Continuous symmetries and approximate quantum error correction

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Quantum error correction and symmetry arise in many areas of physics, including many-body systems, metrology in the presence of noise, fault-tolerant computation, and holographic quantum gravity. Here we study the compatibility of these two important principles. If a logical quantum system is encoded into n physical subsystems, we say that the code is covariant with respect to a symmetry group G if a G transformation on the logical system can be realized by performing transformations on the individual subsystems. For a G-covariant code with G a continuous group, we derive a lower bound on the error correction infidelity following erasure of a subsystem. This bound approaches zero when the number of subsystems n or the dimension d of each subsystem is large. We exhibit codes achieving approximately the same scaling of infidelity with n or d as the lower bound. Leveraging tools from representation theory, we prove an approximate version of the Eastin-Knill theorem: If a code admits a universal set of transversal gates and corrects erasure with fixed accuracy, then, for each logical qubit, we need a number of physical qubits per subsystem that is inversely proportional to the error parameter. We construct codes covariant with respect to the full logical unitary group, achieving good accuracy for large d (using random codes) or n (using codes based on W-states). We systematically construct codes covariant with respect to general groups, obtaining natural generalizations of qubit codes to, for instance, oscillators and rotors. In the context of the AdS/CFT correspondence, our approach provides insight into how time evolution in the bulk corresponds to time evolution on the boundary without violating the Eastin-Knill theorem, and our five-rotor code can be stacked to form a covariant holographic code.

I. INTRODUCTION

Quantum error-correcting codes protect fragile quantum states against noise [1]. If quantum information is cleverly encoded in a highly entangled state of many physical subsystems, then damage inflicted by local interactions with the environment can be reversed by a suitable recovery operation. Aside from their applications to resilient quantum computing, quantum error-correcting codes appear in a wide variety of physical settings where quantum states are delocalized over many subsystems, such as topological phases of matter [2–5] and the AdS/CFT correspondence in holographic quantum gravity [6, 7].

On the other hand, naturally occurring physical systems often respect symmetries, and phases of matter can be classified according to how these symmetries are realized in equilibrium states. Likewise, quantum error-correcting codes often have approximate or exact symmetries with important implications. In the case of a time-translation-invariant many-body system, for example, certain energy subspaces are known to form approximate quantum error-correcting codes [8, 9], which are preserved under time evolution. Limits to sensitivity in quantum metrology are related to the degree of asymmetry of probe states, a notion formalized in the resource theory of asymmetry and reference frames [10, 11]. Thus, reference frame information can be protected against noise using quantum codes with suitable symmetry properties [12]. Furthermore, recent developments in quantum gravity have shown that the AdS/CFT correspondence can be viewed as a quantum error-correcting code which is expected to be compatible with the natural physical symmetries of the system, such as time-translation invariance [7, 13–16]. Finally, the Eastin-Knill theorem [17–19], which complicates the construction of fault-tolerant schemes for quantum computation by forbidding quantum error-correcting codes from admitting a universal set of transversal gates, can be viewed as the statement that finite-dimensional quantum codes which correct erasure have no continuous symmetries [12]. Thus, there are loopholes to the Eastin-Knill theorem that are naturally exploited by holographic theories of quantum gravity. This article provides a detailed quantitative investigation of those loopholes, critically evaluating their potential for application to quantum fault-tolerance.

A continuous symmetry, as opposed to a discrete symmetry, allows for infinitesimally small transformations that are arbitrarily close to the identity operation. Such symmetry transformations are generated by conserved operators called *charges*. For instance, consider a particle in three-dimensional space that we rotate about the Z-axis by an angle θ. Acting on the Hilbert space, this symmetry transformation is represented by a unitary U_θ that is generated by the Z-component of the Hermitian angular momentum operator J_z, i.e., $U_\theta = e^{-iJ_z\theta}$. Crucially, a unitary operation U that is covariant with respect
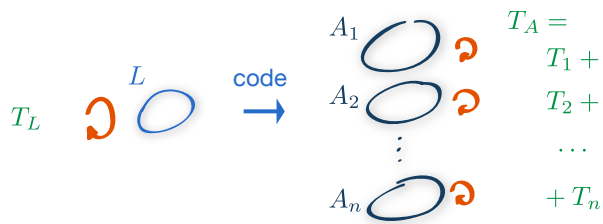
FIG. 1: Quantum information represented on an abstract
logical system L is encoded on several physical subsystems
$A_1 \ldots A_n$ using a code. Suppose that the code is compatible
with a continuous transversal symmetry, for instance, rota-
tions in 3D space. This means that by rotating all individual
physical subsystems one induces the same transformation as if
we had simply rotated the initial logical system L. We show
that such codes necessarily perform poorly as approximate
error-correcting codes against erasures. The reason is that
the code must encode an eigenstate of the logical charge T_L
that generates the symmetry as a codeword that is a global
eigenstate of the corresponding physical charge T_A. Since
the latter is a sum of local terms $T_A = \sum T_i$, and since the
environment is handed the local reduced states of the code-
word, the environment can deduce on average the total charge
of the codeword. Because logical information leaks to the
environment, the code cannot be a good error-correcting code.

to rotations about the Z axis must conserve the physical
quantity J_z. In particular, if the initial state $|\psi\rangle$ is an
eigenstate of J_z with eigenvalue m, then the transformed
state $U|\psi\rangle$ must also be an eigenstate of J_z with the same
eigenvalue (up to a constant shift in all the eigenvalues).

Here, we study the accuracy of quantum error-
correcting codes that are covariant with respect to con-
tinuous symmetries (Fig. 1). Our results build on earlier
work showing that infinite-dimensional covariant quan-
tum codes exist, while finite-dimensional covariant codes
cannot correct erasure errors perfectly [12, 20].

A finite-dimensional error-correcting code that is covari-
ant with respect to a continuous symmetry cannot correct
erasure of a subsystem exactly, because an adversary who
steals the erased subsystem could acquire some informa-
tion about the encoded state, hence driving irreversible
decoherence of the logical quantum information [12, 20].
More concretely, if Π is the projector onto the code space,
then the error-correction conditions [21, 22] state that
any operator O supported on the erased subsystem must
act trivially within the codespace, i.e., $\Pi O \Pi \propto \Pi$. If the
symmetry acts transversally, the corresponding generator
T_A is a sum of strictly local terms, $T_A = \sum T_i$, where
each T_i is supported on a single subsystem. However,
this implies that $\Pi T_A \Pi = \sum \Pi T_i \Pi \propto \Pi$, and hence it
follows from the error-correction condition that any such
T_A must act trivially on the codewords.

Crucially for the considerations in this paper, the above
argument makes two implicit assumptions: that the sum
over i is finite (bounded number of subsystems), and that
the codewords are normalizable (finite-dimensional sub-

systems). If both assumptions are relaxed, then quantum
codes covariant with respect to a continuous symmetry
are possible, as shown in [12]. Our main task in this
paper is to explore quantitatively the case where the
number of subsystems and the dimension of each subsys-
tem are finite, using the tools of approximate quantum
error correction [23–25]. That is, we will quantify the
deviation from perfect correctability in this case, for a
code covariant with respect to a continuous symmetry.
Assuming that the symmetry acts transversally and that
the noise acts by erasing one or more subsystems, we
provide upper bounds on the accuracy of the code, char-
acterized using either the average entanglement fidelity or
the worst-case entanglement fidelity of the error-corrected
state. Our proof strategy is to show that in the presence
of a continuous symmetry, the environment necessarily
learns some information about the logical charge, which
implies that the code necessarily performs imperfectly as
an error-correcting code [25–27]. In fact, some of these
assumptions may be relaxed in our main technical theo-
rem; for instance, the generating charge may be a sum
of k-local terms, instead of a sum of strictly local terms
as for a transversal symmetry action, and the code only
needs to be approximately rather than exactly covariant.

Our lower bound on infidelity vanishes in two interesting
regimes: as the dimension d of the physical subsystems
gets large, or as the number n of physical subsystems
gets large. In these limits we can find error-correcting
codes whose infidelity approximately matches the scaling
of our bound with d or n. We construct explicit examples
based on normalized versions of the rotor code presented
in Ref. [12], and note that codes considered in Ref. [8]
provide further examples. We also discuss a 5-rotor code
that can be stacked to construct a covariant holographic
code [7].

Furthermore, our results provide an approximate ver-
sion of the Eastin-Knill theorem [12, 17–19], which states
that a universal set of transversal logical gates cannot exist
for a finite-dimensional encoding that protects perfectly
against erasure. By applying our bounds and exploiting
the nonabelian nature of the full unitary group on the
logical space, we derive a lower bound on infidelity which
scales as $1/\log d$, where d is the subsystem dimension, for
a code that admits universal transversal logical gates. We
also find that if a code admits a universal set of transver-
sal logical gates, then there are strong lower bounds on
the subsystem dimension d that depend on the code's
infidelity, and which in some regimes are even exponential
in the logical system dimension d_L. Using randomized
code constructions, we prove the existence of codes which
approximately achieve this relationship between d and
d_L. In addition, we exhibit codes with universal transver-
sal logical gates which achieve arbitrarily small infidelity
when the number n of subsystems becomes large with the
logical dimension d_L fixed.

We also provide a general framework for constructing
codes that are covariant with respect to general symmetry
groups, by encoding logical information into the so-called

FIG. 2: Our paper is built around three main technical results. In the presence of a $U(1)$ symmetry, which is implied by any continuous symmetry, we prove a general bound on how well any covariant code can correct against erasures. Our second main contribution is an approximate version of the Eastin-Knill theorem: If the code admits a universal set of transversal logical gates, i.e., if it is covariant with respect to the full unitary group $U(d)$ on the logical space, then our bound can be expressed in terms of the physical subsystem dimensions d_i. Our third main contribution is a general framework for constructing codes that are covariant with respect to any general group G. Along the way, we develop a new criterion for certifying the accuracy of any code, which we use to analyze our examples.

regular representation of the groups. Using this framework we can generalize several widely-known codes (bit-flip, phase-flip, $[[4, 2, 2]]$ code, etc.) to infinite-dimensional covariant codes based on oscillators or rotors.

Finally, we discuss the interpretation of our results in the context of quantum gravity and, in particular, the AdS/CFT correspondence. Time evolution itself provides an example of a symmetry that must be reconciled with the error-correcting properties of the system.

The remainder of the manuscript is organized as follows (Fig. 2). In §II, we summarize our main results. We set up notation in §III and prove a bound on the performance of codes covariant with respect to a $U(1)$ symmetry in §IV. A criterion certifying code performance is derived in §V. In §VI, we apply our bounds and criterion to the following examples of $U(1)$-covariant encodings: an infinite-dimensional rotor extension of the qutrit $[[3, 1, 2]]$ and qubit $[[5, 1, 3]]$ codes as well as a many-body Dicke-state code. We apply our bound to codes admitting universal transversal gates in §VII, discussing a $U(d)$-invariant encoding based on W-states in §VII B. Erasure-correcting codes whose transversal gates form a general group G

are introduced in §VIII. In §IX we study applications to quantum gravity. We conclude with a discussion in §X.

II. SUMMARY OF MAIN RESULTS

A. Bound on the accuracy of codes covariant with respect to a continuous symmetry

Our first main result is a bound on the accuracy of any approximate quantum error-correcting code that is covariant with respect to a continuous symmetry. We consider an encoding map from a logical system L to a physical system A consisting of n subsystems denoted $A_1, A_2, \ldots A_n$. A one-parameter family of continuous unitary symmetries acting on L is generated by the logical charge observable T_L, which corresponds to the physical charge observable T_A acting on A. We assume that the symmetry acts transversally, so that $T_A = \sum_{i=1}^{n} T_i$, where T_i acts on subsystem T_i.

How well does this code protect the logical system against erasure of one of the subsystems? To quantify

the code's performance we may use the *worst-case entanglement fidelity*, where "worst-case" means the minimal fidelity for any entangled state shared by the logical system and a reference system. (See §III for a precise definition.) Then we consider the value f_{worst} of this worst-case entanglement fidelity which is achieved by the best possible recovery map applied after an erasure error. A measure of the residual error after recovery is

$$\epsilon_{\text{worst}} = \sqrt{1 - f_{\text{worst}}^2}. \qquad (1)$$

Our result is a lower bound on ϵ_{worst} which limits the performance of any covariant quantum code:

$$\epsilon_{\text{worst}} \geqslant \frac{\Delta T_L}{2n \max_i \Delta T_i} , \qquad (2)$$

where ΔT denotes the difference between the maximal and minimal eigenvalue of T. That is, the code's accuracy is constrained by the range of charges one wishes to be able to encode, by the size of the charge fluctuations within each subsystem, and by the number of physical subsystems.

We also find that (2) can be generalized in a number of ways. We can express the limit on code performance in terms of other measures besides worst-case entanglement fidelity, such as average entanglement fidelity, or the entanglement fidelity of a fixed input state. We can derive bounds that apply in the case where more than one subsystem is erased, or where the erasure occurs for an unknown subsystem rather than a known subsystem. We can consider cases where the charge distribution for a subsystem has infinite range, but with a normalizable tail. We can also treat the case where the covariance of the code is approximate, or where the physical charge operator is not strictly transversal.

B. Regimes where our bound is circumvented and criterion for code performance

The idea underlying (2) is that for erasure correction to work well one should not be able to learn much about the global value of the charge by performing a local measurement on a subsystem. Hence, to be able to correct the errors to good accuracy, we need either large local charge fluctuations ($\Delta T_i \to \infty$), or many subsystems ($n \to \infty$) so that the global charge is a sum of many local contributions. In fact, codes can be constructed in either limit for which ϵ_{worst} approximately matches the scaling in ΔT_i and n of the lower bound (2).

To study the case of large ΔT_i, we consider a normalized variant of the infinite-dimensional covariant code constructed in [12]. The infinite-dimensional version encodes one logical rotor (with unbounded $U(1)$ charge) in a code block of three rotors. In the modified version of this code, we either truncate the charge of the logical system to $\{-h, -h + 1, \ldots, +h\}$ or use a Gaussian envelope of

width w to normalize the physical codewords. The value of ϵ_{worst} achieved by this code, and our lower bound, both scale like h/w up to a logarithmic factor.

Regarding the limit of a large number of subsystems, we observe that a code discussed in Ref. [8] matches the $1/n$ scaling of our lower bound on ϵ_{worst}. Here the subsystems are qubits, regarded as spin-1/2 particles, and the code space is two-dimensional, spanned by two Dicke states with different values of the total angular momentum J_z along the z-axis. (A Dicke state is a symmetrized superposition of all basis states with a specified J_z). This code is covariant with respect to z-axis rotations by construction, and can be shown to achieve ϵ_{worst} scaling like $1/n$, where n is the number of physical qubits.

A further result of independent interest is a general criterion used in our analysis for certifying the performance of an error-correcting code against arbitrary noise. Stated informally, this criterion asserts that if the reduced density operator on each subsystem is approximately the same for all codewords, and if the environment does not get any information from the off-diagonal terms in the logical density operator, then the code performs well. While this criterion is sufficient to certify the performance of an approximate error-correcting code, it is not necessary—there may be codes achieving small ϵ_{worst} that do not satisfy it.

C. Approximate Eastin-Knill theorem and random $U(d)$-covariant codes

Quantum error-correcting codes are essential for realizing scalable quantum computing using realistic noisy physical gates. In a fault-tolerant quantum computation, logical quantum gates are applied to encoded quantum data, and error recovery is performed repeatedly to prevent errors due to faulty gates from accumulating and producing uncorrectable errors at the logical level. For this purpose, transversal logical gates are especially convenient. For example, if a logical gate on an n-qubit code block can be achieved by applying n single-qubit gates in parallel, then each faulty physical gate produces only a single error in the code block. Nontransversal logical gates, on other hand, either require substantially more computational overhead, or propagate errors more egregiously, allowing a single faulty gate to produce multiple errors in a code block.

A nontrivial transversal logical gate can be regarded as a covariant symmetry operation acting on the code. If all the logical gates in a complete universal gate set could be chosen to be transversal, then the Lie group of transversal logical gates would coincide with the group $U(d_L)$ of unitary gates acting on the d_L-dimensional logical system (up to an irrelevant overall phase). It then follows that *any* generator T_A of $U(d_L)$ acting on the physical system A could be expressed as a sum of terms, where each term in the sum has support on a single subsystem. Unfortunately, the Eastin-Knill theorem rules out this appealing

scenario, if erasure of each subsystem is correctable and the code is finite-dimensional. But now that we have seen that there are parameter regimes in which covariance *can* be compatible with good performance of approximate quantum error-correcting codes, one wonders whether a universal transversal logical gate set is possible after all, at the cost of a small but nonzero value of ϵ_{worst}.

We have found, however, that a fully $U(d_L)$-covariant code requires a value of ϵ_{worst} which scales quite unfavorably with the local subsystem dimension. Leveraging tools from representation theory, we show that the lower bound on ϵ_{worst} becomes

$$\epsilon_{\text{worst}} \geqslant \frac{1}{2n \max_i \ln d_i} + O\left(\frac{1}{nd_L}\right) , \quad (3)$$

where d_i is the dimension of the ith physical subsystem. We also find lower bounds for the local subsystem dimension that depend on the number of logical qubits and the code's infidelity. This result also applies to the case when each gate can be approximated with a discrete sequence of transversal operations to arbitrary accuracy, as in the context of the Solovay-Kitaev theorem.

Furthermore, using randomized constructions, we prove the existence of $U(d_L)$-covariant code families which achieve arbitrarily small infidelity in the limit of large subsystem dimension. In addition, we exhibit a simple $U(d_L)$-covariant code family, whose codewords are generalized W-states, such that ϵ_{worst} approaches zero as the number of subsystems n approaches infinity.

D. Framework for constructing covariant codes

We also develop a general framework for constructing codes that are covariant with respect to any group G admitting a Haar measure, where both the logical system and the physical subsystems transform as the regular representation of G. In this construction, the dimension of each subsystem is the order $|G|$ of the group when G is finite, and infinite when G is a Lie group.

Using this formalism we construct natural generalizations of well-known families of qubit codes, such as the bit-flip and phase-flip codes, with the qubits replaced by $|G|$-dimensional systems. These codes admit transversal logical gates representing each element of G.

When G is a Lie group, the qubits are replaced by infinite-dimensional systems such as rotors or oscillators. These infinite-dimensional codes circumvent the Eastin-Knill theorem—they are covariant with respect to a continuous symmetry group, yet erasure of a subsystem is perfectly correctable.
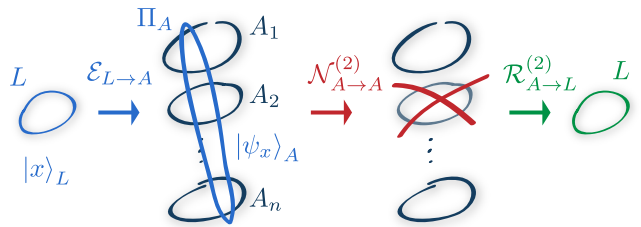

FIG. 3: A code $\mathcal{E}_{L \to A}$ maps a logical state $|x\rangle$ on an abstract logical space L to a state $|\psi_x\rangle_A$ on a physical system A. Here, we consider a physical system composed of several subsystems $A = A_1 \otimes A_2 \otimes \cdots A_n$. The code space, with associated projector Π_A, is the range of the encoding map. The environment acts by erasing a subsystem, represented as a noise channel $\mathcal{N}_{A \to A}^i$. A good error-correcting code is capable of recovering the original logical state $|x\rangle$ from the remaining subsystems, by applying a recovery map $\mathcal{R}_{A \to L}^i$. In our analysis, we assume that the environment chooses randomly which subsystem is erased. A record of which subsystem was chosen is provided, allowing to apply a different recovery map for each erasure situation. The quality of the code is characterized by how close the overall process is to the identity process on the logical system, as measured by either the average or the worst-case entanglement fidelity.

III. SETUP & NOTATION

A. Approximate error correction

Consider a code, which to each logical state $|x\rangle_L$ on some abstract logical system L associates a state $|\psi_x\rangle_{A_1 A_2 \ldots A_n}$ on a physical system A consisting of n subsystems $A = A_1 \otimes A_2 \otimes \cdots \otimes A_n$ (Fig. 3). The span of all codewords $\{|\psi_x\rangle_A\}$ forms the *code subspace*. More generally, we denote by $\mathcal{E}_{L \to A}(\cdot)$ the encoding channel which associates to any logical state the corresponding encoded physical state. In this work, the encoding is usually an isometry, *i.e.*, the encoding itself does not introduce noise into the system.

The noise channel is the process to which the physical system is exposed, which might cause the information encoded in it to get degraded. It is a quantum channel $\mathcal{N}_{A \to B}$ mapping the physical system to physical system B. (The system B might be the same as A, but it might be different; for instance, B might include a register which remembers which type of error occurred or which subsystem was lost.)

To study the approximate error correction properties of a code, we need to quantify the approximation quality using distance measures between states and channels. Proximity between quantum states can be quantified using the trace distance $\delta(\rho, \sigma) = \|\rho - \sigma\|_1/2$, or using the fidelity[1] $F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$ [1]. We need to quantify how

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[1] Throughout this paper, we stick to the convention that the fidelity

close a quantum channel $\mathcal{K}_{L \to L}$ is to the identity channel. Two standard measures to achieve this are the average entanglement fidelity F_e and the worst-case entanglement fidelity F_{worst} [28, 29], defined as

$$F_e^2(\mathcal{K}) = \langle \hat{\phi} | (\mathcal{K} \otimes \mathrm{id})(|\hat{\phi}\rangle\langle\hat{\phi}|) |\hat{\phi}\rangle ; \qquad (4)$$

$$F_{worst}^2(\mathcal{K}) = \min_{|\phi\rangle} \langle\phi| (\mathcal{K} \otimes \mathrm{id})(|\phi\rangle\langle\phi|) |\phi\rangle . \qquad (5)$$

Here the input state appearing in the definition of F_e is $|\hat{\phi}\rangle_{LR} = \sum_{k=0}^{d_L-1} |k\rangle \otimes |k\rangle / \sqrt{d_L}$, the maximally entangled state of L and a reference system R; the system R has the same dimension as L, which we denote by d_L. The optimization in the definition of F_{worst} ranges over all bipartite states of L and R. We may also use the state fidelity $F(\rho, \sigma)$ to compare two channels $\mathcal{K}$ and $\mathcal{K}'$; the entanglement fidelity between $\mathcal{K}$ and $\mathcal{K}'$ for a fixed bipartite input state $|\phi\rangle_{LR}$ is defined as

$$F_{|\phi\rangle}^2(\mathcal{K}, \mathcal{K}') = F^2 \Big((\mathcal{K} \otimes \mathrm{id})(|\phi\rangle\langle\phi|), (\mathcal{K}' \otimes \mathrm{id})(|\phi\rangle\langle\phi|) \Big) ; \qquad (6)$$

thus

$$F_e(\mathcal{K}) = F_{|\hat{\phi}\rangle}(\mathcal{K}, \mathrm{id}), \quad F_{worst}(\mathcal{K}) = \min_{|\phi\rangle} F_{|\phi\rangle}(\mathcal{K}, \mathrm{id}). \quad (7)$$

By optimizing over the input state, we may define $F_{worst}(\mathcal{K}, \mathcal{K}')$, which is closely related to the diamond distance between the channels [28, 29].

We now ask how well one can recover the logical state after the encoding and the application of the noise channel. That is, we seek a completely positive map $\mathcal{R}_{B \to L}$ (the *recovery map*), such that $\mathcal{R}_{B \to L} \circ \mathcal{N}_{A \to B} \circ \mathcal{E}_{L \to A}$ is as close as possible to the identity channel $\mathrm{id}_{L \to L}$. The resilience of a code $\mathcal{E}_{L \to A}$ to errors caused by a noise map $\mathcal{N}_{A \to B}$ is thus quantified by the proximity to the identity channel of the combined process $\mathcal{R}_{B \to L} \circ \mathcal{N}_{A \to B} \circ \mathcal{E}_{L \to A}$ for the best possible recovery map $\mathcal{R}_{B \to L}$. Using either the entanglement fidelity with fixed input $|\phi\rangle_{LR}$ or the worst-case entanglement fidelity measures, the quality of the code $\mathcal{E}_{L \to A}$ under the noise $\mathcal{N}_{A \to B}$ is quantified as

$$f_e(\mathcal{N} \circ \mathcal{E}) = \max_{\mathcal{R}_{B \to L}} F_e(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}) ; \qquad (8a)$$

$$f_{worst}(\mathcal{N} \circ \mathcal{E}) = \max_{\mathcal{R}_{B \to L}} F_{worst}(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}) . \qquad (8b)$$

We will also find it convenient to work with the alternative

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and its derived quantities refer to an amplitude rather than a probability, i.e., we use the convention of ref. [1]. In the literature, the quantity that we denote by F^2 is also referred to as "fidelity," while the quantity we represent by F is sometimes called "root fidelity."

quantities

$$\epsilon_e(\mathcal{N} \circ \mathcal{E}) = \sqrt{1 - f_e^2(\mathcal{N} \circ \mathcal{E})} ; \qquad (9a)$$

$$\epsilon_{worst}(\mathcal{N} \circ \mathcal{E}) = \sqrt{1 - f_{worst}^2(\mathcal{N} \circ \mathcal{E})} , \qquad (9b)$$

which are closely related to the infidelity and Bures distance measures. A code which performs well has $f \approx 1$ and correspondingly $\epsilon \approx 0$.

B. Erasures at known locations

In this work, we consider the noise model consisting of erasures which occur at known locations. (Our bound then naturally applies also to erasures at unknown locations, since the latter are necessarily harder to correct against.) For instance, if the ith physical subsystem is lost to the environment with probability q_i, then the corresponding noise map is

$$\mathcal{N}_{A \to AC}^{(1)}(\cdot) = \sum q_i |i\rangle\langle i|_C \otimes |\phi_i\rangle\langle\phi_i|_{A_i} \otimes \mathrm{tr}_{A_i}(\cdot) , \quad (10)$$

where we have introduced a classical register C which records which one of the n systems was lost, and where $|\phi\rangle_i$ are some fixed states.

One can also consider more general erasure scenarios, where any given combination of subsystems can be lost with a given probability. For instance, one might assume that systems A_1 and A_2 are simultaneously lost with probability $q_{\{1,2\}}$, systems A_2 and A_3 are simultaneously lost with probability $q_{\{2,3\}}$, and systems A_1 and A_3 are lost with probability $q_{\{1,3\}}$. More generally, a combination of subsystems, which we label generically by α, can be lost with probability q_α; we assume we know exactly which systems were lost. The corresponding general noise map is then

$$\mathcal{N}_{A \to AC}(\cdot) = \sum_{\alpha \in K} q_\alpha |\alpha\rangle\langle\alpha|_C \otimes \mathcal{N}_{A \to A}^\alpha(\cdot) ; \qquad (11a)$$

$$\mathcal{N}_{A \to A}^\alpha(\cdot) = |\phi_\alpha\rangle\langle\phi_\alpha|_{A_\alpha} \otimes \mathrm{tr}_{A_\alpha}(\cdot) , \qquad (11b)$$

where the register C encodes the exact locations at which simultaneous erasures have occurred, where A_α denotes the physical systems labeled by α (for instance, if $\alpha = \{2,3\}$ then $A_\alpha = A_2 \otimes A_3$), and where $\{|\phi\rangle_\alpha\}$ are fixed states. The sum ranges over a set K of possible α's corresponding to erasures which may occur. Technically, K is any set of subsets of $\{1, 2, \dots, n\}$. Situations which can be described using this setting include for instance any k consecutive erasures, or the erasure of any k subsystems.

C. Characterization via the environment

A very useful characterization of the quantities (8) is provided by Bény and Oreshkov [25], building upon the decoupling approach to error correction [26]. The

recoverability of the logical information can be characterized by studying how much information is leaked to the environment, as represented by a complementary channel $\widehat{\mathcal{N} \circ \mathcal{E}}$ of $\mathcal{N} \circ \mathcal{E}$. Recall that a *complementary channel* $\hat{\mathcal{F}}_{A \to C}$ of a quantum channel $\mathcal{F}_{A \to B}$ is a channel of the form $\hat{\mathcal{F}}_{A \to C}(\cdot) = \mathrm{tr}_B(W_{A \to BC}(\cdot)W^\dagger)$, where $W_{A \to BC}$ is a Stinespring dilation isometry for the map $\mathcal{F}$, i.e., $\mathcal{F}_{A \to B}(\cdot) = \mathrm{tr}_C(W_{A \to BC}(\cdot)W^\dagger)$. Bény and Oreshkov show that the fidelity with which one can reverse the action of the encoding and the noise is exactly the fidelity of the total complementary channel to a constant channel:

$$f_e(\mathcal{N} \circ \mathcal{E}) = \max_\zeta F_{|\hat{\phi}\rangle}(\widehat{\mathcal{N} \circ \mathcal{E}}, \mathcal{T}_\zeta) \; ; \qquad (12a)$$

$$f_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}) = \max_\zeta \min_{|\phi\rangle} F_{|\phi\rangle}(\widehat{\mathcal{N} \circ \mathcal{E}}, \mathcal{T}_\zeta) \; , \qquad (12b)$$

where $\mathcal{T}_\zeta(\cdot) = \mathrm{tr}(\cdot)\zeta$ is the constant channel outputting the state ζ and where the maximizations range over all quantum states ζ on the output system of $\widehat{\mathcal{N} \circ \mathcal{E}}$.

Now we determine a complementary channel $\widehat{\mathcal{N} \circ \mathcal{E}}$ to the encoding and noise channels. Consider first the single-erasure noise channel (10). A Stinespring dilation of $\mathcal{N}^{(1)}_{A \to AC}$ on two additional systems $C' \otimes E$ is given as $\mathcal{N}^{(1)}_{A \to AC} = \mathrm{tr}_{C'E}(W(\cdot)W^\dagger)$, with

$$W_{A \to ACC'E} =$$
$$\sum \sqrt{q_i}\, |i\rangle_C \otimes |i\rangle_{C'} \otimes |\phi_i\rangle_{A_i} \otimes \mathbb{1}_{A_i \to E} \otimes \mathbb{1}_{A \backslash A_i} \; , \quad (13)$$

where $\mathbb{1}_{A_i \to E}$ is an isometric embedding of A_i into E and $\mathbb{1}_{A \backslash A_i}$ is the identity operator on all systems A except A_i. Now consider a Stinespring dilation of $\mathcal{E}_{L \to A}$ as $\mathcal{E}_{L \to A} = \mathrm{tr}_F(V_{L \to AF}(\cdot)V^\dagger)$. Then, we may take

$$\widehat{\mathcal{N} \circ \mathcal{E}}_{L \to C'EF}(\cdot) = \mathrm{tr}_{AC}(W V (\cdot) V^\dagger W^\dagger)$$
$$= \sum q_i\, |i\rangle\langle i|_{C'} \otimes \mathrm{tr}_{A \backslash A_i}(V(\cdot)V^\dagger) \; , \quad (14)$$

where $\mathrm{tr}_{A \backslash A_i}$ denotes the partial trace over all systems except A_i (the latter is then embedded in the E system). Hence, the complementary channel to the single erasure channel simply gives the erased information to the environment with the corresponding erasure probability. It is straightforward to see that for the more general noise channel (11) a complementary channel is given by

$$\widehat{\mathcal{N} \circ \mathcal{E}}_{L \to C'EF}(\cdot) = \sum q_\alpha\, |\alpha\rangle\langle\alpha|_{C'} \otimes \mathrm{tr}_{A \backslash A_\alpha}(V(\cdot)V^\dagger) \; , \quad (15)$$

where the register C' now remembers which combination of systems were lost. This channel provides the environment with the systems that were erased, where each erasure combination α appears with probability q_α.

D. Covariant codes

The final ingredient we introduce is covariance with respect to a symmetry group (Fig. 1). Let G be any Lie group acting unitarily on the logical and physical systems, with representing unitaries $U_L(g)$ and $U_A(g)$, respectively, for any $g \in G$. A code $\mathcal{E}_{L \to A}$ is *covariant* if it commutes with the group action:

$$\mathcal{E}_{L \to A}\big(U_L(g)(\cdot)U_L^\dagger(g)\big) = U_A(g)\,\mathcal{E}_{L \to A}(\cdot)\,U_A^\dagger(g) \; . \quad (16)$$

On either logical and physical systems, we can expand the unitary action of G in terms of generators of the corresponding Lie algebra, i.e., for a given g there is a generator T_L on L and a generator T_A on A such that

$$U_L(g) = e^{-i\theta T_L} \; ; \qquad U_A(g) = e^{-i\theta T_A} \; , \quad (17)$$

for some $\theta \in \mathbb{R}$ that we can choose to normalize our generators. The generators are Hermitian matrices, and they can be interpreted as physical observables. (For instance, the generators of the rotations in 3-D space are the angular momenta.)

If the encoding map is isometric, $\mathcal{E}_{L \to A}(\cdot) = V_{L \to A}(\cdot)V^\dagger$, then any eigenstate $|t\rangle$ of T_L with eigenvalue t must necessarily be encoded into an eigenstate of T_A with the same eigenvalue t (up to a constant offset). This can be seen as follows. Expanding the condition (16) for small θ yields

$$V[T_L, (\cdot)]V^\dagger = [T_A, V(\cdot)V^\dagger] \; . \quad (18)$$

Let $\{|t, j\rangle_L\}$ be a basis of eigenstates of T_L where t is the eigenvalue and where j is a degeneracy index. Inserting in the place of $(\cdot)$ the operator $|t, j\rangle\langle t', j'|$, we obtain

$$(t - t')|\psi_{t,j}\rangle\langle\psi_{t',j'}| = [T_A, |\psi_{t,j}\rangle\langle\psi_{t',j'}|] \; , \quad (19)$$

where $|\psi_{t,j}\rangle = V|t, j\rangle$. Setting $t = t', j = j'$, we see that $|\psi_{t,j}\rangle$ is necessarily an eigenstate of T_A; let $u_{t,j}$ be its corresponding eigenvalue. Setting $t = t', j \neq j'$ in (19) implies $0 = (u_{t,j} - u_{t,j'})|\psi_{t,j}\rangle\langle\psi_{t,j'}|$ and hence $u_{t,j} = u_{t,j'} =: u_t$. Now (19) tells us for any t, t', j, j' that $t - t' = u_t - u_{t'}$. It follows that $u_t = t - \nu$ for all t, for some constant offset ν; in other words, the codewords must have the same charge as the logical state, except for a possible constant offset ν. We may condense this condition into the constraint $[T_A, VV^\dagger] = 0$ along with the identity

$$V^\dagger T_A V = T_L - \nu \mathbb{1}_L \; . \quad (20)$$

Equivalently, acting with V on (20) we have

$$T_A V = V(T_L - \nu \mathbb{1}). \quad (21)$$

This is a crucial property of covariant codes, and is a central ingredient of the proof of our main result.

Our main result, in its simplified form, further assumes

that the action of the group is transversal on the physical systems, meaning that $U_A(g) = U_1(g) \otimes U_2(g) \otimes \cdots U_n(g)$. In this case, the corresponding generator is strictly local, $T_A = T_1 + T_2 + \cdots + T_n$, where each of the T_i's act only on A_i.

As opposed to covariant isometries, covariant channels in general do not conserve charge since they may exchange charge with the environment. For instance, the fully depolarizing channel is covariant with respect to any symmetry but it changes the charge of its input. Our main result in its fully general form is formulated for approximately charge-conserving channel encodings, which is a superset of covariant isometries.

IV. INACCURACY OF COVARIANT CODES FOR A CONTINUOUS SYMMETRY

Our first main result is a general characterization of how poorly a code necessarily performs against erasures at known locations, given that the code must be covariant with respect to a continuous symmetry. For the sake of clarity, we first present a simplified version of our general bound. Consider an encoding map $|x\rangle_L \to |\psi_x\rangle_A$ with respect to some basis $\{|x\rangle_L\}$, which we may represent by an isometry $V_{L\to A} = \sum_x |\psi_x\rangle_A \langle x|_L$. Denote the corresponding encoding channel by $\mathcal{E}_{L\to A}(\cdot) = V(\cdot)V^\dagger$.

Pick any generator T_L from the Lie algebra of the symmetry acting on L. Let T_i be the corresponding generator acting on the ith physical subsystem A_i, with the total generator on A being $T_A = \sum_i T_i$. As Hermitian matrices, these are quantum mechanical observables whose eigenvalues we may think of as abstract "charges." (These charges might correspond to the component of angular momentum in a given direction, the number of particles, or some other physical quantity.) Crucially, since the code $\mathcal{E}$ is covariant, a logical charge eigenstate $|t\rangle_L$ must be encoded into a codeword $|\psi_t\rangle_A$ which is an eigenvector of T_A with the same eigenvalue t, up to a constant offset ν. Let us assume for simplicity that $\nu = 0$.

Assume the environment erases a subsystem i chosen at random with probability $q_i = 1/n$. Then the environment gets the information represented by the complementary channel (14). That is, if the original state was $|x\rangle_L$, then the environment gets the state $\rho_i^x = \mathrm{tr}_{A\setminus A_i}(|\psi_x\rangle\langle\psi_x|_L)$ on subsystem i with probability $1/n$. Yet, because the charge observable is local, the environment can learn the expectation value of the charge. Indeed, for any $|x\rangle_L$,

$$\mathrm{tr}(T_L|x\rangle\langle x|_L) = \mathrm{tr}(T_A|\psi_x\rangle\langle\psi_x|_A) = \sum_i \mathrm{tr}(T_i\rho_i^x) , \quad (22)$$

where the first equality holds because the code is covariant, and the second because the charge is local. Hence, if we define the observable $Z_{C'E} = n\sum_i |i\rangle\langle i|_{C'} \otimes T_i$ on the environment systems, we have

$$\mathrm{tr}(T_L|x\rangle\langle x|_L) = \mathrm{tr}(Z_{C'E}\,\widehat{\mathcal{N} \circ \mathcal{E}}(|x\rangle\langle x|_L)) , \quad (23)$$

making it clear that the environment can measure the average charge using the information it has available.

Surely, if the charge expectation value leaks to the environment, then the code must be bad. However, the accuracy of the code is measured in terms of an entanglement fidelity (worst-case or fixed input) to the identity channel. Hence, it still remains to relate the accuracy of the code to the environment's ability to access the codeword's total charge. On one hand, we observe that the difference in expectation value of $Z_{C'E}$ on the environment can be translated into a distinguishability of codewords in terms of the trace distance. More precisely and in general, for any two states ρ, σ, if there is an observable Q for which ρ, σ have different expectation values, then $\delta(\rho, \sigma) \geq |\mathrm{tr}(Q\rho) - \mathrm{tr}(Q\sigma)|/(2\|Q\|_\infty)$. In our case, consider two logical charge eigenstates $|\phi_\pm\rangle_L$ corresponding to the maximal and minimal eigenvalues of T_L; then it holds that

$$\delta\left(\widehat{\mathcal{N} \circ \mathcal{E}}(|\phi_-\rangle\langle\phi_-|_L), \widehat{\mathcal{N} \circ \mathcal{E}}(|\phi_+\rangle\langle\phi_+|_L)\right) \geq \frac{\Delta T_L}{2\|Z_{C'E}\|_\infty} , \quad (24)$$

where ΔT_L is the spectral range of T_L, i.e., the difference between the maximal and minimal eigenvalue of T_L. We assume here for simplicity that the maximal and minimal eigenvalues of T_i are equal in magnitude, such that $\Delta T_i = 2\|T_i\|_\infty$; hence $2\|Z_{C'E}\|_\infty = 2n\max_i\|T_i\|_\infty = n\max_i \Delta T_i$. On the other hand, if the environment's states are distinguishable for different codewords, then the accuracy of the code is bad; specifically, we show in the Appendix (Lemma 23) that for any two logical states $|x\rangle_L, |x'\rangle_L$, we have

$$\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}) \geq \frac{1}{2}\delta\left(\widehat{\mathcal{N} \circ \mathcal{E}}(|x\rangle\langle x|_L), \widehat{\mathcal{N} \circ \mathcal{E}}(|x'\rangle\langle x'|_L)\right) . \quad (25)$$

Finally, we have proven our simplified main result.

Theorem 1. *The performance of the covariant code $\mathcal{E}(\cdot) = V(\cdot)V^\dagger$ under the above assumptions, quantified by the worst-case entanglement fidelity, is bounded as follows:*

$$\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}) \geq \frac{1}{2n}\frac{\Delta T_L}{\max_i \Delta T_i} . \quad (26)$$

A similar analysis leads to a bound for the figure of merit ϵ_e based on the average entanglement fidelity,

$$\epsilon_e(\mathcal{N} \circ \mathcal{E}) \geq \frac{1}{n}\frac{\|T_L - \mathrm{tr}(T_L)\mathbb{1}_L/d_L\|_1/(2d_L)}{\max_i \Delta T_i} , \quad (27)$$

The right hand side of (27) is simply a different measure for the spread of eigenvalues; unlike ΔT_L, it takes contributions from all eigenvalues of T_L. The argument of the norm is simply the charge operator T_L with a global shift that makes the operator traceless. Equation (27) is proven as a special case of Theorem 2.

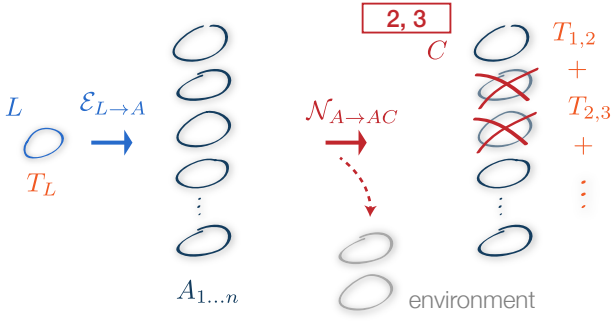In Appendix B, we provide an alternative proof for the

FIG. 4: The general setting of Theorem 2. An approximately charge-conserving encoding maps a logical state onto several physical systems. The noise acts by randomly erasing some subsystems, and storing which systems were erased in a register C. Which combinations of subsystems can be lost and with which probability can be chosen arbitrarily. The continuous symmetry is assumed to have a generator represented by T_L on the logical system and by T_A on the physical systems. We assume that T_A can be written as a sum of terms $T_A = \sum T_\alpha$, where each T_α acts on a combinations of subsystems that could possibly be lost to the environment. For instance, T_A may include a term $T_{3,4,7}$ acting on systems $A_3 A_4 A_7$ only if the noise model is such that the systems $3, 4, 7$ have a nonzero probability of being simultaneously erased.

bound (27) using a different approach: We quantify the information leaked to the environment by studying the connected correlation functions between the subsystems. In fact, we lower bound the sum of the correlation functions between the logical qubit and individual physical subsystems, and since this total correlation is non-zero, we deduce that the environment is correlated with the logical information, which translates to an upper bound on the fidelity of recovery.

In short, a covariant code with respect to a local charge may not perform well for correcting a single erasure at a known location, unless it either encodes the information into large physical systems, with a large range of possible charge values ($\max_i \Delta T_i \to \infty$), or it encodes the information into many physical systems ($n \to \infty$).

The following theorem generalizes Theorem 1 in a number of ways. It allows for the code to only approximately conserve charge, considers erasures affecting multiple systems with arbitrary erasure probabilities, and does not require the charge to be strictly local; finally, it can be applied in situations in which the codewords have most of their weight on a finite charge range (but may have distribution tails extending to arbitrarily large charge values). The setting of Theorem 2 is depicted in Fig. 4.

Theorem 2. Let L and $A = A_1 \otimes \cdots \otimes A_n$ be the logical and physical systems, respectively, and let $\mathcal{E}_{L \to A}$ be any completely positive, trace-preserving map. Consider logical and physical observables T_L and T_A. We assume that:

(a) There is a $\nu \in \mathbb{R}$ and a $\delta \geqslant 0$ such that

$\|(T_L - \nu \mathbb{1}_L) - \mathcal{E}^\dagger(T_A)\|_\infty \leqslant \delta$, i.e., the code is approximately charge-conserving up to a constant shift;

(b) We can write $T_A = \sum_\alpha T_\alpha$, where each term T_α acts on a subset of physical systems labeled by α;

(c) Fixing cut-offs $t_\alpha^\pm$ for each α, there is $\eta \geqslant 0$ such that for any state σ_L, we have

$$\left| \text{tr}\left(\sum (T_\alpha - t_\alpha) \Pi_\alpha^\perp \, \mathcal{E}(\sigma_L) \right) \right| \leqslant \eta \;, \qquad (28)$$

where $\Pi_\alpha^\perp$ projects onto the eigenspaces of T_α whose eigenvalues are outside $[t_\alpha^-, t_\alpha^+]$, and where $t_\alpha = (t_\alpha^- + t_\alpha^+)/2$. That is, when chopping off parts of the codewords exceeding charge $t_\alpha^\pm$ on term α and shifting the charge term to center it around zero, the total average charge chopped off does not exceed η;

(d) The noise acts as per (11) by erasing subsystems labeled by α with probability $q_\alpha > 0$, for each α for which there is a corresponding term T_α in the global generator T_A.

Then the accuracy of the code $\mathcal{E}_{L \to A}$ against the noise $\mathcal{N}$ is bounded as

$$\left. \begin{array}{r} \epsilon_{\text{e}}(\mathcal{N} \circ \mathcal{E}) \\ \langle \epsilon_{\text{e}}(\mathcal{N}^\alpha \circ \mathcal{E}) \rangle_\alpha \end{array} \right\} \geqslant \frac{\|T_L - \mu(T_L)\mathbb{1}_L\|_1/d_L - \delta - \eta}{\max_\alpha(\Delta T_\alpha/q_\alpha)} \;; \tag{29a}$$

$$\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}) \geqslant \frac{\Delta T_L/2 - \delta - \eta}{\max_\alpha(\Delta T_\alpha/q_\alpha)} \;, \tag{29b}$$

where $\Delta T_\alpha = t_\alpha^+ - t_\alpha^-$, where d_L is the dimension of L, where $\langle \cdot \rangle_\alpha = \sum_\alpha q_\alpha(\cdot)$, and where $\mu(T_L)$ is a median eigenvalue of T_L. We define a median eigenvalue of T_L to be a number μ such that the length-d_L vector of eigenvalues of T_L counted with multiplicity has at least $\lceil d_L/2 \rceil$ components that are less than or equal to μ, and at least $\lceil d_L/2 \rceil$ components that are greater than or equal to μ.

Additionally, the first term in the numerator of (29a) may be replaced by $\|T_L - \text{tr}(T_L)\mathbb{1}/d_L\|_1/(2d_L)$.

The bound (29a) is intuitively sensitive to the "average amount of logical charge" in absolute value, up to an arbitrary charge offset; this makes sense since the average entanglement fidelity "only samples the average case." On the other hand, the worst-case entanglement fidelity picks up the worst possible situation, noticing that there are two states with maximally different charges; the bound (29b) reflects that the code will perform the worst for those input states. The median eigenvalue in (29a) appears as an optimal solution to the optimization $\min_\mu \|T_L - \mu \mathbb{1}_L\|_1$. For an operator T that has the same number of positive eigenvalues as negative ones (with multiplicity), such as a component of spin, we can set $\mu(T) = 0$.

For isometric encodings, condition (a) really means that the encoding is approximately covariant. However

our theorem holds more generally for encodings that are not an isometry, as long as they approximately conserve charge. The latter condition is stricter than being covariant. However, an approximately covariant channel encoding that does not approximately preserve charge can still fit in the context of Theorem 2, by explicitly considering instead its covariant Stinespring dilation [30–33] into an ancilla system which is then erased by the environment with certainty as part of the noise channel.

The proof of Theorem 2 is provided in Appendix A. The proof is split into two parts. A first part shows that there exists an observable accessible to the environment which is able to infer the global logical charge to a good approximation. The second part deduces from the existence of such an observable that the code must necessarily have limited performance, as quantified by various entanglement fidelity measures.

V. CRITERION FOR CERTIFYING CODE PERFORMANCE

Here we introduce a criterion that allows us to certify a given encoding as performing accurately as an approximate error-correcting code against any given noise channel, as measured by the worst-case entanglement fidelity. Proving that a code has a good average-case entanglement fidelity (i.e., showing that $\epsilon_e(\mathcal{N} \circ \mathcal{E})$ is small) is perhaps comparatively easier, as one can attempt to guess a suitable recovery map for a maximally entangled input state and directly compute the fidelity of recovery. The method we present provides an upper bound to the stricter measure $\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E})$ and does not require us to come up with explicit recovery procedures.

Intuitively, if we consider erasures at known locations, we can expect that if all local reduced states of codewords look alike independently of the logical information, then the code performs well. That is, if for each individual subsystem each codeword has the same reduced state, then because the environment gets access only to those individual reduced states, it obtains no information about the codeword and the erasure is thus correctable. This intuition is correct in the exact case, but in the approximate case the fact that the entanglement fidelity is defined with a "stabilization" over a reference system poses an additional challenge [34]. Our solution is to consider how logical operators of the form $|x\rangle\langle x'|$ are encoded, where $\{|x\rangle\}$ is any fixed basis of the logical system. In the case of a single erasure at a known location, we define

$$\rho_i^{x,x'} = \text{tr}_{A \setminus A_i}(\mathcal{E}(|x\rangle\langle x'|)) , \qquad (30)$$

noting that $\rho_i^{x,x'}$ is a quantum state if $x = x'$ but is not even necessarily Hermitian for $x \neq x'$. Our criterion then states the following: If the states $\rho_i^{x,x}$ are approximately independent of x, and if each $\rho_i^{x,x'}$ for $x \neq x'$ has a very small norm, then the code is a good approximate error-correcting code against erasure of subsystem i.

Theorem 3. *For any encoding channel $\mathcal{E}$ and for any noise channel $\mathcal{N}$, let $\widehat{\mathcal{N} \circ \mathcal{E}}$ be a complementary channel of $\mathcal{N} \circ \mathcal{E}$. Fixing a basis of logical states $\{|x\rangle\}$, we define*

$$\rho^{x,x'} = \widehat{\mathcal{N} \circ \mathcal{E}}(|x\rangle\langle x'|) . \qquad (31)$$

Assume that there exists a state ζ, as well as constants $\epsilon, \nu \geqslant 0$ such that

$$F(\rho^{x,x}, \zeta) \geqslant \sqrt{1 - \epsilon^2} \qquad (32a)$$

$$\|\rho^{x,x'}\|_1 \leqslant \nu \quad \text{for } x \neq x'. \qquad (32b)$$

Then, the code $\mathcal{E}$ is an approximate error-correcting code with an approximation parameter satisfying

$$\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}) \leqslant \epsilon + d_L \sqrt{\nu} , \qquad (33)$$

where d_L is the logical system dimension.

If one of several noise channels is applied at random but it is known which one occurred, then (33) holds for the overall noise channel if the assumptions above are satisfied for each individual noise channel.

Note that the criterion holds for any arbitrary noise channel, not only for erasures at known locations. The proof of Theorem 3 is given in Appendix C.

Our criterion can be seen as an expression of the approximate Knill-Laflamme conditions [25] in a particular basis, but where we provide simple and practical conditions on how to bound the error parameter ϵ_{worst} of the code.

Our criterion is a sufficient condition for a code to be approximately error-correcting, but the condition is not necessary. When the criterion does not apply we cannot draw any conclusion about the code's performance.

We note that our criterion does not make reference to individual Kraus operators of the noise channel, as the Knill-Laflamme conditions or their approximate verisons do [21, 25]. This property eases its application to large-dimensional physical quantum systems.

VI. EXAMPLES OF COVARIANT CODES

Here we study three classes of covariant codes that illustrate the behavior of our bound in either regimes of large subsystem dimensions, or large number of physical subsystems (Table I).

A. Three-rotor secret-sharing code

In this subsection, we apply our criterion to a truncated version of a code introduced by Hayden *et al.* [12], linking that code to the well-known three-qutrit secret-sharing quantum polynomial code [35, 36]. While illustrating how to use our criterion, it also provides a covariant code

	Covariance	Dimen.	Error correction
$[[3,1,2]]_{\mathbb{Z}}$			
sharp cutoff	$U(1)$	Finite	Approximate
smooth cutoff	$U(1)$	Infinite	Approximate
$[[5,1,3]]_{\mathbb{Z}}$			
qudit version	$\mathbb{Z}_D$	Finite	Exact
smooth cutoff	$U(1)$	Infinite	Approximate
$[[n, a\log n, b\log n]]$			
finite n	$U(1)$	Finite	Approximate

TABLE I: Summary of the codes considered in §VI: the three-rotor secret-sharing code, the five-rotor perfect code, and an n-qubit "thermodynamic code" with codewords consisting of Dicke states (and a, b chosen appropriately).

which performs well in the limit of codewords covering a large range of physical charge on the subsystems.

1. Rotor version of the qutrit secret-sharing code

For our purposes, a quantum rotor (also, an $O(2)$ or planar quantum rotor) is simply a system with a basis $\{|x\rangle\}$ that is labeled by an integer $x \in \mathbb{Z}$ indexing representations of $U(1)$ [37]. Consider the three-rotor code given in Ref. [12] defined by the isometry from L to $A = A_1 \otimes A_2 \otimes A_3$ given as

$$V_{L\to A}: \quad |x\rangle_L \;\to\; \sum_{y\in\mathbb{Z}} |-3y, y-x, 2(y+x)\rangle_A \;, \quad (34)$$

where the states $\{|x\rangle\}$ are eigenstates of the angular momentum operators T_L and $T_A = T_1 + T_2 + T_3$. This code can correct against the loss of any of the three subsystems [12]. Moreover, the code is covariant with respect to the charge T: A logical state $|x\rangle_L$ is mapped onto a codeword with the same total charge x.

Interestingly, this code is a natural rotor generalization of the three-qutrit secret-sharing code [35, 36]. The three-qutrit code maps the basis vectors $|j\rangle_L$ ($j = 0, 1, 2$) of a logical qutrit into the codewords $\sum_k |k, k - j, k + j\rangle$ where the addition is modulo 3. Now, substitute each qutrit subsystem with a rotor. We obtain a code defined by the following encoding map:

$$\tilde{V}_{L\to A}: \quad |x\rangle_L \;\to\; \sum_{y\in\mathbb{Z}} |y, y-x, y+x\rangle \;. \quad (35)$$

This code is not yet covariant with respect to the charge states $|x\rangle$, as the charge of the codeword corresponding to $|x\rangle_L$ is not x. However, we may apply the isometry mapping $|(.)\rangle \to |-3(.)\rangle$ on the first rotor and $|(.)\rangle \to |2(.)\rangle$ on the second, yielding the encoding map (34). (In fact, the code (35) is covariant with respect to a different physical charge generator, $T'_A = -3T_1 + T_2 + 2T_3$, whereas

the code (34) is covariant with respect to the natural physical charge carried by three rotors, $T_A = T_1 + T_2 + T_3$.) In this sense, the code (34) is a natural $U(1)$-covariant generalization of the qutrit secret-sharing code.

In the following sections, we address the problem that the codewords in (34) are not normalizable, by building suitable wave packet states. We normalize the codewords in two different ways: the sharp cutoff selects a range of charges to use for each rotor and discards the rest, while the smooth cutoff imposes a Gaussian envelope on each rotor, thereby keeping all the states but making them less prominent as the charge increases [38] (see also related recent work [39]). Our noise model is one single erasure at a known location with probabilities $q_1, q_2, q_3 = 1/3$, as given by (10).

2. Sharp cutoff

Let us now truncate the logical system L to a dimension of $2h + 1$ for some fixed h, so the charge with respect to which the system is $U(1)$-covariant becomes $T_L = \sum_{x=-h}^{h} x|x\rangle\langle x|_L$. The physical subsystems are truncated in turn to $2m + 1$ dimensions, so there are in total two parameters $\{h, m\}$ that determine the ranges of the logical and physical charges. Normalizing the codewords, the isometry becomes

$$V_{L\to A}^{(m)}: \quad |x\rangle_L \;\to\; \frac{1}{\sqrt{2m+1}} \sum_{y=-m}^{+m} |-3y, y-x, 2(x+y)\rangle \;, \quad (36)$$

for $x \in -h, \cdots, h$.

Since the code is covariant and finite-dimensional, it does not allow for perfect error-correction. In Appendix D, we show that the code has an accuracy parameter which satisfies

$$\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}^{(m)}) \lesssim \sqrt{2}\sqrt{\frac{h}{m}} \;. \quad (37)$$

By comparison, our bound (26) in this case reads

$$\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}^{(m)}) \geqslant \frac{1}{2} \frac{\Delta T_L}{\max_i q_i^{-1} \Delta T_i} \approx \frac{1}{18}\frac{h}{m} \;. \quad (38)$$

There is a difference of a square root between the scaling of our actual code performance and of our bound. This is due to switching between the trace distance and a fidelity-based distance in both of our bounds, and in the way we have applied our criterion to derive (37).

3. Smooth cutoff

We now consider a different approach to normalizing the codewords: by using a Gaussian envelope we can

achieve a "smoother" cut-off in contrast to the sharp cut-off considered above (such an envelope is known to be optimal for finite-sized quantum clocks [40]). We impose an envelope controlled by a parameter $w > 0$ on the code states to make them normalizable. The encoding isometry $V_{L \to A}^{(w)}$ now acts as

$$|x\rangle_L \to \frac{1}{\sqrt{c_w}} \sum_{y=-\infty}^{\infty} e^{-\frac{y^2}{4w^2}} |-3y, y-x, 2(x+y)\rangle, \quad (39)$$

with a normalization factor $c_w = \sum_{y=-\infty}^{\infty} e^{-y^2/(2w^2)}$. Note that the envelope does not disturb the symmetry—the code remains covariant since all of the basis states used to write each logical state still have the same charge. We still consider a $(2h + 1)$-dimensional logical system L in order to see how the bound scales in terms of h/w. Deferring calculations to Appendix D, the present code has an accuracy parameter satisfying

$$\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}^{(w)}) \leqslant \sqrt{1 - e^{-\frac{h^2}{4w^2}}} \approx \frac{h}{2w}. \quad (40)$$

Our bound (26) in this case reads

$$\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}^{(w)}) \gtrsim \frac{h/w}{12\sqrt{2\ln(w/h)}}, \quad (41)$$

where we have kept only the first order in h/w, and where the logarithmic term results from cutting off the infinite tails of our codewords. Hence, we see that the present code achieves approximately the scaling of our bound, as both expressions scale as h/w up to a logarithmic factor.

We may ask for the reason of the discrepancy in the accuracy between the sharp and the smooth cut-off versions of our code. For the sharp cut-off, the error parameter scales as $\epsilon_{\mathrm{worst}} \sim \sqrt{h/w}$, while for the smooth cut-off it scales approximately as $\epsilon_{\mathrm{worst}} \sim h/w$. This can be explained from the following property of the infidelity. Loosely speaking, the error parameter $\epsilon_{\mathrm{worst}}$ is related to how much the local reduced state on a single system varies as a function of the logical state, as measured in terms of the infidelity [this can be seen from (12)]. While in both normalized versions of the above code, using either the sharp or the smooth cut-off, we are careful to ensure that all codewords are close to each other, it turns out that codewords with a sharp cut-off are in a regime where the infidelity is more sensitive to differences than the smooth cut-off. This is because those codewords have incompatible supports. More precisely, for any state ρ, the infidelity $\sqrt{1 - F^2(\rho, \rho + \varepsilon X)}$, for a small perturbation $\rho \to \rho + \varepsilon X$, can grow like the square root of ε if $\rho + \varepsilon X$ has overlap outside of the support of ρ, while it grows linearly in ε in well-behaved cases. The sharp cut-off belongs to the former regime, while in the case of the smooth cut-off the infidelity is better behaved.

B. Five-rotor perfect code

Here, we provide a rotor extension of the five-qubit perfect code [22, 41] that can be tiled to construct holographic codes [7]. While qudit [42] and oscillator [43] extensions have been considered, a rotor extension is not as straightforward because one has to take care of preserving the phases in the code states as needed to error-correct erasures. Our rotor code is the limit of a sequence of qudit codewords whose constituent phases approach multiples of an *irrational* number. This same trick has been used to obtain an irrational magnetic flux via a sequence of rational fluxes in the two-dimensional electron gas problem [44] as well as rotor limits of other Hamiltonians [37]. This limit is meant to be an idealization since there is not enough storage space to measure an irrational number to infinite precision.

Let the dimension D of each of the five physical subsystems be finite for the qudit $[[5, 1, 3]]_{\mathbb{Z}_D}$ code and infinite for the rotor $[[5, 1, 3]]_{\mathbb{Z}}$ code. The general form of the unnormalized encoding for both codes is

$$|x\rangle \to \sum_{j,k,l,m,n \in \mathbb{Z}_D} T_{jklmnx}^{(D)} |j, k, l, m, n\rangle. \quad (42)$$

We introduce the rotor code as a limiting case of the qudit code, obtaining a concise expression for the qudit perfect tensor $T^{(D)}$ in the process.

1. Qudit version

Consider first the known finite-D case, for which[2]

$$T_{jklmnx}^{(D)} = \delta_{x,j+k+l+m+n}^{(D)} \omega^{jk+kl+lm+mn+nj}, \quad (43)$$

where $\delta_{a,b}^{(D)} = 1$ if $a = b$ modulo D and ω is a primitive D-th root of unity. Notice how the above expression makes the cyclic permutation symmetry naturally manifest. The delta function encodes the state label x into the sum of the physical qudit variables, with the key difference from the sharply-cutoff $[[3, 1, 2]]_{\mathbb{Z}}$ code being that the sum is modulo D. This property makes this code exactly error-correcting and *not* covariant with respect to a $U(1)$ symmetry. Instead, this code is covariant with respect to a $\mathbb{Z}_D$ symmetry generated by $Z^{\otimes 5}$, where $Z = \sum_{k \in \mathbb{Z}_D} \omega^k |k\rangle\langle k|$ is the qudit Pauli matrix.

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[2] This formula was obtained by constructing the codespace projection out of powers of products of the code stabilizers, applying it to canonical basis states $|x, 0, 0, 0, 0\rangle$, and calculating the overlap of the resulting codeword with basis states $|j, k, l, m, n\rangle$.

2. Smooth cutoff

To take the qudit-to-rotor limit, pick $\omega = \exp(2\pi i L/D)$ with incommensurate integers $L, D \to \infty$ such that L/D approaches a positive irrational number Φ. The indices in Eq. (42) now range over $\mathbb{Z}$,

$$T_{jklmnx}^{(\infty)} = \delta_{x,j+k+l+m+n} e^{2\pi i \Phi(jk+kl+lm+mn+nj)}, \quad (44)$$

and δ is the usual Kronecker delta function. The final ingredient is to normalize the states, which can be done via a sharp or a smooth cutoff as in the $[[3,1,2]]_{\mathbb{Z}}$ code. We perform the latter using a cyclically-symmetric Gaussian envelope with spread w, prepending $\exp[-\frac{1}{4w^2}(j^2+k^2+l^2+m^2+n^2)]$ to the tensor $T_{jklmn}^{(\infty)}$ in Eq. (42) and then normalizing the codewords. The resulting code is covariant with respect to a $U(1)$ symmetry generated by the total physical charge $T_A = \sum_{i=1}^{5} T_{A_i}$, analogous to the three-rotor code (39). With the addition of the envelope, the resulting tensor becomes approximately perfect. This rotor version can be stacked to form a approximately error-correcting $U(1)$-covariant holographic code in the same way as the qubit perfect tensors were connected in Ref. [7].

One can apply the certification criteria to this code to yield the same scaling as for the three-rotor code (40) for the model of a single erasure (see Appendix D for details),

$$\epsilon_{\text{worst}}(\mathcal{N}_{1 \text{ erasure}} \circ \mathcal{E}^{(w)}) \lesssim \frac{1}{\sqrt{160}} \frac{h}{w}. \quad (45)$$

However, this code is capable of correcting any single-subsystem error, so it can correct for known erasure of any two subsystems. Calculating the bound for the noise channel $\mathcal{N}$ consisting of erasure of any two sites with equal probability yields the same scaling,

$$\epsilon_{\text{worst}}(\mathcal{N}_{2 \text{ erasures}} \circ \mathcal{E}^{(w)}) \lesssim \frac{1}{\sqrt{60}} \frac{h}{w}. \quad (46)$$

The larger coefficient is sensible since a code approximately correcting at most two erasures should be better at correcting only one. In both cases, there are additional corrections of order $O(he^{-cw^2})$ for $c > 0$ arising from a detailed application of our criterion.

C. Thermodynamic codes for $n \to \infty$

We now investigate a class of covariant codes in the limit where the number of subsystems n grows large. We exploit the codes developed in Ref. [8], relevant for quantum computing with atomic ensembles [45].

For these codes the basis vectors for the code space can be chosen to be energy eigenstates of a many-body system, with the property that the reduced state on a subsystem appears to be thermal with a nonzero temperature; we therefore call them *thermodynamic codes*.

This thermal behavior of local subsystems is expected for closed quantum systems that satisfy the eigenstate thermalization hypothesis [46] or dynamical typicality [47, 48]. Energy eigenstates with slightly different values of the total energy also have slightly different values of the locally measurable temperature; thus the identity of a codeword is imperfectly hidden from a local observer, and therefore erasure of a subsystem is imperfectly correctable.

Consider a many-body system, such as a one-dimensional spin chain, and pick out two global energy levels $|E\rangle_A, |E'\rangle_A$ in the middle of the spectrum, with a given energy difference $\Delta E = E' - E$. Assume, in the spirit of the eigenstate thermalization hypothesis, that the reduced states of both $|E\rangle_A$ and $|E'\rangle_A$ on each individual system A_i are approximately thermal. The corresponding temperature scales as $T \propto E/n$ since the temperature is an intensive thermodynamic variable. Then, the temperature difference vanishes for $n \to \infty$, and the resulting reduced thermal states for these two states are very close. Intuitively, this means that if a system A_i is provided to the environment, the latter cannot tell whether the global state is $|E\rangle$ or $|E'\rangle$, and hence the two energy levels form a two-dimensional code space that is approximately error-correcting against erasures at known locations.

For example, consider the code developed in [8, Appendix D], in the context of a 1D translation-invariant Heisenberg spin chain. Here we consider as relevant charge the total magnetization $M = \sum \sigma_Z^i$ of the spin chain. The codewords $|h_m^n\rangle$ in [8, Appendix D] are Dicke states with respect to total magnetization—i.e., they are a superposition of canonical n-spin basis states that all have some fixed magnetization m:

$$|h_m^n\rangle = \binom{n}{n/2 + m/2}^{-1/2} \sum_{\boldsymbol{s}:\, \sum s_j = m} |\boldsymbol{s}\rangle_n. \quad (47)$$

The code is covariant with respect to total magnetization by construction, by defining the magnetization charge operator in the abstract logical system to correspond to the magnetization of the corresponding codeword. The values m are spaced out by steps of $2d+1$, thus ensuring that any errors which change the magnetization by at most $2d$ cannot cause logical bit flips. This trick—using a sufficiently large spacing between codewords so that they are not mapped into each other by errors—has analogues in CSS codes, related multi-qubit codes [49], and bosonic error-correction [50]. However, to show that such errors are indeed correctable, one still has to make sure that expectation values of errors with each codeword do not depend on the codeword in the large-n limit.

In Appendix D, we show that this code's approximation parameter as an approximate error-correcting code against the erasure of a constant number of sites scales as

$$\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}) = O(1/n). \quad (48)$$

On the other hand, our bound (26) also displays the same

scaling,

$$\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}) = \Omega(1/n) . \qquad (49)$$

In consequence, this code has an approximation parameter that displays the same scaling as our bound, meaning that our bound is approximately tight in the regime $n \to \infty$.

VII. APPROXIMATE EASTIN-KNILL THEOREM

Our second main technical contribution is an approximate version of the Eastin-Knill theorem. The Eastin-Knill theorem states that it is not possible for an error-correcting code to admit a universal set of transversal logical gates, imposing severe restrictions on fault-tolerant quantum computation [17]. In fact, an approximate version of the Eastin-Knill theorem naturally follows from our bounds in Theorem 1. This is intuitively seen in the setup of our main theorem depicted in Fig. 1, by choosing the transformation group to be the full unitary group $U(d_L)$ on the logical system: To any logical unitary we require that there correspond an transversal unitary on the physical system that achieves the same logical transformation. Hence, our bound provides a limitation to the accuracy of codes that admit a universal set of transversal logical gates. The goal of this section is to specialize our main bound (26) to this situation, in order to obtain a limit expressed in terms of the dimensions of the local physical subsystems.

There is a subtlety worth noting in the argument above. In the setting of the Eastin-Knill theorem, it is not necessarily required that the mapping of logical to physical unitaries forms an actual representation, i.e., that it is compatible with the group structure. However, it turns out that we may assume this without loss of generality. Intuitively, as long as one can generate logical unitaries that are close to the identity with a transversal physical unitary, one can show that there are corresponding physical generators which span a *bona fide* representation (Appendix E). That is, if a code admits a universal transversal gate set, then it is necessarily covariant with respect to the full logical unitary group for some transversal representation on the physical systems.

The bounds derived in Theorems 1 and 2 cannot in general be directly related to the dimension of the local physical subsystems. Indeed, there is no restriction on how large ΔT_i can be. The only restriction that enters the statement of our main theorem is that a logical charge eigenstate must be mapped onto a global physical eigenstate of the same charge (up to a constant offset); the logical charge operator and the local physical charge operators may otherwise be chosen arbitrarily. For example, the repetition code spanned by $\{|000\rangle, |111\rangle\}$ with logical charge $\delta\sigma_z$, physical charge $M\sigma_z^{(1)} - M\sigma_z^{(2)} + \delta\sigma_z^{(3)}$, and $M \gg \delta$ can have a very large range M of charges on each local physical subsystem despite the systems having only two levels. In the other extreme, a completely degenerate local physical system will have zero charge range despite a possibly huge dimension.

The above observation is an expression of the fact that the covariance is with respect to an abelian symmetry group ($U(1)$). In contrast, for non-abelian Lie groups, one may no longer choose the generators arbitrarily because they have to obey nontrivial commutation relations with each other. Consider for instance a code that is covariant with respect to spin, where the group is $SU(2)$. The three generators of the corresponding Lie algebra, J_x, J_y, and J_z, satisfy the commutation relations $[J_x, J_y] = iJ_z$ along with the corresponding cyclic permutations of x, y, z. We know in the case of $SU(2)$ that the irreducible representations are labeled by a spin quantum number j that is a positive integer or half-integer, that the generator J_z in this representation has nondegenerate eigenvalues $m = -j, -j+1, \ldots, +j$, and hence that the dimension of the irreducible representation labeled by j is $2j+1$. By rotational symmetry, the same holds for any other standard generator in that irreducible representation by choosing an appropriate basis. In other words, if the dimension of a physical subsystem is small, we cannot "fit" a generator on that system with a large range of angular momentum values. More precisely, if T_i^z is the spin generator corresponding to J_z on the ith physical subsystem, the largest irreducible representation that can appear in the action of T_i^z must fit in the physical subsystem, that is, we may not have any j larger than $(d_i - 1)/2$, where d_i is the dimension of the ith physical subsystem, or else the representation is too big. In turn, this bounds the range of J_z charge values as $\Delta T_i^z \leqslant d_i - 1$. Hence, if we encode a qubit using a code that admits a universal set of transversal logical unitaries, we may apply our bound (26), choosing $T_L = J_z = \text{diag}(1/2, -1/2)$ on the logical level with $\Delta T_L = 1$, with the corresponding $\Delta T_i = \Delta T_i^z \leqslant d_i - 1$; we then obtain

$$\epsilon_{\text{worst}}(SU(2)\text{-covariant code}) \geqslant \frac{1}{2n \max_i(d_i - 1)} . \quad (50)$$

Thus, remarkably, the non-abelian nature of the group $SU(2)$ allows us to bound the expression in (26) directly in terms of the dimensions of the local physical subsystems. This is because physically, the generators $J_{x,y,z}$ of $SU(2)$ correspond to rotations around different axes, and the Lie algebra commutation relations require all of them to be of a similar scale. No such requirement was present for $U(1)$ since we were free to rotate around a chosen axis arbitrarily quickly.

In the case of a code that is covariant with respect to $SU(d)$ with $d > 2$, the dependence on the physical subsystem dimensions becomes considerably more restrictive. We provide an overview of our argument, leaving technical details to Appendix E. Irreducible representations, or *irreps*, of $SU(d)$ are indexed by $d-1$ nonnegative integers $(\lambda_1, \lambda_2, \cdots, \lambda_{d-1}) \equiv \lambda$ arranged in decreasing order. These integers determine the largest eigenvalues

of the now $d-1$ commuting generators of $SU(d)$. For $SU(2)$, only one generator J_z is diagonal in the canonical basis and the integer $\lambda = \lambda_1 = 2j$ determines the highest spin attainable in that irrep. For the fundamental representation $\lambda = (1,0)$ of $SU(3)$, the two simultaneously diagonalizable generators are the two Gell-Mann matrices that are diagonal in the canonical basis. Since the entries in λ are decreasing, the largest eigenvalue that any generator could have is λ_1, i.e., $\left\| T_\lambda^{(i)} \right\|_\infty \leqslant \lambda_1$. It turns out that the irrep that minimizes the dimension out of all irreps with fixed λ_1 is the completely symmetric irrep $(\lambda_1, 0, 0, \cdots, 0)$. The dimension of this irrep is the dimension of the symmetric subspace on λ_1 number of d-dimensional systems, which is a polynomial of degree $d-1$ in λ_1. Therefore, in order to fit in a system of dimension d_i, the largest possible λ_1 is of order $O\big(d_i^{1/(d-1)}\big)$. Now, any general representation can be decomposed into irreps, and a generator T is simply $T = \bigoplus T_\lambda$, where T_λ is the corresponding generator for each irrep. We then have $\|T\|_\infty = \max_\lambda \|T_\lambda\|_\infty$. So, if a representation fits in the system dimension d_i, then it cannot contain any irrep λ with λ_1 larger than $O\big(d_i^{1/(d-1)}\big)$. For a code that is covariant with respect to the full unitary group on the logical space, we have $d = d_L$, and picking a simple standard generator for our earlier bound (26), we obtain the following theorem.

Theorem 4 (Approximate Eastin-Knill theorem). *The performance of an $SU(d_L)$-covariant code, quantified by the worst-case entanglement fidelity, is bounded as follows:*

$$\epsilon_{\text{worst}}(SU(d_L)\text{-covariant code})$$
$$\geqslant \frac{1}{2n} \frac{1}{\max_i \ln d_i} + O\left(\frac{1}{n\, d_L}\right) . \quad (51)$$

The following bounds also hold:

$$\max_i \ln d_i \geqslant \frac{\ln(d_L - 1)}{2n\epsilon_{\text{worst}}} - \frac{\ln\big(1 + (2n\epsilon_{\text{worst}})^{-1}\big)}{2n\epsilon_{\text{worst}}} ; \quad (52a)$$

$$\max_i \ln d_i \geqslant (d_L - 1) \ln\left(\frac{1}{2\epsilon_{\text{worst}} n d_L}\right) . \quad (52b)$$

Similar bounds can be obtained for the figure of merit ϵ_{e}, based on the average entanglement fidelity, by making in (51) and (52b) the replacement $\epsilon_{\text{worst}} \to d_L \epsilon_{\text{e}}/2$.

(Proof on page 57.)

In other words, any code that (a) stores a large amount of quantum information, and (b) admits universal transversal gates, has severe restrictions on its ability to recover from erasure errors.

The bound (51) is useful to determine the precision limit of a code that has a universal set of transversal gates. If we imagine that each physical subsystem is composed of m_i qubits lumped together, then the error parameter of the code scales at least inversely in the largest number of qubits m_i that were lumped together. If we set for instance $d_L \sim 10^3$ (10 logical qubits) that are encoded

into n systems consisting of 10 qubits each, i.e., $d_i \sim 10^3$, we obtain the rather prohibitive error parameter $\epsilon_{\text{worst}} \gtrsim 0.14/n$. (This estimate can be improved to $\epsilon_{\text{worst}} \geqslant 0.5/n$ using a tighter bound given in Appendix E.)

The bound (52a) shows that if ϵ_{worst} is kept constant and for $d_L \to \infty$, we must have that d_i grows polynomially in d_L, where the exponent is $1/(2n\epsilon_{\text{worst}})$. If, for instance, we wish to achieve a precision of $\epsilon_{\text{worst}} \sim 10^{-3}$, then we must have the scaling $d_i \sim (d_L)^{500/n}$. Concretely, for $d_L \sim 10^3$ (10 logical qubits) encoded into $n = 10$ subsystems, the physical subsystems need to be of a respectable dimension $d_i \sim 10^{65} \sim 2^{216}$, i.e., the physical subsystem must comprise 216 qubits lumped together.

Our third bound is interesting in the regime of extremely high accuracy. Suppose we wish to accurately resolve individual logical basis states of a highly mixed logical state. The logical information might, for instance, be entangled with a large reference system. In such a situation, we require $\epsilon_{\text{worst}} \lesssim d_L^{-1}$. Bound (52b) then asserts that the physical subsystem dimension must grow *exponentially* in the logical system dimension.

Finally, we note that Eqs. (51), (52a) and (52b) are obtained from a more general, tighter bound on $\max d_i$ which is expressed as a binomial coefficient (see Appendix E for details). In some cases, this bound allows to obtain tighter estimates on the physical dimension of the subsystems.

A. Random Constructions

The bounds of Theorem 4 severely limit the error correction capability of the unitary $SU(d_L)$-covariant codes. We now show that it is possible to find good $SU(d_L)$-covariant codes in regimes of large physical systems that are not excluded by Theorem 4.

The constructions we present are randomized as well as asymptotic in the dimension of the physical subsystems. More precisely, we consider the encoding of one d_L-dimensional Hilbert space $\mathscr{H}_L$ in a physical space which is a tensor product of three Hilbert spaces $\mathscr{H}_A = \mathscr{H}_{A_1} \otimes \mathscr{H}_{A_2} \otimes \mathscr{H}_{A_3}$. The encoding is done via an isometry $V_{L \to A}$, which is $U(d_L)$ covariant: For all $U \in U(d_L)$,

$$V U = r_1(U) \otimes r_2(U) \otimes r_3(U)\, V . \quad (53)$$

Here $r_1, r_2,$ and r_3 are three irreps of $U(d_L)$. Our constructions are randomized in the following way:

- V is chosen randomly from all possible isometries satisfying the covariance condition (53).

- The irreps $r_1, r_2,$ and r_3 are chosen randomly, or at least *generically*. In fact, we only need that the irreducible representation does not belong to a small subset of all possible irreducible representations.

In Appendix F, we use randomized constructions to prove existence of $U(d_L)$-covariant codes with small error (measured by ϵ_e based on the average entanglement fidelity), as summarized in the following theorem:

Theorem 5. *For $d_L \geqslant 4$ and every $\epsilon > 0$, there exists a $U(d_L)$-covariant code with error $\epsilon_e \leqslant \epsilon$ and physical dimensions d_i, $i \in \{1, 2, 3\}$, such that*

$$\max_i \ln d_i \leqslant d_L(d_L - 1) \ln\left(\frac{1}{\epsilon_e}\right) + C_2, \qquad (54)$$

for some C_2 which is only a function of d_L.

It is not clear how to compare the performance of our code given by (54) to our bounds (52) because our nonconstructive proof does not specify the behavior of C_2 as a function of d_L, which is given by details of the representation theory of $U(d_L)$. It remains open whether the lower bound can be strengthened or the constructions can be improved.

Our proof technique does not immediately work for $U(2)$-covariant codes, as it is harder to bound the fluctuations of the fidelity of recovery when the logical Hilbert space is too small. For $U(3)$-covariant codes, our methods lead to codes with a slightly different scaling from Eq. (54). In fact, for the $U(3)$-case, one can provide randomized and *non-asymptotic* constructions (which work for known finite physical dimensions) using the explicit formulas for the Littlewood-Richardson coefficients [51]. These constructions are not included in the present paper, as there is little specific interest in the $d_L = 3$ case.

The proof of Theorem 5 is technical and relies on the representation theory of the unitary group (see Appendix F for details). The proof starts by connecting the average fidelity recovery of erasure of a fixed subsystem to the smoothness of the *Littlewood-Richardson coefficients*. Littlewood-Richardson coefficients are representation theory quantities that count the degeneracy of a particular irrep of $U(d_L)$ in the tensor product of two other irreps, and their smoothness follows from modern results in representation theory of the unitary group [51] (Fig. 5).

B.   Generalized W-state encoding

Here we consider another example of an approximate quantum error-correcting code, covariant with respect to the full unitary group on the logical system. It is based on the W state, and achieves an arbitrarily small ϵ_{worst}. in the limit of a large number of subsystems, $n \to \infty$. The logical system L of dimension d_L is encoded into a physical system composed of n copies of a $(d_L + 1)$-dimensional space, where each subsystem is a copy of the logical system with an additional basis vector $|\perp\rangle$. The encoding is

$$|\psi\rangle_L \to \frac{1}{\sqrt{n}}\big(|\psi, \perp, \ldots, \perp\rangle + |\perp, \psi, \perp, \ldots\rangle +$$
$$\cdots + |\perp, \ldots, \perp, \psi\rangle\big) . \quad (55)$$

Any logical unitary U can be carried out on the encoded state transversally by applying the unitary $U \otimes U \otimes \cdots \otimes U$, where we let U act trivially on the extra state $|\perp\rangle$.
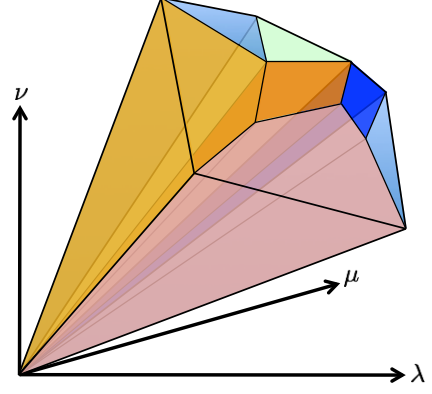

FIG. 5: Smoothness of the Littlewood-Richardson coefficients, required for our proof that random covariant codes can asymptotically correct against errors. A Littlewood-Richardson coefficient $c_{\mu\nu}^\lambda$ is the coefficient that counts the degeneracy of the $U(d_L)$-irrep labeled by the Young diagram λ in the tensor product of two other irreps labeled by μ and ν. The Littlewood-Richardson coefficients are non-zero in a convex cone in the space of (μ, ν, λ). This cone—or *chamber complex*—is divided into several smaller convex cones—or chambers—in which $c_{\mu\nu}^\lambda$ is a polynomial of μ, ν, and λ. Hence, a generic choice of irreps on which we chose a random code will have corresponding coefficients that are smooth, which we show implies good asymptotic performance of the code.

Remarkably, aside from being $U(d_L)$-covariant, this trivial code is also effective against random erasures. Intuitively, this is because the environment will only receive access to the logical state $|\psi\rangle$ with probability $1/n$ if it gets access to a single subsystem; that is, the environment is unlikely to learn anything about the logical information. This can be formalized with a direct application of our criterion (Theorem 3). Given a basis $|x\rangle_L$ of L, the reduced state ρ_1^x on any single physical subsystem of the codeword (55) corresponding to $|x\rangle_L$ is

$$\rho_1^x = \frac{1}{n}|x\rangle\langle x| + \left(1 - \frac{1}{n}\right)|\perp\rangle\langle\perp| , \qquad (56)$$

and thus $F^2(\rho_1^x, |\perp\rangle\langle\perp|) = 1 - (1/n)$ for all x, and it follows that $\sqrt{1 - F^2(\rho_1^x, |\perp\rangle\langle\perp|)} \leqslant \sqrt{2/n} =: \epsilon$. For $x \neq x'$ we have according to (30),

$$\rho_1^{x,x'} = \frac{1}{n}|x\rangle\langle x'| , \qquad (57)$$

and thus $\|\rho_1^{x,x'}\|_1 = 1/n =: \nu$. The corresponding reduced states on the other physical subsystems are the same by symmetry of the codeword (55). Then Theorem 3 asserts that this code has an error parameter that is at most

$$\epsilon_{\text{worst}} \leqslant \frac{\sqrt{2} + d_L}{\sqrt{n}} . \qquad (58)$$

That is, for fixed d_L, the code becomes a good error-correcting code in the limit $n \to \infty$.

In contrast to the thermodynamic codes presented above, this W-state code does not saturate our bound on $\epsilon_{\mathrm{worst}}$, which is inversely proportional to n rather than the square root of n. The reason for this discrepancy is the same as for the difference between a sharp and a smooth cut-off for the three-rotor code, discussed in §VI A 3. Again, here, as n becomes large, the local reduced state grows close to the rank-deficient state $|\perp\rangle\langle\perp|$, which is a regime where the infidelity is particularly sensitive to small perturbations. In contrast, for instance, our thermodynamic codes of §VI C have reduced states that are full-rank, allowing the code to achieve the same scaling as our accuracy bound as $n \to \infty$. While this code does not achieve the same $1/n$ scaling as the thermodynamic codes, it does exhibit covariance with repect to the full logical unitary group $U(d_L)$.

VIII. ERROR-CORRECTING CODES FOR GENERAL GROUPS

In this section, we develop a framework for constructing codes that are covariant with respect to any group G admitting a left- and right-invariant Haar measure, encompassing in particular codes that are based on rotors, oscillators, and qudits. Our construction is based on quantum systems that transform as the *regular representation* of G. Orthonormal basis states $\{|g\rangle\}_{g \in G}$ for this representation are labeled by group elements; if the group has an infinite number of elements, then the quantum system is infinite-dimensional.

A qubit can transform as the regular representation of the group $\mathbb{Z}_2$, and a qudit as the regular representation of $\mathbb{Z}_d$. An oscillator provides a regular representation of the (noncompact) group $\mathbb{R}$, with the group acting by translation in either its position basis $\{|x\rangle\}$ or its momentum basis $\{|p\rangle\}$. Similarly, a rotor provides a regular representation of the group $U(1)$, with orthonormal basis states $\{|e^{i\phi}\rangle\}$; when Fourier transformed, it can transform as a regular representation of $\mathbb{Z}$, where the basis states are the eigenstates of angular momentum $\{|\ell\rangle\}_{\ell \in \mathbb{Z}}$.

For ease of presentation, we will consider codes whose logical system and whose physical subsystems transform as the regular representation of any compact group G, commenting on noncompact groups in §VIII C. Well-known qubit codes such as the bit-flip, phase-flip, and $[[4, 2, 2]]$ codes, naturally extend to this setting. More generally, we will also discuss extensions of the $[[m^2, 1, m]]$ and $[[2m, 2m - 2, 2]]$ qubit codes.

A. Bit- and phase-flip codes

For simplicity, let us review bit-flip and phase-flip codes first. An M-qubit bit-flip encoding copies the logical basis state index $x \in \mathbb{Z}_2$ in each of the M subsystems. An M-qubit phase-flip encoding hides the logical index in the sum of the physical qubit states. Taking $M = 3$ for concreteness, the two encodings are

$$|x\rangle_L^{\mathrm{bit}} \to |x, x, x\rangle \tag{59a}$$

$$|x\rangle_L^{\mathrm{phs}} \to \frac{1}{2} \sum_{y_1, y_2, y_3 \in \mathbb{Z}_2} \delta_{x, y_1+y_2+y_3} |y_1, y_2, y_3\rangle, \tag{59b}$$

where $\delta_{x,y} = 1$ if $x = y$ modulo 2. Bit-flip codes protect against single-qubit shifts $x \to x+1$ while phase-flip codes protect against single-qubit operators which are diagonal in the canonical basis.

By viewing a qubit as a regular representation of the group $G = \mathbb{Z}_2$, we can see how to generalize this construction to other groups. For a finite group G with order $|G|$, consider the $|G|$-dimensional Hilbert space V spanned by $\{|g\rangle \mid g \in G\}$ with inner product $\langle g|h\rangle = \delta_{g,h}$, where $\delta_{g,h} = 1$ if g and h are the same group element and zero otherwise. For compact continuous groups, the Hilbert space is infinite-dimensional, and $\delta_{g,h}$ becomes the Dirac delta function—infinite when $g = h$ and zero otherwise—and sums $\frac{1}{|G|} \sum_{h \in G}$ are replaced by integrals $\int dg$, where dg is the group's normalized Haar measure [52, 53]. We'll write sums below for simplicity, with the understanding that the sum is to be replaced by an integral when G is a compact Lie group.

The respective $M = 3$-subsystem bit- and phase-flip generalizations of Eq. (59) for finite groups are

$$|g\rangle_L^{\mathrm{bit}} \to |g, g, g\rangle \tag{60a}$$

$$|g\rangle_L^{\mathrm{phs}} \to \frac{1}{|G|} \sum_{h_1, h_2, h_3 \in G} \delta_{g, h_1 h_2 h_3} |h_1, h_2, h_3\rangle. \tag{60b}$$

The bit-flip encoding records a group element redundantly, while the phase-flip encoding hides g in a product of three group elements. The error-correction properties of these codes are analogous to those for $G = \mathbb{Z}_2$: the bit-flip codes correct against errors which take individual subsystems into states orthogonal to $|g\rangle$ while phase-flip codes correct against single-subsystem errors diagonal in the $|g\rangle$-basis.

To perform an X-type gate on these codes, introduce left and right-multipliers, $\overrightarrow{X}_g$ and $\overleftarrow{X}_g$, which act as

$$\overrightarrow{X}_g |h\rangle = |gh\rangle \qquad \text{and} \qquad \overleftarrow{X}_g |h\rangle = |hg\rangle. \tag{61}$$

The sets $\{\overrightarrow{X}_g\}_{g \in G}$ and $\{\overleftarrow{X}_g\}_{g \in G}$ are permutation matrices forming the left and right regular representations of G. Note that the arrow points towards h from the side that g acts. Since multiplying from the left commutes with multiplying from the right, the two sets commute with each other.

For the bit-flip code (60a), the logical left multiplication gate

$$\overrightarrow{X}_{L,k}^{\mathrm{bit}} : |g\rangle_L^{\mathrm{bit}} \to |kg\rangle_L^{\mathrm{bit}} \tag{62}$$

can be implemented transversally:

$$\overrightarrow{X}^{\text{bit}}_{L,k} = \overrightarrow{X}_k \otimes \overrightarrow{X}_k \otimes \overrightarrow{X}_k. \qquad (63)$$

For the phase-flip code, which provides no protection against bit-flips at all, logical left multiplication is implemented by acting on a single subsystem:

$$\overrightarrow{X}^{\text{phs}}_{L,k} = \overrightarrow{X}_k \otimes I \otimes I, \qquad (64)$$

where I is the subsystem identity. Similar constructions hold for logical right multipliers.

For continuous G, the code states become nonnormalizable, but the gates work the same way. Therefore, the logical operators $\overrightarrow{X}_{L,k}$ define exact continuous symmetries of theses codes. However, these codes do not correct erasure of a subsystem; rather, each code corrects only a limited set of single-subsystem errors. The same is true for the qubit codes that inspired this construction.

We can concatenate the bit-flip code and the phase-flip code for qubits to obtain Bacon-Shor codes [54, 55], which have the parameters $[[m^2, 1, m]]_{\mathbb{Z}_2}$. This notation means that one logical qubit is encoded in a code block of m^2 physical qubits, and that the code distance is m; hence erasure of any $m-1$ of the qubits can be corrected. Of the codes in this family, the best known are the $[[4, 1, 2]]_{\mathbb{Z}_2}$ error-detecting code [56, 57] and Shor's nine-qubit $[[9, 1, 3]]_{\mathbb{Z}_2}$ error-correcting code [58].

Likewise, by concatenating the G-covariant bit-flip and phase-flip codes, we obtain the G-covariant $[[m^2, 1, m]]_G$ code. For finite G, this is a G-covariant encoding of a $|G|$-dimensional logical system in m^2 $|G|$-dimensional subsystems, protected against erasure of any $m-1$ of the subsystems. If G is a compact Lie group, this code has continuous G symmetry. In that case, as the Eastin-Knill theorem requires, the encoding is infinite-dimensional.

Rather than discussing this generalized Bacon-Shor code construction more explicitly here, in VIII B we'll provide a more detailed discussion of a related code, with two rather than just one $|G|$-dimensional logical subsystems.

B. The $[[4, 2, 2]]_G$ code and its generalizations

There is also a $[[4, 2, 2]]_{\mathbb{Z}_2}$ qubit code [59], which can be extended to a covariant $[[4, 2, 2]]_G$ code, with encoding map

$$|g_1, g_2\rangle_L \to \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g, g^{-1}g_1, gg_2, g^{-1}g_1g_2\rangle. \qquad (65)$$

In fact, the $[[4, 2, 2]]_{\mathbb{Z}_2}$ code can be viewed as a mimimal version of Kitaev's toric code [2], defined by just one plaquette operator and one star operator, and (65) defines the corresponding quantum double code with group G.

Given $l \in G$, the physical operator $I \otimes I \otimes \overleftarrow{X}_l \otimes \overleftarrow{X}_l$ has

the effect of replacing g_2 by g_2l in Eq. (65), hence mapping the logical state to $|g_1, g_2l\rangle_L$. The physical operator $\overrightarrow{X}_l \otimes I \otimes \overrightarrow{X}_l \otimes I$, after a redefinition of the summation variable $(g \to l^{-1}g')$, has the effect of replacing g_1 by lg_1, hence mapping to the logical state to $|lg_1, g_2\rangle_L$. Since the left and right multipliers commute, and both logical operations are transversal, the code is covariant with respect to the group $G \times G$.

Using the quantum error-correction conditions [21, 22], we can check that this code corrects one erasure. Let O_1 be an operator acting on the first subsystem, and consider its matrix element between code states. Plugging into Eq. (65) and contracting indices we find

$$_L\langle g_1, g_2|O_1|g_1', g_2'\rangle_L = \delta_{g_1, g_1'}\delta_{g_2, g_2'} \text{tr}(O_1)/|G| . \qquad (66)$$

This means that the code satisfies the condition for correctability of erasure of the first subsystem. A similar calculation can be performed for operators acting on any of the other subsystems; therefore erasure is correctable for each of the four subsystems.

The $[[4, 2, 2]]_{\mathbb{Z}_2}$ qubit code can be generalized to a $[[2m, 2m-2, 2]]_{\mathbb{Z}_2}$ code, which can also be extended to a covariant $[[2m, 2m-2, 2]]_G$ code for any group G. To understand this construction, first consider a different $[[4, 2, 2]]_G$ code, which has a smaller covariance group than the code described above. Now we use the encoding map

$$|g_1, g_2\rangle_L \to \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g, gg_1, gg_2g_1, gg_2\rangle. \qquad (67)$$

Unlike the previously considered code, this code has the property of being invariant under the action of a "stabilizer" operator $S_l = \overrightarrow{X}_l \otimes \overrightarrow{X}_l \otimes \overrightarrow{X}_l \otimes \overrightarrow{X}_l$ for each $l \in G$. The price we pay for this invariance property is a reduction in the number of independent transversal operations which act nontrivially on the code space. There is no nontrivial symmetry of the code acting from the left, but the operator $I \otimes \overleftarrow{X}_l \otimes \overleftarrow{X}_l \otimes I$ maps $|g_1, g_2\rangle_L$ to $|g_1l, g_2\rangle_L$. Therefore, this code is G-covariant. We can also check that it satisfies the condition for correctability of erasure for each one of the four subsystems.

To illustrate how this code generalizes to a higher-length code with more physical subsystems, we will, to be concrete, describe the corresponding $[[2m, 2m-2, 2]]_G$ code with $m = 4$. This code has the stabilizer $S_l = \overrightarrow{X}_l^{\otimes 8}$ for each $l \in G$, and the encoding map

$$|g_1, g_2, g_3, g_4, g_5, g_6\rangle_L$$
$$\to \frac{1}{\sqrt{|G|}} \sum_{g \in G} S_g|1, g_1, g_2g_1, g_2g_3, g_4g_3, g_4g_5, g_6g_5, g_6\rangle. \qquad (68)$$

Aside from being invariant under the action of S_l, the code has another important property: each codeword is a superposition of states $|h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\rangle$

of the eight physical subsystems having the property $h_1^{-1} h_2 h_3^{-1} h_4 h_5^{-1} h_6 h_7^{-1} h_8 = 1$ (for this to work the code has to have even length). These two properties together suffice to ensure that erasure of each subsystem is correctable.

This code is covariant under the group G^3. The operator

$$I \otimes \overleftarrow{X}_{h_1} \otimes \overleftarrow{X}_{h_1} \otimes \overleftarrow{X}_{h_3} \otimes \overleftarrow{X}_{h_3} \otimes \overleftarrow{X}_{h_5} \otimes \overleftarrow{X}_{h_5} \otimes I \quad (69)$$

acts on the code's basis states according to

$$|g_1, g_2, g_3, g_4, g_5, g_6\rangle_L \rightarrow |g_1 h_1, g_2, g_3 h_3, g_4, g_5 h_5, g_6\rangle_L. \quad (70)$$

In general, the $[[2m, 2m - 2, 2]]_G$ code has a transversal G^{m-1} symmetry, acting similarly.

C. Further extensions and some limitations

One can extend these constructions to noncompact groups. For example, the oscillator $[[9, 1, 3]]_{\mathbb{R}}$ code was noticed early on [43, 60] (see also [61, 62]). Another example is the rotor $[[4, 2, 2]]_{\mathbb{Z}}$ encoding

$$|a, b\rangle \rightarrow \sum_{j,k,l \in \mathbb{Z}} \delta_{a,j+k} \delta_{b,l} |j, k, j + l, k + l\rangle. \quad (71)$$

As done in §VI A, one can impose an envelope so that the codewords are normalizable. In general, a bi-invariant Haar measure is sufficient to perform the left- and right-multiplier transversal gates as well as the error-correction, but one would have to approximate the codewords to avoid infinities due to non-normalizable Haar measures. For the oscillator code $[[4, 2, 2]]_{\mathbb{R}}$, for which the above is an integral over oscillator position states, we additionally need to approximate the position states with a displaced and finitely squeezed vacuum [38]. In other words, noncompactness *and* the continuous nature of the group may each require approximations to achieve normalizability of the codewords.

One may also ask if it is possible to extend the secret-sharing code $[[3, 1, 2]]_{\mathbb{Z}_3}$ from §VI A to a more general group G. An extension does indeed work for $G \in \{\mathbb{R}, \mathbb{Z}, \mathbb{Z}_{2D+1}, U(1)\}$, but the code breaks down at, e.g., $\mathbb{Z}_{2D}$ due to there being a non-measure-zero set of order-two elements in the group. Writing a natural guess for the encoding,

$$|g\rangle_L \rightarrow \frac{1}{\sqrt{|G|}} \sum_{h \in G} |h, gh^{-1}, gh^{-2}\rangle, \quad (72)$$

we see that the third subsystem stores the logical index "in plain sight" whenever $h^2 = 1$. Roughly speaking, for groups with too many such elements, the environment can extract logical information from the code.

IX. SYMMETRIES AND ERROR CORRECTION IN QUANTUM GRAVITY

The interplay between continuous symmetries and quantum error correction has implications for holography and quantum gravity. The AdS/CFT correspondence [63, 64] is a duality between quantum gravity in Anti-de Sitter (AdS) space, and a conformal field theory (CFT) in one fewer spatial dimensions, where the CFT resides on the boundary of the AdS space. It was recently discovered that the duality map from bulk operators to boundary operators may be regarded as the encoding map of a quantum error-correcting code, where the code space is spanned by low-energy states of the CFT. Specifically, local operators deep inside the bulk AdS are encoded as highly nonlocal operators in the boundary CFT which are robust against erasure errors in the boundary theory [6, 13, 65]. Here, we discuss symmetries of this AdS/CFT code. First, we reprise a recent analysis from [15, 16], which rules out exact global symmetries for quantum gravity in the bulk AdS space. Then we explain how our results in this paper clarify the correspondence between time evolution in the bulk and boundary theories.

A. No bulk global symmetries

A longstanding conjecture holds that quantum gravity is incompatible with global symmetry. One argument supporting this claim goes as follows [66, 67]. According to semiclassical theory, which should be reliable for large black holes, the Hawking radiation emitted by a black hole is not affected by the amount of global charge the black hole might have previously consumed. Therefore, a process in which a black hole arises from the gravitational collapse of an object with large charge, and then evaporates completely, will not obey charge conservation.

This argument may not be trustworthy if the symmetry group is a small finite group, in which case the total charge cannot be "large," and any missing charge might reappear in the late stages of black hole evaporation when semiclassical theory does not apply. But recently, Harlow and Ooguri used AdS/CFT technology to show that even discrete global symmetries are disallowed in the bulk [16]. Here we will reprise their argument, expressing it in language that emphasizes the conceptual core of the proof, and that may be more accessible for those familiar with the formalism of quantum error correction.

To quantum coding theorists, it sounds strange to hear that the AdS/CFT code cannot have discrete symmetries, because typical quantum codes do. To illustrate this point we'll revisit a simple quantum-error correcting code that is often used to exemplify the structure of the AdS/CFT code: the three-qutrit code [6], which we already discussed in §VI A 1. This encodes a single logical qutrit in a block of three physical qutrits, and protects against the erasure of any one of the three qutrits.

The three-qutrit code is an example of a stabilizer

code—the code space may be defined as the simultaneous eigenspace of a set of generalized Pauli operators. For a qutrit with basis states $\{|j\rangle, j = 0, 1, 2\}$, the generalized Pauli group is generated by operators X and Z defined by

$$X|j\rangle = |j + 1 \ (\text{mod } 3)\rangle, \quad Z|j\rangle = \omega^j|j\rangle, \quad (73)$$

where $\omega = e^{2\pi i/3}$, which obey the commutation relations

$$ZX = \omega XZ, \quad Z^{-1}X = \omega^{-1}XZ^{-1}. \quad (74)$$

The code space of the three-qutrit code is the simultaneous eigenspace with eigenvalue 1 of the operators

$$S_X = X \otimes X \otimes X, \quad S_Z = Z \otimes Z \otimes Z \quad (75)$$

acting on the three qutrits in the code block. Note that, although X and Z do not commute, S_X and S_Z do commute, and can therefore be simultaneously diagonalized. Any nontrivial weight-one Pauli operator (supported on a single qutrit and distinct from the identity) must fail to commute with at least one of S_X or S_Z. Therefore no nontrivial weight-one operator preserves the code space, which is why erasure of a single qutrit is correctable.

However, there are weight-two Pauli operators that commute with both S_X and S_Z, and therefore preserve the code space; for example,

$$X_L = X \otimes X^{-1} \otimes I, \quad Z_L = Z \otimes I \otimes Z^{-1}. \quad (76)$$

Because they preserve the code space, and act nontrivially on the code space, we say that X_L and Z_L are nontrivial logical operators for this code. Furthermore, X_L and Z_L obey the same commutation relations as X and Z; they generate the logical Pauli group acting on an encoded qutrit. Note that because S_X acts trivially on the code space, the operator X_L, which is supported on the first two qutrits, acts on the code space in the same way as $X_L S_X$, which is supported on the first and third qutrit, and also in the same way as $X_L S_X^{-1}$, which is supported on the second and third qutrit. A similar observation also applies to Z_L and S_Z. This feature illustrates a general property: if O_L is a logical operator, and A is a subset of the qutrits in the code block such that erasure of A is correctable, then we may represent O_L as a physical operator supported on the complementary set A^c.

Our purpose in describing this code is just to point out that the transversal logical operators X_L and Z_L may be viewed as global symmetries of the code. The action of each of these operators on the logical system can be realized as a tensor product of single-qutrit operators. Such a symmetry is what Harlow and Ooguri rule out. We need to understand why their argument applies to the AdS/CFT code, but not to the qutrit code or to other stabilizer codes.

Harlow and Ooguri use special properties of AdS/CFT in two different ways, and their argument proceeds in two steps. The first step (explained in more detail below)

appeals to *entanglement wedge reconstruction*, together with the structure of global symmetries in quantum field theory, to show that any global symmetry acting on the bulk acts transversally on the boundary. That is, the boundary can be expressed as a union of disjoint subregions $\{A_k\}$ such that erasure of each A_k is correctable, and any bulk global symmetry operator U_L, when reconstructed on the boundary, can be expressed as a tensor product $\bigotimes_k W_k$, where W_k is supported on A_k. (Here we ignore a correction factor supported only where the regions touch, which is inessential to the argument.) This is just the property that we have assumed throughout this paper, and which is exemplified by the three-qutrit code discussed above.

The second step of the argument (also explained further below) is the crucial one, which invokes a property of the AdS/CFT code which is not shared by the typical quantum codes which arise in work on fault-tolerant quantum computation. Harlow and Ooguri argue that each W_k is itself a logical operator; that is, each W_k maps the code space to the code space. The essence of this part of the argument is that the code space is the span of low-energy states in the CFT, and the W_k's, perhaps after suitable smoothing, can be chosen so that they do not increase the energy of the CFT by very much. As we have already emphasized, this property does not apply to the three-qutrit code, where X_L is a logical operator, yet its weight-one factors $X \otimes I \otimes I$ and $I \otimes X^{-1} \otimes I$ are not logical. Indeed, because $X \otimes I \otimes I$ changes the eigenvalue of the unitary operator S_Z by the multiplicative factor ω, it maps the code space (the simultaneous eigenspace of S_Z and S_X with eigenvalue 1) to a subspace orthogonal to the code space (the eigenspace of S_Z with eigenvalue ω).

A logical operator supported on a region A, where erasure of A is correctable, must be the logical identity. We can easily see that's true, because otherwise an adversary could steal region A and apply a nontrivial logical operator, altering the encoded state and therefore introducing an uncorrectable error. Now the conclusion of Harlow and Ooguri follows easily. The bulk global symmetry operator U_L is a product of logical operators, each of which is trivial; therefore U_L must be the identity.

As Harlow and Ooguri note (Footnote 68 in [16]), their argument, which excludes discrete symmetries of the AdS/CFT code as well as continuous symmetries, is quite different than the Eastin-Knill argument, which excludes only continuous symmetries of a code. Both arguments apply in a framework where the symmetry of the code can be applied transversally, as a product of local operators. But for the Eastin-Knill argument, there is no need to assume that these local operators preserve the codespace, and therefore the argument applies to general codes. In contrast, Harlow and Ooguri assert that for the AdS/CFT code in particular, the local operators *do* preserve the code space. Therefore, their argument excluding discrete symmetries applies to the AdS/CFT code, but not to the typical codes studied by quantum information theorists.

For completeness, we'll now sketch the two key steps of the Harlow-Ooguri argument in slightly greater detail, starting with the step which shows that a bulk global symmetry acts transversally on the boundary. We begin by noting that a global symmetry in the bulk AdS space implies a corresponding symmetry acting on the boundary; to see this we need only consider the action of the bulk global symmetry on bulk local operators in the limit where the support of the bulk local operators approaches the boundary. Furthermore, a global symmetry operator of the boundary CFT is *splittable*; that is, it can be expressed as a tensor product of many operators, each supported on a small region. In coding theory language, the encoding isometry V which maps bulk to boundary has the property

$$VU_L = U_{\text{CFT}}V, \qquad (77)$$

where U_L is the bulk symmetry operator and U_{CFT} is the corresponding CFT symmetry operator. Because the CFT symmetry is splittable, we may consider decomposing the CFT into small spatial subregions $\{A_k\}$, and infer that

$$U_{\text{CFT}} = \bigotimes_k W_k, \qquad (78)$$

where W_k is a CFT operator supported on A_k.

Next we would like to see that the boundary subregions can be chosen so that erasure of any A_k is correctable. This point is most naturally discussed using the language of operator algebra quantum error correction [6]. We consider the subalgebra $\mathcal{A}$ of logical operators which are supported on a subregion of the bulk. Each logical operator $O_L \in \mathcal{A}$ can be "reconstructed" as a physical operator O_{CFT} acting on the boundary using the encoding isometry V:

$$VO_L = O_{\text{CFT}}V. \qquad (79)$$

What we wish to show is that, for any O_L in $\mathcal{A}$, and for each boundary subregion A_k, the reconstructed boundary operator O_{CFT} can be chosen to have support on the complementary boundary subregion A_k^c. This property ensures that, for the bulk subalgebra $\mathcal{A}$, erasure of boundary region A_k is correctable.

The argument showing that erasure of boundary subregion A_k is correctable is illustrated in Fig. 6. Associated with each boundary subregion A_k is a bulk subregion a_k which is called the entanglement wedge of A_k. The AdS/CFT code has these important properties [6]: (1) A bulk operator supported in bulk subregion a_k can be reconstructed as a boundary operator supported in boundary subregion A_k. This property is called *subregion duality*. (2) Furthermore, a bulk operator supported in the bulk complement a_k^c of bulk subregion a_k can be reconstructed as a boundary operator supported in the boundary complement A_k^c of boundary subregion A_k. This property is called *complementary recovery*.

It follows from complementary recovery that if the bulk

subalgebra $\mathcal{A}$ is supported in a_k^c, then erasure of boundary subregion A_k is correctable for the subalgebra $\mathcal{A}$. This is the key fact that we need. As in Fig. 6, for any fixed subregion a_0 of the bulk, we can choose the decomposition of the boundary into subregions $\{A_k\}$ such that a_0 lies outside the entanglement wedge of each A_k. Therefore, the algebra $\mathcal{A}$ of bulk operators supported on a_0 has the feature that erasure of each A_k is correctable for the algebra $\mathcal{A}$. This completes the first step of the Harlow-Ooguri argument, showing that a bulk global symmetry operator U_L must be transversal in the sense we have assumed in this paper—it factorizes as a tensor product of boundary operators, each of which is supported on a correctable boundary subregion.

Actually, so far we have ignored a subtlety in this argument associated with general covariance in the bulk [16]. Operators acting in the bulk are not really strictly local; rather a bulk "local" operator is accompanied by gravitational *dressing* which connects it to the boundary. This dressing is needed in order to enforce invariance under bulk diffeomorphisms. Because the dressing extends to the boundary, it has support on at least one of the a_k, and its reconstructed counterpart has support on at least one boundary subregion. However, this complication does not invalidate the argument, because the dressing is purely gravitational, and is therefore oblivious to the global charge defined within the bulk subalgebra $\mathcal{A}$.

Now we come to the second part of the Harlow-Ooguri argument, which establishes that the operator W_k supported on boundary subregion A_k is actually a *logical* operator. In the holographic correspondence, the choice of code space is actually rather flexible. One possible procedure [6] is to pick a set of local operators deep in the bulk, corresponding to highly nonlocal operators when reconstructed in the CFT. Then the code space is spanned by polynomials of bounded degree in these operators acting on the CFT vacuum state. The motivation for this choice is that each of the highly nonlocal CFT operators raises the energy of the CFT by only a small amount, hence producing only very weak back reaction on the bulk geometry. Logical operators are those that preserve this low-energy sector of the CFT, and Harlow and Ooguri assert that each operator W_k can be chosen to have this property. Since W_k preserves the code space, and is supported on the correctable boundary subregion A_k, it must act trivially on the code space. This assertion is affirmed if the code's logical operators may be regarded as bulk operators which are supported in a bulk region which is outside the entanglement wedge of the A_k (such as the region a_0 in Fig. 6), since in that case each logical operator can be reconstructed on the complementary boundary region A_k^c, where W_k acts trivially. Therefore, since each W_k is a trivial logical operator, we conclude that the global symmetry operator U_L is the identity acting on the code space.

In this argument, we assumed that subregion duality and complementary recovery are *exact* properties of the AdS/CFT code, and thus inferred that erasure of bound-
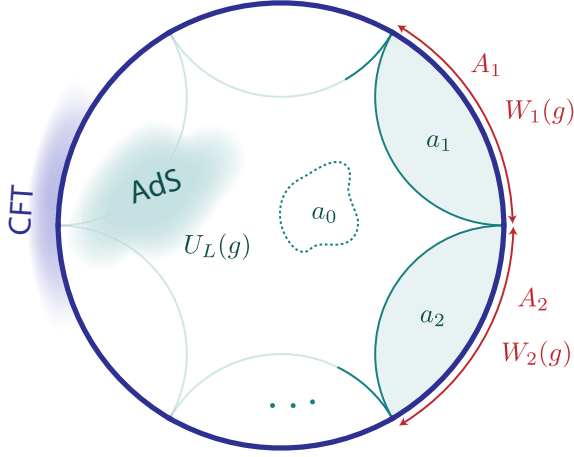
FIG. 6: Nontrivial bulk global symmetries are incompatible with the AdS/CFT quantum error-correcting code. A bulk global symmetry operator $U_L(g)$ corresponds to a boundary global symmetry operator $U_{\mathrm{CFT}}(g)$ which is transversal with respect to the decomposition of the boundary into subregions $\{A_k\}$: $U_{\mathrm{CFT}}(g) = \bigotimes_k W_k(g)$, where $W_k(g)$ is supported on A_k. The bulk subregion a_0 is outside the entanglement wedge a_k of each boundary subregion A_k; therefore erasure of each A_k is correctable for the algebra $\mathcal{A}$ of bulk local operators on a_0. Furthermore, each $W_k(g)$ maps low-energy states of the boundary CFT to low-energy states. This means that $W_k(g)$ is a logical operator which preserves the code subspace of the CFT. A logical operator $W_k(g)$ supported on a correctable boundary subregion A_k must be the logical identity. Therefore the global symmetry operator $U_L(g)$ acts trivially on bulk local operators.

ary region A_k is *exactly* correctable. In fact, though, these properties of the code hold precisely only in the leading order of a systematic expansion in Newton's gravitational constant G_N, and can be modified when corrections higher order in G_N are included. Nevertheless, the conclusion that bulk global symmetries are disallowed continues to hold even when these higher-order corrections are taken into account, assuming the corrections are small. A nontrivial global symmetry operation (if one were allowed), acting on a bulk local operator ϕ, should modify ϕ by an amount $\delta\phi$ which is $O(1)$, independent of G_N. But we have argued that $\delta\phi = 0$ to leading order in G_N (since exact correctability of A_k holds to this order). Higher-order corrections might make an $O(G_N)$ contribution to $\delta\phi$, but these small corrections do not suffice to restore the proper nontrivial action on ϕ of the putative global symmetry.

Now we have found that exact bulk local symmetries cannot occur in AdS/CFT. But what can we say about whether *approximate* discrete global symmetries are allowed? As we've discussed, finite-dimensional quantum error-correcting codes *can* have exact discrete symmetries, even though the AdS/CFT code does not. In this respect, discrete symmetries are essentially different than continu-

ous symmetries, which are disallowed by the Eastin-Knill theorem for any finite-dimensional quantum code that can correct erasure of subsystems exactly. Therefore, we can't expect to make general statements which are directly analogous to Theorem 2 about limitations on approximate *discrete* symmetries that apply to general codes.

Nevertheless, it may be instructive to study further the properties of approximate quantum error-correcting codes which are approximately covariant with respect to a discrete symmetry. In the setting of AdS/CFT, it is of particular interest to consider the case where the local transformations $\{W_k\}$ in Eq. (78) are either precisely or approximately logical.

B. Bulk time evolution

A natural symmetry arising in AdS/CFT is the time-translation invariance of the boundary CFT, which is governed by a local Hamiltonian. Time evolution in the bulk AdS space is a bit subtle because of the general covariance of the bulk theory, but if we fix the gauge by choosing a preferred sequence of bulk time slices, then time evolution in the bulk corresponds to time evolution on the boundary. From the perspective of quantum error correction, this correspondence is puzzling, because covariance of the AdS/CFT code with respect to time evolution seems to be incompatible with perfect correctability of erasure on the boundary [12, 20]. Indeed, the analysis of bulk global symmetries in §IX A, which is applicable to both discrete and continuous symmetries, builds on the observation that a boundary global symmetry operator, when restricted to a correctable boundary subregion, preserves the code space and therefore must be a trivial logical operator. Why can't we apply similar reasoning to the action of the boundary Hamiltonian, concluding (incorrectly) that bulk time evolution is trivial?

The answer hinges on a crucial distinction, emphasized in [15, 16], between global symmetry and *long-range gauge symmetry* in the bulk. As we've noted, a "local" operator in the bulk is not truly local; it requires gravitational dressing connecting it to the boundary. For the analysis of bulk global symmetries, this dressing could be ignored, because the dressing transforms trivially under the global symmetry. For the analysis of bulk time evolution, the dressing cannot be ignored, because the dressing depends on the energy-momentum of a bulk quantum state. It is the nontrivial action of the boundary Hamiltonian on the asymptotic gravitational dressing of bulk "local" operators which is responsible for the bulk time evolution. Furthermore, because the dressing can be detected by localized boundary observers, erasure of boundary subregions can really be corrected only approximately rather than exactly.

Our Theorem 2 clarifies the situation by quantifying the incompatibility between continuous symmetries and error correction. In the regime of sufficiently large physical subsystems, or for a large enough number of subsystems,

covariant codes can provide arbitrarily good protection against erasure errors. The AdS/CFT setting fulfills both of these criteria. The boundary theory is a field theory, which formally has an unbounded number of local physical subsystems. Furthermore, in the "large N" limit of the CFT, which corresponds to semiclassical gravity in the bulk, the Hilbert space dimension of each local subsystem is very large [63].

We note that holographic quantum codes, toy models of the bulk which capture some of the properties of full blown AdS/CFT, have been constructed in which local Hamiltonian evolution in the bulk is realized approximately by a local Hamiltonian in the boundary theory [68]. However, although these codes are approximately covariant, the boundary Hamiltonian is far from uniform.

X. DISCUSSION

In this paper we have studied quantum error-correcting codes that are exactly or approximately covariant with respect to a continuous symmetry group. A special case of our main result applies if the logical charge operator T_L which generates a continuous symmetry is a *transversal* logical operator of the code. This means that the logical system L is encoded in a physical system A which can be decomposed as a tensor product of physical subsystems $\{A_i\}$ such that erasure of each A_i is correctable, and that the physical symmetry generator T_A is a sum $T_A = \sum_i T_i$, such that T_i is a *local* charge operator supported only on subsystem A_i.

The Eastin-Knill theorem [17–19] asserts that no quantum error-correcting code can be covariant with respect to a continuous symmetry if the number of physical subsystems is finite, each subsystem is finite-dimensional, and erasure of each subsystem is exactly correctable. However, it was shown in [12] that this conclusion can be evaded by infinite-dimensional codes. Our main objective here has been to clarify the properties of covariant quantum codes in which the dimension of each physical subsystem is large but finite, and in which the number of subsystems is large but finite.

In Theorem 1, we consider codes that can correct erasure of a subsystem only *approximately*, and we derive a lower bound on the worst case entanglement infidelity ϵ_{worst} that can be achieved by the best recovery map after an erasure. In keeping with the findings of [12], this lower bound approaches zero when the number n of subsystems approaches infinity, or when the fluctuations of the local charge of individual subsystems grow without bound. The idea behind the lower bound is that, if the number of subsystems and the local charge fluctuations are both finite, then some information about the value of the global logical charge is available to an adversary who takes possession of a single physical subsystem, resulting in irreversible decoherence of the logical state. In Theorem 2, we extend the result by relaxing the assumptions. This more general theorem applies when the code is not exactly covariant, when the logical charge operator is not exactly transversal, and when more than one subsystem is erased.

While originally derived in the context of fault-tolerant quantum computing, the Eastin/Knill theorem has a variety of other applications, for example to quantum reference frames and quantum clocks [12, 20] (cf. also recent related work [39]), and to the holographic dictionary relating bulk and boundary physics in the AdS/CFT correspondence [12]. When applied to these settings, our results provide limitations on transmission of reference frames over noisy channels, and help to clarify the relationship between bulk and boundary time evolution for the AdS/CFT quantum code. Our lower bounds on infidelity also apply to the recently discovered quantum codes arising in one-dimensional translation-invariant spin chains [8].

Our main result hinges on an interplay between the noise model and the structure of the local charge observables. Specifically, Theorem 2 applies under the following condition: For any term T_α that appears in the physical charge $T_A = \sum_\alpha T_\alpha$, there is a nonzero probability that *all* physical subsystems supporting T_α are *simultaneously* lost to the environment. One may wonder whether this condition is really necessary—*e.g.*, would a code with a 2-local charge operator be allowed if it could correct only a single erasure? It turns out that such codes do exist, showing that our condition is necessary. As a simple example, the erasure of a single qubit is correctable for the [[4,2,2]] quantum code, but there is also a nontrivial logical operator $Q = X \otimes X \otimes I \otimes I$ supported on the first two qubits [69]. We can exponentiate this 2-local operator to generate a logical rotation of the first logical qubit. This provides an example of a code that is exactly error-correcting against a single located erasure and that is nevertheless exactly covariant with respect to a two-local charge.

In the lower bound (2), the range ΔT_L of the logical charge operator and corresponding range ΔT_i of the physical charge are not directly related to the corresponding system dimensions if the symmetry is abelian. The situation is different when we apply our bound (26) to codes that are covariant with respect to the full unitary group $U(d_L)$. In that case, there is a minimal subsystem dimension for each value of ΔT_i, and Theorem 4 therefore follows from Theorem 1.

While Theorems 1 and 2 pertain to correction of erasure errors, similar conclusions should apply for more general errors. For dephasing errors in particular, the information leaked to the environment can be explicitly characterized in accord with recent results [27, 70].

Our work builds on Ref. [12], where covariant quantum codes arose in the study of reference frames; *i.e.*, asymmetric states which convey "physical" information [10, 71]. As shown in [12], exact error correction of reference frames is impossible for finite-dimensional systems, yet in the real world reference frames are always finite-dimensional and communication channels are always imperfect. Neverthe-

less, in practice we routinely share reference frames over noisy channels, easily reaching agreement about which direction is "up" or what time it is; furthermore quantum technologists can distribute entanglement between nodes of a quantum network, which is possible only if the nodes share a common phase reference. Our results clarify, quantitatively, why accurate communication of reference information is achievable in practice. A quantum reference frame of sufficiently high dimensionality becomes effectively classical, quite robust against the ravages of environment noise. Examples of such systems include highly excited oscillators and rotors, Bose-Einstein condensates, superconductors, and other macroscopic phases of quantum matter.

In metrology, quantum error correction provides a promising tool for improving sensitivity by protecting a probe system against a noisy environment [20, 72–75]. However, there is a delicate balance to achieve between error-correcting against the noise while still being sensitive to the physical observable H one wishes to measure. In order to correct against errors, one needs to encode in an appropriate codespace. Furthermore, in order to measure H, it needs to act nontrivially within that codespace. The ability to measure H directly by local observations corresponds, in the language of this paper, to covariance of the code with respect to the physical charge H. In other words, adapting our setup to one from quantum metrology is straightforward: the goal now is to estimate the continuous parameter ω in $H = \omega T_A$ as accurately as possible while at the same time being able to correct against relevant noise. Recent efforts have determined that it is possible to measure H at the Heisenberg limit using an error-correcting code if H is not a sum of the operators characterizing the correctable noise [76–79]. But if the physical charge T_A is a sum of local charges, the Eastin-Knill theorem poses a challenge to the application of error-correcting techniques; namely, we cannot measure what we can correct. The infinite dimensional counterexamples of [12] show that it is nonetheless possible to correct against local noise *and* admit a charge that is a sum of such noise operators, granted one has non-normalizable codewords. The bounds and example codes of this paper provide a quantitative version of this infinite-dimensional limit.

Our results suggest the possibility that one could sacrifice some error correction precision to achieve better sensitivity with physical, i.e. normalizable, states (cf. also [39]). However, to properly apply our results to quantum metrology, there are some additional steps that need to be taken, which is the subject of ongoing follow-up work. First, since we are trying to measure an unknown parameter (and not necessarily to protect quantum information *per se*), we should account for the fact that a code is only required to reconstruct a logical state that would yield a precise reading of said parameter. Second, our results are stated in terms of the worst-case entanglement fidelity, but for applications to metrology one would prefer different figures of merit, such as the precision at which the probe can

sense magnetic fields, or the ability of a quantum clock to tell time accurately. Finally, it would be desirable to consider noise models that are more relevant to quantum metrology, such as fluctuating background magnetic fields that induce dephasing errors. Bény's characterization of approximate quantum error correction of algebras [80] provides a promising tool for addressing these challenges because one can specify precisely which observables need to be faithfully reproduced after action by the noise and a possible recovery operation.

Approximate quantum error-correcting codes also arise naturally in many-body quantum systems [8, 9]. We anticipate that constraints on correlation functions of many-body quantum states can be derived from the covariance properties of the corresponding codes.

Finally, the interplay of symmetry and quantum error correction has a prominent role in the AdS/CFT holographic correspondence. Although covariance with respect to a continuous symmetry is incompatibile with perfect correctability of erasure of physical subsystems for any finite-dimensional quantum code, nevertheless we expect that in the AdS/CFT code continuous time evolution of the boundary system corresponds to continuous time evolution of the encoded logical bulk system. Our results relieve the tension between these two observations, because near perfect correctability can be achieved if either the number of physical subsystems, or the dimension of each physical subsystem, becomes very large. Both these provisos apply to the continuum limit of a regulated holographic boundary conformal field theory, as the number of lattice sites per unit volume is very large in this limit, and the number of degrees of freedom per site is also very large if semiclassical gravity accurately describes the bulk geometry (the "large-N limit").

Recent results indicate that not just exact continuous symmetries, but also exact discrete symmetries, are incompatible with the quantum error correction properties of the AdS/CFT code [15, 16]. An intriguing topic for further research will be investigation of approximate symmetries, both continuous and discrete, in the context of quantum gravity.

SUPPLEMENTAL MATERIAL

List of Appendices

Appendix A: Proof of our bounds for a covariant code

The proof of Theorem 2 is split into two lemmas. A first lemma deduces that the environment has access to the logical charge, to a good approximation.

Lemma 6. *Under the assumptions of Theorem 2, and following the latter's notation, there exists an observable $Z_{C'E}$ satisfying*

$$\left\|\widehat{\mathcal{N} \circ \mathcal{E}}^{\dagger}(\mathbb{1}_F \otimes Z_{C'E}) - (T_L - \nu'\mathbb{1}_L)\right\|_{\infty} \leqslant \delta + \eta \; ; \tag{A.1}$$

$$\|Z_{C'E}\|_{\infty} \leqslant \max_{\alpha} \frac{\Delta T_{\alpha}}{2q_{\alpha}} \; , \tag{A.2}$$

where $\nu' = \nu + \sum(t_{\alpha}^- + t_{\alpha}^+)/2$ and where the complementary channel $\widehat{\mathcal{N} \circ \mathcal{E}}_{L \to C'EF}$ to the combined encoding and noise is given by (15).

Proof of Lemma 6. Let $\Pi_{\alpha} = \mathbb{1} - \Pi_{\alpha}^{\perp}$ be the projector which projects onto the eigenspaces of T_{α} whose corresponding eigenvalues are in the range $[t_{\alpha}^-, t_{\alpha}^+]$. Recall that $t_{\alpha} = (t_{\alpha}^- + t_{\alpha}^+)/2$ is the midpoint of the interval $[t_{\alpha}^-, t_{\alpha}^+]$. Define the observables $\tilde{T}_{\alpha} = \Pi_{\alpha}(T_{\alpha} - t_{\alpha}\mathbb{1})$, and observe that $\tilde{T}_{\alpha}$ has eigenvalues between $-\Delta T_{\alpha}/2$ and $+\Delta T_{\alpha}/2$, and hence $\|\tilde{T}_{\alpha}\|_{\infty} \leqslant \Delta T_{\alpha}/2$. Define the observable

$$Z_{C'E} = \sum_{\alpha} |\alpha\rangle\langle\alpha|_{C'} \otimes (q_{\alpha}^{-1}\tilde{T}_{\alpha}) \; . \tag{A.3}$$

Then, for any logical state σ_L, and writing $\rho_A = \mathcal{E}(\sigma_L)$,

$$
\begin{aligned}
\operatorname{tr}\left(Z_{C'E} \widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L)\right) &= \sum \operatorname{tr}\left(\tilde{T}_{\alpha} \operatorname{tr}_{A \setminus A_{\alpha}}(\rho_A)\right) = \sum \operatorname{tr}\left(\tilde{T}_{\alpha} \rho_A\right) = \sum \operatorname{tr}\left(\Pi_{\alpha}(T_{\alpha} - t_{\alpha}\mathbb{1}) \rho_A\right) \\
&= \sum \operatorname{tr}\left((\mathbb{1} - \Pi_{\alpha}^{\perp})(T_{\alpha} - t_{\alpha}\mathbb{1}) \rho_A\right) \\
&= \sum \left(\operatorname{tr}(T_{\alpha} \rho_A) - t_{\alpha} - \operatorname{tr}\left(\Pi_{\alpha}^{\perp}(T_{\alpha} - t_{\alpha}\mathbb{1}) \rho_A\right)\right) \\
&= \operatorname{tr}(T_A \rho_A) - \sum t_{\alpha} - \sum \operatorname{tr}\left(\Pi_{\alpha}^{\perp}(T_{\alpha} - t_{\alpha}\mathbb{1}) \rho_A\right) \; , \tag{A.4}
\end{aligned}
$$

thus

$$\mathrm{tr}\Big\{\widehat{\mathcal{N}\circ\mathcal{E}}^\dagger(\mathbb{1}_F\otimes Z_{C'E})\,\sigma_L\Big\} - \Big(\mathrm{tr}(T_A\rho_A) - \sum t_\alpha\Big) = -\sum\mathrm{tr}(\Pi_\alpha^\perp\,(T_\alpha - t_\alpha\mathbb{1})\,\rho_A)\ . \quad \text{(A.5)}$$

Noting that $\mathrm{tr}\big(T_A\mathcal{E}(\sigma_L)\big) - \sum t_\alpha = \mathrm{tr}\big(\big[\mathcal{E}^\dagger(T_A) - \sum t_\alpha\mathbb{1}\big]\,\sigma_L\big)$, we have

$$\Big|\mathrm{tr}\Big\{\Big[\widehat{\mathcal{N}\circ\mathcal{E}}^\dagger(\mathbb{1}_F\otimes Z_{C'E}) - \Big(\mathcal{E}^\dagger(T_A) - \sum t_\alpha\mathbb{1}\Big)\Big]\sigma_L\Big\}\Big| = \Big|\sum\mathrm{tr}\big(\Pi_\alpha^\perp(T_\alpha - t_\alpha\mathbb{1})\rho_A\big)\Big| \leqslant \eta\ , \tag{A.6}$$

where we have used condition (28). Recall that for any Hermitian operator X, we have $\|X\|_\infty = \max_\sigma|\mathrm{tr}(X\sigma)|$ with an optimization over all density matrices σ. Since (A.6) holds for all σ_L, we have

$$\Big\|\widehat{\mathcal{N}\circ\mathcal{E}}^\dagger(\mathbb{1}_F\otimes Z_{C'E}) - \Big(\mathcal{E}^\dagger(T_A) - \sum t_\alpha\mathbb{1}\Big)\Big\|_\infty \leqslant \eta\ . \tag{A.7}$$

Using the approximate charge conservation condition $\|(T_L - \nu\mathbb{1}) - \mathcal{E}^\dagger(T_A)\|_\infty \leqslant \delta$ and the triangle inequality for the infinity norm, we finally obtain

$$\Big\|\widehat{\mathcal{N}\circ\mathcal{E}}^\dagger(\mathbb{1}_F\otimes Z_{C'E}) - (T_L - \nu'\mathbb{1})\Big\|_\infty \leqslant \delta + \eta\ , \tag{A.8}$$

setting $\nu' = \nu + \sum t_\alpha$.

Since the infinity norm picks out the largest eigenvalue in absolute value, we see from (A.3) that $\|Z_{C'E}\|_\infty = \max_\alpha q_\alpha^{-1}\|\tilde{T}_\alpha\|_\infty \leqslant \max_\alpha q_\alpha^{-1}\Delta T_\alpha/2$. ∎

The second part of the proof of Theorem 2 is to deduce from the environment's access to the global charge that the code performs poorly with respect to the various entanglement fidelity measures. We phrase this statement as a more general lemma that applies in fact to any noise model, and can be used to bound the fixed-input entanglement fidelity for any given fixed input state $|\phi\rangle_{LR}$, as long as the environment has access to an observable which yields some information about the logical state. In analogy with ϵ_e and ϵ_worst, we define for any $|\phi\rangle_{LR}$ and for any channel $\mathcal{N}'$,

$$\epsilon_{|\phi\rangle}(\mathcal{N}') = \sqrt{1 - F_{|\phi\rangle}^2(\mathcal{N}', \mathrm{id})}\ . \tag{A.9}$$

This lemma can be seen as a refinement of Bény's characterization of approximate error correction using operator algebras [80]. To formulate the lemma, we define two auxiliary quantities that depend on a state σ and an observable T:

$$C_{\sigma,T}^0 = \big\|\sigma^{1/2}\,(T - \mathrm{tr}(T\sigma)\mathbb{1})\,\sigma^{1/2}\big\|_1\ , \tag{A.10a}$$

$$C_{\sigma,T} = \min_\mu \big\|\sigma^{1/2}\,(T - \mu\mathbb{1})\,\sigma^{1/2}\big\|_1\ , \tag{A.10b}$$

where in the second line the optimization ranges over all $\mu \in \mathbb{R}$. Intuitively, both these quantities $C_{\sigma,T}$ pick up the average charge absolute value (where T is the charge and according to the state σ), up to a constant charge offset μ or $\mathrm{tr}(T\sigma)$. Special cases of these quantities will be discussed in the proof of Theorem 2.

Lemma 7. *Let $(\mathcal{N}\circ\mathcal{E})_{L\to A'}$ be the combined encoding and noise channel with total output system(s) A', where both encoding and noise channels may be any completely positive, trace-*

preserving maps. Let $\widehat{\mathcal{N} \circ \mathcal{E}}_{L \to E'}$ be a complementary channel with combined output system(s) E'. (In the context of Theorem 2, we set $A' = A \otimes C$ and $E' = E \otimes C' \otimes F$, but this lemma holds more generally.) Suppose that there exists observables T_L and $Z_{E'}$ on the input and environment systems respectively, as well as $\nu' \in \mathbb{R}$, $\delta' \geqslant 0$, such that:

$$\left\| \widehat{\mathcal{N} \circ \mathcal{E}}^\dagger(Z_{E'}) - (T_L - \nu' \mathbb{1}_L) \right\|_\infty \leqslant \delta' \ . \tag{A.11}$$

Then, for any $|\phi\rangle_{LR}$, both $\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E})$ and $\epsilon_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E})$ are lower bounded by two different independent bounds:

$$\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}) \geqslant \epsilon_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E}) \geqslant \frac{C_{\phi_L, T_L} - \delta'}{2\|Z_{E'}\|_\infty} \tag{A.12a}$$

$$\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}) \geqslant \epsilon_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E}) \geqslant \frac{\frac{1}{2} C^0_{\phi_L, T_L} - \delta'}{2\|Z_{E'}\|_\infty} \tag{A.12b}$$

Finally, if $\widehat{\mathcal{N} \circ \mathcal{E}}(\cdot) = \sum q_\alpha \widehat{\mathcal{N}_\alpha \circ \mathcal{E}}(\cdot)$ for a probability distribution $\{q_\alpha\}$ and a set of noise channels $\{\mathcal{N}_\alpha\}$, then for any $|\phi\rangle_{LR}$, the same bounds apply to the average of the individual error parameters corresponding to each erasure event:

$$\sum q_\alpha \epsilon_{|\phi\rangle}(\mathcal{N}_\alpha \circ \mathcal{E}) \geqslant \begin{cases} \dfrac{C_{\phi_L, T_L} - \delta'}{2\|Z_{E'}\|_\infty} \\[2ex] \dfrac{\frac{1}{2} C^0_{\phi_L, T_L} - \delta'}{2\|Z_{E'}\|_\infty} \end{cases} \tag{A.13}$$

In summary: There are two figures of merit we are interested in, $\epsilon_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E})$ and $\left\langle \epsilon_{|\phi\rangle}(\mathcal{N}_\alpha \circ \mathcal{E}) \right\rangle_\alpha$, and both are bounded from below by the same bound expressed in terms of the auxiliary quantities (A.10a) and (A.10b).

Proof of Lemma 7. We start by showing the following two statements: For any $|\phi\rangle_{LR}$, and for any state $\zeta_{E'}$, it holds that

$$\delta\left(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'} \otimes \phi_R \right) \geqslant \frac{C_{\phi_L, T_L} - \delta'}{2\|Z_{E'}\|_\infty} \ ; \text{ and} \tag{A.14}$$

$$\delta\left(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \rho_{E'} \otimes \phi_R \right) \geqslant \frac{C^0_{\phi_L, T_L} - 2\delta'}{2\|Z_{E'}\|_\infty} \ , \tag{A.15}$$

where $\rho_{E'} = \widehat{\mathcal{N} \circ \mathcal{E}}(\phi_L)$.

We recall the following expressions for the one-norm of any Hermitian operator A:

$$\|A\|_1 = \max_{\|X\|_\infty \leqslant 1} \operatorname{tr}(XA) \tag{A.16a}$$

$$= \min_{\substack{\Delta_\pm \geqslant 0 \\ A = \Delta_+ - \Delta_-}} \operatorname{tr}(\Delta_+) + \operatorname{tr}(\Delta_-) \ , \tag{A.16b}$$

where the first optimization ranges over operators Hermitian X, and the second over positive semidefinite operators $\Delta_\pm$. We start form the left-hand side of (A.14). By choosing a candidate X in (A.16a) of the form $(Z/\|Z\|_\infty) \otimes X'$ with $\|X'\|_\infty \leqslant 1$, then for any $|\phi\rangle_{LR}$

and for any $\zeta_{E'}$ we have that

$$\frac{1}{2}\big\|\widehat{\mathcal{N}\circ\mathcal{E}}(\phi_{LR}) - \zeta_{E'}\otimes\phi_R\big\|_1$$

$$\geqslant \max_{\|X_R'\|_\infty\leqslant 1}\frac{1}{2\|Z_{E'}\|_\infty}\,\mathrm{tr}\Big\{(Z_{E'}\otimes X_R')\big(\widehat{\mathcal{N}\circ\mathcal{E}}(\phi_{LR}) - \zeta_{E'}\otimes\phi_R\big)\Big\}$$

$$= \max_{\|X_R'\|_\infty\leqslant 1}\frac{1}{2\|Z_{E'}\|_\infty}\,\mathrm{tr}\Big\{\big(\widehat{\mathcal{N}\circ\mathcal{E}}^\dagger(Z_{E'})\otimes X_R'\big)\phi_{LR} - (Z_{E'}\zeta_{E'})\otimes(X_R'\phi_R)\Big\}\,, \quad (A.17)$$

where the optimization ranges over Hermitian operators X_R' on the R system. Making use of the main assumption of this lemma, and restricting the optimization to X_R' such that $\mathrm{tr}(X_R'\phi_R) = 0$ yields

$$(A.17) \geqslant \max_{\substack{\|X_R'\|_\infty\leqslant 1 \\ \mathrm{tr}(X_R'\phi_R)=0}}\frac{1}{\|Z_{E'}\|_\infty}\Big[\mathrm{tr}\big\{(T_L - \nu\mathbb{1}_L)\,\mathrm{tr}_R(X_R'\phi_{LR})\big\} - \delta'\Big]\,, \qquad (A.18)$$

using the fact that if $\|A - B\|_\infty \leqslant \delta'$, then $\mathrm{tr}(AY) \geqslant \mathrm{tr}(BY) - \delta'\,\mathrm{tr}(Y)$ for any Hermitian A, B and positive semidefinite Y, and that furthermore here $\mathrm{tr}(Y) = \mathrm{tr}(X_R'\phi_{LR}) \leqslant \mathrm{tr}(\phi_{LR}) = 1$. Without loss of generality, we may assume that $R \simeq L$ (if R is smaller, then embed it trivially in a larger system of same dimension as L; if R is larger, then remove unused dimensions on which ϕ_R has no support, noting that the support of ϕ_R may not exceed the dimension of L). Let $\{|k\rangle_L\}$, $\{|k\rangle_R\}$ be Schmidt bases of L and R corresponding to $|\phi\rangle_{LR}$, and recall that we have the relations $|\phi\rangle_{LR} = \phi_L^{1/2}|\Phi\rangle_{L:R} = \phi_R^{1/2}|\Phi\rangle_{L:R}$, where $|\Phi\rangle_{L:R} = \sum|k\rangle_L\otimes|k\rangle_R$ and where as before $\phi_L = \mathrm{tr}_R(\phi_{LR})$ and $\phi_R = \mathrm{tr}_L(\phi_{LR})$. Note that for any operator X_R', we have $X_R'|\Phi\rangle_{L:R} = X_L|\Phi\rangle_{L:R}$ where X_L is related to X_R' by a transpose with respect to the bases used to define $|\Phi\rangle_{L:R}$, which implies also $\|X_L\|_\infty = \|X_R\|_\infty$. Consequently, $\mathrm{tr}_R(X_R'\phi_{LR}) = \mathrm{tr}_R(X_R'\phi_L^{1/2}\Phi_{L:R}\phi_L^{1/2}) = \phi_L^{1/2}X_L\phi_L^{1/2}$. Finally, note that $\mathrm{tr}(X_R'\phi_R) = \mathrm{tr}(X_R'\phi_{LR}) = \mathrm{tr}(\phi_L^{1/2}X_L\phi_L^{1/2}) = \mathrm{tr}(X_L\phi_L)$. So we obtain

$$(A.18) = \max_{\substack{\|X_L\|_\infty\leqslant 1 \\ \mathrm{tr}(X_L\phi_L)=0}}\frac{1}{2\|Z_{E'}\|_\infty}\Big[\mathrm{tr}\big(\phi_L^{1/2}(T_L - \nu'\mathbb{1}_L)\phi_L^{1/2}X_L\big) - \delta'\Big]\,. \qquad (A.19)$$

The optimization (A.19) is a semidefinite program, and we proceed to compute its dual program [81]. In terms of the variables $X_L = X_L^\dagger$, $A, B \geqslant 0$, and $\mu \in \mathbb{R}$, and writing for short $T_L' = T_L - \nu'\mathbb{1}_L$, we have

$$\max_{\substack{\|X\|_\infty\leqslant 1 \\ \mathrm{tr}(X\phi_L)=0}}\mathrm{tr}\big[\phi_L^{1/2}T_L'\phi_L^{1/2}X_L\big]$$

$$= \quad \text{maximize}: \quad \mathrm{tr}\big[\phi_L^{1/2}T_L'\phi_L^{1/2}X_L\big] \qquad\qquad (A.20a)$$
$$A: \quad X_L \leqslant \mathbb{1}_L$$
$$B: \quad X_L \geqslant -\mathbb{1}_L$$
$$\mu: \quad \mathrm{tr}(X_L\phi_L) = 0$$

$$= \quad \text{minimize}: \quad \mathrm{tr}(A) + \mathrm{tr}(B) \qquad\qquad (A.20b)$$
$$X_L: \quad \phi_L^{1/2}T_L'\phi_L^{1/2} = \mu\phi_L + A - B\,.$$

Strong duality holds because of Slater's conditions. Indeed $X_L = 0$ is strictly feasible in the

primal problem; the dual is actually also strictly feasible by choosing (say) $\mu = 0$ and A and B to be the positive and negative parts respectively of the Hermitian operator $\phi_L^{1/2} T_L' \phi_L^{1/2}$ plus a constant times the identity. For fixed μ in (A.20b), we recognize the dual semidefinite program for the one-norm of a Hermitian matrix (A.16b), and hence we actually obtain the same expression as in (A.10b),

$$\max_{\substack{\|X\|_\infty \leqslant 1 \\ \mathrm{tr}(X\phi_L)=0}} \mathrm{tr}\big[\phi_L^{1/2}\, T_L'\, \phi_L^{1/2}\, X_L\big] = \min_{\mu \in \mathbb{R}} \big\|\phi_L^{1/2}(T_L' - \mu\mathbb{1})\phi_L^{1/2}\big\|_1 = C_{\phi_L, T_L'} \; . \tag{A.21}$$

Then

$$\text{(A.19)} = \frac{1}{2\|Z_{E'}\|_\infty}\Big(\min_\mu \big\|\phi_L^{1/2}\left(T_L - \nu'\mathbb{1}_L - \mu\mathbb{1}_L\right)\phi_L^{1/2}\big\|_1 - \delta'\Big) = \frac{C_{\phi_L, T_L} - \delta'}{2\|Z_{E'}\|_\infty} \; , \tag{A.22}$$

noting that the constant shift $\nu'\mathbb{1}_L$ can be absorbed into the optimization over μ. This proves (A.14).

Now we show (A.15). Similarly to how we started above, we write

$$\frac{1}{2}\big\|\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}) - \rho_{E'} \otimes \phi_R\big\|_1$$
$$\geqslant \max_{\|X_R'\|_\infty \leqslant 1} \frac{1}{2\|Z_{E'}\|_\infty} \mathrm{tr}\Big\{(Z_{E'} \otimes X_R')\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR} - \phi_L \otimes \phi_R)\big)\Big\}$$
$$= \max_{\|X_R'\|_\infty \leqslant 1} \frac{1}{2\|Z_{E'}\|_\infty} \mathrm{tr}\Big\{\big(\widehat{\mathcal{N} \circ \mathcal{E}}^\dagger(Z_{E'}) \otimes X_R'\big)\big(\phi_{LR} - \phi_L \otimes \phi_R\big)\Big\} \tag{A.23}$$

Define $Z_L = \widehat{\mathcal{N} \circ \mathcal{E}}^\dagger(Z_{E'})$, and using the same procedure to define $|\Phi\rangle_{L:R}$ as above with X_L in one-to-one correspondence with X_R' via the transpose operation and with $\mathrm{tr}(X_L\phi_L) = \mathrm{tr}(X_R\phi_R)$, we obtain

$$\text{(A.23)} = \max_{\|X_L\|_\infty \leqslant 1} \frac{1}{2\|Z_{E'}\|_\infty} \Big[\mathrm{tr}\big\{Z_L\phi_L^{1/2}X_L\phi_L^{1/2}\big\} - \mathrm{tr}\big\{Z_L\phi_L\,\mathrm{tr}(X_L\phi_L)\big\}\Big] \; . \tag{A.24}$$

By assumption, we have $Z_L = \widehat{\mathcal{N} \circ \mathcal{E}}^\dagger(Z_{E'}) = T_L' + \Delta_L$ with $T_L' = T_L - \nu'\mathbb{1}_L$ and $\|\Delta_L\|_\infty \leqslant \delta'$, so this implies that

$$\text{(A.24)} \geqslant \max_{\|X_L\|_\infty \leqslant 1} \frac{1}{2\|Z_{E'}\|_\infty} \Big[\mathrm{tr}\big\{T_L'\phi_L^{1/2}X_L\phi_L^{1/2}\big\} - \mathrm{tr}\big\{T_L'\phi_L\,\mathrm{tr}(X_L\phi_L)\big\} - 2\delta'\Big]$$
$$= \frac{1}{2\|Z_{E'}\|_\infty}\Big[\big\|\phi_L^{1/2}\big(T_L' - \mathrm{tr}(T_L'\phi_L)\mathbb{1}_L\big)\phi_L^{1/2}\big\|_1 - 2\delta'\Big]$$
$$= \frac{1}{2\|Z_{E'}\|_\infty}\Big[\big\|\phi_L^{1/2}\big(T_L - \mathrm{tr}(T_L\phi_L)\mathbb{1}_L\big)\phi_L^{1/2}\big\|_1 - 2\delta'\Big] \; , \tag{A.25}$$

where in the last line we use the fact that $\mathrm{tr}(T_L'\phi_L) = \mathrm{tr}(T_L\phi_L) - \nu'$. This proves (A.15).

Now, following Bény and Oreshkov [25], we have the duality also for a fixed input state, and

there exists a state $\zeta_{E'}$ such that[3]

$$F_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E}, \mathrm{id}) = F(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'} \otimes \phi_R) , \tag{A.26}$$

and thus

$$\epsilon_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E}) = P(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'} \otimes \phi_R) , \tag{A.27}$$

where $P(\sigma, \rho) = \sqrt{1 - F^2(\sigma, \rho)}$ denotes the "purified distance" or "root infidelity" between the two states [82, 83]. Now, using known inequalities between this distance measure and the trace distance [82], we have

$$\epsilon_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E}) = P\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'} \otimes \phi_R\big) \geqslant \delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'} \otimes \phi_R\big) , \tag{A.28}$$

which in combination with (A.14) proves (A.12a). The first part of (A.12a) trivially follows from the fact that $\epsilon_{\mathrm{worst}}(\cdot) = \max_{|\phi\rangle} \epsilon_{|\phi\rangle}(\cdot)$.

From (A.27), and using the fact that the purified distance cannot increase under partial trace, we find with $\rho_{E'} = \widehat{\mathcal{N} \circ \mathcal{E}}(\phi_L)$,

$$P(\rho_{E'}, \zeta_{E'}) \leqslant \epsilon_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E}) . \tag{A.29}$$

By triangle inequality, and using again the known inequality between trace distance and purified distance, we obtain

$$\begin{aligned}
\delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \rho_{E'} \otimes \phi_R\big) &\leqslant P\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \rho_{E'} \otimes \phi_R\big) \\
&\leqslant P\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'} \otimes \phi_R\big) + P\big(\zeta_{E'} \otimes \phi_R, \rho_{E'} \otimes \phi_R\big) \\
&\leqslant 2\,\epsilon_{|\phi\rangle}(\mathcal{N} \circ \mathcal{E}) .
\end{aligned} \tag{A.30}$$

Combining this with (A.15) proves (A.12b).

Now we further assume that $\widehat{\mathcal{N} \circ \mathcal{E}} = \sum q_\alpha \widehat{\mathcal{N}_\alpha \circ \mathcal{E}}$ for some set of α's and a probability distribution $\{q_\alpha\}$. Then as above, invoking Bény and Oreshkov for each α with corresponding optimal states $\zeta_{E'}^\alpha$, we have

$$\begin{aligned}
\sum q_\alpha \epsilon_\phi(\mathcal{N}_\alpha \circ \mathcal{E}) &= \sum q_\alpha P\big(\widehat{\mathcal{N}_\alpha \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'}^\alpha \otimes \phi_R\big) \geqslant \sum q_\alpha \delta\big(\widehat{\mathcal{N}_\alpha \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'}^\alpha \otimes \phi_R\big) \\
&\geqslant \delta\Big(\sum q_\alpha \widehat{\mathcal{N}_\alpha \circ \mathcal{E}}(\phi_{LR}), \sum q_\alpha \zeta_{E'}^\alpha \otimes \phi_R\Big) = \delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta_{E'}' \otimes \phi_R\big) , \quad (\text{A.31})
\end{aligned}$$

using the joint convexity of the trace distance and defining $\zeta_{E'}' = \sum q_\alpha \zeta_{E'}^\alpha$. Directly invoking (A.14) then proves the first bound in (A.13). We also have (A.31) $\geqslant \delta(\rho_{E'}, \zeta_{E'}')$, and hence by triangle inequality

$$\delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \rho_{E'} \otimes \phi_R\big) \leqslant 2 \sum q_\alpha \epsilon_\phi(\mathcal{N}_\alpha \circ \mathcal{E}) . \tag{A.32}$$

Combining with (A.15) then yields the second bound in (A.13). ∎

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[3] The statement with fixed input state is only briefly stated towards the end of their paper, as that claim is in fact easier to prove than their main theorem for the worst-case entanglement fidelity.

We may now combine these two lemmas to finally prove Theorem 2.

Proof of Theorem 2. Thanks to Lemma 6 there exists $Z_{C'E}$ and $\nu' \in \mathbb{R}$ such that

$$\left\|\widehat{\mathcal{N} \circ \mathcal{E}}^\dagger(Z_{C'E}) - (T_L - \nu'\mathbb{1}_L)\right\|_\infty \leqslant \delta + \eta \; ; \tag{A.33a}$$

$$\|Z_{C'E}\|_\infty \leqslant \max_\alpha \frac{\Delta T_\alpha}{2q_\alpha} \; . \tag{A.33b}$$

We may directly plug this observable into Lemma 7 to deduce that the bound (A.12a) applies to our approximately covariant code. We now need to compute the form of the bound for the particular quantities $\epsilon_\mathrm{e}(\mathcal{N} \circ \mathcal{E})$, $\langle\epsilon_\mathrm{e}(\mathcal{N}^\alpha \circ \mathcal{E})\rangle_\alpha$ and $\epsilon_\mathrm{worst}(\mathcal{N} \circ \mathcal{E})$.

First, let $|\phi\rangle_{LR} = |\hat\phi\rangle_{LR}$ be the maximally entangled state between L and $R \simeq L$. Then by definition, and recalling the alternative expression in (A.21) for C_{ϕ_L, T_L} with a maximization, we have

$$C_{\phi_L, T_L} = C_{\mathbb{1}_L/d_L, T_L} = \frac{1}{d_L} \min_\mu \|T_L - \mu\mathbb{1}_L\|_1 = \frac{1}{d_L} \max_{\substack{\|X\|_\infty \leqslant 1 \\ \mathrm{tr}(X)=0}} \mathrm{tr}\big[T_L X\big] \; . \tag{A.34}$$

Let $\mu(T_L)$ denote a median eigenvalue of T_L counted with multiplicity, which implies the following. Let $\{|k\rangle_L\}$ for $k = 1, \ldots, d_L$ be an eigenbasis of T_L with its elements arranged such that the eigenvalues of T_L are nonincreasing in k, $\langle 1|T_L|1\rangle \geqslant \langle 2|T_L|2\rangle \geqslant \cdots \geqslant \langle d_L|T_L|d_L\rangle$. Let

$$P_+ = \sum_{k=1}^{\lfloor d_L/2 \rfloor} |k\rangle\langle k|_L \; ; \qquad\qquad P_- = \sum_{k=\lceil d_L/2 \rceil+1}^{d_L} |k\rangle\langle k|_L \; , \tag{A.35}$$

noting that the two projectors are orthogonal and that $\mathrm{rank}(P_+) = \mathrm{tr}(P_+) = \mathrm{tr}(P_-) = \mathrm{rank}(P_-)$. That is, we divide all basis vectors into two sets of equal size, corresponding to the smallest eigenvalues and the largest eigenvalues respectively, possibly leaving out the middle basis vector if the space dimension is odd. Then, the eigenvalues corresponding to the eigenbasis vectors included in P_+ (respectively, P_-) are all greater than or equal to (respectively less than or equal to) $\mu(T_L)$. If d_L is odd, then the basis vector that was left out corresponds to the eigenvalue $\mu(T_L)$.

Now set $X = P_+ - P_-$, satisfying $\|X\|_\infty \leqslant 1$. We have $\|T_L - \mu(T_L)\mathbb{1}\|_1 = \mathrm{tr}\big[X(T_L - \mu(T_L)\mathbb{1})\big]$: Indeed, the one-norm is equal to the sum of the absolute values of the eigenvalues of its argument, which is precisely taken care of by our careful choice of X. Then $\|T_L - \mu(T_L)\mathbb{1}\|_1 = \mathrm{tr}\big(XT_L\big) - \mu(T_L)\,\mathrm{tr}(X) = \mathrm{tr}\big(XT_L\big)$ because $\mathrm{tr}(X) = 0$ by construction. Now because both $\mu(T_L)$ and X are optimization candidates in (A.34), we have

$$\frac{1}{d_L}\left\|T_L - \mu(T_L)\mathbb{1}\right\|_1 \geqslant C_{\mathbb{1}/d_L, T_L} \geqslant \frac{1}{d_L}\,\mathrm{tr}(XT_L) = \frac{1}{d_L}\left\|T_L - \mu(T_L)\mathbb{1}\right\|_1 \; , \tag{A.36}$$

which implies that $C_{\mathbb{1}/d_L, T_L} = d_L^{-1}\left\|T_L - \mu(T_L)\mathbb{1}\right\|_1$. Lemma 7 states that $\big[C_{\mathbb{1}/d_L, T_L} - (\delta + \eta)\big]/(2\|Z_{C'E}\|_\infty)$ is a lower bound both to $\epsilon_\mathrm{e}(\mathcal{N} \circ \mathcal{E})$ and to $\langle\epsilon_\mathrm{e}(\mathcal{N}^\alpha \circ \mathcal{E})\rangle_\alpha$, which proves (29a) as we recall the property (A.33b).

That the norm term in (29a) can be replaced by $\|T_L - \mathrm{tr}(T_L)\mathbb{1}_L/d_L\|_1/(2d_L)$ follows from the alternative bound in Lemma 7, stating that $\big[(C^0_{\mathbb{1}/d_L, T_L}/2) - (\delta + \eta)\big]/(2\|Z_{C'E}\|_\infty)$

[cf. (A.10a)] is also a lower bound to both error parameters considered in (29a).

For $\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E})$, we get to pick $|\phi\rangle_{LR}$ freely and this will yield a valid bound. Let $|\psi^{\pm}\rangle_L$ be eigenstates of T_L corresponding to the maximal and minimal eigenvalues T_L, respectively, with $\langle\psi^+|T_L|\psi^+\rangle - \langle\psi^-|T_L|\psi^-\rangle = \Delta T_L$. Now choose two arbitrary orthogonal states $|\pm\rangle_R$ on R and set

$$|\phi\rangle_{LR} = \frac{1}{\sqrt{2}}\big[|\psi^+\rangle_L|+\rangle_R + |\psi^-\rangle_L|-\rangle_R\big] \,, \tag{A.37}$$

with $\phi_L = \Pi_L/2$, where we write $\Pi_L = |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|$. Recall the alternative expression in (A.21) for C_{ϕ_L,T_L} with a maximization. We can choose as candidate $X_L = |\psi^+\rangle\langle\psi^+|_L - |\psi^-\rangle\langle\psi^-|_L$, since we have indeed $\text{tr}(\phi_L X_L) = 0$ and $\|X_L\|_\infty \leqslant 1$, and we obtain

$$C_{\phi_L,T_L} \geqslant \frac{1}{2}\,\text{tr}\big(\Pi_L X_L \Pi_L T_L\big) = \frac{\Delta T_L}{2} \,. \tag{A.38}$$

Lemma 7 then asserts that

$$\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}) \geqslant \frac{\Delta T_L/2 - \delta - \eta}{\max_\alpha \Delta T_\alpha/q_\alpha} \,, \tag{A.39}$$

where we recall (A.33b). This proves (29b). ∎

At this point we comment on Condition (28) in the statement of Theorem 2. It may look a bit awkward, but its meaning is intuitively simple: First, we need to shift the charge values to center them at zero for each α for our proof. Second, we need to make sure that if we project any codeword into the given range of physical charge values for each α, then the total error we make when attempting to determine the expectation value of the actual (possibly unbounded) charge observable T_A is small. In practice, this just means that the part of the codewords outside of the given range of charge values only has a small contribution to the total expectation value of charge. For convenience we may use the following simplified criterion, where we simply fix a charge cut-off value t:

Proposition 8. *Consider $V_{L\to A}$, T_L, K and $T_A = \sum_{\alpha\in K} T_\alpha$ as in Theorem 2. Let $t > 0$. Set $t_\alpha^+ = -t_\alpha^- = t$ and define $\Pi_\alpha, \Pi_\alpha^\perp$ as in the statement of Theorem 2. Let $\{|\phi_\alpha^{t',j}\rangle\}$ be an eigenbasis of T_α corresponding to eigenvalues t' with a possible degeneracy index j. Suppose that there is an $\eta' \geqslant 0$ such that for any logical state ψ_L and for any α,*

$$\sum_{t',j:\,|t'|>t} |t'|\,\langle\phi_\alpha^{t',j}|\rho_\alpha|\phi_\alpha^{t',j}\rangle \leqslant \eta' \,, \tag{A.40}$$

where we write $\rho_\alpha = \text{tr}_{A\backslash A_\alpha}(V\psi_L V^\dagger)$ and where the sum ranges over the eigenstate labels (t',j) such that $|t'| > t$. Then, condition (28) is satisfied with $\eta = |K|\eta'$, and furthermore $\Delta T_\alpha = 2t$ for all α.

Proof of Proposition 8. We have $t_\alpha = (t_\alpha^+ + t_\alpha^-)/2 = 0$. For any ψ_L, calculate

$$\left|\sum \text{tr}\big(\Pi_\alpha^\perp T_\alpha V\psi_L V^\dagger\big)\right| \leqslant \sum |\text{tr}\big(\Pi_\alpha^\perp T_\alpha V\psi_L V^\dagger\big)|$$

$$\leqslant \sum_\alpha \left|\sum_{t',j:\,|t'|>t} t'\,\langle\phi_\alpha^{t',j}|\text{tr}_{A\backslash A_\alpha}(V\psi_L V^\dagger)|\phi_\alpha^{t',j}\rangle\right|$$
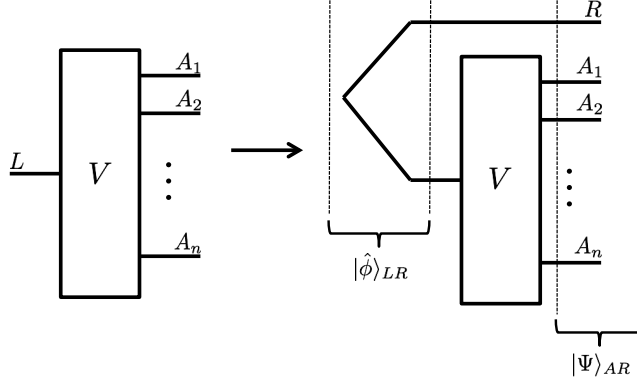
FIG. 7: Depiction of the construction of the state $|\Psi\rangle_{AR}$ by injecting the maximally entangled state $|\hat{\phi}\rangle_{LR}$ into the encoding isometry $V_{L\to A}$.

$$\leqslant \sum_\alpha \sum_{t',j:\ |t'|>t} |t'|\,\langle\phi_\alpha^{t',j}|\,\mathrm{tr}_{A\backslash A_\alpha}(V\psi_L V^\dagger)|\phi_\alpha^{t',j}\rangle$$

$$\leqslant \sum_\alpha \eta' \leqslant |K|\,\eta' \ . \tag{A.41}$$

Note by the way that the left hand side of (A.40) is exactly $\mathrm{tr}(\Pi_\alpha^\perp\,|T_\alpha|V\psi_L V^\dagger)$. ∎

Appendix B: Correlation functions and bounds

In this section we present an alternative strategy for proving the bound (27), by studying the connected correlation functions between the physical subsystems and the logical information.

The covariance of the codes can be seen as a linear constraint, which can be easily employed to obtain a second order constraints. To start, we again assume the simpler case of isometric encoding. We construct the state corresponding to the encoding isometry $V_{L\to A}$ by injecting a maximally entangled state $|\hat{\phi}\rangle_{LR}$ to $V_{L\to A}$ (Fig. 7):

$$|\Psi\rangle_{AR} = V|\hat{\phi}\rangle_{LR} \ . \tag{B.1}$$

We have $T_A|\Psi\rangle_{LA} = T_A V|\hat{\phi}\rangle_{LR} = V(T_L - \nu\mathbb{1}_L)|\hat{\phi}\rangle_{LR}$ for some constant ν. Define $T_R = (T_L - \nu\mathbb{1}_L)^T$ where the transpose is taken as a matrix ignoring the Hilbert space label; this ensures that $(T_L - \nu\mathbb{1}_L)|\hat{\phi}\rangle_{LR} = T_R|\hat{\phi}\rangle_{LR}$. Therefore, the covariance of V translates to the invariance of $|\Psi\rangle$:

$$\left(\sum_{i=1}^n T_{A_i}\right)|\Psi\rangle_{RA} = T_A|\Psi\rangle_{RA} = T_R|\Psi\rangle_{RA} \ . \tag{B.2}$$

We define the *connected correlator* between two operators A, B as

$$\langle A, B \rangle := \mathrm{tr}\,(AB\Psi) - \mathrm{tr}\,(A\Psi)\,\mathrm{tr}\,(B\Psi) \ . \tag{B.3}$$

Consider an arbitrary operator X_R. It be seen from (B.2) that

$$\langle X_R, T_R \rangle = \sum_{i=1}^{n} \langle X_R, T_{A_i} \rangle \ .$$

Using the triangle inequality, we obtain

$$|\langle X_R, T_R \rangle| \leqslant \sum_{i=1}^{n} |\langle X_R, T_{A_i} \rangle| \qquad \text{for all } X_R. \tag{B.4}$$

Although the derivation of (B.4) is very simple, it provides a general lower bound to the amount of correlations between the reference system and the physical subsystems, from which we can draw physical consequences. The correlation functions measure how close the state Ψ_{RA_i} is to the product state $\Psi_R \otimes \Psi_{A_i}$:

$$\begin{aligned}
|\langle X_R, T_{A_i} \rangle| &= |\mathrm{tr}[X_R T_{A_i}(\Psi_{RA_i} - \Psi_R \otimes \Psi_{A_i})]| \\
&\leqslant \|X_R\|_\infty \|T_{A_i}\|_\infty \|\Psi_{RA_i} - \Psi_R \otimes \Psi_{A_i}\|_1 \ ,
\end{aligned} \tag{B.5}$$

where we used Hölder's inequality. We can replace $T_{A_i} \to T_{A_i} - t\mathbb{1}$ in (B.5) without changing the left hand side of the inequality as $\langle X_R, T_{A_i} - t\mathbb{1} \rangle = \langle X_R, T_{A_i} \rangle$:

$$\begin{aligned}
|\langle X_R, T_{A_i} \rangle| &\leqslant \|X_R\|_\infty \|T_{A_i} - t\mathbb{1}\|_\infty \|\Psi_{RA_i} - \Psi_R \otimes \Psi_{A_i}\|_1 \\
&= \frac{1}{2}\|X_R\|_\infty \Delta T_{A_i} \|\Psi_{RA_i} - \Psi_R \otimes \Psi_{A_i}\|_1 \ ,
\end{aligned} \tag{B.6}$$

where the second line follows by a suitable choice of t, and where ΔT_{A_i} is the difference between the maximal and minimal eigenvalue of T_{A_i}.

The accuracy to which the code V can correct against errors is precisely determined by how close Ψ_{RA_i} is to a product state. Indeed, consider the noise channel $\mathcal{N}_{A \to A}^i$ in (11b) that erases the system A_i. By Bény and Oreshkov (12a), we have

$$\begin{aligned}
\epsilon_{\mathrm{e}}(\mathcal{N}^i \circ \mathcal{E}) &= \min_\zeta \sqrt{1 - F^2\big(\widehat{\mathcal{N}^i \circ \mathcal{E}}(\hat{\phi}_{LR}), \zeta \otimes \Psi_R\big)} \\
&\geqslant \min_\zeta \frac{1}{2}\big\|\widehat{\mathcal{N}^i \circ \mathcal{E}}(\hat{\phi}_{LR}) - \zeta \otimes \Psi_R\big\|_1 \\
&= \min_\zeta \frac{1}{2}\big\|\Psi_{RA_i} - \zeta_{A_i} \otimes \Psi_R\big\|_1
\end{aligned} \tag{B.7}$$

where $\widehat{\mathcal{N}^i \circ \mathcal{E}}(\hat{\phi}_{LR}) = \Psi_{RA_i}$ and $\Psi_R = \mathbb{1}_R/d_L$, and where we have used the known relation $\delta(\cdot, \cdot) \leqslant \sqrt{1 - F^2(\cdot, \cdot)}$ between the trace distance and the fidelity. Because the trace distance cannot increase under the partial trace, and if we set ζ_{A_i} to be the optimal state in the expression above, also have $(1/2)\|\Psi_{A_i} - \zeta_{A_i}\|_1 \leqslant \epsilon_{\mathrm{e}}(\mathcal{N}^i \circ \mathcal{E})$ and thus by triangle inequality,

$$\frac{1}{2}\big\|\Psi_{RA_i} - \Psi_{A_i} \otimes \Psi_R\big\|_1 \leqslant 2\epsilon_{\mathrm{e}}(\mathcal{N}^i \circ \mathcal{E}) \ . \tag{B.8}$$

It remains to combine (B.8) with (B.5) and (B.4) and to choose the best possible X_R to get our final result.

Theorem 9. *The individual entanglement fidelities of recovery of a covariant code $\mathcal{E}(\cdot) = V(\cdot)V^\dagger$ against single erasures at known locations satisfy the following inequality:*

$$\frac{1}{2d_L}\left\|T_L - \mathrm{tr}(T_L)\frac{\mathbb{1}}{d_L}\right\|_1 \leqslant \sum_{i=1}^n \Delta T_i \, \epsilon_{\mathrm{e}}(\mathcal{N}^i \circ \mathcal{E}) \; . \tag{B.9}$$

Furthermore, this can be used to show that

$$\epsilon_{\mathrm{e}}(\mathcal{N} \circ \mathcal{E}) \geqslant \frac{1}{2d_L}\frac{\|T_L - \mathrm{tr}(T_L)\mathbb{1}/d_L\|_1}{\max_i q_i^{-1}\Delta T_i} \; . \tag{B.10}$$

Note that $T_L - \mathrm{tr}(T_L)\mathbb{1}/d_L$ is just a shift of T_L by a multiple of identity to make it traceless. Therefore, $\|T_L - \mathrm{tr}(T_L)\mathbb{1}/d_L\|_1$ is a 1-norm measure for the spread of eigenvalues of T_L. The bounds of Theorem 9 and Eq. (27) have a very similar nature.

Proof of Theorem 9.   We start with the correlator in the left hand side of (B.4):

$$\langle X_R, T_R\rangle = \mathrm{tr}(X_R T_R \Psi_R) - \mathrm{tr}(X_R \Psi_R)\,\mathrm{tr}(T_R \Psi_R) = \frac{1}{d_L}\,\mathrm{tr}\left(X_R\left[T_R - \frac{\mathrm{tr}(T_R)}{d_L}\mathbb{1}\right]\right) \; . \tag{B.11}$$

Now, choose the optimal X_R such that $\|X_R\|_\infty \leqslant 1$ and that $\|T_R - \mathrm{tr}(T_R)\mathbb{1}/d_L\|_1 = \mathrm{tr}[X_R(T_R - \mathrm{tr}(T_R)\mathbb{1}/d_L)]$. Plugging into (B.4), and combining with (B.6) and (B.8), immediately gives (B.9).

Furthermore from (B.9) we have

$$\frac{1}{d_L}\left\|T_L - \mathrm{tr}(T_L)\frac{\mathbb{1}}{d_L}\right\|_1 \leqslant \sum (q_i^{-1}\Delta T_i)(q_i \epsilon_{\mathrm{e}}(\mathcal{N}^i \circ \mathcal{E}))$$

$$\leqslant \left(\max_i\,(q_i^{-1}\Delta T_i)\right)\sum q_i \epsilon_{\mathrm{e}}(\mathcal{N}^i \circ \mathcal{E}) \; . \tag{B.12}$$

By convexity of $x \mapsto x^2$, and by Lemma 24, we have

$$\sum q_i \epsilon_{\mathrm{e}}(\mathcal{N}^i \circ \mathcal{E}) \leqslant \sqrt{\sum q_i \epsilon_{\mathrm{e}}^2(\mathcal{N}^i \circ \mathcal{E})} = \epsilon_{\mathrm{e}}(\mathcal{N} \circ \mathcal{E}) \; . \tag{B.13}$$

Combining (B.13) with (B.12) proves (B.10). ∎

Appendix C: Criterion for approximate codes

When we come up with a new code, how can we show that it forms an ϵ-approximate error-correcting code against erasures at known locations? Here we provide a criterion that, when it can be applied, certifies that a given code performs well.

Let L be the logical space and A be the physical space, and consider an encoding operation $\mathcal{E}_{L \to A}$ that can be any completely positive, trace-preserving map. Note that in the case of a more general noise model, A does not necessarily have to be composed of several subsystems.

Consider a collection of noise channels $\{\mathcal{N}^\alpha\}$ and probabilities $\{q_\alpha\}$. We assume that the environment applies a random noise channel from this set with the corresponding probability, while providing a record of which noise channel was applied in a separate register C. The overall noise channel that is applied by the environment is then

$$\mathcal{N}_{A \to AC}(\cdot) = \sum q_\alpha |\alpha\rangle\langle\alpha|_C \otimes \mathcal{N}^\alpha_{A \to A}(\cdot) \ . \tag{C.1}$$

Given complementary channels $\widehat{\mathcal{N}^\alpha \circ \mathcal{E}}$ of $\mathcal{N}^\alpha \circ \mathcal{E}$, we can construct a complementary channel of $\mathcal{N} \circ \mathcal{E}$ as

$$\widehat{\mathcal{N} \circ \mathcal{E}}_{A \to C'E}(\cdot) = \sum q_\alpha |\alpha\rangle\langle\alpha|_{C'} \otimes \widehat{\mathcal{N}^\alpha \circ \mathcal{E}}(\cdot) \ , \tag{C.2}$$

with an additional register C' and where the outputs of the individual complementary channels for each α are embedded into a system E.

We fix any basis $\{|x\rangle_L\}$ of L, and we define for each α the operators

$$\rho^{x,x'}_\alpha = \widehat{\mathcal{N}^\alpha \circ \mathcal{E}}(|x\rangle\langle x'|_L) \ . \tag{C.3}$$

Note that $\rho^{x,x}_\alpha$ is a quantum state for each α and for each x, but that $\rho^{x,x'}_\alpha$ is not necessarily even Hermitian for $x \neq x'$.

For an isometric encoding $\mathcal{E}$, and in the noise $\mathcal{N}$ acts by erasing a collection of subsystems labeled by α and chosen with probability q_α, the operators $\rho^{x,x'}_\alpha$ are simply the reduced operators on the sites labeled by α of the logical operator $|x\rangle\langle x'|$:

$$\rho^{x,x'}_\alpha = \operatorname{tr}_{A \setminus A_\alpha}(\mathcal{E}(|x\rangle\langle x'|)) \ . \tag{C.4}$$

Proposition 10. *Assume that there exists $\nu, \epsilon' \geqslant 0$, and that there exists a quantum state ζ_α for each α, such that for all α,*

$$F(\rho^{x,x}_\alpha, \zeta_\alpha) \geqslant \sqrt{1 - \epsilon'^2} \quad \text{for all } x \ ; \quad \text{and} \tag{C.5a}$$

$$\|\rho^{x,x'}_\alpha\|_1 \leqslant \nu \qquad \text{for all } x \neq x' \ . \tag{C.5b}$$

Then $\mathcal{E}_{L \to A}$ is an approximate error-correcting code against the noise $\mathcal{N}$, with approximation parameter

$$\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}) \leqslant \epsilon' + d_L \sqrt{\nu} \ , \tag{C.6}$$

where d_L is the dimension of the logical system L.

Proof of Proposition 10. Let

$$\zeta_{C'E} = \sum_\alpha q_\alpha |\alpha\rangle\langle\alpha|_{C'} \otimes \zeta_\alpha \ . \tag{C.7}$$

Using the Bény-Oreshkov property (12), the proof strategy is to find a lower bound to the entanglement fidelity of the channel $\widehat{\mathcal{N} \circ \mathcal{E}}$ to the constant channel $\mathcal{T}_\zeta$ outputting the state $\zeta_{C'E}$ defined above.

Consider a reference system $R \simeq L$, and let $\{|x\rangle_R\}$ be any fixed basis of R. Let $|\Phi\rangle_{L:R} = \sum_x |x\rangle_L \otimes |x\rangle_R$. For any state $|\sigma\rangle_{LR}$, there exists a complex matrix B_R such that $|\sigma\rangle_{LR} =$

$B_R |\Phi\rangle_{L:R}$ and $\sigma_R = \mathrm{tr}_L(\sigma_{LR}) = B_R B_R^\dagger$ (choose $B_R = \sum_{x,x'} \langle x, x' | \sigma \rangle_{LR} |x'\rangle\langle x|_R$). Note that $\|B_R B_R^\dagger\|_\infty = \|B_R^\dagger B_R\|_\infty \leqslant 1$. We have

$$
\begin{aligned}
(\widehat{\mathcal{N} \circ \mathcal{E}} \otimes \mathrm{id}_R)(\sigma_{LR}) &= B_R \, \widehat{\mathcal{N} \circ \mathcal{E}}(\Phi_{L:R}) \, B_R^\dagger \\
&= \sum_\alpha q_\alpha |\alpha\rangle\langle\alpha|_{C'} \otimes (B_R \, \widehat{\mathcal{N}^\alpha \circ \mathcal{E}}(\Phi_{L:R}) \, B_R^\dagger) \\
&= \sum_{\alpha, x, x'} q_\alpha \, |\alpha\rangle\langle\alpha|_{C'} \otimes (B_R \, \rho_{ER}^\alpha \, B_R^\dagger) \,,
\end{aligned}
\tag{C.8}
$$

where we have defined for each α the positive semidefinite operator

$$
\rho_{ER}^\alpha = \widehat{\mathcal{N}^\alpha \circ \mathcal{E}}(\Phi_{L:R}) = \sum_{x,x'} \rho_\alpha^{x,x'} \otimes |x\rangle\langle x'|_R \,.
\tag{C.9}
$$

While the ρ_{ER}^α's are positive semidefinite, they are not normalized to unit trace as proper quantum states. Recalling that the fidelity is jointly concave, we have

$$
\begin{aligned}
F\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_{LR}), \zeta_{C'E} \otimes \sigma_R\big) &= F\Big(\sum_\alpha q_\alpha |\alpha\rangle\langle\alpha|_{C'} \otimes (B_R \rho_{ER}^\alpha B_R^\dagger), \sum_\alpha q_\alpha |\alpha\rangle\langle\alpha|_{C'} \otimes \zeta_\alpha \otimes \sigma_R\Big) \\
&\geqslant \sum_\alpha q_\alpha \, F\big(B_R \rho_{BR}^\alpha B_R^\dagger, \zeta_\alpha \otimes \sigma_R\big) \,.
\end{aligned}
\tag{C.10}
$$

At this point, we define for each α the positive semidefinite operator

$$
\tilde{\rho}_{ER}^\alpha = \sum_x \rho_\alpha^{x,x} \otimes |x\rangle\langle x|_R \,.
\tag{C.11}
$$

Note that $B_R \tilde{\rho}_{ER}^\alpha B_R^\dagger$ is a quantum state, because $\mathrm{tr}(B_R \tilde{\rho}_{ER}^\alpha B_R^\dagger) = \sum_x \mathrm{tr}(B_R |x\rangle\langle x| B_R^\dagger) = \mathrm{tr}(B_R B_R^\dagger) = 1$. In fact, the quantum states $B_R \tilde{\rho}_{ER}^\alpha B_R^\dagger$ and $B_R \rho_{ER}^\alpha B_R^\dagger$ are close in trace distance:

$$
\begin{aligned}
\big\| B_R (\rho_{ER}^\alpha - \tilde{\rho}_{ER}^\alpha) B_R^\dagger \big\|_1 &= \Big\| B_R \Big(\sum_{x \neq x'} \rho_\alpha^{x,x'} \otimes |x\rangle\langle x'| \Big) B_R^\dagger \Big\|_1 \\
&= \Big\| \sum_{x \neq x'} \rho_\alpha^{x,x'} \otimes (B_R |x\rangle\langle x'| B_R^\dagger) \Big\|_1 \\
&\leqslant \sum_{x \neq x'} \big\| \rho_\alpha^{x,x'} \big\|_1 \cdot \big\| B_R |x\rangle\langle x'| B_R^\dagger \big\|_1 \\
&\leqslant \sum_{x \neq x'} \big\| \rho_\alpha^{x,x'} \big\|_1 \leqslant d_L^2 \, \nu \,,
\end{aligned}
\tag{C.12}
$$

using our assumption (C.5b), and noting that $\|B_R |x\rangle\langle x'| B_R^\dagger\|_1 \leqslant \|B_R|x\rangle\|_1 \|\langle x'|B_R^\dagger\|_1 = \|\langle x|B_R^\dagger\|_1 \|\langle x'|B_R^\dagger\|_1 \leqslant 1$ because $\mathrm{tr}\sqrt{\langle x|B_R^\dagger B_R|x\rangle} \leqslant 1$. Recalling the relation $P(\cdot,\cdot) \leqslant \sqrt{2\delta(\cdot,\cdot)} = \sqrt{\|(\cdot) - (\cdot)\|_1}$ between the purified distance $P(\cdot,\cdot) = \sqrt{1 - F^2(\cdot,\cdot)}$ and the trace distance, we have

$$
P(B_R \, \rho_{ER}^\alpha \, B_R^\dagger, B_R \, \tilde{\rho}_{ER}^\alpha \, B_R^\dagger) \leqslant d_L \sqrt{\nu} \,.
\tag{C.13}
$$

On the other hand, using again the joint concavity of the fidelity, we have

$$
F\big(B_R \, \tilde{\rho}_{ER}^\alpha \, B_R^\dagger, \zeta_\alpha \otimes \sigma_R\big) = F\Big(\sum_x \rho_\alpha^{x;x} \otimes (B_R |x\rangle\langle x| B_R^\dagger), \sum_x \zeta_\alpha \otimes (B_R |x\rangle\langle x| B_R^\dagger)\Big)
$$

$$\geqslant \sum_x \langle x| B^\dagger B |x\rangle_R \, F\left(\rho_\alpha^{x,x} \otimes \frac{B_R |x\rangle\langle x| B_R^\dagger}{\langle x| B^\dagger B |x\rangle_R}, \zeta_\alpha \otimes \frac{B_R |x\rangle\langle x| B_R^\dagger}{\langle x| B^\dagger B |x\rangle_R}\right)$$

$$= \sum_x \langle x| B^\dagger B |x\rangle_R \, F(\rho_\alpha^{x,x}, \zeta_\alpha)$$

$$\geqslant \sum_x \langle x| B^\dagger B |x\rangle_R \, \sqrt{1 - \epsilon'^2}$$

$$\geqslant \sqrt{1 - \epsilon'^2} \,, \tag{C.14}$$

recalling our assumption (C.5a) and using the fact that $\mathrm{tr}(B^\dagger B) = \mathrm{tr}(BB^\dagger) = 1$; hence

$$P\big(B_R \, \tilde\rho_{ER}^\alpha \, B_R^\dagger, \zeta_\alpha \otimes \sigma_R\big) \leqslant \epsilon' \,. \tag{C.15}$$

By triangle inequality for the purified distance, we have

$$P\big(B_R \rho_{BR}^\alpha B_R^\dagger, \zeta_\alpha \otimes \sigma_R\big) \leqslant P\big(B_R \, \rho_{ER}^\alpha \, B_R^\dagger, B_R \, \tilde\rho_{ER}^\alpha \, B_R^\dagger\big) + P\big(B_R \, \tilde\rho_{ER}^\alpha \, B_R^\dagger, \zeta_\alpha \otimes \sigma_R\big)$$

$$\leqslant d_L \sqrt{\nu} + \epsilon' \,. \tag{C.16}$$

Returning to (C.10), we now have $F\big(B_R \rho_{BR}^\alpha B_R^\dagger, \zeta_\alpha \otimes \sigma_R\big) \geqslant \sqrt{1 - (d_L \sqrt{\nu} + \epsilon')^2}$ and hence

$$F\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_{LR}), \zeta_{C'E} \otimes \sigma_R\big) \geqslant \sqrt{1 - (d_L \sqrt{\nu} + \epsilon')^2} \,. \tag{C.17}$$

As this holds for any $|\sigma\rangle_{LR}$, we deduce that

$$f(\mathcal{N} \circ \mathcal{E}) \geqslant \sqrt{1 - (d_L \sqrt{\nu} + \epsilon')^2} \,, \tag{C.18}$$

which implies

$$\epsilon(\mathcal{N} \circ \mathcal{E}) \leqslant d_L \sqrt{\nu} + \epsilon' \,. \qquad \blacksquare$$

Appendix D: Calculations for covariant code examples

D.1. Three-rotor secret-sharing code

Sharp cutoff. We complete the exposition in the main text in §VI A by calculating the approximation parameter $\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}^{(m)})$ of the constructed code.

The strategy is to apply Proposition 10. First write the operators (C.3) in our situation,

$$\rho_i^{x,x'} = \mathrm{tr}_{A \setminus A_i}(V |x\rangle\langle x'| V^\dagger) \,. \tag{D.1}$$

We need to show that $\rho_i^{x,x}$ is approximately constant of x and that $\rho_i^{x,x'}$ is very small for $x \neq x'$. The latter condition turns out to be simple: for any i and for any $x \neq x'$, we will see that $\rho_i^{x,x'} = 0$; hence we may take $\nu = 0$ in Proposition 10.

For each i, we would like to show that there exists a state ζ_i such that $\rho_i^{x,x}$ is close to ζ_i in fidelity distance for each x. We choose to work with the trace distance instead, and deduce that the states are close in fidelity using the relation $F(\cdot, \cdot) \geqslant \sqrt{1 - 2\delta(\cdot, \cdot)}$ between
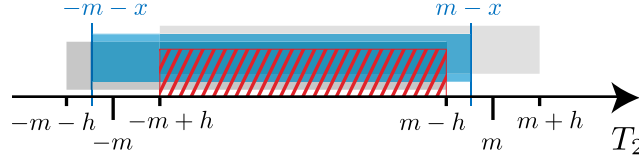
FIG. 8: Finding the "common minimal operator" for the different $\rho_2^{x,x}$'s. The solid rectangles illustrate the spectra of the different $\rho_2^{x,x}$. The eigenvalues are all equal and the rectangles are displaced vertically for readability. The hatched region corresponds to a good choice for τ_2.

the fidelity and the trace distance. We bound the trace distance as follows. For each i, we find a positive semidefinite operator τ_i with the property tht $\rho_i^{x,x} \geqslant \tau_i$ for all x. This implies that $\rho_i^{x,x} = \tau_i + \Delta_i^x$ for some positive semidefinite operators Δ_i^x with $\mathrm{tr}(\Delta_i^x) = 1 - \mathrm{tr}(\tau_i)$. Define $\zeta_i = \tau_i + \xi_i$, for any freely chosen $\xi_i \geqslant 0$ with $\mathrm{tr}(\xi_i) = 1 - \mathrm{tr}(\tau_i)$. Then, we have $\rho_i^{x,x} - \zeta_i = \Delta_i^x - \xi_i$, and $\delta(\rho_i^{x,x}, \zeta_i) = (1/2)\|\rho_i^{x,x} - \zeta_i\|_1 \leqslant (1/2)(\mathrm{tr}(\Delta_i^x) + \mathrm{tr}(\xi_i)) = 1 - \mathrm{tr}(\tau_i)$. To summarize: If we find, for each i, an operator $\tau_i \geqslant 0$ with $\rho_i^{x,x} \geqslant \tau_i$ for all x, then we can deduce that there are states ζ_i such that

$$F(\rho_i^{x,x}, \zeta_i) \geqslant \sqrt{1 - \epsilon'^2} \, , \tag{D.2}$$

where $\epsilon' = \min_i \sqrt{2(1 - \mathrm{tr}(\tau_i))}$.

We may calculate the corresponding operators $\rho_i^{x,x'}$, starting with $i = 1$:

$$
\begin{aligned}
\rho_1^{x,x'} &= \mathrm{tr}_{A\backslash A_1}(V|x\rangle\langle x'|V^\dagger) \\
&= \frac{1}{2m+1} \sum_{y,y'=-m}^{m} |-3y\rangle\langle-3y'|\, \delta_{y-x,y'-x'}\, \delta_{2(x+y),2(x'+y')} \\
&= \frac{\delta_{x,x'}}{2m+1} \sum_{y=-m}^{m} |-3y\rangle\langle-3y|_{A_1} \, ,
\end{aligned}
\tag{D.3}
$$

since the two Kronecker deltas force $x' = x$ and $y' = y$. Similarly, we have

$$\rho_2^{x,x'} = \frac{\delta_{x,x'}}{2m+1} \sum_{y=-m}^{m} |y-x\rangle\langle y-x|_{A_2} \tag{D.4}$$

$$\rho_3^{x,x'} = \frac{\delta_{x,x'}}{2m+1} \sum_{y=-m}^{m} |2(x+y)\rangle\langle 2(x+y)|_{A_3} \, . \tag{D.5}$$

First of all, for each of $i = 1, 2, 3$ we have that $\rho_i^{x,x'} = 0$ if $x \neq x'$. Then, we have that $\rho_1^{x,x}$ is already independent of x, so we may choose $\tau_1 = \rho_1^{1,1} = \rho_1^{x,x} \, \forall \, x$. Next, $\rho_2^{x,x}$ is diagonal, with constant diagonal elements $1/(2m+1)$ at states $-m-x, -m-x+1, \ldots, m-x$. We may thus choose

$$\tau_2 = \frac{1}{2m+1} \sum_{u=-m+h}^{m-h} |u\rangle\langle u| \, , \tag{D.6}$$

such that $\rho_2^{x,x} \geqslant \tau_2$ for all x (Fig. 8). Finally, $\rho_3^{x,x}$ is also diagonal with elements $1/(2m+1)$

at states $-2m + 2x, -2m + 2x + 2, \ldots, 2m + 2x$. Similarly we may choose

$$\tau_3 = \frac{1}{2m+1} \sum_{u=-m+h}^{m-h} |2u\rangle\langle 2u| \,, \tag{D.7}$$

which guarantees that $\rho_3^{x,x} \geqslant \tau_3$ for each x. We have

$$\begin{aligned}
\mathrm{tr}(\tau_1) &= 1 \,; \\
\mathrm{tr}(\tau_2) &= \frac{2(m-h)+1}{2m+1} = 1 - \frac{2h}{2m+1} \,; \\
\mathrm{tr}(\tau_3) &= 1 - \frac{2h}{2m+1} \,,
\end{aligned} \tag{D.8}$$

so we may set according to the above $\epsilon' = \sqrt{4h/(2m+1)}$. According to Proposition 10, the code $V_{L\to A}^{(m)}$ is an approximate quantum error-correcting code with

$$\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}^{(m)}) \leqslant \epsilon' \,. \tag{D.9}$$

We have $1/(2m+1) \approx 1/2m$, and to first order in h/m, we have

$$\epsilon(\mathcal{N} \circ \mathcal{E}^{(m)}) \lesssim \sqrt{2}\sqrt{\frac{h}{m}} \,. \tag{D.10}$$

So our codes become good in the limit $h/m \to 0$.

To compare with our bound (26), we choose $q_1 = q_2 = q_3 = 1/3$ and note that $\delta = 0$, $\eta = 0$, and $\Delta T_L = 2h$. Also, we have

$$\Delta T_1 = 2 \cdot 3m \,; \qquad \Delta T_2 = 2(m+h) \,; \qquad \Delta T_3 = 4(m+h) \,; \tag{D.11}$$

so, for $m \gg h$, we have $\max_i q_i^{-1} \Delta T_i \approx 18m$. Our bound then reads

$$\epsilon(\mathcal{N} \circ \mathcal{E}) \geqslant \frac{1}{2} \frac{\Delta T_L}{\max_i q_i^{-1} \Delta T_i} \approx \frac{1}{2} \frac{2h}{18m} = \frac{1}{18} \frac{h}{m} \,. \tag{D.12}$$

Smooth cutoff.   Again, we make use of Proposition 10. First, we compute the normalization factor as

$$c_w = \sum_{y=-\infty}^{\infty} e^{-\frac{y^2}{2w^2}} = \sum_{y=-\infty}^{\infty} \left(e^{-\frac{1}{2w^2}}\right)^{(y^2)} = \vartheta_3\left(0, e^{-\frac{1}{2w^2}}\right) \,, \tag{D.13}$$

where $\vartheta_3(z, q)$ is Jacobi's theta function.[4] A straightforward observation is that $c_w \geqslant 1$ (the term $y = 0$ in the sum is already equal to one).

---

[4] See DLMF: *http://dlmf.nist.gov/20*. Our notation follows DLMF's notation.

We need to determine the operators $\rho_{1,2,3}^{x,x'}$. We have

$$\rho_1^{x,x'} = \operatorname{tr}_{A \setminus A_1}(V|x\rangle\langle x'|V^\dagger)$$

$$= c_w^{-1} \sum_{y,y'=-\infty}^{\infty} e^{-\frac{y^2}{4w^2} - \frac{y'^2}{4w^2}} |-3y\rangle\langle -3y'| \, \delta_{y-x,y'-x'} \, \delta_{2(x+y),2(x'+y')}$$

$$= \frac{\delta_{x,x'}}{c_w} \sum_{y=-\infty}^{\infty} e^{-\frac{y^2}{2w^2}} |-3y\rangle\langle -3y|_{A_1} \, . \tag{D.14}$$

Similarly, for the second and third systems,

$$\rho_2^{x,x'} = \frac{\delta_{x,x'}}{c_w} \sum_{y=-\infty}^{\infty} e^{-\frac{(y+x)^2}{2w^2}} |y\rangle\langle y|_{A_2} \tag{D.15}$$

$$\rho_3^{x,x'} = \frac{\delta_{x,x'}}{c_w} \sum_{y=-\infty}^{\infty} e^{-\frac{(y-x)^2}{2w^2}} |2y\rangle\langle 2y|_{A_3} \, . \tag{D.16}$$

Hence, we have $\|\rho_1^{x,x'}\|_1 = \|\rho_2^{x,x'}\|_1 = \|\rho_3^{x,x'}\|_1 = 0$ for all $x \neq x'$, so the conditions (C.5b) are satisfied with $\nu = 0$.

Now we need to verify the conditions (C.5a). For the first system, $\rho_1^{x,x}$ doesn't depend on x, so choosing $\zeta_1 = \rho_1^{0,0}$ we have $P(\rho_1^{x,x}, \zeta_1) = 0$ for all x. For the second system, we choose $\zeta_2 = \rho_2^{0,0}$ and calculate

$$F(\rho_2^{x,x}, \zeta_2) = \sum_{y=-\infty}^{\infty} \sqrt{\frac{1}{c_w} e^{-\frac{(y+x)^2}{2w^2}}} \sqrt{\frac{1}{c_w} e^{-\frac{y^2}{2w^2}}} = \frac{1}{c_w} \sum_{y=-\infty}^{\infty} e^{-\frac{(y+x)^2+y^2}{4w^2}} \geqslant e^{-\frac{h^2}{8w^2}} \, , \tag{D.17}$$

where the calculation of the last inequality is carried out below in Lemma 11. Hence

$$P(\rho_2^{x,x}, \zeta_2) \leqslant \sqrt{1 - e^{-\frac{h^2}{4w^2}}} = \frac{h}{2w} \sqrt{1 + O\left(\left(\frac{h}{w}\right)^2\right)} = \frac{h}{2w} + O\left(\left(\frac{h}{w}\right)^3\right) \, . \tag{D.18}$$

Now, we look at the third system. Defining $\zeta_3 = \rho_3^{0,0}$, we have

$$F(\rho_3^{x,x}, \zeta_3) = \sum_{y=-\infty}^{\infty} \sqrt{\frac{1}{c_w} e^{-\frac{(y-x)^2}{2w^2}}} \sqrt{\frac{1}{c_w} e^{-\frac{y^2}{2w^2}}} = \frac{1}{c_w} \sum_{y=-\infty}^{\infty} e^{-\frac{(y-x)^2+y^2}{4w^2}} \geqslant e^{-\frac{h^2}{8w^2}} \, , \tag{D.19}$$

invoking again the calculation in Lemma 11. Hence

$$P(\rho_3^{x,x}, \zeta_3) \leqslant \sqrt{1 - e^{-\frac{h^2}{4w^2}}} = \frac{h}{2w} + O\left(\left(\frac{h}{w}\right)^3\right) \, . \tag{D.20}$$

We are now in position to apply our criterion. Proposition 10 tells us that

$$\epsilon(\mathcal{N} \circ \mathcal{E}^{(w)}) \leqslant \sqrt{1 - e^{-\frac{h^2}{4w^2}}} = \frac{h}{2w} + O\left(\left(\frac{h}{w}\right)^3\right) \, . \tag{D.21}$$

Hence, our code's performance scales as $(1/2)(h/w)$. For instance, it performs well in the limit $h/w \to 0$, for instance in the limit $w \to \infty$ with a constant h.

Let's now see how our bound applies to our code (we need the more general bound, because

we are dealing with infinite-dimensional systems with an unbounded charge observable). We need to cut off tails of the codeword states on the physical systems to make the range of charge values finite. Choose cut-offs $W_1, W_2, W_3 \geqslant 0$ for each physical system. We would like to compute an upper bound to $\sum \psi_x \psi_{x'}^* \, \mathrm{tr}(\Pi_i^\perp \rho_i^{x,x'}) = \sum |\psi_x|^2 \, \mathrm{tr}(\Pi_i^\perp \rho_i^{x,x})$, where $\Pi_i^\perp$ projects outside of the cut-off region. We have

$$\mathrm{tr}(\Pi_1^\perp \rho_1^{x,x}) = c_w^{-1} \sum_{|3y|>W_1} e^{-\frac{y^2}{2w^2}} \leqslant c_w^{-1} \sum_{|y|>\lfloor W_1/3 \rfloor} e^{-\frac{y^2}{2w^2}} \leqslant \frac{2}{c_w} \frac{w^2}{\lfloor W_1/3 \rfloor} e^{-\frac{(\lfloor W_1/3 \rfloor)^2}{2w^2}} \ , \quad \text{(D.22)}$$

where the bound is calculated in Lemma 12 below. Then,

$$\mathrm{tr}(\Pi_2^\perp \rho_2^{x,x}) = c_w^{-1} \sum_{|y|>W_2} e^{-\frac{(y+x)^2}{2w^2}} \leqslant \frac{2}{c_w} \frac{w^2}{W_2-|x|} e^{-\frac{(W_2-|x|)^2}{2w^2}} \leqslant \frac{2}{c_w} \frac{w^2}{W_2-h} e^{-\frac{(W_2-h)^2}{2w^2}} \ . \quad \text{(D.23)}$$

Similarly,

$$\mathrm{tr}(\Pi_3^\perp \rho_3^{x,x}) = c_w^{-1} \sum_{|2y|>W_3} e^{-\frac{(y-x)^2}{2w^2}} \leqslant \frac{2}{c_w} \frac{w^2}{\lfloor W_3/2 \rfloor - h} e^{-\frac{(\lfloor W_3/2 \rfloor - h)^2}{2w^2}} \ . \quad \text{(D.24)}$$

Hence, choosing $W_1 = W_2 = W_3 =: W$ with $W \geqslant 2h$ and choosing for simplicity W as a multiple of 6, we have $\lfloor W_1/3 \rfloor = W/3 \geqslant (1/3)(W-2h)$, as well as $W_2 - h \geqslant W - 2h$ and also $\lfloor W_3/2 \rfloor - h = (1/2)(W-2h)$; furthermore $W - 2h \geqslant (1/2)(W-2h) \geqslant (1/3)(W-2h)$. Then,

$$\left| \sum_i \mathrm{tr}(\Pi_i^\perp \rho_i^{x,x}) \right| \leqslant \frac{2}{c_w} \frac{w^2}{(1/3)(W-2h)} e^{-\frac{(\frac{1}{3}(W-2h))^2}{2w^2}} + \frac{2}{c_w} \frac{w^2}{W-2h} e^{-\frac{(W-2h)^2}{2w^2}}$$

$$+ \frac{2}{c_w} \frac{w^2}{(1/2)(W-2h)} e^{-\frac{(\frac{1}{2}(W-2h))^2}{2w^2}}$$

$$\leqslant \frac{12}{c_w} \frac{w^2}{W-2h} e^{-\frac{(W-2h)^2}{18w^2}} \leqslant \frac{12\,w^2}{W-2h} e^{-\frac{(W-2h)^2}{18w^2}} =: \eta \ , \quad \text{(D.25)}$$

recalling that $c_w \geqslant 1$. Also, $\Delta T_i = 2W_i = 2W$ by construction. Furthermore $\Delta T_L = 2h$ and $\delta = 0$. So, our bound reads (assuming that the noise erasure probabilities are $q_1 = q_2 = q_3 = 1/3$)

$$\epsilon(\mathcal{N} \circ \mathcal{E}) \geqslant \frac{1}{2} \frac{1}{(\max_i q_i^{-1}) \cdot 2W} [2h - 2\eta] = \frac{1}{2} \frac{1}{6W} \left[2h - \frac{24\,w^2}{W-2h} e^{-\frac{(W-2h)^2}{18w^2}} \right]$$

$$\approx \frac{1}{2} \frac{1}{6W} \left[2h - \frac{24\,w^2}{W} e^{-\frac{W^2}{18w^2}} \right] = \frac{h}{6W} - \frac{4\,w^2}{W^2} e^{-\frac{W^2}{18w^2}} \ . \quad \text{(D.26)}$$

considering the regime $W \gg h$, i.e., $W - 2h \approx W$. Now, if we choose the cutoff $W = \beta w$ to be proportional to w, then we can write our bound as a function of h/w:

$$\epsilon(\mathcal{N} \circ \mathcal{E}) \gtrsim \frac{1}{6\beta} \frac{h}{w} - \frac{4\,e^{-\beta^2/8}}{\beta^2} \ . \quad \text{(D.27)}$$

The second term is exponentially suppressed in β; so choosing β only very moderately large, we get a bound which is effectively proportional to h/w with a proportionality constant $1/(6\beta)$.

Now we find a suitable β to plug into (D.27) to get a bound in terms of h/w only. If we attempt to minimize the bound (D.27), we get as minimization condition

$$0 = \frac{\partial}{\partial\beta}(\text{bound}) = -\frac{1}{6\beta^2}\frac{h}{w} + \frac{8\,e^{-\beta^2/8}}{\beta^3} + \frac{e^{-\beta^2/8}}{\beta} = \frac{1}{\beta^2}\left[-\frac{h}{6w} + \frac{8+\beta^2}{\beta}e^{-\beta^2/8}\right]. \quad \text{(D.28)}$$

Writing $z = \beta^2/4$ (i.e., $\beta = 2\sqrt{z}$) we obtain $h/(6w) = e^{-z/2}(4+2z)/(\sqrt{z})$; the square of this equation gives

$$\frac{h^2}{36w^2} = \left[4z + 16 + \frac{16}{z}\right]e^{-z}. \quad \text{(D.29)}$$

To render this equation tractable, and since we only have to come up with an approximate educated guess for β, we may simplify this equation by keeping the leading term, expecting that z should be moderately large, yielding

$$\left(\frac{h}{12w}\right)^2 \approx ze^{-z}. \quad \text{(D.30)}$$

The solution to the equation $x^2 = ze^{-z}$ is given by the Lambert W function[5] with $z = -W(-x^2)$. Using the expansion of the negative branch W_{m} of the function near $z \to -\infty$, we have[6] $-W_{\mathrm{m}}(-x^2) \approx \ln(1/x^2)$, and hence we may select $z \approx \ln\left((12w/h)^2\right) = 2\ln(12w/h)$. This in turn yields the educated guess $\beta = 2\sqrt{2\ln(12w/h)}$ to plug into (D.27), and the bound becomes

$$\epsilon(\mathcal{N}\circ\mathcal{E}) \gtrsim \frac{h}{12w}\left[\frac{1}{\sqrt{2\ln(12w/h)}} - \frac{1}{2\ln(12w/h)}\right] \approx \frac{h/w}{12\sqrt{2\ln(w/h)}}, \quad \text{(D.31)}$$

using $\sqrt{\ln(12w/h)} = \sqrt{\ln(w/h) + \ln(12)} \approx \sqrt{\ln(w/h)}$.

Lemma 11. *We have for integer x, h, with $|x| \leqslant h$ and with h even,*

$$\frac{1}{c_w}\sum_{y=-\infty}^{\infty} e^{-\frac{(y\pm x)^2 + y^2}{4w^2}} \geqslant e^{-\frac{h^2}{8w^2}}. \quad \text{(D.32)}$$

Proof of Lemma 11. First, we may assume without loss of generality that we have the "+" case in the exponent (or else simply send $x \to -x$). Completing the square, we have $(y+x)^2 + y^2 = 2y^2 + 2xy + x^2 = 2(y+x/2)^2 + x^2/2$, and hence

$$\frac{1}{c_w}\sum_{y=-\infty}^{\infty} e^{-\frac{(y+x)^2 + y^2}{4w^2}} = \frac{1}{c_w}\sum_{y=-\infty}^{\infty} e^{-\frac{(y+x/2)^2}{2w^2} - \frac{x^2}{8w^2}} = \frac{e^{-\frac{x^2}{8w^2}}}{c_w}\sum_{y=-\infty}^{\infty} e^{-\frac{(y+x/2)^2}{2w^2}}. \quad \text{(D.33)}$$

At this point we need to distinguish the case where x is even from the case where x is odd.

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[5] https://dlmf.nist.gov/4.13
[6] https://dlmf.nist.gov/4.13.E11

Assuming first that x is even, we may redefine $y \to y + x/2$ in the summation and we have

$$\text{(D.33)} \ [x \text{ even}] = \frac{e^{-\frac{x^2}{8w^2}}}{c_w} \sum_{y=-\infty}^{\infty} e^{-\frac{y^2}{2w^2}} = \frac{e^{-\frac{x^2}{8w^2}}}{c_w} \cdot c_w = e^{-\frac{x^2}{8w^2}} \geqslant e^{-\frac{h^2}{8w^2}} \ , \tag{D.34}$$

recalling that $|x| \leqslant h$. In the case that x is odd, we need to work a little bit more; we may redefine $y \to y + (x-1)/2$, and we have

$$\text{(D.33)} \ [x \text{ odd}] = \frac{e^{-\frac{x^2}{8w^2}}}{c_w} \sum_{y=-\infty}^{\infty} e^{-\frac{(y+\frac{1}{2})^2}{2w^2}} = \frac{e^{-\frac{x^2}{8w^2}}}{c_w} \vartheta_2\left(0, e^{-\frac{1}{2w^2}}\right) \ , \tag{D.35}$$

using another theta function corresponding to this type of summation. Lemma 13 shows that $\vartheta_2\left(0, e^{-1/(2w^2)}\right) \geqslant e^{-1/(8w^2)} \vartheta_3\left(0, e^{-1/(2w^2)}\right)$, and so we have

$$\text{(D.35)} \geqslant e^{-\frac{x^2+1}{8w^2}} \geqslant e^{-\frac{(|x|+1)^2}{8w^2}} \geqslant e^{-\frac{h^2}{8w^2}} \ , \tag{D.36}$$

where we have assumed that h is even, and so $|x| + 1 \leqslant h$. ∎

Lemma 12. *We have, for $W \geqslant 0$,*

$$\sum_{|y|>W} e^{-\frac{(y\pm x)^2}{2w^2}} \leqslant 2\,\frac{w^2}{W-|x|}\,e^{-\frac{(W-|x|)^2}{2w^2}} \ . \tag{D.37}$$

Proof of Lemma 12. Assume $x \geqslant 0$, or else redefine $x \to -x$. We have

$$\sum_{|y|>W} e^{-\frac{(y\pm x)^2}{2w^2}} \leqslant 2 \cdot \sum_{y>W-x} e^{-\frac{y^2}{2w^2}} \leqslant 2 \cdot \sum_{y\geqslant W-x+1} e^{-\frac{y^2}{2w^2}} \leqslant 2 \int_{W-x}^{\infty} dy\, e^{-\frac{y^2}{2w^2}} \ , \tag{D.38}$$

where the integral is necessarily an overestimation of the sum, as the sum can be seen as an integral of a step function, where each step is specified at the right edge by the value of the integrand function; this step function lies beneath the actual decreasing function $e^{-y^2/(2w^2)}$. Setting $t = y/(w\sqrt{2})$,

$$\text{(D.38)} = 2 \int_{\frac{W-x}{w\sqrt{2}}}^{\infty} dt\, w\sqrt{2}\, e^{-t^2} = w\,\sqrt{2\pi}\,\frac{2}{\sqrt{\pi}} \int_{\frac{W-x}{w\sqrt{2}}}^{\infty} dt\, e^{-t^2} = w\,\sqrt{2\pi}\,\mathrm{erfc}\left(\frac{W-x}{w\sqrt{2}}\right) \ . \tag{D.39}$$

We use the known bound[7]

$$\mathrm{erfc}(z) \leqslant \frac{e^{-z^2}}{z\sqrt{\pi}} \ , \tag{D.40}$$

leading to

$$\text{(D.39)} \leqslant 2\,\frac{w^2}{W-x}\,e^{-\frac{(W-x)^2}{2w^2}} \ . \tag{∎}$$

---

[7] See for instance *http://dlmf.nist.gov/7.8.E4* or *http://mathworld.wolfram.com/Erfc.html*

Finally, we prove a property of the theta functions that we used above.

Lemma 13. *Let $0 < q \leqslant 1$, and let $z \in \mathbb{C}$ with $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) \geqslant 0$. Then*

$$\vartheta_3(z, q) \geqslant \vartheta_3(0, q) \ . \tag{D.41}$$

Furthermore, we have

$$\vartheta_2(0, q) \geqslant q^{1/4}\, \vartheta_3(0, q) \ . \tag{D.42}$$

Proof of Lemma 13.   We start by proving (D.41). Writing $q = e^{i\pi\tau}$ with $\mathrm{Re}(\tau) = 0$ and $\mathrm{Im}(\tau) \geqslant 0$, we have[8]

$$\vartheta_3(z, q) = \vartheta_3(0, q) \cdot \prod_{n=1}^{\infty} \frac{\cos\left((n - \tfrac{1}{2})\pi\tau + z\right) \cos\left((n - \tfrac{1}{2})\pi\tau - z\right)}{\cos^2\left((n - \tfrac{1}{2})\pi\tau\right)} =: \vartheta_3(0, q) \cdot \prod_{n=1}^{\infty} a_n \ . \tag{D.43}$$

We will show that the product is greater than 1, by showing that $a_n \geqslant 1$ for each n. We have

$$a_n = \frac{\cos(i(a + b)) \, \cos(i(a - b))}{\cos^2(i\, a)} \ , \tag{D.44}$$

defining $a, b \geqslant 0$ as $a = (n - 1/2)\, \pi\, \mathrm{Im}(\tau)$ and $b = \mathrm{Im}(z)$. Since $\cos(i\,\varphi) = \cosh(\varphi)$, we have

$$a_n = \frac{\cosh(a + b) \, \cosh(a - b)}{\cosh^2(a)} \ . \tag{D.45}$$

(By the way, this is another way of seeing that $\vartheta_3(z, q)$ must be real and positive, since all the a_n are real positive and $\vartheta_3(0, q)$ is real positive as given by its series representation. Recall that z is pure imaginary with $\mathrm{Im}(z) \geqslant 0$, and that $0 < q \leqslant 1$.) With the usual properties of the hyperbolic functions, we have

$$
\begin{aligned}
\cosh(a + b) \cosh(a - b) &= [\cosh(a)\cosh(b) + \sinh(a)\sinh(b)][\cosh(a)\cosh(b) - \sinh(a)\sinh(b)] \\
&= \cosh^2(a)\cosh^2(b) - \sinh^2(a)\sinh^2(b) \\
&= \cosh^2(a)\cosh^2(b)\left[1 - \tanh^2(a)\tanh^2(b)\right] \\
&\geqslant \cosh^2(a)\cosh^2(b)\left[1 - \tanh^2(b)\right] = \cosh^2(a) \ ,
\end{aligned} \tag{D.46}
$$

using $\tanh(a) \leqslant 1$ and $1/\cosh^2(b) = 1 - \tanh^2(b)$. Hence finally, $a_n \geqslant 1$. This proves (D.41).

To prove (D.42), we invoke the following property of the theta functions,[9] valid for any $q = e^{i\pi\tau}$,

$$\vartheta_2(0, q) = q^{1/4} \vartheta_3\left(\frac{1}{2}\pi\tau, q\right) \ . \tag{D.47}$$

———

[8] See *http://dlmf.nist.gov/20.5.E7*, Eq. (20.5.7)
[9] See *http://dlmf.nist.gov/20.2.E12*, Eq. (20.2.12)

For $0 < q \leqslant 1$, necessarily τ is pure imaginary with $\text{Im}(\tau) \geqslant 0$; we may thus invoke (D.41), which proves (D.42). ∎

D.2. Five-rotor perfect code

The normalized encoding for this code is

$$|x\rangle \rightarrow \frac{1}{\sqrt{c_{w,x}}} \sum_{j,k,l,m,n\in\mathbb{Z}} e^{-\frac{1}{4w^2}\left(j^2+k^2+l^2+m^2+n^2\right)} T_{jklmnx}^{(\infty)} |j,k,l,m,n\rangle , \qquad (D.48)$$

where $c_{w,x}$ is the normalization and $T^{(\infty)}$ is defined in Eq. (44).

Single erasure. We first calculate $\rho_\ell^{x,x}$ for $\ell \in \{1,2,3,4,5\}$ and then outline why $||\rho_\ell^{x,x'\neq x}||_1 = O(e^{-cw^2})$. By the cyclic permutation symmetry of the code, we only have to calculate $\rho_1^{x,x}$. Performing the partial trace and simplifying all Kronecker delta functions leaves us with the diagonal reduced density matrix

$$\rho_1^{x,x} = \frac{1}{c_{w,x}} \sum_{j\in\mathbb{Z}} \left(\sum_{k,l,m\in\mathbb{Z}} e^{-\frac{1}{2w^2}\left(j^2+k^2+l^2+m^2+[j+k+l+m-x]^2\right)} \right) |j\rangle\langle j| \qquad (D.49)$$

Now we apply the Poisson summation formula,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} dx e^{2\pi i n x} f(x) , \qquad (D.50)$$

to each of the three sums above. Typically, the $n = 0$ term on the right-hand-side is dominant (i.e., the leading order contribution in the large-w limit), and taking only this term is equivalent to approximating the sum with a Gaussian integral. Each of the remaining terms suppressed as $O(e^{-cw^2})$, where c is a positive constant increasing with n. Because c increases with n, the $n+1$-th term is subleading with respect to the nth term. Thus, the entire sum of exponentially suppressed terms can itself be bounded by an exponential (e.g., $e^{-2x} + e^{-3x} < e^{-x}$ for $x > 1$). We omit these corrections and focus on the dominant term $k = l = m = 0$ after having applied Poisson summation to Eq. (D.49):

$$\rho_1^{x,x} \sim \frac{\sqrt{2}\pi^{3/2} w^3}{c_{w,x}} \sum_{j\in\mathbb{Z}} e^{-\frac{5j^2-2jx+x^2}{8w^2}} |j\rangle\langle j| . \qquad (D.51)$$

Forcing $\text{Tr}\{\rho_1^{x,x}\} = 1$ and once again approximating the resulting sum with an integral solves for the normalization $c_{w,x}$ in the large w limit. Plugging that back into the above equation and simplifying produces

$$\rho_1^{x,x} \sim \sqrt{\frac{5}{2\pi}} \frac{1}{2w} \sum_{j\in\mathbb{Z}} e^{-\frac{(x-5j)^2}{40w^2}} |j\rangle\langle j| . \qquad (D.52)$$

Now we calculate the fidelity of the above state to $\rho_1^{0,0}$. Using the fact that the states commute with each other, taking the square root of each entry in the resulting diagonal

matrix, and applying Poisson summation yields

$$F^2\left(\rho_1^{x,x}, \rho_1^{0,0}\right) = \sqrt{\frac{5}{2\pi}}\frac{1}{2w}\sum_{j\in\mathbb{Z}}e^{-\frac{50j^2-10jx+x^2}{80w^2}} \sim e^{-\frac{x^2}{160w^2}} \geqslant e^{-\frac{h^2}{160w^2}} . \tag{D.53}$$

Plugging this into the infidelity yields the result (45).

Returning to the $x \neq x'$ case, we show why those cases do not significantly contribute. The reduced density matrix is of the form

$$\rho_1^{x,x'} = \frac{1}{\sqrt{c_{w,x}c_{w,x'}}}\sum_{j\in\mathbb{Z}}\left(\sum_{k,l,m\in\mathbb{Z}}\gamma_{j,k,l,m}^{x,x'}\right)|j\rangle\langle j+x-x'|$$

where $\gamma_{j,k,l,m}^{x,x'}$ is a product of a Gaussian in the variables k,l,m (just like the $x = x'$ case above) and a phase $\propto 2\pi\Phi$ (which goes away when $x = x'$). We first apply Poisson summation to the internal three sums and evaluate the normalizations in the large-w limit. In this case, the centers of the Gaussians in the k,l,m-sum depend on Φ and the dominant term on the right-hand-side of Eq. (D.50) may not longer be the center-of-mass term $n = 0$. We will however set Φ to be an irrational number close to zero from now on, i.e., taking $\Phi \ll 1$ while making sure that $\Phi w \to \infty$. This makes sure that the center-of-mass mode is dominant. Writing the norm and applying Poisson summation to the remaining sum reveals

$$\|\rho_1^{x,x'}\|_1 \sim \sqrt{\frac{5}{2\pi}}\frac{1}{2w}e^{-2\pi^2\Phi^2w^2\left(x-x'\right)^2}\sum_{j\in\mathbb{Z}}e^{-\frac{25j^2-30jx+20jx'+13x^2-20xx'+8x'^2}{40w^2}}$$

$$= O\left(e^{-2\pi^2\left(x-x'\right)^2\Phi^2w^2}\right) . \tag{D.54}$$

We see that the one-norm is exponentially suppressed in w^2 for the off-diagonal (i.e., $x \neq x'$) reduced matrices.

Two erasures. We first calculate $\rho_{\ell,\ell'}^{x,x}$ for $\ell,\ell' \in \{1,2,3,4,5\}$ and then argue that $\|\rho_{\ell,\ell'}^{x,x'\neq x}\|_1 = O(e^{-cw^2})$. Due to the cyclic permutation symmetry, we only need to calculate $\rho_{1,2}^{x,x}$ and $\rho_{1,3}^{x,x}$. Performing the partial trace, simplifying the Kronecker delta functions, plugging in the normalization, and applying Poisson summation yields

$$\rho_{1,2}^{x,x} \sim \sqrt{\frac{5}{3}}\frac{1}{2\pi w^2}\sum_{j,k\in\mathbb{Z}}e^{-\frac{10j^2+5jk+10k^2-5(j+k)x+x^2}{15w^2}}|j,k\rangle\langle j,k| \sim \rho_{1,3}^{x,x} . \tag{D.55}$$

In other words, both $\rho_{1,2}^{x,x}$ and $\rho_{1,3}^{x,x}$ are identical in the large w limit. Note that, unlike $\rho_1^{x,x}$, these matrices have off-diagonal elements that are exponentially suppressed in w^2. These elements have been ignored above, but we mention them in the $x \neq x'$ case below. Taking the fidelity between $\rho_{\ell,\ell'}^{x,x}$ and $\rho_{\ell,\ell'}^{0,0}$ as before yields the result $F^2(\rho_{1,2}^{x,x}, \rho_{1,2}^{0,0}) \geqslant e^{-\frac{h^2}{60w^2}}$ claimed in (46).

The $x \neq x'$ case is more difficult this time because the unapproximated reduced density

matrix no longer has just one nonzero diagonal. Without any approximations, it is

$$\rho_{1,2}^{x,x'} = \sqrt{\frac{5}{3}} \frac{1}{2\pi w^2} \sum_{j,j',k \in \mathbb{Z}} \left(\sum_{l,m \in \mathbb{Z}} \gamma_{j,j',k,l,m}^{x,x'} \right) |j\rangle\langle j'| \otimes |k\rangle\langle k + x' - x + j - j'| . \qquad (D.56)$$

Applying Poisson summation to the internal two sums for $x \neq x'$ reveals that all matrix elements are exponentially suppressed with w^2,

$$\sum_{l,m \in \mathbb{Z}} \gamma_{j,j',k,l,m}^{x,x'} = \sqrt{\frac{5}{3}} \frac{1}{2\pi w^2} O \left(e^{-\frac{4}{3}\pi^2 \left[3(j'-j)^2 - 3(j'-j)(x'-x) + (x'-x)^2 \right] \Phi^2 w^2} \right) . \qquad (D.57)$$

However, there are particular values of (j, j') for which the function in the exponent above is minimized; we select those and show that the trace norm is exponentially suppressed in w^2. For even $x - x'$, the band at $j' = j - \frac{x-x'}{2}$ decays the slowest. Ignoring all other bands and calculating the trace norm yields

$$\|\rho_{1,2}^{x,x'}\|_1 = O \left(e^{-\frac{1}{3}\pi^2 (x-x')^2 \Phi^2 w^2} \right) . \qquad (D.58)$$

For odd $x - x'$, there are two bands $j' = j - \frac{x-x'\pm 1}{2}$ whose entries decay the slowest. Calculating the square root of $\rho_{1,2}^{x,x'} \rho_{1,2}^{x,x'\dagger}$ is more difficult since the resulting matrix is tri-diagonal. However, ignoring the off-diagonal entries, taking the square root, and bounding the resulting integral still yields exponential scaling with w^2.

D.3. Thermodynamic codes

Here, we carry out the calculations that are relevant for §VI C of the main text.

The operators $\rho_d^{m,m}$ (reduced states on d consecutive sites) are provided as:

$$\rho_d^{m,m} = \sum_{r=-d}^{d} K_{r,d,m}^N |h_r^d\rangle\langle h_r^d|_d , \qquad (D.59)$$

with

$$K_{r,d,m}^N = \frac{\binom{d}{d/2 + r/2} \binom{N-d}{(N-d)/2 + (m-r)/2}}{\binom{N}{N/2 + m/2}} . \qquad (D.60)$$

The fidelity between two states which commute reduces to the Bhattacharyya coefficient (the classical version of the fidelity):

$$F(\rho_d^{m,m}, \rho_d^{0,0}) = \sum_{r=-d}^{d} \sqrt{K_{r,d,m}^N} \sqrt{K_{r,d,0}^N} . \qquad (D.61)$$

The complicated calculation is deferred to Lemma 14 below, which gives us:

$$F(\rho_d^{m,m}, \rho_d^{0,0}) \geqslant 1 - O(N^{-2}) . \tag{D.62}$$

This, in turn, tells us that

$$P(\rho_d^{m,m}, \rho_d^{0,0}) \leqslant O(N^{-1}) . \tag{D.63}$$

The "logical off-diagonal" terms $\rho_d^{m,m'}$ for $m \neq m'$ are exactly zero, because we made sure to space out the codewords in magnetization by $2d + 1$, following the construction of ref. [8].

Hence, applying Proposition 10, we see that our code is an AQECC against the erasure of d consecutive sites, with

$$\epsilon(\mathcal{N} \circ \mathcal{E}) \leqslant O(N^{-1}) . \tag{D.64}$$

This matches exactly the scaling of our bound (26).

Lemma 14. *There exists a constant $D_{d,m}$ of N such that (for constant d, m):*

$$F(\rho_d^{m,m}, \rho_d^{0,0}) \geqslant 1 - \frac{D_{d,m}}{N^2} + O(N^{-3}) . \tag{D.65}$$

Proof of Lemma 14. We use Stirling's formula up to order $1/N^2$:

$$\ln(N!) = N \ln(N) - N + \frac{1}{2} \ln(2\pi N) + \frac{1}{12N} + O(N^{-3}) , \tag{D.66}$$

(noting that there is in fact no term of order $1/N^2$). Now, for any x, ignoring terms of order $O(N^{-3})$, we have:

$$\ln \binom{N}{N/2 + x/2} = N \ln(N) + \frac{\ln(2\pi)}{2} + \frac{\ln(N)}{2} + \frac{1}{12N} + O(N^{-3})$$
$$- \left[\left(\frac{N}{2} + \frac{x}{2} \right) \ln \left(\frac{N}{2} + \frac{x}{2} \right) + \frac{\ln(2\pi)}{2} + \frac{1}{2} \ln \left(\frac{N}{2} + \frac{x}{2} \right) + \frac{1}{6(N + x)} \right]$$
$$- \left[\left(\frac{N}{2} - \frac{x}{2} \right) \ln \left(\frac{N}{2} - \frac{x}{2} \right) + \frac{\ln(2\pi)}{2} + \frac{1}{2} \ln \left(\frac{N}{2} - \frac{x}{2} \right) + \frac{1}{6(N - x)} \right] .$$

Using the expansions

$$\ln \left(\frac{N}{2} \pm \frac{x}{2} \right) = \ln(N) - \ln(2) + \ln \left(1 \pm \frac{x}{N} \right)$$
$$= \ln(N) - \ln(2) \pm \frac{x}{N} - \frac{x^2}{2N^2} \pm \frac{x^3}{3N^3} + O(N^{-4}) ; \tag{D.67}$$

$$\frac{1}{N \pm x} = \frac{1}{N} \left(\frac{1}{1 \pm x/N} \right) = \frac{1}{N} \mp \frac{x}{N^2} + O(N^{-3}) , \tag{D.68}$$

one continues, still keeping all the terms up to order $1/N^2$:

$$\ln \binom{N}{N/2 + x/2} = N \ln(2) - \frac{\ln(N)}{2} + \ln \left(\frac{2}{\sqrt{2\pi}} \right) - \frac{1}{4N} - \frac{x^2}{2N} + \frac{x^2}{2N^2} + O(N^{-3}) .$$

Now we may apply this to calculate $\ln\left(\sqrt{K_{r,d,m}^N K_{r,d,0}^N}\right)$, using the fact that $\ln(N - d) = \ln(N) + \ln(1 - d/N) = \ln(N) - d/N - d^2/(2N^2) + O(N^{-3})$:

$$
\ln\sqrt{K_{r,d,m}^N K_{r,d,0}^N}
$$

$$
= \ln\binom{d}{d/2 + r/2} + \frac{1}{2}\ln\left(\frac{N - d}{(N - d)/2 + (m - r)/2}\right)
$$

$$
+ \frac{1}{2}\ln\left(\frac{N - d}{(N - d)/2 - r/2}\right) - \frac{1}{2}\ln\left(\frac{N}{N/2 + m/2}\right) - \frac{1}{2}\ln\binom{N}{N/2}
$$

$$
= \ln\binom{d}{d/2 + r/2} - d\ln(2)
$$

$$
+ \frac{1}{2}\left\{ -\frac{1}{2}\left[\ln(N) - \frac{d}{N} - \frac{d^2}{2N^2}\right] - \frac{1}{4}\left(\frac{1}{N} + \frac{d}{N^2}\right) - \frac{(m - r)^2}{2}\left(\frac{1}{N} + \frac{d}{N^2}\right) + \frac{(m - r)^2}{2N^2}\right\}
$$

$$
+ \frac{1}{2}\left\{ -\frac{1}{2}\left[\ln(N) - \frac{d}{N} - \frac{d^2}{2N^2}\right] - \frac{1}{4}\left(\frac{1}{N} + \frac{d}{N^2}\right) - \frac{r^2}{2}\left(\frac{1}{N} + \frac{d}{N^2}\right) + \frac{r^2}{2N^2}\right\}
$$

$$
- \frac{1}{2}\left\{ -\frac{1}{2}\ln(N) - \frac{1}{4N} - \frac{m^2}{2N} + \frac{m^2}{2N^2}\right\} - \frac{1}{2}\left\{ -\frac{1}{2}\ln(N) - \frac{1}{4N}\right\} + O(N^{-3})
$$

$$
= \ln\left(2^{-d}\binom{d}{d/2 + r/2}\right) + \frac{d}{2N} + \frac{A_{m,r}}{N} + \frac{B_{d,m,r}}{N^2} + O(N^{-3}) ,
$$

with

$$
A_{m,r} = \frac{1}{2}r(m - r) ; \tag{D.69a}
$$

$$
B_{d,m,r} = \frac{1}{4}\left[d^2 - d(1 + m^2 + 2r^2 - 2mr) + 2r^2 - 2mr\right] . \tag{D.69b}
$$

Using $0 \leqslant |r| \leqslant d$, write

$$
B_{d,m,r} \geq \frac{1}{4}\left[d^2 - d(1 + m^2 + 2d^2 + 2|m|d) - 2|m|d\right]
$$

$$
\geq \frac{1}{4}\left[-2d^3 - d^2(2|m| - 1) - d(1 + m^2 + 2|m|)\right] =: -\frac{1}{4}C_{d,m} .
$$

Then,

$$
F(\rho_d^{m,m}, \rho_d^{0,0}) \geqslant e^{d/(2N)} 2^{-d} \sum_{r=-d}^d \binom{d}{d/2 + r/2}\exp\left\{\frac{A_{m,r}}{N} + \frac{B_{d,m,r}}{N^2} + O(N^{-3})\right\}
$$

$$
\geqslant \exp\left\{-\frac{C_{d,m}}{4N^2} + O(N^{-3})\right\} e^{d/(2N)} 2^{-d} \sum_{r=-d}^d \binom{d}{d/2 + r/2}\exp\left\{\frac{A_{m,r}}{N}\right\} .
$$

$$
\tag{D.70}
$$

Recall the identities

$$
\sum_{r=-d}^d \binom{d}{d/2 + r/2} = \sum_{k=0}^d \binom{d}{k} = 2^d ; \tag{D.71a}
$$

$$\sum_{k=0}^{d} \binom{d}{k} k = d\, 2^{d-1} \; ; \tag{D.71b}$$

$$\sum_{k=0}^{d} \binom{d}{k} k^2 = (d + d^2)\, 2^{d-2} \; . \tag{D.71c}$$

We have $\exp\{A_{m,r}/N\} = 1 + r(m-r)/(2N) + r^2(m-r)^2/(8N^2) + O(N^{-3}) \geqslant 1 + r(m-r)/(2N) + O(N^{-3})$. Replacing the summation index r by $k = (d+r)/2 = 0, 1, \ldots, d$, we calculate

$$\begin{aligned}
&\frac{2^{-d}}{2N} \sum_{r=-d}^{d} \binom{d}{d/2 + r/2} r(m-r) \\
&= \frac{2^{-d}}{2N} \sum_{k=0}^{d} \binom{d}{k} (2k-d)(m-2k+d) \\
&= \frac{2^{-d}}{2N} \left[(-d\,m - d^2) \sum_{k=0}^{d} \binom{d}{k} + (2m + 2d + 2d) \sum_{k=0}^{d} \binom{d}{k} k - 4 \sum_{k=0}^{d} \binom{d}{k} k^2 \right] \\
&= \frac{2^{-d}}{2N} \left[(-d\,m - d^2)\, 2^d + (2m + 2d + 2d) \frac{d}{2} 2^d - 4 \frac{d + d^2}{4} 2^d \right] \\
&= -\frac{d}{2N} \; ,
\end{aligned}$$

and then

$$2^{-d} \sum_{r=-d}^{d} \binom{d}{d/2 + r/2} \exp\left\{ \frac{A_{m,r}}{N} \right\} \geq 2^{-d} \sum_{r=-d}^{d} \binom{d}{d/2 + r/2} \left(1 + \frac{r(m-r)}{2N} + O(N^{-3}) \right)$$

$$= 1 - \frac{d}{2N} + O(N^{-3}) \; .$$

Finally, plugging into (D.70) gives us

$$F(\rho_d^{m,m}, \rho_d^{0,0}) \geqslant \left\{ 1 - \frac{C_{d,m}}{4N^2} + O(N^{-3}) \right\} \left\{ 1 + \frac{d}{2N} + \frac{d^2}{8N^2} + O(N^{-3}) \right\} \left\{ 1 - \frac{d}{2N} + O(N^{-3}) \right\}$$

$$\geqslant 1 - \frac{C_{d,m}}{4N^2} - \frac{d^2}{8N^2} + O(N^{-3}) \; , \tag{D.72}$$

so we may define $D_{d,m} = C_{d,m}/4 + d^2/8$, proving the claim. $\blacksquare$

Appendix E: Proof of the approximate Eastin-Knill theorem

E.1. Equivalence of the existence of a universal transversal gate set and the $U(d_L)$-covariance property of the code

First, we show that the setting of the Eastin-Knill theorem is equivalent to studying the $U(d_L)$-covariance property of the corresponding code. More precisely, we show that given

a code $V_{L \to A}$, if there exists a mapping u of logical unitaries U_L to transversal physical unitaries $u(U_L) = U_1 \otimes \cdots \otimes U_n$ satisfying $V^\dagger u(U_L)V = U_L$ for all U_L, where u does not even have to be continuous, then the code $V_{L \to A}$ is necessarily covariant with respect to the full unitary group on the logical space.

The statement is pretty intuitive, because given any rule that maps logical unitaries to physical transversal unitaries, we can compose the physical unitaries corresponding to different logical unitaries, and presumably generate a *bona fide* representation by starting from a minimal generating set of unitaries. This intuition proves correct, though it is not immediately clear if the mapping generated in this way is continuous. Here we provide a derivation that smooths out these technical details.

Proposition 15. *Let $V_{L \to A}$ be any code, with $A = A_1 \otimes \cdots \otimes A_n$. Suppose that for each unitary U_L on L there exists a transversal unitary $U_A = u(U_L) = u_1(U_L) \otimes \cdots \otimes u_n(U_L)$ such that $V^\dagger u(U_L)V = U_L$ for all U_L. Then there exists a mapping u' that maps any U_L to a transversal physical unitary $u'(U_L) = u'_1(U_L) \otimes \cdots \otimes u'_n(U_L)$ such that*

- *u' is continuous;*

- *for all U_L, $V^\dagger u'(U_L)V = V^\dagger u(U_L)V = U_L$; and*

- *for any U_L, U'_L, we have $u'(U_L U'_L) = u'(U_L)\, u'(U'_L)$.*

Proof of Proposition 15. Observe first that for all U_L, because $u(U_L)$ implements a logical unitary, it must fix the code space $\Pi = VV^\dagger$. Hence, we must necessarily have $[u(U_L), VV^\dagger] = 0$ for all U_L.

Let $T_L^{(j)}$ be a basis of the Lie algebra $\mathfrak{u}(d_L)$ of $U(d_L)$. Let (ϑ_k) be a sequence of positive reals converging to zero, and for each j, consider the sequence of transversal physical unitaries $\big(u\big(e^{-i\vartheta_k T_L^{(j)}}\big)\big)_k$. Let $\delta > 0$. Since the sequence of unitaries is supported on a compact set, it admits a convergent subsequence and hence, there exist ϑ', ϑ'' such that $|\vartheta' - \vartheta''| \leqslant \delta$ and $\big\| u\big(e^{-i\vartheta' T_L^{(j)}}\big) - u\big(e^{-i\vartheta'' T_L^{(j)}}\big) \big\|_\infty \leqslant \delta$. We define $\tilde{U}^{(j)} = u^\dagger\big(e^{-i\vartheta'' T_L^{(j)}}\big)\, u\big(e^{-i\vartheta' T_L^{(j)}}\big)$, which then satisfies

$$\big\| \tilde{U}^{(j)} - \mathbb{1} \big\|_\infty \leqslant \delta \ . \tag{E.1}$$

We also have that $V^\dagger \tilde{U}^{(j)} V = V^\dagger u^\dagger\big(e^{-i\vartheta'' T_L^{(j)}}\big) VV^\dagger u\big(e^{-i\vartheta' T_L^{(j)}}\big)V = e^{i\vartheta'' T_L^{(j)}} e^{-i\vartheta' T_L^{(j)}} = e^{-i\alpha T_L^{(j)}}$, with $\alpha = \vartheta' - \vartheta''$, where we recall that $[u(U_L), VV^\dagger] = 0$ for any U_L. Define

$$\tilde{U}_i^{(j)} = e^{-i\chi_i^{(j)}} u_i^\dagger\big(e^{-i\vartheta'' T_L^{(j)}}\big)\, u_i\big(e^{-i\vartheta'(T_L^{(j)})}\big) \ ; \qquad T_i^{(j)} = \alpha^{-1}\, i \log\big(\tilde{U}_i^{(j)}\big) \ , \tag{E.2}$$

where $\chi_i^{(j)}$ is chosen such that there exists an eigenvector $|\chi_i^{(j)}\rangle$ of $\tilde{U}_i^{(j)}$ with eigenvalue exactly equal to one. Choosing $\chi^{(j)}$ in $[-\pi, \pi[$ such that $\chi^{(j)} = \sum_i \chi_i^{(j)} \mod 2\pi$, we then have $\tilde{U}_1^{(j)} \otimes \cdots \otimes \tilde{U}_n^{(j)} = e^{-i\chi^{(j)}} \tilde{U}^{(j)}$. Recall that for any operator X, we have $\|X\|_\infty = \max_{|\phi\rangle, |\psi\rangle} \operatorname{Re} \operatorname{tr}\{|\psi\rangle\langle\phi|\, X\}$ where the optimization ranges over vectors satisfying $\||\phi\rangle\|, \||\psi\rangle\|_\infty \leqslant 1$. We have using (E.1) and for a suitably chosen phase $e^{-i\xi}$ that

$$\delta \geqslant \operatorname{Re} \operatorname{tr}\Big\{ e^{-i\xi} \Big(\bigotimes |\chi_i^{(j)}\rangle\langle\chi_i^{(j)}|\Big)\big[\tilde{U}^{(j)} - \mathbb{1}\big] \Big\}$$
$$= \operatorname{Re}\Big\{ e^{-i\xi} \Big(\bigotimes \langle\chi_i^{(j)}|\Big)\Big[\Big(\bigotimes e^{i\chi_i^{(j)}} \tilde{U}_i^{(j)}\Big) - \mathbb{1}\Big]\Big(\bigotimes |\chi_i^{(j)}\rangle\Big) \Big\}$$

$$= \mathrm{Re}\Big\{e^{-i\xi}\left(e^{i\chi^{(j)}} - 1\right)\Big\}$$
$$= \left|e^{i\chi^{(j)}} - 1\right| \, , \tag{E.3}$$

where the phase $e^{-i\xi}$ is chosen such that $e^{-i\xi}\big(e^{i\chi^{(j)}} - 1\big)$ is real positive. This implies that $\delta \geqslant \mathrm{Im}\{e^{i\chi^{(j)}}\}$ and $\mathrm{Re}\{e^{i\chi^{(j)}}\} \geqslant 1 - \delta$, which implies in turn $|\chi^{(j)}| \leqslant \arcsin(\delta) \leqslant 2\delta$. This also implies that $\big\|e^{-i\chi^{(j)}}\tilde{U}^{(j)} - \tilde{U}^{(j)}\big\|_\infty = \big|e^{-i\chi^{(j)}} - 1\big|\big\|\tilde{U}^{(j)}\big\|_\infty \leqslant \delta$, and hence by triangle inequality

$$\big\|e^{-i\chi^{(j)}}\tilde{U}^{(j)} - \mathbb{1}\big\|_\infty \leqslant \big\|e^{-i\chi^{(j)}}\tilde{U}^{(j)} - \tilde{U}^{(j)}\big\|_\infty + \big\|\tilde{U}^{(j)} - \mathbb{1}\big\|_\infty \leqslant 2\delta \, . \tag{E.4}$$

Similarly, for each i, we have $\big\|\tilde{U}_i^{(j)} - \mathbb{1}\big\|_\infty = \mathrm{Re}\,\mathrm{tr}\big[|\psi_i\rangle\langle\phi_i|(\tilde{U}_i^{(j)} - \mathbb{1})\big]$ for some $|\psi_i\rangle, |\phi_i\rangle$, and hence

$$\begin{aligned}
2\delta &\geqslant \big\|e^{-i\chi^{(j)}}\tilde{U}^{(j)} - \mathbb{1}\big\|_\infty \\
&\geqslant \mathrm{Re}\,\mathrm{tr}\Bigg\{\bigg[|\psi_i\rangle\langle\phi_i| \otimes \bigotimes_{i'\neq i}|\chi_{i'}^{(j)}\rangle\langle\chi_{i'}^{(j)}|\bigg]\bigg(\bigotimes\tilde{U}_i^{(j)} - \mathbb{1}\bigg)\Bigg\} \\
&= \mathrm{Re}\,\mathrm{tr}\Bigg\{\bigg[|\psi_i\rangle\langle\phi_i| \otimes \mathbb{1}\bigg]\bigg[\tilde{U}_i^{(j)} \otimes \bigotimes_{i'\neq i}|\chi_{i'}^{(j)}\rangle\langle\chi_{i'}^{(j)}| - \mathbb{1} \otimes \bigotimes_{i'\neq i}|\chi_{i'}^{(j)}\rangle\langle\chi_{i'}^{(j)}|\bigg]\Bigg\} \\
&= \mathrm{Re}\,\mathrm{tr}\big\{|\psi_i\rangle\langle\phi_i|(\tilde{U}_i^{(j)} - \mathbb{1})\big\} = \big\|\tilde{U}_i^{(j)} - \mathbb{1}\big\|_\infty \, . \tag{E.5}
\end{aligned}$$

This implies that all eigenvalues of $\tilde{U}_i^{(j)}$ are δ-close to one, and hence the corresponding phases are all close to zero; more precisely, every eigenvalue $e^{i\gamma}$ of $\tilde{U}_i^{(j)}$ satisfies $2\delta \geqslant |e^{i\gamma} - 1|$; by the same reasoning as above, the statements $2\delta \geqslant \mathrm{Im}\{e^{i\gamma}\}$ and $\mathrm{Re}\{e^{i\gamma}\} \geqslant 1 - 2\delta$ imply that $|\gamma| \leqslant \arcsin(2\delta) \leqslant 4\delta$, and hence

$$\big\|\alpha T_i^{(j)}\big\|_\infty \leqslant 4\delta \, . \tag{E.6}$$

Now we set $T_A^{(j)} = \sum_i T_i^{(j)} - (\chi^{(j)}/\alpha)\,\mathbb{1}$, which is a sum of local terms. We then have $e^{-i\alpha T_A^{(j)}} = e^{i\chi^{(j)}}\,e^{-i\alpha T_1^{(j)}} \otimes \cdots \otimes e^{-i\alpha T_n^{(j)}} = e^{i\chi^{(j)}}\tilde{U}_1^{(j)} \otimes \cdots \otimes \tilde{U}_n^{(j)} = \tilde{U}^{(j)}$, with also $\big\|\alpha T_A^{(j)}\big\|_\infty \leqslant 4n\delta + 2\delta$. For any U_L, the unitary $u(U_L)$ commutes with the code space Π, and therefore $[\tilde{U}^{(j)}, \Pi] = 0$. Furthermore, since $e^{-i\alpha T_A^{(j)}}$ and $T_A^{(j)}$ share the same eigenspaces, we also have that $[T_A^{(j)}, \Pi] = [T_A^{(j)}, VV^\dagger] = 0$. We thus have $e^{-i\alpha T_L^{(j)}} = V^\dagger\tilde{U}^{(j)}V = V^\dagger e^{-i\alpha T_A^{(j)}}V = e^{-i\alpha V^\dagger T_A^{(j)}V}$, and thus by taking the logarithm, we obtain $V^\dagger T_A^{(j)}V = T_L^{(j)}$, where no ambiguities arise from taking the logarithm since the operators $\alpha T_A^{(j)}$ and $\alpha T_L^{(j)}$ have small norm, controlled by a suitably small choice of δ, and hence do not straddle the branch cut.

We can then define the mapping $u'(U_L)$ as the Lie group representation of $U(d_L)$ generated by the operators $T_A^{(j)}$ that span the corresponding Lie algebra. More explicitly, for any U_L, we may write $i\log(U_L) = \sum c_j T_L^{(j)}$ for some unique set of real coefficients $c_j = c_j(U_L)$ (up the zero-measure set of unitaries that have an eigenvalue that coincides with the logarithm branch cut), and we set

$$u'(U_L) = \exp\Big(-i\sum c_j(U_L)\,T_A^{(j)}\Big) \tag{E.7}$$

where c_j are the unique coefficients of the expansion of $i \log(U_L)$ in terms of the $T_L^{(j)}$ as defined above. We have that $u'(U_L)$ is transversal because each $T_A^{(j)}$ is a sum of local terms. Then we also have for any U_L that $V^\dagger u'(U_L) V = V^\dagger \exp\left(-i \sum c_j(U_L) T_A^{(j)}\right) V = \exp\left(-i \sum c_j(U_L) V^\dagger T_A^{(j)} V\right) = \exp\left(-i \sum c_j(U_L) T_L^{(j)}\right) = U_L$. Because u' is a Lie group representation, it is continuous and compatible with the group structure. ∎

E.2. Proof of the approximate Eastin-Knill bound

Recall that each irrep of $U(d_L)$ is represented by a *Young diagram* λ, where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{d_L})$, and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{d_L} = 0$, and $\lambda_i \in \mathbb{Z}$. The dimension of each irrep is given by the Weyl dimension formula, which for the $U(d_L)$ group is equal to the Schur polynomial S_λ evaluated at the vector $(1, 1, \cdots, 1)$. More explicitly, it can be evaluated to

$$D_\lambda = \prod_{1 \leqslant i < j \leqslant d_L} \frac{\lambda_i - \lambda_j + j - i}{j - i} \ . \tag{E.8}$$

To derive Theorem 4 we need to first prove few intermediate results. First, we will prove a bound on D_λ, based on λ_1.

Lemma 16. *The symmetric representation has the minimal dimension among the representations with fixed λ_1. More precisely, the following inequality holds,*

$$D_{Sym^{\lambda_1}} = \binom{d_L - 1 + \lambda_1}{d_L - 1} \leqslant D_\lambda \ , \tag{E.9}$$

where λ is a representation of $U(d_L)$ and $D_{Sym^{\lambda_1}}$ is the dimension of the symmetric representation with the Young diagram $\lambda = (\lambda_1, 0, 0, \cdots, 0)$.

Proof of Lemma 16. Suppose that $\lambda_1 = l$. We use the dimension formula Eq. (E.8). Consider the logarithm of the dimension, which is (up to a fixed constant) equal to:

$$f(\lambda_2, \cdots, \lambda_{d_L - 1}) = \sum_{1 \leqslant i < j \leqslant d_L} \log(\lambda_i - \lambda_j + j - i) \ . \tag{E.10}$$

Note that we fix $\lambda_1 = l$ and $\lambda_{d_L} = 0$, so they do not appear as parameters of f. Also, the vector $\hat\lambda = (\lambda_2, \cdots, \lambda_{d_L - 1})$ is an integer vector in the simplex Δ with d_L extremal points $\hat{v}_i \in \mathbb{R}^{d_L - 1}$, where $\hat{v}_0 = (0, 0, \cdots, 0)$, $\hat{v}_1 = (l, 0, \cdots, 0)$,$\cdots$, and $\hat{v}_{d_L - 1} = (l, l, \cdots, l)$.

We first extend the function f to all of the real points in Δ, and show that f is a concave function inside Δ. This would show that the minimum of f is attained at one of its extremal points.

A direct computation of the Hessian of f, reveals that for $2 \leqslant r, s \leqslant d_L - 1$,

$$H_{r,s} = \delta_{rs} \left[- \sum_{1 \leqslant i \leqslant d_L, i \neq s} K_{is} \right] + (1 - \delta_{rs}) K_{rs} \ , \tag{E.11}$$

where $K_{rs} = 1/(\lambda_s - \lambda_r + r - s)^2$. One can see that if $w = \sum_{2 \leqslant i \leqslant d_L - 1} \alpha_i e_i$ is an arbitrary

vector, then

$$w^\dagger H w = - \left(\sum_{2 \leqslant i \leqslant d_L - 1} |\alpha_i|^2 (K_{1i} + K_{id_L}) + \sum_{2 \leqslant i < j \leqslant d_L - 1} K_{ij} |\alpha_i - \alpha_j|^2 \right) . \tag{E.12}$$

This is a negative number, and shows that f is strictly concave. Therefore, the minimum of f is attained on one of the extremal point $\hat{v}_i$, $0 \leqslant i \leqslant d_L - 1$. Using the Weyl-dimension formula we have,

$$f(\hat{v}_i) = \prod_{j=0}^{i} \frac{\binom{l+d_L-1-j}{l}}{\binom{l+j}{l}} . \tag{E.13}$$

One can easily see that $f(\hat{v}_i)$ is increasing for $i \leqslant (d_L - 1)/2$ and decreasing for $i \geqslant (d_L - 1)/2$. Therefore, its minimum is attained at $f(\hat{v}_0) = f(\hat{v}_{d_L-1})$. ∎

Consider a fixed element in the Cartan subalgebra of $\mathfrak{su}(d_L)$, a $d_L \times d_L$ matrix $T = \mathrm{diag}(1, 0, 0, 0, \cdots, -1)$, and T_λ, the corresponding generator in the representation given by the Young diagram λ. We have the following lemma:

Lemma 17. *It holds that* $\|T_\lambda\|_\infty \leqslant \lambda_1$.

Proof of Lemma 17. A basis for the representation λ is given by different semi-standard fillings of the Young diagram λ with numbers $1 \cdots d_L$. If we indicate fillings of the λ by m_λ, then $\{|m_\lambda\rangle\}$ forms a basis for the representation λ. Although this is not an orthogonal basis, if the number content of m_λ and m'_λ are different then $|m_\lambda\rangle$ and $|m'_\lambda\rangle$ are orthogonal. This basis diagonalizes T_λ.

In particular, if $\#_i m_\lambda$ indicates the number of times that i appears in the filling m_λ, then $\langle m_\lambda | T_\lambda | m_\lambda \rangle = \#_1 m_\lambda - \#_d m_\lambda$. This immediately leads to the conclusion that the eigenvalues of T_λ are $\#_1 m_\lambda - \#_d m_\lambda$, for different fillings m_λ.

For any semi-standard filling of the Young diagrams, the numbers are strictly increasing in the columns. Therefore, $\#_i m_\lambda \leqslant \lambda_1$, as there are no repeats in the columns. So we showed that eigenvalues of T_λ are between $-\lambda_1$ and λ_1, which completes the proof. ∎

We are now in a position to prove Theorem 4. First, however, we prove a version of Theorem 4 that provides a stronger bound expressed as a binomial coefficient, which we will use to prove the bounds stated in Theorem 4.

Theorem 18. *Let $V_{L \to A}$ be an isometry that is covariant with respect to the full $SU(d_L)$ group on the logical space, and write $\mathcal{E}(\cdot) = V(\cdot)V^\dagger$. Consider the single erasure noise model represented by $\mathcal{N}$ in (10) with equal erasure probabilities, $q_i = 1/n$ for all i. Then*

$$\max_i d_i \geqslant \binom{d_L - 1 + \lceil (2n\epsilon_{\mathrm{worst}}(\mathcal{N} \circ \mathcal{E}))^{-1} \rceil}{d_L - 1} . \tag{E.14}$$

In terms of the average entanglement fidelity measure, the bound reads instead

$$\max_i d_i \geqslant \binom{d_L - 1 + \lceil (nd_L \epsilon_{\mathrm{e}}(\mathcal{N} \circ \mathcal{E}))^{-1} \rceil}{d_L - 1} . \tag{E.15}$$

The bound in Theorem 18 allows to derive slightly stronger bounds than those obtained from the simplified expressions in Theorem 4. For instance, suppose that $d_L = d_i$ as in the examples given in the main text. The binomial coefficient $\binom{a+b}{b}$ is increasing in b, which can be seen using the recurrence relation $\binom{a+b+1}{b+1} = \frac{a+b+1}{b+1}\binom{a+b}{b} \geqslant \binom{a+b}{b}$. Also, the binomial coefficient $\binom{a+b}{b}$ for $b \geqslant 2$ satisfies $\binom{a+b}{b} \geqslant \binom{a+2}{2} = (a+2)(a+1)/2 \geqslant a+2$ (assuming $a \geqslant 1$). Hence, if $d_L = d_i$, then condition (E.14) implies that $\lceil (2n\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}))^{-1}\rceil \leqslant 1$, because otherwise we would have $\binom{d_L - 1 + \lceil (2n\epsilon_{\text{worst}}(\mathcal{N}\circ\mathcal{E}))^{-1}\rceil}{d_L - 1} \geqslant \binom{d_L - 1 + 2}{2} \geqslant d_L + 1$. This implies that, for $d_L = d_i$, we must have $\epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E}) \geqslant 1/(2n)$.

Proof of Theorem 18. Combining Lemma 16 and Lemma 17, we get

$$D_\lambda \geqslant \binom{d_L - 1 + \lceil \|T_\lambda\|_\infty \rceil}{d_L - 1} \tag{E.16}$$

Now, we return to the original problem of approximate Eastin-Knill theorem, where the group $SU(d_L)$ acts on physical subsystems. We fix the generator $T = \text{diag}(1, 0, 0, 0, \cdots, -1)$ of $\mathfrak{su}(d_L)$, and let T_i be the corresponding generator acting on the subsystem i. Let $T_i = \bigoplus_\lambda T_\lambda$ be the decomposition of T_i with respect to the decomposition of the representation on subsystem i, and assume that $\hat{\lambda}(i)$ is the Young diagram in this direct sum with the largest $\|T_\lambda\|_\infty$. Therefore, $\|T_i\|_\infty = \|T_{\hat{\lambda}(i)}\|_\infty$, and we have:

$$d_i \geqslant D_{\hat{\lambda}(i)} \geqslant \binom{d_L - 1 + \lceil \|T_{\hat{\lambda}(i)}\|_\infty \rceil}{d_L - 1} = \binom{d_L - 1 + \lceil \|T_i\|_\infty \rceil}{d_L - 1}. \tag{E.17}$$

This implies

$$\max_i d_i \geqslant \max_i \binom{d_L - 1 + \lceil \|T_i\|_\infty \rceil}{d_L - 1} = \binom{d_L - 1 + \lceil \max_i \|T_i\|_\infty \rceil}{d_L - 1}. \tag{E.18}$$

Let i' denote the index of the subsystem that maximizes $\|T_i\|_\infty$, such that our bound (26) with $\Delta T_L = 2$ and $\Delta T_i \leqslant 2\|T_i\|_\infty$ reads

$$\epsilon_{\text{worst}} \geqslant \frac{1}{2n\|T_{i'}\|_\infty}, \tag{E.19}$$

noting that $\max_i \Delta T_i \leqslant 2\max_i\|T_i\|_\infty = 2\|T_{i'}\|_\infty$, and writing $\epsilon_{\text{worst}} = \epsilon_{\text{worst}}(\mathcal{N} \circ \mathcal{E})$ as a shorthand. Therefore, $\|T_{i'}\|_\infty \geqslant (2n\epsilon_{\text{worst}})^{-1}$, and we obtain

$$\max_i d_i \geqslant \binom{d_L - 1 + \lceil (2n\epsilon_{\text{worst}})^{-1}\rceil}{d_L - 1}. \tag{E.20}$$

If we had used the bound (29a) instead of (26), we would have instead of (E.19) that

$$\epsilon_{\text{e}} \geqslant \frac{1}{nd_L\|T_{i'}\|_\infty}, \tag{E.21}$$

and we can perform the replacement $\epsilon_{\text{worst}} \to (d_L/2)\,\epsilon_{\text{e}}$ in the argument above. ∎

Proof of Theorem 4. We use the following standard inequality of binomial coefficients. For

integers $a, b > 0$, we have the two lower bounds

$$\binom{a+b}{a} \geqslant \begin{cases} \left(1 + \frac{a}{b}\right)^b \\ \left(1 + \frac{b}{a}\right)^a , \end{cases} \tag{E.22}$$

noting that $\binom{a+b}{b} = \binom{a+b}{a}$. Consider (E.14), with $a = d_L - 1$ and $b = \lceil (2n\epsilon_{\text{worst}})^{-1} \rceil$. The first lower bound gives us

$$\max_i \ln(d_i) \geqslant b \ln\left(1 + \frac{d_L - 1}{b}\right) \geqslant b \ln \frac{d_L - 1}{b} \geqslant \frac{1}{2n\epsilon_{\text{worst}}} \ln\left[\frac{d_L - 1}{(2n\epsilon_{\text{worst}})^{-1} + 1}\right] , \tag{E.23}$$

and hence

$$\max_i \ln(d_i) \geqslant \frac{1}{2n\epsilon_{\text{worst}}} \ln(d_L - 1) - \frac{\ln\left(1 + (2n\epsilon_{\text{worst}})^{-1}\right)}{2n\epsilon_{\text{worst}}} . \tag{E.24}$$

This proves (52a).

Applying the second bound in (E.22) to (E.14), we obtain

$$\max_i d_i \geqslant \left[\frac{d_L - 1 + \lceil (2n\epsilon_{\text{worst}})^{-1} \rceil}{d_L - 1}\right]^{d_L - 1}$$

$$= \exp\left\{ (d_L - 1) \ln\left(1 + \frac{\lceil (2n\epsilon_{\text{worst}})^{-1} \rceil}{d_L - 1}\right)\right\} . \tag{E.25}$$

We can rearrange (E.25) to

$$\exp\left\{\frac{\max_i \ln(d_i)}{d_L - 1}\right\} - 1 \geqslant \frac{\lceil (2n\epsilon_{\text{worst}})^{-1} \rceil}{d_L - 1} \geqslant \frac{(2n\epsilon_{\text{worst}})^{-1}}{d_L - 1} , \tag{E.26}$$

which in turn implies

$$\epsilon_{\text{worst}} \geqslant \frac{1}{2n(d_L - 1)} \left[\max_i\left(\exp\left\{\frac{\ln(d_i)}{d_L - 1}\right\} - 1\right)\right]^{-1} . \tag{E.27}$$

Henceforth we let i denote the index of the physical subsystem with largest dimension, i.e., $d_i = \max_{i'} d_{i'}$. For large d_L, we have

$$(d_L - 1)\left(\exp\left\{\frac{\ln(d_i)}{d_L - 1}\right\} - 1\right) = \ln(d_i) + O\left(\frac{\ln^2(d_i)}{d_L}\right) = \ln(d_i)\left[1 + O\left(\frac{\ln(d_i)}{d_L}\right)\right] , \tag{E.28}$$

and thus

$$\epsilon_{\text{worst}} \geqslant \frac{1}{2n \max_i \ln(d_i)}\left[1 + O\left(\frac{\ln(d_i)}{d_L}\right)\right] = \frac{1}{2n \max_i \ln(d_i)} + O\left(\frac{1}{nd_L}\right) , \tag{E.29}$$

which is the desired bound (51). The bound (52b) follows from (E.25) by noting that $\lceil (2n\epsilon_{\text{worst}})^{-1} \rceil \geqslant (2n\epsilon_{\text{worst}})^{-1}$ and that $\log(1 + x) \geqslant \log(x)$.

The alternative expressions for ϵ_{e} follow from the use of the bound (E.15), following the same steps as above while effecting the replacement $\epsilon_{\text{worst}} \to d_L \epsilon_{\text{e}}/2$. ∎

Appendix F: Circumventing the Eastin-Knill theorem by randomized constructions

F.1.   Randomized constructions: Overview

The proof of Theorem 5 is technical, and relies on the recent developments in the representation theory of $U(d)$, and new counting formulas for the *Littlewood-Richardson* coefficients. Here, we sketch the proof strategy, and refer the reader to Appendix F for the technical details.

Although our randomized constructions do not properly work for producing good $U(2)$-covariant codes,[10] for the $U(3)$ case we can find explicit (non-asymptotic) bounds with a slightly different scaling. There, one can benefit from the fact that the fusion rules of $U(3)$ representation theory are known [51, Section 5]. We will not discuss $U(3)$ case further, and will focus on $d_L \geqslant 4$ for the rest of this section.

Consider codes that map logical information on the Hilbert space $\mathscr{H}_L$ to three physical subsystems $\mathscr{H}_A = \mathscr{H}_{A_1} \otimes \mathscr{H}_{A_2} \otimes \mathscr{H}_{A_3}$, and denote by d_i the dimension of $\mathscr{H}_{A_i}$. In order to precisely define what we mean by the random isometry V, consider the state corresponding to V (similar to what we did in the analysis of correlation functions in Appendix B). The corresponding state, $|\Psi\rangle$, lives on $\mathscr{H}_R \otimes \mathscr{H}_{A_1} \otimes \mathscr{H}_{A_2} \otimes \mathscr{H}_{A_3}$, where as before $\mathscr{H}_R \simeq \mathscr{H}_L$ is a reference system. The covariance of V translates to the invariance of $|\Psi\rangle$:

$$\left[\overline{U} \otimes r_1(U) \otimes r_2(U) \otimes r_3(U)\right] |\Psi\rangle_{RA_1A_2A_3} = |\Psi\rangle_{RA_1A_2A_3} \text{ for all } U \in U(d_L). \qquad \text{(F.1)}$$

Therefore, $\Psi_{RA_1A_2A_3}$ lives on an invariant subspace of the unitary group. The projector to this invariant subspace is given by

$$\Pi_{RA_1A_2A_3} = \int dU\, \overline{U} \otimes r_1(U) \otimes r_2(U) \otimes r_3(U) . \qquad \text{(F.2)}$$

We denote by $d_P = \text{tr}\left(\Pi_{RA_1A_2A_3}\right)$ the dimension of the invariant subspace. Further, define $\Pi_{RA_i} := \text{tr}_{A \backslash A_i}\left(\Pi_{RA_1A_2A_3}\right)$ and $\Pi_{\widehat{RA_i}} := \text{tr}_{RA_i}\left(\Pi_{RA_1A_2A_3}\right)$.

Now, we can chose the state $|\Psi\rangle_{RA_1A_2A_3}$ randomly from $\Pi_{RA_1A_2A_3}$, and *define* V to be the corresponding isometry, i.e., $V_{L \to A_1A_2A_3} := \langle\Phi|_{LR}|\Psi\rangle_{RA_1A_2A_3}$, where $|\Phi\rangle = \sum|k\rangle_L|k\rangle_R$ for some standard choice of bases on $\mathscr{H}_L$ and $\mathscr{H}_R$.

As in Appendix B, we consider single erasures at known locations, i.e., the noise channel is given by $\mathcal{N}(\cdot) = \sum q_i|i\rangle\langle i|_C \otimes \mathcal{N}^i(\cdot)$, where $\mathcal{N}^i$ erases the i-th system as per (11). If the isometry V is chosen at random in the space of covariant isometries, then on average, the fidelity of recovery of the code defined by the isometry is lower bounded as follows.

Lemma 19. *Suppose that the covariant isometry V is chosen randomly as above. Then, the infidelity of the code after erasure of subsystem $i \in \{1, 2, 3\}$, averaged over all covariant*

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[10] More precisely, our techniques do not lead to proper lower bounds for the fidelity of recovery of random $U(2)$-covariant codes, but this might only be caused by not lower bounding the fidelity of recovery with strong enough inequalities.

isometries, satisfies the following inequality:

$$\frac{1}{2}\,\mathbb{E}[\epsilon_{\mathrm{e}}^2(\mathcal{N}^i \circ \mathcal{E})] \leqslant \frac{1}{2}\left\|\frac{\Pi_{RA_i}}{d_P} - \frac{\mathbb{1}_{RA_i}}{d_L d_i}\right\|_1 + \frac{1}{2}\sqrt{d_L d_i}\sqrt{\frac{\mathrm{tr}\big(\Pi_{\widehat{RA_i}}^2\big)}{d_P^2}}\ . \tag{F.3}$$

Intuitively, Lemma 19 states that in order to get good quantum codes we need to do the followings:

1. Control the constant offset, $\left\|d_P^{-1}\Pi_{RA_i} - \mathbb{1}_{RA_i}/(d_R d_i)\right\|_1$. This can be achieved by making sure that Π_{RA_i} is close to a multiple of identity.

2. Control the *fluctuations* by minimizing $d_p^{-2}\,\mathrm{tr}\big(\Pi_{\widehat{RA_i}}^2\big)$. Note that this is the purity of density matrix $\Pi_{\widehat{RA_i}}/d_p$, so it would be small if $\Pi_{\widehat{RA_i}}$ is close to a multiple of a projector.

Lemma 19 is how far we can go without discussing the detailed representation theory of $U(d_L)$. From now on, we focus on analyzing Π_{RA_i} and $\Pi_{\widehat{RA_i}}$.

Without loss of generality assume that $i = 1$. Also, suppose λ, μ, ν are the Young diagrams defining the irreducible representations r_1, r_2 and r_3. Similarly, $r_{e_1}(U) = U$, where $e_1 = (1, 0, 0, \cdots, 0)$ is the Young diagram of the standard representation. Now, we use representation theory techniques to explicitly compute Π_{RA_1} and $\Pi_{\widehat{RA_1}} = \Pi_{A_2 A_3}$.

The degeneracies of fusion of different irreps of $U(d_L)$ are known, and specified by the so called Littlewood-Richardson coefficients $c_{\mu\nu}^\theta$:

$$r_\mu \otimes r_\nu = \bigoplus_\theta r_\theta \otimes I_{c_{\mu\nu}^\theta}\ . \tag{F.4}$$

A specific case of this formula which is also applicable to our analysis is a version *Pieri formula* (See Appendix A.1 of [84]): If e_i is the i-th computation basis vector, then

$$\bar{r}_{e_1} \otimes r_\lambda = \bigoplus_{i \in \mathcal{I}} r_{\lambda - e_i} \tag{F.5}$$

where $\mathcal{I} \in \{1, 2, \cdots, d_L\}$ is the index set that $\lambda - e_i$ is a valid Young diagram, i.e., a non-increasing sequence. In particular, if λ is strictly decreasing then $\mathcal{I} = \{1, 2, \cdots, d_L\}$. This relation can be derived by either directly applying the Littlewood-Richardson rule [84, Appendix A.1], or starting from the standard Pieri formula and dualizing representations. With this, we have

$$\Pi_{RA_1 A_2 A_3} = \int dU (\bigoplus_{i \in \mathcal{I}} r_{\lambda - e_i}(U)) \otimes (\bigoplus_\theta r_\theta(U) \otimes I_{c_{\mu\nu}^\theta}). \tag{F.6}$$

From the Schur orthogonality relations for compact groups (Peter-Weyl theorem), we have that $\int dU [\mathrm{tr}\,\overline{r_\beta(U)}]r_\alpha(U) = \delta_{\alpha\beta}I_\alpha/d_\alpha$. Applying this to Eq. (F.6) we get the following

explicit relations:

$$\Pi_{RA_1} = \bigoplus_{i \in \mathcal{I}} \frac{c_{\mu\nu(\lambda-e_i)}}{d_{\lambda-e_i}} I_{d_{\lambda-e_i}}, \tag{F.7}$$

$$\Pi_{A_2 A_3} = \bigoplus_{i \in \mathcal{I}} \frac{1}{d_{\lambda-e_i}} I_{d_{\lambda-e_i}} \otimes I_{c_{\mu\nu(\lambda-e_i)}}, \tag{F.8}$$

where $c_{\mu\nu\lambda} := c_{\mu\nu}^{\overline{\lambda}}$, and $\overline{\lambda}$ is the dual of λ. Recall that in order for the random codes to perform well, we need that Π_{RA_1} and $\Pi_{A_2 A_3}$ to be close to multiples of projectors. Equations (F.7) and (F.8) show that to achieve this we only need $\frac{c_{\mu\nu(\lambda-e_i)}}{d_{\lambda-e_i}}$ and $\frac{1}{d_{\lambda-e_i}}$ to be almost constants as i varies. The following lemma makes this observation quantitative:

Lemma 20. *Suppose that $0 \leqslant \delta \leqslant 1/2$ is a real number such that for all $i \in \mathcal{I}$,*

$$1 - \delta \leqslant \frac{c_{\mu\nu(\lambda-e_i)}}{c_{\mu\nu\lambda}} \leqslant 1 + \delta \tag{F.9}$$

$$1 - \delta \leqslant \frac{d_{\lambda-e_i}}{d_\lambda} \leqslant 1 + \delta, \tag{F.10}$$

then,

$$\frac{1}{2} \mathbb{E}[\epsilon_{\mathrm{e}}^2(\mathcal{N}^1 \circ \mathcal{E})] \leqslant 4\delta + \frac{5}{2\sqrt{c_{\mu\nu\lambda}}} . \tag{F.11}$$

(Proof on page 63.)

Lemma 20 demonstrates that in order to get useful lower bounds on the fidelity of the codes, one has to show that d_λ and $c_{\mu\nu\lambda}$ are stable under perturbations by e_i. We construct our irreps such that they achieve this stability.

Define $|\lambda| := \sum_i \lambda_i$ for arbitrary Young diagram λ. It is known that if $|\mu| + |\nu| + |\lambda| \neq 0$, then $c_{\mu\nu\lambda} = 0$. Now, the construction is as follows: Fix a triplet of Young diagrams $(\hat{\mu}, \hat{\nu}, \hat{\lambda})$ such that $|\hat{\mu}_i| + |\hat{\nu}_i| + |\hat{\lambda}_i| = 0$ and set

$$(\mu, \nu, \lambda) = (N\hat{\mu} + e_1, N\hat{\nu}, N\hat{\lambda}), \tag{F.12}$$

for large values of N. We used $N\hat{\mu} + e_1$ instead of $N\hat{\mu}$ is to ensure that $|\mu| + |\nu| + |\lambda - e_i| = 0$ as we need $c_{\mu\nu(\lambda-e_i)}$ to be non-zero.

Showing smoothness of d_λ is much simpler, because by the *Weyl dimension formula* (see [84, Section 15.3]) it is polynomial in $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{d_L})$. So by basic Taylor expansion we have $d_{\lambda+e_i} = d_\lambda + \partial d_\lambda/\partial \lambda_i + 1/2\, \partial^2 d_\lambda/\partial \lambda_i^2 + \cdots$. Note that the total degree of the terms in the sum decreases by differentiation. Hence, d_λ is the dominant term in the expansion of $d_{\lambda+e_i}$ and other terms are lower order in N. Therefore, there exist N_0' and C_0' such that for $N \geqslant N_0'$,

$$1 - \frac{C_0'}{N} \leqslant \frac{d_{\lambda-e_i}}{d_\lambda} \leqslant 1 + \frac{C_0'}{N}. \tag{F.13}$$

The Littlewood-Richardson coefficients are much more complicated. They can be computed using efficient algorithms, such as the Littlewood-Richardson rule, but there is no explicit formula. In fact, they are specific cases of the called *Kronecker coefficients* whose computation is known to be NP-hard [85]. However, a series of new developments in the representation theory of the unitary group has revealed interesting polynomiality properties for the LR coefficients.

It is known that $c_{\mu\nu\lambda}$ as a function of μ, ν and λ (in the $3d_L - 1$ dimensional subspace constrained by the condition $|\mu| + |\nu| + |\lambda| = 0$) is non-zero if and only if (μ, ν, λ) is in a particular convex cone. This cone, or *chamber complex*, is then divided to several sub-cones or *chambers*. In Ref. [51] it is shown that $c_{\mu\nu\lambda}$ is a polynomial within each chamber. See Fig. 5.

We chose $(\hat{\mu}, \hat{\nu}, \hat{\lambda})$ such that it is in the interior of one of the chambers, and $c_{N\hat{\mu}, N\hat{\nu}, N\hat{\lambda}}$ is not constant. If N is large enough, $c_{\mu\nu(\lambda - e_i)}$ will remain in the interior of the same chamber for all i, and are described by the same polynomial. Hence, similar to d_λ, we have:

$$1 - \frac{C_0''}{N} \leqslant \frac{c_{\mu\nu(\lambda - e_i)}}{c_{\mu\nu\lambda}} \leqslant 1 + \frac{C_0''}{N} \ , \tag{F.14}$$

where $N \geqslant N_0''$ and for some N_0'' and C_0''. Clearly, these bounds the smoothness required for Lemma 20 to work, and therefore we get our main theorem. Their detailed proof will be in Appendix F.

Theorem 21. *Suppose that $d_L \geqslant 4$. There exist Young diagrams $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\nu}$, an integer N_0, and a constant C_0, such that if $(\mu, \nu, \lambda) = (N\hat{\mu} + e_1, N\hat{\nu}, N\hat{\lambda})$ and V is a random covariant isometry in the sense of (53), we have,*

$$\epsilon_e(\mathcal{N} \circ \mathcal{E}) \leqslant \frac{C_0}{\sqrt{N}} \quad for \quad N \geqslant N_0 \ . \tag{F.15}$$

For these constructions, we have,

$$\epsilon_e(\mathcal{N} \circ \mathcal{E}) \leqslant C_1 \left(\max_i d_i \right)^{-\frac{1}{d_L(d_L - 1)}} \ . \tag{F.16}$$

(Proof on page 65.)

Finally, Theorem 5 follows immediately from Theorem 21.

F.2.   Randomized constructions: Detailed proofs

First, we prove Lemma 19.

Proof of Lemma 19.   First, we express the error-correcting accuracy of the code V according to the average entanglement fidelity in terms of the distance of the codewords to a maximally mixed state, including the reference system. By Bény/Oreshkov (12a) (choosing $\zeta_E = \mathbb{1}_{A_i}/d_i$), we have

$$f_e(\mathcal{N}^i \circ \mathcal{E}) \geqslant F\big(\widehat{\mathcal{N}^i \circ \mathcal{E}}(\hat{\phi}_{LR}), \zeta_E \otimes \hat{\phi}_R \big) = F\left(\Psi_{RA_i}, \frac{\mathbb{1}_{A_i}}{d_i} \otimes \frac{\mathbb{1}_R}{d_L} \right) \ , \tag{F.17}$$

and hence

$$\epsilon_e^2(\mathcal{N}^i \circ \mathcal{E}) \leqslant 1 - F^2\big(\Psi_{RA_i}, \frac{\mathbb{1}_{RA_i}}{d_L d_i} \big) \leqslant \left\| \Psi_{RA_i} - \frac{\mathbb{1}_{RA_i}}{d_L d_i} \right\|_1 \ , \tag{F.18}$$

where we recall the usual relations between trace distance and the fidelity.

We denote by $\mathbb{E}$ the averaging over all possible invariant states $\Psi_{RA_1 A_2 A_3}$. Taking an average

over (F.18) gives us

$$\frac{1}{2}\,\mathbb{E}(\epsilon_{\mathrm{e}}^2(\mathcal{N}^i \circ \mathcal{E})) \leqslant \frac{1}{2}\,\mathbb{E}\,\|\Psi_{RA_i} - \tau_{RA_i}\|_1\;,\qquad\qquad (\mathrm{F.19})$$

where we write $\tau_{RA_i} = \mathbb{1}_{RA_i}/(d_L d_i)$. Applying triangle inequality, Cauchy-Schwarz inequality, and the concavity of square root gives us (see Ref. [26] for similar calculations),

$$\begin{aligned}
\frac{1}{2}\,\mathbb{E}(\epsilon_{\mathrm{e}}^2(\mathcal{N}^i \circ \mathcal{E})) &\leqslant \frac{1}{2}\,\mathbb{E}\,\|\Psi_{RA_i} - \tau_{RA_i}\|_1\\
&\leqslant \frac{1}{2}\|\mathbb{E}\,\Psi_{RA_i} - \tau_{RA_i}\|_1 + \frac{1}{2}\,\mathbb{E}\|\Psi_{RA_i} - \mathbb{E}\,\Psi_{RA_i}\|_1\\
&\leqslant \frac{1}{2}\|\mathbb{E}\,\Psi_{RA_i} - \tau_{RA_i}\|_1 + \frac{1}{2}\sqrt{d_R d_i\left(\mathrm{tr}[\mathbb{E}\,\Psi_{RA_i}^2] - \mathrm{tr}[(\mathbb{E}\,\Psi_{RA_i})^2]\right)}.\quad(\mathrm{F.20})
\end{aligned}$$

Now, consider the rank-d_P projector to the invariant space $\Pi_{RA_1 A_2 A_3}$ that we constructed in §VII A. Define $L : \mathbb{C}^{d_P} \to \mathscr{H}_A \otimes \mathscr{H}_R$ be the isometry mapping to the invariant space, which satisfies

$$L^\dagger L = \mathbb{1}_{d_P}\;;\;\text{ and }\qquad\qquad LL^\dagger = \Pi_{RA_1 A_2 A_3}\;.\qquad\qquad (\mathrm{F.21})$$

We can define $|\Psi\rangle_{RA} = L|\chi\rangle$, where $|\chi\rangle$ is a random state in $\mathbb{C}^{d_p}$. Then,

$$\mathbb{E}\,\Psi_{RA_i} = \mathbb{E}\,\mathrm{tr}_{AA_i}\big(L\chi L^\dagger\big) = \mathrm{tr}_{\widehat{RA_i}}\big(L(\mathbb{E}\,\chi)L^\dagger\big) = \frac{1}{d_P}\,\mathrm{tr}_{\widehat{RA_i}}\big(LL^\dagger\big) = \frac{\Pi_{RA_i}}{d_P}\;,\qquad(\mathrm{F.22})$$

where we used $\mathbb{E}\,\chi = \mathbb{1}/d_P$. For simplicity, we henceforth set $i = 1$ without loss of generality. If $\mathcal{F}_{A_2 A_3}$ is flip operator swapping two copies of the Hilbert space $\mathscr{H}_{A_2 A_3} = \mathscr{H}_{A_2} \otimes \mathscr{H}_{A_3}$, we have

$$\begin{aligned}
\mathbb{E}\,\mathrm{tr}[\Psi_{RA_1}^2] &= \mathbb{E}\,\mathrm{tr}[\Psi^{\otimes 2}\mathcal{F}_{A_2 A_3}] = \mathrm{tr}[L^{\otimes 2}\,\mathbb{E}\,\chi^{\otimes 2}L^{\dagger \otimes 2}\mathcal{F}_{A_2 A_3}]\\
&= \mathrm{tr}[L^{\otimes 2}\frac{I + \mathcal{F}}{d_P(d_P + 1)}L^{\dagger \otimes 2}\mathcal{F}_{A_2 A_3}] = \frac{1}{d_P(d_P + 1)}\,\mathrm{tr}[\Pi_{RA}^{\otimes 2}(\mathcal{F}_{RA_1} + \mathcal{F}_{A_2 A_3})]\\
&= \frac{\mathrm{tr}(\Pi_{RA_1}^2) + \mathrm{tr}(\Pi_{A_2 A_3}^2)}{d_P(d_P + 1)}\;.
\end{aligned}\qquad(\mathrm{F.23})$$

Substituting into (F.20) and applying basic inequalities lead to,

$$\begin{aligned}
\frac{1}{2}\,\mathbb{E}(\epsilon_{\mathrm{e}}^2(\mathcal{N}^1 \circ \mathcal{E})) &\leqslant \frac{1}{2}\left\|\frac{\Pi_{RA_i}}{d_P} - \tau_{RA_1}\right\|_1 + \frac{1}{2}\frac{\sqrt{d_R d_1}}{d_P}\sqrt{\frac{1}{1 + 1/d_P}\,\mathrm{tr}(\Pi_{A_2 A_3}^2) - \frac{1}{d_P + 1}\,\mathrm{tr}(\Pi_{RA_1}^2)}\\
&\leqslant \frac{1}{2}\left\|\frac{\Pi_{RA_1}}{d_P} - \tau_{RA_1}\right\|_1 + \frac{1}{2}\frac{\sqrt{d_R d_1}}{d_P}\sqrt{\mathrm{tr}(\Pi_{A_2 A_3}^2)}\;,\qquad(\mathrm{F.24})
\end{aligned}$$

which is the desired formula. $\blacksquare$

Next, we prove Lemma 20.

Proof of Lemma 20. For simplicity of exposition, define two probability distributions

$p, q : \mathcal{I} \to \mathbb{R}_{\geqslant 0}$,

$$p_i = \frac{c_{\mu\nu(\lambda - e_i)}}{d_P} \; ; \qquad\qquad q_i = \frac{d_{\lambda - e_i}}{d_R d_1} \; . \qquad\qquad \text{(F.25)}$$

From Lemma 19, we have,

$$\frac{1}{2}\,\mathbb{E}(\epsilon_e^2(\mathcal{N}^i \circ \mathcal{E})) \leqslant \frac{1}{2}\left\|\frac{\Pi_{RA_i}}{d_P} - \tau_{RA_i}\right\|_1 + \frac{1}{2}\sqrt{d_L d_1}\sqrt{\frac{\operatorname{tr}\Pi_{A_2 A_3}^2}{d_P^2}} \; . \qquad \text{(F.26)}$$

We would like to bound both terms on the right hand side of (F.26). We have

$$\left\|\frac{\Pi_{RA_i}}{d_P} - \tau_{RA_i}\right\|_1 = \sum_{i \in \mathcal{I}} d_{\lambda - e_i}\left|\frac{c_{\mu\nu(\lambda - e_i)}}{d_P d_{\lambda - e_i}} - \frac{1}{d_R d_1}\right| = \sum_{i \in \mathcal{I}} |p(i) - q(i)| \; . \qquad \text{(F.27)}$$

Also,

$$\operatorname{tr}\!\big(\Pi_{A_2 A_3}^2\big) = \sum_{i \in \mathcal{I}} (d_{\lambda - e_i} c_{\mu\nu(\lambda - e_i)})\frac{1}{d_{\lambda - e_i}^2} = \frac{d_P}{d_R d_1}\sum_{i \in \mathcal{I}} \frac{p(i)}{q(i)} \; . \qquad \text{(F.28)}$$

Now, the condition of the lemma can be written as

$$1 - \delta \leqslant \frac{p_i}{c_{\mu\nu\lambda}/d_P} \leqslant 1 + \delta \; . \qquad\qquad \text{(F.29)}$$

By summing over i, we get,

$$\frac{1}{|\mathcal{I}|(1 + \delta)} \leqslant \frac{c_{\mu\nu\lambda}}{d_P} \leqslant \frac{1}{|\mathcal{I}|(1 - \delta)} \; . \qquad\qquad \text{(F.30)}$$

With some algebra, we obtain

$$\left|p_i - \frac{1}{|\mathcal{I}|}\right| \leqslant \left|p_i - \frac{c_{\mu\nu\lambda}}{d_P}\right| + \left|\frac{c_{\mu\nu\lambda}}{d_P} - \frac{1}{|\mathcal{I}|}\right| \leqslant \frac{\delta}{|\mathcal{I}|(1 - \delta)} + \frac{\delta}{|\mathcal{I}|(1 - \delta)} = \frac{4\delta}{|\mathcal{I}|} \; . \qquad \text{(F.31)}$$

Similarly, $\left|q_i - 1/|\mathcal{I}|\right| \leqslant 4\delta/|\mathcal{I}|$. Therefore,

$$\left\|\frac{\Pi_{RA_i}}{d_P} - \tau_{RA_i}\right\|_1 = \sum_{i \in \mathcal{I}} |p_i - q_i| \leqslant \sum_{i \in \mathcal{I}}\left|p_i - \frac{1}{|\mathcal{I}|}\right| + \left|q_i - \frac{1}{|\mathcal{I}|}\right| \leqslant 8\delta \; . \qquad \text{(F.32)}$$

On the other hand, $p_i \leqslant (1 + \delta)\frac{c_{\mu\nu\lambda}}{d_P}$, and $1/q_i \leqslant \frac{d_R d_1}{d_\lambda (1 - \delta)}$. Now, we get that $p_i/q_i \leqslant (1 + \delta)^2/(1 - \delta)^2$. So,

$$\operatorname{tr}\!\big(\Pi_{A_2 A_3}^2\big) = \frac{d_P}{d_R d_1}\sum_{i \in \mathcal{I}} \frac{p_i}{q_i} \leqslant \frac{d_P |\mathcal{I}|}{d_R d_1}\left(\frac{1 + \delta}{1 - \delta}\right)^2$$

$$\leqslant \frac{d_P^2}{d_R d_1 c_{\mu\nu\lambda}}\frac{(1 + \delta)^2}{(1 - \delta)^3} \leqslant 5^2 \frac{d_P^2}{d_R d_1 c_{\mu\nu\lambda}} \; . \qquad \text{(F.33)}$$

Substituting in the formula for the fidelity completes the proof. ∎

Next, we would like to prove our main theorem on random constructions, Theorem 21. Before that, we need to show that the Littlewood-Richardson coefficients can grow significantly with the size the Young diagrams. This is the content of next lemma:

Lemma 22. *In the chamber complex of Littlewood-Richardson coefficients discussed in §VII A, there are chambers in which $c_{\mu\nu\lambda}$ is a polynomial of degree $\binom{d_L-1}{2}$*

Proof of Lemma 22. Consider the following relation for the Littlewood-Richardson coefficients, derived by comparing dimensions:

$$d_\mu d_\nu = \sum_\lambda c_{\mu\nu\lambda} d_\lambda. \tag{F.34}$$

Define the average of Littlewood-Richardson coefficients weighted by the dimension d_λ, i.e.,

$$\bar{c} = \frac{\sum_\lambda c_{\mu\nu\lambda} d_\lambda}{\sum_\lambda d_\lambda} . \tag{F.35}$$

Also, assume that the number of λ's where $c_{\mu\nu\lambda} \neq 0$ is $N_{\mu\nu}$ and the average dimension of d_λ, averaged over such λ's is,

$$\bar{d} = \frac{\sum_{\lambda \text{ where } c_{\mu\nu\lambda} \neq 0} d_\lambda}{N_{\mu\nu}} . \tag{F.36}$$

Now (F.34) becomes

$$\frac{d_\mu d_\nu}{N_{\mu\nu}\bar{d}} = \bar{c} . \tag{F.37}$$

Consider the case where $\mu = N\mu_0$ and $\nu = N\nu_0$, for some fixed μ_0 and ν_0 and large N. It is known that the dimension of the chamber complex is $3d_L - 1$, see, e.g., Proposition 1 in [86]. Therefore, as two d_L dimensional axis are fixed by μ and ν, the section of the cone corresponding to $c_{\mu\nu\lambda} \neq 0$ is $d_L - 1$ dimensional, and therefore $N_{\mu\nu} = O(N^{d_L-1})$. From the Weyl dimension formula, it is known that d_μ, d_ν , and $\bar{d}$ are all $O\left(N^{d_L(d_L-1)/2}\right)$. So,

$$\bar{c} = O\left((N^{(d_L-1)(d_L-2)/2}\right) . \tag{F.38}$$

This shows that there exists at least one chamber whose polynomial is at least degree $\binom{d_L-1}{2}$. On the other hand, it is known that degree of the polynomials are bounded above by $\binom{d_L-1}{2}$ (see Corollary 4.2 in [51]). This completes the proof. ∎

Proof of Theorem 21. We start from Eqs. (F.13) and (F.14). If we set $C_0 = \max(C_0', C_0'')$ and $N_0 = \max(N_0', N_0'')$, we have

$$1 - \frac{C_0}{N} \leqslant \frac{d_{\lambda-e_i}}{d_\lambda} \leqslant 1 + \frac{C_0}{N} ; \tag{F.39a}$$

$$1 - \frac{C_0}{N} \leqslant \frac{c_{\mu\nu(\lambda-e_i)}}{c_{\mu\nu\lambda}} \leqslant 1 + \frac{C_0}{N} . \tag{F.39b}$$

Further, suppose that $\hat{\mu}, \hat{\nu}$, and $\hat{\lambda}$ where chosen such that $c_{\mu\nu\lambda}$ grows superlinearly as a function of N. This is possible for $d_L \geqslant 4$ as a result of Lemma 22. Using this fact and

Lemma 20, we get

$$\mathbb{E}(\epsilon_{\rm e}^2(\mathcal{N}^1 \circ \mathcal{E})) = O(1/N) . \tag{F.40}$$

In fact, the same relation holds for $\epsilon_{\rm e}(\mathcal{N}^2 \circ \mathcal{E})$ and $\epsilon_{\rm e}(\mathcal{N}^3 \circ \mathcal{E})$, and using the Markov inequality and the union bound we can show that there exists $\hat{\mu}, \hat{\nu}$, and $\hat{\lambda}$ for which

$$\max\big(\epsilon_{\rm e}^2(\mathcal{N}^1 \circ \mathcal{E}), \epsilon_{\rm e}^2(\mathcal{N}^2 \circ \mathcal{E}), \epsilon_{\rm e}^2(\mathcal{N}^3 \circ \mathcal{E})\big) = O(1/N) . \tag{F.41}$$

As a consequence, and using Lemma 24, we get (F.15). The second equation, (F.16), follows from (F.15) using the Weyl dimension which indicates that $d_i = O\left(N^{d_L(d_L-1)/2}\right)$. ∎

Appendix G: Some general lemmas

A first lemma relates the correctability of the code to the environment's ability to distinguish two states in terms of the trace distance.

Lemma 23. *For any encoding channel $\mathcal{E}$ and noise channel $\mathcal{N}$, and for any two logical states σ_L, σ_L', and if $\widehat{\mathcal{N} \circ \mathcal{E}}$ is a complementary channel of $\mathcal{N} \circ \mathcal{E}$, we have that*

$$\epsilon_{\rm worst}(\mathcal{N} \circ \mathcal{E}) \geqslant \frac{1}{2} \delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L), \widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L')\big) . \tag{G.1}$$

Proof of Lemma 23. Let ζ be the state achieving the optimum in (12b). We have

$$\begin{aligned}
\epsilon_{\rm worst}^2(\mathcal{N} \circ \mathcal{E}) &= 1 - f_{\rm worst}^2(\mathcal{N} \circ \mathcal{E}) \\
&= 1 - \min_{\phi_{LR}} F^2(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \mathcal{T}_\zeta(\phi_{LR})) \\
&= \max_{\phi_{LR}}\big[1 - F^2(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta \otimes \phi_R)\big] \\
&\geqslant \max_{\phi_{LR}} \delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\phi_{LR}), \zeta \otimes \phi_R\big)^2 ,
\end{aligned} \tag{G.2}$$

recalling that the trace distance obeys $\delta(\rho, \sigma) \leqslant \sqrt{1 - F^2(\rho, \sigma)}$ (see, e.g., [82]). Choosing the optimization candidates $\sigma_L \otimes |0\rangle\langle0|_R$ and $\sigma_L' \otimes |0\rangle\langle0|_R$ in the last inequality, we obtain both

$$\epsilon_{\rm worst}(\mathcal{N} \circ \mathcal{E}) \geqslant \delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L), \zeta\big) ; \tag{G.3}$$

$$\epsilon_{\rm worst}(\mathcal{N} \circ \mathcal{E}) \geqslant \delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L'), \zeta\big) . \tag{G.4}$$

Hence, by triangle inequality,

$$\delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L), \widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L')\big) \leqslant \delta\big(\widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L), \zeta\big) + \delta\big(\zeta, \widehat{\mathcal{N} \circ \mathcal{E}}(\sigma_L')\big) \leqslant 2\,\epsilon_{\rm worst}(\mathcal{N} \circ \mathcal{E}) . ∎$$

The following lemma relates the global fidelity of the code to the fidelities corresponding to the correction of individual errors. Note that we do not necessarily expect a similar result

to hold for the worst-case entanglement fidelity, because the worst-case input state might be different for each erasure event.

Lemma 24. *Let $\mathcal{N}_{A\to A}^{\alpha}$ and $\mathcal{N}_{A\to AC}(\cdot) = \sum q_\alpha \mathcal{N}^\alpha(\cdot) \otimes |\alpha\rangle\langle\alpha|_C$ correspond to a noise model of erasures at known locations, as given in (11). Then, for any $|\phi\rangle_{LR}$, the average entanglement fidelity of the code with respect to $|\phi\rangle_{LR}$ is directly related to the individual fidelities of recovery for each possible erasure:*

$$f_{|\phi\rangle}^2(\mathcal{N} \circ \mathcal{E}) = \sum q_\alpha f_{|\phi\rangle}^2(\mathcal{N}^\alpha \circ \mathcal{E}) \ , \tag{G.5}$$

and consequently,

$$\epsilon_{|\phi\rangle}^2(\mathcal{N} \circ \mathcal{E}) = \sum q_\alpha \epsilon_{|\phi\rangle}^2(\mathcal{N}^\alpha \circ \mathcal{E}) \ . \tag{G.6}$$

Proof of Lemma 24. The average entanglement fidelity associated with the different noise channels can be written as:

$$f_{|\phi\rangle}^2(\mathcal{N} \circ \mathcal{E}) = \max_{\mathcal{R}} \langle\phi|_{LR} \big[\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}(\phi_{LR})\big]|\phi\rangle_{LR} \ ; \tag{G.7a}$$

$$f_{|\phi\rangle}^2(\mathcal{N}^\alpha \circ \mathcal{E}) = \max_{\mathcal{R}^\alpha} \langle\phi|_{LR} \big[\mathcal{R}^\alpha \circ \mathcal{N}^\alpha \circ \mathcal{E}(\phi_{LR})\big]|\phi\rangle_{LR} \ , \tag{G.7b}$$

where the optimizations range over recovery channels $\mathcal{R}_{AC\to L}$ and $\mathcal{R}_{A\to L}^\alpha$, respectively. We have

$$\max_{\mathcal{R}} \langle\phi|_{LR} \big[\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}(\phi_{LR})\big]|\phi\rangle_{LR}$$

$$= \max_{\mathcal{R}} \sum q_\alpha \langle\phi|_{LR} \big[\mathcal{R}\big(|\alpha\rangle\langle\alpha|_C \otimes \big(\mathcal{N}^\alpha \circ \mathcal{E}\big)(\phi_{LR})\big)\big]|\phi\rangle_{LR}$$

$$\leqslant \sum q_\alpha \max_{\mathcal{R}} \langle\phi|_{LR} \big[\mathcal{R}\big(|\alpha\rangle\langle\alpha|_C \otimes \big(\mathcal{N}^\alpha \circ \mathcal{E}\big)(\phi_{LR})\big)\big]|\phi\rangle_{LR}$$

$$\leqslant \sum q_\alpha \max_{\mathcal{R}_{A\to L}^\alpha} \langle\phi|_{LR} \big[\mathcal{R}^\alpha\big(\big(\mathcal{N}^\alpha \circ \mathcal{E}\big)(\phi_{LR})\big)\big]|\phi\rangle_{LR} \ , \tag{G.8}$$

showing that

$$f_{|\phi\rangle}^2(\mathcal{N} \circ \mathcal{E}) \leqslant \sum q_\alpha f_{|\phi\rangle}^2(\mathcal{N}^\alpha \circ \mathcal{E}) \ . \tag{G.9}$$

Physically, the reverse inequality follows because a global recovery strategy is to measure the register containing the record that indicates which error occurred, and to apply the optimal recovery strategy corresponding to that error. Specifically, if $\mathcal{R}_{A\to L}^\alpha$ are optimal choices in (G.7b) for each α, then we define

$$\mathcal{R}_{AC\to L}(\cdot) = \sum \mathcal{R}_{A\to L}^\alpha\big(\langle\alpha|(\cdot)|\alpha\rangle_C\big) \ . \tag{G.10}$$

Then,

$$f_{|\phi\rangle}^2(\mathcal{N} \circ \mathcal{E}) \geqslant \langle\phi|_{LR} \big[\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}(\phi_{LR})\big]|\phi\rangle_{LR}$$

$$= \sum q_\alpha \langle\phi|_{LR} \big[\mathcal{R}\big(|\alpha\rangle\langle\alpha|_C \otimes \big(\mathcal{N}^\alpha \circ \mathcal{E}\big)(\phi_{LR})\big)\big]|\phi\rangle_{LR}$$

$$= \sum q_\alpha \langle\phi|_{LR} \big[\mathcal{R}^\alpha\big(\big(\mathcal{N}^\alpha \circ \mathcal{E}\big)(\phi_{LR})\big)\big]|\phi\rangle_{LR}$$

$$= \sum q_\alpha f_{|\phi\rangle}^2 (\mathcal{N}^\alpha \circ \mathcal{E}) \ , \tag{G.11}$$

as claimed. ∎

The following lemma is a technical consequence of the concavity of the fidelity function.

Lemma 25. *Let ρ, σ be two (normalized) quantum states. Let $\tau \geqslant 0$ with $\rho \geqslant \tau$. Then*

$$F(\rho, \sigma) \geqslant \mathrm{tr}(\tau)\, F\Big(\frac{\tau}{\mathrm{tr}(\tau)}, \sigma\Big) \ . \tag{G.12}$$

Proof of Lemma 25. Since $\rho \geqslant \tau$, we have $\rho - \tau =: \Delta \geqslant 0$. Then $\rho = \tau + \Delta = \mathrm{tr}(\tau)\,\frac{\tau}{\mathrm{tr}(\tau)} + \mathrm{tr}(\Delta)\,\frac{\Delta}{\mathrm{tr}(\Delta)}$, and by concavity of the fidelity,

$$F(\rho, \sigma) = F\Big(\mathrm{tr}(\tau)\,\frac{\tau}{\mathrm{tr}(\tau)} + \mathrm{tr}(\Delta)\,\frac{\Delta}{\mathrm{tr}(\Delta)}, \sigma\Big) \geqslant \mathrm{tr}(\tau)\, F\Big(\frac{\tau}{\mathrm{tr}(\tau)}, \sigma\Big) + \mathrm{tr}(\Delta)\, F\Big(\frac{\Delta}{\mathrm{tr}(\Delta)}, \sigma\Big) \ . \tag{G.13}$$

The claim follows by noting that $\mathrm{tr}(\Delta)\, F\big(\Delta/\mathrm{tr}(\Delta), \sigma\big) \geqslant 0$. ∎

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