{|\varphi_i\rangle\}_{i=0}^{K-1}$, then the outer code A is taken to be an $((n',K',d'))_K$ code, i.e., a subspace $A \subset \mathcal{H}_K^{\otimes n'}$. The concatenated code Q_c is constructed in the following way: for any codeword $|\phi\rangle = \sum_{i_1\ldots i_{n'}} \alpha_{i_1\ldots i_{n'}} |i_1 \ldots i_{n'}\rangle$ in A, replace each basis vector $|i_j\rangle$ (where $i_j = 0,\ldots,K-1$ for $j=1,\ldots,n'$) by a basis vector $|\varphi_{i_j}\rangle$ in B, i.e.,

$$|\phi\rangle \mapsto |\tilde{\phi}\rangle = \sum_{i_1\ldots i_{n'}} \alpha_{i_1\ldots i_{n'}} |\varphi_{i_1}\rangle \ldots |\varphi_{i_{n'}}\rangle, \qquad (1)$$

so the resulting code Q_c is an $((nn', K', \delta))_q$ code, and the distance δ of Q_c is at least dd', for examples, see [1, 12].

In its simplest version, a generalized concatenated quantum code is also constructed from two quantum codes, an *outer* code A and an *inner* code B which is an $((n, K, d))_q$ code. The inner code B is further partitioned into r mutually orthogonal subcodes $\{B_i\}_{i=0}^{r-1}$, i.e.

$$B = \bigoplus_{i=0}^{r-1} B_i, \tag{2}$$

and each B_i is an $((n, K_i, d_i))_q$ code, with basis vectors $\{|\varphi_{i,j}\rangle\}_{j=0}^{K_i-1}$, and $i = 0, \ldots, r-1$.

Now choose the outer code A to be an $((n', K', d'))_r$ quantum code in the Hilbert space $\mathcal{H}_r^{\otimes n'}$. While for concatenated quantum codes each basis state $|i\rangle$ of the space $\mathcal{H}_r$ is replaced by a basis state $|\varphi_i\rangle$ of the inner code, for a generalized concatenated quantum code Q_{gc} the basis state $|i\rangle$ is mapped to the subcode B_i of the inner code. For simplicity we assume that all subcodes B_i are of equal dimension, i.e., $K_1 = K_2 = \ldots = K_r = R$. Then the dimension of the resulting code Q_{gc} is $\mathcal{K} = K' R^{n'}$, i.e., for each of the n' coordinates of the outer code, the dimension $\mathcal{K}$ is increased by the factor R. For a codeword $|\phi\rangle = \sum_{i_1 \ldots i_{n'}} \alpha_{i_1 \ldots i_{n'}} |i_1 \ldots i_{n'}\rangle$ of the outer code and a basis state $|j_1 \ldots j_{n'}\rangle$ (where $j_l = 0, \ldots, R-1$ for $l = 1, \ldots, n'$) of the space $\mathcal{H}_R^{\otimes n'}$, the encoding is given by the following mapping:

$$|\phi\rangle |j_1 \ldots j_{n'}\rangle \mapsto \sum_{i_1 \ldots i_{n'}} \alpha_{i_1 \ldots i_{n'}} |\varphi_{i_1, j_1}\rangle \ldots |\varphi_{i_{n'}, j_{n'}}\rangle. \tag{3}$$

Note that the special case when $R = 1$ corresponds to concatenated quantum codes. The resulting code Q_{gc} has parameters $((nn', \mathcal{K}, \delta))_q$ where the distance δ is at least $\min\{dd', d_i\}$. If some of the K_is differ, the calculation of the dimension is more involved.

CWS-GCQC From now on we restrict ourselves in constructing some special kind of quantum codes, namely, CWS codes. CWS codes include all the stabilizer codes and many good nonadditive codes [6], so it is a large class of quantum codes. The advantage of the CWS framework is that the problem of constructing quantum codes is reduced to the construction of some classical codes correcting certain error patterns induced by a graph. So the point of view of constructing these codes could be fully classical. For simplicity we only consider nondegenerate codes here.

A nondegenerate $((n, K, d))_q$ CWS codes Q_{CWS} is fully characterized by a graph $\mathcal{G}$ and a classical code $\mathcal{C}$ [6, 7, 8], and for simplicity we only consider q a prime power. For any graph $\mathcal{G}$ of n vertices, there exists a unique stabilizer code $((n, 1, d_{\mathcal{G}}))$ defined by $\mathcal{G}$ (called the graph state of $\mathcal{G}$). We call the distance $d_{\mathcal{G}}$ the graph distance of $\mathcal{G}$. For constructing a nondegenerate CWS code, we require that the distance of the code be $\leq d_{\mathcal{G}}$. Then any quantum error E acting on Q_{CWS} can be transformed into a classical error by a mapping $Cl_{\mathcal{G}}(E)$ whose image is an n-bit string. The nondegenerate code Q_{CWS} detects the error set $\mathcal{E}$ if and only if $\mathcal{C}$ detects $Cl_{\mathcal{G}}(\mathcal{E})$ [6, 7, 8].

We take the inner code B to be an $((n, K, d))_q$ nondegenerate CWS code, constructed by a graph $\mathcal{G}$ and a classical code $\mathcal{B}$. Furthermore, we decompose B as $B = \bigoplus_{i=0}^{r-1} B_i$ such that each B_i is an $((n, K_i, d_i))_q$ CWS code constructed from $\mathcal{G}$. The basis vectors of each B_i can be represented by classical codewords of a code $\mathcal{B}_i = \{\mathbf{b}_{i,j}\}_{j=1}^{K_i}$. Then consequently, the classical code $\mathcal{B}$ has a partition $\mathcal{B} = \bigcup_{i=0}^{r-1} \mathcal{B}_i$.

Now we take the outer code A to be an $((n', K', d' = 1))_r$ code in the Hilbert space $\mathcal{H}_r^{\otimes n}$, which is constructed from a classical $(n', K', d_c)_r$ code $\mathcal{A}$ over an alphabet of size r, of length n', size K', and distance d_c in the following way: the basis vector $|\psi_{i_1 \ldots i_{n'}}\rangle$ of A is given by

$$|\psi_{i_1 \ldots i_{n'}}\rangle = |i_1 \ldots i_{n'}\rangle, \ \forall (i_1 \ldots i_{n'}) \in \mathcal{A}^{n'}. \tag{4}$$

Denote the generalized concatenated code obtained from A and B by Q_{gc}. It is straightforward to see Q_{gc} is also a CWS code, where the corresponding graph is given by n' disjoint copies of the graph $\mathcal{G}$. The corresponding classical code $\mathcal{C}_{gc}$ is a classical generalized concatenated code with inner code $\mathcal{B} = \bigcup_{i=0}^{r-1} \mathcal{B}_i$ and outer code $\mathcal{A}$. The minimum distance of Q_{gc} is at least $\min\{d, d_i, d_{\mathcal{G}}\}$. However, the following statement provides an improved lower bound.

Main Result: The minimum distance of Q_{gc} is given by $\min\{dd_c, d_i, d_{\mathcal{G}}\}$.

We will not give a technical detailed proof of this result here. Instead, since the proof idea can be illustrated clearly with a simple example, we will analyze such an example, which also illustrates a systematical method of constructing good nonadditive quantum codes that outperform the best stabilizer codes.

Good Nonadditive Codes We start taking the subcode B_0 of the inner code B to be the well-known $((5, 2, 3))_2$ code, the shortest one-error-correcting quantum code. As a CWS code, this code can be constructed by a pentagon graph as well as a classical code $\mathcal{B}_0 = \{00000, 11111\}$. Further details can be found in [6], here we just focus on the classical error patterns given by the mapping $Cl_{\mathcal{G}}$. Since the pentagon has graph distance 3, the CWS code B_0 has distance at least 3 if $\mathcal{B}_0$ detects up to two errors with the error patterns induced by the pentagon. The induced error patterns are given by the following strings

$$\begin{aligned} Z &: \{10000, 01000, 00100, 00010, 00001\}, \\ X &: \{01001, 10100, 01010, 00101, 10010\}, \\ Y &: \{11001, 11100, 01110, 00111, 10011\}. \end{aligned} \tag{5}$$

It is straightforward to check that $\mathcal{B}_0$ indeed detects two of these errors.

The classical code $\mathcal{B}_0$ is linear, so we can choose 15 disjoint proper cosets, e.g., $\mathcal{B}_1 = \{00001, 11110\}$ and $\mathcal{B}_{15} = \{01111, 10000\}$. Combining these classical codes with the pentagon gives us the CWS codes B_i, each of which is a $((5, 2, 3))_2$ quantum code. The union $\mathcal{B} = \bigcup_{i=0}^{15} \mathcal{B}_i$ of all cosets is a classical $(5, 32, 1)_2$ code which consists of all 5-bit strings. Combining $\mathcal{B}$ with a pentagon gives us the CWS quantum inner code B which is a $((5, 32, 1))_2$ quantum code. It can be decomposed as $B = \bigoplus_{i=0}^{15} B_i$.

For the outer code we take a quantum code A which corresponds to a classical code $\mathcal{A} = (3, 16, 3)_{16}$, i.e., a distance three code over $GF(16)$ of length 3. Hence the basis of A is given by $|i_1i_2i_3\rangle$ where $(i_1i_2i_3)$ is one of the 16 codewords of $\{\mathtt{000}, \mathtt{111}, \ldots, \mathtt{aaa}, \ldots, \mathtt{fff}\}$ of $\mathcal{A}$. Here we use the hexadecimal notation to denote the 16 symbols of the alphabet $GF(16)$.

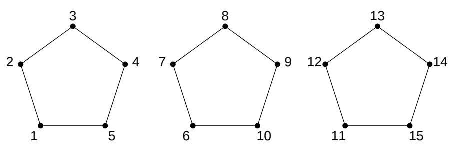

FIG. 1: Three pentagons: graph with 15 vertices.

Now we construct the GCQC $Q_{gc}^{\{15\}}$ of length 15 from A and B in the following way: first, due to the product state form of the basis of A, we choose the corresponding graph $\mathcal{G}^{\{15\}}$ to be by three disjoint pentagons, as shown in FIG. 1. We denote this graph by $\mathcal{G}^{\{15\}}$. The distance of the graph state corresponding to $\mathcal{G}^{\{15\}}$ is still 3. So from these three pentagons we can obtain a nondegenerate CWS quantum code whose distance is at most 3. The error patterns induced by the mapping $Cl_{\mathcal{G}^{\{15\}}}$ given by this 15 vertex graph are simply the strings from Eq. (5) on the coordinates 1–5 (or 6–10, or 11–15) and zeros on the other coordinates. For instance, 10000 in Eq. (5) gives rise to three strings of length 15 which are 100000000000000, 000001000000000, and 000000000010000. In total there are 45 strings in the induced error set of three pentagons corresponding to the 45 single-qubit errors on 15 qubits.

Now we need to figure out what the corresponding classical code $\mathcal{C}_{gc}^{\{15\}}$ is. We know that it is the generalized concatenated code with inner code $\mathcal{B} = \bigcup_{i=0}^{15} \mathcal{B}_i$ and outer code $\mathcal{A}$. To see how this works explicitly, consider the first codeword $\mathbf{a}_0 = 000$ of $\mathcal{A}$. Each of the three zeros is replaced by the code $B_0 = \{00000, 11111\}$, i. e, $(\mathbf{a}_0, j_1, j_2, j_3)$ (where $j_l = 0, 1$) will be mapped to one of the 8 codewords of $\mathcal{C}_{gc}^{\{15\}}$, which are strings of length 15, given by 000000000000000, 000000000011111, 000001111100000, 000001111111111, 111110000000000, 111110000011111, 111111111100000, 111111111111111. Similarly, any other codeword $\mathbf{a}_i$ of $\mathcal{A}$ will be mapped to 2^3 codewords in $\mathcal{C}_{gc}^{\{15\}}$ obtained by concatenating three codewords of B_i. The size of $\mathcal{C}_{gc}^{\{15\}}$ is then $2^3 \times 16 = 2^7$.

We now show that the distance of $Q_{gc}^{\{15\}}$ is 3. To see this, we only need to show that $\mathcal{C}_{gc}^{\{15\}}$ detects up to two errors of the error patterns induced by three pentagons. This is clear via the following two observation: i) $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}_{gc}^{\{15\}}$ correspond to different codewords of the outer code $\mathcal{A}$: since the pentagons are disjoint, and $\mathcal{A}$ has distance 3, at least 3 strings in the induced error patterns are needed to transform $\mathbf{c}_1$ to $\mathbf{c}_2$. ii) $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}_{gc}^{\{15\}}$ correspond to same codewords of the outer code $\mathcal{A}$: since at least 3 strings in the induced error patterns are needed to transform codewords in $\mathcal{B}_i$, at least 3 strings in the induced error are needed to transform $\mathbf{c}_1$ to $\mathbf{c}_2$.

Now one can generalize the construction of $Q_{gc}^{\{15\}}$ to the case of more than three pentagons. Suppose we use n' pentagons to construct single-error-correcting CWS codes, then we observe the following

Fact 1 *Choose the inner code as* $B = \bigcup_{i=0}^{15} B_i$ *with each* B_i *a* $((5, 2, 3))_2$ *quantum code, and the outer code* A *corresponding to the classical code* $\mathcal{A}$ *with parameters* $(n', K', 3)_{16}$*, then the resulting GCQC* Q_{gc} *is a* $((5n', 2^{n'}K', 3))_2$ *binary quantum code.*

This indicates that if we have a good classical code over $GF(16)$ of distance 3, then we may systematically construct good quantum codes via the generalized concatenation method described above.

Example 1 *Using the quantum code corresponding to the classical Hamming code with parameters* $(17, 16^{15}, 3)_{16}$ *as the outer code, then by* ***Fact 1*** *we get a quantum code with parameters* $((85, 2^{77}, 3))_2$*, which is a quantum Hamming code [2]. If we properly choose the labeling of the subcodes* B_i *by elements of* $GF(16)$*, the correponding classical code is linear [19], and hence this quantum code is a stabilizer code [6].*

If we take a quantum code corresponding to a good nonlinear classical code as the outer code, then we can construct a good nonadditive quantum code [6]. Here we give examples of such a good quantum codes which are constructed using a good nonlinear classical codes. Those nonlinear codes are obtained via the following classical construction, called 'subcode over subalphabet' (see [19, Lemma 3.1]).

Fact 2 *If there exists an* $(n, K, d)_q$ *code, then for any* $s < q$*, there exists an* $(n', K', d)_s$ *code with size at least* $K(s/q)^n$*.*

Example 2 *It is known that there is a classical Hamming code with parameters* $(18, 17^{16}, 3)_{17}$*. Therefore, using* ***Fact 2*** *there is a* $(18, \lceil \frac{16^{18}}{17^2} \rceil, 3)_{16}$ *code. Then the resulting quantum code has parameters* $((90, 2^{81.825}, 3))_2$*. For a binary quantum code with* $n = 90$ *and* $d = 3$*, the* *quantum Hamming bound ($K \leq q^n/((q^2-1)n+1)$, see [2]) gives $K < 2^{81.918}$, and the linear programing bound (see [2]) gives $K < 2^{81.879}$. So the best stabilizer quantum code can only be $((90, 2^{81}, 3))_2$. Hence our simple construction gives a nonadditive single-error-correcting quantum code which outperforms any possible stabilizer codes. This is the first such example given by construction, not by numerical search.*

Example 3 *The similar CWS-GCQC idea works also for the nonbinary case using the nonbinary CWS construction [8]. Take the inner code to be a union of* 81 *mutually orthogonal* $((10, 729, 3))_3$ *codes that is constructed from a graph that is a ring of ten vertices [20]. Choose the outer code as the quantum code corresponding to the classical* $(84, \lceil \frac{81^{84}}{83^2} \rceil, 3)_{81}$*, which is obtained from the Hamming code* $(84, 83^{82}, 3)_{83}$*. Then the resulting quantum code has parameters* $((840, 3^{831.955}, 3))_3$*. For a ternary quantum code with* $n = 840$ *and* $d = 3$*, the Hamming bound gives* $K < 3^{831.978}$*, and the linear programing bound gives* $K < 3^{831.976}$*, so the best stabilizer code can only be* $((840, 3^{831}, 3))_3$*. This is the first known nonbinary nonadditive code which outperforms any stabilizer codes.*

It is straightforward to generalize the above construction for binary and ternary codes to build good nonadditive quantum codes in Hilbert space $H_q^{\otimes n}$ for any prime power q. For this, we take the inner code B_0 as the perfect quantum Hamming code $((q^{n_s}, q^{n_s-2s}, 3))_q$ in $H_q^{\otimes n}$ of length $n_s = (q^{2s}-1)/(q^2-1)$. The full space $B = ((n_s, q^{n_s}, 1))_q$ can be decomposed as the sum of q^{2s} orthogonal translates of B_0. The outer quantum code is then corresponding to a classical code over an alphabet of size $Q = q^{2s}$ given by **Fact 2**, i.e., the classical code is obtained from the P-ary Hamming code $[L_i, L_i - i, 3]_P$ where P is the least prime power exceeding Q, and $L_i = (P^i-1)/(P-1)$. The result is the code $V_{si} = ((N_{si}, M_{si}, 3))_q$ with length $N_{si} = L_i n_s = (P^i-1)(Q-1)/(q^2-1)(P-1)$ and dimension $M_{si} \geq q^{N_{si}}/P^i$.

The number of different errors we want to deal with is $(q^2-1)N_{si}+1 > Q^i = q^{si}$ for $P > Q$ and $i > 1$. By the quantum Hamming bound $K \leq q^{N_{si}}/((q^2-1)N_{si}+1) < q^{N_{si}}/Q^i$, the dimension of any stabilizer code (including degenerate codes) is upper bounded by $K \leq q^{N_{si}-2si-1}$. Hence for any prime power P with $Q^i < P^i < qQ^i$, the dimension M_{si} is strictly larger than $q^{N_{si}-2si-1}$, i.e., our codes are better than any stabilizer codes. Moreover, we have $q^{N_{si}}/P^i \leq M_{si} \leq q^{N_{si}}/Q^i$. Since $Q/P \to 1$ for $s \to \infty$ [19], these families of nonadditive codes asymptotically achieve the quantum Hamming bound.

Discussion We have introduced the concept of GCQC, which is a systematic construction of good QECCs, both stabilizer codes and nonadditive codes. One way of generalizing the concatenation of Eq. (3) is to put some constraints on the additional degrees of freedom $|j_1 \dots j_{n'}\rangle$ by using a second outer code. Additionally, one can recursively decompose the codes B_i in the decomposition (2) of the inner code, which leads to a more general construction of GCQC with which more good quantum codes can be constructed (see [21]). While the nonadditive codes of this letter tighten the gap between lower and upper bounds for the dimension of the codes, we believe that in general the GCQC construction gives a promising way for further constructing new quantum codes of good performance, and we hope that this generalized concatenation technique will also shed light on improvements of fault-tolerant protocols.

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