Atom in a Box, Dauger Research,http://daugerresearch.com/orbitals/index.shtml

Shor’s Quantum Error Correcting Code

by Joel Frederico

Need for Error Correction

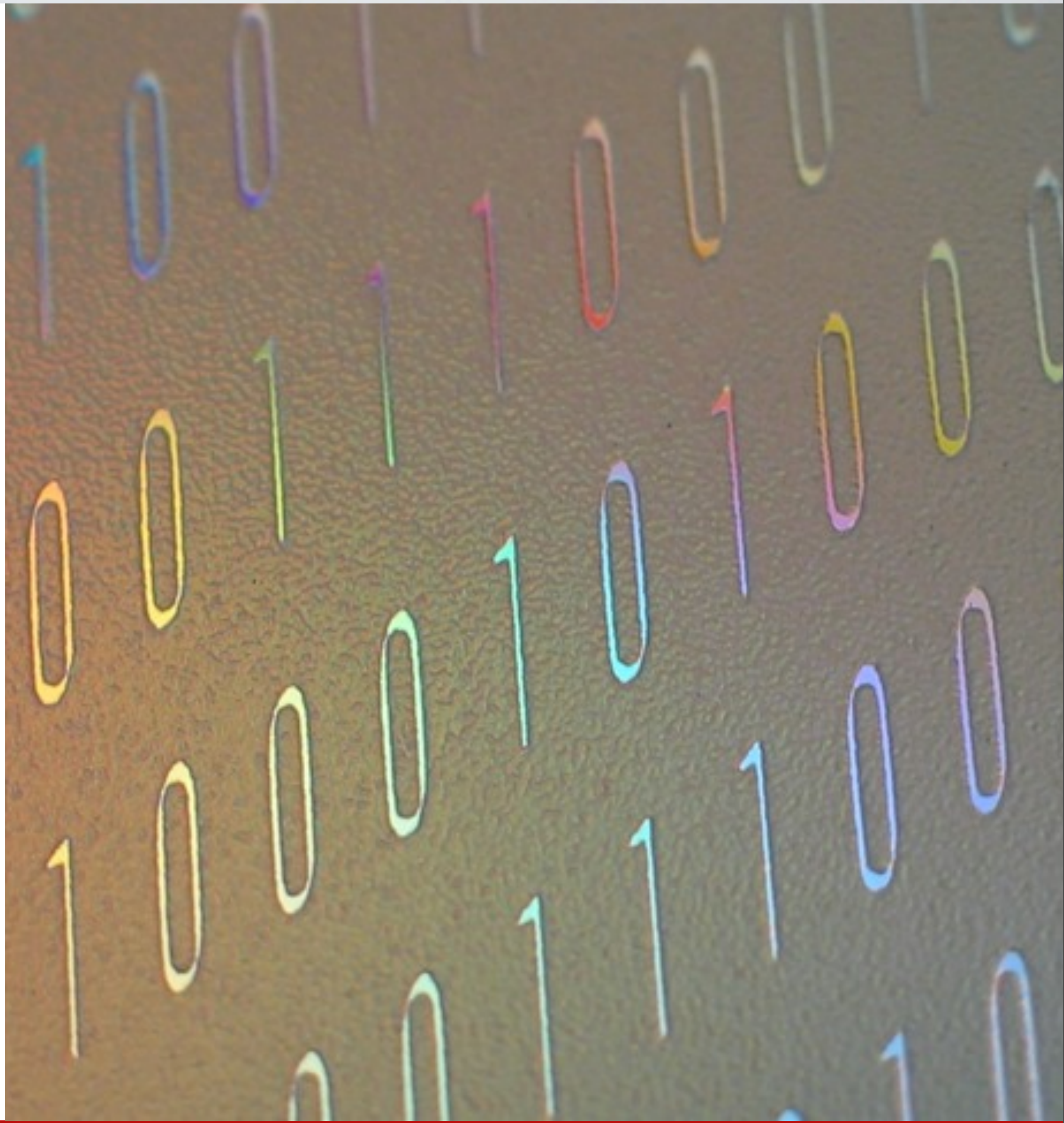
- There is always noise
- Classical systems
  - Mechanical errors
  - Electrical noise
  - Cosmic rays and alpha decay!
- Quantum systems
  - Hamiltonians need to be precise
  - Environment terms perturb system
  - Decoherence

Joel Frederico

U.S. DEPARTMENT OF ENERGY
SLAC NATIONAL ACCELERATOR LABORATORY
STANFORD UNIVERSITY

Storing Information: Bits

- Classical bits (cbits)
  - 0’s and 1’s states
  - Magnetic domains
  - Measurement determines which state
- Quantum bits (qubits)
  - Assign different states |0> and |1>
  - Photons used frequently
  - Superposition stores information!
  - Measurement destroys states (performed only at the end)
  - Decoherence destroys states (...or does it?)

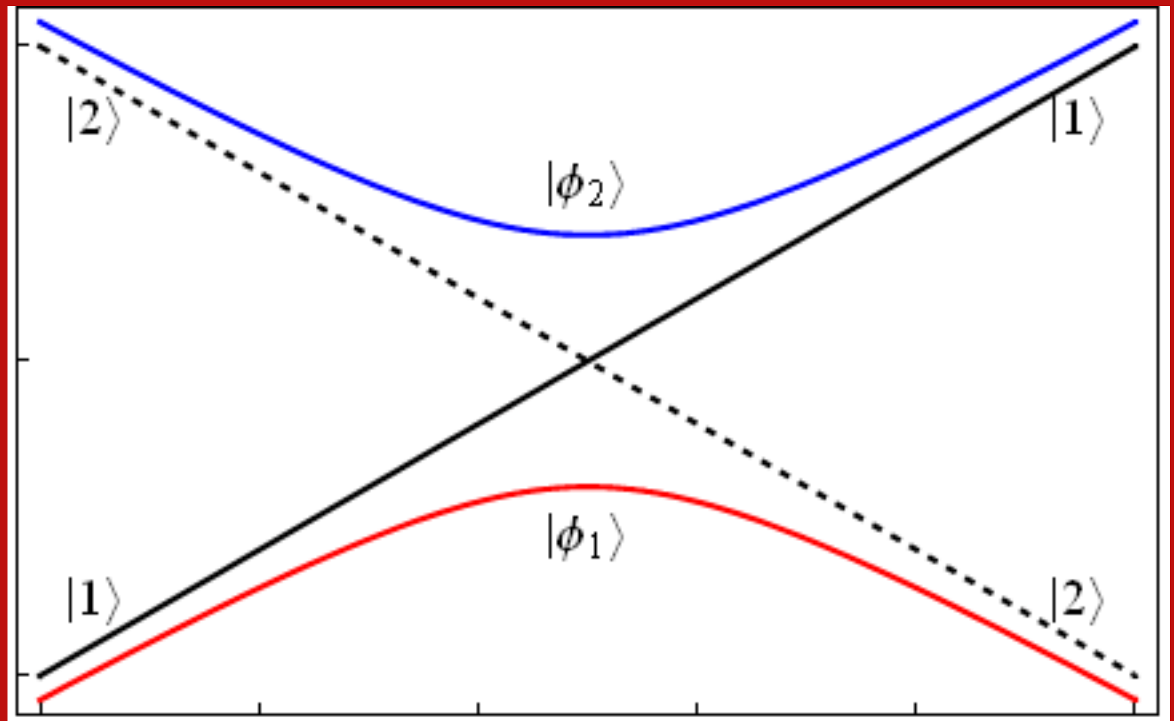


Decoherence: An Example

- Imagine spin up/down system in a superposition: $\alpha|0\rangle + \beta|1\rangle$
- Each state evolves according to

$$\exp(-iE_{0/1}/\hbar\omega t)$$

- Perturbing terms in Hamiltonian adiabatically transfer solution
  - Energy shifts
  - States get out of phase
- Perturbing terms change instantly
  - Different eigensystem evolution
  - Superposition coefficients can change
- Information is lost



Classical Solution: Codes

- Definitions:
  - Code: A system of error correction
  - Codeword: A configuration of substates that represents a macrostate
- Example: Repetition Code
  - Codewords: 0 → 000, 1 → 111
  - Assume low error rate: (probability of two errors is negligible)
  - One error: 000 becomes 010
  - Obviously corrects to the nearest codeword: 010 → 000 → 0
  - Use most frequent state, error corrected

Quantum Problems

- Limitations
  - No-cloning theorem: impossible to use simple repetition code
  - Must account for dephasing and coefficient errors
  - Cannot measure the state
- Strategy
  - Instead, spread out information
  - Measure decoherence (not state), removing environmental entanglement
  - Restore original state

Shor’s Solution: Measure Decoherence

- What if:
  - Only one qubit decoheres,
  - One of the two types of errors is corrected with a repetition code,
  - And two layers correct all errors?
- Turns out this works
  - Decoherence = unknown error occurred
  - Project into subspaces that indicate error
  - Projection collapses indeterminate error to determinate error
  - Correct now-known error

One layer

- Decoherence of the first qubit of an arbitrary state:

$$|\psi_1\rangle = \alpha|\underline{0}\rangle + \beta|\underline{1}\rangle = \alpha|000\rangle + \beta|111\rangle$$

$$\begin{aligned}|e_0\rangle|\psi_1\rangle &\overset{U(t)}{\mapsto} \alpha(|a_0\rangle|0\rangle + |a_1\rangle|1\rangle)|00\rangle + \beta(|a_2\rangle|0\rangle + |a_3\rangle|1\rangle)|11\rangle \\ &= \alpha|a_0\rangle|000\rangle + \beta|a_3\rangle|111\rangle \\ &\quad + \alpha|a_1\rangle|100\rangle + \beta|a2\rangle|011\rangle\end{aligned}$$

- Project into subspaces:
  - S1 = {A|000> + B|111> ∀ A, B}
  - S2 = {A|100> + B|011> ∀ A, B}
  - If found in S2, measure with bit-flip operator $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$
    - Other subspaces indicate which qubit to apply bit-flip operator to, e.g. S3 = {A|010> + B|101> ∀ A, B}
    - New state: $|\psi_1\rangle = \alpha|a_0\rangle|000\rangle + \beta|a_3\rangle|111\rangle$

Adding a Layer

- Instead of |0> and |1>, three non-physical qubits in the Walsh-Hadamard basis:

$$|\psi_1\rangle = \alpha_1|a_4\rangle|{+}{+}{+}\rangle + \beta_1|a_5\rangle|{-}{-}{-}\rangle$$

- Each Walsh-Hadamard qubit is represented by three physical |0> or |1> qubits

$$|+\rangle = |\underline{0}\rangle + |\underline{1}\rangle = |000\rangle + |111\rangle$$
$$|-\rangle = |\underline{0}\rangle - |\underline{1}\rangle = |000\rangle - |111\rangle$$

- Decoherence in the physical qubits can be corrected as before
- This results in: $\left(|e_0\rangle|\underline{0}\rangle = |e_0\rangle|000\rangle\right) \overset{U'(t)}{\mapsto} \left(|a_0'\rangle|000\rangle = |a_0'\rangle|\underline{0}\rangle\right)$

$$\left(|e_0\rangle|\underline{1}\rangle = |e_0\rangle|111\rangle\right) \overset{U'(t)}{\mapsto} \left(|a_3'\rangle|000\rangle = |a_3'\rangle|\underline{1}\rangle\right)$$

- How does this affect |ψ>?

Final Correction

- To be general (only assuming the first qubit decoheres):

$$|e_0\rangle|\psi_1\rangle \overset{U'(t)}{\mapsto} \alpha(a_0|0\rangle + a_3|1\rangle)|{+}{+}\rangle + \beta(a_0|0\rangle - a_3|1\rangle)|{-}{-}\rangle$$
$$= \alpha\left(\frac{a_0 - a_3}{2}|+\rangle + \frac{a_0 + a_3}{2}|-\rangle\right)|{+}{+}\rangle$$
$$+ \beta\left(\frac{a_0 + a_3}{2}|+\rangle + \frac{a_0 - a_3}{2}|-\rangle\right)|{-}{-}\rangle$$
$$= \alpha|b_0\rangle|{+}{+}{+}\rangle + \beta|b_3\rangle|{-}{-}{-}\rangle + \alpha|b_1\rangle|{-}{+}{+}\rangle + \beta|b_2\rangle|{+}{-}{-}\rangle$$

- Project into subspaces again (again, other subspaces indicate errors in other qubits):
  - S1 = {A|+++> + B| − − − > ∀ A, B}, implies $(b_1 = b_2 = 0) \Rightarrow a_0 = -a_3$
  - S2 = {A| − ++> + B|+ − − > ∀ A, B}, implies $(b_0 = b_3 = 0) \Rightarrow a_0 = a_3$
  - Apply a Walsh-Hadamard-flip operator $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to correct S2
  - Final state: $|\psi_1\rangle = \alpha|a_0\rangle|{+}{+}{+}\rangle + \beta|a_0\rangle|{-}{-}{-}\rangle$, fully corrected!

What have we done?

- By projecting into subspaces?
  - Determined where decoherence has occured
  - Collapsed decoherence to a specific kind
    - Bit flip errors resolved in the first layer
    - Phase flip errors resolved in the second layer
- By using two layers?
  - Spread out information among 9 physical qubits
  - Allowed decoherence measurement while leaving state alone

Calderbank-Shor-Steane (CSS) Codes

- The two-layer approach is general
- A repetition code was used in both layers
- More complex classical codes can be used
  - Increased efficiency
  - Recover from more errors
- Shor's Code leads to this approach
- Need appropriate basis choice
  - Codes benefit from other bases
  - Must be chosen intelligently
- CSS codes studied more widely

Limitations and Problems

- Correlated errors
  - Qubits may always decohere in pairs, triples, etc.
  - Intuition dictates separated qubits
  - Error mechanisms may dictate qubit density (cosmic ray/shower affects all qubits in a radius?)
  - Must understand particular system
- Correction procedure may have errors
  - Use multiple layers
  - Recognize there will be an error rate
  - Objective: keep error rate low

Conclusion

- Shor's Code proved no-cloning surmountable
- Provided basis for creating new codes
- Brought quantum computation into the realm of the possible
- Provides insight into the challenges of quantum computation
- Questions?

Citations

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