References

[1] G. K. Huth and C. L. Weber, "Minimum weight convolutional codewords of finite length," *IEEE Trans. Inform. Theory*, vol. IT-22, pp. 243–246, Mar. 1976.

[2] F. Hemmati and D. J. Costello Jr., "Asymptotically catastrophic convolutional codes," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 298–304, May 1980.

[3] A. D. Wyner and R. B. Ash, "Analysis of recurrent codes," *IEEE Trans. Inform. Theory*, vol. IT-9, pp. 143–156, 1963.

[4] Ø. Ytrehus, "A note on high rate binary convolutional codes," Dep. Informatics, Univ. of Bergen, Bergen, Norway, Tech. Rep. 68, Aug. 1992.

[5] ——, "Ascetic convolutional codes," in *Proc. 33rd Allerton Conf. on Communication, Control, and Computing*, Oct. 1995, pp. 382–390.

[6] G. D. Forney, Jr., "Convolutional codes I: Algebraic structure," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 720–738, Nov. 1970.

[7] E. Paaske, "Short binary convolutional codes with maximal free distance for rates $2/3$ and $3/4$," *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 683–689, Sept. 1974.

[8] G. D. Forney, Jr., "Structural analysis of convolutional codes via dual codes," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 512–518, July 1973.

[9] R. M. Karp, "A characterization of the minimum cycle mean in a digraph," *Discr. Math.*, vol. 23, pp. 309–311, 1978.

Rational Rate Punctured Convolutional Codes for Soft-Decision Viterbi Decoding

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***Abstract*—We present rational rate k/n punctured convolutional codes (n up to 8, $k = 1, \cdots, n-1$, and constraint length ν up to 8) with good performance. Many of these codes improve the free distance and (or) weight spectra over previously reported codes with the same parameters. The tabulated codes are found by an exhaustive (or a random) search.**

***Index Terms*— Maximum-likelihood decoding complexity, punctured convolutional codes.**

I. Introduction

Punctured k/n, $n = k+1, k = 2, 3, \cdots$ convolutional codes were introduced by Cain *et al.* [1] to simplify maximum-likelihood (Viterbi) decoding. Using the notation in [1], a punctured convolutional encoder is defined by a set of n generator polynomials $P^1(D), P^2(D), \cdots, P^n(D)$, where

$$P^j(D) = p_0^j + p_1^j D + \cdots + p_\nu^j D^\nu$$

and ν is the constraint length of the punctured encoder.

A punctured convolutional encoder where all generators $P^j(D)$ have zero delay and maximum degree $p_0^j = p_\nu^j = 1$, for $1 \le j \le n$, is called an antipodal punctured code. This class of punctured codes was investigated extensively because of the hypothesis that the best punctured codes are always antipodal ones.

Punctured codes are obtained by periodically deleting bits from low-rate $1/n_0, 2 \le n_0 \le n$ convolutional codes. At first, only $1/n_0$, $n_0 < n$ low-rate codes have been used for puncturing or, in other words, punctured codes have been constructed with a restriction on the number of different code generators $P^j(D)$ (see, for example, [2]–[4]). However, in [5] convolutional codes were searched without the restriction on the number of different generators and they may be regarded as obtained from $1/n_0$, $n_0 = n$ low-rate code.

Manuscript received September 2, 1996; revised November 8, 1996.

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Publisher Item Identifier S 0018-9448(97)03686-9.

TABLE I
Performance of Rate $3/7$, $\nu = 2$ Convolutional Codes with Different Branch Partitions

Branch partition	d_f	Spectra
1,1,5	4	t=1 4 2 . . . , f=1 8 3 . . .
1,2,4	5	t=3 4 5 . . . , f=3 8 15 . . .
1,3,3	5	t=2 3 6 . . . , f=2 6 16 . . .
2,2,3	6	t=6 0 22 . . . , f=9 0 71 . . .

Previous papers dealing with the problem of punctured codes were focused on codes with rate $k/(k+1)$, $k = 2, 3, \cdots$, and it was assumed that outputs corresponding to $P^1(D)$ and $P^2(D)$ were transmitted on the same branch in the code trellis and the outputs corresponding to each of the other generators were successively transmitted on separate branches. Tables of the best rate $k/(k+1)$, $k = 2, 3, \cdots$ punctured codes are presented in [5]. Convolutional codes with rational rates other than $1/n$ and $k/(k+1)$ were searched only in the framework of regular convolutional codes [6].

To provide punctured convolutional code of rate k/n, $k = 2, 3, \cdots, n > k$ we consider all possible partitions of n by numbers $l_1, \cdots, l_k$, $0 < l_i < n$. For example, to obtain $k/n = 3/5$ we introduce two partitions $5 = 2+2+1$ and $5 = 3+1+1$. If the first partition is used, outputs corresponding to $P^1(D)$ and $P^2(D)$ are transmitted on the same branch of the trellis; outputs corresponding to $P^3(D)$ and $P^4(D)$ are transmitted on the other branch; and the output generated by $P^5(D)$ is transmitted on the separate branch. The second partition unites outputs produced by $P^1(D)$, $P^2(D)$, and $P^3(D)$.

In this correspondence some new $k/(k+1)$ punctured codes better than codes previously tabulated are listed. We also have found new punctured codes of rational rates other than $k/(k+1)$. We demonstrate some new examples of rate $(kl)/(nl)$ punctured convolutional codes which are superior to the corresponding rate k/n codes. These results are presented in the form of the extended tables of the best known punctured codes of rates k/n, $k = 1 \cdots 7$, $n = 2 \cdots 8$, $k < n$, with constraint length $\nu \le 8$. More than 75% of the codes presented are new.

We also study some properties of rational rate punctured convolutional codes. We show that antipodal codes are not always optimal. The dependence of punctured code performance upon the output bit distribution over trellis branches (branch partition) is investigated. We show that the optimal branch partition for the given code rate does not exist. We also improve the existing tables of rate $1/n$, $n = 3, 4$ codes and present tables of new rate $1/n$, $n = 5 \cdots 8$ codes.

II. Search Technique

We looked for optimal codes in the sense of maximizing d_f and minimizing f. Here and below d_f denotes the free distance of the code, $f = \{f_i\}, i = 1, 2, \cdots$, where f_i denotes the total weight of all paths with weight $d_f + i - 1$. More exactly, we chose codes with maximum d_f and among codes with equal d_f we preferred the code having the best coefficient f_1. For codes with equal d_f and the same f_1 we chose the code having better second coefficient f_2, and so on, up to the tenth coefficient.

TABLE II
CONVOLUTIONAL CODES OF RATE $l/(2l),\ l = 1,\cdots,4$

ν	l	d_f	Generators	Spectra $\frac{t_1,t_2,\cdots}{f_1,f_2,\cdots}$
2[o]	1	5	(7,5)	1,2,4,8,16,32,64,128,256,512 1 4 12 32 80 192 448 1024 2304 5120
3[o]	1	6	(17,15)	1 3 5 11 25 55 121 267 589 1299 2 7 18 49 130 333 836 2069 5060 12255
4[o]	1	7	(31,27)	2 3 4 16 37 68 176 432 925 2156 4 12 20 72 225 500 1324 3680 8967 22270
4[s]	3	7	(27,23),(33,21),(37,27)	5 11 21 46 110 259 583 1344 3152 7289 8 36 83 211 634 1675 4185 10816 27934 70105
4[l]	4	8	(33,25),(33,25),(37,35),(37,27)	49 0 0 0 1352 0 0 0 38717 0 178 0 0 0 9368 0 0 0 392654 0
5[o]	1	8	(65,57)	1 8 7 12 48 95 281 605 1272 3334 2 36 32 62 332 701 2342 5503 12506 36234
5[s]	3	8	(67,51),(67,51),(67,55)	2 18 25 49 113 250 756 1682 3881 9223 2 79 105 305 783 1853 6469 15486 39938 102870
6[o]	1	10	(155,117)	11 0 38 0 193 0 1331 0 7275 0 36 0 211 0 1404 0 11633 0 77433 0
6[s]	2	10	(155,177),(145,113)	20 0 77 0 453 0 2621 0 14776 0 69 0 473 0 3442 0 24227 0 163182 0
6[s]	3	10	(147,125),(155,117),(173,135)	25 0 139 0 638 0 3714 0 22274 0 81 0 835 0 5063 0 34582 0 245908 0
7[o]	1	10	(345,237)	1 6 12 26 52 132 317 730 1823 4446 2 22 60 148 340 1008 2642 6748 18312 48478
7[s]	2	11	(323,247),(335,257)	13 32 52 108 255 639 1582 3784 9224 22246 47 166 330 750 2073 5714 15414 39728 104476 271160
8[o]	1	12	(657,435)	11 0 50 0 286 0 1630 0 9639 0 33 0 281 0 2179 0 15035 0 105166 0
8[s]	2	12	(657,515),(773,517)	13 41 55 124 314 723 1856 4475 10706 25972 48 205 372 918 2514 6511 18340 48081 123802 321734

[o] Code found by Odenwalder [13].
[s] New code found in this study by an exhaustive search.
[l] Code found by Lee [5].

TABLE III
CONVOLUTIONAL CODES OF RATE $(l/3l),\ l = 1,2$

ν	l	d_f	Generators	Spectra $\frac{t_1,t_2,\cdots}{f_1,f_2,\cdots}$
2[o]	1	8	(7,7,5)	2 0 5 0 13 0 34 0 89 0 3 0 15 0 58 0 201 0 655 0
3[o]	1	10	(17,15,13)	3 0 2 0 15 0 24 0 87 0 6 0 6 0 58 0 118 0 507 0
4[o]	1	12	(37,33,25)	5 0 3 0 13 0 62 0 108 0 12 0 12 0 56 0 320 0 693 0
5[o]	1	13	(75,53,47)	1 3 6 4 5 12 14 33 66 106 1 8 26 20 19 62 86 204 420 710
6[s]	1	15	(155,127,117)	3 3 6 9 4 18 35 45 77 153 7 8 22 44 22 94 219 282 531 1104
6[s]	2	15	(155,137,117),(165,133,117)	4 9 12 16 21 32 49 89 188 306 8 26 42 70 116 182 268 562 1240 2117
7[o]	1	16	(357,251,233)	1 0 8 0 24 0 51 0 133 0 1 0 24 0 113 0 287 0 898 0
7[s]	2	16	(255,237,233),(347,255,237)	1 10 13 10 17 35 63 96 165 280 1 29 46 38 75 189 368 585 1093 1988
8[s]	1	18	(637,567,515)	1 5 8 14 11 21 43 54 119 215 2 15 36 72 54 123 282 388 872 1705

[o] Code found by Odenwalder [13].
[s] New code found in this study by an exhaustive search.

To compute code spectra we used the Balakirsky algorithm [7] which was generalized and represented in the matrix form in [8]. The search was performed without any restriction on the number of different generator polynomials and all possible branch partitions were analyzed. Moreover, we looked for the codes with good performance among cancellable rate (kl/nl rate) codes and took into account nonantipodal codes as well.

To speed up an exhaustive search we deleted all duplicate codes (obtained by reversing the columns of a generator matrix) and catastrophic codes. For fast rejecting of bad codes in the first phase

TABLE IV
CONVOLUTIONAL CODES OF RATE $l/(4l)$, $l = 1, 2$

ν	l	d_f	Generators	Spectra $\frac{t_1,t_2,\ldots}{f_1,f_2,\ldots}$
2^s	1	10	(7,7,5,5)	1 0 2 0 4 0 8 0 16 0
				1 0 4 0 12 0 32 0 80 0
2^s	2	10	(7,7,5,5),(7,7,7,5)	1 3 2 2 6 7 9 15 21 28
				1 5 5 6 19 29 41 73 119 174
3^l	1	13	(17,15,13,13)	2 1 0 3 1 4 8 4 15 16
				4 2 0 10 3 16 34 18 77 84
4^s	1	16	(37,33,27,25)	4 0 2 0 4 0 15 0 30 0
				8 0 7 0 17 0 60 0 140 0
5^s	1	18	(77,67,55,51)	3 0 3 0 9 0 13 0 26 0
				5 0 9 0 34 0 59 0 142 0
6^s	1	20	(171,155,127,117)	2 0 6 0 7 0 15 0 25 0
				3 0 17 0 32 0 66 0 130 0
7^s	1	22	(353,335,277,231)	1 2 2 2 5 8 10 14 14 18
				2 4 4 6 18 32 50 78 82 92
8^s	1	24	(765,671,513,473)	1 0 6 0 8 0 23 0 37 0
				1 0 15 0 33 0 111 0 210 0

[l] Code found by Larsen [12].
[s] New code found in this study by an exhaustive search.

TABLE V
CONVOLUTIONAL CODES OF RATE $1/5$

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\ldots}{f_1,f_2,\ldots}$
2^s	13	(7,7,7,5,5)	1 1 0 1 3 1 1 6 5 2
			1 2 0 2 9 4 3 24 25 10
3^s	16	(17,17,15,13,13)	1 2 0 0 2 1 2 6 5 3
			2 4 0 0 6 3 8 24 20 13
4^s	20	(37,35,33,27,25)	4 0 0 0 3 0 5 0 8 0
			8 0 0 0 10 0 20 0 32 0
5^s	22	(77,73,67,55,45)	1 2 1 0 2 2 3 6 2 3
			2 4 2 0 6 6 12 24 8 15
6^s	25	(175,171,155,127,113)	1 2 1 1 3 2 5 5 7 8
			1 4 3 2 9 8 21 24 33 38
7^s	28	(375,317,271,265,233)	2 0 4 0 3 0 12 0 17 0
			3 0 11 0 9 0 47 0 89 0
8^r	31	(635,565,533,467,771)	2 2 2 3 2 4 3 6 17 12
			4 4 6 8 8 18 13 26 77 64

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

of the code analysis we used the following tests:

1) For a punctured code given by its generators $P^{(j)}(D), j = 1, \cdots, n$ we compute an estimate of the free distance by the formula

$$\hat{d}_f = \sum_{i=1}^{n} w(P^{(i)}(D))$$

where $w(a)$ denotes the Hamming weight of the sequence a. If $\hat{d}_f$ is less than the free distance of the known punctured code with the same parameters we skip the considered code.

2) Further, if all $w(P^{(j)}(D)), i = 1, \cdots, n$ are even, the corresponding convolutional code is skipped as a catastrophic code.

3) After that we construct other estimates for d_f based on calculating weights of linear combinations of the code generators and their shifts. We also compute estimates for spectrum coefficients.

TABLE VI
CONVOLUTIONAL CODES OF RATE $1/6$

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\ldots}{f_1,f_2,\ldots}$
2^s	16	(7,7,7,7,5,5)	2 0 0 0 5 0 0 0 13 0
			3 0 0 0 15 0 0 0 58 0
3^s	20	(17,17,15,15,13,13)	3 0 0 0 2 0 0 0 15 0
			6 0 0 0 6 0 0 0 58 0
4^s	24	(37,35,33,27,27,25)	4 0 1 0 0 0 4 0 5 0
			8 0 3 0 0 0 15 0 21 0
5^s	27	(77,73,67,55,53,51)	2 2 0 0 1 4 1 1 4 1
			4 4 0 0 3 14 3 4 16 4
6^s	30	(175,173,157,131,127,115)	1 2 1 0 1 2 2 5 1 2
			2 4 2 0 2 6 8 19 4 8
7^r	34	(265,365,367,331,231,277)	3 0 2 0 5 0 6 0 13 0
			5 0 5 0 17 0 22 0 55 0
8^r	37	(655,717,751,513,737,531)	1 2 4 1 1 1 2 9 4 3
			1 4 12 4 5 2 6 38 20 14

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE VII
CONVOLUTIONAL CODES OF RATE $1/7$

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\ldots}{f_1,f_2,\ldots}$
2^s	18	(7,7,7,7,5,5,5)	1 0 1 0 1 0 3 0 2 0
			1 0 2 0 2 0 9 0 7 0
3^s	23	(17,17,15,15,13,13,13)	2 1 0 0 0 2 0 1 1 4
			4 2 0 0 0 6 0 4 3 16
4^s	28	(37,35,35,33,27,27,25)	4 0 0 0 2 0 2 0 2 0
			8 0 0 0 6 0 8 0 8 0
5^s	32	(77,73,67,65,57,55,45)	4 0 0 0 0 0 5 0 6 0
			8 0 0 0 0 0 16 0 24 0
6^s	36	(173,165,157,137,135,131,115)	4 0 0 0 3 0 0 0 12 0
			8 0 0 0 8 0 0 0 46 0
7^r	40	(335,257,345,237,337,265,233)	4 0 0 0 7 0 1 0 8 0
			8 0 0 0 21 0 6 0 35 0
8^r	44	(763,737,675,557,551,531,523)	4 0 4 0 3 0 4 0 5 0
			8 0 14 0 12 0 15 0 21 0

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE VIII
CONVOLUTIONAL CODES OF RATE $1/8$

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\ldots}{f_1,f_2,\ldots}$
2^s	21	(7,7,7,7,7,5,5,5)	1 1 0 0 0 1 3 1 0 0
			1 2 0 0 0 2 9 4 0 0
3^s	26	(17,17,17,15,15,13,13,13)	1 2 0 0 0 0 2 0 0 1
			2 4 0 0 0 0 6 0 0 3
4^s	32	(37,37,35,33,33,27,25,25)	4 0 0 0 1 0 0 0 5 0
			8 0 0 0 4 0 0 0 18 0
5^s	36	(77,73,67,65,57,55,47,45)	2 0 2 0 0 0 2 0 3 0
			3 0 5 0 0 0 6 0 10 0
6^r	40	(153,145,173,175,127,135,111,157)	1 0 3 0 0 0 4 0 2 0
			1 0 7 0 0 0 13 0 7 0
7^r	45	(235,267,255,351,323,317,257,277)	1 2 1 0 4 2 0 1 0 3
			1 4 3 0 12 8 0 2 0 14
8^r	50	(563,757,551,665,771,753,651,517)	3 0 3 0 1 0 5 0 3 0
			5 0 10 0 2 0 16 0 16 0

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE IX
CONVOLUTIONAL CODES OF RATE $2l/3l, l = 1, 2$

ν	l	d_f	Generators	Spectra $\frac{t_1,t_2,\cdots}{f_1,f_2,\cdots}$
2^c	1	3	7,(5,7)	1 4 14 40 116 339 991 2897 8468 24752 1 10 54 226 856 3072 10647 35998 119478 390904
2^s	2	4	7,(5,7),(5,7),7	23 0 182 0 1852 0 17968 0 175645 0 66 0 1108 0 15749 0 201210 0 2429022 0
3^l	2	4	15,(11,17)	2 11 34 109 366 1244 4228 14328 48500 164182 5 41 193 808 3299 13191 51573 197848 748104 2797609
3^s	1	4	17,(13,15),11,(11,17)	4 20 67 234 827 2945 10630 38234 137259 492258 6 78 445 2010 8712 36764 153391 626771 2521149 10012726
4^l	1	5	25,(23,35)	5 18 54 193 714 2578 9295 33508 120801 435731 15 88 370 1640 7116 29942 123443 499706 1999757 7927310
4^s	2	5	25,(31,23),27,(31,27)	10 40 122 402 1511 5565 19876 71418 258189 932189 28 204 840 3318 14709 63006 256026 1027664 4108317 16271464
5^l	1	6	65,(57,75)	16 0 182 0 2700 0 36893 0 514307 0 56 0 1301 0 27620 0 499380 0 8624252 0
5^r	2	6	55,(57,75),55,(71,47)	17 76 231 838 3724 14246 55661 219394 865531 3397054 86 430 2210 9234 48784 211996 958057 4187826 18343641 78801888
6^c	1	6	147, (135,147)	1 17 59 175 668 2638 9976 37654 142203 536424 1 81 402 1487 6793 31018 132403 560393 2351179 9739403
7^c	1	8	337,(251,337)	66 0 706 0 10727 0 155705 0 2257451 0 395 0 6695 0 135288 0 2449036 0 42657107 0
8^s	1	8	625,(577,711)	9 58 161 566 2251 8668 33592 128238 490356 1879346 38 416 1404 5994 27194 118184 515022 2169506 9090826 37921186

s New code found in this study by an exhaustive search.
r New code found in this study by a random search.
c Code found by Cain [1].
l Code found by Lee [5].

TABLE X
CONVOLUTIONAL CODES OF RATE $2/5$

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\cdots}{f_1,f_2,\cdots}$
2^s	6	(5,7),(7,7,5)	1 4 5 8 17 29 50 93 168 299 1 8 14 30 77 157 314 654 1317 2592
3^s	8	(13,17),(17,11,13)	5 0 13 0 45 0 145 0 487 0 9 0 42 0 200 0 829 0 3354 0
4^s	9	(21,27),(33,37,25)	3 5 6 9 22 40 71 152 260 478 5 12 20 32 104 202 381 962 1782 3526
5^s	11	(55,73),(73,75,51)	6 10 14 22 32 73 154 323 618 1089 14 32 70 124 176 460 1084 2420 5064 9612
6^s	12	(157,165),(157,127,111)	2 13 23 18 12 110 202 201 496 1353 3 43 106 119 64 691 1388 1712 4301 11338
7^s	14	(333,275), (357,245,235)	15 0 45 0 151 0 510 0 1951 0 42 0 219 0 953 0 3768 0 16604 0
8^r	15	(633,565),(575,613,571)	11 20 22 40 80 167 314 534 1061 2238 43 78 106 226 514 1168 2420 4322 9143 20604

s New code found in this study by an exhaustive search.
r New code found in this study by a random search.

Note that all the above mentioned tests are very simple and their complexity is proportional to the constraint length. On the other hand, they reject more than 99% of codes that should be analyzed. So the exponentional number of computations which is necessary to derter-mine d_f and spectra has been performed for less than 1% of codes.

The best punctured convolutional codes found for rates $k/(lk)$, $k = 1, \cdots, 4$, $l = 2, 3, 4$ are presented in Tables II–IV. In Tables V–VIII the best convolutional codes for rates $1/n, n = 5, \cdots, 8$ are listed. Table IX contains the best punctured convolutional codes for rates $(2l)/(3l), l = 1, 2$. Codes of rate $2/n, n = 5, 7$ are given in Tables X and XI. Tables XII–XV contain codes of rates $3/n$, $n = 4, 5, 7, 8$, respectively. The best codes found for rates $4/n$, $n = 5, 7$, $5/n$, $n = 6, 7, 8$, $6/7$, and $7/8$ are listed in Tables XVI–XXII.

The codes of rate k/n are given by n generator polynomials. In Tables II–XXII octal representation of the generator polynomials is used. The numbers in parentheses correspond to the generator polynomials providing outputs transmitted on the same branch of the trellis (code partition). For example, the punctured code of rate $2/3$ given by the following polynomials $P^{(1)}(D) = 1 + D + D^2$, $P^{(2)}(D) = 1 + D^2$, $P^{(3)}(D) = 1 + D + D^2$ is presented in the form $7, (5, 7)$.

TABLE XI
CONVOLUTIONAL CODES OF RATE 2/7

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\dots}{f_1,f_2,\dots}$
2[s]	9	(7,5,5),(7,7,7,5)	3 3 3 8 9 11 25 35 46 88
			5 6 9 32 41 56 147 226 322 678
3[s]	11	(17,15,13),(17,15,13,11)	2 2 2 4 4 7 14 15 22 43
			2 4 6 10 12 26 54 64 102 216
4[s]	13	(35,27,23),(37,33,25,23)	2 3 5 3 4 10 15 25 29 55
			2 6 17 12 16 36 67 116 141 312
5[s]	16	(73,65,57),(77,73,55,45)	8 0 13 0 21 0 42 0 132 0
			16 0 50 0 94 0 215 0 730 0
6[r]	17	(145,135,117),(175,145,133,117)	2 3 6 10 13 18 20 26 47 84
			2 6 20 40 57 90 112 130 277 528
7[r]	19	(331,375,253),(251,363,267,355)	3 6 4 9 14 20 27 27 68 93
			6 15 12 36 59 90 145 146 375 566
8[r]	21	(531,723,647),(771,673,455,567)	5 8 6 9 11 15 37 51 78 124
			13 29 18 38 54 72 201 305 488 811

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XII
CONVOLUTIONAL CODES OF RATE 3/4

ν	d_f	Generators	Spectra
2[c]	3	7,5,(7,5)	6 23 80 290 1068 3945 14538 53532 197148 726135
			15 104 540 2557 11534 50352 214435 896374 3693934 15051008
3[c]	4	15,17,(17,15)	29 0 532 0 10059 0 190112 0 3593147 0
			124 0 4504 0 125991 0 3153896 0 74212497 0
4[l]	4	31,31,(35,23)	5 42 134 662 3643 16585 79605 393135 1889651 9120711
			10 290 1188 7174 48976 262074 1439629 8036900 43076925 229217547
5[l]	5	65,65,(47,61)	13 71 326 1626 8320 42351 215475 1095716 5570379 28322278
			51 474 2978 18918 116366 690938 4023083 23045360 130315745 729441322
6[l]	6	117,173,(165,127)	45 109 844 3444 20880 100121 543248 2745946 14482759 74507558
			276 843 9588 44046 326876 1756787 10912170 60965550 356096818 1996399542
7[r]	6	323,241,(357,235)	10 86 385 1896 10015 52767 276377 1445764 7558454 39541590
			35 649 3929 23650 147068 901169 5371039 31472833 182241700 1046107232
8[r]	6	521,747,(631,417)	1 44 216 985 5254 27868 146064 768120 4039071 21234125
			1 312 2086 12040 76631 469504 2800390 16522861 96257137 555436184

[c] Code found by Cain [1].
[r] New code found in this study by a random search.
[l] Code found by Lee [5].

TABLE XIII
CONVOLUTIONAL CODES OF RATE 3/5

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\dots}{f_1,f_2,\dots}$
2[s]	4	7,(7,5),(7,5)	3 12 24 56 145 376 964 2448 6227 15860
			4 32 104 312 968 2928 8588 24680 70052 196856
3[s]	4	15,(15,11),(15,13)	1 5 14 40 102 292 835 2300 6446 18131
			1 10 47 185 596 2056 6858 21613 68289 213679
4[s]	6	25,(35,21),(37,27)	18 0 139 0 1210 0 10594 0 92572 0
			54 0 796 0 9666 0 109250 0 1170371 0
5[s]	6	47,(47,65),(45,57)	1 16 40 98 299 919 2941 8936 26655 81651
			1 51 225 682 2274 8522 31116 105316 350300 1179116
6[s]	8	171,(117,155),(115,133)	36 0 283 0 2673 0 25673 0 244794 0
			142 0 2014 0 25464 0 309326 0 3562781 0
7[r]	8	313,(275,353),(347,331)	5 34 96 253 701 2275 7302 22529 70019 217293
			18 181 678 2114 6675 24746 88050 298552 1015540 3421803
8[r]	8	657,(723,537),(731,527)	1 12 41 140 376 1111 3495 11055 34828 107798
			3 59 219 972 3184 10912 38129 134355 467158 1576979

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XIV
CONVOLUTIONAL CODES OF RATE 3/7

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\cdots}{f_1,f_2,\cdots}$
2^s	6	(5,7),(5,7),(5,7,7)	6 0 22 0 74 0 265 0 972 0 9 0 71 0 373 0 1751 0 7861 0
3^s	7	(13,11),(17,15),(17,15,13)	3 7 12 14 31 74 132 241 480 973 4 17 39 53 142 378 772 1552 3401 7572
4^s	8	(33,25),(37,33),(35,31,27)	1 9 10 16 42 70 137 286 579 1182 1 22 38 72 199 395 854 1963 4310 9502
5^s	10	(75,53),(75,57),(71,53,45)	11 0 46 0 137 0 679 0 2568 0 22 0 232 0 816 0 5085 0 22291 0
6^r	12	(135,171),(177,115),(165,155,133)	32 0 66 0 314 0 1299 0 5361 0 103 0 343 0 2084 0 10143 0 48636 0
7^r	12	(317,235),(357,253),(315,357,225)	2 20 30 31 78 195 356 697 1592 3251 4 72 136 177 452 1287 2652 5687 14124 30859
8^r	14	(757,551),(551,657),(755,457,611)	25 0 77 0 333 0 1294 0 5597 0 81 0 397 0 2257 0 10289 0 52032 0

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XV
CONVOLUTIONAL CODES OF RATE 3/8

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\cdots}{f_1,f_2,\cdots}$
2^s	6	(5,3,7),(5,7,7),(5,7)	1 4 5 8 12 22 37 59 112 191 1 6 14 24 47 106 192 349 709 1318
3^s	8	(17,15,13),(17,15,13),(13,11)	2 7 3 10 19 27 68 83 154 313 2 17 7 32 68 121 350 440 934 2029
4^s	10	(35,27,23),(37,33,25),(33,23)	8 0 19 0 57 0 196 0 628 0 15 0 71 0 255 0 1097 0 4221 0
5^s	12	(75,57,47),(73,65,57),(65,47)	16 0 37 0 84 0 33 0 1242 0 39 0 164 0 485 0 2086 0 9028 0
6^r	13	(123,137,175),(147,135,173),(151,133)	6 15 18 24 45 91 150 275 539 988 15 52 80 129 271 597 999 2013 4290 8188
7^r	14	(255,351,373),(233,375,271),(227,357)	5 13 18 25 42 76 153 291 500 943 13 37 68 128 228 460 973 1977 3792 7560
8^r	16	(647,775,423),(733,665,745),(627,705)	28 0 43 0 160 0 555 0 1892 0 95 0 230 0 1017 0 4032 0 15886 0

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XVI
CONVOLUTIONAL CODES OF RATE 4/5

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\cdots}{f_1,f_2,\cdots}$
2^y	2	5,5,7,(7,5)	1 12 54 253 1198 5648 26689 126130 595973 2816035 1 36 311 2119 12989 75386 422652 2311208 12405034 65628595
3^l	3	15,11,11,(11,17)	6 32 185 1030 5745 32204 180171 1008281 5643351 31582834 11 184 1627 12094 85568 578261 3788098 24302635 153364002 955380916
4^s	4	27,33,27,(37,05)	30 126 815 4822 29046 174460 1048673 6299761 159 990 9076 66149 482470 3378525 23254193 157267471
5^l	4	75,75,71,(67,41)	4 46 295 1832 11910 76572 496157 3203569 11 297 2876 23759 192413 1475175 11117934 81828702
6^l	5	145,113,153,(113,145)	22 146 920 5983 39409 260246 1714477 11291651 99 1184 10987 89453 708470 5487822 41494416 308387277
7^s	6	247,217,257,(233,361)	134 0 6010 0 262004 0 11603310 0 1015 0 86430 0 5363399 0 308528378 0
8^r	6	421,437,663,(711,557)	55 254 1747 10245 70326 453674 3025187 20000817 364 2355 22079 156048 1283476 9549045 72702219 539377032

[y] Code found by Yasuda *et al.* [3].
[l] Code found by Lee [5].
[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XVII
CONVOLUTIONAL CODES OF RATE 4/7

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\dots}{f_1,f_2,\dots}$
2[s]	4	7,(7,5),(7,5),(7,5)	3 9 21 48 114 289 720 1796 4447 11011
			4 19 72 224 658 1918 5451 15325 42391 115965
3[s]	5	13,(17,11),(17,13),(17,11)	5 14 36 91 234 626 1688 4543 12170 32533
			8 43 158 514 1600 4910 14990 45198 134438 395343
4[s]	6	31,(31,25),(35,25),(37,35)	8 30 52 145 427 1180 3255 8965 25077 68886
			18 129 262 948 3175 9854 31231 94729 291849 877005
5[s]	7	53,(71,53),(67,45),(65,57)	9 35 91 188 530 1670 4753 13619 38608 108320
			21 183 619 1379 4647 16894 52739 167408 524146 1597876
6[r]	8	127,(165,103),(103,175),(165,173)	18 48 97 279 776 2362 6726 18877 54114 154324
			66 259 679 2204 6961 23531 75882 233216 728538 2253646
7[r]	8	343,(271,365),(267,355),(245,317)	1 26 60 140 400 1117 3318 9578 27508 79153
			1 124 358 1020 3347 10672 35384 112671 354074 1108194
8[r]	9	533,(777,511),(707,655),(735,547)	6 34 63 165 538 1493 4247 12490 35997 103653
			18 166 417 1344 4600 14926 46737 151722 478709 1495918

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XVIII
CONVOLUTIONAL CODES OF RATE 5/6

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\dots}{f_1,f_2,\dots}$
2[y]	2	5,5,5,7,(7,5)	2 26 133 711 4066 22967 129044 726302 4089234 23018636
			2 111 982 7281 52086 355925 2351248 15195363 96588467 605898000
3[s]	3	17,13,13,15,(13,17)	15 96 601 3963 26039 170868 1121515 7360948
			61 686 6257 53004 426069 3309892 25092563 186800019
4[l]	4	25,27,31,37,(31,35)	111 0 5628 0 291695 0 15122002 0
			754 0 74393 0 5682302 0 389260277 0
5[l]	4	73,47,75,67,(73,51)	15 138 993 7841 61322 478265 3739326 29216709
			58 1090 11475 119518 1158329 10779875 97863757
6[r]	4	107,107,145,173,(177,125)	3 69 552 4134 33240 267081 2136422 17089419
			5 588 6932 68236 678845 6485600 60201390 548105220
7[r]	5	237,237,257,277,(327,275)	29 268 2027 16112 131033 1058057 8541423
			202 2860 29542 299853 2936137 27716864 256460355
8[r]	5	635,605,727,557,(461,727)	11 148 1110 8857 70776 576629 4684287
			66 1667 16384 167138 1600256 15269020 142035181

[y] Code found by Yasuda *et al.* [3].
[l] Code found by Lee [5].
[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XIX
CONVOLUTIONAL CODES OF RATE 5/7

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\dots}{f_1,f_2,\dots}$
2[s]	3	7,(7,5),7,7,(7,5)	4 25 69 285 1021 4100 15084 58335 218958 838180
			6 92 375 2172 9157 44163 186693 826495 3461555 14703642
3[s]	4	15,(15,11),13,17,(17,11)	17 49 205 773 3141 12764 50932 204294 818585 3283749
			64 237 1596 7181 35316 164877 753517 3396031 15103051 66577343
4[s]	4	33,(33,21),31,23,(33,25)	2 27 109 445 1955 8638 38596 172629 769654 3427398
			2 105 665 4014 22564 119296 623571 3206722 16172876 80381293
5[r]	5	75,(57,75),57,55,(67,51)	10 45 202 875 3890 17374 77126 343427 1529138 6806601
			28 245 1555 8512 45998 241361 1233278 6201481 30775873 151137287
6[r]	6	121,(157,115),173,167,(161,127)	48 0 978 0 18870 0 382064 0 7661173 0
			223 0 9011 0 247794 0 6517621 0 160995297 0
7[r]	6	311,(215,367),243,227,(367,211)	10 65 233 1044 4840 21813 98435 445282 2012747 9102918
			45 500 2046 11546 63532 325384 1662087 8404538 41869085 207107670
8[r]	6	645,(531,647),521,761,(657,421)	2 34 129 576 2649 11836 53986 245388 1114467 5065538
			4 228 1137 6165 33581 175490 905998 4597694 23109295 115130348

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XX
CONVOLUTIONAL CODES OF RATE 5/8

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\ldots}{f_1,f_2,\ldots}$
2	4	7,7,(7,5),(7,5),(7,5)	11 25 55 165 481 1421 4026 11519 33029 94789 22 85 292 1043 3448 11741 38104 122285 388098 1220713
3[s]	4	17,(15,13),13,(17,11),(13,11)	1 14 41 114 340 1050 3344 10491 32675 101899 1 37 176 697 2566 9302 33951 120625 420997 1453503
4[s]	6	35,(35,27),33,(37,23),(35,27)	46 0 379 0 3880 0 39408 0 400587 0 175 0 2704 0 37820 0 487364 0 6031776 0
5[r]	6	75,(47,65),77,(77,45),(77,51)	10 32 88 304 1026 3379 10996 37049 123127 406164 24 152 566 2266 9302 35361 131394 493925 1820868 6591006
6[r]	7	133,(131,163),133,(123,171),(171,103)	22 65 176 626 2061 6932 23306 78181 263870 885132 94 400 1316 5588 22001 83961 311371 1158417 4302017 15720820
7[r]	8	237,(325,267),365,(367,223),(205,323)	52 0 508 0 5689 0 63894 0 718745 0 237 0 3897 0 58595 0 839660 0 11429083 0
8[r]	8	657,(575,717),435,(663,427),(751,623)	9 49 131 428 1408 4705 16058 53837 179989 604121 35 268 953 3681 13944 53444 204115 757704 2784146 10170572

[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XXI
CONVOLUTIONAL CODES OF RATE 6/7

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\ldots}{f_1,f_2,\ldots}$
2[y]	2	5,5,5,5,7,(7,5)	4 39 237 1574 10540 69904 464351 3086199 5 186 1982 18139 154364 1248256 9773159 74779002
3[s]	2	11,15,13,11,17,(15,11)	1 23 166 1266 9926 77849 611345 4802421 1 99 1369 15594 161619 1580595 14877079 136241270
4[l]	3	31,35,35,21,21,(33,25)	6 77 696 5884 51187 447084 3899292 34001078 11 592 8326 96478 1060924 11184679 114340727 1143583363
5[l]	4	51,77,51,75,67,(43,65)	39 350 3524 28432 283504 2469003 23463299 223 3954 56337 589151 7119903 73395325 803977591
6[l]	4	113,157,107,165,145,(117,155)	7 158 1420 12795 120424 1131071 10603037 36 1426 19735 237614 2759684 30848277 335743127
7[r]	4	253,257,317,307,227,(363,347)	4 119 1067 9582 90915 860833 8125929 10 1341 17245 197995 2280858 25464500 276876174
8[r]	5	625,613,415,751,455,(533,605)	39 376 3460 32956 313727 2990946 28543781 314 4886 60717 721002 8232447 91703539 1000882165

[y] Code found by Yasuda *et al.* [3].
[l] Code found by Lee [5].
[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

TABLE XXII
CONVOLUTIONAL CODES OF RATE 7/8

ν	d_f	Generators	Spectra $\frac{t_1,t_2,\ldots}{f_1,f_2,\ldots}$
2[y]	2	5,5,5,5,5,7,(7,5)	6 66 444 3296 25164 189152 1421519 10700202 8 393 4344 43084 412051 3746196 32997600 284670109
3[s]	2	11,17,13,11,11,17,(15,11)	2 32 276 2580 23558 213496 1942689 17685630 2 160 2690 36622 439416 4947140 53708624 567996598
4[s]	3	21,27,37,23,27,21,(35,23)	12 154 1546 14978 149512 1492311 37 1480 22199 290717 3640770 43636938
5[l]	4	57,45,43,57,53,61,(51,77)	144 0 16474 0 1800805 0 1104 0 308236 0 52508002 0
6[r]	4	131,103,145,117,177,153,(151,117)	31 393 4141 43413 473725 5099137 227 5304 78831 1060216 14096751 179195313
7[r]	4	337,311,253,235,253,311,(235,311)	13 228 2369 25380 277166 3018851 94 2681 41037 573245 7712490 99796234
8[r]	4	733,613,437,535,765,771,(733,551)	4 115 1173 12262 135283 1487738 11 1531 21566 291845 3907212 50798571

[y] Code found by Yasuda *et al.* [3].
[l] Code found by Lee [5].
[s] New code found in this study by an exhaustive search.
[r] New code found in this study by a random search.

III. Properties of Punctured Convolutional Codes

We discovered some properties of punctured convolutional codes. First, it was found that the universal optimal branch partition does not exist.

As follows from the description above, a branch partition is determined by an ordered set of integers $(l_1, \cdots, l_k)$ such that

$$\sum_{i=1}^{k} l_i = n.$$

Two branch partitions are equivalent if one of them is a cyclic shift of the other.

In Table I, performance of rate $3/7$, $\nu = 2$ codes with different branch partitions are presented. Here, $t = \{t_i\}$, $i = 1, 2, \cdots$ are coefficients of series expansion for the probability of the first error event and t_i denotes the total number of all paths with weight $d_f + i - 1$. It follows from this example that code performance strictly depends on branch partition.

In Table IX we present the optimal rate $4/6$ code with $\nu = 2$, $d_f = 4$ having branch partition $(1, 2, 2, 1)$. Any other partition generates a code with $d_f = 3$. Nevertheless, the optimal branch partition for all rate $4/6$ codes of constraint length $\nu = 3, \cdots, 8$ is $(1, 2, 1, 2)$. Thus generally speaking, the universal optimal branch partition does not exist.

Secondly, we have found that rate $4/5$, $\nu = 4$ code presented in Table XVI has better spectrum than any antipodal code with the same decoding complexity although all known optimal punctured convolutional codes presented in [5] and in other papers are antipodal.

Thirdly, new results concerning cancellable rate punctured codes were obtained. In [9], some rate $k/(2k)$, $k = 2, 3, 4, 5$ convolutional codes with $\nu = 4$ which are superior (in the sense of minimizing weight spectrum coefficients) to the best rate $1/2$ codes with the same d_f and ν were found. In [10], a rate $4/8$ code with $\nu = 4$ having $d_f = 8$ is given (the best $1/2$ code with $\nu = 4$ has $d_f = 7$). In [11], a rate $2/3$ time-varying unit memory convolutional code with $d_f = 4$ superior to the best rate $2/3$ unit memory time-invariant convolutional code having $d_f = 3$ is presented.

We have performed a search of good cancellable rate $(kl)/(nl)$ convolutional codes over punctured convolutional codes without restrictions on the number of different generator polynomials. We considered rate $(kl)/(nl) = 2/4, 3/6, 4/8, 4/6, 6/8$ codes. The codes found have been included in the tables of rate k/n codes if they had better free distance d_f or better spectrum coefficients, or, more precisely, if f_1/l was less than f_1 for the corresponding rate k/n code. Results are presented in Tables II, III, IV, IX.

Note that we found arate $2/4$ code with $\nu = 7$, $d_f = 11$ conjectured by Mooser in [9].

IV. Concluding Remarks

New rational rates punctured convolutional codes are listed. It is found that the code performance strictly depends on the branch partition and that the optimal branch partition may depend on the code constraint length. We presented an example of a nonantipodal code that is better than any antipodal code with the same code rate and the same decoding complexity. We tabulated some cancellable rate convolutional codes with larger free distance than corresponding time-invariant codes with the same decoding complexity.

References

[1] J. B. Cain, G. C. Clark, Jr., and J. M. Geist, "Punctured convolutional codes of rate $(n-1)/n$ and simplified maximum likelihood decoding," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 97–100, Jan. 1979.

[2] K. J. Hole, "New short constraint length rate $(N-1)/N$ punctured convolutional codes for soft-decision Viterbi decoding," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1079–1081, Sept. 1988.

[3] Y. Yasuda, K. Kashiki, and Y. Hirata, "High-rate convolutional codes for soft decision Viterbi decoding," *IEEE Trans. Commun.*, vol. COM-32, pp. 315–319, Mar. 1984.

[4] D. Haccoun and G. Begin, "Further results on high-rate punctured convolutional codes for Viterbi and sequential decoding," *IEEE Trans. Commun.*, vol. 38, pp. 1922–1929, Nov. 1990.

[5] P. J. Lee, "Constructions of rate $(n-1)/n$ punctured convolutional codes with minimal required SNR criterion," *IEEE Trans. Commun.*, vol. 36, pp. 1171–1173, Oct. 1988.

[6] D. G. Daut, J. W. Modestino, and L. D. Wismer, "New short constraint length convolutional codes constructions for selected rational rates," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 794–800, Sept. 1982.

[7] V. B. Balakirsky, "Estimations of transfer functions of noncatastrofic convolutional encoders," *IEEE Trans. Inform. Theory*, vol. 42, pp. 1014–1021, May 1996.

[8] A. N. Trofimov and B. D. Kudryashov, "Distance spectra and upper bounds on error probability for trellis codes," *IEEE Trans. Inform. Theory*, vol. 41, pp. 561–572, Mar. 1995.

[9] M. Mooser, "Some periodic convolutional codes better than any fixed code," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 750–751, Sept. 1983.

[10] P. J. Lee, "There are many good periodically time-varying convolutional codes," *IEEE Trans. Inform. Theory*, vol. 35, pp. 460–463, Mar. 1989.

[11] R. Palazzo, "A time-varying convolutional encoder better than the best time-invariant encoder," *IEEE Trans. Inform. Theory*, vol. 39, pp. 1109–1110, May 1993.

[12] K. J. Larsen, "Short convolutional codes with maximal free distance for rates $1/2$, $1/3$ and $1/4$," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 371–373, May 1973.

[13] J. P. Odenwalder, "Optimal decoding of convolutional codes," Ph.D. dissertation, Dept. Elec. Eng., Univ. of California, Los Angeles, Jan. 1970.