CS70–Spring 2013 — Homework 10

Felix He, SID 303308****

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Collaborators: None

1. Using Chebyshev's inequality

1. The variance of binomial distribution is $\sigma^2 = np(1-p)$. So according to Chebyshev's inequality we have

$$P[|X - \mu| \geq c] \leq \frac{Var(X)}{c^2}$$
$$P[|X - \frac{n}{2}| \geq \sqrt{n}] \leq \frac{np(1-p)}{(\sqrt{n}^2)} \leq \frac{1}{4} \quad (p = \frac{1}{2})$$

   Same inequality applies for $5\sqrt{n}$

2. Similarly

$$P[|X \geq 11] = P[X - 1 \geq 10] \leq P[|X - 1| \geq 10] \leq \frac{1}{10^2} = \frac{1}{100}$$

Estimating pi

1. To estimate pi, we actually estimate $\frac{\text{Area of Circle}}{\text{Area of Square}}$, which is $\frac{\pi r^2}{1} = \frac{\pi}{4}$.

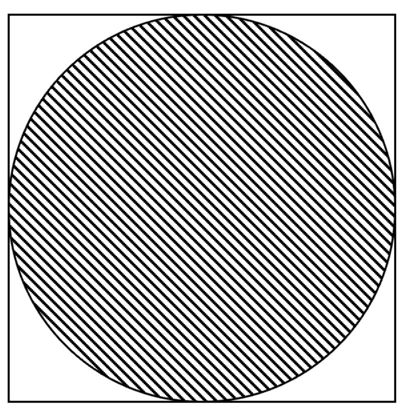

Figure 1: estimating π

The code is as below, we can see the value of $\pi \approx 3.142$

Listing 1: Monte Carlo Estimation of π

```
import numpy as np

# Generating random points within the unit square (0,0) to (1,1)
np.random.seed(42)  # For reproducibility
total = 20000000
points = np.random.rand(total, 2)  # points, each with two
   coordinates (x, y)

count = 0
for i in points:
    if ((i[0] -  0.5)**2 + (i[1] -  0.5)**2 <= 0.5**2):
        count += 1
ratio = count / total
print(ratio * 4)
```

3. True/false

1. False.
   Suppose $X = -1$ or 1 with equal probability $\frac{1}{2}$. Let Y be X^2, which is always 1. Y is entirely determined by X. However, $E[X] = 0$, $E[Y] = 1$ $E[XY] = E[X \times X^2] = E[X^3] = 0$, $E[XY] = E[X]E[Y] = 0$. But X and Y are not independent.

2. True

$$
\begin{aligned}
E[XY] &= \sum_a \sum_b ab \times P[X = a, Y = b] \\
&= \sum_a \sum_b ab \times P[X = a] \times P[Y = b] \\
&= \left(\sum_a \times P[X = a] \right) \times \left(\sum_b \times P[X = b] \right) \\
&= E[X] \times E[Y]
\end{aligned}
$$

3. False
   Let $X = -1$ or 1 with equal probability $\frac{1}{2}$, $Y = X$, $E[XY] = E[X^2] = 1 \neq E[X]E[Y] = 0$

4. False.
   Say $p = 3$, then $X = 0, 1, 2$ each with probability of $\frac{1}{3}$. $Y = 0, 1, 2$ each with probability of $\frac{1}{3}$. And we can see $S = X + Y = 0, 1, 2; 1, 2, 0; 2, 0, 1$ each with probability of $\frac{1}{9}$. So $E[S] = 1 \neq E[X] + E[Y] = 2$

4. Those 3407 Votes

1.

$$E[B] = np = 432286 * 0.03 = 1296.86$$
$$Var[B] = np * (1-p) = 1292.97$$

2.

$$P[B \geq 3407] = P[B - 2110 \geq 1480] \leq P[|B - 2110| \geq 2110] \leq \frac{Var[B]}{2110^2} = 0.00029$$

The Buchanan's vote is not significant.

3. We can use Union bound or direct calculation for that

$$P[\text{At least 1 county} \geq 3407] = 1 - P[\text{No counties} \geq 3407] = 1 - (1-b)^6 7 \approx 0.02$$

It's much greater than 0.00029.

4. The bound would increase.

$$P[B - \mu \geq c] \leq P[|B - \mu| \geq c] \leq \frac{Var[B]}{c^2}$$

The overall p would increase in this case, since for the 80% $P[B] = 0.003$, for the 20% the $P[B] = \frac{1}{6}$. $E[B] = np$ would increase, $Var[B] = np(1-p)$ would increase. and c in the above inequality would decrease. So the overall bound would increase.

5. Parameter Inference

Part 1 Geometric distribution

1.

$$P[X_i] = (1-p)^{x_i - 1} p$$

2.

$$P[p] = \prod_{i=1}^{m} (1-p)^{x_i - 1} p$$

3. To maximize $P[p]$, we can maximize $\log P[p]$.
To prove as a lemma that x maximizes $f(x)$ if and only if x maximizes $\log f(x)$, we can simply prove by taking first and second derivative of $\log f(x)$ to show the bigger $f(x)$, the bigger $\log f(x)$.
Likelihood Function:

$$P[p] = \prod_{i=1}^{m} (1-p)^{x_i - 1} \cdot p$$

Log-Likelihood Function:

$$\log P[p] = \sum_{i=1}^{m} ((x_i - 1)\log(1-p) + \log p)$$

Derivative of Log-Likelihood:

$$\frac{\partial}{\partial p} \log P[p] = \sum_{i=1}^{m} \left(\frac{x_i - 1}{1-p} - \frac{1}{p} \right)$$

Setting the Derivative to Zero:

$$\begin{aligned}
0 &= -\sum_{i=1}^{m} x_i \frac{1}{1-p} + m\frac{1}{1-p} + m\frac{1}{p} \\
0 &= -p\sum_{i=1}^{m} x_i + pm + (1-p)m \\
p &= \frac{m}{\sum_{i=1}^{m} x_i}
\end{aligned}$$

4. It intuitively makes sense because it's the inverse of average number of trials to achieve success. Actually if you have m to be very large, which means many experiments, then p to maximize your observation is in fact the inherent p of the geometric distribution.

Part 2 Binomial distribution

1. $$P[x_i] = C_n^{x_i}(1-p)^{n-x_i}p^{x_i}$$

2. $$P = \prod_{i=1}^{m} C_n^{x_i}(1-p)^{n-x_i}p^{x_i}$$

3. $$\begin{aligned}
\log P &= \sum_{i=1}^{m} \log C_n^{x_i} + (n-x_i)\log(1-p) + x_i \log p \\
&= \sum_{i=1}^{m} \log C_n^{x_i} + \sum_{i=1}^{m}(n-x_i)\log(1-p) + \sum_{i=1}^{m} x_i \log p
\end{aligned}$$

   Take derivative of $\log P$ and make it to 0. We have:

$$\begin{aligned}
\frac{d\log P}{dp} &= 0 \\
-\frac{\sum_{i=1}^{m}(n-x_i)}{1-p} + \frac{\sum_{i=1}^{m} x_i}{p} &= 0 \\
p &= \frac{\sum_{i=1}^{m} x_i}{mn}
\end{aligned}$$

4. So it intuitively makes sense. $p = \frac{\text{\# total successful trials}}{\text{\# total trials}}$

6. Best Question NA

1.

$$\begin{aligned} E[IE] &= ((C_0+0.25)\times 2.5\times(A_0+70)) + ((1-C_0-0.25)\times(A_0+70)) \\ &= (1.5A_0C_0 + 1.375A_0 + 105C_0 + 96.25)S_0 \end{aligned}$$

2.

$$\begin{aligned} E[BT] &= C_0(A_0+100)\times 2 + (1-C_0)(A_0+100) \\ &= (A_0C_0 + A_0 + 100C_0 + 100)S_0 \end{aligned}$$

3. Take $C = 25 + A_0$

$$\begin{aligned} h_0 &= H'_o \\ h_1 &= 0.95h_0 - C \\ h_n &= 0.95h_n - C \end{aligned}$$

Then we can get

$$h_n = 0.95^n H'_0 - 20C(1-0.95^n)$$

We solve for $h_n = 0$ to get n.

$$n = \frac{1}{\log 0.95}\log(\frac{20(25+A_0)}{H'_0 + 20(25+A_0)})$$

Then the DPS

$$\begin{aligned} DPS[BoRK] &= \frac{H'_0}{n}(S_0+0.4) \\ &= \frac{H'_0}{\frac{1}{\log 0.95}\log(\frac{20(25+A_0)}{H'_0+20(25+A_0)})}(S_0+0.4) \end{aligned}$$

4.

$$A = \begin{cases} DPS[BoRK] & \text{with probability } (1-C_0) \\ 2DPS[BoRK] & \text{with probability } C_0 \end{cases}$$

So

$$\begin{aligned} E[BoRk] &= DPS[BoRK]\times(1-C_0) + 2DPS[BoRK]\times C_0 \\ &= (1+C_0)DPS[BoRK] = (1+C_0)\frac{H'_0}{\frac{1}{\log 0.95}\log(\frac{20(25+A_0)}{H'_0+20(25+A_0)})}(S_0+0.4) \end{aligned}$$

$$\begin{aligned} E[IE] &= (1.5A_0C_0 + 1.375A_0 + 105C_0 + 96.25)S_0 \\ E[BT] &= (A_0C_0 + A_0 + 100C_0 + 100)S_0 \\ E[BoRk] &= (1+C_0)\frac{H'_0}{\frac{1}{\log 0.95}\log(\frac{20(25+A_0)}{H'_0+20(25+A_0)})}(S_0+0.4) \end{aligned}$$

5. 1. When H_0' is relatively small and S_0 relatively large, then BoRk has no much advantage. IE and BT are similar, high A_0' is good for IE. So IE is optimal if $A_0 = 100, C_0 = 0, H_0' = 100, S_0 = 1$. (233.75 200 183.09)
2. Similarly, when A_0 is low, and C_0 is low, it's good for BT. So if $A_0 = 0, H_0' = 100, C_0 = 0, S_0 = 1$, BT is optimal. (96.25 100 39.392)
3. So we know if H_0' is relatively high and S_0 relatively small, then BoRK has advantage. So if $A_0 = 0, H_0' = 1000, C_0 = 0, S_0 = 0.1$, optimal for BoRk. (9.625 10.0 23.34)

Below is the pseudocode.

```python
import math

# Variables
A0 = 0 # Example value for base attack damage
C0 = 0  # Example value for critical strike chance
S0 = 0.1  # Example value for attack speed
H0_prime = 1000  # Example value for enemy health

# Calculation for Infinity Edge
E_IE = (1.5 * A0 * C0 + 1.375 * A0 + 105 * C0 + 96.25) * S0
print("Expected DPS with Infinity Edge:", E_IE)

# Calculation for Bloodthirster
E_BT = (A0 * C0 + A0 + 100 * C0 + 100) * S0
print("Expected DPS with Bloodthirster:", E_BT)

# Calculating the logarithmic term
log_base = math.log(0.95, 10)
log_ratio = math.log((20 * (25 + A0)) / (H0_prime + 20 * (25
    + A0)), 10)

# DPS calculation for Blade of the Ruined King
inverse_log_scale = 1 / log_base
E_BoRK = (1 + C0) * (H0_prime / (inverse_log_scale *
   log_ratio)) * (S0 + 0.4)

print("Expected DPS with Blade of the Ruined King:", E_BoRK)
```