1. **Triangle Inequality**

Recall the triangle inequality, which states that for real numbers x_1 and x_2,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

Use induction to prove the generalized triangle inequality:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Answer: We use induction on $n \geq 2$. The base case $n = 2$ is the usual triangle inequality. Assume the inequality holds for some $n \geq 2$ (this is the inductive hypothesis). For $n+1$, we can write:

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &\leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}| && \text{(by the usual triangle inequality)} \\ &\leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}| && \text{(by the induction hypothesis).} \end{aligned}$$

This completes the induction.

2. **Power Inequality**

Use induction to prove that for all integers $n \geq 1$, $2^n + 3^n \leq 5^n$.

Answer: We use induction on n. The base case $n = 1$ is true because $2 + 3 = 5$. Assume the inequality holds for some $n \geq 1$. For $n+1$, we can write:

$$2^{n+1} + 3^{n+1} = 2 \cdot 2^n + 3 \cdot 3^n < 3 \cdot 2^n + 3 \cdot 3^n = 3(2^n + 3^n) \overset{(*)}{\leq} 3 \cdot 5^n < 5 \cdot 5^n = 5^{n+1},$$

where the inequality (*) follows from the induction hypothesis. This completes the induction.

3. **Convergence of Series**

Use induction to prove that for all integers $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2.$$

Hint: Strengthen the induction hypothesis to $\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2 - \frac{1}{\sqrt{n}}$.

Answer: We use induction on n. The base case $n = 1$ is true because $1/3 < 1$. Assume the inequality holds for some $n \geq 1$. For $n+1$, by the inductive hypothesis, we have that

$$\sum_{k=1}^{n+1} \frac{1}{3k^{3/2}} = \sum_{k=1}^{n} \frac{1}{3k^{3/2}} + \frac{1}{3(n+1)^{3/2}} \leq 2 - \frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}}.$$

Thus, to prove our claim, it suffices to show that

$$-\frac{1}{\sqrt{n}}+\frac{1}{3(n+1)^{3/2}} \leq -\frac{1}{\sqrt{n+1}}. \tag{1}$$

This is a purely arithmetic problem and there are multiple ways to proceed.

Notice that to prove the inequality (1), it suffices to show that

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \geq \frac{1}{3(n+1)^{3/2}} = \frac{1}{3(n+1)\sqrt{n+1}},$$

which is equivalent to showing that

$$\frac{\sqrt{n+1}}{\sqrt{n}} - 1 = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} \geq \frac{1}{3(n+1)}.$$

So we want to show

$$\frac{\sqrt{n+1}}{\sqrt{n}} \geq \frac{1}{3(n+1)} + 1 = \frac{3n+4}{3n+3},$$

and squaring both sides means this is equivalent to

$$\frac{n+1}{n} \geq \frac{(3n+4)^2}{(3n+3)^2}.$$

At this point we cross-multiply, so we just need to show that

$$(n+1)(3n+3)^2 \geq n(3n+4)^2.$$

This is something that can be easily seen by expanding both sides and canceling terms, so we have shown Equation (1).

This computation allows us to conclude that

$$\sum_{k=1}^{n+1} \frac{1}{3k^{3/2}} = \sum_{k=1}^{n} \frac{1}{3k^{3/2}} + \frac{1}{3(n+1)^{3/2}} \leq 2 - \frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}} \overset{(1)}{\leq} 2 - \frac{1}{\sqrt{n+1}},$$

where we have used equation (1) for the last inequality. This concludes the induction.

4. Grid Induction

A bug lives on the grid $\mathbb{N}^2$. He starts at some location $(i, j) \in \mathbb{N}^2$, and every second he does one of the following (if possible):

(i) Jump one inch down to $(i, j-1)$, as long as $(i, j-1) \in \mathbb{N}^2$.

(ii) Jump one inch left to $(i-1, j)$, as long as $(i-1, j) \in \mathbb{N}^2$.

For example, if the bug is at $(5,0)$, then his only option is to jump left to $(4,0)$. Prove that no matter where the bug starts and how the bug jumps, he will always reach $(0,0)$ in finite time.

Answer: We prove a slightly stronger statement that if the bug starts at $(i, j) \in \mathbb{N}^2$, then he always reaches $(0,0)$ in $n = i + j$ seconds. We use induction on n. The base case $n = 0$ is trivial. Assume the statement holds for some $n \in \mathbb{N}$, and suppose the bug starts at some $(i, j) \in \mathbb{N}^2$ with $i + j = n + 1$. After one second, the bug will either be at $(i, j-1)$ or $(i-1, j)$. In either case, since $i + j - 1 = n$, the induction hypothesis tells us that the bug will reach $(0,0)$ after n additional seconds. Thus, the bug starting from (i, j) will reach $(0,0)$ in $1 + n = i + j$ seconds.

Supplemental Problems

5. Divergence of Harmonic Series

You may have seen the *harmonic series* $1+\frac{1}{2}+\frac{1}{3}+\cdots$ in calculus. We will prove that the harmonic series diverges, i.e., the sum approaches infinity.

Let $H_j = \sum_{k=1}^{j} \frac{1}{k}$. Use induction to show that for all integers $n \geq 0$, $H_{2^n} \geq 1+\frac{n}{2}$, thus showing that H_j must grow unboundedly as $j \to \infty$.

Answer: We use induction on n. The base case $n=0$ is true because $H_1 = 1$. Assume the statement holds for some $n \in \mathbb{N}$. That is, our inductive hypothesis is that $H_{2^n} \geq 1+\frac{n}{2}$. For $n+1$, we can group the terms in $H_{2^{n+1}}$ into two parts:

$$H_{2^{n+1}} = \underbrace{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n}}_{=H_{2^n} \geq 1+\frac{n}{2}} + \underbrace{\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^{n+1}}}_{\geq \frac{1}{2^{n+1}}+\frac{1}{2^{n+1}}+\cdots+\frac{1}{2^{n+1}} = \frac{2^n}{2^{n+1}}} \geq \left(1+\frac{n}{2}\right)+\frac{1}{2} = 1+\frac{n+1}{2}.$$

Here, the first 2^n terms of $H_{2^{n+1}}$ was simply a copy of H_{2^n}, and we applied the inductive hypothesis on these terms. For each of the final 2^n terms of $H_{2^{n+1}}$, we observed that each term was greater than $\frac{1}{2^{n+1}}$; the total of these 2^n terms is therefore greater than $\frac{2^n}{2^{n+1}} = \frac{1}{2}$.

This concludes the inductive step and proves the desired statement via induction.

6. Fibonacci Expansion

The *Fibonacci numbers* are defined recursively by $F_1 = F_2 = 1$, and $F_k = F_{k-1}+F_{k-2}$ for $k \geq 3$.

Prove that every positive integer n has a binary expansion in the Fibonacci basis that does not use two consecutive Fibonacci numbers, i.e., we can write:

$$n = c_k \cdot F_k + c_{k-1} \cdot F_{k-1} + \cdots + c_2 \cdot F_2 + c_1 \cdot F_1$$

for some $k \in \mathbb{N}$ and $c_1, \ldots, c_k \in \{0,1\}$ with the property that $c_i \cdot c_{i+1} = 0$ for all $1 \leq i \leq k-1$.

For example, we could write $6 = F_1+F_3+F_4$, but this uses consecutive Fibonacci numbers. We can write it instead as $6 = F_1+F_5$, which satisfies the desired property.

Answer: We use *strong* induction on n. For the base case $n=1$, we can write $1 = F_1$ (or $1 = F_2$). Assume the statement holds for all integers between 1 and n. For $n+1$, let k denote the largest integer such that $F_k \leq n+1$; note that we necessarily have $k \geq 2$. If $n+1 = F_k$, then we are done. Otherwise, let $n' = n+1-F_k$, so $1 \leq n' \leq n$. Using the inductive hypothesis, we can write

$$n' = c_\ell \cdot F_\ell + c_{\ell-1} \cdot F_{\ell-1} + \cdots + c_2 \cdot F_2 + c_1 \cdot F_1 \tag{2}$$

for some $\ell \in \mathbb{N}$ and $c_1, \ldots, c_\ell \in \{0,1\}$ with the property that $c_i \cdot c_{i+1} = 0$ for all $1 \leq i \leq \ell-1$. Without loss of generality we may also assume $c_\ell = 1$, for otherwise we can remove the ℓ-th term from (2). Recalling our definition of n', we can now write

$$n+1 \;=\; F_k + n' \;=\; \underbrace{1 \cdot F_k}_{c_k=1} + \underbrace{1 \cdot F_\ell}_{c_\ell=1} + c_{\ell-1} \cdot F_{\ell-1} + \cdots + c_2 \cdot F_2 + c_1 \cdot F_1. \tag{3}$$

To finish the proof, we claim that $\ell \leq k-2$. Suppose the contrary that $\ell \geq k-1$; then from (3) we get

$$n+1 \geq F_k + F_\ell \geq F_k + F_{k-1} = F_{k+1},$$

contradicting our choice of k as the largest integer satisfying $F_k \leq n+1$. This completes the inductive step.