1. Conditional probability

I have a bag containing either a \$5 bill (with probability $1/3$) or a \$10 bill (with probability $2/3$). I then add a \$5 bill to the bag, so it now contains two bills. The bag is shaken, and you randomly draw a bill from the bag, which turns out to be a \$5 bill. If a second student draws the remaining bill from the bag, what is the probability that it too is a \$5 bill?

Answer: Let A be the event that the bill that was initially in the bag was a \$5 bill, and B the event that the bill that you drew was a \$5 bill. We want to compute the conditional probability $\Pr[A \mid B]$. Note that

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A] = \frac{1}{3} \cdot 1 = \frac{1}{3},$$

$$\Pr[\overline{A} \cap B] = \Pr[\overline{A}] \cdot \Pr[B \mid \overline{A}] = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

Hence,

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[A \cap B]}{\Pr[A \cap B] + \Pr[\overline{A} \cap B]} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} = \frac{1}{2}.$$

2. Weather forecast

Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year, so the prior probability of rain is just $5/365$. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 5% of the time. What is the probability that it will rain on the day of Marie's wedding?

Answer: Let R be the event that it rains, and W be the event that the weatherman predicts rain. Using the Bayesian Inference Rule:

$$\Pr[R \mid W] = \frac{\Pr[R \cap W]}{\Pr[W]} = \frac{\Pr[R]\Pr[W \mid R]}{\Pr[W]} = \frac{\Pr[R]\Pr[W \mid R]}{\Pr[W \cap R] + \Pr[W \cap \overline{R}]}$$

$$= \frac{\Pr[R]\Pr[W \mid R]}{\Pr[R]\Pr[W \mid R] + \Pr[\overline{R}]\Pr[W \mid \overline{R}]} = \frac{\frac{5}{365} \cdot 0.9}{\frac{5}{365} \cdot 0.9 + \frac{360}{365} \cdot 0.05} = 0.2$$

So maybe Marie doesn't have to cancel her wedding plans after all!

3. Prosecutor's Paradox

A murder has been committed in a city. The police are confident that the murderer must be one of the one million adult residents of the city and its surrounding area, but they initially have no reason to suspect anyone ahead of anyone else. The only piece of evidence is a DNA sample obtained from the scene. During a routine check, this sample is found to match the DNA of a man in the city (which had been collected for unrelated reasons), and the man is put on trial for murder. At the trial, an expert witness testifies that if the man is innocent, the probability of a DNA match is $1/10,000$, and as a result the man is convicted. You may assume that, if the man is guilty, then his DNA matches with probability 1.

1. Let M denote the event that the DNA matches, and I the event that the man is innocent. From the above data, write down $\Pr[I]$, $\Pr[\overline{I}]$, $\Pr[M \mid I]$, and $\Pr[M \mid \overline{I}]$.

   Answer:
   $$\Pr[I] = \frac{999999}{1000000}$$
   $$\Pr[\overline{I}] = \frac{1}{1000000}$$
   $$\Pr[M \mid I] = \frac{1}{10000}$$
   $$\Pr[M \mid \overline{I}] = 1$$

2. Use the Inference Rule to compute the probability that the man is innocent, given that the DNA matched. Do you think the man should have been convicted?

   Answer: Using the Bayesian Inference Rule, we get
   $$\Pr[I \mid M] = \frac{\Pr[I]\Pr[M \mid I]}{\Pr[I]\Pr[M \mid I] + \Pr[\overline{I}]\Pr[M \mid \overline{I}]} = \frac{\frac{999999}{1000000}\cdot\frac{1}{10000}}{\frac{999999}{1000000}\cdot\frac{1}{10000} + \frac{1}{1000000}\cdot 1} = \frac{999999}{999999+10000} \approx 0.99.$$
   So obviously the man shouldn't have been convicted because there is a 99% chance that he's innocent given the evidence.
   This is an example of the so-called "Prosecutor's Paradox", whereby a judge or jury confuses $\Pr[M \mid I]$ (the probability that the evidence points to an innocent person) with $\Pr[I \mid M]$ (the probability that the accused is innocent given the evidence), and assumes that if the former is small then the latter is also small.

4. Independence

Suppose two fair dice are thrown. Let A be the event that the number on the first die is odd, B the event that the number on the second die is odd, and C the event that the sum of the two numbers is odd.

1. Show that the three events A,B,C are pairwise independent (i.e., each pair (A,B), (B,C), and (A,C) is independent).

   Answer: It is obvious that A and B are independent and that $\Pr[A] = \Pr[B] = \frac{1}{2}$. Moreover, note that the sum of the two numbers is odd if and only if exactly one of them is odd, so $\Pr[C] = \Pr[A \cap \overline{B}] + \Pr[\overline{A} \cap B] = \Pr[A]\cdot\Pr[\overline{B}] + \Pr[\overline{A}]\cdot\Pr[B] = \frac{1}{2}$.
   Also, $\Pr[B \cap C] = \Pr[B]\cdot\Pr[C \mid B] = \Pr[B]\cdot\Pr[\overline{A} \mid B] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. (Given B, event C is equivalent to $\overline{A}$.) Since $\Pr[B]\cdot\Pr[C] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$, we see that B and C are independent.
   Similarly, $\Pr[A \cap C] = \Pr[A]\cdot\Pr[C \mid A] = \Pr[A]\cdot\Pr[\overline{B} \mid A] = \frac{1}{2}\cdot\frac{1}{2} = \Pr[A]\cdot\Pr[C]$, which shows that A and C are independent.

2. Show that the events A,B,C are not mutually independent.

   Answer: Note that $\Pr[A \cap B \cap C] = 0$, whereas $\Pr[A]\cdot\Pr[B]\cdot\Pr[C] = \frac{1}{8}$. Hence A,B,C are not mutually independent.