Quantum Fluctuations of the Superconducting Cosmic String

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Quantum fluctuations of the proposed superconducting string with Bose charge carriers are studied in terms of the vortices on the string world sheet. In the thermodynamical limit, we find that they appear in the form of free vortices rather than as bound pairs. This fluctuation mode violates the topological conservation law on which superconductivity is based. However, this limit may not be reached. The critical size of the superconducting string is estimated as a function of the coupling constants involved.

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A charged Higgs boson coupled to the electromagnetic field can obviously not have a nonvanishing vacuum expectation value, since then the vacuum would be a superconducting medium and the photon would be massive. However, as pointed out recently by Witten,[1] the expectation value of this Higgs field could have an inhomogeneous distribution, which vanishes everywhere except on a lower-dimensional object such as a string. He showed that under certain conditions, a cosmic string created in the early Universe could indeed support such a distribution of the Higgs field and argued that such a string could then be superconducting. Cosmological consequences based on this observation were explored by Osteriker, Thompson, and Witten.[2]

This paper discusses the quantum fluctuations of such a superconducting string. As common in many systems with broken symmetries, there are topological excitations which tend to disorder the system. These fluctuations are particularly important for two-dimensional systems with a O(2) symmetry. For example, the topological vortices in a superfluid film can be created via thermal fluctuations. At low enough temperatures, these vortices can only occur in tightly bound pairs and do not change the long-distance characteristics of the system. However, as the temperature is increased to a critical value, these pairs dissociate into free vortices and they drive a phase transition which destroys the superfluidity. The phase transition driven by the vortices is commonly known as the Kosterlitz-Thouless transition.[3] Similar phase transitions also occur in superconducting films under more restrictive conditions.[4] I investigate the effect of quantum fluctuations in terms of the vortices on the world sheet of the superconducting string in a similar fashion. Unlike the instantons of the Abelian Higgs model in $1+1$ dimensions,[5] these vortices are not localized in space-time, the action (or let us call it the creation energy) of a single vortex diverges with the size of the system and vortices interact with each other through a long-ranged potential. However, as a result of the effect of the electromagnetic screening, I find that the creation energy of a single vortex diverges less rapidly than the entropy (in fact, the creation energy $\sim \ln\ln L$, while the entropy $\sim \ln L$, L being the linear size of the system). Free energy therefore favors the creation of free vortices. In addition to this observation, I have performed a renormalization-group analysis (similar to that of Kosterlitz[6]) which incorporates the long-ranged interaction of the vortices and have found that, indeed, a condensate of free vortices with finite density is present in the ground state. The density of the free vortices serves as a "disorder parameter"[7] of the system and its nonvanishing means that the system is in a disordered phase. However, although the entropy always outweighs the creation energy in the thermodynamical limit $L \to \infty$, the critical size L_c at which the entropy starts to dominate the creation energy could be extremely large depending on the coupling constants involved. The critical size is estimated at the end of this paper.

Let us start by recalling some basic facts about the proposed superconducting string with Bose charge carriers.[1] One considers a $U(1)\otimes\tilde{U}(1)$ gauge field theory, with gauge fields A_μ and R_μ and Higgs fields σ and ϕ, interacting according to the following Lagrangean:

$$L = -\tfrac{1}{4}F_{\mu\nu}^2 - \tfrac{1}{4}R_{\mu\nu}^2 + |D_\mu\phi|^2 + |D_\mu\phi|^2 - \tfrac{1}{8}\lambda(|\phi|^2-\mu^2)^2 - \tfrac{1}{4}\tilde{\lambda}|\sigma|^4 - f|\sigma|^2|\phi|^2 + m^2|\sigma|^2, \tag{1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu - \partial_\nu A_\mu$, $R_{\mu\nu}=\partial_\mu R_\nu - \partial_\nu R_\mu$, $D_\mu\sigma=(\partial_\mu + ieA_\mu)\sigma$, and $D_\mu\phi=(\partial_\mu + igR_\mu)\phi$.

In the range of parameters given by $\mu^2, m^2>0$ and $f\mu^2 - m^2>0$, the $\tilde{U}(1)$ symmetry is broken ($|\langle\phi\rangle| = \mu$), while the electromagnetic U(1) symmetry is unbroken ($|\langle\sigma\rangle| = 0$). However, there exists a classical solution to the broken $\tilde{U}(1)$ theory in the form of a string,[8] where the Higgs field ϕ vanishes in the core of the string and approaches μ at infinity. In this case, as shown in Ref. 1, the Higgs field σ has the opposite behavior: It vanishes everywhere except in the core of the string. Provided that the amplitude of the σ field in the string varies smoothly over the range of the coherence length $1/m$ [m being the mass parameter of the σ field, see (1)], the dynamics of the low-lying phase excitations $\theta(x_3,x_0)$ of the σ field can be treated by consideration of the following effective action:

$$I=\int d^4x[-\tfrac{1}{4}F_{\mu\nu}^2+K\delta^2(x_1,x_2)(\partial_i\theta+eA_i)^2], \qquad (2)$$

where $\mu=0,1,2,3$ and $i=0,3$, and without loss of generality, I have assumed that the string is directed along the third axis. K is a constant estimated to be $K\approx 1/\bar{\lambda}$.

The way to see whether the string is superconducting is to observe that the model defined by (2) admits a topological invariant

$$N=\frac{1}{2\pi}\oint dx_3\frac{\partial\theta}{\partial x_3}. \qquad (3)$$

Since θ is an angular parameter, N so defined is always an integer. In Ref. 1 it was shown that in a sector with nonzero N, the ground state is current carrying; if N is conserved, the current cannot decay and the string is therefore superconducting. However, even through the topologically trivial phase fluctuations cannot change the quantum number N, there are topological vortex excitations which violate the conservation of N, as was originally remarked in Ref. 1. A vortex on the string world sheet is a singularity where the amplitude of the σ field vanishes and the phase becomes ill-defined. A vortex with unit vorticity can change N by one unit. Therefore, N is only conserved if these vortices form tightly bound objects with zero net vorticity, since at large distances these objects are indistinguishable from the topologically trivial phase fluctuations. The question about the conservation of N is hence reduced to a statistical-mechanics problem of the vortex gas: Do the vortices appear in the form of tightly bound objects or do they form a gas of free vortices?

As we shall study the imaginary-time propagation of the string, let us consider the Euclidean version of (2) by a Wick rotation ($x_0=ix_4$). In the Lorentz gauge $\partial^\mu A_\mu=0$, the components A_1 and A_2 are decoupled and can be set equal to zero without loss of generality. The Eculidean version of (2) is then given by

$$I_E=\int d^4x[-\tfrac{1}{2}A_i\Delta A_i+K\delta^2(x_1,x_2)(\partial_i\theta+eA_i)^2], \qquad (4)$$

where $\Delta\equiv\sum_{\mu=1}^4\partial^2/\partial x_\mu^2$ and from now on the index i refers to $i=3,4$.

In order to study the effective interaction of the vortices, we first integrate out A_i to get an effective action for θ. The equation of motion of A_i is given by

$$\Delta A_i=2Ke\delta^2(x_1,x_2)(\partial_i\theta+eA_i)\equiv J_i, \qquad (5)$$

or in Fourier space

$$A_i(q)=-(1/q^2)J_i(q). \qquad (6)$$

Let us denote $q\equiv(q_1,q_2,q_3,q_4)$ and $p\equiv(q_3,q_4)$. As a result of the δ function in the definition of $J_i(x)$, $J_i(q)$ is actually a function $J_i(p)$ of q_3 and q_4 only. Integration of (6) over q_1 and q_2 yields

$$\int dq_1dq_2A_i(q)=-J_i(p)\int dq_1dq_2\frac{1}{q_1^2+q_2^2+q_3^2+q_4^2}\approx 2\pi J_i(p)\ln(|p|r_0), \qquad (7)$$

where r_0 is the thickness of the string which serves as an ultraviolet cutoff of the integral. Defining $A_i(p)\equiv\int d^4x\,e^{ixq}A_i(x)\delta^2(x_1,x_2)$ we find that the left-hand side of (7) is just $(2\pi)^2A_i(p)$. Therefore

$$2\pi A_i(p)=2Ke[\Phi_i(p)+eA_i(p)]\ln(|p|r_0), \qquad (8)$$

where $\Phi_i\equiv\partial_i\theta$. Solving this equation we obtain

$$eA_i(p)=\frac{\alpha\ln(|p|r_0)}{1-\alpha\ln(|p|r_0)}\Phi_i(p),$$
$$J_i(p)=\frac{2Ke}{1-\alpha\ln(|p|r_0)}\Phi_i(p), \qquad (9)$$

where $\alpha\equiv Ke^2/\pi$.

These relations are to be contrasted with the usual type of Higgs mechanism, where $A_i(p)$ has a simple pole at the mass of the vector boson or the inverse London penetration depth. This usual behavior is nothing but the Meissner screening effect. In the present case, however, the "photon mass" cannot result simply because both coupling constants K and e are dimensionless. The physical reason behind this is rather clear: The electromagnetic field extends over four-dimensional space-time, while the matter field is restricted to the two-dimensional world sheet. The effect of the induced currents in screening the electromagnetic field is therefore much weaker and, in fact, only logarithmic.

The singularity in (9) appears at momenta greater than the physical cutoff $1/r_0$ and so will not concern us here. From (9) one can easily find the effective action for θ:

$$I_E=K\int\frac{d^2p}{(2\pi)^2}\frac{p^2}{1-\alpha\ln(|p|r_0)}\theta(p)\theta(-p). \qquad (10)$$

A vortex (with infinitesimal core radius) located at $\bar{x}$ is defined by

$$\nabla^2\theta(x)=-2\pi n\delta^2(x-\bar{x}), \qquad (11)$$

or by the fact that θ changes by $2\pi n$ along any contour enclosing $\bar{x}$:

$$\oint d\mathbf{x}\cdot\nabla\theta=2\pi n. \qquad (12)$$

The integer n is called the vorticity. The loop integral in (12) can be taken along the boundary of the string world sheet. Identifying both ends of the string and using (3) and (12) one finds that

$$N(x_4=\infty)-N(x_4=-\infty)=n; \tag{13}$$

therefore, a vortex with vorticity n changes the winding number N by n units.

Given the effective action (10), the action of a single vortex is very easy to compute. From (11) we have $\theta(p)=2\pi n e^{-ip\bar{x}}/p^2$. Substituting this into (10) yields

$$I_0=2\pi n^2 K\int \frac{dp}{p}\frac{1}{1-\alpha\ln(pr_0)} \approx \frac{2\pi^2 n^2}{e^2}\ln\left[1+\alpha\ln\frac{L}{r_0}\right]. \tag{14}$$

The ultraviolet cutoff of this integral is the core radius of the vortex $a\simeq 1/m$, the scale over which the amplitude of the σ field varies, while the infrared cutoff is simply L, the linear extension of the string world sheet. I have assumed that a and r_0 are of comparable length. The way I_0 diverges with L is a particular consequence of the effective logarithmic screening. If perfect Meissner screening would take place (like in the Abelian Higgs model in $1+1$ dimensions[5]), I_0 would remain finite. If there were no screening at all, I_0 would diverge like $\ln L$.[3] In the present case, while the creation energy I_0 of a free vortex diverges as $\ln\ln L$, its "entropy" is simply $S=\ln(L/a)^2$. The entropy term arises in the path integral in the integration over the zero mode, namely, the position of the vortex. The contribution of a single vortex to the partition function is

$$Z=D_0 D e^{-I_0}, \tag{15}$$

where D_0 is the zero-mode contribution, $D_0=(L/a)^2$, and D is the fluctuation determinant. By exponentiating D_0, we obtain the entropy term.

Comparing I_0 with S one sees that in the large-L limit, the entropy term always outweighs the creation energy. The free energy $F=-\ln Z=I_0-S$ is therefore always lowered by the creation of vortices. This simple argument strongly suggests that a gas of free vortices is the favorable configuration. However, this argument is only heuristic in the sense that it only involves a single vortex, while vortices actually have long-ranged interactions. In Ref. 6 Kosterlitz performed a detailed renormalization-group analysis to support a similar simple argument for the two-dimensional XY model. In order to carry out a similar analysis, let us first find the partition function of the vortex gas. A multivortice configuration is defined by

$$\nabla^2\theta(x)=-2\pi\textstyle\sum_i n_i\delta^2(x-x_2), \tag{16}$$

where n_i is the vorticity and x_i is the location of the ith vortex. From (16) it follows that $\theta(p)=\sum_i 2\pi n_i \times\exp(-ipx_i)/p^2$. Substituting this into (10) one obtains the partition function of a vortex gas:

$$I=(2\pi)^2 K\textstyle\sum_{i,j}n_i n_j G(x_i-x_j), \tag{17}$$

where

$$G(x)=\int\frac{d^2p}{(2\pi)^2}\frac{1}{1-\alpha\ln(|p|r_0)}\frac{e^{-ipx}}{p^2}. \tag{18}$$

Only overall neutral configurations $\sum_i n_i=0$ contribute to the partition function, since nonneutral configurations have infinite action associated with them. For $\alpha=0$, this reduces to the partition function of a two-dimensional Coulomb gas, interacting with a logarithmic potential.

In order to find out about the presence of free vortices, one adds a chemical-potential term $-\ln y\sum_i n_i^2$ to (17) which controls the density of vortices. Our question can therefore be formulated in a more precise way: Is the added chemical-potential term a relevant or an irrelevant perturbation in the sense of the renormalization-group flow? The partition function (17) only reveals the degrees of freedom at the scale of the ultraviolet cutoff a; at this scale, it is impossible to distinguish tightly bound vortices from the free ones. To see their differences, one has to go to larger and larger scale by the procedure of coarse graining, or integrating out small distance fluctuations. If vortices are tightly bound, coarse graining would decrease their effective density, since at distances larger than their separation, they are indistinguishable from topologically trivial phase fluctuations; y and therefore the effective density of vortices scale to zero. On the other hand, if there is a gas of *free* vortices present, coarse graining will only increase their effective density. Vortices becomes the dominant configurations at large distances.

I have carried out a renormalization-group analysis to the lowest order in perturbation theory in y, with only vortices of unit vorticity included. The calculation follows the approach of Kogut.[9] The idea is to map the partition function of the vortex gas to a field-theoretical model by means of a duality transformation and then carry out a standard momentum-shell integration. (For details of the calculation, see Zhang.[10]) To the *lowest* order, the renormalization-group equation that I obtained is of the following form:

$$dK=-\frac{K^2e^2}{\pi}\frac{da}{a}, \qquad dy=-y\frac{da}{a}\left[\frac{2\pi K}{1+\alpha\ln(a/r_0)}-2\right], \tag{19}$$

where a is the renormalization scale. The constants e and r_0 are unrenormalized. This renormalization-group equation possesses a line of fixed points $y=0$. This agrees with the common knowledge that for $y=0$, only topologically trivial phase fluctuations are present and they lead to algebraically decaying correlation functions

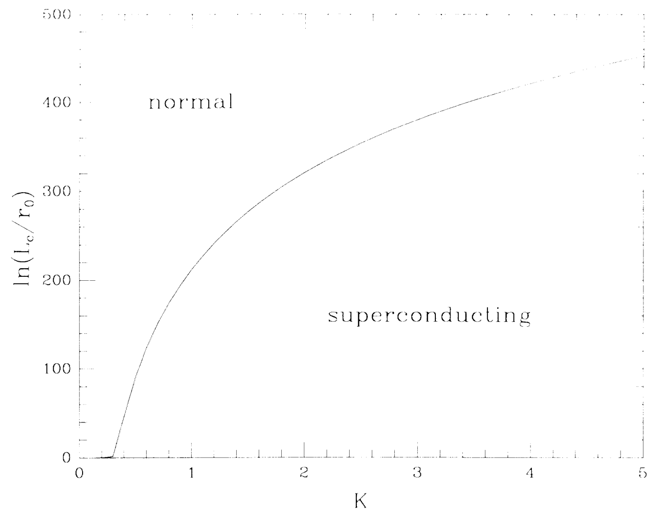

FIG. 1. Critical size of the superconducting string as a function of the coupling constant K.

in two dimensions. For $e=0$ (recall that $\alpha = Ke^2/\pi$), (19) reduces to the Kosterlitz-Thouless case:

$$dy = -y\frac{da}{a}(2\pi K - 2), \quad y \propto a^{2-2\pi K}. \tag{20}$$

This indicates that for $K < 1/\pi$, the line of fixed points is infrared unstable, and the perturbation is relevant, while for $K > 1/\pi$ the line of fixed points is stable. In the present case, from (19)

$$\frac{1}{K} = \frac{1}{K_0} + \frac{e^2}{\pi}\ln\left(\frac{a}{r_0}\right), \qquad y \propto a^2\left[1 + \frac{2K_0 e^2}{\pi}\ln\left(\frac{a}{r_0}\right)\right]^{-\pi^2/e^2}, \tag{21}$$

where K_0 is the value of K at the scale $a = r_0$. For large a, y always increases as one scales towards large distances and the entire line of fixed points is infrared unstable. Vortices are always relevant!

In conclusion, I therefore find that the renormalization-group analysis confirms the simple argument based on creation energy versus entropy: In the thermodynamical limit $L \to \infty$, vortices are free rather than bound into pairs. However, this limit may not be reached in the actual systems. The critical size L_c can be estimated to be the scale at which the entropy starts to dominate the creation energy, namely,

$$\frac{2\pi^2}{e^2}\ln\left[1 + \frac{Ke^2}{\pi}\ln\left(\frac{L_c}{r_0}\right)\right] = \ln\left(\frac{L_c}{r_0}\right)^2. \tag{22}$$

The solution of Eq. (22) is plotted in the Fig. 1. For astrophysical applications, L/r_0 is typically the ratio of the galaxy size with respect to the grand unification scale, which is about e^{100}. For strings of this size, only the ones with $K > 0.6$ are superconducting. Fortunately, the recent analysis by Hill, Hodges, and Turner[11] showed that large K values indeed occupy most of the parameter space admitting classical solutions. (Although, there is no constraint on the values of K from classical physics.) Superconducting strings with large K values have L_c exceeding their physical size and are therefore stable against quantum fluctuations.

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