CS70–Spring 2013 — Homework 10

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Collaborators: None

1. Introductions

1.

2.

3. Let X^k represents the number of students end up in groups of size k. Then $E(X_i^k)$ represents student i ends up in groups of size k.

$$\begin{aligned} E(X^k) &= E(X_1^k) + E(X_2^k) + \ldots + E(X_n^k) \\ &= \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \\ &= 1 \end{aligned}$$

If sum the results for all $1 \leq k \leq n$, the result would be n.

$$E(X) = \sum_{k=1}^{n} E(X^k)$$

$E(X)$ is the expected number of students end up in all the groups of size from 1 to n, which must be the total number of students in the class.

4. We already know the expected number of students in group of size k, then how many expected such groups should there be? We should divide by the group size k to get the result.

$$E(G_k) = \frac{\text{Expected number of students in group of size k}}{k} = \frac{1}{k}$$

5. The key observation is we can only have at most one group with size more than half

$$\begin{aligned} \text{P(a group of more than half)} &= \mathbb{E}[\mathcal{Z}] \\ \mathcal{Z} : \text{the number of groups of size} \geq \frac{n}{2} \\ \mathcal{Z} &= \sum_{k \geq \frac{n}{2}} \mathcal{Z}_k \\ \mathcal{Z}_k : \text{number of groups of size } k \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[\mathcal{Z}] &= \sum_{k \geq \frac{n}{2}} \mathbb{E}[\mathcal{Z}_k] \\
&= \sum_{k \geq \frac{n}{2}} \mathrm{P}(\mathcal{Z}_k = 1) \\
&= \sum_{k \geq \frac{n}{2}} \frac{1}{k} \\
&= \begin{cases} H_n - H_{\frac{n}{2}}, & \text{if } n \text{ is even} \\ H_n - H_{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

6. n is even,

$$
\begin{aligned}
H_n - H_{n/2} &= \ln n + \gamma - \ln n + \ln 2 - \gamma \\
&= \ln 2
\end{aligned}
$$

   n is odd,

$$
\begin{aligned}
H_n - H_{\lfloor \frac{n}{2} \rfloor} &= \ln n + \gamma - \ln n - 1 + \ln 2 - \gamma \\
&= \ln \frac{n}{n-1} + \ln 2 \\
&= \ln 2
\end{aligned}
$$

2. Round the Clock

1. Expected waiting time is $E[W] = \frac{24}{8} = 3$.
   Suppose we arrange 3 meals in order after they are chosen based on their time from midnight to 11pm. And their time is denoted as X_1, X_2, X_3. Then the average waiting time is

$$
\begin{aligned}
E[W] &= \frac{E(X_2 - X_1) + E(X_3 - X_2) + E(24 + X_1 - X_3)}{3} \\
&= \frac{E(X_2) - E(X_1) + E(X_3) - E(X_2) + E(24) + E(X_3) - E(X_2)}{3} \\
&= \frac{24}{3} \\
&= 8
\end{aligned}
$$

2. This case it's similar to the question 1 part 1. Instead of having 3 meals, we have 4 meals and we pretend the call is one of the meal. So the question would become if I have 4 meals a day, what's the expected waiting time after I have 1 meal. Similarly we have

$$
\begin{aligned}
E[W] &= \frac{E(X_2 - X_1) + E(X_3 - X_2) + E(X_4 - X_3) + E(24 + X_4 - X_1)}{3} \\
&= \frac{E(X_2) - E(X_1) + E(X_3) - E(X_2) + E(24) + E(X_1) - E(X_4)}{4} \\
&= \frac{24}{4} \\
&= 6
\end{aligned}
$$

3. Expectation of Geometric Distribution

1.

$$S = \sum_{i=1}^{\infty} ir^{i-1} = 1 + 2r + 3r^2 + \ldots + ir^{i-1}$$
$$rS = r + 2r^2 + 3r^3 + \ldots + ir^i$$

$S - rS$, we have

$$(1-r)S = r + r^2 + r^3 + \ldots + r^{i-1} + ri^r$$
$$(1-r)S = \frac{1 \cdot (1-r^i)}{1-r} + ir^i$$

Since $-1 < r < 1$, RHS approaches to

$$(1-r)S = \frac{1}{1-r}$$
$$S = \frac{1}{(1-r)^2}$$

2.

$$E(X) = \sum_{i=1}^{\infty} iPr[X = i] = \sum_{i=1}^{\infty} i(1-p)^{i-1}p = p\sum_{i=1}^{\infty} i(1-p)^{i-1}$$

According Equation(1), let $r = 1 - p$, we know

$$E(X) = p\sum_{i=1}^{\infty} i(1-p)^{i-1} = (1-r)\frac{1}{(1-r)^2} = p\frac{1}{(1-(1-p))^2} = \frac{1}{p}$$

4. Surviving Challenge

It's worth betting.
Let $X_i = 1$ be the event that survive in round i, o.w $E(X_i) = 0$ then expected number of survival rounds in total is

$$E[X] = E[X_1] + E[X_2] + \ldots + E[X_i]$$

And

$$Pr[X_i = 1] = Pr[X_{i-1} = 1] \times \frac{(\#cardsLeft - 3times(round - 1))}{\#cardsLeft}$$

Then we can calculate $E(X) = 4.7$, which is > 4, so it's worth betting. The detailed calculation is as below

Round	**Cards Left**	**Factor**	**Probability of Survival**
1	52	52	1
2	51	48	0.941176471
3	50	44	0.828235294
4	49	40	0.676110444
5	48	36	0.507082833
6	47	32	0.345247886
7	46	28	0.210150887
8	45	24	0.112080473
9	44	20	0.05094567
10	43	16	0.018956528
11	42	12	0.005416151
12	41	8	0.00105681
13	40	4	0.000105681
14	39	0	0

Table 1: Expected Rounds of Survival Calculation