

---

CS 70Discrete Mathematics and Probability TheorySpring 2015VaziraniDiscussion 9M

---

### 1. Sanity check!

At UC Berkeley, each student has an 8-digit ID number.

1. How many ID cards can be formed if repetition of digits is allowed?

**Answer:**  $10^8$ .

2. How many ID cards can be formed if repetition of digits is not allowed?

**Answer:**  $\frac{10!}{2!}$ .

### 2. West Dakotan license plates

In the fictional state of West Dakota, license plates consist of 5 characters, each of which must be either a letter (one of the 26 from A through Z) or a digit (one of the 10 from 0 through 9). Beyond this, there are no further restrictions. How many possible West Dakotan license plates...

- a. contain only letters?

**Answer:**  $26^5$ .

- b. have exactly three letters and two numbers?

**Answer:** We first choose where the three letters will be ( $\binom{5}{3}$ ), and then pick the letters ( $26^3$ ) and numbers ( $10^2$ ). The answer is  $\binom{5}{3} \cdot 26^3 \cdot 10^2$ .

- c. contain the string ABC?

**Answer:** We have to pick two additional characters and decide where they will be in relation to ABC. There are three possibilities: xyABC, xABCy, and ABCxy. Therefore, the answer is  $3 \cdot 36^2$ .

- d. have at least two of the same character?

**Answer:** We first count the total number of plates ( $36^5$ ) and then subtract the number of plates that do not have a repeated character ( $\frac{36!}{31!}$ ). The answer is  $36^5 - \frac{36!}{31!}$ .

### 3. Poker hands

Let's play a version of poker where each hand consists of 6 cards. As usual, the order of cards in each hand doesn't matter. Count the number of hands that have:

- a. 3 distinct pairs. E.g. (1H, 1D, 2C, 2S, 3D, 3S) but not (1H, 1D, 1C, 1S, 2D, 2S)

**Answer:**

We first choose 3 distinct numbers (out of 13 possible choices) to assign to each of the three pairs:  $\binom{13}{3}$ . Then for each pair, choose 2 suits from the 4 possible suits:  $\binom{4}{2}$ . The answer is  $\binom{13}{3} \cdot \binom{4}{2}^3$ .

- b. Exactly 1 triple and 1 pair. E.g. (1H, 1D, 1C, 2S, 2D, 4S) but not (1H, 1D, 1C, 2S, 2D, 2C) or (1H, 1D, 1C, 1S, 2D, 2C).

First choose 3 distinct numbers (out of 13) without order, and then count the number of ways to designate one to be the triple, one to be the pair and one to be the single. Then pick the suits for each of them.

**Answer:**

Choose any 3 distinct numbers without order:  $\binom{13}{3}$ . Number of ways to designate one to be the triple, one to be the pair and one to be the single:  $3!$ . Pick the suits for the triple, the pair, and the single:  $\binom{4}{3} \cdot \binom{4}{2} \cdot \binom{4}{1}$ . Final answer:  $13 \cdot 12 \cdot 11 \cdot \binom{4}{3} \cdot \binom{4}{2} \cdot \binom{4}{1}$ .

- c. The same question as above, but with a different method:

First choose 3 distinct numbers *with* order. Let these numbers be  $a, b$  and  $c$ . Now let the deck of cards be of the form  $a\_ , a\_ , a\_ , b\_ , b\_ , c\_ ,$  where the blanks represent suits. Now count the number of ways to choose the suits.

**Answer:**

Choose 3 distinct numbers with order:  $13 \cdot 12 \cdot 11$ . Pick the suits for the triple, the pair, and the single:  $\binom{4}{3} \cdot \binom{4}{2} \cdot \binom{4}{1}$ . Final answer:  $13 \cdot 12 \cdot 11 \cdot \binom{4}{3} \cdot \binom{4}{2} \cdot \binom{4}{1}$ .

- d. Do these two methods give the same answer?

**Answer:**

Yes. Even though we initially pick numbers without order in b., we immediately induce an ordering on them by assigning triple, pair, and single to each of them.

#### 4. Stars and bars

- a. Suppose you have seven stars as below:

\*\*\*\*\*

We want to place two bars between these stars, so as to divide them into three groups. For instance, we can do:

★★|★|★★★★

In this case, Group 1 contains two stars, Group 2 one star, and Group 3 four stars. We are also allowed to have a group that contains no stars, e.g., the following example places all seven stars in Group 3:

||\*\*\*\*\*

How many ways are there to divide seven stars into three groups?

**Answer:** There are total of nine objects and we need to choose where the two bars will be, so the number of possibilities is  $\binom{9}{2}$ .

- b. How many solutions does

$$x_0 + x_1 + x_2 = 10$$

have, if all  $x$ 's must be *positive* integers?

**Answer:** This is equivalent to asking how many ways there are to divide 10 stars into 3 groups, where each group must contain at least one star. So we begin by putting one star in each group, then the problem reduces to counting the number of ways to put 7 stars in 3 groups. The answer is still  $\binom{9}{2}$ .