CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani Discussion 4M

1. **Build-up error**

   What is wrong with the following "proof"?

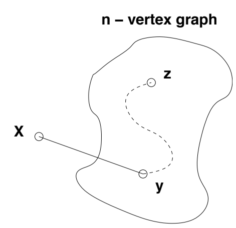
   False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

   Proof: We use induction on the number of vertices $n \geq 1$.

   Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

   Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

   Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



   All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$. $\square$

2. **Odd degree vertices**

   Claim: Let $G = (V,E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

   Prove the claim above using:

(i) Induction on $m = |E|$ (number of edges)

(ii) Induction on $n = |V|$ (number of vertices)

(iii) Well-ordering principle

(iv) Direct proof (e.g., counting the number of edges in G)

3. Minimum connectivity

Suppose you have n nodes, and you want to put edges between them to make the resulting graph connected. What is the minimum number of edges that you need? In this problem, we show that the answer is $n-1$. (*Note:* The case when G has $n-1$ edges is called a *tree*, which is a minimally connected graph on n vertices. Trees have many useful properties that we will explore further in tomorrow's lecture.)

Prove that if G is a connected graph on n vertices, then G has at least $n-1$ edges.