CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani Discussion 2M

1. Distributing the quantifiers

Let $P(x)$ and $Q(x)$ denote some propositions involving x. For each statement below, either prove that the statement is correct or provide a counterexample if it is false.

(a) $\forall x\,(P(x) \vee Q(x))$ is equivalent to $(\forall x, P(x)) \vee (\forall x, Q(x))$.

(b) $\forall x\,(P(x) \wedge Q(x))$ is equivalent to $(\forall x, P(x)) \wedge (\forall x, Q(x))$.

(c) $\exists x\,(P(x) \vee Q(x))$ is equivalent to $(\exists x, P(x)) \vee (\exists x, Q(x))$.

(d) $\exists x\,(P(x) \wedge Q(x))$ is equivalent to $(\exists x, P(x)) \wedge (\exists x, Q(x))$.

Answer:

(a) False. One direction of implication is true, namely (prove this as an exercise)

$$(\forall x, P(x)) \vee (\forall x, Q(x)) \quad \text{implies} \quad \forall x\,(P(x) \vee Q(x)).$$

However, the converse is false, so the two statements are not equivalent. As a counterexample, take the universe to be $\mathbb{R}$, take $P(x)$ to be the statement "$x \geq 0$," and $Q(x)$ to be the statement "$x < 0$." Then $\forall x(P(x) \vee Q(x))$ is true, but $(\forall x, P(x)) \vee (\forall x, Q(x))$ is false.

(b) True. Recall that to prove $A \Leftrightarrow B$, we have to prove both $A \Rightarrow B$ and $B \Rightarrow A$.

Suppose the first statement $\forall x\,(P(x) \wedge Q(x))$ is true. This means for all x, $P(x) \wedge Q(x)$ is true, so $P(x)$ and $Q(x)$ are both true. Thus, the statement $(\forall x,\, P(x))$ is true, and similarly the statement $(\forall x,\, Q(x))$ is true, so the conjunction $(\forall x, P(x)) \wedge (\forall x, Q(x))$ is true.

Conversely, suppose the second statement $(\forall x, P(x)) \wedge (\forall x, Q(x))$ is true. This means for all x, both $P(x)$ is true and $Q(x)$ is true, which implies $P(x) \wedge Q(x)$ is true. Thus, the first statement $\forall x\,(P(x) \wedge Q(x))$ is true. This completes the proof of the equivalence.

(c) True. We can prove the equivalence directly as in part (b). Alternatively, we can use our result in part (b) as follows. We know from part (b) that for any propositions $\tilde{P}(x)$ and $\tilde{Q}(x)$, the following is true:

$$\forall x\,(\tilde{P}(x) \wedge \tilde{Q}(x)) \;\text{ is equivalent to }\; (\forall x, \tilde{P}(x)) \wedge (\forall x, \tilde{Q}(x)).$$

Recall that $A \Leftrightarrow B$ is equivalent to $\neg A \Leftrightarrow \neg B$, so we can write the equivalence above as

$$\neg\big(\forall x\,(\tilde{P}(x) \wedge \tilde{Q}(x))\big) \;\text{ is equivalent to }\; \neg\big((\forall x, \tilde{P}(x)) \wedge (\forall x, \tilde{Q}(x))\big).$$

After simplifying the negations, we arrive at

$$\exists x\,(\neg\tilde{P}(x) \vee \neg\tilde{Q}(x)) \;\text{ is equivalent to }\; (\exists x, \neg\tilde{P}(x)) \vee (\exists x, \neg\tilde{Q}(x)).$$

Finally, let us choose $\tilde{P}(x)$ to be $\neg P(x)$ and $\tilde{Q}(x)$ to be $\neg Q(x)$, so we obtain

$$\exists x\,(P(x) \vee Q(x)) \;\text{ is equivalent to }\; (\exists x, P(x)) \vee (\exists x, Q(x)),$$

which is what we want to prove.

(d) False. One direction of implication is true, namely (prove this as an exercise)

$$\exists x\,(P(x) \wedge Q(x)) \quad \text{implies} \quad (\exists x, P(x)) \wedge (\exists x, Q(x)).$$

However, the converse is false. Take the same counterexample as in part (a). Then $(\exists x, P(x)) \wedge (\exists x, Q(x))$ is true but $\exists x\,(P(x) \wedge Q(x))$ is false.

2. Pigeonhole Principle

Prove that if you put $n+1$ apples into n boxes in any way you like, then at least one box must contain at least 2 apples.

Answer: Suppose the contrary that all boxes contain at most 1 apple. Then there are at most n apples in the n boxes, contradicting our assumption that there are $n+1$ apples in total.

3. Proof by contraposition

Let x be a positive real number. Prove that if x is irrational (i.e., not a rational number), then $\sqrt{x}$ is also irrational.

Answer: We prove the contrapositive. Assume $\sqrt{x}$ is rational, so we can write $\sqrt{x} = m/n$ for some integers m, n. Then $x = m^2/n^2$, which is also a rational number.

4. Proof by cases

A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Answer: Let $n = m^2$ for some integer m. Since n is odd, m is odd, so it is of the form $m = 4\ell+1$ or $m = 4\ell+3$ for some integer ℓ. We consider both cases:

1. If $m = 4\ell+1$ for some integer ℓ, then $m^2 = 16\ell^2 + 8\ell + 1 = 8(2\ell^2 + \ell) + 1$.
2. If $m = 4\ell+3$ for some integer ℓ, then $m^2 = 16\ell^2 + 24\ell + 9 = 8(2\ell^2 + 3\ell + 1) + 1$.

In all cases, we have $m^2 = 8k+1$ for some integer k.

5. Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.

Answer: Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.