Erasure vs Percolation thresholds for topological codes

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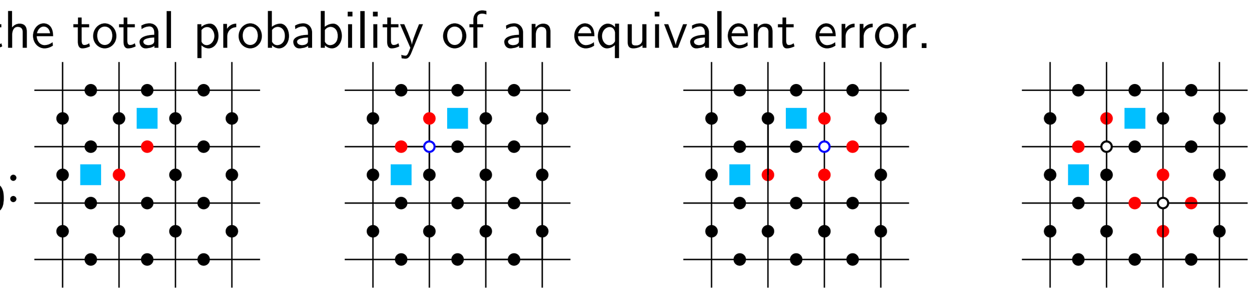
Motivation: map Ising model ↔ ML decoding

★ Maximum likelihood decoding threshold

- Independent X/Z noise, $p_X = p_Z = p$.
- CSS code $\mathcal{Q}$ $[[n,k,d]]$ with n-column generator matrices: $PQ^T = 0 \bmod 2$.
- Number of encoded qubits $k = n - \operatorname{rank}_2 P - \operatorname{rank}_2 Q$.
- Z-codeword c: $Pc^T = 0 \bmod 2$; c linearly independent from rows of Q.
- Degenerate errors: $e \simeq e' = e + \alpha Q$. Identical effect on the code. Given e, let $Z_0(e)$ be the total probability of an equivalent error.

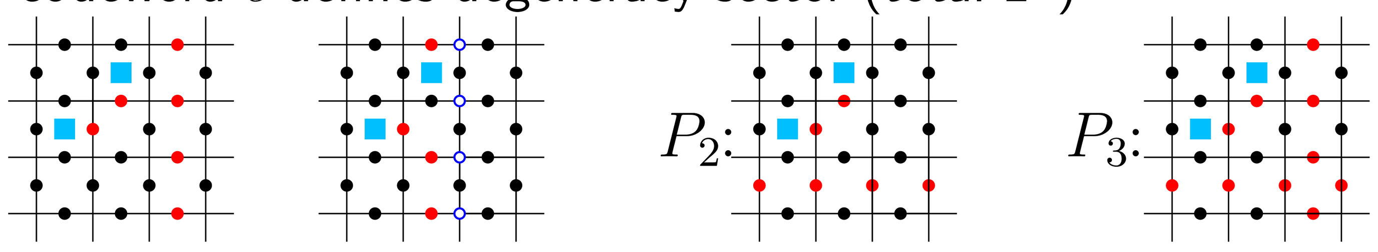
P_0:



$p^2(1-p)^{n-2}$ $p^2(1-p)^{n-2}$ $p^4(1-p)^{n-4}$ $p^6(1-p)^{n-6}$

- Each codeword c defines degeneracy sector (total 2^k)

P_1: P_2: P_3:



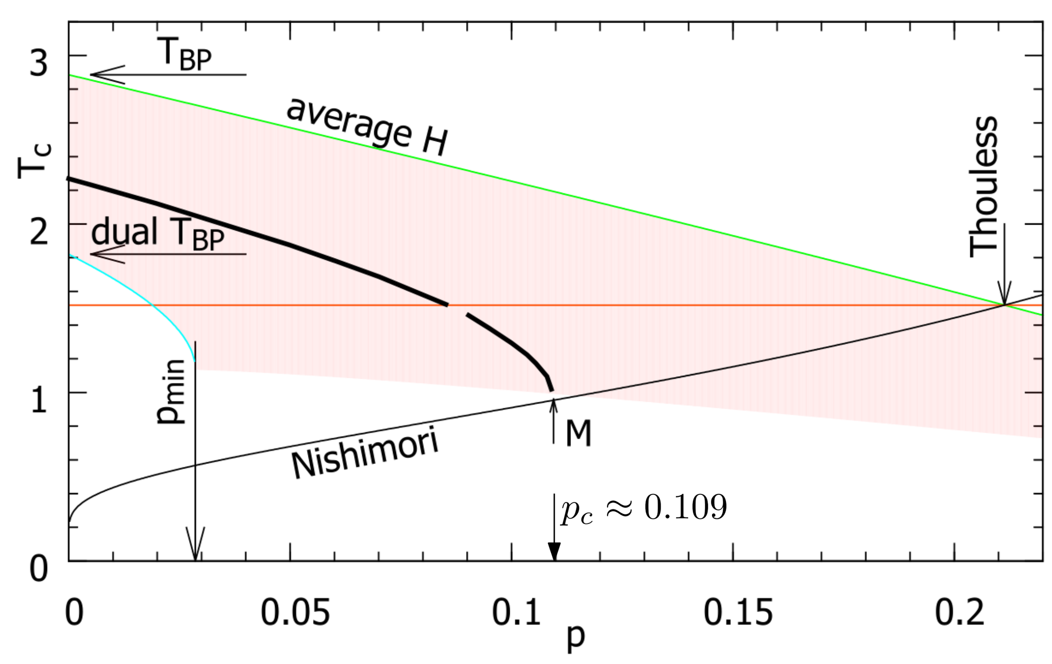
$p^6(1-p)^{n-6}$ $p^4(1-p)^{n-4}$ $p^6(1-p)^{n-6}$ $p^{10}(1-p)^{n-10}$

- Let $Z_{\text{tot}}(e) \equiv \sum_c Z_0(e+c)$ be the probability of the same syndrome as e.
- Given p, average maximum likelihood (ML) decoding success probability

$$\mathbb{P}_p = \left\langle \frac{Z_0(e)}{Z_{\text{tot}}(e)} \right\rangle_e, \quad Z_{\text{tot}}(e) = \sum_c Z_0(e+c).$$

- $Z_0(e)$: partition function of a random-bond Ising model, at $e^{-2\beta} = \dfrac{p}{1-p}$
- Temperatures β^{-1} away from Nishimori line: suboptimal decoding.
- Consider a code sequence $\mathcal{Q}_t$, $t \in \mathbb{N}$.
- ML decoding threshold p_c for code sequence: $\limsup\limits_{t\to\infty} \mathbb{P}_p(\mathcal{Q}_t) = 1$, $p < p_c$.

Phase diagram of the square-lattice random-bond Ising model, where p is the probability of a flipped bond. Black line denotes the numerical boundary $T_c(p)$ of the ferromagnetic phase, $p_{\min}$, T_{BP}, and "dual T_{BP}" are various analytical bounds for the phase transition.



★ Approach of Kovalev and LPP [QIC 15, 825 ('15)]

For Ising models associated with a sequence of codes, define

- "decodable phase" via $\left\langle \dfrac{Z_0(e)}{Z_{\text{tot}}(e)} \right\rangle_e \to 1$ at $t \to \infty$.
- Free energy increment: $\beta \Delta F_c \equiv \langle \ln Z_0(e) \rangle_e - \langle \ln Z_0(e+c) \rangle_e$.
- Defect tension $\lambda_c \equiv d_c^{-1} \Delta F_c$, where $d_c \equiv \min\limits_\alpha \operatorname{wgt}(c + \alpha Q)$ is the min weight.

It is easy to check that:

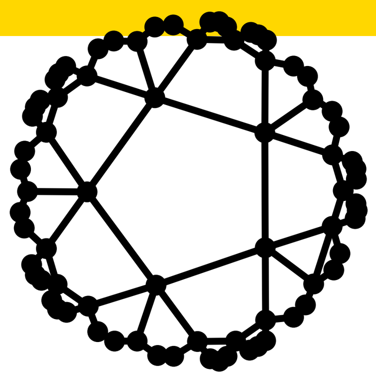
- In the decodable phase $\Delta F_c \to \infty$, $\forall c \not\simeq 0$.
- $\Rightarrow$ if k is finite and code distance $d = \min_c d_c$ diverges with n, or for a finite-R code family with power-law distance scaling: $[\lambda_c > 0,\ \forall c \not\simeq 0] \Rightarrow$ decodable
- In the decodable phase for a code family with rate R:
  average defect tension satisfies $\bar\lambda \equiv \langle \lambda_c \rangle_{c \not\simeq 0} \ge \beta^{-1} R \ln 2$.
- We thus expect: $\begin{cases} \text{for } R = 0,\ T_{\text{dec}}(p) \ge T_c^{(\text{tension})}(p), \\ \text{for } R > 0,\ T_{\text{dec}}(p) \le T_c^{(\text{tension})}(p). \end{cases}$

Here $T_c^{(\text{tension})}(p)$ is the critical temperature for non-zero defect tension $\lambda_c > 0$.

Question: What is the general relation between the decodable and the conventionally defined ferromagnetic phase?

- Under what conditions they coincide?
- Under what conditions they differ?
- In particular, for finite-rate codes.

Important examples: hyperbolic surface codes.
E.g., codes from $\{5,5\}$ hyperbolic tilings: $[[30,8,3]]$, $[[40,10,4]]$, $[[80,18,5]]$, $[[150,32,6]]$, $[[900,182,8]]$, ...



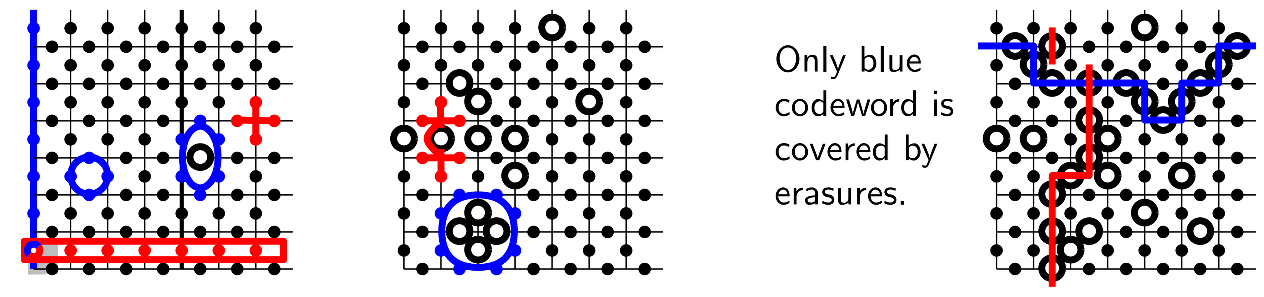
- Highly-symmetric, finite rate family
- Logarithmic distance scaling

Erasures in surface codes ↔ edge percolation

★ Erasure threshold for a sequence of codes

- Independent erasure errors with probability p on a CSS code $\mathcal{Q}$.
- Block Error Rate (BLER): $H_p(\mathcal{Q}) \equiv \mathbb{P}_p$(a codeword of $\mathcal{Q}$ is fully covered)
- Consider a sequence of CSS codes $\mathcal{Q}_t$, $t \in \mathbb{N}$.
- Erasure threshold p_E: guarantees that $\limsup\limits_{t\to\infty} H_p(\mathcal{Q}_t) = 0$ at $p < p_E$.
- Same configurations as in edge percolation on graph $\mathcal{G}$

toric code $[[98,2,7]]$



Only blue codeword is covered by erasures.

★ Edge percolation on an infinite graph $\mathcal{G}(\mathcal{V},\mathcal{E})$

- p_c: formation of an infinite cluster. Order parameter $\theta_i = \mathbb{P}_p(|\mathcal{C}_i| = \infty)$, $i \in \mathcal{V}$.
- p_T: divergence of the cluster susceptibility $\chi_i \equiv \mathbb{E}_p(|\mathcal{C}_i|)$. Transitive: $p_T = p_c$.
- p_u: uniqueness transition (infinite cluster unique for $p > p_u$).
- An expander graph $\mathcal{G}$ has non-zero Cheeger constant $h_{\mathcal{G}} = \min\limits_{|\mathcal{W}| \le |\mathcal{V}|/2} \dfrac{|\partial \mathcal{W}|}{|\mathcal{W}|}$.
- Transitive expander: $p_c < p_u$. Transitive amenable ($h_{\mathcal{G}} = 0$) graph: $p_c = p_u$.
- Connectivity (corr. function) $\tau_{ij} \equiv \mathbb{P}_p(i \in \mathcal{C}_j)$. For $p > p_u$, $\tau_{ij} \ge \theta_i \theta_j$.

★ Linker cluster expansion

For a graph $\mathcal{G}$, consider the quantity $K_i(h,p;\mathcal{G}) = \sum_v e^{hv} v^{-1} f_v^{(i)}(p)$, where $f_v^{(i)}(p) = \sum_{e,b} N_{v,e,b}^{(i)} p^e (1-p)^b$ is the probability that the vertex i is in a cluster with v vertices, and $N_{v,e,b}^{(i)}$ is the number of containing-i clusters with v vertices, e edges, and b boundary edges (incident on one of the vertices in the cluster). $K_i(0,p) = \mathbb{E}_p\left(|\mathcal{C}_i|^{-1}\right)$ is the expected inverse size of a cluster connected to $i \in \mathcal{V}$. $K_i(h,p)$ is an analog of the free energy for percolation:

Statement 1. $\partial_h K_i(h,p)\big|_{h=0} = 1 - \theta_i(p)$, $\partial_h^2 K_i(h,p)\big|_{h=0} = \chi_i(p)$.

Statement 2. *On a transitive graph, at $p < p_c$, the cluster probabilities $f_v(p)$ decay exponentially with v* $\Rightarrow$ *$K_i(0,p)$ is analytic for $p < p_c$.*

Statement 3. *Consider a pair of mutually dual locally planar graphs, $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{P})$ and $\widetilde{\mathcal{G}}(\widetilde{\mathcal{V}} = \mathcal{P}, \mathcal{E}, \widetilde{\mathcal{P}} = \mathcal{V})$. Expected number of homologies covered on $\mathcal{G}$ at p is*

$$\mathbb{E}_p(\#\hom(\mathcal{G})) = 1 - |\mathcal{V}| + p|\mathcal{E}| + \sum_{i \in \mathcal{V}} K_i(0,p;\mathcal{G}) - \sum_{j \in \widetilde{\mathcal{V}}} K_j(0,1-p;\widetilde{\mathcal{G}}).$$

★ Results

Consider a sequence of locally planar transitive graphs $\mathcal{G}_t$, locally convergent to a transitive graph $\mathcal{G}$ (balls of increasing radius r_t in $\mathcal{G}_t$ and $\mathcal{G}$ are isomorphic).

Theorem 1. *(**a**) Generally, $p_E \le p_c$. (**b**) The inequality is saturated if distances scale as a power of block length, $d \ge A n^\alpha$, $A > 0$ and $\alpha > 0$.*

Proof outline. (**a**) Assume $p_E > p_c$, and choose p in between. For such a p, only homologically trivial clusters are covered at large t. These can be mapped $1:1$ to those at $\mathcal{G}$. Any finite cluster on $\mathcal{G}$ will be encountered at large enough t, and these are all likely to appear, thus $\theta(p) = 0$. Contradiction. (**b**) Use exponential decay of $f_v(p)$ for $p < p_c$, and a bound on $N_{v,e,b}$. ■

Theorem 2. *When distance scales logarithmically with block length, $d \ge B \log n$, $B > 0$, $p_E > 0$.*

Conjecture 3. *When distance scales logarithmically with block length, $d \le B' \log n$, $B' > 0$, $p_E < p_c$. With sublogarithmic distance, $p_E = 0$.*

Example 1. *Consider rectangular toric code with dimensions $t \times L_t$.*

- *Infinite graph limit = square lattice; percolation threshold $p_c = 1/2$.*
- *For $L_t = 2^{t \ln t}$, $p_E = 0$.* $\quad \mathbb{P}_{\text{vert}} \ge 1 - (1-p^t)^{L_t} \ge 1 - e^{-p^t L_t} \to 1, \quad \forall p > 0.$
- *For $L_t = m^t$, $p_E \le 1/m$.*

For a sequence of locally planar transitive graphs $\mathcal{G}_t$, locally convergent to a transitive graph $\mathcal{G}$, introduce the homological order parameter, η_p, the expected fraction of homologies asymptotically covered by erasures (the corresponding limit exists). Define the corresponding critical probability p_H, s.t. $\eta_p = 0$ at $p < p_H$, and $\eta_p > 0$ at $p > p_H$. Clearly, $p_H \ge p_E$; $p_H = p_E$ for sequences with bounded k.

Theorem 4 (Delfosse and Zemor, 2010). *Generally, for a sequence with unbounded distance and finite rate, $p_H \ge p_c$.*

Theorem 5. *For a sequence of hyperbolic surface codes converging to $\{\ell, m\}$ tilings of the infinite hyperbolic plane, $1/\ell + 1/m < 1$, $p_H = p_c$.*

Proof outline. From Statement 3, $\dfrac{2R}{\ell}\eta_p = K(0,p;\mathcal{G}) - K(0,1-p;\widetilde{\mathcal{G}}) + \dfrac{2}{\ell}p - 1$.
Use analyticity of $K(0,p)$ for $p < p_c$ and $p > 1/(h_{\mathcal{G}} + 1)$. ■

Work in progress: • Extend to $\mathbb{Z}_q$ models; • Code families with provable analyticity of $K(h,p)$ and free energy density $f(p,h,T)$ (cf. Lee-Yang zeros); • Critical scaling, ...