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Correlation inequalities for multicomponent ferromagnets

Kei-ichi Kondo and Yūki Sugiyama
Department of Physics, Faculty of Science, Nagoya University, Chikusa-ku, Nagoya 464, Japan

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A systematic method to obtain a series of new correlation inequalities for a class of two-component vector spin systems is presented. These correlation inequalities are applied to lattice scalar field models with two-body, anisotropic or isotropic ferromagnetic interactions and interaction potential of even polynomials, especially the $\lambda(|\Phi|^2)^2$ model, which includes the plane rotator model as a special case. Here an external field $\mathbf{h} = (h,H)$, $h \geqslant 0$, $H \geqslant 0$, is present. The possible way to extend our method to the N-component ($N \geqslant 3$) case is also discussed.

I. INTRODUCTION

Correlation inequalities are a very powerful tool in rigorous studies of quantum field theory and statistical mechanics.[1,2] For a set of lattice sites Λ, let $A = \{A_i\}_{i\in\Lambda}$ be a family of non-negative integers, only a finite number of which are nonzero. Define $\varphi^A = \Pi_{i\in\Lambda}\varphi_i^{A_i}$ and $|A| = \Sigma_{i\in\Lambda}A_i$, where the φ_i are real-valued scalar fields. For appropriate "ferromagnetic" expectations $\langle\cdot\rangle$, the first and second GKS inequalities[1] read

$$\langle\varphi^A\rangle \geqslant 0,\qquad \langle\varphi^A;\varphi^B\rangle \equiv \langle\varphi^A\varphi^B\rangle - \langle\varphi^A\rangle\langle\varphi^B\rangle \geqslant 0, \tag{1.1}$$

for *all A,B*.

But, it is unknown, in general, whether similar correlation inequalities hold for twice and more-times-truncated expectations, although they would have many important applications in rigorous studies of quantum field theory and statistical mechanics.[2,3] For nearest neighbor ferromagnetic interactions, e.g., the correlation inequality $\langle\varphi^A;\varphi^B;\varphi^C\rangle \geqslant 0$ (which has *not* been proved) would imply the convexity of the susceptibility and the monotonicity of the specific heat w.r.t. (with respect to) the inverse temperature J in the high temperature region $J < J_c$ for $|A| = |B| = |C| = 2$ (Ref. 3), and the convexity of the magnetization w.r.t. J for $|A| = 1$, $|B| = |C| = 2$ in the presence of the magnetic field. For the twice-truncated expectation, the following correlation inequality (called the new Lebowitz inequality[4]) is known to hold, for *all A,B,C*;

$$\langle\varphi^A;\varphi^B;\varphi^C\rangle \geqslant -2\min[\langle\varphi^A\rangle\langle\varphi^B;\varphi^C\rangle,\langle\varphi^B\rangle\langle\varphi^A;\varphi^C\rangle, \langle\varphi^C\rangle\langle\varphi^A;\varphi^B\rangle], \tag{1.2}$$

where

$$\langle\varphi^A;\varphi^B;\varphi^C\rangle \equiv \langle\varphi^A\varphi^B\varphi^C\rangle - \langle\varphi^A\rangle\langle\varphi^B\varphi^C\rangle - \langle\varphi^B\rangle\langle\varphi^A\varphi^C\rangle - \langle\varphi^C\rangle\langle\varphi^A\varphi^B\rangle + 2\langle\varphi^A\rangle\langle\varphi^B\rangle\langle\varphi^C\rangle. \tag{1.3}$$

Yet, it is incorrect to expect that $\langle\varphi^A;\varphi^B;\varphi^C\rangle \geqslant 0$ holds for all A,B,C. In fact, for $|A| = |B| = |C| = 1$, the GHS inequality holds,[5] namely

$$\langle\varphi_i;\varphi_j;\varphi_k\rangle \leqslant 0, \tag{1.4}$$

which implies the concavity of the magnetization and monotonicity of the susceptibility w.r.t. the magnetic field h. Moreover, for the four-times-truncated expectation, we have the Lebowitz inequality[6]

$$\langle\varphi_i;\varphi_j;\varphi_k;\varphi_l\rangle \leqslant 0 \tag{1.5}$$

in the absence of external magnetic field.

In the earlier work,[7] we presented a series of correlation inequalities for higher-times-truncated expectations. For example, in the presence of an external field, we proved

$$\langle\varphi_i;\varphi_j;\varphi_k;\varphi_l\rangle \geqslant -4\langle\varphi_i;\varphi_j\rangle\langle\varphi_k;\varphi_l\rangle, \tag{1.6a}$$

$$\langle\varphi_i;\varphi_j;\varphi_k;\varphi_l\rangle \leqslant -4\langle\varphi_i\rangle\langle\varphi_j;\varphi_k;\varphi_l\rangle. \tag{1.6b}$$

Note that the inequality (1.6b) reduces to (1.5) in the absence of an external field. Inequalities of this type [i.e., (1.2), (1.6a), and (1.6b)] turn out to be very useful in the rigorous study of the continuum limit and the critical behavior of the broken-symmetry lattice scalar field models.[8] But they are restricted to one-component ferromagnets whose single spin measures belong to the EMN (Ellis–Monroe–Newman) class.[9–12]

The extension of correlation inequalities to multicomponent ferromagnets has been performed by several authors.[13–26] For two-component spin systems where the spin is denoted as $\Phi = (\varphi,\xi)$, $|\Phi| = 1$, Monroe[14] presented the following correlation inequalities of GKS type:

$$\langle\varphi^A;\varphi^B\rangle \geqslant 0,\quad \langle\xi^A;\xi^B\rangle \geqslant 0,\quad \langle\varphi^A;\xi^B\rangle \leqslant 0. \tag{1.7}$$

We would like to obtain explicit correlation inequalities for the three- and four-times- (at least) truncated expectations for multicomponent ferromagnets whose spin variables Φ are allowed to take *unbounded* values. Although the broken-symmetry scalar $\lambda\Phi^4$ theory plays an important role in the Higgs mechanism,[27] it has been recently proved[8] that (under reasonable assumptions) the continuum scalar $\lambda(\varphi^4)_d$ field theory obtained from the corresponding lattice regularized model by taking the continuum limit is trivial in $d > 4$ dimensions, if one adopts the renormalization condition that the vacuum expectation value of the single (renormalized) scalar field remains finite and nonzero in this limit. However, the result is restricted to the one-component model.[8] This work has begun with the motivation of extending the triviality proof of broken-symmetry $\lambda(\varphi^4)_d$ field theory to the two- and more-component cases.

In Sec. II, we present a strategy based on duplicated variables for proving correlation inequalities. In Sec. III, we consider the anisotropic case. Section IV is devoted to the

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isotropic case. In this paper, explicit correlation inequalities are derived only for a class of two-component ferromagnets. In the final section, we discuss the possibility of extending our method to N-component ($N \geqslant 3$) ferromagnets. The explicit correlation inequalities involving three- and four-times-truncated expectations are listed in Appendix A for the anisotropic case and in Appendix B for the isotropic case.

II. TWO-COMPONENT MODEL

In this section we consider the two-component scalar field or spin model on a lattice Λ that consists of a set of N sites in d-dimensional space. To each site we associate a two-dimensional vector spin $\Phi = (\varphi,\xi)$. The Hamiltonian is of the form

$$\mathscr{H}_\Gamma(\Phi) = -\sum_{i,j:i<j=1}^{N} (J_{ij}\varphi_i\varphi_j + K_{ij}\xi_i\xi_j) - \sum_{j=1}^{N}(h_j\varphi_j + H_j\xi_j), \tag{2.1}$$

where $\Gamma \equiv \{J_{ij},K_{ij},h_j,H_j\}$ and $J_{ij},K_{ij},h_j,H_j \geqslant 0$ for all i and j. Note that the ferromagnetic interaction strength J_{ij} and the external fields h_j,H_j are made to vary from bond to bond and from site to site, respectively. Let a finite family of real-valued random variables $\Phi = \{\Phi_j \in \mathbb{R}^2; i = 1,...,N\}$ be distributed by the measure μ on $(\mathbb{R}^2)^N$ given by

$$d\mu_\Gamma(\Phi) = Z_\Gamma^{-1} \exp[-\mathscr{H}_\Gamma(\Phi)] \prod_{j=1}^{N} d\nu(\varphi_j), \tag{2.2}$$

where Z_Γ is the partition function that guarantees the normalization $\int d\mu_\Gamma(\Phi) = 1$ and $d\nu(\varphi_j)$ is the single spin measure,

$$d\nu(\Phi) = d\Phi \exp[-V(\Phi)], \quad d\Phi = d\varphi\, d\xi, \tag{2.3}$$

whose explicit form is specified below.

Before proceeding to find new correlation inequalities for two-component vector spin systems, we should explain the connection of our paper with that of Monroe.[14] Monroe considered the spin Φ with unit length $|\Phi| \equiv (\varphi^2 + \xi^2)^{1/2} = 1$, whose distribution over the unit circle is given by $f(\Phi)$ assumed to be even, i.e., $f(-\Phi) = f(\Phi)$. Our model allows the spin to be unbounded and, if desired, the fixed-length case is recovered by adopting $V(\Phi) = \lambda(|\Phi|^2 - 1)^2$ and taking the limit $\lambda \to \infty$. So our method covers a more general class of two-component ferromagnets than those encompassed by Monroe.

Consider the duplicate system whose random variables Φ and $\tilde{\Phi} \equiv (\tilde{\varphi},\tilde{\xi})$ are independently, identically distributed according to μ.

Introducing the vector notation

$$\vec{\varphi} \equiv (\varphi,\tilde{\varphi}), \quad \vec{\xi} \equiv (\xi,\tilde{\xi}), \qquad \vec{\Phi} \equiv (\vec{\varphi},\vec{\xi}), \quad \vec{H} \equiv (h,\tilde{h},H,\tilde{H}), \tag{2.4}$$

the sum of the original and duplicated Hamiltonian can be written as

$$\mathscr{H}_\Gamma(\Phi) + \mathscr{H}_\Gamma(\tilde{\Phi}) = -\sum_{i,j:i<j}^{N} (J_{ij}\vec{\varphi}_i\cdot\vec{\varphi}_j + K_{ij}\vec{\xi}\cdot\vec{\xi}) - \sum_{j=1}^{N} \vec{H}_j\cdot\vec{\Phi}_j. \tag{2.5}$$

For an orthogonal matrix T, define

$$\vec{X} \equiv T\vec{\Phi}, \tag{2.6}$$

where we have defined the variables

$$\vec{X} \equiv (\vec{x},\vec{y}), \quad \vec{x} \equiv (x,\tilde{x}), \vec{y} \equiv (y,\tilde{y}). \tag{2.7}$$

(I) If T is taken to be the direct sum of two orthogonal matrices A,B such that

$$T = \begin{vmatrix} A & O \\ O & B \end{vmatrix}, \tag{2.8}$$

then we obtain

$$\mathscr{H}_\Gamma(\Phi) + \mathscr{H}_\Gamma(\tilde{\Phi}) = -\sum_{i,j:i<j}^{N} (J_{ij}\vec{x}_i\cdot\vec{x}_j + K_{ij}\vec{y}_i\cdot\vec{y}_j) - \sum_{j=1}^{N} (T\vec{H}_j)\cdot\vec{X}_j, \tag{2.9}$$

where $\vec{x} = A\vec{\varphi}$ and $\vec{y} = B\vec{\xi}$.

(II) If the ferromagnetic interaction is of isotropic type: $J_{ij} = K_{ij}$, then, for any orthogonal matrix T,

$$\mathscr{H}_\Gamma(\Phi) + \mathscr{H}_{\tilde{\Gamma}}(\tilde{\Phi}) = -\sum_{i,j:i<j=1}^{N} J_{ij}\vec{X}_i\cdot\vec{X}_j - \sum_{j=1}^{N} (T\vec{H})\cdot\vec{X}_j. \tag{2.10}$$

For the multi-index $\mathbf{P}(a) = \{P_j(a)\}_{j\in\Lambda}$ (a collection of non-negative integers), define

$$\{(\vec{X})^{(a)}\}^{\mathbf{P}(a)} = \prod_{j=1}^{N} \{(\vec{X}_j)^{(a)}\}^{P_j(a)}, \tag{2.11}$$

where a labels the components of $\vec{\mathbf{X}}$.

Definition 1: Let Φ and $\tilde{\Phi}$ be two independent copies of a random variable Φ distributed by ν. Now we define the class $\mathfrak{g}$ of single spin measures by

$$\mathfrak{g} \equiv \left\{\nu;\, E_0\left[\prod_{a=1}^{4} \{(\vec{X})^{(a)}\}^{\mathbf{P}(a)}\right] \geqslant 0, \quad \text{for all } \mathbf{P}(a) \geqslant 0,\ a = 1,...,4\right\}, \tag{2.12}$$

where we have defined the unnormalized expectation

$$E_0[F(\Phi,\tilde{\Phi})] = \int F(\Phi,\tilde{\Phi}) \prod_{i=1}^{N} d\nu(\Phi_i) \prod_{j=1}^{N} d\nu(\tilde{\Phi}_j). \tag{2.13}$$

Now we look for an orthogonal matrix T that satisfies the conditions (2.12). First, we restrict the potential V to be of the Φ^4 type, namely,

$$V_4(\Phi) = \lambda(\varphi^2 + \xi^2)^2 + \mu(\varphi^2 + \xi^2), \quad \lambda \geqslant 0, \quad \mu \in \mathbb{R}. \tag{2.14}$$

In general, however, we suppose V is a polynominal of degree D, that is,

$$V_D(\Phi) = \sum_{n=1}^{D} \lambda_{2n}(|\Phi|^2)^n, \quad |\Phi|^2 \equiv \varphi^2 + \xi^2. \tag{2.15}$$

Then we define

$$\mathfrak{V}(\Phi,\tilde{\Phi}) \equiv V_D(\Phi) + V_D(\tilde{\Phi}) = \sum_{n=1}^{D} \lambda_{2n}[(|\Phi|^2)^n + (|\tilde{\Phi}|^2)^n], \tag{2.16}$$

which is rewritten in terms of the variable X by substituting the relation $\vec{\Phi} \equiv T^t \vec{X}$ (T^t: transposed matrix of T): $\mathfrak{V}_D(\Phi,\tilde{\Phi}) = W_D(X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$, and is decomposed into a sum of (even terms in each $X^{(a)}$) and (odd terms in each $X^{(a)}$). The even terms cause no problem. The odd terms must be ferromagnetic.

Define

$$P_{2n}(\Phi,\tilde{\Phi}) \equiv (|\Phi|^2)^n + (|\tilde{\Phi}|^2)^n. \tag{2.17}$$

Since the orthogonality of T implies

$$P_2(\Phi,\tilde{\Phi}) = (X^{(1)})^2 + (X^{(2)})^2 + (X^{(3)})^2 + (X^{(4)})^2, \tag{2.18}$$

it suffices to check the ferromagnetic character of $P_{2n}(\Phi,\tilde{\Phi})$ for $n \geqslant 2$.

Finally, note that

$$\begin{aligned} d\nu(\Phi) d\nu(\tilde{\Phi}) &= d\varphi\, d\xi\, d\tilde{\varphi}\, d\tilde{\xi} \exp[-\{V(\varphi,\xi) + V(\tilde{\varphi},\tilde{\xi})\}] \\ &= dx\, dy\, d\tilde{x}\, d\tilde{y} \exp[-W(\vec{x},\vec{y})] \det \left| \frac{\partial(\vec{\varphi},\vec{\xi})}{\partial(\vec{x},\vec{y})} \right| \\ &= dx\, dy\, d\tilde{x}\, d\tilde{y} \exp[-W(\vec{x},\vec{y})] \det T^t. \end{aligned} \tag{2.19}$$

Thus we must check the following points: (I) the orthogonality of T, (II) $\det T^T = \det T > 0$, (III) ferromagnetic character of the odd terms in $W(\vec{x},\vec{y})$, and (IV) $(T\vec{H}_j)^{(a)} \geqslant 0$ for all $a = 1,...,4$ (in the presence of external fields).

The requirements (II) and (III) are sufficient to conclude that $\nu \in \mathfrak{g}$. Therefore, the requirements (I) and (IV) imply that, for all $\mathbf{P}(a) \geqslant 0$ ($a = 1,...,4$),

$$\int d\mu_\Gamma(\Phi) d\mu_\Gamma(\tilde{\Phi}) \prod_{a=1}^{4} \{(T\vec{\Phi})^{(a)}\}^{\mathbf{P}(a)} \geqslant 0, \tag{2.20}$$

provided that $\nu \in \mathfrak{g}$ (see, e.g., Secs. 4.3 and 4.7 of Ref. 2).

For a given set of multi-index $\mathbf{P}(a) \equiv \{P_i(a)\}_{i\in\Lambda}$ ($a = 1,...,4$), the corresponding correlation inequalities are obtained by explicitly expanding the product

$$\begin{aligned} \prod_{a=1}^{4} \{(T\vec{\Phi})^{(a)}\}^{\mathbf{P}(a)} &= \prod_{i,j,k,l=1}^{N} \{(T\vec{\Phi}_i)^{(1)}\}^{P_i(1)} \{(T\vec{\Phi}_j)^{(2)}\}^{P_j(2)} \\ &\quad \times \{(T\vec{\Phi}_k)^{(3)}\}^{P_k(3)} \{(T\vec{\Phi}_l)^{(4)}\}^{P_l(4)}, \end{aligned}$$

where

$$\begin{aligned} (T\vec{\Phi}_j)^{(a)} &\equiv \sum_{b=1}^{4} T_{ab} \vec{\Phi}_j^{(b)} \\ &= T_{a1}\varphi_j + T_{a2}\tilde{\varphi}_j + T_{a3}\xi_j + T_{a4}\tilde{\xi}_j, \end{aligned} \tag{2.21}$$

and rewriting each term of the result in terms of the normalized expectation

$$\langle(\cdot)\rangle \equiv \int d\mu_\Gamma(\Phi)(\cdot), \tag{2.22}$$

according to the rule that, for $F(\Phi) = \varphi^A \xi^B$ and $G(\tilde{\Phi}) \equiv \tilde{\varphi}^C \tilde{\xi}^D$,

$$\int d\mu_\Gamma(\Phi) d\mu_\Gamma(\tilde{\Phi}) F(\Phi) G(\tilde{\Phi}) = \langle F(\Phi)\rangle \langle G(\Phi)\rangle.$$

Here it should be remarked that, in this procedure, there appear only terms that are a product of at most two expectations, since we have prepared the twofold duplicated system; but see Sec. V. Subsequently, we rewrite them in terms of the truncated expectations; see Appendixes A and B.

Finally, note that the measure $d\mu_\Gamma(\Phi) d\mu_\Gamma(\tilde{\Phi})$ is invariant under the transformation of exchanging the original and duplicated variables

$$\varphi_i \to \tilde{\varphi}_i \text{ and } \xi_j \to \tilde{\xi}_j, \text{ for all } i,j \in \Lambda, \tag{2.23}$$

and, if the external field is zero, under the following independent four transformations:

$$\varphi_i \to -\varphi_i, \quad \xi_j \to -\xi_j, \quad \tilde{\varphi}_k \to -\tilde{\varphi}_k, \quad \tilde{\xi}_l \to -\tilde{\xi}_l. \tag{2.24}$$

III. THE ANISOTROPIC CASE

In this section, we look for an orthogonal matrix satisfying the above conditions (I)–(IV) such that T is of block diagonal form entailing two 2×2 orthogonal matrices. Any 2×2 orthogonal matrix can be written as

$$A = \begin{vmatrix} \alpha & \beta \\ -\beta & \alpha \end{vmatrix} \quad \text{or} \quad B = \begin{vmatrix} \alpha & \beta \\ \beta & -\alpha \end{vmatrix}, \tag{3.1}$$

where $\alpha^2 + \beta^2 = 1$. Here note that $\det A = 1$, but $\det B = -1$.

Then the 4×4 matrix T is obtained as follows.

<u>Case (1):</u>

$$T = \begin{vmatrix} A & 0 \\ 0 & \tilde{A} \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \tilde{\alpha} & \tilde{\beta} \\ 0 & 0 & -\tilde{\beta} & \tilde{\alpha} \end{vmatrix}. \tag{3.2}$$

Indeed, the requirement (II) is satisfied, since

$$\det T^t = \tfrac{1}{4}(\alpha^2 + \beta^2)(\tilde{\alpha}^2 + \tilde{\beta}^2) > 0.$$

By explicit calculations, we obtain

$$\begin{aligned} P_4(\Phi,\tilde{\Phi}) = \tfrac{1}{2}\{&(X^{(1)})^4 + (X^{(2)})^4 + (X^{(3)})^4 + (X^{(4)})^4 \\ &+ 6(X^{(1)})^2(X^{(2)})^2 + 2(X^{(1)})^2(X^{(3)})^2 \\ &+ 2(X^{(1)})^2(X^{(4)})^2 + 2(X^{(2)})^2(X^{(3)})^2 \\ &+ 2(X^{(2)})^2(X^{(4)})^2 + 6(X^{(3)})^2(X^{(4)})^2\} \\ &+ 4\alpha\beta\tilde{\alpha}\tilde{\beta} X^{(1)}X^{(2)}X^{(3)}X^{(4)}. \end{aligned} \tag{3.3}$$

Then the requirement (III) forces us to take

$$\alpha\beta\tilde{\alpha}\tilde{\beta} = -1 (\leqslant 0), \tag{3.4}$$

which is sufficient to conclude that $\nu \in \mathfrak{g}$. Then we have

$$\begin{aligned} T\vec{H} = 1/\sqrt{2}(&(\alpha+\beta)h, (\alpha-\beta)h, \\ &(\tilde{\alpha}+\tilde{\beta})H, (\tilde{\alpha}-\tilde{\beta})H). \end{aligned} \tag{3.5}$$

Let $\alpha,\beta,\tilde{\alpha},\tilde{\beta} = \pm 1$. Then one has

$$\begin{aligned} T\vec{H} = 1/\sqrt{2}(&(\alpha+\beta)h, (\alpha-\beta)h, \\ &\tilde{\alpha}(1-\alpha\beta)H, \tilde{\alpha}(1+\alpha\beta)H). \end{aligned} \tag{3.6}$$

(i) The case of $\alpha\beta = 1$: $T\vec{H} = \sqrt{2}(\alpha h, 0, 0, \tilde{\alpha}H)$. Now the requirement $(T\vec{H})^{(a)} \geqslant 0$ for all $a = 1,...,4$ leads to the final result

$$\alpha = 1, \quad \tilde{\alpha} = 1, \quad \beta = 1, \quad \tilde{\beta} = -1. \tag{3.7}$$

(ii) The case of $\alpha\beta = -1$: in this case $T\vec{H} = \sqrt{2}(0, \alpha h, \tilde{\alpha}H, 0)$. Then we obtain

$$\alpha=1,\quad \tilde\alpha=1,\quad \beta=-1,\quad \tilde\beta=1. \tag{3.8}$$

Case (2):

$$T=\begin{vmatrix} B & 0\\ 0 & \tilde B\end{vmatrix}=\frac{1}{\sqrt2}\begin{vmatrix} \alpha & \beta & 0 & 0\\ \beta & -\alpha & 0 & 0\\ 0 & 0 & \tilde\alpha & \tilde\beta\\ 0 & 0 & \tilde\beta & -\tilde\alpha\end{vmatrix}. \tag{3.9}$$

Then

$$\det T'=(\alpha^2+\beta^2)(\tilde\alpha^2+\tilde\beta^2)/4>0.$$

Also in this case, the odd term has the same form as above. So we can proceed as before and obtain the following results.

(i) The case of $\alpha\beta=1$:

$$T\vec H=\sqrt2(\alpha h,0,0-\tilde\alpha H).$$

Then $\alpha=1, \tilde\alpha=-1, \beta=1, \tilde\beta=1$.

(ii) The case of $\alpha\beta=-1$: in this case $T\vec H=\sqrt2(0,-\alpha h,\tilde\alpha H,0)$. Then we obtain

$$\alpha=-1,\quad \tilde\alpha=1,\quad \beta=1,\quad \tilde\beta=1.$$

Case (3):

$$T=\begin{vmatrix} A & 0\\ 0 & \tilde B\end{vmatrix}=\frac{1}{\sqrt2}\begin{vmatrix} \alpha & \beta & 0 & 0\\ -\beta & \alpha & 0 & 0\\ 0 & 0 & \tilde\alpha & \tilde\beta\\ 0 & 0 & \tilde\beta & -\tilde\alpha\end{vmatrix}, \tag{3.10}$$

which is orthogonal, but yields

$$\det T'=(\alpha^2+\beta^2)(\tilde\alpha^2+\tilde\beta^2)/4<0.$$

Hence this choice of T contradicts requirement (II).

In this paper, explicit correlation inequalities are obtained for the matrix T case (1) (i):

$$T=\frac{1}{\sqrt2}\begin{vmatrix} 1 & 1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 1\end{vmatrix}. \tag{3.11}$$

Then one finds

$$\begin{aligned}\int d\mu_\Gamma(\Phi)d\mu_\Gamma(\tilde\Phi)\prod_{a=1}^{4}\{(T\vec\Phi)^{(a)}\}^{\mathbf P(a)} &=\int d\mu_\Gamma(\Phi)d\mu_\Gamma(\tilde\Phi)\prod_{i,j,k,l=1}^{N}(\varphi_i+\tilde\varphi_i)^{P_i(1)}\\ &\times(-\varphi_j+\tilde\varphi_j)^{P_j(2)}(\xi_k-\tilde\xi_k)^{P_k(3)}(\xi_l+\tilde\xi_l)^{P_l(4)}.\end{aligned} \tag{3.12}$$

Combining this with the invariance of the measure $d\mu_\Gamma(\Phi)d\mu_\Gamma(\tilde\Phi)$ under the transformation (2.23), nontrivial correlation inequalities are obtained only if

$$\sum_{i=1}^{N}P_j(2)\ \text{and}\ \sum_{k=1}^{N}P_k(3)\ \text{are both even or both odd.} \tag{3.13}$$

By performing the transformations (2.24) simultaneously, it is easy to see that all the possible choices of T above produce the same set of correlation inequalities as those obtained by, e.g., (3.11). So all cases are exhausted by considering (3.11).

For the matrix T of (3.2), one has

$$\begin{aligned}|\Phi|^2&=\vec X\cdot\vec X-2\alpha\beta X^{(1)}X^{(2)}-2\tilde\alpha\tilde\beta X^{(3)}X^{(4)},\\ |\Phi|^2&=\vec X\cdot\vec X+2\alpha\beta X^{(1)}X^{(2)}+2\tilde\alpha\tilde\beta X^{(3)}X^{(4)},\end{aligned} \tag{3.14}$$

where we have defined the inner product

$$\vec X\cdot\vec X\equiv(X^{(1)})^2+(X^{(2)})^2+(X^{(3)})^2+(X^{(4)})^2. \tag{3.15}$$

It is not difficult to show that the condition (3.4) is sufficient to guarantee the ferromagnetic character (III) of the polynomial $P_{2n}(\Phi,\tilde\Phi)$ for all $n\geqslant2$. So the correlation inequalities presented below hold also for

$$V_D(\Phi)=\sum_{n=1}^{D}\lambda_{2n}(|\Phi|^2)^n,$$

$$\text{if } \lambda_{2n}\geqslant0\quad(n\geqslant2)\ \text{and}\ \lambda_2\equiv\mu\in\mathbf R. \tag{3.16}$$

We define $N(a)\equiv\Sigma_{i\in\Lambda}P_i(a)$ and the index $N\equiv(N(1),N(2);N(3),N(4))$. Then, for example, we obtain

$$\langle\varphi_i;\xi_j\rangle\leqslant0, \tag{3.17}$$

for (0,1;1,0), which is a special case of (1.7), and

$$\langle\varphi_i;\xi_j\rangle\geqslant-2\langle\varphi_i\rangle\langle\xi_j\rangle, \tag{3.18}$$

for (1,0;0,1). For more correlation inequalities, see Appendix A. The inequalities (25), (28), and (29) in Ref. 14 correspond to (1,2;0,0), (1,0;2,0), and (0,1;1,1), respectively.

Especially, in the case that $h\equiv0$ (but $H>0$), our inequalities have the following form:

(1,1;1,0) [the special case of (1.7)],

$$\langle\varphi_i\varphi_j;\xi_k\rangle\leqslant0; \tag{3.19}$$

(0,2;0,1), (2,0;0,1),

$$\langle\varphi_i\varphi_j;\xi_k\rangle\geqslant-2\langle\varphi_i\varphi_j\rangle\langle\xi_k\rangle; \tag{3.20}$$

(0,0;2,1),

$$\langle\xi_i;\xi_j;\xi_k\rangle\geqslant-2\langle\xi_i;\xi_j\rangle\langle\xi_k\rangle; \tag{3.21}$$

(2,2;0,0),

$$\langle\varphi_i;\varphi_j;\varphi_k;\varphi_l\rangle\geqslant-2\langle\varphi_i\varphi_j\rangle\langle\varphi_k\varphi_l\rangle; \tag{3.22}$$

(1,1;1,1),

$$\langle\varphi_i\varphi_j;\xi_k;\xi_l\rangle\leqslant-2\langle\varphi_i\varphi_j;\xi_k\rangle\langle\xi_l\rangle; \tag{3.23}$$

(0,2;2,0), (2,0;2,0),

$$\langle\varphi_i\varphi_j;\xi_k;\xi_l\rangle\geqslant-2\langle\varphi_i\varphi_j\rangle\langle\xi_k;\xi_l\rangle; \tag{3.24}$$

(0,2;0,2),

$$\begin{aligned}\langle\varphi_i\varphi_j;\xi_k;\xi_l\rangle\geqslant&-2\langle\varphi_i\varphi_j;\xi_k\rangle\langle\xi_l\rangle;\\ &-2\langle\varphi_i\varphi_j;\xi_l\rangle\langle\xi_k\rangle\\ &-2\langle\varphi_i\varphi_j\rangle\langle\xi_k;\xi_l\rangle\\ &-4\langle\varphi_i\varphi_j\rangle\langle\xi_k\rangle\langle\xi_l\rangle\end{aligned} \tag{3.25}$$

$$\begin{aligned}\geqslant&-2\langle\varphi_i\varphi_j\rangle\langle\xi_k;\xi_l\rangle\\ &-4\langle\varphi_i\varphi_j\rangle\langle\xi_k\rangle\langle\xi_l\rangle;\end{aligned} \tag{3.26}$$

where we have used (3.19),

(0,0;2,2),

$$\begin{aligned}\langle\xi_i;\xi_j;\xi_k;\xi_l\rangle\geqslant&-2\langle\xi_i;\xi_j;\xi_k\rangle\langle\xi_l\rangle\\ &-2\langle\xi_i;\xi_j;\xi_l\rangle\langle\xi_k\rangle\\ &-2\langle\xi_i;\xi_j\rangle\langle\xi_k;\xi_l\rangle\\ &-4\langle\xi_i;\xi_j\rangle\langle\xi_k\rangle\langle\xi_l\rangle.\end{aligned} \tag{3.27}$$

IV. THE ISOTROPIC CASE

Consider now an orthogonal matrix T given as the tensor product

$$T = A \otimes \widetilde{A} = \frac{1}{2}\begin{vmatrix} \alpha\tilde{\alpha} & \alpha\widetilde{\beta} & \beta\tilde{\alpha} & \beta\widetilde{\beta} \\ -\alpha\widetilde{\beta} & \alpha\tilde{\alpha} & -\beta\widetilde{\beta} & \beta\tilde{\alpha} \\ -\beta\tilde{\alpha} & -\beta\widetilde{\beta} & \alpha\tilde{\alpha} & \alpha\widetilde{\beta} \\ \beta\widetilde{\beta} & -\beta\tilde{\alpha} & -\alpha\widetilde{\beta} & \alpha\tilde{\alpha} \end{vmatrix}, \tag{4.1}$$

where A and $\widetilde{A}$ are the 2×2 orthogonal matrices (3.1). However, an explicit calculation shows that

$$\begin{aligned} P_4(\Phi,\widetilde{\Phi}) = \tfrac{1}{2}\{ & (X^{(1)})^4 + (X^{(2)})^4 + (X^{(3)})^4 + (X^{(4)})^4 \\ & + 6(X^{(1)})^2(X^{(2)})^2 + 2(X^{(1)})^2(X^{(3)})^2 \\ & + 2(X^{(1)})^2(X^{(4)})^2 + 2(X^{(2)})^2(X^{(3)})^2 \\ & + 2(X^{(2)})^2(X^{(4)})^2 + 6(X^{(3)})^2(X^{(4)})^2\} \\ & + 4X^{(1)}X^{(2)}X^{(3)}X^{(4)}. \end{aligned} \tag{4.2}$$

Note that, in this expansion, the quantities $\alpha, \beta, \tilde{\alpha}, \widetilde{\beta}$ to be specified do not appear anywhere. Moreover, for both $T = A \otimes \widetilde{B}$ and $T = B \otimes \widetilde{B}$, $P_4(\Phi,\widetilde{\Phi})$ has the same form as (4.2). Hence the requirement (III) cannot be satisfied for these 4×4 matrices T of the tensor-product type.

Next we try to look for an orthogonal matrix T of the form

$$T = \alpha\mathbf{1} + \beta S, \tag{4.3}$$

where $\mathbf{1}$ is the unit matrix. Note that

$$\begin{aligned} TT' &= (\alpha\mathbf{1} + \beta S)(\alpha\mathbf{1} + \beta S') \\ &= \alpha^2\mathbf{1} + \alpha\beta(S + S') + \beta^2 SS'. \end{aligned}$$

Hence for $TT' = 1$ to hold, it is sufficient that S satisfy the conditions

$$S = -S', \quad SS' = 1, \quad \alpha^2 + \beta^2 = 1. \tag{4.4}$$

For example, the 2×2 matrix A is recovered from

$$S = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad T = \begin{vmatrix} \alpha & \beta \\ -\beta & \alpha \end{vmatrix}, \quad \alpha^2 + \beta^2 = 1.$$

For the 4×4 matrix, we set

$$T = \frac{1}{2}\begin{vmatrix} a & b & c & d \\ -b & a & e & f \\ -c & -e & a & g \\ -d & -f & -g & a \end{vmatrix} = \frac{1}{2}a\mathbf{1} + S. \tag{4.5}$$

The condition that all the off-diagonal elements of AA' vanish turns out to be satisfied if we take

$$f = -cde \quad \text{and} \quad g = bde, \tag{4.6}$$

provided that $b,c,d,e,f,g = +1$ or -1.

In this reduction, we obtain

$$\begin{aligned} P_4(\Phi,\widetilde{\Phi}) = \tfrac{1}{2}\{ & (X^{(1)})^4 + (X^{(2)})^4 + (X^{(3)})^4 + (X^{(4)})^4 \\ & + 2(2 - abce)(X^{(1)})^2(X^{(2)})^2 \\ & + 2(X^{(1)})^2(X^{(3)})^2 \\ & + 2(2 + abce)(X^{(1)})^2(X^{(4)})^2 \\ & + 2(2 + abce)(X^{(2)})^2(X^{(3)})^2 \\ & + 2(X^{(2)})^2(X^{(4)})^2 \\ & + 2(2 - abce)(X^{(3)})^2(X^{(4)})^2\} \\ & - 4abcdX^{(1)}X^{(2)}X^{(3)}X^{(4)}. \end{aligned} \tag{4.7}$$

Then the requirement (ferromagnetic interaction)

$$abcd = 1 \tag{4.8}$$

is sufficient to conclude that $\nu \in g$. Finally we have

$$T = \frac{1}{2}\begin{vmatrix} a & b & c & abc \\ -b & a & e & -abe \\ -c & -e & a & ace \\ -abc & -abe & -ace & a \end{vmatrix}. \tag{4.9}$$

Note that $\det T' = 1$, and in addition,

$$\begin{aligned} T\vec{H} = \tfrac{1}{2}[& (a+b)h + c(1+ab)H, \\ & (a-b)h + e(1-ab)H, \\ & -(c+e)h + a(1+ce)H, \\ & ab(-c+e)h + a(1-ce)H]. \end{aligned} \tag{4.10}$$

Consider now the case $ab = 1$, $ce = 1$ for which

$$\begin{aligned} T\vec{H} &= (ah + cH, 0, -ch + aH, 0) \\ &= a(h + (c/a)H, 0 - (c/a)h + H, 0). \end{aligned}$$

For $ac = 1$, $T\vec{H} = a(h + H, 0, -h + H, 0)$, the requirement (IV) is satisfied if we take $a = 1$, $b = 1$, $c = 1$, $e = 1$, provided that $H > h > 0$. For $ac = -1$, $T\vec{H} = a(h - H, 0, h + H, 0)$, then the requirement (IV) is satisfied if $a = 1$, $b = 1$, $c = -1$, $e = -1$, provided that $h > H > 0$. Other cases are discussed in a similar manner, and the results are summarized as seen in Table I. Here the requirement (IV) is satisfied for cases from 1–4 in the region $H > h > 0$, and for the cases from 5–8 in the region $h > H > 0$.

We present correlation inequalities for the matrix T (under the condition $H > h > 0$):

$$T = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{vmatrix}. \tag{4.11}$$

By the same argument as that in the previous section, nontrivial correlation inequalities are obtained only if

$$\sum_{j=1}^{N} P_j(2) \text{ and } \sum_{k=1}^{N} P_k(3) \text{ are both even or both odd.} \tag{4.12}$$

As in the anisotropic case, one can check that all the possible

TABLE I. Allowed sets of matrix elements for the matrix (4.5) which satisfies all the requirements (I)–(IV); we assumed that $a = +1$.

	a	b	c	d	e	f	g	ab	ce	$abce$
1	+	+	+	+	+	−	+	+	+	+
2	+	+	+	+	−	+	−	+	−	−
3	+	−	+	−	+	+	+	−	+	−
4	+	−	−	+	+	+	−	−	−	+
5	+	+	−	−	−	+	+	+	+	+
6	+	+	−	−	+	−	−	+	−	−
7	+	−	−	+	−	−	+	−	+	−
8	+	−	+	−	−	−	−	−	−	+

choices of T above produce the same set of correlation inequalities as those obtained by, e.g., (4.11).

By explicit calculations, one may show that the condition (4.8) that guarantees the ferromagnetic character (III) of the polynominal $P_n(\Phi,\widetilde{\Phi})$, for $n=2,4$, is sufficient to satisfy the requirement (III) for $n=3$, but fails for $n=4$. So the correlation inequalities presented below hold also for the

$$V_6(\Phi)=\eta(|\Phi|^2)^3+\lambda(|\Phi|^2)^2+\mu|\Phi|^2$$

model ($\eta,\lambda\geqslant 0, \mu\in\mathbb{R}$), together with the $\lambda(|\Phi|^2)^2$ model.

For example, we have the following correlation inequalities corresponding to the multi-index $(N(1),\ldots,N(4))$:

(0,1,1,0),

$$\langle\varphi_i;\varphi_j\rangle-\langle\varphi_i;\xi_j\rangle+\langle\xi_i;\varphi_j\rangle-\langle\xi_i;\xi_j\rangle\geqslant 0; \tag{4.13}$$

(1,0,0,1),

$$\begin{aligned}&\langle\varphi_i;\varphi_j\rangle+2\langle\varphi_i\rangle\langle\varphi_j\rangle-\langle\varphi_i;\xi_j\rangle\\&\quad-2\langle\varphi_i\rangle\langle\xi_j\rangle-\langle\xi_i;\varphi_j\rangle-2\langle\xi_i\rangle\langle\varphi_j\rangle\\&\quad+\langle\xi_i;\xi_j\rangle+2\langle\xi_i\rangle\langle\xi_j\rangle\leqslant 0.\end{aligned} \tag{4.14}$$

Further explicit correlation inequalities are presented in Appendix B, for the case of $h\equiv 0$. For the isotropic case, in contrast with the anisotropic case, we can obtain upper bounds on $\langle\xi_i;\xi_j;\xi_k\rangle$. For example, corresponding to the multi-index (0,1,1,1), we have

$$\begin{aligned}\langle\xi_i;\xi_j;\xi_k\rangle\leqslant&-2\langle\xi_i;\xi_j\rangle\langle\xi_k\rangle\\&+\langle\varphi_i\varphi_j;\xi_k\rangle+\langle\varphi_i\varphi_k;\xi_j\rangle\\&-\langle\xi_i;\varphi_j\varphi_k\rangle+2\langle\varphi_i\varphi_j\rangle\langle\xi_k\rangle.\end{aligned} \tag{4.15}$$

Note here that in the right-hand side of this inequality the first three terms are nonpositive and the remaining two are non-negative.

V. ON THE EXTENSIONS TO N-COMPONENT MODELS ($N\geqslant 3$)

In this section we discuss the extension of our method to get more correlation inequalities for N-component ferromagnets ($N\geqslant 3$). Let us consider R-fold replicated systems ($R\geqslant 2$). The case $N=2$, $R=2$ has been already considered in the previous sections. Let us put $F=N\times R$. In the following we try to find an $F\times F$ ($F>4$) orthogonal matrix T that satisfies the requirement (III). If we could find such a matrix T, all the remaining steps are carried out easily, as exemplified in the preceding sections.

First, we consider the two-component model, but we increase R to obtain correlation inequalities for higher- (at most R)-times-truncated expectations. This might enable us to obtain, e.g., the GHS inequality for two-component ferromagnets: $\langle\xi_i;\xi_j;\xi_k\rangle\leqslant 0$.[21]

(i) The case $N=2$, $R=3$: For example, consider the orthogonal 6×6 matrix

$$T(6)=\begin{vmatrix}\vartheta^{(1)} & 0\\ 0 & \vartheta^{(2)}\end{vmatrix},$$

where $\vartheta^{(a)}$ is a 3×3 orthogonal matrix, e.g.,

$$\vartheta=\begin{vmatrix}1 & 0 & 0\\ 0 & \alpha & \beta\\ 0 & -\beta & \alpha\end{vmatrix},\quad \alpha^2+\beta^2=1.$$

But, by explicit calculations, it turns out that all the 6×6 matrices T of this type do not satisfy the requirement (III).

(ii) The case $N=2$, $R=4$: A candidate is the following 8×8 matrix:

$$T(8)=\begin{vmatrix}A^{(1)} & 0 & 0 & 0\\ 0 & A^{(2)} & 0 & 0\\ 0 & 0 & A^{(3)} & 0\\ 0 & 0 & 0 & A^{(4)}\end{vmatrix},$$

where each $A^{(a)}$ is a 2×2 orthogonal matrix of the form (3.1). This matrix is obviously orthogonal, but does not satisfy the requirement (III). Furthermore, the orthogonal matrix

$$\mathrm{T}(8)=\begin{vmatrix}T^{(1)} & 0\\ 0 & T^{(2)}\end{vmatrix}$$

made from the 4×4 matrix T in (4.9) cannot satisfy the requirement (III), either.

Next we consider the N-component case, $N\geqslant 3$ with $R=2$. A possible (orthogonal) matrix T is obtained as the direct sum

$$T=\begin{vmatrix}A^{(1)} & & & \\ & A^{(2)} & & \\ & & \ddots & \\ & & & A^{(N)}\end{vmatrix},$$

where each $A^{(a)}$ is a 2×2 orthogonal matrix as in (3.1). But a matrix of this type fails to satisfy the requirement (III) for $N=3$ and 4.

The above consideration forces us to take T not of the diagonal form. As a first step, we tried to look for an $F\times F$ ($F\geqslant 5$) orthogonal matrix T with elements $T_{ab}=+1$ or -1. In the range $5\leqslant F\leqslant 7$, we exhausted all the possible cases and obtained the result that there exist no orthogonal matrices T which take values only $+1$ or -1 as their matrix elements. For $F=8$, however there exist such orthogonal matrices. We expect that some of them satisfy all the requirements (I)–(IV). Work to check them is now in progress. We hope that the results can be reported in a subsequent paper. The applications of the new correlation inequalities obtained in this paper to the study of the critical behavior of multicomponent ferromagnets will be presented elsewhere.

APPENDIX A: NEW CORRELATION INEQUALITIES (ANISOTROPIC CASE)

In what follows we enumerate correlation inequalities together with the index $N\equiv(N(1),N(2),N(3),N(4))$:

(0,1;1,0),

$$\langle\varphi_i;\xi_j\rangle\leqslant 0; \tag{A1}$$

(1,0;0,1),

$$\langle\varphi_i;\xi_j\rangle\geqslant-2\langle\varphi_i\rangle\langle\xi_j\rangle; \tag{A2}$$

(1,2;0,0),

$$\langle\varphi_i;\varphi_j;\varphi_k\rangle\geqslant-2\langle\varphi_i\rangle\langle\varphi_j;\varphi_k\rangle; \tag{A3}$$

(1,1;1,0),

$$\langle\varphi_i;\varphi_j;\xi_k\rangle\leqslant-2\langle\varphi_i\rangle\langle\varphi_j;\xi_k\rangle; \tag{A4}$$

(0,2;0,1),

$$\langle \varphi_i;\varphi_j;\xi_k \rangle \geqslant -2\langle \varphi_i;\varphi_j \rangle \langle \xi_k \rangle; \tag{A5}$$

(0,1;1,1)

$$\langle \varphi_i;\xi_j;\xi_k \rangle \leqslant -2\langle \varphi_i;\xi_j \rangle \langle \xi_k \rangle; \tag{A6}$$

(1,0;2,0),

$$\langle \varphi_i;\xi_j;\xi_k \rangle \geqslant -2\langle \varphi_i \rangle \langle \xi_j;\xi_k \rangle; \tag{A7}$$

(0,0;2,1),

$$\langle \xi_i;\xi_j;\xi_k \rangle \geqslant -2\langle \xi_i;\xi_j \rangle \langle \xi_k \rangle; \tag{A8}$$

(2,0;0,1),

$$\begin{aligned}\langle \varphi_i;\varphi_j;\xi_k \rangle \geqslant & -2\langle \varphi_i \rangle \langle \varphi_j;\xi_k \rangle \\ & -2\langle \varphi_j \rangle \langle \varphi_i;\xi_k \rangle \\ & -2\langle \varphi_i;\varphi_j \rangle \langle \xi_k \rangle \\ & -4\langle \varphi_i \rangle \langle \varphi_j \rangle \langle \xi_k \rangle; \end{aligned} \tag{A9}$$

(1,0;0,2),

$$\begin{aligned}\langle \varphi_i;\xi_j;\xi_k \rangle \geqslant & -2\langle \varphi_i \rangle \langle \xi_j;\xi_k \rangle \\ & -2\langle \xi_j \rangle \langle \varphi_i;\xi_k \rangle \\ & -2\langle \varphi_i;\xi_j \rangle \langle \xi_k \rangle \\ & -4\langle \varphi_i \rangle \langle \xi_j \rangle \langle \xi_k \rangle; \end{aligned} \tag{A10}$$

(2,2;0,0),

$$\begin{aligned}\langle \varphi_i;\varphi_j;\varphi_k;\varphi_l \rangle \geqslant & -2\langle \varphi_i \rangle \langle \varphi_j;\varphi_k;\varphi_l \rangle \\ & -2\langle \varphi_j \rangle \langle \varphi_i;\varphi_k;\varphi_l \rangle \\ & -2\langle \varphi_i;\varphi_j \rangle \langle \varphi_k;\varphi_l \rangle \\ & -4\langle \varphi_i \rangle \langle \varphi_j \rangle \langle \varphi_k;\varphi_l \rangle; \end{aligned} \tag{A11}$$

(2,1;1,0),

$$\begin{aligned}\langle \varphi_i;\varphi_j;\varphi_k;\xi_l \rangle \leqslant & -2\langle \varphi_i \rangle \langle \varphi_j;\varphi_k;\xi_l \rangle \\ & -2\langle \varphi_j \rangle \langle \varphi_i;\varphi_k;\xi_l \rangle \\ & -2\langle \varphi_i;\varphi_j \rangle \langle \varphi_k;\xi_l \rangle \\ & -4\langle \varphi_i \rangle \langle \varphi_j \rangle \langle \varphi_k;\xi_l \rangle; \end{aligned} \tag{A12}$$

(0,3;1,0),

$$\begin{aligned}\langle \varphi_i;\varphi_j;\varphi_k\xi_l \rangle \leqslant & -2\langle \varphi_i;\varphi_j \rangle \langle \varphi_k;\xi_l \rangle \\ & -2\langle \varphi_i;\varphi_k \rangle \langle \varphi_j;\xi_l \rangle \\ & -2\langle \varphi_j;\varphi_k \rangle \langle \varphi_i;\xi_l \rangle; \end{aligned} \tag{A13}$$

(1,1;1,1),

$$\begin{aligned}\langle \varphi_i;\varphi_j;\xi_k;\xi_l \rangle \leqslant & -2\langle \varphi_i;\varphi_j;\xi_k \rangle \langle \xi_l \rangle \\ & -2\langle \varphi_i \rangle \langle \varphi_j;\xi_k;\xi_l \rangle \\ & -2\langle \varphi_j;\xi_k \rangle \langle \varphi_i;\xi_l \rangle \\ & -4\langle \varphi_j;\xi_k \rangle \langle \varphi_i \rangle \langle \xi_l \rangle; \end{aligned} \tag{A14}$$

(0,2;2,0),

$$\begin{aligned}\langle \varphi_i;\varphi_j;\xi_k\xi_l \rangle \geqslant & -2\langle \varphi_i;\varphi_j \rangle \langle \xi_k;\xi_l \rangle \\ & -2\langle \varphi_i;\xi_k \rangle \langle \varphi_j;\xi_l \rangle \\ & -2\langle \varphi_j;\xi_l \rangle \langle \varphi_j;\xi_k \rangle; \end{aligned} \tag{A15}$$

(0,2;0,2),

$$\begin{aligned}\langle \varphi_i;\varphi_j;\xi_k;\xi_l \rangle \geqslant & -2\langle \varphi_i;\varphi_j;\xi_k \rangle \langle \xi_l \rangle \\ & -2\langle \varphi_i;\varphi_j;\xi_l \rangle \langle \xi_k \rangle \\ & -2\langle \varphi_i;\varphi_j \rangle \langle \xi_k;\xi_l \rangle \\ & -4\langle \varphi_i;\varphi_j \rangle \langle \xi_k \rangle \langle \xi_l \rangle; \end{aligned} \tag{A16}$$

(2,0;2,0),

$$\begin{aligned}\langle \varphi_i;\varphi_j;\xi_k\xi_l \rangle \geqslant & -2\langle \varphi_i \rangle \langle \varphi_j;\xi_k;\xi_l \rangle \\ & -2\langle \varphi_j \rangle \langle \varphi_i;\xi_k;\xi_l \rangle \\ & -2\langle \varphi_i;\varphi_j \rangle \langle \xi_k;\xi_l \rangle \\ & -4\langle \varphi_i \rangle \langle \varphi_j \rangle \langle \xi_k;\xi_l \rangle; \end{aligned} \tag{A17}$$

(0,1;3,0),

$$\begin{aligned}\langle \varphi_i;\xi_j;\xi_k;\xi_l \rangle \leqslant & -2\langle \varphi_i;\xi_j \rangle \langle \xi_k;\xi_l \rangle \\ & -2\langle \varphi_i;\xi_k \rangle \langle \xi_j;\xi_l \rangle \\ & -2\langle \varphi_i;\xi_l \rangle \langle \xi_j;\xi_k \rangle; \end{aligned} \tag{A18}$$

(0,1;1,2),

$$\begin{aligned}\langle \varphi_i;\xi_j;\xi_k;\xi_l \rangle \leqslant & -2\langle \varphi_i;\xi_j;\xi_k \rangle \langle \xi_l \rangle \\ & -2\langle \varphi_i;\xi_j;\xi_l \rangle \langle \xi_k \rangle \\ & -2\langle \varphi_i;\xi_j \rangle \langle \xi_k;\xi_l \rangle \\ & -4\langle \varphi_i;\xi_j \rangle \langle \xi_k \rangle \langle \xi_l \rangle; \end{aligned} \tag{A19}$$

(0,0;2,2),

$$\begin{aligned}\langle \xi_i;\xi_j;\xi_k;\xi_l \rangle \geqslant & -2\langle \xi_i;\xi_j;\xi_k \rangle \langle \xi_l \rangle \\ & -2\langle \xi_i;\xi_j;\xi_l \rangle \langle \xi_k \rangle \\ & -2\langle \xi_i;\xi_j \rangle \langle \xi_k;\xi_l \rangle \\ & -4\langle \xi_i;\xi_j \rangle \langle \xi_k \rangle \langle \xi_l \rangle. \end{aligned} \tag{A20}$$

APPENDIX B: NEW CORRELATION INEQUALITIES (ISOTROPIC CASE)

The correlation inequalities are obtained for the matrix T (4.11) that allows us to include the external field (h,H), provided that $H \geqslant h \geqslant 0$. However, the full form is rather complicated. So, in the following, we present them in the case of $h \equiv 0$, $H \geqslant 0$:

(0,1,1,0),

$$\langle \varphi_i \varphi_j \rangle \geqslant \langle \xi_i;\xi_j \rangle; \tag{B1}$$

(1,0,0,1),

$$\langle \varphi_i \varphi_j \rangle \leqslant \langle \xi_i;\xi_j \rangle + 2\langle \xi_i \rangle \langle \xi_j \rangle; \tag{B2}$$

(0,1,1,1),

$$\begin{aligned}\langle \xi_i;\xi_j;\xi_k \rangle \leqslant & -2\langle \xi_i;\xi_j \rangle \langle \xi_k \rangle \\ & -\langle \xi_i;\varphi_j;\varphi_k \rangle \\ & +\langle \varphi_i \varphi_j;\xi_k \rangle \\ & +\langle \varphi_i \varphi_k;\xi_j \rangle \\ & +2\langle \varphi_i \varphi_j \rangle \langle \xi_k \rangle; \end{aligned} \tag{B3}$$

(1,1,1,0),

$$
\begin{aligned}
\langle \xi_i;\xi_j;\xi_k\rangle \leqslant & -2\langle \xi_i\rangle\langle \xi_j;\xi_k\rangle \\
& +\langle \xi_i;\varphi_j\varphi_k\rangle \\
& -\langle \varphi_i\varphi_j;\xi_k\rangle \\
& +\langle \varphi_i\varphi_k;\xi_j\rangle \\
& +2\langle \xi_i\rangle\langle \varphi_j\varphi_k\rangle;
\end{aligned}
\tag{B4}
$$

(1,2,0,0),

$$
\begin{aligned}
\langle \xi_i;\xi_j;\xi_k\rangle \geqslant & -2\langle \xi_i\rangle\langle \xi_j;\xi_k\rangle \\
& -\langle \xi_i;\varphi_j\varphi_k\rangle \\
& -\langle \varphi_i\varphi_j;\xi_k\rangle \\
& -\langle \varphi_i\varphi_k;\xi_j\rangle \\
& -2\langle \xi_i\rangle\langle \varphi_j\varphi_k\rangle;
\end{aligned}
\tag{B5}
$$

(1,0,2,0),

$$
\begin{aligned}
\langle \xi_i;\xi_j;\xi_k\rangle \geqslant & -2\langle \xi_i\rangle\langle \xi_j;\xi_k\rangle \\
& -\langle \xi_i;\varphi_j\varphi_k\rangle \\
& +\langle \varphi_i\varphi_j;\xi_k\rangle \\
& +\langle \varphi_i\varphi_k;\xi_j\rangle \\
& -2\langle \xi_i\rangle\langle \varphi_j\varphi_k\rangle;
\end{aligned}
\tag{B6}
$$

(0,2,0,1),

$$
\begin{aligned}
\langle \xi_i;\xi_j;\xi_k\rangle \geqslant & -2\langle \xi_i;\xi_j\rangle\langle \xi_k\rangle \\
& +\langle \xi_i;\varphi_j\varphi_k\rangle \\
& -\langle \varphi_i\varphi_j;\xi_k\rangle \\
& +\langle \varphi_i\varphi_k;\xi_j\rangle \\
& -2\langle \varphi_i\varphi_j\rangle\langle \xi_k\rangle;
\end{aligned}
\tag{B7}
$$

(0,0,2,1),

$$
\begin{aligned}
\langle \xi_i;\xi_j;\xi_k\rangle \geqslant & -2\langle \xi_i;\xi_j\rangle\langle \xi_k\rangle \\
& -\langle \xi_i;\varphi_j\varphi_k\rangle \\
& -\langle \varphi_i\varphi_j;\xi_k\rangle \\
& -\langle \varphi_i\varphi_k;\xi_j\rangle \\
& -2\langle \varphi_i\varphi_j\rangle\langle \xi_k\rangle;
\end{aligned}
\tag{B8}
$$

(2,0,0,1),

$$
\begin{aligned}
\langle \xi_i;\xi_j;\xi_k\rangle \geqslant & -2\langle \xi_i;\xi_j\rangle\langle \xi_k\rangle \\
& -2\langle \xi_i;\xi_k\rangle\langle \xi_j\rangle \\
& -2\langle \xi_j;\xi_k\rangle\langle \xi_i\rangle \\
& +\langle \xi_i;\varphi_j\varphi_k\rangle \\
& +\langle \xi_j;\varphi_i\varphi_k\rangle \\
& -\langle \varphi_i\varphi_j;\xi_k\rangle \\
& -2\langle \varphi_i\varphi_j\rangle\langle \xi_k\rangle \\
& +2\langle \varphi_i\varphi_k\rangle\langle \xi_j\rangle \\
& +2\langle \varphi_j\varphi_k\rangle\langle \xi_i\rangle \\
& -4\langle \xi_i\rangle\langle \xi_j\rangle\langle \xi_k\rangle;
\end{aligned}
\tag{B9}
$$

(1,0,0,2),

$$
\begin{aligned}
\langle \xi_i;\xi_j;\xi_k\rangle \geqslant & -2\langle \xi_i;\xi_j\rangle\langle \xi_k\rangle \\
& -2\langle \xi_i;\xi_k\rangle\langle \xi_j\rangle \\
& -2\langle \xi_j;\xi_k\rangle\langle \xi_i\rangle \\
& -\langle \xi_i;\varphi_j\varphi_k\rangle \\
& +\langle \xi_j;\varphi_i\varphi_k\rangle \\
& +\langle \varphi_i\varphi_j;\xi_k\rangle \\
& +2\langle \varphi_i\varphi_j\rangle\langle \xi_k\rangle \\
& +2\langle \varphi_i\varphi_k\rangle\langle \xi_j\rangle \\
& -2\langle \varphi_j\varphi_k\rangle\langle \xi_i\rangle \\
& -4\langle \xi_i\rangle\langle \xi_j\rangle\langle \xi_k\rangle;
\end{aligned}
\tag{B10}
$$

(0,0,0,3),

$$
\begin{aligned}
\langle \xi_i;\xi_j;\xi_k\rangle \geqslant & -2\langle \xi_i;\xi_j\rangle\langle \xi_k\rangle \\
& -2\langle \xi_i;\xi_k\rangle\langle \xi_j\rangle \\
& -2\langle \xi_j;\xi_k\rangle\langle \xi_i\rangle \\
& -\langle \xi_i;\varphi_j\varphi_k\rangle \\
& -\langle \xi_j;\varphi_i\varphi_k\rangle \\
& -\langle \varphi_i\varphi_j;\xi_k\rangle \\
& -2\langle \varphi_i\varphi_j\rangle\langle \xi_k\rangle \\
& -2\langle \varphi_i\varphi_k\rangle\langle \xi_j\rangle \\
& -2\langle \varphi_j\varphi_k\rangle\langle \xi_i\rangle \\
& -4\langle \xi_i\rangle\langle \xi_j\rangle\langle \xi_k\rangle;
\end{aligned}
\tag{B11}
$$

(1,1,1,1),

$$
\begin{aligned}
&\langle \varphi_i;\varphi_j;\varphi_k;\varphi_l\rangle + \langle \varphi_i;\varphi_j;\xi_k;\xi_l\rangle \\
&\quad -\langle \varphi_i;\xi_j;\varphi_k;\xi_l\rangle - \langle \varphi_i;\xi_j;\xi_k;\varphi_l\rangle \\
&\quad -\langle \xi_i;\varphi_j;\varphi_k;\xi_l\rangle - \langle \xi_i;\varphi_j;\xi_k;\varphi_l\rangle \\
&\quad +\langle \xi_i;\xi_j;\varphi_k;\varphi_l\rangle + \langle \xi_i;\xi_j;\xi_k;\xi_l\rangle \\
&\quad -2\langle \xi_i\rangle\langle \varphi_j;\xi_k;\varphi_l\rangle + 2\langle \xi_i\rangle\langle \xi_j;\varphi_k;\varphi_l\rangle \\
&\quad -2\langle \xi_i\rangle\langle \varphi_j;\varphi_k;\xi_l\rangle + 2\langle \xi_i\rangle\langle \xi_j;\xi_k;\xi_l\rangle \\
&\quad +2\langle \varphi_i;\varphi_i;\xi_k\rangle\langle \xi_l\rangle - 2\langle \varphi_i;\xi_j;\varphi_k\rangle\langle \xi_l\rangle \\
&\quad -2\langle \xi_i;\varphi_j;\varphi_k\rangle\langle \xi_l\rangle + 2\langle \xi_i;\xi_j;\xi_k\rangle\langle \xi_l\rangle \\
&\quad +2\langle \varphi_i;\varphi_l\rangle\langle \varphi_j;\varphi_k\rangle - 2\langle \varphi_i;\varphi_j\rangle\langle \xi_k;\xi_l\rangle \\
&\quad -2\langle \xi_i\rangle\langle \xi_l\rangle\langle \varphi_i;\varphi_k\rangle + 2\langle \xi_i;\xi_l\rangle\langle \xi_j;\xi_k\rangle \\
&\quad -4\langle \xi_i\rangle\langle \varphi_j;\varphi_k\rangle\langle \xi_l\rangle + 4\langle \xi_i\rangle\langle \xi_j;\xi_k\rangle\langle \xi_l\rangle < 0;
\end{aligned}
\tag{B12}
$$

$(0,2,2,0)$,

$$\begin{aligned}
&\langle\varphi_i;\varphi_j;\varphi_k;\varphi_l\rangle + \langle\varphi_i;\varphi_j;\xi_k;\xi_l\rangle \\
&\quad - \langle\varphi_i;\xi_j;\varphi_k;\xi_l\rangle - \langle\varphi_i;\xi_j;\xi_k;\varphi_l\rangle \\
&\quad - \langle\xi_i;\varphi_j;\varphi_k;\xi_l\rangle - \langle\xi_i;\varphi_j;\xi_k;\varphi_l\rangle \\
&\quad + \langle\xi_i;\xi_j;\varphi_k;\varphi_l\rangle + \langle\xi_i;\xi_j;\xi_k;\xi_l\rangle \\
&\quad + 2\langle\varphi_i;\varphi_j\rangle\langle\varphi_k;\varphi_l\rangle + 2\langle\varphi_i;\varphi_l\rangle\langle\varphi_j;\varphi_k\rangle \\
&\quad + 2\langle\varphi_i;\varphi_k\rangle\langle\varphi_j;\varphi_l\rangle \\
&\quad + 2\langle\varphi_i;\varphi_j\rangle\langle\xi_k;\xi_l\rangle - 2\langle\varphi_i;\varphi_k\rangle\langle\xi_j;\xi_l\rangle \\
&\quad - 2\langle\varphi_i;\varphi_l\rangle\langle\xi_j;\xi_k\rangle - 2\langle\varphi_j;\varphi_k\rangle\langle\xi_i;\xi_l\rangle \\
&\quad - 2\langle\varphi_j;\varphi_l\rangle\langle\xi_i;\xi_k\rangle + 2\langle\varphi_k;\varphi_l\rangle\langle\xi_i;\xi_j\rangle \\
&\quad + 2\langle\xi_i;\xi_j\rangle\langle\xi_k;\xi_l\rangle + 2\langle\xi_i;\xi_k\rangle\langle\xi_j;\xi_l\rangle \\
&\quad + 2\langle\xi_i;\xi_l\rangle\langle\xi_j;\xi_k\rangle \geqslant 0. \qquad \text{(B13)}
\end{aligned}$$

Other cases are omitted.

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