CS70–Spring 2013 — Homework 1

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Collaborators: None

1. Proof by induction

- base case: $n = 2$, $s_n = 1 - \frac{1}{2} = \frac{1}{2}$
- inductive hypothesis: $n = k$, $s_k = (1 - \frac{1}{2}) \times (1 - \frac{1}{3}) \times \cdots \times (1 - \frac{1}{k})$ and $s_k = \frac{1}{k}$
- inductive step: $n = k+1$, $s_{k+1} = (1-\frac{1}{2}) \times (1-\frac{1}{3}) \times \cdots \times (1-\frac{1}{k}) \times (1-\frac{1}{k+1})$, $s_k = \frac{1}{k} \times (1-\frac{1}{k+1}) = \frac{1}{k} \times \frac{k}{k+1} = \frac{1}{k+1}$, so it holds for $n = k + 1$

2.Another induction proof

- base case: $n = 0$, $a_n = 3^2 + 4^1 = 13$
- inductive hypothesis: 13 divides a_k for $n = k$, which means $a_k = 13d, \forall d \in \mathbb{Z}$
- inductive step: $n = k+1$, $a_{k+1} = 3^{k+3}, 4^{2k+3}$ $a_{k+1} - 3a_k = 3^{k+3} + 4^{2k+3} - 3^{k+3} - 3 \times 4^{2k+1} = 16 \times 4^{2k+1} - 3 \times 4^{2k+1} = 13 \times 4^{2k+1}$. And since $a_k = 13d$, we have $a_{k+1} = 13 \times 4^{2k+1} + 13d$, $\forall d \in \mathbb{Z}$. So 13 divides a_{k+1} also.

3. Tower of Brahma

Think it in this way, you have n plates on the needles. Then to move these n plates to another needle, you need to move all the $n - 1$ plates on top of the last one to one of the other two needles first and then move the last plate to another needle, and then put back all the $n - 1$ plates on top of the biggest plates. So suppose the number of moves to move n plates is denoted by $F(n)$, $F(n) = 1 + 2F(n - 1)$. We check some number to see the trends.

$$\begin{aligned}
&F(1) = 1 \\
&F(2) = 1 + 2F(1) = 3 \\
&F(3) = 1 + 2F(2) = 7 \\
&F(4) = 1 + 2F(3) = 15 \\
&\vdots \\
&F(n) = 1 + 2F(n - 1)
\end{aligned}$$

we can guess $F(n) = 2^n - 1$.
Then can prove it using induction.

- base case: $n = 1$, just need one move, $F(1) = 2^1 - 1 = 1$
- inductive hypothesis: $n = k$, $F(k) = 2^k - 1$ holds
- inductive step: $n = k + 1$, we can move the k plates above the bottom one to another needle first taking $F(k)$ moves, then move the bottom plate to another empty needle, which takes 1 move, then move the k plates back to the biggest plate, which takes $F(k)$ moves also, so $F(k + 1) = 2F(k) - 1 = 2 \times (2^k - 1) + 1 = 2^{k+1} - 1$. So the statement is true

4.The proof of the π is in the eating

Some thoughts on this: we can first assume $n = k + 1$ case, then remove the largest one to get $n = k$ case, which we assume can be placed in order according inductive hypothesis. The reason that why we do this instead of increment from $n = k$ case is we insert the plus 1 pizza doughs, and we cannot assume it to be the largest, it need to be random size compared with the stack, which making it harder to prove. It's easier to place the largest one at the bottom, isolate it and use the inductive hypothesis to say the $n = k$ doughs can also be placed in order.

- base case: $n = 1$, it's true without doing anything
- inductive hypothesis: the stack can be placed in order for $n = k$ plates.
- inductive step: first assume $n = k + 1$, then we remove the largest plate, we know the rest $n = k$ can be placed in order with the move. Now we put back the largest plate to its place. 1. We first place the spatula on the largest plate say the m one, then the m plate and all plates above it got flipped. 2. place the spatula on the bottom plate, then the whole stack got turned over and the largest plate m is at the bottom. now the $m + 1$ to $k + 1$ plates is in original order, the 1 to $m - 1$ plates are in reverse order 3. flip the 1 to $m - 1$ plates again. Now all plates above the largest plate (the m extra one plate) is in its original order. And according to the inductive hypothesis we know we can flip these k plates to an ordered stack.

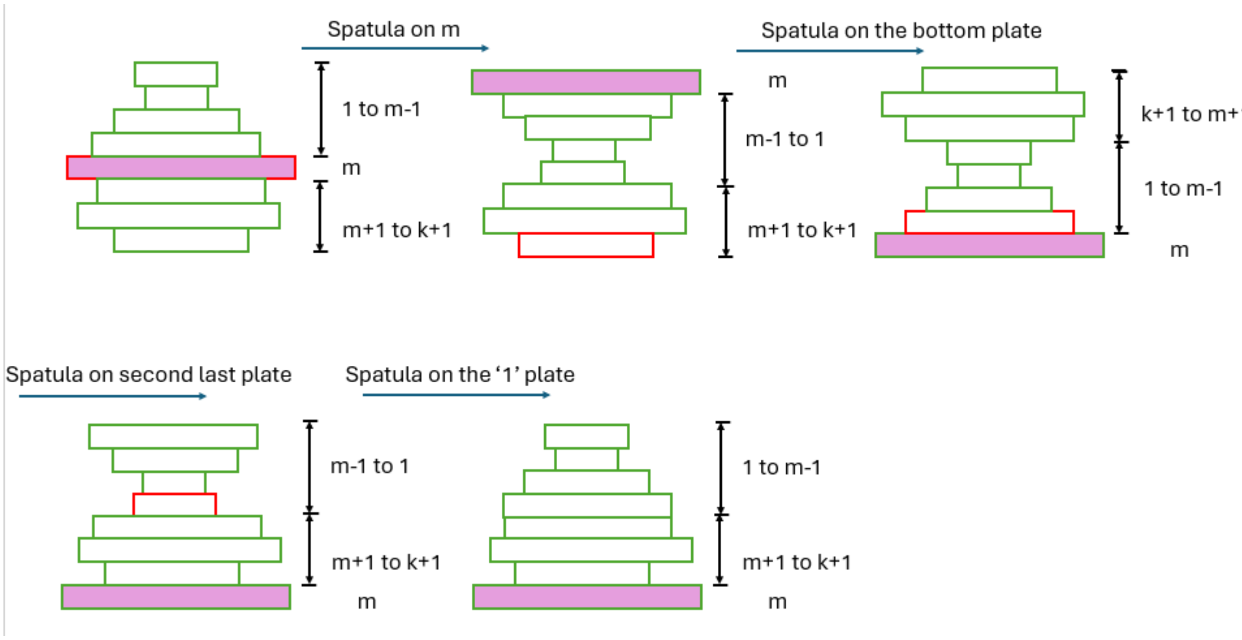

Figure 1:

5.Boring Birds

Choosing the base case is a little bit tricky?

The idea is you can also reduce the number of yellow birds to 0, then the red birds to 0 or 1, then the black birds to 0 so that the game is ended.

- base case:
  1. the number of yellow, red, black birds is (1, 0, 0), it's valid you take the move (i) then take the move (ii) leaving only 1 red bird, the game ends.
  2. the number of yellow, red, black birds is (0, 1, 0). No move can take, ends.
  3. the number of yellow, red, black birds is (0, 2, 0). take move (ii), then 7 move (iii), ends.
  4. the number of yellow, red, black birds is (0, 0, 1). take move (iii), ends

- inductive hypothesis:
  Using strong hypothesis, the game can ended with finite number of birds, such that the number of yellow, red, black birds (n_1, n_2, n_3) satisfied $n_1 \leq k_1 \wedge n_2 \leq k_2 \wedge n_3 \leq k_3$, $\forall k_1, k_2, k_3 \geq 1$

- inductive step:
  1. $(n_1, n_2, n_3) = (k_1+1, k_2, k_3)$, if $k_2 \geq 1$, we can take move (i), then 2 move (ii), 14 move (iii), after that we can get $(n_1, n_2, n_3) = (k_1, k_2 - 1, k_3)$, then according to inductive hypothesis, the game can end with finite number of moves with these number of birds.
  2. $(n_1, n_2, n_3) = (k_1, k_2 + 1, k_3)$, we take one move (ii) then 7 move (ii), we can reach $(n_1, n_2, n_3) = (k_1, k_2, k_3)$.
  3. $(n_1, n_2, n_3) = (k_1, k_2, k_3 + 1)$, we take one move (iii) then we can reach $(n_1, n_2, n_3) = (k_1, k_2, k_3)$
  So if statement holds.

6. You be the grader

1. F. base case is wrong, $n = 1$ is even.

2. T.

3. T.

4. F. "max(x, y) = k+1. Then it follows that max(x-1, y-1) = k". The inductive step is using some property of the real **max** function. However, this **max** here you can take it as just a function "**F**". So we are trying to prove $\forall x, y, n \in \mathbb{N}, if F(x, y) = n, then x \leq y$, there is no such property "F(x, y) = k+1. Then it follows that F(x-1, y-1) = k".