CS70–Spring 2013 — Homework 1

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Collaborators: None

1. Getting started

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2. Triksy Profs

Ask either of the professor: which door would the your brother say if I were to ask which door leads to the treasure. Choose the opposite door of the answer

The answer will always point to the wrong door.
case 1. if I ask the one who always tell the truth, then he would tell you the wrong door that his brother who lies would tell you
case 2. if I ask the one who always lie, then he would tell you the opposite door that his brother who tell the truth.
So either case would lead to the wrong door, you just choose the opposite door to the answer then you can get the treasure

The key is you need to ask a question that "ask" both brothers, '$+-$'='$-$', '$-+$'='$-$', this is how you always get the '-'(wrong) door

3.Exclusive OR

Table 1: exclusive or

P	Q	$P \oplus Q$	$(P \wedge \neg Q) \vee (\neg P \wedge Q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

Therefore, $P \oplus Q$ is logically equivalent to $(P \wedge \neg Q) \vee (\neg P \wedge Q)$

4.Implications

(a) True, hypothesis is true, conclusion is also true

(b) True, hypothesis is false, false hypothesis always implies true

(c) false, hypothesis is true, conclusion is false. so the implication is false

(d) True, hypothesis is false, false hypothesis always implies true

5. Chasing chains of consequences

10 not in S means 2×5 not in S, so 5 not in S. Similarly 12 not in S implies 6 not in S, then it implies 3 is not in S. 11 is prime. So the set S does not contains any number that is $2^n \times 3$ or $2^n \times 5$ or $2^n \times 11$.

6. Practice with quantifiers

(a) true, hypothesis is false

(b) true, hypothesis is false

(c) false, $x = 2$

(d) true, x can only be 0 or 1

(e) true, proved by induction

7. Working with quantifiers

(a) for all prime numbers and 0, 1 $P(n)$ holds true. we can see when $P(n)$ is false by taking negation

$$\begin{aligned}
\neg P(n) &= \exists i \in \mathbb{N} \neg(i < n \Rightarrow (\forall j \in \mathbb{N}.(j = n \vee n \neq i \cdot j))) && \text{by definition} \\
&= \exists i \in \mathbb{N}(i < n \wedge \neg(\forall j \in \mathbb{N}.(j = n \vee n \neq i \cdot j))) && \text{Implication negation, } \neg(P \Rightarrow Q) \equiv P \wedge \neg Q \\
&= \exists i \in \mathbb{N}(i < n \wedge (\exists j \in \mathbb{N}.\neg(j = n \vee n \neq i \cdot j))) && \text{De Morgan's laws, } \neg\forall \equiv \exists\neg \\
&= \exists i \in \mathbb{N}(i < n \wedge (\exists j \in \mathbb{N}.(j \neq n \wedge n = i \cdot j))) && \text{Negating the disjunction}
\end{aligned}$$

so $P(n)$ is not true $\iff$ when n can be factorialized by i and j where $i < n$ and $j \neq n$, which means n is not prime. And also 0, 1 also holds true for $P(n)$ here.

(b)

$$\neg\forall i \in \mathbb{N}\, \neg\exists j \in \mathbb{N}\, \exists k \in \mathbb{N} \neg\forall l \in \mathbb{N}\, f(i, j) \neq g(k, l)$$

applying DeMorgan's Law

$$\neg\forall x \equiv \exists x \neg$$
$$\neg\exists x \equiv \forall x \neg$$

we have

$$
\begin{aligned}
&\neg\forall i \in \mathbb{N}\, \neg\exists j \in \mathbb{N}\, \exists k \in \mathbb{N} \neg\forall l \in \mathbb{N}\, f(i,j) \neq g(k,l) \equiv \\
&\exists i \in \mathbb{N}\, \neg(\neg\exists j \in \mathbb{N}\, \exists k \in \mathbb{N} \neg\forall l \in \mathbb{N}\, f(i,j) \neq g(k,l)) \equiv \\
&\exists i \in \mathbb{N}\, \exists j \in \mathbb{N}\, \exists k \in \mathbb{N} \neg\forall l \in \mathbb{N}\, f(i,j) \neq g(k,l) \equiv \\
&\exists i \in \mathbb{N}\, \exists j \in \mathbb{N}\, \exists k \in \mathbb{N} \exists l \in \mathbb{N}\, \neg(f(i,j) \neq g(k,l)) \equiv \\
&\exists i \in \mathbb{N}\, \exists j \in \mathbb{N}\, \exists k \in \mathbb{N} \exists l \in \mathbb{N}\, f(i,j) = g(k,l)
\end{aligned}
$$

8. A few proofs

(a) True. we first prove if n^2 is even, then n is even using contraposition (if n is odd which means $n = 2k+1 (k \in \mathbb{Z}), n^2 = (2k+1)^2 = 4k(k+1)+1$, n^2 is odd).

True. Then use direct proof. Since n is even means $n = 2k(k \in \mathbb{Z})$, then $n^5 = 2(2^4k^5)$. Therefore n^5 is even.

(b) Use direct proof. $\forall n \in \mathbb{N}$, $n(n-1)$ is even since either n is even or odd then $n-1$ must be odd or even. Since $n(n-1)$ is even, an even number add 3 is odd (even + odd = odd).

(c) True. Use contraposition. $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$, then we can write it in this way $\neg Q = x < 10 \wedge y < 10$, then $x + y < 20$, $\neg P$ is True. So we prove the original statement

(d) False. Prove by cases. $\sqrt{2}$ is irrational, $(\sqrt{2})^2 = 2$ which is rational. So the statement is false.

9. You be the grader

F.
The reason is we want to prove when $n^2 \leq n$ is true, $(n+1)^2 \leq n+1$ is true. $(n+1)^2 \leq n+1$ is something we are trying to prove true, we cannot assume it's true, then working backwards. We cannot assume something to be true to prove it true.