CS 70	Discrete Mathematics and Probability Theory	
Spring 2015	Vazirani	Discussion 8M

1. **Sanity check!**

   1. Is $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = n^2$ an injection (1-1)? Briefly justify.

      Answer: Yes. One way to illustrate is by drawing the 1-1 mapping from n to n^2.
      More formally, we can show that the preimage is unique by showing that $m \neq n \implies f(m) \neq f(n)$. We can do a proof by showing its contrapositive $f(m) = f(n) \implies m = n$.

      $$\begin{aligned} f(m) = f(n) &\implies m^2 = n^2 \\ &\implies m^2 - n^2 = 0 \\ &\implies (m-n)(m+n) = 0 \\ &\implies m = \pm n \end{aligned}$$

      Since n can't be negative, we have an injection.

   2. Is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 + 1$ an surjection (onto)? Briefly justify.

      Answer: We show that for any value of y, there always exists a corresponding input x. It's easy to see that $x = \sqrt[3]{y-1}$. Thus for any value of y, there exists this value of x which maps to it.

2. **Natural Numbers**

   Prove that for any integer $k \geq 1$, $\mathbb{N}^k$ is countably infinite.

   Answer: Proceed by induction on k.

   Base Case: $k = 1$.
   In this case, $\mathbb{N}^k = \mathbb{N}$, so it is by definition countably infinite.

   Inductive Hypothesis: Assume $\mathbb{N}^k$ is countably infinite.

   Inductive Step: We need to prove that $\mathbb{N}^{k+1}$ is countably infinite. Since $\mathbb{N}^k$ is countably infinite, there exists some bijection $f : \mathbb{N} \to \mathbb{N}^k$. Then we can write every element in $\mathbb{N}^{k+1}$ as a pair $(f(m), n)$ for some unique $m, n \in \mathbb{N}$, where $f(m) \in \mathbb{N}^k$ represents the first k entries and $n \in \mathbb{N}$ represent the $(k+1)^{\text{th}}$ entry. Using this, we define a bijection $g : \mathbb{N} \to \mathbb{N}^{k+1}$ using the following pattern:

   $$0 \to (f(0), 0), \;\; 1 \to (f(1), 0), \;\; 2 \to (f(1), 1), \;\; 3 \to (f(0), 1), \;\; \ldots$$

   This is essentially the same as the spiral argument from the notes, but restricted to the first quadrant.

3. **Bitstrings**

   Use diagonalization to show that set of all infinite-length bitstrings is not countable.

   Answer: We can show that the set of all infinite-length binary strings is uncountable using a diagonalization argument, similar to the one in the notes. Suppose the contrary that the set of all infinite-length bitstrings is countable, so we can enumerate it as follows:

$$\begin{array}{lllllll}
1 & \longleftrightarrow 0 & 1 & 1 & 0 & 1\ldots \\
2 & \longleftrightarrow 0 & 1 & 0 & 0 & 1\ldots \\
3 & \longleftrightarrow 1 & 1 & 1 & 0 & 0\ldots \\
4 & \longleftrightarrow 0 & 0 & 1 & 0 & 1\ldots \\
5 & \longleftrightarrow 1 & 0 & 1 & 1 & 1\ldots
\end{array}$$

We can construct a new infinite-length bitstring by flipping each bit down the diagonal, giving us a new infinite-length bitstring that is not in our enumeration (shown in red above), a contradiction. Therefore, the number of infinite-length binary strings is uncountable.

4. Countable Programs

Prove that the set of all programs is countably infinite.

Answer: Since assembly language is a complete programming language, we can express any program in assembly. The length of a program is finite, so it can be expressed in as a finite bitstring. The set of all finite binary strings is countably infinite, and the set of all programs is no larger than the set of finite binary strings, so the set of all programs is countably infinite.