1. **Distributing the quantifiers**

   Let $P(x)$ and $Q(x)$ denote some propositions involving x. For each statement below, prove that the statement is correct or provide a counterexample.

   (a) $\forall x\,(P(x) \vee Q(x))$ is equivalent to $(\forall x, P(x)) \vee (\forall x, Q(x))$.

   (b) $\forall x\,(P(x) \wedge Q(x))$ is equivalent to $(\forall x, P(x)) \wedge (\forall x, Q(x))$.

   (c) $\exists x\,(P(x) \vee Q(x))$ is equivalent to $(\exists x, P(x)) \vee (\exists x, Q(x))$.

   (d) $\exists x\,(P(x) \wedge Q(x))$ is equivalent to $(\exists x, P(x)) \wedge (\exists x, Q(x))$.

2. **Pigeonhole principle**

   Prove that if you put $n+1$ apples into n boxes in any way you like, then at least one box must contain at least 2 apples.

3. Proof by contraposition

Let x be a positive real number. Prove that if x is irrational (i.e., not a rational number), then $\sqrt{x}$ is also irrational.

4. Proof by cases

A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

5. Numbers of friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.