Many-body quantum magic

Zi-Wen Liu[1, *] and Andreas Winter[2, 3, †]

[1] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[2] *ICREA—Institució Catalana de Recerca i Estudis Avançats, Pg. Lluis Companys, 23, ES-08001 Barcelona, Spain*
[3] *Física Teòrica: Informació i Fenòmens Quàntics, Departament de Física, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain*

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Magic (non-stabilizerness) is a necessary but "expensive" kind of "fuel" to drive universal fault-tolerant quantum computation. To properly study and characterize the origin of quantum "complexity" in computation as well as physics, it is crucial to develop a rigorous understanding of the quantification of magic. Previous studies of magic mostly focused on small systems and largely relied on the discrete Wigner formalism (which is only well behaved in odd prime power dimensions). Here we present an initiatory study of the magic of genuinely many-body quantum states (with focus on the important case of many qubits) at a quantitative level. We first address the basic question of how magical a many-body state can be, and show that the maximum magic of an n-qubit state is essentially n, simultaneously for a range of natural resource measures. As a corollary, we show that the resource theory of magic with stabilizer-preserving free operations is asymptotically reversible. We then show that, in fact, almost all n-qubit pure states have magic of nearly n. In the quest for explicit, scalable cases of highly entangled states whose magic can be understood, we connect the magic of hypergraph states with the second-order nonlinearity of their underlying Boolean functions. Next, we go on and investigate many-body magic in practical and physical contexts. We first consider a variant of measurement-based quantum computation (MBQC) where the client is restricted to Pauli measurements, in which magic is a necessary feature of the initial "resource" state. We show that n-qubit states with nearly n magic, or indeed almost all states, cannot supply nontrivial speedups over classical computers. We then present an example of analyzing the magic of "natural" condensed matter systems. We apply the Boolean function techniques to derive explicit bounds on the magic of the ground states of certain 2D symmetry-protected topological (SPT) phases, and comment on possible further connections between magic and the quantum complexity of matter.

I. INTRODUCTION

The Clifford group and the closely associated stabilizer formalism [1, 2] are a central notion in quantum computation and many related areas. The Clifford group on n qubits is defined as the normalizer of the n-qubit Pauli (aka discrete Weyl) group, and is generated by the Hadamard gate H, CNOT gate, and $\pi/4$ phase gate P. Then the quantum states prepared by Clifford gates from a canonical trivial state are called stabilizer states. The celebrated Gottesman–Knill theorem [2, 3] states that a quantum computation with only Clifford or stabilizer components can be efficiently simulated by a classical computer, or in other words, cannot supply desired quantum computational advantages. Moreover, the logical Clifford operations can be easily implemented in a fault-tolerant fashion (e.g. transversally) in stabilizer quantum error correction codes and are thus considered low-cost for fault tolerance, but for non-Clifford ones this is not always the case. Indeed, the Eastin–Knill theorem [4] indicates that no code can implement a universal gate set transversally. To sum up, the non-stabilizerness, which is now commonly aliased "magic", represents a precious "resource" that is needed to empower practical quantum computation. A rigorous, quantitative understanding of magic would play key roles in the study of quantum computational complexity and advantages in many ways. For example, a direction of great recent interest is to link magic measures to the costs of classical simulation algorithms [5–10]. Indeed, the quantification of other important quantum resource features such as entanglement [11, 12] and coherence [13, 14] has been a characteristic research line of quantum information, which help understand and characterize "quantumness" in various scenarios.

Previous studies on magic largely focused on small or uncorrelated systems, and little is known about how it behaves in entangled many-body states. In particular, the number of stabilizer states grows very rapidly and their geometric structures become highly complicated as one increases the size of the system, which makes the calculation or even numerical analysis of magic measures on large states difficult in general. The discrete Wigner formalism [15, 16], on which many studies of magic are based (see e.g. [8, 17–19]), is easier to deal with and allows for computable magic measures, but only nicely defined and connected to the magic theory for qudits of odd prime dimensions. It would be important to explore the little understood many-entangled-qubit setting. For example, some fundamental questions are: How "magical" can many-body quantum states be and typically is? How

* zliu1@perimeterinstitute.ca
† andreas.winter@uab.cat

can we calculate the magic of many-body states?

Another important motivation comes from a physics perspective. A theme of many-body physics is to characterize or classify different phases of matter according to their physical features such as symmetry, magnetism or superconductivity, through certain quantities like order parameters. A new perspective that has drawn great interest in recent years is to investigate the "quantum complexity" of phases, which is encoded in e.g. the cost of probing or simulating them and their computational power. Important relevant topics include, among others, the computational universality in measurement-based quantum computing (MBQC) [20–23], sign problems for Monte Carlo methods (see e.g. [24–26]), etc. Can we find some "order parameters" that are helpful in probing these computation-related features of many-body systems? Given the fundamental connection between magic and quantum computation, exploring the roles of many-body magic would be a promising direction to go.

In this work, we investigate many-body magic at a quantitative level from both the mathematical and physical perspectives. We first give an overview of magic measures induced from general resource theories, and briefly discuss their relations with several other known magic measures. Importantly, we give an argument about a "range" of resource measures, showing that any measure that satisfies certain consistency conditions in terms of state transformation is sandwiched in between the min-relative entropy of resource and the free robustness. That is, the min-relative entropy of magic and the free robustness of magic are in some sense the "extremes" of the family of (suitably regularized) magic measures. We show that the roof values of such consistent magic measures an n-qubit state are essentially n. An interesting implication is that the resource theory of magic is asymptotically reversible under stabilizer-preserving free operations, thus lifting the ϵ-resource-generation subtlety in the general theory of reversibility [27]. We also show the magic measures typically take value very close to n on an n-qubit pure state, which resembles the situation of entanglement (the well-known Page's theorem and its variants [28–32]). Then, we turn to the quest for explicit methods for analyzing the many-body magic of certain states. In this work we consider the family of hypergraph states [33], which are widely relevant in quantum error correction and fault-tolerance, MBQC, and quantum many-body physics etc. We find that the magic of hypergraph states can be understood via analyzing the second-order nonlinearity of their underlying Boolean functions, thus establishing connections between many-body magic, and Boolean function analysis and coding theory. Next, we make some initial observations about many-body magic in regard to quantum computation and condensed matter physics. First, the questions of how powerful a many-body state is for quantum computation directly arise from measurement-based quantum computation (MBQC) [20, 21], one of the standard models of quantum computation. Here we suggest considering a variant of MBQC which we call Pauli MBQC, where the client is allowed to make Pauli measurements and thus the magic of the computation is completely isolated to the offline-prepared resource state. We find that many-body states with nearly n magic, and indeed almost all states, cannot supply nontrivial speedups over classical computers. Like entanglement for the conventional MBQC [34, 35], here most resource states are too magical for universality, which highlights the curious phenomenon that too much computational resource could actually harm the computational power. This is a no-go result in the high-magic regime, and eventually we would like to find more fine-grained relations between magic and computational power. Then we take a first look at the magic of many-body systems of interest in condensed matter physics. An interesting case is the symmetry-protected topological (SPT) phases, where magic is expected to be a key feature that supports nontrivial physics in beyond 1D [36]. As a demonstration, we employ the Boolean function techniques to derive explicit bounds on the magic of certain representative 2D SPT ground states defined on different lattices, based on that they take the form of hypergraph states. A general observation is that the magic of such SPT states is weak (although generically necessary [36]), which seems to go hand in hand with that they are short-range entangled and is consistent with the Pauli MBQC universality known for some models we studied. We shall lastly give some discussions on possible further relations of many-body magic and the many facets of the quantum complexity of phases of matter. With these attempts, we hope to raise further interest in the characterization and application of many-body magic in condensed matter physics, and stimulate further explorations into the connections between quantum computation, complexity, and condensed matter.

II. MAGIC: RESOURCE THEORY AND MEASURES

Here we review the magic measures we mainly consider in this paper, which are rooted in the resource theory framework, and summarize their relations with other useful measures studied in the literature.

We first formally define notations. The Clifford group on n-qubits $\mathcal{C}_n$ is defined as the normalizer of the n-qubit Pauli group $\mathcal{P}_n$ composed of tensor products of I, X, Y, Z on n qubits with phases $\pm 1, \pm i$:

$$\mathcal{C}_n = \{U : UWU^\dagger \in \mathcal{P}_n, \forall W \in \mathcal{P}_n\}.$$

Then the pure stabilizer states are generated by Clifford group elements acting on the trivial computational basis state $|0\rangle^{\otimes n}$. We denote by STAB the convex hull of all stabilizer states, whose extreme points are precisely the pure stabilizer states. Hence, STAB is a convex polytope inside the set of states, denoted $\mathcal{S}$, in dimension $4^n - 1$ for n qubits, and with $2^{\Theta(n^2)}$ vertices. It is a highly symmetric object, with the Clifford group acting multiply transitively on the vertex set.

The stabilizer polytope STAB is considered to be the set of resource free states in the theory of magic, motivated the Gottesman–Knill theorem that states that its elements evolving under Clifford unitaries are efficiently simulable by classical means, and thus are not universal for quantum computation (unless BQP = BPP). However, adding a non-Clifford unitary gate restores universality, and it was observed by Bravyi and Kitaev [37] that the same results when a pure "magic" state is made available. There are many ways of measuring the degree of magic in a state. From the meta-theory of general resource theories, we have straightaway several standard measures of magic, that satisfy fundamental properties such as monotonicity under free (STAB-preserving, which include Clifford) operations, faithfulness, etc:

- Max-relative entropy of magic:

$$\mathfrak{D}_{\max}(\rho) = \min_{\sigma \in \mathrm{STAB}} D_{\max}(\rho \| \sigma),$$

  with $D_{\max}(\rho\|\sigma) := \log \min\{\lambda : \rho \leq \lambda\sigma\}$. This measure is also known as log-generalized robustness, $\mathfrak{D}_{\max}(\rho) = \log(1 + R_g(\rho))$, where

$$R_g(\rho) = \min s \geq 0 \text{ s.t. } \rho \in (1+s)\,\mathrm{STAB} - s\,\mathcal{S}.$$

- Min-relative entropy of magic:

$$\mathfrak{D}_{\min}(\rho) = \min_{\sigma \in \mathrm{STAB}} D_{\min}(\rho \| \sigma),$$

  with $D_{\min}(\rho\|\sigma) := -\log \operatorname{Tr} \Pi_\rho \sigma$. For a pure state $|\psi\rangle$, $\mathfrak{D}_{\min}(\psi) = -\log \max_{\phi \in \mathrm{STAB}} |\langle\psi|\phi\rangle|^2$.

- Free robustness of magic:

$$R(\rho) = \min s \geq 0 \text{ s.t. } \rho \in (1+s)\,\mathrm{STAB} - s\,\mathrm{STAB}.$$

  The log-free-robustness is $LR(\rho) = \log(1 + R(\rho))$.

In Appendix A, we give an argument supporting that "good" magic measures satisfying certain consistency conditions induced from state transformability are sandwiched between min-relative entropy of magic $\mathfrak{D}_{\min}$ and free robustness of magic LR.

These should be compared to other recently studied important measures, of which we consider the following.

- Stabilizer extent [6]: $\xi(\rho) = 2^{\mathfrak{D}_{\max}(\rho)} = 1 + R_g(\rho)$
- Stabilizer fidelity [6]: for pure ψ, $F(\psi) = 2^{-\mathfrak{D}_{\min}(\psi)}$
- Stabilizer rank [5, 6, 38]: smooth version bounded by $\chi^\epsilon(\psi) \leq 1 + \xi(\psi)/\epsilon^2 = 1 + 2^{\mathfrak{D}_{\max}(\rho)}/\epsilon^2$
- Negativity/mana (for odd prime power dimensions) [18, 39] : $\mathscr{M}(\rho) \leq LR(\rho) + 1$. See Appendix B for some details.

In the present paper we focus on multi-qubit systems, but it is worth noting that the Pauli group, and hence the Clifford group as its normalizer, generalize to arbitrary local dimension d, the theory being algebraically most satisfying if d is a prime power. In the appendix we present some considerations for odd prime power dimensions. In odd dimensions, a necessary (but for mixed states not sufficient) condition for a state being in STAB is that it has non-negative discrete Wigner function [16, 17]. The so-called mana measures how much negativity the Wigner function has [18]. Building on this, the so-called thauma measures [19] are also defined by minimum divergences (such as the max- and min-relative entropies above), but for odd prime power dimensions, relative to positive semidefinite matrices with non-negative discrete Wigner function.

III. BEHAVIORS OF MAGIC MEASURES

As mentioned, the behaviors of magic measures on general, entangled many-body states is little understood before. Here we address the questions of their roof and typical values. It is known that the maximum value of the min- and max-relative entropy of magic over product states (and indeed over all fully separable states) is

$$\mathfrak{D}_{\max}(\mathrm{SEP}) = \mathfrak{D}_{\min}(\mathrm{SEP}) = \log(3-\sqrt{3})n \approx 0.34n,$$

attained on the tensor product of the "golden state" $G = \frac{1}{2}(I + \frac{X+Y+Z}{\sqrt{3}})$ [40], due to weak additivity. Note that these measures carry fundamental operational interpretations in terms of value in transformations. How large can they get when we consider general states?

First, observe that the value of $\mathfrak{D}_{\max}$ or log-generalized-robustness (and so of all entropic measures) is capped at n:

Theorem 1. *On an n-qubit system,* $\max_\rho \mathfrak{D}_{\max}(\rho) \leq n$.

Proof. The generalized robustness of magic is upper bounded by the generalized robustness of coherence, since STAB contains all diagonal density matrices. The maximum value of the log-robustness of coherence is n [41]. □

The free robustness could in general be much larger than the generalized robustness or even infinite (e.g. in coherence theory). But here we find that LR is virtually also bounded above by n:

Theorem 2. *For any n-qubit state ρ, $R(\rho) \leq \sqrt{2^n(2^n+1)}$. Hence,* $\max_\rho LR(\rho) \leq n + 2^{-n-1}$.

Proof. The free robustness is a linear program (LP),

$$1 + R(\rho) = \min \sum_{\phi \in \mathrm{STAB}} |c_\phi| \text{ s.t. } \rho = \sum_{\phi \in \mathrm{STAB}} c_\phi \phi,$$

where c_ϕ are real coefficients. Its dual LP is well known:

$$1 + R(\rho) = \max \operatorname{Tr} \rho A \text{ s.t. } \forall \phi \in \mathrm{STAB}\ |\operatorname{Tr} \phi A| \leq 1,$$

where the maximum runs over Hermitian matrices A.

Thus,

$$\max_\rho 1 - R(\rho) = \max \|A\| \text{ s.t. } \forall\phi \in \mathrm{STAB}\ |\operatorname{Tr}\phi A| \leq 1.$$

We expand A in the Pauli basis, $A = \sum_P \alpha_P P$, so that

$$\|A\|^2 \leq \operatorname{Tr} A^2 = 2^n \sum_P \alpha_P^2. \tag{1}$$

On the other hand, a pure stabilizer state ϕ is given by an abelian subgroup G of the Pauli group, of cardinality 2^n, and a character $\chi : G \to \pm 1$:

$$\phi = 2^{-n} \left(\mathbb{1} + \sum_{P \in G \setminus \mathbb{1}} s_G(P)\chi(P)P \right),$$

where s_G is a characteristic sign function of the abelian subgroup. Thus, for a dual feasible A,

$$\operatorname{Tr}\phi A = \alpha_{\mathbb{1}} + \sum_{P \in G\setminus\mathbb{1}} s_G(P)\chi(P)\alpha_P \leq 1. \tag{2}$$

Now note that $\left[\sqrt{\frac{1}{2^n}}\chi(P)\right]_{P,\chi}$ is a unitary matrix, and so

$$\begin{aligned}\sum_{P\in G} \alpha_P^2 &= \sum_{P\in G} (s_G(P)\alpha_P)^2 \\ &= 2^{-n} \sum_\chi \left(\sum_{P \in G} s_G(P)\chi(P)\alpha_P \right)^2 \leq 1,\end{aligned}$$

the latter because of Eq. (2).

Now, we use the fact [42] that the Pauli group modulo phases is a union of $2^n + 1$ stabilizer subgroups that intersect only in the identity: $\widetilde{\mathcal{P}}_n \setminus \mathbb{1} = \bigcup_{j=0}^{2^n} G_j \setminus \mathbb{1}$. This allows us to obtain from the last equation, by summing over j,

$$\begin{aligned}\sum_P \alpha_P^2 &\leq (2^n+1)\alpha_{\mathbb{1}}^2 + \sum_{P\neq\mathbb{1}} \alpha_P^2 \\ &= \sum_{j=0}^{2^n} \sum_{P\in G_j} \alpha_P^2 \leq 2^n + 1.\end{aligned}$$

Together with Eq. (1), we get $\|A\|^2 \leq 2^n(2^n+1)$, concluding the proof. $\square$

Remark. Observe that in the proof we did not actually use the set of all stabilizer states, only the $2^n(2^n+1)$ states from a complete set of mutually unbiased bases. Nevertheless the bound is already very good.

This result indicates in a rough sense that STAB occupies the whole state space quite well, so that optimizing over all states in the definition of robustness does not help much as compared to optimizing over STAB only. While in the resource theory of entanglement, there are several studies into the relative volume of the separable states, starting with [43], we are not aware of similar results for STAB.

Now that we established the generalized and free log-robustness of an n-qubit state are upper bounded by essentially n, as are all other measures of present interest, we turn to finding highly magical states, i.e. lower bounds in the maximum value of a given measure. To start, we show that the min-relative entropy of magic (and hence of all entropic measures) of a Haar-random state typically gets close to n, which is nearly maximal.

Theorem 3. *Let $|\psi\rangle$ be a random n-qubit state drawn from the Haar measure. Then for any $n \geq 6$,*

$$\Pr\{\mathfrak{D}_{\min}(\psi) < n - 2\log n - 0.63\} < \exp(-n^2). \tag{3}$$

Proof. This result is a nonasymptotic variant of [6, Claim 2]. Let $|\phi\rangle$ be any n-qubit state. For Haar-random $|\psi\rangle$, the probability density function of $\alpha = |\langle\phi|\psi\rangle|^2$ is given by $p(\alpha) = (2^n-1)(1-\alpha)^{2^n-2}$ (see e.g. [44–46]). So the cumulative distribution function is given by $\Pr\{|\langle\phi|\psi\rangle|^2 \geq \beta\} = (1-\beta)^{2^n-1} \leq \exp(-(2^n-1)\beta)$.

By the union bound, we have

$$\Pr\left\{\max_{\phi\in\mathrm{STAB}} |\langle\phi|\psi\rangle|^2 \geq \epsilon\right\} \leq |\mathrm{STAB}_n| \cdot \exp(-(2^n-1)\epsilon), \tag{4}$$

where $|\mathrm{STAB}_n|$ is the cardinality of the set of n-qubit pure stabilizer states. It is known [47] that

$$|\mathrm{STAB}_n| = 2^n \prod_{k=0}^{n-1} \left(2^{n-k}+1\right). \tag{5}$$

It can be verified that $|\mathrm{STAB}_n| = 2^{c_n n^2}$ with c_n monotonically decreasing with n (asymptotically, $|\mathrm{STAB}_n| = 2^{(1/2+o(1))n^2}$). Note that $c_6 \approx 0.784$, so for $n \geq 6$, $|\mathrm{STAB}_n| < 2^{0.78n^2}$. Continuing Eq. (4), for $n \geq 6$ and $\epsilon > 0$,

$$\begin{aligned}\Pr\left\{\max_{\phi\in\mathrm{STAB}} |\langle\phi|\psi\rangle|^2 \geq \epsilon\right\} &< 2^{0.78n^2} \cdot \exp(-(2^n-1)\epsilon) \\ &< \exp(0.54n^2 - (2^n-1)\epsilon) \\ &\leq \exp(0.54n^2 - 2^{n+\log\epsilon}).\end{aligned}$$

By the definition of $\mathfrak{D}_{\min}$, the above translates to

$$\Pr\{\mathfrak{D}_{\min}(\psi) \leq \gamma\} < \exp(0.54n^2 - 2^{n-\gamma}). \tag{6}$$

In order for the r.h.s. to be $\leq \exp(-n^2)$, we need $0.54n^2 - 2^{n-\gamma} \leq -n^2$, which implies that

$$\gamma \geq n - 2\log n - \log(0.54+1) > n - 2\log n - 0.63,$$

so the claimed bound follows. $\square$

Remark. We state the result for $n \geq 6$, simply because for $n < 6$, it turns out that $n - 2\log n - c_n' < 0$ where c_n' is the best corresponding constant emerging from the same derivation, so that the induced bounds are trivial.

The situation is reminiscent to the well-studied case of entanglement, where the Haar-random values of corresponding measures are nearly maximal [28, 31, 32, 34]).

Then an interesting question is when do (approximate) unitary t-designs generate such nearly maximal magic, with high probability. It is recently shown by Haferkamp *et al.* [48] shows that $\widetilde{O}(t^4 \log(1/\epsilon))$ single-qubit non-Clifford gates (independent of n) are sufficient to form ϵ-approximate unitary t-designs for sufficiently large n. Since each single-qubit gate can only generate constant magic (in terms of all consistent magic measures), this result indicates that approximate designs of order at least $t = \widetilde{\Omega}(n^{1/4})$ (treating ϵ as a constant) are needed to guarantee nearly maximal, or indeed even linear magic. In light of a conceptually similar result for entanglement that unitary designs or order $\approx n$ generate nearly maximal min-entanglement entropy [31, 32], we further conjecture that unitary $O(n)$-designs are sufficient to achieve nearly maximal $\mathfrak{D}_{\min}$.

A closely related conclusion is the following.

Theorem 4. *For any n,*

$$\max_{\rho \in \mathcal{S}(\mathcal{H}_2^{\otimes n})} \mathfrak{D}_{\min}(\rho) > n - 2\log n + 0.96. \tag{7}$$

For sufficiently large n, the bound can be improved to

$$\max_{\rho \in \mathcal{S}(\mathcal{H}_2^{\otimes n})} \mathfrak{D}_{\min}(\rho) > n - 2\log n - \log\ln 2 + 1 + \epsilon, \tag{8}$$

for any $\epsilon > 0$.

Proof. Following the derivation of Eq. (6) in the proof of Theorem 3, we obtain

$$\Pr\{\mathfrak{D}_{\min}(\psi) \leq \gamma\} < 2^{c_n n^2} \exp(-2^{n-\gamma}), \tag{9}$$

where $c_n = \log(|\mathrm{STAB}_n|)/n^2$. Hence, as long as

$$\gamma < n - 2\log n - \log(c_n \ln 2), \tag{10}$$

it holds that $\Pr\{\mathfrak{D}_{\min}(\psi) \leq \gamma\} < 1$ and thus $\max_\psi \mathfrak{D}_{\min}(\psi) > \gamma$. For $n \geq 7$, it holds that $c_n < 0.74$, and thus $\max_\psi \mathfrak{D}_{\min}(\psi) > n - 2\log n - \log(0.74\ln 2) > n - 2\log n + 0.96$. Recall that $\mathfrak{D}_{\min}(G^{\otimes n}) = \log(3-\sqrt{3})n \gtrsim 0.34n$, where $G = \frac{1}{2}(I + \frac{X+Y+Z}{\sqrt{3}})$. For $n < 7$, it can be verified that $n - 2\log n + 0.96 < 0.34n$ holds. Hence the first claimed bound follows.

To obtain the second bound for large n, recall that $c_n = 1/2 + o(1)$ as $n \to \infty$ [47] and apply it to Eq. (10). Plugging this into Eq. (10) leads us to the claimed bound. □

In conclusion, we see that the maximum values of all consistent magic measures are approximately n for n-qubit states. The results have interesting implications to the reversibility of magic state transformation. We say a theory is reversible if resource states can be transformed back and forth using the free operations without loss. Ref. [27] showed that reversibility holds asymptotically, i.e. in the i.i.d. limit, for general resource theories satisfying several natural axioms, if the set of free operations not only includes all resource non-generating operations, but one allows approximately non-generating operations, which is a bit unsatisfying from a fundamental conceptual point of view. It would be very interesting to know whether this enlargement of the set of free operations is necessary for asymptotic reversibility. More generally, by results in [40], for theories including magic, if $LR(\rho)$ and $\mathfrak{D}_{\min}(\rho)$ are asymptotically equal to n for some sequence of n-qubit "currency" states ρ_n, for large n, then the two rates, of distillation and of formation, using these states as currency, are asymptotically identical, meaning that the resource theory is asymptotically reversible, without the need of ϵ-generating transformations. Thus our results indicate that magic theory is asymptotically reversible, with vanishingly small rate loss in the asymptotic i.i.d. limit, under the set of STAB-preserving operations.

Here we briefly discuss some implications on the cost of classical simulation of certain quantum computations, a research direction of great recent interest. In particular, assuming that Clifford operations are "free", one is interested in how the cost of certain algorithms scales with the amount of magic. Here one considers the general quantum computation model built upon Clifford operations (gates, measurements), and magic states which are used e.g. to emulate certain non-Clifford gate (such as the standard case of using $|T\rangle$-states to emulate T-gates in Clifford+T circuits) by state injection gadgets [49], or as the resource state of Pauli MBQC (see Sec. V). For example, there are two leading methods: (i) Stabilizer decomposition, for which the cost is determined by the (smooth) stabilizer rank [6, 38, 50]; (ii) Quasiprobability method based on stabilizer pseudomixture (also applies to mixed states), for which the cost is determined by the free robustness of magic [7]. Improvements over brute-force simulation rely on very special input magic states that admit "good" decompositions, such as an array of $|T\rangle$-states when we know the Clifford+T circuit representation or the T-count is low. The fact that almost all states must have $LR \approx n$ and maximal stabilizer rank (because lower-rank states are only a finite number of lower-dimensional manifolds, which form a measure-zero set) tells us that these simulation methods typically give us no improvement (even in the exponent) over brute-force methods. Interestingly, for Pauli MBQC, this indicates that most input states have too much magic so that the decomposition methods have almost maximal simulation cost but the computation can actually be well simulated by trivial distributions (see Sec. V).

IV. HYPERGRAPH STATES

AND BOOLEAN FUNCTIONS

The preceding analysis shows that there exist n-qubit states, in fact almost all of them with respect to the Haar measure, for which all magic measures are close to the maximum values. But we lack explicit constructions for highly magical states. Here we go in this direction by looking at hypergraph states, which are generalizations of graph states that possess highly flexible entanglement structures determined by an underlying hypergraph.

We first formally define graph and hypergraph states. Graph states constitute an important family of many-body quantum states that plays key roles in various areas of quantum information, such as quantum error correction and MBQC. Given a graph $G = \{V, E\}$ defined by a set of n vertices V and a set of edges E, the corresponding n-qubit graph state is given by

$$|\Psi_G\rangle := \prod_{\substack{i_1,i_2\in V\\ \{i_1,i_2\}\in E}} CZ_{i_1i_2} H^{\otimes n}|0\rangle^{\otimes n}. \tag{11}$$

Note that all gates are in the Clifford group, so graph states are stabilizer states. Conversely, it is known that every stabilizer state is equivalent to a graph state, up to a tensor product of local Clifford unitaries. Graph states thus already exhibit rich entanglement structures, indeed the same as general stabilizer states, which include most quantum error correcting codes known. More generally, one can define *hypergraph states* [33] based on hypergraphs, where the hyperedges may contain $k \geq 2$ vertices and represent $C^{k-1}Z$ gates that acts Z on one of the qubits conditioned on the $k-1$ others being 1 (which are no longer Clifford). That is, given a hypergraph $\widetilde{G} = \{V, E\}$ defined by a set of n vertices V and a set of hyperedges E, the corresponding n-qubit hypergraph state is given by

$$|\Psi_{\widetilde{G}}\rangle := \prod_{\substack{i_1,\cdots,i_k\in V\\ \{i_1,\cdots,i_k\}\in E}} C^{k-1}Z_{i_1\cdots i_k} H^{\otimes n}|0\rangle^{\otimes n}. \tag{12}$$

An important observation is that the hypergraph (including graph) states admit representations in terms of Boolean functions:

$$|\Psi\rangle = 2^{-n/2}\sum_{x\in\mathbb{Z}_2^n}(-1)^{f(x)}|x\rangle, \tag{13}$$

where $f(x) : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is a Boolean function

$$f(x) = \sum_{\substack{i_1,\cdots,i_k\in V\\ \{i_1,\cdots,i_k\}\in E}} x_{i_1}\cdots x_{i_k}, \tag{14}$$

which we call the characteristic function of the hypergraph state $|\Psi\rangle$. Each f corresponds to a hypergraph state (modulo a global phase) and there are 2^{2^n-1} possibilities [33]. For a graph state, $f(x) = \sum_{\substack{i_1,i_2\in V\\ \{i_1,i_2\}\in E}} x_{i_1}x_{i_2}$ is a function with only quadratic terms, where each term corresponds to an edge, and there are now only $2^{\binom{n}{2}}$ possibilities. Any quadratic characteristic function (which may additionally include linear terms x_i) induces a stabilizer state because a term x_i simply corresponds to a Pauli-Z gate on the i-th qubit. Call such states induced by quadratic characteristic functions *quadratic states* and denote the set of quadratic states Q. Note that although the set of quadratic states does not include all stabilizer states (with additional local H and P freedom; see Eq. (18)), that is, $Q \subset \mathrm{STAB}$, the size of Q is close to that of STAB, both scaling roughly as $2^{n^2/2}$ asymptotically.

This formalism allows us to analyze certain magic properties of hypergraph states through Boolean functions. Given two hypergraph states $|\Psi\rangle$ and $|\Psi'\rangle$ with characteristic functions f and f' respectively, we have

$$\langle\Psi|\Psi'\rangle = 2^{-n}\sum_{x\in\mathbb{Z}_2^n}(-1)^{f(x)+f'(x)} = 1 - 2^{1-n}\mathrm{wt}(f+f'), \tag{15}$$

where $\mathrm{wt}(f)$ denotes the Hamming weight of f, i.e. the number of 1's in the truth table of f. Therefore, $\mathrm{wt}(f + f')$ (also called the Hamming distance between f and f') essentially counts the number of non-collisions between f and f'. Here, we are interested in the minimization of $\mathrm{wt}(f + f')$ over all quadratic f' (or equivalently the second-order binary Reed-Muller code $RM(2, n)$), namely the *second-order nonlinearity* or *nonquadraticity* of f, formally defined as

$$\chi(f) := \min_{f'\in RM(2,n)} \mathrm{wt}(f + f'). \tag{16}$$

This generalizes the well studied nonlinearity of Boolean functions (see also e.g. bent functions), which has important applications in cryptography, coding theory etc [51]. This leads to lower bounds on the maximum overlap between $|\Psi\rangle$ with stabilizer states, because a quadratic characteristic function induces stabilizer state as argued. Using Eq. (15), we obtain the following bound on $\mathfrak{D}_{\min}$ in terms of nonquadraticity:

$$\mathfrak{D}_{\min}(\Psi) \leq -\log\max_{q\in Q}|\langle\Psi|q\rangle|^2 = -2\log(1 - 2^{1-n}\chi(f)). \tag{17}$$

It is known that all stabilizer states can be generated by single-qubit Clifford gates acting on graph states (which form a subset of Q) [52]. Note that Q is closed under single-qubit Pauli operators up to global phases. So any pure stabilizer state $|s\rangle$ takes the form

$$|s\rangle = \bigotimes_{i\in\mathcal{I}} P_i \bigotimes_{j\in\mathcal{J}} H_j|q\rangle, \tag{18}$$

where $|q\rangle \in Q$ is a quadratic state, and $\mathcal{I}, \mathcal{J}$ are respectively the set of indices of qubits that P, H act on. Here we conjecture that such local P, H gates do not lead to significantly larger overlap with hypergraph states, and consequently that their maximum overlap or $\mathfrak{D}_{\min}$ with respect to Q is essentially the same as with STAB (note that Q is almost as large as STAB) and Eq. (17) is an equality. We leave a more careful analysis of this problem for future work.

The above technique allows us to obtain bounds on the magic of generally highly entangled of arbitrary size, from the nonquadraticity of Boolean functions. In Sec. VI, we shall use this technique to analyze certain physically motivated hypergraph states, which serve as explicit examples. Also, results on Boolean functions and Reed-Muller codes lead to several general understandings. The maximum possible nonquadraticity χ is equivalent to the covering radius of the second-order Reed-Muller code $RM(2,n)$, denoted by $r(RM(2,n))$. Determining the covering radii for codes is an important but generally difficult task. For general n, there are only bounds known for $r(RM(2,n))$. The best upper bound to our knowledge is from [53],

$$\max \chi(f) \equiv r(RM(2,n)) \leq 2^{n-1} - \frac{\sqrt{15}}{2}2^{n/2} + O(1). \tag{19}$$

Thus by Eq. (17),

$$\mathfrak{D}_{\min}(\Psi) \leq n - \log 15 + o(1). \tag{20}$$

We learn from this bound that for any hypergraph state $|\Psi\rangle$, the min-relative entropy of magic $\mathfrak{D}_{\min}(\Psi)$ is upper bounded by $n - \log 15 \approx n - 3.9$ in the large-n limit. We also have lower bounds coming from simple covering arguments [54]:

$$\max \chi(f) \geq 2^{n-1} - \frac{\sqrt{\ln 2}}{2} n 2^{n/2} + O(1), \tag{21}$$

which implies that there exists hypergraph state $|\Psi\rangle$ such that

$$-\log \max_{q\in Q} |\langle\Psi|q\rangle|^2 \geq n - 2\log n - \log\ln 2 + o(1). \tag{22}$$

Recall that Q and STAB are very similar sets, and that we believe that the left hand side is close to $\mathfrak{D}_{\min}$ especially in the present high-magic regime. This bound is very close to the Haar-random value in Theorem 3, but unfortunately is also not constructive. To our knowledge, the Boolean function with the largest nonquadraticity in the literature is the modified Welch function (see e.g. [55]) defined as $f_{\mathrm{W}}(x) = \mathrm{tr}(x^{2^r+3})$ where $r = \frac{n+1}{2}, n$ odd. We have $\chi(f_{\mathrm{W}}) \approx 2^{n-1} - 2^{(3n-1)/4}$, so for the corresponding hypergraph state Ψ_{W} it holds that

$$\mathfrak{D}_{\min}(\Psi_{\mathrm{W}}) \leq -\log \max_{q\in Q} |\langle\Psi_{\mathrm{W}}|q\rangle|^2 \approx 0.5n. \tag{23}$$

Although we know that the typical magic of a random state is close to n, we do not yet have specific constructions of many-body states with such high magic. The situation is reminiscent to e.g. the superadditivity of classical capacity [56], which is shown for certain random ensembles, but no deterministic construction is known.

V. PAULI MEASUREMENT BASED QUANTUM COMPUTATION

Measurement based quantum computation (MBQC) [20, 21] is a profound and promising model for quantum computation, where one prepares a many-body entangled state offline, and then executes the computation by a sequence of local measurements adaptively determined by a classical computer on this resource state.

This model is naturally tied to resource theory as it essentially formalizes quantum computation as free online manipulations of a resource state. Standard MBQC only allows single-qubit measurements, so entanglement among qubits in the initial state becomes the key resource feature, and a core line of study is to understand the degree of entanglement that supports universal quantum computation (see e.g. [34, 35, 57, 58]).

Here we consider a variant where one is restricted to measuring mutually compatible Pauli observables (including multi-qubit ones such as $X\otimes X$, which cover entangled measurements, for the greatest generality), which we call Pauli MBQC. This is desirable both practically and conceptually, in a similar spirit as the magic state distillation plus injection method [37, 49] for circuit models. Clearly, we would like the online procedures to be simple and fault-tolerant. Moreover, the "magic" of the computation is now isolated to offline resource state preparation, which paves the way for understanding and analyzing the genuine "quantumness" in MBQC models. As a comparison, the standard MBQC using cluster states [20, 57] require online measurements in "magical" bases since cluster states are stabilizer states, leaving certain computational non-classicality in the online part.

More generally, in this Pauli MBQC setting, it is clear that many-body magic states are necessary, due to the Gottesman-Knill theorem. Known resource states for Pauli MBQC include a hypergraph state introduced by Takeuchi, Morimae and Hayashi [59], and the Miller–Miyake state [60] (which will be discussed in the next section). A central question is whether all magic resource states can supply significant speedups over classical algorithms, or support universal quantum computation. In the case of standard MBQC, it is known that resource states with "too much" entanglement (and thereby most states) are not useful for quantum speedups [34, 35]. Here we show that similar rules also hold for Pauli MBQC and magic states, by adapting the arguments in [34].

Theorem 5. *Pauli MBQC with any n-qubit resource state $|\Psi\rangle$ with $\mathfrak{D}_{\min}(\Psi) \geq n - O(\log n)$ cannot achieve superpolynomial speedups over* BPP *machines (classical randomized algorithms) for problems in* NP.

Proof. First note that all Pauli observables have eigenvalues ± 1, and those defined nontrivially on multiple qubits (joint measurements) have degenerate eigenstates. Suppose we measure k (mutually compatible) observables, labeled by $P_i, i = 1, ..., k$. The measurement outcome of P_i as a binary variable $y_i = \pm 1$, so the collective outcome can be represented by a string $y = y_1, ..., y_k$ with 2^k possible values, each of which corresponds to a subspace of the entire Hilbert space. The probability of obtaining y is given by

$$p(y) = \mathrm{Tr}(\Pi_y |\Psi\rangle\langle\Psi|), \tag{24}$$

where Π_y is the projector onto the subspace corresponding to y. Notice that Π_y takes the form

$$\Pi_y = \sum_{j=1}^{2^{n-k}} |s_j\rangle\langle s_j|, \tag{25}$$

where $\{s_j\}, j = 1, ..., 2^{n-k}$ is the set of stabilizer states that are stabilized by $\{y_i P_i\}, i = 1, ..., k$. There are 2^{n-k} such stabilizer states because each measurement halves the dimension. Therefore, we have

$$p(y) = \sum_{j=1}^{2^{n-k}} |\langle s_j|\Psi\rangle|^2 \leq 2^{n-k-\mathfrak{D}_{\min}(\Psi)}, \tag{26}$$

by using standard properties of the trace function and the definition of $\mathfrak{D}_{\min}$. Suppose the algorithm succeeds with probability $\geq 2/3$, that is, let G be the set of strings leading to valid solutions, then $\sum_{y\in G} p(y) \geq 2/3$. Therefore, the size of G obeys

$$|G| \geq 2^{-n+k+\mathfrak{D}_{\min}(\Psi)+1}/3. \tag{27}$$

As a result, one can simulate in polynomial time the above procedure by a classical randomized algorithm, namely in BPP. More specifically, one generates k uniformly random bits from an i.i.d. source and feeds it into the classical control of the computation to see whether it succeeds; this checking step takes time poly(n) due to the NP assumption. If it fails, generate another random string and check again. The probability that the algorithm still has not succeeded after t repetitions satisfies

$$p_f = (1 - |G|/2^k)^t \leq \left(1 - \frac{2^{-n+\mathfrak{D}_{\min}(\Psi)+1}}{3}\right)^t. \tag{28}$$

So to achieve success probability $\geq 2/3$, namely $p_f \leq 1/3$, the number of repetitions needed satisfies

$$t \leq 3\log 3 \cdot 2^{n-\mathfrak{D}_{\min}(\Psi)-1}. \tag{29}$$

When $\mathfrak{D}_{\min}(\Psi) \geq n - O(\log n)$, it can be directly seen that t is upper bounded by poly(n). Multiplying the checking time, it can be concluded that the total runtime of this classical simulation is upper bounded by poly(n). □

That is, if the resource states is too "magical", any Pauli measurement scheme will produce outcomes that are too uniform across all possible ones, which can in some sense be well approximated by classical random bits. Indeed, the known examples of Pauli MBQC such as Takeuchi–Morimae–Hayashi [59] and Miller–Miyake [60] are based on resource states prepared by Clifford+CCZ circuits with specific structures, which is expected to have low magic (see Sec. VI).

Combining with results in Sec. III, we see that almost all states are useless for Pauli MBQC in a strong sense:

Corollary 6. *The fraction of states (w.r.t. Haar measure) that are useful for Pauli MBQC is exponentially small in n.*

VI. QUANTUM PHASES OF MATTER

The Clifford group and stabilizer formalism have become standard notions and tools in recent studies of condensed matter physics, but so far there is little discussion on their physical relevance and the role of magic, especially at a quantitative level. Here we would like to present some basic discussions and results on the magic of certain many-body systems of interest from a phase of matter perspective, in the hope of stimulating further explorations in this direction. This section can also be viewed as a case study of the techniques introduced in Sec. IV for analyzing many-body magic.

Here we consider the symmetry-protected topological (SPT) phases, which has drawn great interest in the condensed matter community (see e.g. [61, 62] for introductions) and in particular widely considered as candidate many-body resource states for MBQC (see e.g. [22] for a review). It is realized that certain classes of nontrivial SPT phases in $\geq$ 2D must contain magic that is "robust" in a sense [36], indicating that on top of entanglement, magic may be a characteristic feature that underlies the physics of such systems. Here we showcase how to apply the Boolean function techniques introduced in Sec. IV to representative 2D SPT states. For example, it is known that the Levin–Gu [63] and Miller–Miyake [60] models have ground states that are hypergraph states prepared by Clifford+CCZ circuits defined on corresponding lattices, so that the characteristic functions of these ground states are restricted to cubic ones, namely third order Reed-Muller codes $RM(3, n)$. For concreteness, think about the well-known Levin–Gu state $|\Psi_{\mathrm{LG}}\rangle$ [63] defined on the 2D triangular lattice (see Fig. 1), which takes the form

$$|\Psi_{\mathrm{LG}}\rangle = U_{CCZ} U_{CZ} U_Z H^{\otimes n} |0\rangle^{\otimes n}, \tag{30}$$

where U_{CCZ}, U_{CZ}, U_Z are respectively composed of CCZ, CZ, Z gates acting on all triangles, edges, and vertices. More generally, consider third-order hypergraph states

$$|\hat{\Psi}\rangle = U_{CCZ}|\Phi\rangle, \quad |\Phi\rangle \in Q, \tag{31}$$

defined on 2D triangulated lattices (such as the ordinary triangular lattice and the Union Jack lattice, as depicted in Fig. 1), where U_{CCZ} represents CCZ gates acting on

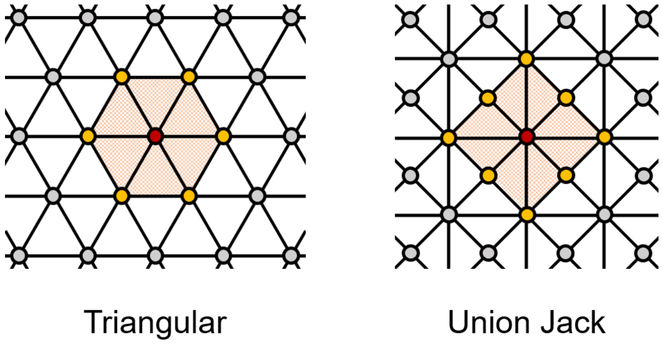

FIG. 1. 2D triangulated lattices. The shaded area represents a unit cell, based on which we decompose the underlying Boolean functions of the systems and derive bounds on their nonquadracity (details in Appendix C).

all triangles. Note that the Clifford+CCZ preparation circuits of such states are in the third-level of the Clifford hierarchy [49]. Such states are called "Clifford magic states" in Ref. [6] and are shown to have the property that the "stabilizer extent", $\xi(\Psi) := \min \|c\|_1^2$ where c is the amplitude vector of a decomposition into pure stabilizer states, is equal to $2^{\mathfrak{D}_{\min}(\Psi)}$ due to convex duality. It is known that the logarithm of stabilizer extent $\log \xi$ and max-relative entropy monotone $\mathfrak{D}_{\max}$ (and thus also generalized robustness) are equivalent [40, 64]. Therefore we have the collapse property $\mathfrak{D}_{\max}(\hat{\Psi}) = \mathfrak{D}_{\min}(\hat{\Psi})$. Using techniques from Refs. [65, 66], we rigorously prove the following crude bounds for the two example lattices (which hold for both open and periodic boundary conditions):

- Triangular lattice: $\mathfrak{D}_{\max}(\hat{\Psi}) = \mathfrak{D}_{\min}(\hat{\Psi}) < 0.56n$.
- Union Jack lattice: $\mathfrak{D}_{\max}(\hat{\Psi}) = \mathfrak{D}_{\min}(\hat{\Psi}) < 0.46n$.

Roughly, our approach is to find proper decompositions of the cubic characteristic functions based on cell structures of the underlying lattices (as illustrated in Fig. 1), which allow us to bound its distance from certain quadratic functions (and thus the nonquadraticity). See Appendix C for technical details of the derivation. Note that, we expect the above bounds to be loose, and it can likely be shown that $\mathfrak{D}_{\max}(\hat{\Psi}) = \mathfrak{D}_{\min}(\hat{\Psi}) \leq \left(2 - \frac{2}{3}\log 6\right) n \lesssim 0.28n$ for all regular triangulated lattices (see also Appendix C for more detailed discussions and ways to improve the bounds), which is achieved by disjoint CCZ gates (namely, $CCZ^{\otimes \frac{n}{3}}$) because $\mathfrak{D}_{\max}(CCZ|{+}{+}{+}\rangle)/3 = \mathfrak{D}_{\min}(CCZ|{+}{+}{+}\rangle)/3 = \log(16/9)/3 = 2 - \frac{2}{3}\log 6$ [6]. Also note that the maximum product-state value is $\log(3-\sqrt{3})n \approx 0.34n$, where each qubit is the golden state $\frac{1}{2}(I + \frac{X+Y+Z}{\sqrt{3}})$. So an observation is that although the CCZ gates can generate rich entanglement structures that supply interesting topological properties, the many-body magic of the corresponding SPT states is rather weak and not quite different from states without nontrivial entanglement, although it is generically necessary [36]. This makes the role of magic more curious. Note that e.g. the fixed point of Miller–Miyake model on the Union Jack lattice (which satisfies Eq. (31)) is known to be universal for Pauli MBQC [67], so the bound is consistent with Theorem 5. Nevertheless, note that the magic of such many-body states is still an extensive quantity, i.e. scales with the system size n. A simple argument is that one can do Pauli measurements on vertices in a periodic manner (Fig. 2 illustrates the case of Union Jack lattice; the idea can be generalized to other regular lattices), which leaves a periodic array of $O(n)$ uncoupled CCZ blocks, each containing a certain amount of magic. Note that a characteristic feature of SPT phases is that

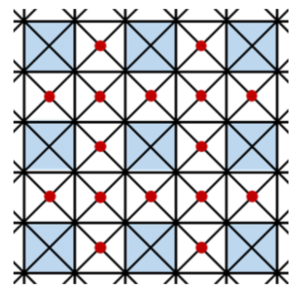

FIG. 2. Extensiveness of magic. After measuring the red vertices by Pauli observables, the system is left with decoupled CCZ blocks (colored in blue).

they are short-range entangled, which accords with the rather weak magic. It also indicates that for SPT phases, the method of calculating the magic of small lattices and then "scale up" the results may help approximate the magic of the whole system well. For future work, it would be particularly interesting to look into long-range entangled, intrinsically topologically ordered systems like topological codes.

We anticipate that the study of many-body magic will provide a new and useful perspective on characterizing and classifying quantum phases of matter. Note that the family of hypergraph states can exhibit very rich many-body features, so the Boolean function techniques could be more widely useful in this regard. A natural direction is to further explore the connections between magic and computational complexity or power of phases. For example, a direct question following the above discussions is whether magic can be used to diagnose whether the phase is universal for Pauli MBQC, or more generally certain notions of quantum "computational phase transitions". In particular, noting that the above studied Miller–Miyake and Levin–Gu models are known to be universal on the Union Jack lattice but likely not universal on the triangular lattice [22, 60], we wonder if more refined analysis on the scaling factors and robustness properties of magic associated with different lattices helps understand how many-body magic is connected to MBQC power. On the other hand, magic determines the cost of many standard methods for preparing and simulating the systems and could plausibly be connected to related problems like the notorious sign problem in various forms (see e.g. [24–26]). For example, the extensive property directly indicates that the run time of the quasi-probability sampling algorithm of Howard and Campbell [7] is exponential. Also, as recently found in Ref. [68], the many-body magic (mana of qutrit systems in their example) interestingly peaks at the phase transition of certain models, indicating strong relevance of magic in many-body physics (also see discussions in Ref. [68]). We finally refer readers to Ref. [36], which shows necessity and robustness of magic throughout certain types of $\geq$ 2D SPT phases, and contains more results and discussions about magic from condensed matter perspectives, in relation to symmetries, sign problems, MBQC, and more.

VII. CONCLUDING REMARKS

In this work, we formally studied the magic of many-body states from multiple aspects, and propose it as a possibly useful probe of many-body quantum complexity. We found that magic is a highly nontrivial theory with complicated mathematical structures, so that the calculation and analysis of many-body magic measures are in general very difficult. Our results indicate an interesting interplay with entanglement worth being further studied: although magic and entanglement are two disparate notions, they may be correlated in some way in the highly entangled regime. For example, we now know that states are typically almost maximally entangled and magical at the same time, but do highly magical states have to be highly entangled in some sense, or vice versa? On a related note, the problem of explicitly constructing maximally magical states with respect to any of the measures we investigated, is still wide open.

As often is the case in resource theories, some of the most interesting quantifiers are hard to compute, and even their upper and lower bounds may present serious computational challenges. In the case of magic, the difficulty of calculations scales badly with n because of the exponential growth of the set of stabilizer states (despite our observation that free and generalized robustness are linear programs). The search for easier bounds thus remains highly important. For certain condensed matter systems it may be sufficient to calculate values for small lattices, but the general case remains to be explored.

With the present work, we hope to raise further interest in these questions, and the approach to many-body physics by magic in general. Indeed, as discussed earlier, many-body magic could be very relevant to the characterization of quantum complexity of phases of matter, such as the cost of simulating certain phases and the computational power of them.

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Appendix A: Range of consistent resource measures

The following argument goes some way to justify that the min-relative entropy and the free robustness in some sense define a sensible range of resource measures.

Consider a theory with finite free robustness (i.e. satisfying condition FFR defined in [40]), such as magic theory in which we are interested here. By the definition of free robustness, there exists a free state $\delta \in \mathcal{F}$ such that $\frac{1}{1+R(\phi)}\phi + \frac{R(\phi)}{1+R(\phi)}\delta \in \mathcal{F}$. Consider the following cptp map

$$\mathcal{E}(\omega) = (\operatorname{Tr}\psi\omega)\phi + (1 - \operatorname{Tr}\psi\omega)\delta, \tag{A1}$$

where ψ and ϕ are pure states. Notice that $\mathcal{E}(\psi) = \phi$. It can be verified that if $\frac{1-\operatorname{Tr}\psi\omega}{\operatorname{Tr}\psi\omega} \geq R(\phi)$, that is,

$$\operatorname{Tr}\psi\omega \leq 2^{-LR(\phi)}, \tag{A2}$$

then $\mathcal{E}(\omega) \in \mathcal{F}$. When $\mathfrak{D}_{\min}(\psi) \geq LR(\phi)$, then for any $\omega \in \mathcal{F}$ Eq. (A2) holds and thus $\mathcal{E}(\omega) \in \mathcal{F}$. That is, $\mathcal{E}$ is a resource non-generating operation. To summarize, the condition $\mathfrak{D}_{\min}(\psi) \geq LR(\phi)$ implies that there must exist a resource non-generating operation that accomplishes the one-shot transformation $\psi \to \phi$.

The consistency argument goes as follows. Pick a standard reference resource measure f_r such as relative entropy of resource (or others satisfying $\mathfrak{D}_{\min} \leq f_r \leq LR$). It plays the role of fixing a standard normalization in order to e.g. avoid ambiguities about constants. Based on this, consider the following consistency condition of resource measure f: If the transformation $\psi \to \phi$ by free operations is possible, then

$$f(\psi) \geq f_r(\phi) \quad \text{and} \quad f_r(\psi) \geq f(\phi). \tag{A3}$$

If these are not satisfied then f may be regarded inconsistent with the reference measure f_r since comparisons with f_r will rule out feasible free transformations.

This consistency condition implies $\mathfrak{D}_{\min} \leq f \leq LR$. Suppose there exists ψ such that $f(\psi) < \mathfrak{D}_{\min}(\psi)$. Then there must exist some ϕ such that $f(\psi) < LR(\phi) \leq$

$\mathfrak{D}_{\min}(\psi)$, so $\psi \to \phi$ is feasible but $f(\psi) < f_r(\phi)$, violating the first consistency condition. Similarly, suppose there exists ψ' such that $f(\psi') > LR(\psi')$. Then there must exist some ϕ' such that $f(\psi') > \mathfrak{D}_{\min}(\phi') \geq LR(\psi')$, so $\phi' \to \psi'$ is feasible but $f_r(\phi') < f(\psi')$, violating the second consistency condition.

We also refer readers to Refs. [69, 70] for other arguments about ranges of resource measures.

Appendix B: Wigner negativity and robustness

Here we discuss the relations between computable magic measures based on Wigner negativity and free robustness, which have not explicitly appeared in the literature before.

Here consider odd prime power dimension $D = d^n$, for which the stabilizer formalism in terms of discrete Wigner function is well defined. Detailed introductions can be found in the literature, e.g. [18]. Here we are interested in the widely-used magic measures defined as follows. Define the phase space point operators as

$$A_{\mathbf{0}} = \frac{1}{d^n}\sum_{\boldsymbol{u}} T_{\boldsymbol{u}}, \text{ and } A_{\boldsymbol{u}} = T_{\boldsymbol{u}} A_{\mathbf{0}} T_{\boldsymbol{u}}^\dagger, \tag{B1}$$

where $T_{\boldsymbol{u}}$ are the discrete Heisenberg-Weyl (generalized Pauli) operators:

$$T_{\boldsymbol{u}} = \omega^{-\frac{a_1 a_2}{2}} Z^{a_1} X^{a_2}, \quad \boldsymbol{u} = (a_1, a_2) \in \mathbb{Z}_d \times \mathbb{Z}_d \tag{B2}$$

where $\omega = e^{2\pi i/d}$. For composite systems,

$$T_{(a_1,a_2)\oplus(b_1,b_2)\oplus\cdots\oplus(u_1,u_2)} = T_{(a_1,a_2)} \otimes T_{(b_1,b_2)} \otimes \cdots \otimes T_{(u_1,u_2)}. \tag{B3}$$

Then for state ρ in dimension D, the corresponding discrete Wigner quasi-probability representation is given by

$$W_\rho(\boldsymbol{u}) := \frac{1}{d^n} \operatorname{Tr} A_{\boldsymbol{u}} \rho. \tag{B4}$$

It is known that pure stabilizer states are precisely those pure states that have non-negative Wigner functions (discrete Hudson's theorem) [16]. This motivates us to use the negative values of the Wigner functions to measure magic. Define $\mathcal{W}$ to be the set of all real-valued functions v on phase space points $\boldsymbol{u}$ with the normalisation $\sum_{\boldsymbol{u}} v(\boldsymbol{u}) = 1$. This is an affine space containing all Wigner functions W_ρ. In this space, identify the convex cone of non-negative functions,

$$\mathcal{W}_+ := \{v \in \mathcal{W} : \forall \boldsymbol{u}\ v(\boldsymbol{u}) \geq 0\}, \tag{B5}$$

which by definition contains all non-negative Wigner functions, in particular those W_σ of mixtures of stabilizer states $\sigma \in \mathrm{STAB}$.

Definition 7 (Sum negativity)**.** The *sum negativity* (or Wigner negativity) of a state ρ is defined as

$$\begin{aligned}\mathcal{N}(\rho) &:= \sum_{\boldsymbol{u}: W_\rho(\boldsymbol{u})<0} |W_\rho(\boldsymbol{u})| \\ &= \frac{1}{2}\left(\sum_{\boldsymbol{u}} |W_\rho(\boldsymbol{u})| - 1\right).\end{aligned}$$

Theorem 8. *For all states ρ, $\mathcal{N}(\rho) \leq R(\rho)$.*

Proof. The definition of the Wigner negativity is evidently equivalent to

$$\mathcal{N}(\rho) = \min s \text{ s.t. } W_\rho = (1+s)v - sv',\ v, v' \in \mathcal{W}_+.$$

The optimal functions are $v \propto (W_\rho)_+ = \max\{W_\rho, 0\}$ and $v' \propto (W_\rho)_- = \min\{W_\rho, 0\}$.

On the other hand, the definition of the free robustness is equivalent to

$$R(\rho) = \min s \text{ s.t. } W_\rho = (1+s)W_\sigma - sW_{\sigma'},\ \sigma, \sigma' \in \mathrm{STAB},$$

because the Wigner function is an isomorphism, so the above condition is just one way to express $\rho = (1+s)\sigma - s\sigma'$. Since all $W_\sigma, W_{\sigma'} \in \mathcal{W}_+$, the former minimization is a relaxation of the latter, hence the claim follows. □

Ref. [18] also defined an additive version of the sum negativity:

Definition 9 (Mana)**.** The *mana* of state ρ is defined as $\mathcal{M}(\rho) := \log\left(\sum_{\boldsymbol{u}} |W_\rho(\boldsymbol{u})|\right) = \log(2\mathcal{N}(\rho)+1)$.

Corollary 10. *For all states ρ, $\mathcal{M}(\rho) < LR(\rho) + 1$.*

Proof. Elementary:

$$\begin{aligned}\mathcal{M}(\rho) &= \log(2\mathcal{N}(\rho)+1) \\ &\leq \log(2R(\rho)+1) \\ &< \log(2R(\rho)+2) = LR(\rho)+1,\end{aligned}$$

by Theorem 8. □

Therefore, each upper bound on LR also give bounds on mana.

This positive Wigner simplex contains the stabilizer polytope and is geometrically simpler, so the associated measures are easier to compute. Note that Theorem 8.4 of [71] gives a lower bound of $\mathcal{M}$ in terms of $\mathfrak{D}_{\min}$ that works for small $\mathcal{M}$. In general can we lower bound $\mathcal{M}$ in terms of other smaller measures associated with STAB?

Also note that the negativity and free robustness measures are related to the runtimes or other costs of quasiprobability methods of classical simulation [7, 39].

Appendix C: Clifford+CCZ circuits, cubic Boolean functions, and SPT phases

Here we provide technical details and extensive discussions on how to employ coding theory tools to analyze the magic of Clifford+CCZ circuits of certain structures which correspond to 2D SPT phases of interest, supplementing Sec. VI.

Let $f \in RM(3,n)$ be a cubic Boolean function with n variables. Suppose there is a set of indices $\mathcal{X} = \{x_{c_1}, \cdots, x_{c_s}\}$ such that every cubic term involves one variable from $\mathcal{X}$, that is, f takes the following form

$$f(x) = \sum_{i=1}^{s} x_{c_i} q_i + q, \tag{C1}$$

where $q_i \in RM(2, n-s)$ is the quadratic function associated with x_{c_i} so that $\sum_{i\in\{1,\cdots,s\}} x_{c_i} q_i$ is the cubic part of f, and q is quadratic (containing linear terms as well). We call Eq. (C1) an order-s decomposition of f. Given some quadratic function $q = xQx^T$, let $2h_q$ be the rank of the symplectic matrix $Q + Q^T$. Following the arguments in Section 4 of Ref. [65], we see that there exists a quadratic function $\tilde{q}$ (given in Theorem 3) such that

$$\mathrm{wt}(f+\tilde{q}) = 2^{n-1} - \sum_{r\in\mathbb{F}_2^s} 2^{n-s-1-h_{\sum_{i=1}^s r_i q_i}} \tag{C2}$$

$$\leq 2^{n-1} - \sum_{r\in\mathbb{F}_2^s} 2^{n-s-1-\sum_{i=1}^s r_i h_{q_i}} \tag{C3}$$

$$= 2^{n-1} - 2^{n-s-1}\prod_{i=1}^{s}\left(1+2^{-h_{q_i}}\right). \tag{C4}$$

Therefore, given an order-s decomposition of f, we have the following bound for its nonquadraticity

$$\chi(f) \leq 2^{n-1} - 2^{n-s-1}\prod_{i=1}^{s}\left(1+2^{-h_{q_i}}\right), \tag{C5}$$

and thus for the corresponding state $|\Psi_f\rangle$, we have

$$\mathfrak{D}_{\max}(\Psi_f) = \mathfrak{D}_{\min}(\Psi_f) \leq -2\log(1-2^{1-n}\chi(f)) \tag{C6}$$

$$\leq 2s - 2\log\prod_{i=1}^{s}\left(1+2^{-h_{q_i}}\right). \tag{C7}$$

by Eq. (17).

We now apply the above general technique to hypergraph states on 2D triangulated lattices (depicted in Fig. 3) given by Eq. (31). As we shall see, the decomposition of the cubic characteristic functions is based on the lattice structure.

First consider the triangular lattice. As illustrated in Fig. 3, consider the lattice to be a tiling of unit cells, each of which is a hexagon (such as the shaded one). Label the center vertex of each cell as x_i, which is involved in 6 CCZ gates (cubic terms) with the 6 boundary vertices $(x_{i1}, \cdots, x_{i6})$ of the cell. So the characteristic function can be expressed as

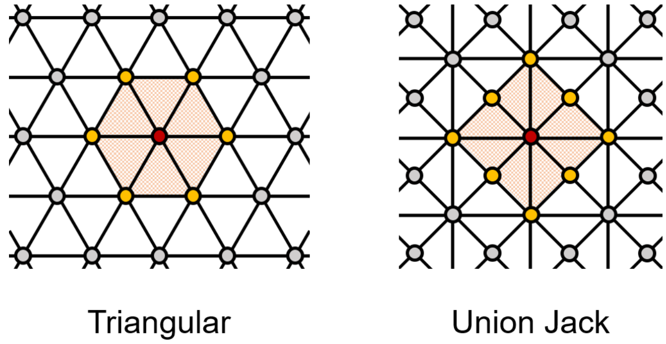

FIG. 3. 2D triangulated lattices. For each lattice, the shaded area is the unit cell we choose; the red vertex is the center vertex that only belongs to its cell; the yellow vertices are shared with neighboring cells. The Boolean functions are decomposed according to the cells.

$$f_{\mathrm{Tri}}(x) = \sum_i x_i(x_{i1}x_{i2} + x_{i2}x_{i3} + x_{i3}x_{i4} + x_{i4}x_{i5} + x_{i5}x_{i6} + x_{i6}x_{i1}) + q, \tag{C8}$$

where q is quadratic, with further constraints that each boundary vertex is shared by 3 neighboring hexagons, so the corresponding variables are actually the same one. Also, boundary vertices never serve as a center vertex. For simplicity, we make a minor assumption that the lattice is consisted of complete cells. If we consider periodic boundary conditions, then for m cells, there are $6m/3 = 2m$ boundary vertices in total. Therefore, we have $n = m + 2m = 3m$, so Eq. (C8) gives an order-$s = m = n/3$ decomposition. Also note that $q_i = x_{i1}x_{i2} + x_{i2}x_{i3} + x_{i3}x_{i4} + x_{i4}x_{i5} + x_{i5}x_{i6} + x_{i6}x_{i1}$, so correspondingly

$$Q_i = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix}, \tag{C9}$$

and thus

$$h_{q_i} = \frac{1}{2}\mathrm{rank}(Q_i + Q_i^T) = 3. \tag{C10}$$

Plugging everything into Eq. (C4), we obtain

$$\mathfrak{D}_{\max}(\Psi_{f_{\mathrm{Tri}}}) = \mathfrak{D}_{\min}(\Psi_{f_{\mathrm{Tri}}}) \leq \left(\frac{2}{3} - \frac{2}{3}\log\frac{9}{8}\right)n \tag{C11}$$

$$\lesssim 0.56n. \tag{C12}$$

For open boundary conditions, the difference is that there are $O(\sqrt{n})$ boundary vertices that are shared by less than

3 cells, which leads to $s = n/3 - O(\sqrt{n})$. As a result, the bound is modified by $-O(\sqrt{n})$, and Eq. (C12) still holds.

For the Union Jack lattice, we follow a similar procedure. Now we define the unit cell to be a square involving 9 vertices as illustrated in Fig. 3, each of which has 1 center vertex involved in 8 CCZ gates (cubic terms) with the 8 boundary vertices shared with neighboring cells. Again, assume that the lattice is consisted of complete cells. Consider periodic boundary conditions. Note that among the 8 boundary vertices, 4 in the corner are shared by 4 cells, and 4 on the edge are shared by 2 cells. So for m cells, there are $4m/4 + 4m/2 = 3m$ boundary vertices in total. So $n = m + 3m = 4m$ and thus this cell structure gives $s = n/4$. It can also be verified that $h_{q_i} = 4$. So for the corresponding states $|\Psi_{f_{\rm UJ}}\rangle$ we have

$$\mathfrak{D}_{\max}(\Psi_{f_{\rm UJ}}) = \mathfrak{D}_{\min}(\Psi_{f_{\rm UJ}}) \leq \left(\frac{1}{2} - \frac{1}{2}\log\frac{17}{16}\right) n \tag{C13}$$

$$\lesssim 0.46n, \tag{C14}$$

by Eq. (C4). Similarly, the open boundary conditions lead to a $-O(\sqrt{n})$ correction. Note that for the Union Jack lattice another natural definition of the unit cell is the small square with 1 center vertex and 4 corner vertices. However, it can be verified that this cell structure gives $s = \frac{1}{2}$, which leads to a bound worse than Eq. (C14).

Finally, we note that we expect the constant factors in the above bounds to be loose, although they already imply that the many-body magic of corresponding SPT phases are much weaker than typical states. Below we outline two promising paths towards improved bounds: (i) In the above method, the inequality (C3)) can be loose, because the shared terms in q_i's corresponding to neighboring cells will be cancelled out in the summation, which leads to the general effect that $h_{\sum q_i} < \sum h_{q_i}$. Therefore a direct possibility is a more refined calculation of Eq. (C2) by analyzing $h_{\sum_{i=1}^{s} r_i q_i}$. (ii) Show that the characteristic cubic functions are *separable* [65, 66], meaning that in the decomposition with the smallest possible $\mathcal{X}$ under all affine transformations of the variables, each cubic term involves exactly one variable from $\mathcal{X}$. Then, by [65, Thm. 4] the maximum nonquadracity of such separable cubic functions is

$$\max_{f\in RM(3,n), f \text{ separable}} \chi(f) = 2^{n-1} - \frac{1}{2}6^{\lfloor n/3 \rfloor}, \tag{C15}$$

leading to the bound for corresponding third-order hypergraph states $|\hat{\Psi}\rangle$:

$$\mathfrak{D}_{\max}(\hat{\Psi}) = \mathfrak{D}_{\min}(\hat{\Psi}) \leq \left(2 - \frac{2}{3}\log 6\right) n \lesssim 0.28n, \tag{C16}$$

which is already attained by $CCZ^{\otimes \frac{n}{3}}|+\ldots+\rangle$, which does not have interesting entanglement. Note that most cubic functions studied before are indeed separable [65, 72, 73], so the above strong upper bound may hold quite generally for third-order hypergraph states. We believe that the above cases indeed have separable characteristic functions since the cubic terms are highly regular.

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