CS70–Spring 2013 — Homework 3

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Collaborators: None

1. Simple recurrence relations

Induction is not provided here. Basically, you can check the base case where $n = 1$, then assume $n = k$ holds true for the expressions, then use the relation between $k + 1$ and k, and the expression for $n = k$ to check the expression hold true for $n = k + 1$

1. $a_n = 2^{n-1}$
2. $b_n = 2^{n+1} - 1$
3. $c_n = 2$
4. $d_n = 3^{n-1} \times 2 + (\frac{3^{n-1}-1}{2} \times 4)$
5. $e_n = 23^{n-1} \times 100 + (\frac{23^{n-1}-1}{2} \times 13)$ (similar to d_n)

2. Make it stronger

1. if the induction hypothesis is $a_k \leq 3^{2^k}$, we have

$$a_{k+1} = 3 \times a_k^2 \leq 3 \times (3^{2^k})^2 = 3^{2^{k+1}+1}$$

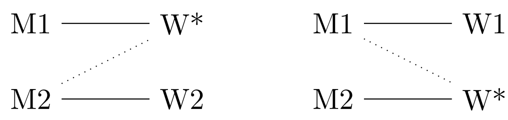
   we can only get $a_{k+1} \leq 3^{2^{k+1}+1}$ but not $a_{k+1} \leq 3^{2^{k+1}}$

2. base case: $a_1 = 1 \leq 3^2$
   IH: $a_k \leq 3^{2^k-1}$
   IS: $a_{k+1} = 3 \times a_k^2 \leq 3 \times (3^{2^k-1})^2 = 3^{2^{k+1}-1}$, so the statement holds.

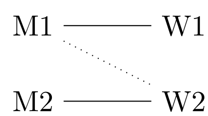
3. One side wins, the other loses

I guess the context omits that the two proposals are stable.

1. proof by contradiction. if both men $M1$ and $M2$ match to the same woman $W*$. It implies that in one proposals, $M1 - -W*$ and in another proposals, $M2 - -W*$. But $W*$ must have a higher preference for either $M1$ or $M2$. It will produce a rogue couple in either of the proposals.



2. proof by contradiction. If it's not stable it means there exists a rogue couple. For example, rogue couple between $M1$ and $W2$. It means the preference of $M1 : W2 > W1$



   And to produce the above results, in two proposals, there are $M1 - W1$ and $M1 - W3$. And from above we know the preference of $M1 : W1 > W3$. Since $W2 > W1$, we can get $W2 > W3$. So in proposal 2, rogue couple between $M1$ and $W2$ exists.

3. proof by contradiction. Suppose in one proposal $P1$ where $(M2, W2), (M1, W)$ and in another proposal $P2$ where $(M2, W), (M1, W3)$, if for $W : M1 > M2$, then we match in the final set is $(W, M1)$, and we know if this is the pair, for $M1 : W > W3$, this would form a rouge couple $(M1, W)$ in $P2$. Same case for $W : M2 > M1$, rouge couple $(M2, W)$ in $P1$

4. **proof by induction**.
   Prove the Matchmaker's scheme is optimal for men and pessimal for women in all possible n pairings.
   base case: when $n = 1$, it's true, we just use this pair as the optimal pairing for men.
   IH: always an optimal pairing exist for $n \leq k$
   IS: $n = k + 1$, we know $n = k$, we got a optimal pairing from it. then we can use Mr. Matchmaker's scheme to produce an optimal scheme for men from the previous optimal pairing for $n = k$ and the extra one pairing. And it's pessimal for women as we proved in part3.

4. I'm too good to marry

1. for n men and n women, we insert n 'single' mates for men and n 'single' mate for women. And the 'single' mate always have the corresponding man or woman as their highest preference. X_n represents M_n's preference for being single, Y_n represents W_n's preference for being single.

$$
\begin{aligned}
&M_1 : W_1, W_2, \ldots, X_1, \ldots, W_n, X_2, \ldots \\
&M_2 : W_1, W_2, \ldots, X_2, \ldots, W_n, X_1, \ldots \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
&X_1 : M_1, \ldots \\
&X_2 : M_2, \ldots \\
&\vdots \\
&X_n : M_n, \ldots
\end{aligned}
$$

$$W_1 : M_1, M_2, \ldots, Y_1, \ldots, M_n, Y_2, \ldots$$
$$W_2 : M_1, M_2, \ldots, Y_2, \ldots, W_n, Y_1, \ldots$$
$$\vdots$$

$$Y_1 : W_1, \ldots$$
$$Y_2 : W_2, \ldots$$
$$\vdots$$
$$Y_n : W_n, \ldots$$

We can use the proof we learned in TMA (Girl's improvement lemma and contradiction) to prove this is also stable. Since the you can take the 'single' mate as the same with the real mate. We just double the size of the problem from $n * n$ to $2n * 2n$

2. Suppose the in pairing $P1$, $(M1, X1)$, $M1$ is being single. Then for the sake of contradiction, we assume there is another stable pairing $P2$, $(M1, W^*)$ where $M1$ is matched with a woman. Then according to the Mr. MatchMaker, we have $(M1, W^*)$ in $P3$, and we know $M1 : W^* > X1$, otherwise rouge couple in $P2$. Since it's men's optimal, we know in $P3$, the number of single men is decreasing. (only improvement from single to matched. if a man is matched, it cannot be single in another pair, o.w rogue couple in the original pair we proved not possible in part 1).
And since the MatchMaker Scheme is woman pessimal. So we know in $P3$, the number of single woman is at least increasing or remains the same. (cannot improve from single to matched). but the number of single men and single women must remains the same. Therefore it's a contradiction.
Same for woman, a woman can never improve from single to matched (if $W_n : M^* \geq Y_n$) according to the woman pessimal conclusion.

5. Modular decomposition of modular arithmetic

1. As follow

	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Table 1: Addition

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	0	3	0
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Table 2: Multiply

2. appears exactly once in this order $(0, 0), (1, 1), (0, 2), (1, 0), (0, 1), (1, 2)$

3. if $a \equiv c\,(mod\,m)$ and $b \equiv d\,(mod\,m)$, then $a+b \equiv c+d\,(mod\,m)$ and $a \times b \equiv c \times d\,(mod\,m)$.

+	(0;0)	(1;1)	(0;2)	(1;0)	(0;1)	(1;2)
(0;0)	(0;0)	(1;1)	(0;2)	(1;0)	(0;1)	(1;2)
(1;1)	(1;1)	(0;2)	(1;0)	(0;1)	(1;2)	(0;0)
(0;2)	(0;2)	(1;0)	(0;1)	(1;2)	(0;0)	(1;1)
(1;0)	(1;0)	(0;1)	(1;2)	(0;0)	(1;1)	(0;2)
(0;1)	(0;1)	(1;2)	(0;0)	(1;1)	(0;2)	(1;0)
(1;2)	(1;2)	(0;0)	(1;1)	(0;2)	(1;0)	(0;1)

Table 3: PairAddition

×	(0;0)	(1;1)	(0;2)	(1;0)	(0;1)	(1;2)
(0;0)	(0;0)	(0;0)	(0;0)	(0;0)	(0;0)	(0;0)
(1;1)	(0;0)	(1;1)	(0;2)	(1;0)	(0;1)	(1;2)
(0;2)	(0;0)	(0;2)	(0;1)	(0;0)	(0;2)	(0;1)
(1;0)	(0;0)	(1;0)	(0;0)	(1;0)	(0;0)	(1;0)
(0;1)	(0;0)	(0;1)	(0;2)	(0;0)	(0;1)	(0;2)
(1;2)	(0;0)	(1;2)	(0;1)	(1;0)	(0;2)	(1;1)

Table 4: PairMultiplication

6. Power in modular arithmetic

Use the exponentiantion of the modular arithmetic. If we want to get the final 2 digits, then we need $mod\,100$, $m = 100$.
Use the recursive call $mod - exp(x, y, m)$, we have

n^{th} *call*	x	y	m	return value
1	3	32	100	21^2 mod 100 = 41
2	3	16	100	61^2 mod 100 = 21
3	3	8	100	81^2 mod 100 = 61
4	3	4	100	9^2 mod 100 = 81
5	3	2	100	3^2 mod 100 = 9
6	3	1	100	$3 \cdot 1^2$ mod 100 = 3
7	3	0	100	1

Table 5: PowerArithmetic

The return order is from 7 to 1. So 3^{32} mod $100 \equiv 41$ mod 100