Commun. math. Phys. 22, 89–103 (1971)


# Correlation Inequalities on Some Partially Ordered Sets

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Received March 2, 1971

**Abstract.** We prove that increasing functions on a finite distributive lattice are positively correlated by positive measures satisfying a suitable convexity property. Applications to Ising ferromagnets in an arbitrary magnetic field and to the random cluster model are given.

## 1. Introduction

Recently, Griffiths obtained remarkable inequalities for the correlations of Ising ferromagnets with two-body interactions [1]. These inequalities were subsequently generalized to a larger class of spin systems [2, 5]. An apparently unrelated inequality for the probabilities of certain events in a percolation model had been derived earlier by Harris [6, Lemma (4.1)]. While Harris' inequality seems to have drawn less attention than it deserves, Griffiths' inequalities have received several applications of physical interest, and give useful information on the existence of the infinite volume limit and on the problem of phase transitions. Most interesting for the applications is the second inequality, which states that any two observables $f$ and $g$ in a suitably chosen class have positive correlations, or more precisely that their thermal averages, defined with a suitably restricted Hamiltonian, satisfy:

$$\langle fg\rangle - \langle f\rangle \langle g\rangle \geqq 0 . \tag{1.1}$$

One of the simplest situations where a property of this type holds is the following. Let $\Gamma$ be a finite totally ordered set, let $\mu$ be a positive measure on $\Gamma$. Define, for any function $f$ on $\Gamma$

$$\langle f\rangle = Z^{-1} \sum_{x\in\Gamma} \mu(x)\, f(x) \tag{1.2}$$

where

$$Z = \sum_{x \in \Gamma} \mu(x)\,. \tag{1.3}$$

If $f$ and $g$ are increasing real functions on $\Gamma$, then clearly:

$$\langle fg\rangle - \langle f\rangle\langle g\rangle = (2Z^2)^{-1} \sum_{x,y\in\Gamma} \mu(x)\mu(y)(f(x)-f(y))(g(x)-g(y)) \geqq 0\,. \tag{1.4}$$

It is natural to wonder whether something remains of this property when $\Gamma$ is only partially ordered. In the present paper, we shall generalize the positivity of correlations expressed by (1.4) to the case where $\Gamma$ is a finite distributive lattice [9] and where $\mu$ satisfies a suitable convexity condition. This result also generalizes in a natural manner that obtained by two of us [8] for the random cluster model, which includes both Griffiths' second inequality with two-body interactions and Harris' inequality as special cases.

In Section 2, we recall the relevant lattice-theoretic notions, state and prove the main result. We also show that the sufficient condition thereby obtained for (1.4) to hold is by no means necessary. Section 3 is devoted to some applications, including Ising spin systems and the percolation and random-cluster models.

## 2. Correlations on a Finite Distributive Lattice

We recall that a partially ordered set $\Gamma$ is a *lattice* if any two elements $x$ and $y$ in $\Gamma$ have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$. A subset $\Gamma'$ of a lattice $\Gamma$ is called a *sublattice* of $\Gamma$ if for any $x$ and $y$ in $\Gamma'$, $x \wedge y$ and $x \vee y$ also lie in $\Gamma'$. $\Gamma'$ is then itself a lattice with the order relation and lattice operations induced by $\Gamma$. A subset $\Gamma'$ of a lattice is called a *semi-ideal* of $\Gamma$ if for any $x \in \Gamma'$ and $y \in \Gamma$ such that $y \leqq x$, $y$ also lies in $\Gamma'$. A semi-ideal need not be a lattice. The length of a totally ordered set of $n$ elements is defined to be $n-1$; the *length* $l(\Gamma)$ of a lattice $\Gamma$ is defined as the least upper bound of the lengths of the totally ordered subsets of $\Gamma$. A finite non-void lattice has a least element $O$ and a greatest element $I$. A minimal element $x \neq O$ of a lattice is called an *atom*. A lattice is called *distributive* if the operations $\wedge$ and $\vee$ satisfy either of the following two equivalent conditions

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{for all} \quad x, y, z \quad \text{in} \quad \Gamma\,,$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \text{for all} \quad x, y, z \quad \text{in} \quad \Gamma\,.$$

A sublattice of a distributive lattice is also distributive.

A real function on a partially ordered set $\Gamma$ will be called *increasing* (resp. *decreasing*), if for any ordered pair $x \leqq y$ of elements of $\Gamma$, $f(x) \leqq f(y)$ (resp. $f(x) \geqq f(y)$).

Let $\Gamma$ be a partially ordered set and $\mu$ a positive measure on $\Gamma$. For any function $f$ on $\Gamma$, define $\langle f\rangle$ by (1.2, 3). When it is necessary to refer explicitly to $\mu$ and $\Gamma$, the average (1.2) will be denoted by $\langle f;\mu,\Gamma\rangle$. Our main result is the following:

**Proposition 1.** *Let $\Gamma$ be a finite distributive lattice. Let $\mu$ be a positive measure on $\Gamma$ satisfying the following condition:*

(A) *For all $x$ and $y$ in $\Gamma$,*

$$\mu(x\wedge y)\,\mu(x\vee y)\geqq\mu(x)\,\mu(y)\,. \tag{2.1}$$

*Let $f$ and $g$ be both increasing (or decreasing) functions on $\Gamma$. Then*

$$\langle fg\rangle-\langle f\rangle\langle g\rangle\geqq 0\,. \tag{2.2}$$

Before turning to the proof of the proposition we first pave the way with some elementary remarks. Let $\mu$ be a positive measure satisfying condition (A), and let $\Gamma_0\subset\Gamma$ be the support of $\mu$:

$$\Gamma_0=\{x\in\Gamma:\mu(x)>0\}\,. \tag{2.3}$$

If $x\in\Gamma_0$ and $y\in\Gamma_0$, then by (2.1) $\mu(x\wedge y)\,\mu(x\vee y)>0$; this implies both $\mu(x\wedge y)>0$ and $\mu(x\vee y)>0$, or $x\wedge y\in\Gamma_0$ and $x\vee y\in\Gamma_0$. Therefore, by definition $\Gamma_0$ is a sublattice of $\Gamma$; as such it is both finite and distributive. Furthermore, if (2.1) holds for all $x$ and $y$ in $\Gamma_0$, then it holds for all $x$ and $y$ in $\Gamma$. In fact, (2.1) is non-trivial only for $\mu(x)\,\mu(y)>0$, i.e. for $x\in\Gamma_0$ and $y\in\Gamma_0$.

The average (1.2) depends only on the restriction of $f$ to $\Gamma_0$. If $f$ is increasing on $\Gamma$, a fortiori it is increasing on $\Gamma_0$. It follows from the preceding remarks that it is sufficient to prove the proposition for the case where $\Gamma_0=\Gamma$, i.e. where the measure $\mu$ is strictly positive, which we assume from now on.

If $\Gamma$ consists of one element, (2.2) is trivially satisfied as an equality; $l(\Gamma)$ is 0 in that case. If the number of elements of $\Gamma$ is larger than one, the length $l(\Gamma)$ of $\Gamma$ is at least one, and $\Gamma$ contains at least one atom. The proof of the proposition goes by induction on the length of $\Gamma$, starting from $l(\Gamma)=0$, and makes use of the following lemma:

**Lemma.** *Let $\Gamma$ be a finite distributive lattice with an atom $a$, let $\Gamma'_a$, $\Gamma''_a$ and $\Gamma_a$ be the sets $\{x\in\Gamma:x\geqq a\}$, $\{x\in\Gamma:x\ngeqq a\}$ and $\{x\in\Gamma:x=x'\vee a$ with $x'\in\Gamma''_a\}$, respectively. Then* (1) *$\Gamma'_a$, $\Gamma''_a$ and $\Gamma_a$ are finite distributive lattices;* (2) *$\Gamma''_a$ and $\Gamma_a$ are isomorphic;* (3) *$\Gamma_a$ is a semi-ideal of $\Gamma'_a$.*

*Proof of the Lemma.* (1) For all $x$ and $y$ in $\Gamma'_a$ we have $x\geqq a$, $y\geqq a$ and hence, by definition, $x\wedge y\geqq a$. On the other hand, $x\vee y\geqq x\geqq a$. So both $x\wedge y$ and $x\vee y$ belong to $\Gamma'_a$, i.e. $\Gamma'_a$ is a sublattice of $\Gamma$. Secondly, for any $x\in\Gamma''_a$ we have $x\wedge a<a$, and therefore, since $a$ is an atom,

$x \wedge a = O$; conversely, if $x \wedge a = O$ then $x \in \Gamma_a''$. For all $x$ and $y$ in $\Gamma_a''$ we have $(x \wedge y) \wedge a = x \wedge (y \wedge a) = x \wedge O = O$. On the other hand, $(x \vee y) \wedge a = (x \wedge a) \vee (y \wedge a) = O \vee O = O$. Therefore $\Gamma_a''$ is a sublattice of $\Gamma$. Finally, if $x', y' \in \Gamma_a''$, $x = x' \vee a$ and $y = y' \vee a$, then $x \wedge y = (x' \vee a) \wedge (y' \vee a) = (x' \wedge y') \vee a$ and $x \vee y = (x' \vee a) \vee (y' \vee a) = (x' \vee y') \vee a$. So both $x \vee y$ and $x \wedge y$ lie in $\Gamma_a$, i.e. $\Gamma_a$ is a sublattice of $\Gamma$. As $\Gamma$ is finite and distributive, so are $\Gamma_a'$, $\Gamma_a''$ and $\Gamma_a$.

(2) For $x \in \Gamma_a''$ we have $x \vee a \in \Gamma_a$. Conversely, for $x \in \Gamma_a$ there exists by definition an element $x' \in \Gamma_a''$ such that $x = x' \vee a$. Suppose that also $x = x'' \vee a$ with $x'' \in \Gamma_a''$. Then $x' = x' \wedge (x' \vee a) = x' \wedge (x'' \vee a) = (x' \wedge x'') \vee (x' \wedge a) = (x' \wedge x'') \vee O = x' \wedge x''$, and by symmetry $x'' = x' \wedge x''$. Hence $x' = x''$, i.e. $x'$ is determined uniquely (see also Ref. [9], p. 12, Theorem 10), and the mapping $x \to x \vee a$ from $\Gamma_a''$ onto $\Gamma_a$ is one-to-one. We have seen that $x \wedge y \to (x \wedge y) \vee a$ and $x \vee y \to (x \vee y) \vee a$. The mapping is therefore an isomorphism (Ref. [9], p. 24).

(3) Consider an element $x = x' \vee a$ of $\Gamma_a$. For $y \in \Gamma_a'$ such that $y \leqq x$ we have $(y \wedge x') \vee a = (y \vee a) \wedge (x' \vee a) = y \wedge x = y$; so $y = y' \vee a$ with $y' = y \wedge x' \leqq x'$, and therefore $y' \in \Gamma_a''$. Thus, by definition $y \in \Gamma_a$, and $\Gamma_a$ is semi-ideal of $\Gamma_a'$. If the greatest element of $\Gamma_a''$ is denoted by $I''$, the greatest element of $\Gamma_a$ is $I'' \vee a$, and for any $x \in \Gamma_a$ the corresponding element in $\Gamma_a''$ is $x' = x \wedge I''$.

*Proof of Proposition 1.* Suppose that the proposition holds for any lattice of length $\leqq n-1$, and let $\Gamma$ be a lattice of length $n \geqq 1$ and $\mu$ a strictly positive measure on $\Gamma$. Let $f$ and $g$ be increasing functions on $\Gamma$. We consider the quantity

$$\begin{aligned}\Omega &= Z^2(\langle fg \rangle - \langle f \rangle \langle g \rangle) \\ &= \sum_{x, y \in \Gamma} \mu(x)\,\mu(y)\,(f(x)\,g(x) - f(x)\,g(y)) .\end{aligned} \tag{2.4}$$

Let $a$ be an atom of $\Gamma$, and denote by $\Sigma'$ and $\Sigma''$ the sum over all elements of $\Gamma_a'$ and the sum over all elements of $\Gamma_a''$, respectively. We can rewrite $\Omega$ as follows:

$$\begin{aligned}\Omega = & \sum_x{}' \sum_y{}' \mu(x)\,\mu(y)\,(f(x)\,g(x) - f(x)\,g(y)) \\ & + \sum_x{}'' \sum_y{}'' \mu(x)\,\mu(y)\,(f(x)\,g(x) - f(x)\,g(y)) \\ & + \sum_x{}' \sum_y{}'' \mu(x)\,\mu(y)\,(f(x)\,g(x) - f(x)\,g(y) + f(y)\,g(y) - f(y)\,g(x)) .\end{aligned} \tag{2.5}$$

Since $\mu$ satisfies (2.1) on $\Gamma$, it also satisfies (2.1) on the sublattices $\Gamma_a'$, $\Gamma_a''$ and $\Gamma_a$. Furthermore $f$ and $g$ are increasing on $\Gamma_a'$, $\Gamma_a''$ and $\Gamma_a$. Since $l(\Gamma_a') = n-1$ and $l(\Gamma_a'') \leqq n-1$, the first two sums in (2.5) are positive by the induction hypothesis. Using again this hypothesis to obtain a lower bound for the first and third term in the last sum in (2.5) we obtain

$$\Omega \geqq \frac{\Sigma'\mu f\,\Sigma'\mu g\,\Sigma''\mu}{\Sigma'\mu} + \frac{\Sigma'\mu\,\Sigma''\mu f\,\Sigma''\mu g}{\Sigma''\mu} - \Sigma'\mu f\,\Sigma''\mu g - \Sigma'\mu g\,\Sigma''\mu f\,,$$

$$\Omega \geqq (\Sigma'\mu\,\Sigma''\mu)^{-1}(\Sigma'\mu f\,\Sigma''\mu - \Sigma'\mu\,\Sigma''\mu f)(\Sigma'\mu g\,\Sigma''\mu - \Sigma'\mu\,\Sigma''\mu g)\,, \tag{2.6}$$

where the summation variables have been omitted for brevity. We shall now show that, again by virtue of the induction hypothesis, both the second and the third factor in the right-hand member of (2.6) are positive, or equivalently:

$$\langle f;\mu,\Gamma_a''\rangle \leqq \langle f;\mu,\Gamma_a'\rangle\,, \tag{2.7}$$

and similarly for $g$. From this it follows that $\Omega \geqq 0$, which implies the proposition. This part of the proof proceeds in two steps: we shall show that

$$\langle f;\mu,\Gamma_a''\rangle \leqq \langle f;\mu,\Gamma_a\rangle \leqq \langle f;\mu,\Gamma_a'\rangle\,. \tag{2.8}$$

To prove the first inequality in (2.8), we observe that for all $x\in\Gamma_a''$ and $y\in\Gamma_a''$ such that $y\leqq x$ condition (2.1) implies

$$\mu(x)\,\mu(y\vee a)\leqq \mu(x\wedge(y\vee a))\,\mu(x\vee(y\vee a)) = \mu(y)\,\mu(x\vee a)\,. \tag{2.9}$$

Therefore, if we define $\mu_a(x)=\mu(x\vee a)$ for all $x\in\Gamma_a''$, the function $\mu/\mu_a$ is decreasing on $\Gamma_a''$ (notice that by assumption $\mu_a>0$). On the other hand, the function $f_a(x)=f(x\vee a)$ is increasing on $\Gamma_a''$, since $f$ is increasing on $\Gamma_a$. It then follows from the induction hypothesis that on $\Gamma_a''$ with the measure $\mu_a$ (which satisfies (2.1)), the functions $\mu/\mu_a$ and $f_a$ are negatively correlated, or equivalently

$$\Sigma''\mu_a\,\Sigma''\mu f_a \leqq \Sigma''\mu\,\Sigma''\mu_a f_a\,. \tag{2.10}$$

Since $f$ is increasing on $\Gamma$, $f\leqq f_a$ on $\Gamma_a''$. From this and from (2.10) the first inequality in (2.8) follows.

Next we observe that

$$\langle f;\mu,\Gamma_a\rangle = \frac{\langle f\chi;\mu,\Gamma_a'\rangle}{\langle \chi;\mu,\Gamma_a'\rangle}\,, \tag{2.11}$$

where $\chi$ is the characteristic function of $\Gamma_a$. Since $\Gamma_a$ is a semi-ideal of $\Gamma_a'$, $\chi$ is decreasing on $\Gamma_a'$, and hence

$$\langle f\chi;\mu,\Gamma_a'\rangle \leqq \langle f;\mu,\Gamma_a'\rangle\,\langle \chi;\mu,\Gamma_a'\rangle\,, \tag{2.12}$$

which by (2.11) immediately implies the second inequality in (2.8). This completes the proof of Proposition 1.

Proposition 1 provides us with a sufficient condition for increasing functions on $\Gamma$ to have positive correlations. The following argument shows that this condition is not necessary as soon as the length of $\Gamma$ is larger than 2. Let $\mu$ be a positive measure on $\Gamma$. Define $\Omega$ again by (2.4). $\Omega$ is a quadratic form with no diagonal elements with respect to the $\mu(x)$. Take $f$ and $g$ increasing and let $\Omega_1$ be the contribution to $\Omega$ of those terms which contain $\mu(I)$ or $\mu(O)$. Then

$$\begin{aligned}\Omega_1 &= \mu(I) \sum_{x \neq I} \mu(x)\,(f(I)-f(x))\,(g(I)-g(x)) \\ &\quad + \mu(O) \sum_{x \neq O, I} \mu(x)\,(f(x)-f(O))\,(g(x)-g(O)) \\ &\geqq \mu(I)\,\mu(O)\,(f(I)-f(O))\,(g(I)-g(O))\,.\end{aligned}$$

On the other hand,

$$\Omega - \Omega_1 = \tfrac{1}{2} \sum_{x,y}{}' \mu(x)\,\mu(y)\,(f(x)-f(y))\,(g(x)-g(y))\,,$$

where the sum $\Sigma'$ runs over all $(x, y)$ such that $x \neq O$, $x \neq I$, $y \neq O$, $y \neq I$. Therefore

$$|\Omega - \Omega_1| \leqq (f(I)-f(O))\,(g(I)-g(O))\,\tfrac{1}{2} \sum_{x,y}{}' \mu(x)\,\mu(y)\,.$$

Another sufficient condition to ensure that increasing functions on $\Gamma$ have positive correlation is therefore:

$$2\mu(I)\,\mu(O) \geqq \sum_{x,y}{}' \mu(x)\,\mu(y)\,. \tag{2.13}$$

If $l(\Gamma) = 2$, $\Gamma$ can be shown to have at most two atoms. If $\Gamma$ has one atom and length 2 it is totally ordered; in that case (2.2) holds for any $\mu$.
If $\Gamma$ has two atoms and length 2, (2.13) reduces to (2.1). For $l(\Gamma) \geqq 3$, however, (2.13) holds for $\mu(I)\,\mu(O)$ sufficiently large, whatever relations may exist between the $\mu(x)$ for intermediate $x$.

If $\Gamma$ is the lattice $\mathscr{P}(X)$ of subsets of the set $X = \{a, b, c\}$, ordered by inclusion (see next section), one obtains by elementary calculation the following necessary and sufficient set of conditions, which lacks the simplicity of (2.1) or (2.13):

$$\begin{gathered}(\mu(ab)+\mu(ac)+\mu(abc))\,\mu(\emptyset) \geqq (\mu(b)+\mu(c)+\mu(bc))\,\mu(a) \\ (\mu(ab)+\mu(abc))\,(\mu(\emptyset)+\mu(c)) \geqq (\mu(b)+\mu(bc))\,(\mu(a)+\mu(ac)) \\ \mu(abc)\,(\mu(\emptyset)+\mu(b)+\mu(c)) \geqq \mu(bc)\,(\mu(a)+\mu(ab)+\mu(ac))\,,\end{gathered} \tag{2.14}$$

and all conditions obtained from these by arbitrary permutations of $a, b, c$. If we denote disjoint union (and in general addition mod 2, or symmetric difference) by $+$ we can write these conditions in the following general form:

$$\sum_{R\in\Delta_i^+}\sum_{S\in\Delta_i^-}\mu(R+a_i)\,\mu(S)\geqq\sum_{R\in\Delta_i^+}\sum_{S\in\Delta_i^-}\mu(R)\,\mu(S+a_i)\,,\tag{2.15}$$

where $a_i=a,b$ or $c$, $\Delta_i^-$ is an arbitrary semi-ideal of $\mathscr{P}(X\backslash a_i)$, and $\Delta_i^+=\mathscr{P}(X\backslash a_i)\backslash\Delta_i^-$.

## 3. Applications

Consider a finite set $X$ and the set $\mathscr{P}(X)$ of subsets of $X$. If $\mathscr{P}(X)$ is partially ordered by inclusion it is a distributive lattice for this order, the operations $\wedge$ and $\vee$ becoming intersection $\cap$ and union $\cup$ in that case. Taking for $\Gamma$ any distributive sublattice of $\mathscr{P}(X)$ and applying Proposition 1, we obtain the following proposition:

**Proposition 1′.** *Let $X$ be a finite set, $\Gamma$ a sublattice of $\mathscr{P}(X)$, $\mu$ a positive measure on $\Gamma$ satisfying the following condition:*

(A′) *For all $R$ and $S$ in $\Gamma$,*

$$\mu(R\cap S)\,\mu(R\cup S)\geqq\mu(R)\,\mu(S)\,.\tag{3.1}$$

*Let $f$ and $g$ be both increasing (or decreasing) functions on $\Gamma$. Then*

$$\langle fg\rangle-\langle f\rangle\langle g\rangle\geqq0\,.$$

Conversely, if we know Proposition 1′ to hold, Proposition 1 follows immediately from a theorem in lattice theory ([9], p. 59, Theorem 3, Corollary 2) which states that any distributive lattice of length $n$ is isomorphic to a lattice of subsets of a set $X$ of $n$ elements. Therefore, although Proposition 1′ refers to a special class of lattices, it is in fact completely equivalent to Proposition 1.

In the applications we shall restrict ourselves to the case where $\Gamma=\mathscr{P}(X)$ for some $X$. Again, this does not imply a loss of generality, because (a) any measure $\mu$ on a sublattice $\Gamma'$ of $\mathscr{P}(X)$ which satisfies (2.1) can be extended to a measure on $\mathscr{P}(X)$ satisfying (2.1) by defining $\mu(R)=0$ for $R\in\mathscr{P}(X)\backslash\Gamma'$; (b) any increasing function $f$ on $\Gamma'$ can be extended to an increasing function on $\mathscr{P}(X)$ by defining $f(R)=f(R^+)$ for $R\in\mathscr{P}(X)\backslash\Gamma'$, where $R^+$ is the least upper bound of $R$ in $\Gamma'$; (c) averages do not change under this extension.

In order to see more clearly what is achieved by Proposition 1 or 1′ we first give a more explicit description of the set $\mathscr{L}$ of real increasing functions on $\mathscr{P}(X)$. $\mathscr{L}$ is a convex cone, and contains the one-dimensional vector space of constant functions. Constant functions do not contribute to correlations and we eliminate them by a normalization: we impose

$f(\emptyset)=0$. It is furthermore convenient to take a section of the remaining cone by the hyperplane $f(X)=1$. Let therefore $\mathscr{L}_0$ be the set of real increasing functions such that $f(\emptyset)=0$, $f(X)=1$. $\mathscr{L}_0$ is convex, closed under multiplication, and globally invariant under the transformation $f(R)\to\bar{f}(R)=1-f(\bar{R})$, where $\bar{R}=X\backslash R$. It is convenient to introduce the family of functions $n_P\in\mathscr{L}_0$, defined for any $P\subset X$, $P\neq\emptyset$, by

$$\begin{aligned} n_P(R)&=1 \quad \text{if} \quad R\supset P\\ &=0 \quad \text{if} \quad R\not\supset P\,. \end{aligned} \tag{3.2}$$

The functions $n_P$ satisfy

$$\left.\begin{aligned} &n_P n_Q = n_{P\cup Q}\\ &n_P(R)=n_{\bar{R}}(\bar{P})\\ &n_P(R)\,n_P(S)=n_P(R\cap S)\,. \end{aligned}\right\} \tag{3.3}$$

Consider furthermore for any non-void semi-ideal $\Delta^-$ of $\mathscr{P}(X)$ the characteristic function of its complement $\Delta^+=\mathscr{P}(X)\backslash\Delta^-$:

$$\begin{aligned} \chi_{\Delta^+}(R)&=1 \quad \text{if} \quad R\in\Delta^+\\ &=0 \quad \text{if} \quad R\notin\Delta^+\,. \end{aligned} \tag{3.4}$$

The functions $\chi_{\Delta^+}$ belong also to $\mathscr{L}_0$ and satisfy

$$\chi_{\Delta_1^+}\,\chi_{\Delta_2^+}=\chi_{\Delta_1^+\cap\Delta_2^+}\,. \tag{3.5}$$

The set of functions $n_P$ is a subset of the set of functions $\chi_{\Delta^+}$. In fact, if $\Gamma_P'=\{R\in\mathscr{P}(X): R\supset P\}$, $\Gamma_P'$ is the complement of a semi-ideal of $\mathscr{P}(X)$, then $\Gamma_{P\cup Q}'=\Gamma_P'\cap\Gamma_Q'$, and

$$n_P=\chi_{\Gamma_P'}\,. \tag{3.6}$$

Conversely, it is easy to see that any function $\chi_{\Delta^+}$ can be expressed in terms of the functions $n_{P_i}$, where the sets $P_i$ $(i\in J)$ are the minimal sets in $\Delta^+$:

$$\chi_{\Delta^+}=\sum_{J'\subset J,\,J'\neq\emptyset}(-1)^{|J'|+1}\prod_{i\in J'}n_{P_i} \tag{3.7}$$

where for any finite set $J$ we denote by $|J|$ the number of elements in $J$. The functions $\chi_{\Delta^+}$ can be shown to form the extremal points of $\mathscr{L}_0$. The content of (2.2) for increasing functions is then exhausted by taking for $f$ and $g$ all possible functions $\chi_{\Delta^+}$.

We now look for measures on $\mathscr{P}(X)$ that satisfy condition (A'). Define $\lambda(R)$ $(-\infty\leqq\lambda(R)<+\infty)$ by $\mu(R)=\exp(\lambda(R))$. Then (A') becomes:

(B) For all $R$ and $S$ subsets of $X$:

$$\lambda(R\cap S)+\lambda(R\cup S)\geqq\lambda(R)+\lambda(S)\,. \tag{3.8}$$

Let $\mathscr{M}$ be the set of all real functions $\lambda$ on $\mathscr{P}(X)$ that satisfy (3.8). $\mathscr{M}$ is a convex cone; in other words, if $\mu_1$ and $\mu_2$ satisfy (3.1) then $\mu = \mu_1^{\alpha_1}\mu_2^{\alpha_2}$ satisfies (3.1) for all real positive $\alpha_1$ and $\alpha_2$. $\mathscr{M}$ is globally invariant under the transformation $\lambda(R) \to \bar{\lambda}(R) = \lambda(\bar{R})$. Condition (B) expresses that the function $\lambda$ is convex in a suitable sense. For functions of a real variable, convexity is roughly equivalent to increase of the first derivative, and to the positivity of the second derivative. The analogue of this property in the present case is the following. We assume for simplicity that $\lambda$ is finite everywhere. Then (B) is equivalent to either of the following two conditions:

(C) For all $r \in X$, the function $\lambda(R+r) - \lambda(R)$ is an increasing function of $R \subset X \backslash r$.

(D) For all $r \in X$, $s \in X$, $s \neq r$, and for all $R \subset (X \backslash r \backslash s)$, the following quantity is positive

$$\lambda(R+r+s) + \lambda(R) - \lambda(R+r) - \lambda(R+s) \geqq 0\,. \tag{3.9}$$

The functions $n_P$ defined by (3.1) are easily seen to satisfy condition (B). Moreover, if $P$ reduces to a point, $P = \{r\}$, then $n_r$ satisfies (B) with equality for all $(R, S)$. It is therefore convenient to decompose any function $\lambda$ on $\Gamma$ on the basis on the $n_P$:

$$\lambda(R) = \sum_P \varphi(P)\, n_P(R) = \sum_{P \subset R} \varphi(P) \tag{3.10}$$

where the $\varphi(P)$ are obtained by inverting (3.10):

$$\varphi(P) = \sum_{R \subset P} (-)^{|P|-|R|}\, \lambda(R) \tag{3.11}$$

Condition (D) is then expressed as the following condition on $\varphi$:

(E) For all $r \in X$, $s \in X$, $r \neq s$, and for all $R \subset (X \backslash r \backslash s)$, the following quantity is positive:

$$\sum_{P \subset R} \varphi(P+r+s) \geqq 0\,. \tag{3.12}$$

Condition (E) contains no restriction on the one-body term $\varphi(r)$. An interesting special case of (E) is obtained by taking $\varphi(P) \geqq 0$ for all $P$ with $|P| \geqq 2$. On the other hand, if $\varphi(P)$ vanishes for $|P| > 2$, condition (E) reduces to the condition $\varphi(r, s) \geqq 0$ for all $(r, s)$. If $\varphi(P)$ vanishes for all $|P| > 3$, $E$ reduces to the following condition: For all $r \neq s$,

$$\varphi(r, s) \geqq \sum_{t \neq r, s} \mathrm{Max}(0, -\varphi(r, s, t))\,. \tag{3.13}$$

We now describe some applications of the previous results to physical systems.

## 1. Lattice Gas and Ising Spin System

We interpret $X$ as a set of sites which can be either occupied or empty, and $n_r$ as the number of particles on site $r$. We take the Hamiltonian of the system to be $H = -\lambda$ with $\lambda$ defined by (3.10) and $\varphi$ satisfying condition (E). This describes a lattice gas with many-body interactions, a special case being that where all the interactions are attractive ($\varphi(P) \geqq 0$ for $|P| \geqq 2$). The correlation functions are $\varrho(R) = \langle n_R \rangle$. We then obtain, among others, inequalities of the type

$$\frac{\partial \langle n_R \rangle}{\partial \varphi(S)} = \langle n_R n_S \rangle - \langle n_R \rangle \langle n_S \rangle \geqq 0\,, \tag{3.14}$$

for all values of the inhomogeneous "chemical potential" $\varphi(r)$.

The same system can be interpreted as an Ising spin system. With each site $r \in X$ is associated a spin variable $\sigma_r = 2n_r - 1$, and for all $R \subset X$, we define $\sigma_R$ by:

$$\sigma_R = \prod_{r \in R} \sigma_r\,.$$

The Hamiltonian is then rewritten as:

$$H = -\sum_P \varphi(P)\, n_P = -\sum_R J(R)\, \sigma_R + \text{constant}\,. \tag{3.15}$$

The relation between $\varphi$ and $J$ is easily obtained by expanding $\sigma_R$ as a function of the $n_P$ and conversely. One finds:

$$\left.\begin{aligned} J(R) &= \sum_{P \supset R} 2^{-|P|}\, \varphi(P) \\ \varphi(P) &= \sum_{R \supset P} 2^{|P|} (-)^{|R|-|P|}\, J(R)\,. \end{aligned}\right\} \tag{3.16}$$

Condition (E) can be easily expressed in terms of $J$. Substituting (3.16) into (3.12), we obtain, for $R$, $r$ and $s$ disjoint:

$$\sum_{P \subset R} 2^{|P|} \sum_{\substack{S \supset P \\ S \not\ni r, s}} (-)^{|S|-|P|}\, J(S+r+s) \geqq 0\,. \tag{3.17}$$

The sum over $P$ is trivial and we obtain:

$$\sum_{S \not\ni r,s} J(S+r+s)\, (-)^{|S|+|R \cap S|} \geqq 0\,. \tag{3.18}$$

This inequality should hold for all $R$ not containing $r$ or $s$, or equivalently for all $R$. Changing the notation from $R$ to $\bar{R}$ and remembering that $\sigma_S(R) = (-)^{|R \cap S|}$, we see that condition (E) can be rewritten as follows:

(F) For all $r \neq s$, the following function on $\Gamma$ is positive:

$$\sum_{S \not\ni r,s} J(S+r+s)\, \sigma_S \geqq 0\,. \tag{3.19}$$

Notice that condition (F) is invariant under the change of $J(S)$ into $(-)^{|S|}J(S)$, because $\sigma_S(R)=(-)^{|S|}\sigma_S(\bar{R})$. In particular, when reinterpreting a lattice gas as an Ising spin system, condition (E) becomes the same condition on $J$, whether we associate occupied sites with up spins or down spins.

For two-body interactions, ($J(S)=0$ for $|S|>2$), (F) reduces to the condition $J(r,s)\geqq 0$ for all $(r,s)$. With two- and three-body interactions, ($J(S)=0$ for $|S|>3$), (F) reduces to the following condition:

For all $r \neq s$:

$$J(r,s) \geqq \sum_{t \neq r,s} |J(r,s,t)| . \tag{3.20}$$

We now compare the present results with those obtained in earlier works [2]. There, one assume that $J(R)\geqq 0$ for all $R$. For $|R|\geqq 2$, this is similar to, but not equivalent to our condition (F). For instance, with 2 and 3 body interactions, (3.20) implies a stronger restriction than $J_2\geqq 0$, but on the other hand does not contain any condition on the sign of $J_3$. The most interesting difference is that we have no restriction on the one-body potential, and therefore on the magnetic field $h(r)=J(r)$. For instance, with $H$ defined by (3.15) and $J$ satisfying (F), we obtain for any pair of sites $(r,s)$ and any value of the (inhomogeneous) magnetic field:

$$\frac{\partial\langle\sigma_r\rangle}{\partial h(s)} = \langle\sigma_r\sigma_s\rangle - \langle\sigma_r\rangle\langle\sigma_s\rangle = 4(\langle n_r n_s\rangle - \langle n_r\rangle\langle n_s\rangle) \geqq 0 . \tag{3.21}$$

The class $\mathscr{L}$ of functions allowed for $f, g$ in Proposition 1′ is different from the class $\mathscr{Q}$ used in [2], which is the convex cone generated by the $\sigma_R$. For instance, if $r$ and $s$ are two different sites, then $\sigma_r\sigma_s$ lies in $\mathscr{Q}$ but not in $\mathscr{L}$, while $n_r+n_s-n_{rs}=\frac{1}{4}(\sigma_r+\sigma_s-\sigma_r\sigma_s)+Ct$ lies in $\mathscr{L}$ but not in $\mathscr{Q}$. Note however that the functions $n_R$ belong both to $\mathscr{Q}$ and $\mathscr{L}$.

Special cases of the preceding results can be obtained by the methods of Ref. [5]. In fact, the phase space $\Gamma$ is the cartesian product $Z_2^X$ where $Z_2=\{0,1\}$. $Z_2$ is totally ordered and can be considered as a special case of example (3) in [5], with the function $g$ being taken as $n(n(0)=0, n(1)=1)$. Propositions 3 and 5 in [5] then imply (3.14) in the case where $\varphi(P)\geqq 0$ for all $|P|\geqq 2$. The results of the present paper are more general, because here we can accommodate more general functions than linear combinations of the $n_R$ with positive coefficients, for the observables and the exponent of the Boltzmann factor.

The previous considerations extend straightforwardly to more general lattice gases where one allows more than one particle on each site, and by an easy limiting process, to lattice gases without hard cores.

## 2. *Random-cluster Model*

We interpret $X$ as the set of edges of a finite graph $G$. The set of vertices of the graph is denoted by $V$, the incidence relation, which associates two vertices with each edge, by $i$, and the graph by $G=(V,X,i)$. The vertices associated with $r\in X$ are called the ends of $r$. We shall describe several measures on $\Gamma=\mathscr{P}(X)$ which satisfy (3.1).

(a) For any $r\in X$ let $p_r$ and $q_r$ be real positive numbers. Take

$$\mu(R)=\prod_{r\in R} p_r \prod_{s\in\bar{R}} q_s\,. \tag{3.22}$$

Obviously, (3.1) is satisfied as an equality. If the $p_r$ and $q_r$ are restricted to the interval [0, 1] and satisfy $p_r+q_r=1$ for all $r\in X$, $\mu$ is a probability measure on $\mathscr{P}(X)$. A graph provided with this probability measure is called a *percolation model*. Harris' lemma, mentioned in the introduction, constitutes a specialization to this case of Proposition 1′, applied to the infinite quadratic lattice graph (or rather a sufficiently large finite subgraph of it), the functions $f$ and $g$ being arbitrary characteristic functions of the type (3.4).

Consider next for any $R\subset X$ the graph $G_R=(V,R,i)$, obtained from $G$ by omitting all edges not in $R$. By a *cluster* of $G_R$ we shall understand a maximal connected subgraph of $G_R$.

For any subgraph $G'=(U,S,i)$, with $U\subset V$, $S\subset X$, of $G$ let

$$\begin{aligned}\gamma_{G'}(R)&=1 \quad\text{if}\quad G' \quad\text{is a cluster of}\quad G_R\\ &=0 \quad\text{if}\quad G' \quad\text{is not a cluster of}\quad G_R\,.\end{aligned} \tag{3.23}$$

Define $\mu(R)=\exp\lambda(R)$ with

$$\lambda(R)=\sum_{G'}\varphi(G')\,\gamma_{G'}(R) \tag{3.24}$$

where for any $G'$, $\varphi(G')$ is a real number, and where the sum runs over all connected subgraphs of $G$ (or, equivalently, over all subgraphs of $G$). In order that $\lambda(R)$ satisfies (3.9), $\varphi(G)$ has to satisfy the condition

(G) For all $r\in X$, $s\in X$, $r\neq s$, and for all $R\subset(X\backslash r\backslash s)$,

$$\sum_{G'}\varphi(G')\,[\gamma_{G'}(R+r+s)+\gamma_{G'}(R)-\gamma_{G'}(R+r)-\gamma_{G'}(R+s)]\geqq 0\,. \tag{3.25}$$

We shall now discuss several examples of functions $\varphi(G')$ satisfying condition (G).

(b) Let $\xi$ be a real function on the edge set $X$, and take for $G'=(U,S,i)$: $\varphi(G')=\sum_{r\in S}\xi(r)$. Then $\lambda(R)=\sum_{r\in R}\xi(r)$, and (3.9) is satisfied as an equality. With $\xi(r)=\log p_r-\log q_r$, the measure is proportional to that of example (a). A special case is that where all $\xi(r)$ are equal to a real number $c$, then $\lambda(R)=c|R|$.

(c) Let $\psi$ be a real function on the vertex set $V$, and take for $G'=(U,S,i)$: $\varphi(G')=\sum_{v\in U}\psi(r)$. Then $\lambda(R)=\sum_{v\in V}\psi(v)=$ constant, so that (3.9) is trivially satisfied for any choice of $\psi(v)$.

(d) Let $c$ be a real number and take $\varphi(G')=c$ for all $G'$. Then $\lambda(R)=c\gamma(R)$, where $\gamma(R)=\sum_{G'}\gamma_{G'}(R)$ is the total number of clusters of $G_R$. It is easily seen that $\gamma(R+r+s)+\gamma(R)-\gamma(R+r)-\gamma(R+s)$ equals 1 if $G_R$ contains two clusters $G_1'$ and $G_2'$ such that in $G_{R+r+s}$ both $r$ and $s$ have one end in $G_1'$ and one end in $G_2'$, and equals 0 otherwise. Hence (3.9) is satisfied for $c\geqq 0$. The corresponding measure is

$$\mu(R)=\kappa^{\gamma(R)} \quad \text{with} \quad \kappa=\exp c\geqq 1\,. \tag{3.26}$$

(e) Take $\varphi(G')=\psi(v)$ if $G'$ consists of a single vertex $v$ ("isolated vertex"), $\varphi(G')=0$ otherwise. Then $\gamma_{G'}(R+r+s)+\gamma_{G'}(R)-\gamma_{G'}(R+r)-\gamma_{G'}(R+s)$ equals 1 if $G'$ is an isolated vertex of $G_R$ which is neither an isolated vertex of $G_{R+r}$ nor one of $G_{R+s}$, and equals 0 otherwise. $\lambda(R)$ satisfies (3.9) if $\psi(v)\geqq 0$ for all $v$ in $V$. The measure is then

$$\mu(R)=\prod_v{}^{(is)}\exp\psi(v)\,, \tag{3.27}$$

where the product is over the isolated vertices of $G_R$.

(f) Let $\{c_n\}$, $(n=1,2,\ldots)$ be a set of real numbers, and let $n(G')$ be the number of vertices of $G'$. Take $\varphi(G')=c_{n(G')}$. It can be shown that $\varphi$ satisfies (3.25) if the $c_n$ satisfy the following relations:

$$\left.\begin{aligned} c_{2n}&\leqq 2c_n\\ 2c_{n+1}&\leqq c_{n+2}+c_n\end{aligned}\right\}\ \text{for } n\geqq 1\,. \tag{3.28}$$

A few choices of $c_n$ which satisfy (3.28) are: $c_n=n$ (this is the above-mentioned trivial example (c) with $\psi(v)=1$ for all $v$ in $V$), $c_n=c\geqq 0$ (example (d)), $c_n=\delta_{n1}$ (example (e) with $\psi(v)=1$), $c_n=n^{-\alpha}$, where $\alpha$ is a positive real number.

If $\mu_1$ and $\mu_2$ are two measures satisfying (3.1), the product $\mu_1\mu_2$ is also a measure satisfying (3.1). We discuss two examples of such a product measure.

(g) Take $\lambda(R)=c\,\omega(R)$, where $c$ is a positive real number and $\omega(R)$ is the number of independent cycles (cyclomatic number) of $G_R$. According to Euler's formula we have $\omega(R)=|R|-|V|+\gamma(R)$. The corresponding measure is therefore the product of the measures discussed under (b) (with $\xi(r)=c$ for all $r\in X$), (c) (with $\psi(v)=-c$) and (d).

(h) Take the product of the measures (3.22) with $0 \leqq p_r \leqq 1$, $q_r = 1 - p_r$, and (3.26):

$$\mu(R) = \kappa^{\gamma(R)} \prod_{r \in R} p_r \prod_{s \in \bar{R}} q_r \,. \tag{3.29}$$

A graph provided with this measure is called a random-cluster model. In a previous paper by two of us [7] it was shown that for $\kappa = 2$ the function $Z$ which by (1.3) corresponds to the measure (3.29) is equal (apart from a trivial factor) to the partition function of an Ising model with $V$ as the set of sites and with a Hamiltonian containing only ferromagnetic pair interactions:

$$H = -\sum_{r \in X} J_r \sigma_{v(r)} \sigma_{v'(r)} \,, \tag{3.30}$$

$$J_r = -\tfrac{1}{2} \log q_r \tag{3.31}$$

where the vertices $v(r)$ and $v'(r)$ are the ends of the edge $r$ in the graph $G$. Similarly, for any $W \subset V$ the Ising model spin correlation function $\langle \sigma_W \rangle_{Is}$ (where in order to avoid confusion the averages for the Ising model discussed before are denoted by $\langle\ \rangle_{Is}$) is equal to the expectation value under the measure (3.29) with $\kappa = 2$ of a quantity $\varepsilon_W$ defined by

$$\begin{aligned} \varepsilon_W(R) &= 1 \quad \text{if each cluster of } G_R \text{ contains an even number} \\ &\qquad\ \ \text{(possibly zero) of vertices from } W \\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{3.32}$$

that is,

$$\langle \sigma_W \rangle_{Is} = \langle \varepsilon_W ; \mu \rangle \,. \tag{3.33}$$

Now it is obvious that for any pair $(W_1, W_2)$ of subsets of $V$ we have

$$\varepsilon_{W_1 + W_2} \geqq \varepsilon_{W_1} \varepsilon_{W_2} \,, \tag{3.34}$$

where $W_1 + W_2$ is the symmetric difference of $W_1$ and $W_2$. Since the functions $\varepsilon_W$ are increasing on $\mathscr{P}(X)$, Proposition 1′ can be applied, giving

$$\langle \varepsilon_{W_1} \varepsilon_{W_2} \rangle \geqq \langle \varepsilon_{W_1} \rangle \langle \varepsilon_{W_2} \rangle \,. \tag{3.35}$$

From (3.33), (3.34) and (3.35) it follows that

$$\langle \sigma_{W_1 + W_2} \rangle_{Is} \geqq \langle \sigma_{W_1} \rangle_{Is} \langle \sigma_{W_2} \rangle_{Is} \,,$$

which is the second Griffiths-Kelly-Sherman inequality for the case where there are only pair interactions.

*Acknowledgements.* For two of us (C.M.F. and P.W.K.), this work is part of the research program of the Foundation for Fundamental Research on Matter (F.O.M.). This work arose from discussions which took place while one of us (P.W.K.) was the guest of the I.H.E.S. in Bures. We are grateful to Professors R. B. Griffiths and S. Sherman for critical remarks.

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