CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani Discussion 1W

1. Truth tables

Use truth tables to show the following identities (note that the first two are known as *De Morgan's Laws*):

1. $\neg(A \vee B) \equiv \neg A \wedge \neg B$.
2. $\neg(A \wedge B) \equiv \neg A \vee \neg B$.
3. $A \iff B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$.
4. $(A \Rightarrow (B \Rightarrow C)) \vee (B \Rightarrow (A \wedge C)) \equiv \neg A \vee \neg B \vee C$.

Answer:

Here is the truth table for part 4. The two columns in bold are the two original statements. Note that both columns have equal entries, which demonstrates the identity.

A	B	C	$\neg\mathbf{A} \vee \neg\mathbf{B} \vee \mathbf{C}$	$B \Rightarrow C$	$A \Rightarrow (B \Rightarrow C)$	$A \wedge C$	$B \Rightarrow (A \wedge C)$	$(\mathbf{A} \Rightarrow (\mathbf{B} \Rightarrow \mathbf{C})) \vee (\mathbf{B} \Rightarrow (\mathbf{A} \wedge \mathbf{C}))$
T	T	T	**T**	T	T	T	T	**T**
T	T	F	**F**	F	F	F	F	**F**
T	F	T	**T**	T	T	T	T	**T**
T	F	F	**T**	T	T	F	T	**T**
F	T	T	**T**	T	T	F	F	**T**
F	T	F	**T**	F	T	F	F	**T**
F	F	T	**T**	T	T	F	T	**T**
F	F	F	**T**	T	T	F	T	**T**

2. Writing in propositional logic

For each of the following sentences, translate the sentence into propositional logic using the notation introduced in class, and write its negation.

1. The square of a nonzero integer is positive.
2. There are no integer solutions to the equation $x^2 - y^2 = 10$.
3. There is one and only one real solution to the equation $x^3 + x + 1 = 0$.
4. For any two distinct real numbers, we can find a rational number in between them.

Answer:

1. We can rephrase the sentence as "if n is a nonzero integer, then $n^2 > 0$," which we can write as

$$\forall n \in \mathbb{Z}, (n \neq 0) \Rightarrow (n^2 > 0).$$

An equivalent way to write this (using the fact that $A \Rightarrow B$ is equivalent to $\neg A \vee B$) is

$$\forall n \in \mathbb{Z}, (n = 0) \vee (n^2 > 0).$$

The latter is easier to negate, and its negation is given by

$$\exists n \in \mathbb{Z}, (n \neq 0) \wedge (n^2 \leq 0).$$

2. We can write the sentence as
$$\forall x, y \in \mathbb{Z},\, x^2 - y^2 \neq 10.$$
The negation is
$$\exists x, y \in \mathbb{Z},\, x^2 - y^2 = 10.$$

3. For simplicity, let $p(x)$ denote the polynomial $p(x) = x^3 + x + 1$. We can rephrase the sentence as "there is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x." In symbols:
$$\exists x \in \mathbb{R},\, (p(x) = 0) \wedge (\forall y \in \mathbb{R},\, (p(y) = 0) \Rightarrow (x = y)).$$
Its negation is given by
$$\forall x \in \mathbb{R},\, (p(x) \neq 0) \vee (\exists y \in \mathbb{R},\, (p(y) = 0) \wedge (x \neq y)).$$

4. We can rephrase the sentence as "if x and y are distinct real numbers, then there is a rational number z between x and y." In symbols:
$$\forall x, y \in \mathbb{R},\, (x \neq y) \Rightarrow (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$
Equivalently,
$$\forall x, y \in \mathbb{R},\, (x = y) \vee (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$
The negation is
$$\exists x, y \in \mathbb{R},\, (x \neq y) \wedge (\forall z \in \mathbb{Q}, ((x \geq z) \vee (y \leq z)) \wedge ((y \geq z) \vee (x \leq z))).$$

3. Implication

Which of the following implications are true? Give a counterexample for each false assertion.

1. $\forall x \forall y\, P(x,y)$ implies $\forall y \forall x\, P(x,y)$.
2. $\exists x \exists y\, P(x,y)$ implies $\exists y \exists x\, P(x,y)$.
3. $\forall x \exists y\, P(x,y)$ implies $\exists y \forall x\, P(x,y)$.
4. $\exists x \forall y\, P(x,y)$ implies $\forall y \exists x\, P(x,y)$.

Answer:

1. True. The first statement "$\forall x \forall y\, P(x,y)$" means for all x and y in our universe, the proposition $P(x,y)$ holds. The second statement "$\forall y \forall x\, P(x,y)$" has the same meaning, so they are in fact equivalent (the implication goes both ways). In general, you can interchange the order of any *consecutive* sequence of $\forall$.
2. True. Both statements mean there exist x and y in our universe that make $P(x,y)$ true, so both statements are equivalent. In general, you can interchange the order of any *consecutive* sequence of $\exists$.
3. False. Take the universe to be $\mathbb{R}$ (or any set with at least 2 elements), and take $P(x,y)$ to be the statement "$x = y$." Then the first statement "$\forall x \exists y\, P(x,y)$" claims for all $x \in \mathbb{R}$ we can find $y \in \mathbb{R}$ such that $x = y$, which is true because we can take y to be x. However, the second statement "$\exists y \forall x\, P(x,y)$" claims there exists $y \in \mathbb{R}$ such that $x = y$ for all $x \in \mathbb{R}$, which is false because a real number y cannot simultaneously be equal to all other real numbers x. Thus, the implication is false.

4. True. Suppose the first statement "$\exists x \forall y\, P(x,y)$" is true, which means there is a special element $x^* \in \mathbb{R}$ such that $P(x^*, y)$ is true for all $y \in \mathbb{R}$. The second statement claims that for all $y \in \mathbb{R}$ we can find an element $x \in \mathbb{R}$ (which may depend on y) such that $P(x,y)$ is true. But from our first statement we know that we can choose the same value $x = x^*$ for all y. We conclude that the implication holds. However, the implication is only one way. In particular, note that part 4 is the converse to part 3, which we have seen is false.

For Problems 4 and 5, see Discussion 2M handout.