Threshold for subsystem codes

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(Dated: January 31, 2018)

I. DISCUSSION FROM 4/12/17

Statement: consider a BS code and a Shor's code of the same size. Shown: for the BS code, gauge group generators, stabilizer generators, logicals; for the S code, only stabilizer generators.

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BS: X.. ZZ   ZZ ..XXX      ...XX       S: ZZ..XXX
    X        ZZ ..XXX      .                ..XXX
             ZZ            Z
             ...           Z
```

BS-equivalent X errors: same parity in each column. S-equivalent X errors: any pair of rows can be flipped. This preserves mismatches between pairs of qubits in the same row.

Clearly, in this case Shor's code is way better (on average), although there are error combinations which can confuse it.

II. MESSAGE FROM 12/15/2016

Please see theorem 3.1 in[1] (and the proof)

Some related results are in[2] and in[3]

The bond algebra considered by Fernandez and Slawny and others have matrix multiplication from a different side; it should not matter in the case of perfectly measured syndrome. However, I think this will be relevant with syndrome errors (fault-tolerant case).

Statement: consider a subsystem code $Q_1 = Q(S, G)$ with gauge group G and stabilizer group S, an equal-size stabilizer code $Q_0 = Q(G')$ where a commuting subgroup of G, $S \subset G' \subset G$ (elements of G' do not necessarily commute with those of G), is used to define a stabilizer, and a bigger stabilizer code $Q_2 = Q(S)$ (all gauge qubits become logical). Channel setting: $p_c(Q_1) = p_c(Q_0) \geq p_c(Q_2)$. (is this true?) How about FT setting? Here, threshold for Q_1 and Q_2 may not necessarily be equal.

III. PROBLEM SETUP

Consider an n-qubit subsystem code[4,5] with the gauge group $\mathcal{G} \subset \mathcal{P}_n$, and the Abelian stabilizer group $\mathcal{S} \subseteq \mathcal{G}$. Denote the corresponding generator matrices $G = (G_X, G_Z)$ and $S = (S_X, S_Z)$. We have $G * S^T = G\widetilde{S}^T = G_X S_Z^T + G_Z S_X^T = 0$. We have $S = QG$, where Q is some binary matrix. It is not necessary that all rows in G and S be independent. Assume matrix dimensions $r_i \times n_i$ and row weights w_i, where $i \in \{G, S, Q\}$. We have $n_S = n_G = 2n$.

Rows of S generate bare stabilizer group, while rows of G generate dressed stabilizer generators. It is easy to see that non-commuting operators in G can be eliminated in pairs (those correspond to logical generators of gauge qubits), thus $\operatorname{rank} G = 2k_G + \operatorname{rank} S$, and $k_G + k = n - \operatorname{rank} S$. This gives the number of encoded qubits

$$k = n - \operatorname{rank} S - (\operatorname{rank} G - \operatorname{rank} S)/2 = n - \frac{\operatorname{rank} G + \operatorname{rank} S}{2}.$$

The distance of the subsystem code is

$$d = \min_{\alpha}\{\mathrm{wgt}_4(e) : Se^T = 0\}, \quad \mathrm{wgt}_4(e) \equiv \mathrm{wgt}(e_x|e_z).$$

where | represents a binary OR operation on the components of the two vectors.

IV. ML DECODING (2016-11-17

Important: matrix Q should have low weight rows and columns, otherwise we will generate too large error.

Probabilities for a given syndrome: involve summation over all vectors from the gauge group.

The dual matrix: stabilizer plus codewords (all that commutes with $\mathcal{G}$).

V. DISCUSSION 1/31/2018

BP works correctly now. Discussed the output, optimization of the threshold so that the residual error vector has the smallest weight (separately at each N).

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[1] R. Fernåndez and J. Slawny, "Inequalities and many phase transitions in ferromagnetic systems," Comm. Math. Phys. **121**, 91–120 (1989).

[2] Joel L. Lebowitz and David Ruelle, "Phase transitions with four-spin interactions," Communications in Mathematical Physics **304**, 711–722 (2011).

[3] C. Gruber and J. L. Lebowitz, "On the equivalence of different order parameters and coexistence of phases for Ising ferromagnet. II," Communications in Mathematical Physics **59**, 97–108 (1978).

[4] David Poulin, "Stabilizer formalism for operator quantum error correction," Phys. Rev. Lett. **95**, 230504 (2005).

[5] Dave Bacon, "Operator quantum error-correcting subsystems for self-correcting quantum memories," Phys. Rev. A **73**, 012340 (2006).