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Correlations in Ising Ferromagnets. I*

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The following results are proved for a system of Ising spins $\sigma_i = \pm 1$ in zero magnetic field coupled by a purely ferromagnetic interaction of the form $-\Sigma_{i<j} J_{ij}\sigma_i\sigma_j$ with $J_{ij} \geq 0$, for arbitrary crystal lattice and range of interaction: (1) The binary correlation functions $\langle\sigma_k\sigma_l\rangle$ are always nonnegative ($\langle\ \rangle$ denotes a thermal average). (2) For arbitrary i, j, k, and l, $\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle \geq \langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_l\rangle$. Consequences of these results, in particular the second, are: (i) $\langle\sigma_k\sigma_l\rangle$ never decreases if any J_{ij} is increased. (ii) If an Ising model with ferromagnetic interactions exhibits a long-range order, this long-range order increases if additional ferromagnetic interactions are added. This last fact may be used to prove the existence of long-range order in a large class of two- and three-dimensional Ising lattices with purely ferromagnetic interactions of bounded or unbounded range.

I. INTRODUCTION

CONSIDER a finite system of Ising spins $\sigma_i = \pm 1$ with a Hamiltonian

$$\mathcal{H} = -\sum_{i<j}^{N} J_{ij}(\sigma_i\sigma_j - 1), \tag{1}$$

where for every pair $i \neq j$

$$0 \leq J_{ij} = J_{ji} < \infty. \tag{2}$$

That is, all interactions are ferromagnetic, favoring parallel alignment of spins. The thermal average of an operator $\mathcal{O}$ is defined by

$$\langle\mathcal{O}\rangle = \mathrm{Tr}\,[\mathcal{O}\exp(-\beta\mathcal{H})]/Z, \tag{3}$$

where

$$Z = \mathrm{Tr}\,[\exp(-\beta\mathcal{H})] \tag{4}$$

is the partition function, and the inverse temperature $\beta = (kT)^{-1}$ is always positive. As all interactions favor parallel alignment, the following result is not surprising.

Theorem 1: For the system described by (1) and (2) and any pair k, l,

$$\langle\sigma_k\sigma_l\rangle \geq 0. \tag{5}$$

Also it seems intuitively plausible that increasing the ferromagnetic interaction between any pair of spins tends to enhance the tendency of other pairs to line up parallel, a result embodied in Theorem 2.

Theorem 2: For the system described by (1) and (2), and where k, l, m, n denote any four spins (not necessarily all different), the following is true:

$$\beta^{-1}\,\partial\langle\sigma_k\sigma_l\rangle/\partial J_{mn} = \langle\sigma_k\sigma_l\sigma_m\sigma_n\rangle - \langle\sigma_k\sigma_l\rangle\langle\sigma_m\sigma_n\rangle \geq 0. \tag{6}$$

Further, the result (6) still holds when J_{kl} or J_{mn} (of both) is negative (we suppose *all other* J_{ij} are nonnegative).

Section II contains the straightforward proof of Theorem 1 together with definitions and notation useful in discussing Theorem 2. The latter is proved in Sec. III with assistance from two lemmas in Appendix A. An immediate consequence of Theorem 2, with proof in Sec. III, is found in Theorem 3.

Theorem 3: For the system described in (1) and (2), and where k, l, and n denote any three spins, the following relation holds:

$$\langle\sigma_k\sigma_n\rangle \geq \langle\sigma_k\sigma_l\rangle\langle\sigma_l\sigma_n\rangle \tag{7}$$

and it is unnecessary to assume that J_{kl} and J_{ln} are nonnegative.

Some applications of Theorems 2 and 3 to the problem of long-range order in various types of Ising ferromagnets are found in Sec. IV. We hope to present others in a future publication. The principal utility of these theorems seems to lie in applications where the results, just as the theorems themselves, are intuitively very reasonable, but formally difficult to prove. We feel the results merit publication because at the present time the statistical theory of phase transitions, in which the Ising model has played a major role, is seriously restricted by a lack of exact solutions for even relatively simple models. Various approximation methods are of much value, at least in regions removed from the critical point, but there is increasing evidence that they are not adequate to answer many questions of theoretical interest. In the absence of exact solutions (and even if they were available), precise mathematical results may be useful for gaining insight into the behavior of various

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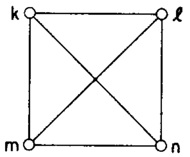

FIG. 1. Complete diagram for a system of 4 spins.

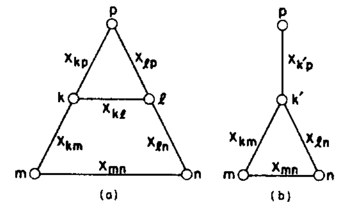

FIG. 2. Illustration of the effect of combining spins k and l by letting X_{kl} go to zero. The result before combination is shown in (a) and the result after combination in (b). The factor $X_{k'p}$ is equal to $X_{kp}\cdot X_{lp}$.

models.[1] We hope our results may make some contribution toward this end.

II. DEFINITIONS, NOTATION, AND THE PROOF OF THEOREM 1

For conceptual purposes it is convenient to represent Ising spins as small circles in a diagram connected with lines or bonds, the bond between spins k and l representing the term $-J_{kl}(\sigma_k\sigma_l-1)$ in (1). An example with 4 spins is shown in Fig. 1. With each bond we associate a factor (Boltzmann factor)

$$X_{kl}=\exp(-2\beta J_{kl}) \tag{8}$$

representing the contribution of the bond to the partition function when $\sigma_k\sigma_l=-1$. In fact, the partition function is simply a sum of terms which are polynomials in the $\{X_{ij}\}$, with any given X_{kl} occurring to the zeroth or first power. For J_{ij} satisfying (2) we have

$$0<X_{ij}\leq 1. \tag{9}$$

If J_{kl} vanishes, i.e., $X_{kl}=1$, we erase the corresponding bond in the diagram. Another important operation is that of taking the limit $J_{kl}\to\infty$ or $X_{kl}\to 0$, which we call "combining" spins k and l. The effect of this operation on the partition function is easily verified: k and l may now be treated as a single spin, say k'. Further, the factors $X_{k'm}$ are simply given as products

$$X_{k'm}=X_{km}X_{lm} \tag{10}$$

(that is, $J_{k'm}=J_{km}+J_{lm}$). Note that if both X_{km} and X_{lm} satisfy (9), so does $X_{k'm}$. That is, the ferromagnetic nature of all bonds is preserved when two spins are combined. An example is shown in Fig. 2.

We use the same diagram to represent both the Hamiltonian (1) and the associated partition function (4). In connection with the latter it is convenient to introduce *restricted* partition functions in which instead of summing over all configurations, as in (4), one sums only over those in which certain spins have specified values. For example, $Z(k+)$, represented in Fig. 3(a), is a restricted partition function in which

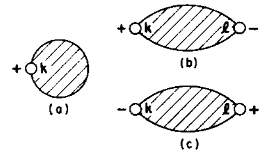

FIG. 3. Diagrams illustrating various restricted partition functions.

only configurations with $\sigma_k=+1$ are included in the sum (4). Figure 3(b) represents the restricted partition function $Z(k+\ l-)$ which, because (1) is invariant under time reversal (unaltered if each σ_i is replaced by $-\sigma_i$) is identical with $Z(k-\ l+)$ illustrated in Fig. 3(c). In these figures all other spins plus connecting bonds are, for brevity, represented by a cross-hatched region or "blob."

We add two terms to complete our notational and diagrammatic machinery. A diagram (Hamiltonian or partition function) is *complete* if every spin is joined to every other spin by a bond; that is, $J_{ij}\neq 0$ for any pair $i\neq j$. A diagram is *connected* if one can get from any spin to any other spin by passing along bonds from spin to spin.

We now prove Theorem 1. In terms of restricted partition functions it suffices to show that

$$\begin{aligned}\tfrac{1}{2}Z\langle\sigma_k\sigma_l\rangle&=\tfrac{1}{2}[Z(k+\ l+)+Z(k-\ l-)-Z(k+\ l-)\\&\quad-Z(k-\ l+)]=Q\\&=[Z(k+\ l+)-Z(k-\ l+)]\geq 0,\end{aligned} \tag{11}$$

where we have used time reversal symmetry [e.g., $Z(k+\ l+)=Z(k-\ l-)$] to simplify the expression for Q.

For a system containing only two spins i and j, Q is simply $1-X_{ij}$ and (11) is obviously true. Now let us proceed by induction. Suppose (11) holds for any system of N spins described by (1) and (2). Let us add one spin, k, to this system, initially connecting it by a single bond to a spin m as shown in Fig. 4(a). Q is a linear function of the factor X_{km}, so it suffices to check (11) at $X_{km}=1$ and $X_{km}=0$. In the former case k is disconnected from the diagram containing l, so $Z(k+\ l+)=Z(k-\ l+)$ and Q vanishes. In the

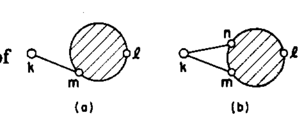

FIG. 4. Diagram illustrating proof of Theorem 1.

[1] For example, the very powerful results of T. D. Lee and C. N. Yang [Phys. Rev. 87, 410 (1952)] on the zeros of the Ising model partition function have provided information of great importance about the behavior of such models in a magnetic field, even though an exact solution to the statistical problem (in two and three dimensions) is still lacking.

latter case, $X_{km} = 0$ combines spins k and m which reduces our problem to N spins, for which Q is nonnegative by the induction hypothesis.

Suppose next that k is connected by two bonds to spins m and n as shown in Fig. 4(b). Q is linear in X_{kn}, and for $X_{kn} = 1$ the (kn) bond disappears and we have the problem considered in the preceding paragraph. But setting $X_{kn} = 0$ reduces the system to one of N spins, and thus the nonnegativity of Q is assured.

Clearly the same technique works as more and more bonds are added joining k to the original system of N spins. There is no difficulty if a bond is added directly connecting k and l. Thus the positivity of Q for all systems containing $N+1$ spins is ensured, given its positivity for all systems of N spins, and our proof is complete.

III. PROOF OF THEOREMS 2 AND 3

Initially we assume that k, l, m, and n all denote different spins; the case where two or more are identical is considered later. We rewrite the requirement (6) in terms of restricted partition functions as follows

$$\begin{aligned} Z^2[\langle\sigma_k\sigma_l\sigma_m\sigma_n\rangle - \langle\sigma_k\sigma_l\rangle\langle\sigma_m\sigma_n\rangle]/8 \\ = \tfrac{1}{4}[(a+b+c+d-e-f-g-h) \\ \times (a+b+c+d+e+f+g+h) \\ -(a+b+e+f-c-d-g-h) \\ \times (a+b+g+h-c-d-e-f)] \\ = F = (a+b)(c+d)-(e+f)(g+h) \geq 0, \end{aligned} \tag{12}$$

where

$$\begin{aligned} a &= Z(k+\ l+\ m+\ n+), & b &= Z(k+\ l+\ m-\ n-), \\ c &= Z(k+\ l-\ m+\ n-), & d &= Z(k+\ l-\ m-\ n+), \\ e &= Z(k+\ l+\ m+\ n-), & f &= Z(k+\ l+\ m-\ n+), \\ g &= Z(k+\ l-\ m+\ n+), & h &= Z(k+\ l-\ m-\ n-), \end{aligned} \tag{13}$$

and we have made free use of time-reversal invariance to replace, for example, $Z(k-\ l+\ m-\ n-)$ by g.

In order to gain insight into the algebraic structure of F, we consider first a simple example: the system illustrated in Fig. 1, a saturated diagram with four spins. Direct calculation yields

$$F = X_{kl}X_{mn}[X_{kn}X_{lm}(1-X_{km}^2)(1-X_{ln}^2) + X_{km}X_{ln}(1-X_{kn}^2)(1-X_{lm}^2)], \tag{14}$$

a quantity obviously nonnegative for all X_{ij} between 0 and 1.

Note that F is the sum of terms with the structure gG where g, the "linear term,"[2] is a simple product of X's and linear in any particular X_{ij}. G, on the other hand, is a polynomial in which any X, if it occurs at all, appears as X^2. Such polynomials we call *quadratic terms*. A more precise definition of a *linear term* is the following: Let W be a set of distinct X's (note that X_{ij} and X_{ji} are considered equivalent) containing at least one member. The linear term g associated with W is simply the product of all X's appearing in W. No additional numerical factors are permitted. For example, $X_{12}X_{13}$ is by our definition a linear term, and $2X_{12}X_{13}$ is not. The latter is of the form gG, with $G = 2$ the "quadratic term."

Any restricted or unrestricted partition function is the sum of linear terms plus a constant (which may be zero). Thus F, the sum of products of pairs of restricted partition functions, may be written as a sum of terms each of which is either constant or the product of X's, some of which occur linearly and some quadratically. After classifying different terms according to the set W_p of linear factors, we may add up all terms with the same W_p and write the sum as g_pG_p, where g_p is the (unique) linear term associated with W_p, and G_p is a quadratic term. (Clearly it is not possible for a particular X to appear both in g_p and G_p.) Some quadratic terms occur without linear factors and we denote their sum by G_o. Thus F has the form

$$F = G_o + \sum_{p=1} g_pG_p, \tag{15}$$

where $g_p \neq g_q$ for $p \neq q$. In general the G's will not have the simple form found in (14).

Provided the bonds kl and mn are present, F is always the product of $X_{kl}X_{mn}$ times a quantity not containing these factors, just as in the example (14). That this is true in general follows from (12) and the observation [see the definition (13)] that c, d, e, and f each contain X_{mn} to the first power while a, b, g, and h do not contain it at all. Similarly X_{kl} occurs to the first power in c, d, g, and h, and is absent from a, b, e, and f. The fact that X_{kl} and X_{mn} are simply multiplicative factors in F is the reason we do not need to require in Theorem 2 that J_{kl} and J_{mn} be nonnegative.

Consider next the example shown in Fig. 5 consisting of two disconnected diagrams A and B which,

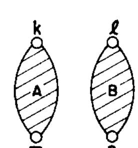

FIG. 5. Special case investigated in connection with Theorem 2.

[2] They could, more properly, be called multilinear.

apart from the fact that one contains the spins k and m and the other the spins l and n, are wholly arbitrary. Let Z_A and Z_B denote the [restricted] partition functions for these diagrams. Each term in (13) may be expressed as a suitable product; for example,

$$f = Z_A(k+\ m-)Z_B(l+\ n+). \tag{16}$$

Inserting these in (12) and making free use of the time reversal symmetry $[Z_A(k+\ m+) = Z_A(k-\ m-)$, etc.] we have

$$F = [Z_A(k+\ m+)^2 - Z_A(k+\ m-)^2] \times [Z_B(l+\ n+)^2 - Z_B(l+\ n-)^2], \tag{17}$$

which is nonnegative by Theorem 1 [see (11)]. We later need the following result:

Lemma 1: If systems A and B in Fig. 5 are both connected systems and the corresponding F given by (17) is decomposed in the form (15), then G_o is nonnegative.

Let F^A and F^B denote the first and second factors on the right-hand side of (17). In analogy with (15) let us decompose F^A as

$$F^A = G_o^A + \sum_{p=1} g_p^A G_p^A \tag{18}$$

and F^B in similar fashion. Since none of the X's appearing in F^A appear in F^B and vice versa, G_o is simply the product $G_o^A G_o^B$. We need only prove that G_o^A is positive—the same proof suffices for G_o^B—in order to prove Lemma 1.

We may write (see Appendix A)

$$Z_A(k+\ m+) = \sum_p r_p; \quad Z_B(k+\ m-) = \sum_q t_q, \tag{19}$$

where each r_p is either 1 or a linear term and the same is true of the t's. By Lemma A1 of Appendix A, for $p \neq p'$, r_p does not contain the same factors as $r_{p'}$, so that $r_p r_{p'}$ will always contain a linear term and can make no contribution to G_o^A. The same holds for the t's. We conclude that

$$G_o^A = \sum_p r_p^2 - \sum_q t_q^2 = \tilde{Z}_A(k+\ m+) - \tilde{Z}_A(k+\ m-), \tag{20}$$

where by $\tilde{Z}_A$ we mean the (restricted) partition function for a system $\tilde{A}$ obtained from A by multiplying by 2 every J_{ij} which occurs in A. The result of this process is to replace every X_{ij} by $X_{ij}{}^2$. Since $\tilde{A}$ contains only ferromagnetic bonds, Theorem 1 applies and, by (11), the expression (20) must be nonnegative. This completes the proof of Lemma 1.

We now prove Theorem 2 for a complete system containing N spins, assuming that k, l, m, and n are distinct spins. The decomposition (15) for F lacks the term G_o, since, as noted above, F contains the linear factors $X_{kl}X_{mn}$. Choose a particular p, say $p = 2$, and set all the factors in W_2 (the set of X's in g_2) equal to 1 everywhere in the expression for F. The result, F', corresponds to a diagram in which every bond corresponding to some X in W_2 has been erased. This diagram, according to Lemma A2 of Appendix A, consists of two disconnected pieces, A and B, each of which is complete. Since X_{kl} belongs to W_2, it is evident that spins k and l cannot both belong to system A or both to system B, for then one or the other of these systems would be incomplete The same holds for spins m and n. Several possibilities remain; without loss of generality we may assume the one shown in Fig. 5.

Of course, F' may be decomposed in the form (15) as

$$F' = G_o' + \sum_p g_p' G_p'. \tag{21}$$

We now assert that G_o' and G_2 are identical. It is clear that setting all the X's in W_2 equal to one does not alter G_2, and thus G_2 is a quadratic term appearing in F' with no linear term as a factor. However, for $p \neq 2$, g_p' (that is, the term obtained from g_p by setting all X's in W_2 equal to one) contains at least one of the X_{ij}. This follows from part (ii) of Lemma A2 in Appendix A. Thus, in fact, G_2 is the *only* quadratic term appearing in F' without a linear term as a factor and must be identical with G_o'. But the latter is nonnegative by Lemma 1 above.

A similar argument works for any G_p in the decomposition (15) of F for a complete system. But if every G_p is nonnegative, so is F, which completes our proof. The same result holds for an incomplete system, since we need only take the limit of setting certain X's equal to 1, and F is a continuous function of the X's.

We next consider the case where spins k, l, m, and n are not all distinct. If k and l are the same, $\sigma_k\sigma_l$ becomes $\sigma_k^2 = 1$ and (6) simply vanishes. The case where m and n are identical is similarly uninteresting. The case where $l = m$ can be considered by taking the limit $X_{lm} \to 0$, that is, by combining the spins. In this case (6) becomes

$$\langle\sigma_k\sigma_n\rangle - \langle\sigma_k\sigma_l\rangle\langle\sigma_l\sigma_n\rangle \geq 0 \tag{22}$$

or, in other words, we have proved Theorem 3.

IV. APPLICATION: LONG-RANGE ORDER IN ISING FERROMAGNETS

The phase transition which occurs as the temperature is lowered in zero magnetic field for an Ising ferromagnet on a square lattice with nearest-neighbor interactions results in (among other things) the appearance of "long-range order"[3] which we define (in general) as

$$L = \liminf (r_{ij} \to \infty) \lim (N \to \infty) \langle \sigma_i \sigma_j \rangle_N, \quad (23)$$

where the $N \to \infty$ limit implies some "sensible" means of defining a correlation function as the number of spins N tends to infinity.[4] In the limit inferior as $r_{ij} \to \infty$ we allow the direction of the vector joining the two spins to vary, though Schultz, Mattis, and Lieb[3] have shown that the result is independent of direction for the Ising ferromagnet mentioned above.

An obvious application of Theorem 2 is the following: Given an Ising model A with purely ferromagnetic interactions, the long-range order L is never less for a model B obtained from A by adding ferromagnetic bonds. Further, the transition temperature (Curie point) of B, which we define as the highest temperature at which long-range order appears, is not less than that of A.

Thus suppose, for example, that we have a two-dimensional square Ising lattice with a ferromagnetic nearest-neighbor interaction, and also ferromagnetic interactions, of arbitrary magnitude, with second, third, and fourth nearest neighbors. This model must (to no one's great surprise!) exhibit long-range order at any temperature below the Curie temperature obtained by Onsager.[5] Or, as another example, consider the particular case of long-range interactions (decreasing exponentially in one of the lattice directions) for which Kac and Thompson[6] have recently shown that a two-dimensional Ising model exhibits long-range order at sufficiently low temperatures. Since the potential is obtained by adding ferromagnetic terms to a case with ferromagnetic interactions between nearest neighbors, the existence of long-range order at low enough temperatures follows at once from Theorem 2.

As another application, we note that the existence of long-range order for the Ising ferromagnet in a two-dimensional square lattice with nearest-neighbor interactions at sufficiently low temperatures implies the same for the corresponding three-dimensional simple cubic lattice. Suppose that spin i is located at $(0, 0, 0)$ and j at (n, m, p)—the three numbers giving x, y, and z coordinates. Let spin k be located at $(n, q, 0)$. By Theorem 3,

$$\langle \sigma_i \sigma_j \rangle \geq \langle \sigma_i \sigma_k \rangle \langle \sigma_k \sigma_j \rangle. \quad (24)$$

But at sufficiently low temperatures $\langle \sigma_i \sigma_k \rangle$ is bounded from below[7] since both spins lie in a plane perpendicular to the z axis, and similarly $\langle \sigma_k \sigma_j \rangle$, since both spins lie in a plane perpendicular to the x axis. We know that long-range order exists for such planar lattices, and the fact that they form portions of three dimensional lattices merely implies that the additional ferromagnetic interactions present serve to enhance (by Theorem 2) or, at the least, not decrease, the correlation functions calculated for planar lattices alone.

This last result is, once again, not unexpected, especially since the presence of spontaneous magnetization in the simple cubic lattice described can be proved by using a simple argument given by Peierls,[8] a rigorous version of which was developed by the author[9] and independently by Dobrushin.[10] The power of Theorem 2 is, we believe, illustrated in the fact that one can proceed immediately from the two- to the three-dimensional case with no need of invoking any new combinatorial argument. And, of course, the cubic lattice with ferromagnetic nearest-neighbor and next-nearest-neighbor interactions, or interactions decreasing as $1/r^4$, or a multitude of other cases, are known immediately to display long-range order at low enough temperatures.

APPENDIX. PARTITION FUNCTIONS FOR COMPLETE SYSTEMS

The partition function Z associated with any diagram [or Hamiltonian of the form (1)] is obtained as follows. A configuration γ denotes a division of indices labeling different spins into two disjoint complementary sets $U(\gamma)$ and $D(\gamma)$. For $j \in U(\gamma)$, $\sigma_j = +1$ ("up") and for $k \in D(\gamma)$, $\sigma_k = -1$ ("down"). Configurations γ and γ' are distinct if and only if $D(\gamma) \neq D(\gamma')$ [or, the equivalent, $U(\gamma) \neq U(\gamma')$]. We now define

$$Z = \sum_\gamma Z_\gamma, \quad (A1)$$

[3] T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).

[4] See R. B. Griffiths, J. Math. Phys. **8**, 484 (1967) (following paper) for an approach which works for an Ising ferromagnet, and M. E. Fisher, J. Math. Phys. **6**, 1643 (1965) for a more general procedure.

[5] L. Onsager, Phys. Rev. **65**, 117 (1944).

[6] M. Kac and C. J. Thompson, Proc. Natl. Acad. Sci. U.S. **55**, 676 (1966). A recent note from these authors indicates that the proof as published is not correct and will require modification.

[7] In accordance with our definition (23) we must assume that spins i and k are sufficiently far apart, and similarly k and j. This may be accomplished by a proper choice of q.

[8] R. Peierls, Proc. Cambridge Phil. Soc. **32**, 477 (1936).

[9] R. B. Griffiths, Phys. Rev. **136**, A437 (1964).

[10] R. L. Dobrushin, Teoriya Veroyatnostei Primeneniya **10**, 209 (1965).

where

$$Z_\gamma = \prod_{i \in D(\gamma)} \prod_{j \in U(\gamma)} X_{ij} \tag{A2}$$

and, in an unsaturated diagram, X_{ij} is set equal to 1 for absent bonds. Each Z_γ is either 1 or a linear term as defined in Sec. III.

Lemma A1: A restricted partition function Z' (that is, with the value of one or more of the σ's specified) for a connected diagram has the form

$$Z' = \sum_\gamma{}' g_\gamma \tag{A3}$$

with g_γ (either 1 or a linear term) $\neq g_\eta$ for $\gamma \neq \eta$. The prime denotes a summation over all configurations satisfying the restriction.

The proof is almost obvious. We know that at least one σ has a specified value, say $\sigma_o = +1$. In a configuration γ we can determine the value (± 1) of any spin σ_i connected to σ_o by a bond by observing whether X_{oi} is present or absent in g_γ. The values of still other spins connected by bonds to these σ_i may be determined by repeating this process, and eventually the configuration γ is uniquely determined from a knowledge of g_γ, since the diagram is connected. [We remark that the lemma holds for the unrestricted partition function for a connected diagram if a factor of 2 is placed in front of the summation in (A3).]

Lemma A2: Let $Z^{(1)}$ and $Z^{(2)}$ be two restricted or unrestricted partition functions (they may be identical, or there may be different restrictions in the two cases) corresponding to the same complete diagram. Suppose the product is decomposed in the form (15):

$$Z^{(1)}Z^{(2)} = G_o + \sum_{p=1} g_p G_p . \tag{A4}$$

The sets W_p corresponding to the linear terms g_p (see Sec. III)are, of course, distinct: $W_p \neq W_{p'}$ for $p \neq p'$.

(i) If all the X's in a *particular* W are set equal to 1 and the corresponbing bonds in the diagram erased, the resulting diagram consists of two disconnected pieces, each of which is complete.

(ii) For $p \neq p'$, W_p is not a subset of $W_{p'}$.

To prove part (i) we consider a particular term $Z_\gamma Z_{\gamma'}$ [see the definition (A2)] in the product $Z^{(1)}Z^{(2)}$, where γ and γ' are configurations permitted by the restrictions (if any) for Z_1 and Z_2, respectively. Now if $D(\gamma)$ is identical with either $D(\gamma')$ or $U(\gamma')$, then $Z_\gamma = Z_{\gamma'}$ and the product contains no linear term. When $D(\gamma)$ is not identical with $D(\gamma')$ or with $U(\gamma')$, we have a situation illustrated schematically in Fig. 6, where the horizontal line indicates the division of indices into $U(\gamma)$ and $D(\gamma)$ [U and D for short] and the vertical into $U(\gamma')$ and $D(\gamma')$ [U' and D' for short]. We now ask, which X's occur linearly in the product $Z_\gamma Z_{\gamma'}$? That is, which X's occur in one factor but not in the other? There are four possibilities: X_{ij} occurs linearly if (a) $i \in D \cap D'$, $j \in U \cap D'$; (b) $i \in U \cap D'$, $j \in U \cap U'$; (c) $i \in D \cap U'$, $j \in U \cap U'$; (d) $i \in D \cap D'$, $j \in D \cap U'$. These bonds, represented schematically in Fig. 6, constitute the set W for the term $Z_\gamma Z_{\gamma'}$.

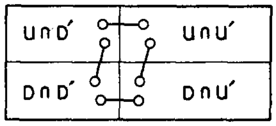

FIG. 6. Schematic diagram illustrating the division of all spin indices (represented by the complete rectangle) into sets according to two configurations γ and γ'. The small circles connected by straight lines indicate bonds whose factors enter linearly in the product $Z_\gamma Z_{\gamma'}$.

If we erase all bonds corresponding to X's in W, it is evident from Fig. 6 that the set of spins splits up into two disconnected sets,

$$\mathrm{A} = (D \cap D') \cup (U \cap U')$$

and

$$\mathrm{B} = (D \cap U') \cup (U \cap D').$$

That is, there are no bonds connecting the systems A and B. On the other hand, none of the bonds connecting two spins within A has been erased, nor any of the bonds connecting two spins within B. Therefore both A and B are complete. It is easily verified that if D is not identical to D' or to U', neither A nor B is a null set. This completes the proof of part (i).

To prove part (ii), assume that W_p is a proper subset of $W_{p'}$ (they cannot be identical for $p \neq p'$). In the initially complete diagram, erase all bonds corresponding to X's in W_p. The result, as we have just shown, is two disconnected systems A and B, each of which is complete. But if instead we were to erase all bonds corresponding to X's in the larger set $W_{p'}$, we would erase not only all the bonds connecting A and B, but additional bonds as well. That is, we would erase some of the bonds within A or within B. This would leave one or both systems incomplete in contradiction with part (i) of the lemma.