q-ary duality

I. DUALITY

f_α is a function of α. $\tilde{f}_\beta = \sum_{\alpha=0}^{q-1} \frac{\omega^{\alpha\beta} \cdot f_\alpha}{\sqrt{q}}$

$\sum_\beta \frac{\tilde{f}_\beta \omega^{-\alpha\beta}}{\sqrt{q}} = \frac{1}{q}\sum_\beta \omega^{-\alpha\beta}(\sum_j \omega^{j\beta} f_j) = \frac{1}{q}\sum_j \sum_\beta \omega^{(-\alpha+j)\beta} f_j = \frac{1}{q}\sum_j q\delta(j,\alpha) f_j = f_\alpha$

$Z = \sum_{\{s_i\}} \prod_b \omega^{\sum_i s_i \Theta_{ib} m_b} e^{-K\delta(\sum_i s_i\Theta_{ib} - e_b, 0)}$,where $K = \beta J$

Take $\alpha_b = \sum_i s_i \Theta_{ib} - e_b$ as a variable, and do the Fourier transformation on the function $f_{\alpha_b} = e^{-K\delta(\alpha_b, 0)}$.

$Z = \sum_{\{s_i\}} \prod_b \omega^{\sum_i s_i\Theta_{ib} m_b} \sum_{\beta_b} \frac{\tilde{f}_\beta \omega^{-(\sum_i s_i \Theta_{ib} - e_b)\beta_b}}{\sqrt{q}}$
$= q^{-N_b/2} \sum_{\{s_i\}} \prod_b \sum_{\beta_b} \tilde{f}_{\beta_b} \omega^{\sum_i s_i \Theta_{ib}(m_b - \beta_b) + e_b\beta_b}$
$= q^{-N_b/2} \sum_{\{s_i\}} \sum_{\{\beta_b\}} \prod_b \tilde{f}_{\beta_b} \omega^{\sum_i s_i \Theta_{ib}(m_b - \beta_b) + e_b\beta_b}$
$= q^{-N_b/2} \sum_{\{s_i\}} \sum_{\{\beta_b\}} (\prod_b \tilde{f}_{\beta_b}) \omega^{\sum_b \sum_i s_i \Theta_{ib}(m_b - \beta_b) + e_b\beta_b}$
$= q^{-N_b/2} \sum_{\{\beta_b\}} (\prod_b \tilde{f}_{\beta_b}) \sum_{\{s_i\}} \omega^{\sum_b \sum_i s_i \Theta_{ib}(m_b - \beta_b) + e_b\beta_b}$

All the terms that $\sum_b \Theta_{ib}(m_b - \beta_b) \neq 0$ will be zero. So all the non zero terms must satisfy $\sum_b \Theta_{ib}(m_b - \beta_b) = 0$. We can rewrite $\beta_b = m_b - \sum_i \sigma_i \tilde{\Theta}_{ib}$, where $\tilde{\Theta}_{ib}$ is the dual matrix of Θ_{ib}, $\Theta\tilde{\Theta}^T = 0$.

Using Smith normal form, we can write $\Theta = UDV$, where $\det V = \pm 1$ and $\det U = \pm 1$ and D has the form:

$$\begin{bmatrix} d_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_k & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

where d_i are integers and $0 < d_1 \leq d_2 \leq \cdots \leq d_k < q$, each d_i is a factor of d_{i+1} and d_k divides q. Here $k = \mathrm{rank}(\Theta)$.

So, the dual of D is

$$\tilde{D} = \begin{bmatrix} \frac{q}{d_1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{q}{d_2} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{q}{d_k} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

and if $d_i = 1$, $\frac{q}{d_i} = 0$.

Since $D\tilde{D}^T = 0$ and V is invertible, we find the dual of Θ to be $\tilde{\Theta} = \tilde{D}(V^{-1})^T$

So we have

$Z = q^{-N_b/2} \sum_{\{\beta_b = m_b - \sigma_i \tilde{\Theta}_{ib}\}} (\prod_b \tilde{f}_{\beta_b}) q^{N_b} \omega^{\sum_b e_b \beta_b}$
$= q^{N_b/2} \sum_{\{\beta_b = m_b - \sigma_i \tilde{\Theta}_{ib}\}} (\prod_b \tilde{f}_{\beta_b}) \omega^{\sum_b e_b \beta_b}$

For any vector $\boldsymbol{\sigma}$, if $\boldsymbol{v}\tilde{\Theta} = 0$, $\boldsymbol{\sigma}\tilde{\Theta} = (\boldsymbol{\sigma} + \boldsymbol{v})\tilde{\Theta}$. The total number of non-zero vectors $\{\boldsymbol{v}\}$ is $\frac{q^k}{\prod_i^k d_i}$.

$Z = \frac{q^{\frac{N_b}{2}-k}}{\prod_i^k d_i} \sum_{\{\sigma_i\}} \prod_b \tilde{f}_{m_b - \sigma_i \tilde{\Theta}_{ib}} \omega^{e_b(m_b - \sum_i \sigma_i \tilde{\Theta}_{ib})}$
$= \frac{q^{\frac{N_b}{2}-k}}{\prod_i^k d_i} \omega^{e_b m_b} \sum_{\{\sigma_i\}} \prod_b \tilde{f}_{m_b - \sigma_i \tilde{\Theta}_{ib}} \omega^{-\sum_i \sigma_i \tilde{\Theta}_{ib} e_b}$

Duality: $\{s_i\} \to \{\sigma_i\}$, $\Theta \to \tilde{\Theta}$, $e_b \to m_b$, $m_b \to -e_b$, and an extra term $\omega^{e_b m_b}$.