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A Construction of MDS Quantum Convolutional Codes

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Abstract In this paper, two new families of MDS quantum convolutional codes are constructed. The first one can be regarded as a generalization of [36, Theorem 6.5], in the sense that we do not assume that $q \equiv 1 \pmod 4$. More specifically, we obtain two classes of MDS quantum convolutional codes with parameters: (i) $[(q^2+1, q^2-4i+3, 1; 2, 2i+2)]_q$, where $q \geq 5$ is an odd prime power and $2 \leq i \leq (q-1)/2$; (ii) $[(\frac{q^2+1}{10}, \frac{q^2+1}{10} - 4i, 1; 2, 2i+3)]_q$, where q is an odd prime power with the form $q = 10m + 3$ or $10m + 7$ ($m \geq 2$), and $2 \leq i \leq 2m - 1$.

Keywords Quantum convolutional codes · Convolutional codes · Constacyclic codes

1 Introduction

Quantum block codes are used to protect quantum information over noisy quantum channels. Many works have been done for the constructions of good quantum error-correcting codes (e.g. see [1–7]). Quantum convolutional coding theory provides a different paradigm for coding quantum information and has numerous benefits for quantum communication [18–21]. For example, the convolutional structure is useful for a quantum communication scenario where a sender possesses a stream of qubits to send to a receiver.

The first important quantum block code construction is that of [1–3], which yields the commonly called Calderbank Shor Steane (CSS) construction. In contrast to quantum block

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codes, the construction for a CSS quantum convolutional code is similar to that for the block case, except that we import classical convolutional codes rather than classical block codes [20, Chap 9]. Forney et al. [21] provided many constructions of CSS quantum convolutional codes from classical binary convolutional codes. A Calderbank-Rains-Shor-Sloane (CRSS) quantum convolutional code was obtained from a classical convolutional code over $\mathbb{F}_4$ [20, Chap 9]. Many classes of quantum convolutional codes have been constructed (e.g. see [22–37]).

Almeida and Palazzo Jr. in [23] obtained a quantum convolutional code with parameters $[(4, 1, 3)]$ (memory $m = 3$). Tan and Li in [31] constructed quantum convolutional codes through LDPC codes. Very recently, La Guardia [35–37] applied the methods presented by Piret in [38] and then generalized by Aly et al. in [26], to construct classical and MDS quantum convolutional codes; Chen et al. in [32] constructed some families of nonbinary quantum convolutional codes by using negacyclic codes.

Motivated by [36], two new families of MDS quantum convolutional codes are constructed in this paper. The first one can be regarded as a generalization of [36, Theorem 6.5], in the sense that we do not assume that $q \equiv 1 \pmod 4$. More specifically, we obtain two classes of MDS quantum convolutional codes with parameters: (i) $[(q^2+1, q^2-4i+3, 1; 2, 2i+2)]_q$, where $q \geq 5$ is an odd prime power and $2 \leq i \leq (q-1)/2$; (ii) $[(\frac{q^2+1}{10}, \frac{q^2+1}{10} - 4i, 1; 2, 2i+3)]_q$, where q is an odd prime power with the form $q = 10m+3$ or $10m+7$ ($m \geq 2$), and $2 \leq i \leq 2m-1$.

The paper is organized as follows. In Section 2, we recall basic notation and necessary facts about constacyclic codes, classical convolutional codes and MDS quantum convolutional codes. In Section 3, we propose constructions of new families of MDS quantum convolutional codes derived from constacyclic codes.

2 Background

In this section, we recall basic notation and necessary facts which are important to the constructions of quantum convolutional codes. We adopt the notation in [36].

2.1 Classical Convolutional Codes

As mentioned in Section 1, quantum convolutional codes can be constructed from classical convolutional codes. In this subsection we present a brief review of classical convolutional codes. Let $G(D) = (g_{ij}) \in \mathbb{F}_{q^2}[D]^{k\times n}$, where $\mathbb{F}_{q^2}[D]^{k\times n}$ denotes the set of all $k \times n$ matrices with entries in $\mathbb{F}_{q^2}[D]$; $G(D)$ is called *basic* if it has a polynomial right inverse. A basic generator matrix is called *reduced* if the overall constraint length $\gamma = \sum_{i=1}^{k}\gamma_i$ has the smallest value among all basic generator matrices, where $\gamma_i = \max_{1\leq j\leq n}\{\deg g_{ij}\}$. In this case the overall constraint length γ will be called the *degree* of the resulting code.

Definition 2.1 *(See [25]) A convolutional code V with parameters $(n, k, \gamma; \mu, d_f)_{q^2}$ is a submodule of $\mathbb{F}_{q^2}[D]^n$ generated by a reduced basic matrix $G(D) = (g_{ij}) \in \mathbb{F}_{q^2}[D]^{k\times n}$, $V = \{\mathbf{u}(D)G(D) \mid \mathbf{u}(D) \in \mathbb{F}_{q^2}[D]^k\}$, where n is the length, k is the dimension, $\gamma = \sum_{i=1}^{k}\gamma_i$ is the degree, $\mu = \max_{1\leq i\leq k}\{\gamma_i\}$ is the memory and $d_f = wt(V) = min\{wt(\mathbf{v}(D)) \mid \mathbf{v}(D) \in V, \mathbf{v}(D) \neq 0\}$ is the free distance of the code. Here, $wt(\mathbf{v}(D)) = \sum_{i=1}^{n} wt(v_i(D))$, where $wt(v_i(D))$ is the number of nonzero coefficients of $v_i(D)$.*

The Hermitian inner product on $\mathbb{F}_{q^2}[D]^n$ is defined as $\langle \mathbf{u}(D) \mid \mathbf{v}(D)\rangle_h = \sum_i \mathbf{u}_i \cdot \mathbf{v}_i^q$, where $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_{q^2}^n$ and $\mathbf{v}_i^q = (v_{1i}^q, v_{2i}^q, \cdots, v_{ni}^q)$. The Hermitian dual of the code V is defined by

$$V^{\perp_h} = \left\{\mathbf{u}(D) \in \mathbb{F}_{q^2}[D]^n \;\middle|\; \langle \mathbf{u}(D) \mid \mathbf{v}(D)\rangle_h = 0 \text{ for all} \mathbf{v}(D) \in V\right\}.$$

We can construct convolutional codes from block codes. Let $\mathcal{C}$ be an $[n, k, d]_{q^2}$ linear code with parity check matrix H. Split H into $\mu + 1$ disjoint submatrices H_i such that

$$H = \begin{pmatrix} H_0 \\ H_1 \\ \vdots \\ H_\mu \end{pmatrix} \tag{2.1}$$

where each H_i has n columns. We then have the polynomial matrix

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D + \cdots + \tilde{H}_\mu D^\mu \tag{2.2}$$

where the matrices $\tilde{H}_i$ for all $1 \le i \le \mu$, are derived from the respective matrices H_i by adding zero-rows at the bottom in such a way that the matrix $\tilde{H}_i$ has κ rows in total. Here κ is the maximal number of rows among the matrices H_i, $1 \le i \le \mu$. It is well known that $G(D)$ generates a convolutional code with κ rows, and that μ is the memory of the resulting convolutional code.

Theorem 2.2 *(See* [26, Theorem 3]*) Suppose that $\mathcal{C}$ is a linear code over $\mathbb{F}_{q^2}$ with parameters $[n, k, d]_{q^2}$ and assume also that $H \in \mathbb{F}_{q^2}^{(n-k)\times n}$ is a parity check matrix for $\mathcal{C}$ partitioned into submatrices $H_0, H_1, \cdots, H_\mu$ as in* (2.1) *such that $\kappa = rkH_0$ and $rkH_i \le \kappa$ for $1 \le i \le \mu$ and consider the polynomial matrix $G(D)$ as in* (2.2)*. Then we have:*

(1) *The matrix $G(D)$ is a reduced basic generator matrix.*
(2) *If $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$, then the convolutional code $V = \{\mathbf{u}(D)G(D) \mid \mathbf{u}(D) \in \mathbb{F}_{q^2}[D]^{n-k}\}$ satisfies $V \subseteq V^{\perp_h}$.*
(3) *If d_f and $d_f^{\perp_h}$ denote the free distances of V and $V^{\perp_h}$ respectively, d_i denotes the minimum distance of the code $\mathcal{C}_i = \{\mathbf{v} \in \mathbb{F}_{q^2}^n \mid \mathbf{v}\tilde{H}_i^t = 0\}$ and $d^{\perp_h}$ is the minimum distance of $\mathcal{C}^{\perp_h}$, then one has* $\min\{d_0 + d_\mu, d\} \le d_f^{\perp_h} \le d$ *and $d_f \ge d^{\perp_h}$.*

Theorem 2.2 suggests that one can obtain classical convolutional codes through linear codes over $\mathbb{F}_{q^2}$. Constacyclic codes constitute a remarkable generalization of cyclic codes, hence form an important class of linear codes in the coding theory. In this paper, we apply Theorem 2.2 to constacyclic codes. The necessary notations and results about constacyclic codes are reviewed in the next subsection.

2.2 Constacyclic Codes

Since we will work with codes endowed with the Hermitian inner product, we need to consider codes over $\mathbb{F}_{q^2}$, where $\mathbb{F}_{q^2}$ denotes the finite field with q^2 elements. Let $\mathbb{F}_{q^2}^* = \mathbb{F}_{q^2} \setminus \{0\}$. For $\lambda \in \mathbb{F}_{q^2}^*$, we denote by $r = \mathrm{ord}(\lambda)$ the order of λ in the cyclic group $\mathbb{F}_{q^2}^*$, i.e., r is the smallest positive integer such that $\lambda^r = 1$. Then r is a divisor of $q^2 - 1$, and λ is called a *primitive rth root of unity*.

Starting from this section till the end of this paper, we assume that n is a positive integer relatively prime to q. A λ*-constacyclic code* $\mathcal{C}$ of length n over $\mathbb{F}_{q^2}$ is an ideal of the quotient ring $\mathbb{F}_{q^2}[X]/\langle X^n - \lambda\rangle$, where $\lambda \in \mathbb{F}_{q^2}^*$ (e.g., see [39] or [40]). It is well known that a unique monic polynomial $g(X) \in \mathbb{F}_{q^2}[X]$ can be found such that $g(X) \mid (X^n - \lambda)$ and $\mathcal{C} = \langle g(X)\rangle = \{f(X)g(X) \mid f(X) \in \mathbb{F}_{q^2}[X]\}$. In this case, $g(X)$ is called the *generator polynomial* of $\mathcal{C}$.

Assume that $\lambda \in \mathbb{F}_{q^2}^*$ is a primitive rth root of unity. As mentioned before, r is a divisor of q^2-1. In particular, $\gcd(r, q) = 1$, so $\gcd(rn, q) = 1$. We denote by $\ell = \mathrm{ord}_{rn}(q^2)$, i.e., ℓ is the smallest positive integer such that $rn \mid (q^{2\ell} - 1)$. Then there exists a primitive rnth root of unity $\beta \in \mathbb{F}_{q^{2\ell}}$ such that $\beta^n = \lambda$. The roots of $X^n - \lambda$ are precisely the elements β^{1+ri} for $0 \le i \le n-1$. Set $\theta_{r,n} = \{1 + ri \mid 0 \le i \le n-1\}$. The *defining set* of a constacyclic code $\mathcal{C} = \langle g(X)\rangle$ of length n is the set $Z = \{j \in \theta_{r,n} \mid \beta^j$ is a root of $g(X)\}$. It is easy to see that the defining set Z is a union of some q^2-cyclotomic cosets modulo rn and $\dim_{\mathbb{F}_{q^2}}(\mathcal{C}) = n - |Z|$ (see [41] or [16]). Since $\ell = \mathrm{ord}_{rn}(q^2)$, it follows that the size of each q^2-cyclotomic cosets modulo rn is a divisor of ℓ (e.g. see [42, Theorem 4.1.4]).

The following theorem gives the BCH bound for constacyclic codes (see [41, Theorem 4.1]).

Theorem 2.3 *(**The BCH bound for constacyclic codes**) Let $\mathcal{C}$ be a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$, where $\lambda \in \mathbb{F}_{q^2}$ is a primitive rth root of unity. Suppose $\ell = \mathrm{ord}_{rn}(q^2)$. Let $\beta \in \mathbb{F}_{q^{2\ell}}$ be a primitive rnth root of unity such that $\beta^n = \lambda$. Assume that the generator polynomial of $\mathcal{C}$ has roots that include the set $\{\beta\zeta^i \mid i_1 \le i \le i_1 + d - 2\}$, where $\zeta = \beta^r$. Then the minimum distance of $\mathcal{C}$ is at least d.*

The *Hermitian inner product* on $\mathbb{F}_{q^2}^n$ is defined as

$$(\mathbf{x}, \mathbf{y})_h = x_0y_0^q + x_1y_1^q + \cdots + x_{n-1}y_{n-1}^q,$$

where $\mathbf{x} = (x_0, x_1, \cdots, x_{n-1}) \in \mathbb{F}_{q^2}^n$ and $\mathbf{y} = (y_0, y_1, \cdots, y_{n-1}) \in \mathbb{F}_{q^2}^n$. For a linear code $\mathcal{C}$ of length n over $\mathbb{F}_{q^2}$, the *Hermitian dual code* of $\mathcal{C}$ is defined as

$$\mathcal{C}^{\perp_h} = \left\{\mathbf{x} \in \mathbb{F}_{q^2}^n \,\middle|\, \sum_{i=0}^{n-1} x_i y_i^q = 0, \quad \text{for any} \mathbf{y} \in \mathcal{C}\right\}.$$

If $\mathcal{C} \subseteq \mathcal{C}^{\perp_h}$, then $\mathcal{C}$ is called a (Hermitian) self-orthogonal code. Conversely, if $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$, we say that $\mathcal{C}$ is a (Hermitian) dual-containing code. For a λ-constacyclic code $\mathcal{C}$ of length n over $\mathbb{F}_{q^2}$, it is shown that $\mathcal{C}^{\perp_h}$ is a λ^{-q}-constacyclic code; further, $\lambda = \lambda^{-q}$ precisely when $r \mid (q + 1)$ ([41, Lemma 2.1(ii)]).

The following results are useful.

Lemma 2.4 *(See* [16, Lemma 2.2]*) Let $\lambda \in \mathbb{F}_{q^2}^*$ be a primitive rth root of unity. Assume that $\mathcal{C}$ is a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set Z. Then $\mathcal{C}$ is a dual-containing code if and only if $Z \bigcap Z^{-q} = \emptyset$, where $Z^{-q} = \{-qz \pmod{rn} \mid z \in Z\}$.*

Lemma 2.5 *(See* [36, Theorem 5.4] *or* [37, Theorem 4.2] *) Let $\lambda \in \mathbb{F}_{q^2}$ be a primitive rth root of unity. Suppose $\ell = \mathrm{ord}_{rn}(q^2)$. Take a primitive rnth root of unity $\beta \in \mathbb{F}_{q^{2\ell}}$ such that $\beta^n = \lambda$. Assume that $\mathcal{C}$ is a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set*

$Z = \bigcup_{i=b}^{\delta-2} C_{1+ri}$, *where b is a nonnegative integer and* C_{1+ri}, $b \le i \le \delta - 2$, *are distinct* q^2*-cyclotomic cosets modulo rn. Then a parity check matrix of* $\mathcal{C}$ *can be obtained from the matrix*

$$H_{\mathcal{C}} = \begin{pmatrix} 1 & \beta^{1+rb} & \beta^{2(1+rb)} & \dots & \beta^{(n-1)(1+rb)} \\ 1 & \beta^{1+r(b+1)} & \beta^{2(1+r(b+1))} & \dots & \beta^{(n-1)(1+r(b+1))} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta^{1+r(\delta-3)} & \beta^{2(1+r(\delta-3))} & \dots & \beta^{(n-1)(1+r(\delta-3))} \\ 1 & \beta^{1+r(\delta-2)} & \beta^{2(1+r(\delta-2))} & \dots & \beta^{(n-1)(1+r(\delta-2))} \end{pmatrix}$$

by expanding each entry as a column vector (containing ℓ *rows) with respect to certain* $\mathbb{F}_{q^2}$*-basis of* $\mathbb{F}_{q^{2\ell}}$ *and then removing any linearly dependent rows.*

2.3 Quantum Convolutional Codes

A quantum convolutional code is defined through its stabilizer, which is a subgroup of the infinite version of the Pauli group, consisting of tensor products of generalized Pauli matrices acting on a semi-infinite stream of qudits. The stabilizer can be defined by a stabilizer matrix of the form

$$S(D) = (X(D) \mid Z(D)) \in \mathbb{F}_q[D]^{(n-k)\times 2n}$$

satisfying $X(D)Z(1/D)^t - Z(D)X(1/D)^t = 0$. Let C be a quantum convolutional code defined by a full-rank stabilizer matrix $S(D)$ given above. Then C has parameters $[(n, k, \mu; \gamma, d_f)]_q$, where n is the frame size, k is the number of logical qudits per frame, $\mu = \max_{1\le i\le n-k, 1\le j\le n} \{\max\{\deg X_{ij}(D), \deg Z_{ij}(D)\}\}$, is the memory, d_f is the free distance and γ is the degree of the code.

The next result enables us to construct convolutional stabilizer codes from classical convolutional codes.

Lemma 2.6 *Let V be an* $(n, (n-k)/2, \gamma; \mu)_{q^2}$ *convolutional code satisfying* $V \subseteq V^{\perp_h}$. *Then there exists an* $[(n, k, \mu; \gamma, d_f)]_q$ *convolutional stabilizer code, where* $d_f = wt(V^{\perp_h} \setminus V)$.

Lemma 2.7 *(See* [25] *or* [36]*) (**Quantum Singleton bound**) The free distance of an* $[(n, k, \mu; \gamma, d_f)]_q$, $\mathbb{F}_{q^2}$*-linear pure convolutional stabilizer code is bounded by*

$$d_f \le \frac{n-k}{2}\left(\left\lfloor \frac{2\gamma}{n+k} \right\rfloor + 1\right) + \gamma + 1.$$

A quantum convolutional code achieving this quantum Singleton bound is called an *maximum-distance-separable (MDS) quantum convolutional code*.

3 Code Constructions

Thereafter, we always assume that q is an odd prime power. In this section, firstly, we use constacyclic codes of lengths $n = q^2 + 1$ and $n = \frac{q^2+1}{10}$ (assume further that $10 \mid (q^2 + 1)$) respectively to construct classical convolutional codes. Consequently, two classes of MDS quantum convolutional codes are derived from these parameters.

3.1 MDS Quantum Convolutional Codes of Length $q^2 + 1$

The main result of this subsection is Theorem 3.4, which generates a family of MDS quantum convolutional codes. The following results are useful to the proof of Theorem 3.4.

Lemma 3.1 *Let $n = q^2 + 1$, $r = q + 1$ and $s = 1 + r\frac{q-1}{2} = \frac{q^2+1}{2} = \frac{n}{2}$, where q is an odd prime power. Then $\theta_{r,n} = \{1 + ri \mid 0 \le i \le n - 1\}$ is a disjoint union of q^2-cyclotomic cosets modulo rn:*

$$\theta_{r,n} = C_s \bigcup C_{1+r(\frac{q-1}{2}+\frac{q^2+1}{2})} \bigcup_{i=1}^{s-1} C_{s-ri}$$

where $C_s = \{s\}$, $C_{1+r(\frac{q-1}{2}+\frac{q^2+1}{2})} = \{1 + r(\frac{q-1}{2} + \frac{q^2+1}{2})\}$ and $C_{s-ri} = \{s - ri, s + ri\}$ for $1 \le i \le s - 1$.

Proof Note that $rn = (q + 1)(q^2 + 1)$, $rn \nmid (q^2 - 1)$ and $rn \mid (q^4 - 1)$, so $\mathrm{ord}_{rn}(q^2) = 2$. We then know that every q^2-cyclotomic coset modulo rn has one or two elements. A straightforward calculation shows that $q^2(1 + ri) \equiv 1 + r(q - 1 - i) \pmod{rn}$ for any integer i. In particular, $q^2(1 + r\frac{q-1}{2}) \equiv 1 + r\frac{q-1}{2} \pmod{rn}$ and $q^2(1 + r(\frac{q-1}{2} + \frac{q^2+1}{2})) \equiv 1 + r(\frac{q-1}{2} - \frac{q^2+1}{2}) \equiv 1 + r(\frac{q-1}{2} + \frac{q^2+1}{2}) \pmod{rn}$, which gives

$$C_{1+r\frac{q-1}{2}} = \left\{1 + r\frac{q-1}{2}\right\} \quad \text{and} \quad C_{1+r(\frac{q^2+1}{2}+\frac{q-1}{2})} = \left\{1 + r(\frac{q-1}{2} + \frac{q^2+1}{2})\right\}.$$

Clearly, $q^2(s - ri) \equiv s + ri \pmod{rn}$ for any integer i. For $1 \le i \le s - 1$, $s - ri \not\equiv s + ri \pmod{rn}$. Thus $C_{s-ri} = \{s - ri, s + ri\}$ for $1 \le i \le s - 1$. It is easy to see that $C_s \ne C_{1+r(\frac{q-1}{2}+\frac{q^2+1}{2})}$. We want to prove that the q^2-cyclotomic cosets C_{s-ri}, $1 \le i \le s - 1$, are distinct. Suppose otherwise that two integers i, j with $1 \le i \ne j \le s - 1$ can be found such that $\{s - ri, s + ri\} = C_{s-ri} = C_{s-rj} = \{s - rj, s + rj\}$. It is obvious that $s - ri \not\equiv s - rj \pmod{rn}$, which forces $s - ri \equiv s + rj \pmod{rn}$. This leads to $n \mid (i + j)$, which is a contradiction. Finally, it is easy to see that the size of the union of these q^2-cyclotomic cosets is equal to n. This completes the proof. □

Lemma 3.2 *Let q be an odd prime power and $\lambda \in \mathbb{F}_{q^2}$ be a primitive $(q+1)$th root of unity. Let $s = \frac{q^2+1}{2}$. If $\mathcal{C}$ is a λ-constacyclic code of length $q^2 + 1$ over $\mathbb{F}_{q^2}$ with defining set*

$$Z = \bigcup_{j=0}^{\delta} C_{s-rj} = \{s - r\delta, s - r(\delta - 1), \cdots, s - r, s, s + r, \cdots, s + r\delta\}, 0 \le \delta \le \frac{q-1}{2}, \tag{3.1}$$

then $\mathcal{C}$ is a $[q^2 + 1, q^2 - 2\delta, 2\delta + 2]$ MDS code satisfying $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.

Proof By Lemma 3.1, one gets $|Z| = 2(\delta+1)-1 = 2\delta+1$. We then see that $d(C) = 2\delta+2$ by the BCH bound for constacyclic codes (see Lemma 2.3) and the Singleton bound for linear codes. It follows that $\mathcal{C}$ is a $[q^2 + 1, q^2 - 2\delta, 2\delta + 2]$ MDS code. We need to show that $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.

By Lemma 2.4, it is enough to prove that $Z \bigcap Z^{-q} = \emptyset$. Suppose otherwise that $Z \bigcap Z^{-q} \ne \emptyset$, i.e. two integers i, j with $0 \le i, j \le \delta$ can be found such that $-qC_{s-ri} =$

C_{s-rj}. Thus, $-qC_{s-ri} = \{-q(s-ri), -q(s+ri)\} = C_{s-rj} = \{s-rj, s+rj\}$. Two cases may occur at this point:

(i) $-q(s-ri) \equiv s-rj \pmod{rn}$. After expanding and reducing this equation, we obtain $qi+j \equiv s \pmod{n}$. Since $0 \le qi+j \le q\delta+\delta \le q\frac{q-1}{2} + \frac{q-1}{2} = \frac{q^2-1}{2} < n$ and $0 < s < n$, it follows that $qi+j = s$. However, $qi+j \le \frac{q^2-1}{2} < s = \frac{q^2+1}{2}$. This is a contradiction.

(ii) $-q(s-ri) \equiv s+rj \pmod{rn}$. Similarly, we obtain $qi \equiv s+j \pmod{n}$. Clearly, $0 \le qi \le \frac{q^2-q}{2} < n$ and $0 < s+j < n$. Thus $qi = s+j$. However, $qi \le \frac{q^2-q}{2} < s+j$. This is a contradiction. □

Using Lemma 2.5, Theorem 2.2 and Lemma 3.1, we obtain the following classical convolutional codes.

Lemma 3.3 *Let $n = q^2+1$, where $q \ge 5$ is an odd prime power. Let i be an integer with $2 \le i \le \frac{q-1}{2}$. Then there exists a classical convolutional code V with parameters $(n, 2i-1, 2; 1, \ge n-2i)_{q^2}$; the free distance of $V^{\perp_h}$ is exactly equal to $2i+2$. Furthermore, V satisfies $V \subseteq V^{\perp_h}$.*

Proof Let $r = q+1$ and $\lambda \in \mathbb{F}_{q^2}$ be a primitive rth root of unity. Assume that β is a primitive rnth root of unity in some extension field of $\mathbb{F}_{q^2}$ such that $\beta^n = \lambda$. Since $\mathrm{ord}_{rn}(q^2) = 2$, it follows that $\beta \in \mathbb{F}_{q^4}$. Let $s = \frac{q^2+1}{2}$ and i be an integer with $2 \le i \le (q-1)/2$. Let $\mathcal{C}$ be a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $Z = \bigcup_{b=0}^{i} C_{s-rb}$. It follows from Lemma 2.5 that a parity check matrix of $\mathcal{C}$, denoted by $N_{\mathcal{C}}$, can be obtained from the following matrix

$$H_{\mathcal{C}} = \begin{pmatrix} 1 & \beta^{s} & \beta^{2s} & \cdots & \beta^{(n-1)s} \\ 1 & \beta^{s-r} & \beta^{2(s-r)} & \cdots & \beta^{(n-1)(s-r)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta^{s-r(i-1)} & \beta^{2(s-r(i-1)))} & \cdots & \beta^{(n-1)(s-r(i-1))} \\ 1 & \beta^{s-ri} & \beta^{2(s-ri)} & \cdots & \beta^{(n-1)(s-ri)} \end{pmatrix}$$

by expanding each entry as a column vector (containing 2 rows) with respect to certain $\mathbb{F}_{q^2}$-basis of $\mathbb{F}_{q^4}$ and then removing any linearly dependent rows. Therefore, $N_{\mathcal{C}}$ has rank $2i+1$, implying that $\mathcal{C}$ is an MDS code with parameters $[n, n-2i-1, 2i+2]$. Consequently, $\mathcal{C}^{\perp_h}$ is also an MDS code with parameters $[n, 2i+1, n-2i]$.

Now let $\mathcal{C}_0$ be a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $Z_0 = \bigcup_{b=0}^{i-1} C_{s-rb}$. Similar reasoning shows that $\mathcal{C}_0$ is an MDS code with parameters $[n, n-2i+1, 2i]$, and that $\mathcal{C}_0^{\perp_h}$ is an MDS code with parameters $[n, 2i-1, n-2i+2]$. Further, a parity check matrix of $\mathcal{C}_0$, denoted by $N_{\mathcal{C}_0}$, can be obtained from the following matrix

$$H_{\mathcal{C}_0} = \begin{pmatrix} 1 & \beta^{s} & \beta^{2s} & \cdots & \beta^{(n-1)s} \\ 1 & \beta^{s-r} & \beta^{2(s-r)} & \cdots & \beta^{(n-1)(s-r)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta^{s-r(i-1)} & \beta^{2(s-r(i-1)))} & \cdots & \beta^{(n-1)(s-r(i-1))} \end{pmatrix}$$

by expanding each entry as a column vector (containing 2 rows) with respect to the $\mathbb{F}_{q^2}$-basis of $\mathbb{F}_{q^4}$ and then removing any linearly dependent rows (This has been done, since $H_{\mathcal{C}_0}$ is a submatrix of $H_{\mathcal{C}}$). In particular, $N_{\mathcal{C}_0}$ has rank $2i-1$.

Next let $\mathcal{C}_1$ be a λ-constacyclic code of length n over $\mathbb{F}_{q^2}$ with defining set $Z_0 = C_{s-ri}$. Thus $\mathcal{C}_1$ has parameters $[n, n-2, \geq 2]$. A parity check matrix, denoted by $N_{\mathcal{C}_1}$, is given by expanding the entries of the matrix

$$H_{\mathcal{C}_1} = \left[1, \beta^{s-ri}, \beta^{2(s-ri)}, \cdots, \beta^{(n-1)(s-ri)}\right]$$

with respect to β (This has been done, since $H_{\mathcal{C}_1}$ is a submatrix of $H_{\mathcal{C}}$). According to Theorem 2.2 (1), a convolutional code V is obtained which is generated by the reduced basic generator matrix

$$G(D) = \tilde{N}_{\mathcal{C}_0} + \tilde{N}_{\mathcal{C}_1} D$$

where $\tilde{N}_{\mathcal{C}_0} = N_{\mathcal{C}_0}$ and $\tilde{N}_{\mathcal{C}_1}$ is derived from $N_{\mathcal{C}_1}$ by adding zero-rows at the bottom such that the rows of $\tilde{N}_{\mathcal{C}_1}$ is exactly equal to the number of rows of $N_{\mathcal{C}_0}$. It follows from Theorem 2.2 that V is a convolutional code of dimension $2i-1$, degree 2, memory 1 and free distance $\geq n-2i$. For the free distance of $V^{\perp_h}$, we have that $\min\{\geq 2i+2, 2i+2\} \leq d_f^{\perp_h} \leq 2i+2$ which forces $d_f^{\perp_h} = 2i+2$.

Finally, it follows from Lemma 3.2 that $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$, which gives $V \subseteq V^{\perp_h}$ by Theorem 2.2 (2). This completes the proof. □

We are now in a position to show the main result of this subsection.

Theorem 3.4 *Let $n = q^2+1$, where $q \geq 5$ is an odd prime power. Let i be an integer with $2 \leq i \leq (q-1)/2$. Then there exist MDS quantum convolutional codes with parameters $[(n, n-4i+2, 1; 2, 2i+2)]_q$.*

Proof By Lemma 3.3, we have constructed a convolutional code V with parameters $(n, 2i-1, 2; 1, \geq n-2i)_{q^2}$; furthermore, V satisfies $V \subseteq V^{\perp_h}$. Now $n = q^2+1$, $\gamma = 2$ and $\mu = 1$. Let k be an integer satisfying $\frac{n-k}{2} = 2i-1$. Thus $k = n-4i+2$. Note that $\mathrm{wt}(V^{\perp_h}) = 2i+2$ and $\mathrm{wt}(V) \geq n-2i$. It is clear that $n-2i > 2i+2$, which shows $d_f = \mathrm{wt}(V^{\perp_h} \setminus V) = 2i+2$. Using Lemma 2.6, there exists an $[(n, n-4i+2, 1; 2, 2i+2)]_q$ convolutional stabilizer code. Finally, we show that the resulting convolutional stabilizer code attains the Quantum Singleton bound (see Lemma 2.7):

$$\frac{n-k}{2}\left(\left\lfloor \frac{2\gamma}{n+k} \right\rfloor + 1\right) + \gamma + 1 = (2i-1)\cdot(0+1) + 2 + 1 = 2i+2 = d_f. \quad \square$$

Example 3.5 *In Table 1, we list some MDS quantum convolutional codes obtained from Theorem 3.4 for $q = 7, 11, 13, 19$ and 23.*

3.2 MDS Quantum Convolutional Codes of Length $\frac{q^2+1}{10}$

Let q be an odd prime power such that $10 \mid (q^2+1)$, i.e., q has the form $10m+3$ or $10m+7$, where m is a positive integer. Let $n = \frac{q^2+1}{10}$, $s = \frac{q^2+1}{2}$ and $r = q+1$. It is clear that $s \equiv 1 \pmod r$, which implies that $s \pmod{rn} \in \theta_{r,n} = \{1+ri \mid 0 \leq i \leq n-1\}$. As in the previous subsection, we need the following lemmas.

Table 1 MDS Quantum Convolutional Codes

q	$[(q^2+1, q^2-4i+3, 1; 2, 2i+2)]_q$	$2 \le i \le \frac{q-1}{2}$
7	$[(50, 52-4i, 1; 2, 2i+2)]_7$	$2 \le i \le 3$
11	$[(122, 124-4i, 1; 2, 2i+2)]_{11}$	$2 \le i \le 5$
13	$[(170, 172-4i, 1; 2, 2i+2)]_{13}$	$2 \le i \le 6$
19	$[(362, 364-4i, 1; 2, 2i+2)]_{19}$	$2 \le i \le 9$
23	$[(530, 532-4i, 1; 2, 2i+2)]_{23}$	$2 \le i \le 11$

Lemma 3.6 *Assume that q is an odd prime power with $10 \mid (q^2+1)$. Let $n = \frac{q^2+1}{10}$, $s = \frac{q^2+1}{2}$ and $r = q+1$. Then $\theta_{r,n} = \{1 + ri \,|\, 0 \le i \le n-1\}$ is a disjoint union of q^2-cyclotomic cosets modulo rn:*

$$\theta_{r,n} = C_s \bigcup \left(\bigcup_{k=0}^{\frac{n-1}{2}-1} C_{s-(q+1)(\frac{n-1}{2}-k)} \right).$$

Proof Observe that $q^4 \equiv 1 \pmod{rn}$, which implies that each q^2-cyclotomic coset modulo rn contains one or two elements. Now,

$q^2(1+(q+1)j) = q^2+q^2(q+1)j = q^2+(q^2+1-1)(q+1)j \equiv q^2-(q+1)j \pmod{rn}$.

It is clear that for $0 \le j \le n-1$, $1+(q+1)j \equiv q^2-(q+1)j \pmod{rn}$ if and only if $j = \frac{q-1}{2} + dn$ (d is an integer), which forces $d = 0$ and hence $j = \frac{q-1}{2}$. This shows that $s = 1 + (q+1)\frac{q-1}{2} = \frac{q^2+1}{2}$ is the unique element of $\theta_{r,n}$ with $q^2 s \equiv s \pmod{rn}$. To complete the proof, it suffices to show that for any $0 \le i \ne j \le \frac{n-1}{2}-1$, $C_{s-(q+1)(\frac{n-1}{2}-i)} = \{s-(q+1)(\frac{n-1}{2}-i), s+(q+1)(\frac{n-1}{2}-i)\}$ and $C_{s-(q+1)(\frac{n-1}{2}-j)}$ are distinct. Suppose otherwise that $C_{s-(q+1)(\frac{n-1}{2}-i)} = C_{s-(q+1)(\frac{n-1}{2}-j)}$ for some $0 \le i \ne j \le \frac{n-1}{2}-1$. If $s-(q+1)(\frac{n-1}{2}-i) \equiv s-(q+1)(\frac{n-1}{2}-j) \pmod{rn}$, then $i \equiv j \pmod n$ which is impossible; If $s-(q+1)(\frac{n-1}{2}-i) \equiv s+(q+1)(\frac{n-1}{2}-j) \pmod{rn}$, then $i+j \equiv -1 \pmod n$ which is a contradiction. □

Let $\lambda \in \mathbb{F}_{q^2}$ be a primitive rth root of unity, and let $\beta \in \mathbb{F}_{q^4}$ be a primitive rnth root of unity such that $\beta^n = \lambda$. Let $\mathcal{C}$ be a λ-constacyclic code of length $n = \frac{q^2+1}{10}$ over $\mathbb{F}_{q^2}$ with defining set

$$Z = \bigcup_{j=0}^{2m-1} C_{s-(q+1)(\frac{n-1}{2}-j)}. \tag{3.2}$$

We then know from Lemma 3.6 that Z is a disjoint union of q^2-cyclotomic cosets modulo rn with $|Z| = 4m$. Moreover, we assert that the minimum distance of $\mathcal{C}$ is exactly equal to $4m+1$. To see this, observe that

$$Z = \left\{ s + r\left(\frac{n-1}{2} - 2m + 1\right), s + r\left(\frac{n-1}{2} - 2m + 2\right), \cdots, s + r\left(\frac{n-1}{2} - 1\right), \right.$$
$$\left. s + r\frac{n-1}{2}, s - r\frac{n-1}{2}, s - r(\frac{n-1}{2} - 1), \cdots, s - r(\frac{n-1}{2} - 2m + 1) \right\}.$$

A simple calculation shows that $s + r\frac{n-1}{2} + r \equiv s - r\frac{n-1}{2} \pmod{rn}$. By the BCH bound for constacyclic codes, $\mathcal{C}$ is an MDS code with parameters $[n, n-4m, 4m+1]$.

The next result shows that $\mathcal{C}$ is a dual-containing code.

Lemma 3.7 *Assume that q is an odd prime power with the form $10m+3$ or $10m+7$, where m is a positive integer. Let $n = \frac{q^2+1}{10}$, $s = \frac{q^2+1}{2}$ and $r = q+1$. Let $\mathcal{C}$ be a λ-constacyclic code of length $n = \frac{q^2+1}{10}$ over $\mathbb{F}_{q^2}$ with defining set as in (3.2), where $\lambda \in \mathbb{F}_{q^2}$ is a primitive rth root of unity. Then $\mathcal{C}$ is a dual-containing code.*

Proof We have to prove that $Z \bigcap Z^{-q} = \emptyset$. We just give a proof for the case $q = 10m+3$. The case for $q = 10m+7$ is proved similarly. Suppose there exist integers j, k with $0 \leq j, k \leq 2m-1$ such that $C_{-q(s-(q+1)(\frac{n-1}{2}-j))} = C_{s-(q+1)(\frac{n-1}{2}-k)}$. Write $j = j_1 m + j_0$ and $k = k_1 m + k_0$, where $j_1, k_1 \in \{0, 1\}$ and $0 \leq j_0, k_0 < m$. Let $j_0' = m - j_0$ and $k_0' = m - k_0$, and so $0 < j_0', k_0' \leq m$.

Case I. $-q(s-(q+1)(\frac{n-1}{2}-j)) \equiv s-(q+1)(\frac{n-1}{2}-k) \pmod{rn}$. After routine computations, we obtain

$$-\frac{q+1}{2} \equiv qj + k \pmod{n}. \tag{3.3}$$

Now $qj+k = (10m+3)(j_1 m + j_0) + k_1 m + k_0 = 10 j_1 m^2 + (10 j_0 + 3 j_1 + k_1)m + 3 j_0 + k_0 = 10 j_1 m^2 + (10m - 10 j_0' + 3 j_1 + k_1)m + 3m - 3 j_0' + m - k_0'$.

Assume $qj + k < n$.

If $j_1 = 0$, it follows from (3.3) that

$$(10m - 10 j_0' + k_1)m + 3m - 3 j_0' + m - k_0' = n - \frac{q+1}{2} = 10m^2 + m - 1.$$

This leads to

$$(k_1 - 10 j_0' + 4)m = m + 3 j_0' + k_0' - 1,$$

which is a contradiction, since $(k_1 - 10 j_0' + 4)m < 0$ and $m + 3 j_0' + k_0' - 1 > 0$.

If $j_1 = 1$, then

$$10m^2 + (10m - 10 j_0' + 3 + k_1)m + 3m - 3 j_0' + m - k_0' = 10m^2 + m - 1,$$

or equivalently, $10 j_0' m + k_0' = 1 + (k_1 + 10m + 6)m - 3 j_0'$. Now, $10 j_0' m + k_0' \leq 10m^2 + m$, but $1 + (k_1 + 10m + 6)m - 3 j_0' > 10m^2 + 3m$, which is a contradiction.

Assume $qj + k > n$.

If $j_1 = 0$, then $qj + k = (10m+3) j_0 + k < (10m+3)m + 2m = 10m^2 + 5m < n$; this is impossible.

If $j_1 = 1$, we claim that $qj + k - n = (k_1 + 10 j_0 - 3)m + k_0 + 3 j_0 - 1 < n$; this is because $(k_1 + 10 j_0 - 3)m + k_0 + 3 j_0 - 1 \leq (1 + 10m - 10 - 3)m + m - 1 + 3(m-1) - 1 < n$. From (3.3) again, we have $(k_1 + 10 j_0 - 3)m + k_0 + 3 j_0 - 1 = n - \frac{q+1}{2} = 10m^2 + m - 1$, or equivalently, $(k_1 - 10 j_0')m = k_0' + 3 j_0'$. This is a contradiction, because $k_0' > 0$, $j_0' > 0$ and $k_1 - 10 j_0' < 0$.

Case II. $-q(s-(q+1)(\frac{n-1}{2}-j)) \equiv s+(q+1)(\frac{n-1}{2}-k) \pmod{rn}$. After routine computations, we get

$$-\frac{q-1}{2} \equiv qj - k \pmod{n}. \tag{3.4}$$

As we did previously, $qj - k = (10m+3)(j_1 m + j_0) - k_1 m - k_0 = 10 j_1 m^2 + (10 j_0 + 3 j_1 - k_1)m + 3 j_0 - k_0$.

If $j_1 = 0$, then $qj - k \leq (10m + 3)(m - 1) < n$. When $0 < qj - k < n$, by (3.4), $10j_0m+3j_0-k_1m-k_0 = 10m^2+m$, which is equivalent to $10j_0'm-m+3j_0'+k_1m-k_0' = 0$. This is impossible, since $10j_0'm - m + 3j_0' + k_1m - k_0' > 10m - m - m > 0$. When $qj - k < 0$ (Clearly, $0 < k - qj < n$), we obtain $5m + 1 = \frac{q-1}{2} = k - qj$, which is a contradiction since $k < 2m$.

If $j_1 = 1$ and $j_0 = 0$, we have $qj - k = (10m + 3)m - k < n$. Using (3.4), we get $k = 2m$, also a contradiction.

If $j_1 = 1$ and $j_0 > 0$, we then know that $qj-k = 10m^2+(10j_0+3-k_1)m+3j_0-k_0 > n$. On the other hand, $qj - k - n = 10m^2 - 10j_0'm - 3j_0' - k_1m - m + k_0' - 1 < n$. Applying (3.4) again, we obtain $-10j_0'm - 3j_0' - k_1m - 2m + k_0' - 1 = 0$. This is impossible, because $-10j_0'm - 3j_0' - k_1m - 2m + k_0' - 1 < 0$. □

The proof of next lemma is quite similar to that of Lemma 3.3, so we omit its proof.

Lemma 3.8 *Assume that q is an odd prime power with the form $10m + 3$ or $10m + 7$, where $m \geq 2$ is a positive integer. Let i be an integer with $2 \leq i \leq 2m - 1$ (This requires $m \geq 2$). Then there exists a classical convolutional code V with parameters $(n, 2i, 2; 1, \geq n - 2i - 1)_{q^2}$; the free distance of $V^{\perp_h}$ is exactly equal to $2i + 3$. Furthermore, V satisfies $V \subseteq V^{\perp_h}$.*

Combining Lemma 2.6 with Lemma 3.8, we obtain the following result.

Theorem 3.9 *Assume that q is an odd prime power with the form $10m + 3$ or $10m + 7$, where $m \geq 2$ is a positive integer. Let $n = \frac{q^2+1}{10}$ and i be an integer with $2 \leq i \leq 2m - 1$ (This requires $m \geq 2$). Then there exist MDS quantum convolutional codes with parameters $[(n, n - 4i, 1; 2, 2i + 3)]_q$.*

Proof By Lemma 3.8, we have constructed a convolutional code V with parameters $(n, 2i, 2; 1, \geq n - 2i + 1)_{q^2}$; furthermore, V satisfies $V \subseteq V^{\perp_h}$. Now $n = \frac{q^2+1}{10}$, $\gamma = 2$ and $\mu = 1$. Let k be an integer satisfying $\frac{n-k}{2} = 2i$. Thus $k = n - 4i$. Note that $\mathrm{wt}(V^{\perp_h}) = 2i + 3$ and $\mathrm{wt}(V) \geq n - 2i - 1$. Since $n - 2i - 1 > 2i + 3$, which gives $d_f = \mathrm{wt}(V^{\perp_h} \setminus V) = 2i + 3$. Using Lemma 2.6, there exists an $[(n, n - 4i, 1; 2, 2i + 3)]_q$ convolutional stabilizer code. Finally, we show that the resulting convolutional stabilizer code attains the Quantum Singleton bound (see Lemma 2.7):

$$\frac{n-k}{2}\left(\left\lfloor \frac{2\gamma}{n+k} \right\rfloor + 1\right) + \gamma + 1 = 2i \cdot (0 + 1) + 2 + 1 = 2i + 3 = d_f. \qquad \square$$

Example 3.10 *In Table 2, we list some MDS quantum convolutional codes obtained from Theorem 3.9.*

4 Codes Comparisons

In this section we compare the parameters of the new convolutional codes with the ones exhibited in [36]. Recall that the generalized Singleton bound of an $(n, k, \gamma; \mu, d_f)_{q^2}$ convolutional code is given by $d_f \leq (n - k)[\lfloor \gamma/k \rfloor + 1] + \gamma + 1$. Thus the discussion on the goodness of the code focuses on the code length, the dimension, free distance and the

Table 2 MDS Quantum Convolutional Codes

m	q	$[((q^2+1)/10, (q^2+1)/10-4i, 1; 2, 2i+3)]_q$	$2 \leq i \leq 2m-1$
2	23	$[(53, 53-4i, 1; 2, 2i+3)]_{23}$	$2 \leq i \leq 3$
2	27	$[(73, 73-4i, 1; 2, 2i+3)]_{27}$	$2 \leq i \leq 3$
3	37	$[(137, 137-4i, 1; 2, 2i+3)]_{37}$	$2 \leq i \leq 5$

degree of the code. Note that the new codes and those shown in [36] have the same degree and memory, respectively. Therefore, in order to perform the comparisons, we only consider the code length, the dimension and the free distance as measure as goodness.

• The first new families of MDS quantum convolutional codes can be regarded as a generalization of [36, Theorem 6.5], in the sense that we drop the numerical constraint $q \equiv 1 \pmod 4$.

• Through comparing the second new families of MDS quantum convolutional codes with that presented in [36], we can see that the code lengths are distinct. More precisely, assume that there exists two odd prime power q, q_0 with the forms $q = 10m+3$ or $10m+7$ ($m \geq 2$), and $q_0 = 4\ell + 1$ such that $\frac{q^2+1}{10} = q_0^2 + 1$. Then $\frac{(10m+3)^2+1}{10} = (4\ell+1)^2 + 1$ or $\frac{(10m+7)^2+1}{10} = (4\ell+1)^2 + 1$, i.e., $10m^2 + 6m + 1 = 16\ell^2 + 8\ell + 2$ or $10m^2 + 14m + 5 = 16\ell^2 + 8\ell + 2$. This is a contradiction. Similar reasoning shows that the second new families of MDS quantum convolutional codes is different from the codes shown in [36].

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