CS 70	Discrete Mathematics and Probability Theory	
Spring 2015	Vazirani	Discussion 9W

1. Sanity check!

Suppose you have a biased coin, with outcomes H and T, with the probability of heads $\Pr[H] = \frac{3}{4}$ and the probability of tails $\Pr[T] = \frac{1}{4}$. Suppose you perform an experiment in which you toss the coin 3 times — an outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

a. What is the *sample space* for your experiment?

Answer: $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

b. Which of the following are examples of *events*? Select all that apply.

(i) $\{(H,H,T),(H,H),(T)\}$
(ii) $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
(iii) $\{(T,T,T)\}$
(iv) $\{(T,T,T),(H,H,H)\}$
(v) $\{(T,H,T),(H,H,T)\}$

Answer: (ii), (iii), (iv), (v).

c. What is the probability of the outcome H,H,T?

Answer: $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$

d. What is the probability of the event that your outcome has exactly two heads?

Answer: $\omega \in \{(H,H,T),(H,T,H),(T,H,H)\}$. The probability = $3 \cdot \frac{9}{64} = \frac{27}{64}$.

2. Rain and Wind

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in CS70, you are curious to play around with these numbers. Find the probability that

a. A given day is windy and rainy.

Answer:

Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $\Pr(R|W) = 0.3$, $\Pr(R|W^C) = 0.8$ and $\Pr(W) = 0.2$. Then probability that a given day is both rainy and windy is $\Pr(R \cap W) = \Pr(R|W)\Pr(W) = 0.3 \cdot 0.2 = 0.06$

b. It rains on a given day.

Answer:

Probability that it rains on a given day is $\Pr(R) = \Pr(R|W)\Pr(W) + Pr(R|W^C)Pr(W^C) = 0.3 \cdot 0.2 + 0.8 \cdot 0.8 = 0.7$

c. Exactly one of two days is rainy. (Assume that the two days are independent.)

Answer:

Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the days are independent, $Pr(R_1) = Pr(R_2) = Pr(R)$. The desired probability is $Pr(R_1)Pr(R_2^C) + Pr(R_1^C)Pr(R_2) = 2 * 0.7 * 0.3 = 0.42$

d. A non-rainy day is also non-windy.

Answer:

Probability that a non-rainy day is non-windy is $Pr(W^C|R^C) = \frac{Pr(W^C \cap R^C)}{Pr(R^C)} = \frac{Pr(R^C|W^C)(W^C)}{Pr(R^C)} = \frac{0.2 \cdot 0.8}{0.3} = \frac{8}{15}$

3. Dice

Suppose we roll three fair 6-sided dice. Each one of the $6^3 = 216$ possible outcomes is assumed to be equally likely.

a. What is the probability that we get three distinct numbers?

Answer:

The number of ways that we can get three distinct numbers is $6 \cdot 5 \cdot 4 = 120$. The desired probability is $\frac{120}{216} = \frac{5}{9}$.

b. Given that we got three distinct numbers, find the conditional probability that one of them was a six.

Answer:

Let A be the event that we get three distinct numbers, and B the event that we get at least one six. The probability that we want is $\Pr[B|A] = \frac{\Pr[A \cap B]}{\Pr(A)}$. We know that $\Pr(A) = \frac{120}{216}$, so what is $\Pr(A \cap B)$? The number of ways of choosing three distinct numbers that contain a six is 3 (pick where the six will be) times $5 \cdot 4$ (pick the other two numbers). Therefore, $\Pr(A \cap B) = \frac{3 \cdot 5 \cdot 4}{216} = \frac{60}{216}$ and $\Pr[B|A] = \frac{60}{120} = \frac{1}{2}$.

4. Shooting Range

You and your friend are at a shooting range. You ran out of bullets. Your friend still has two bullets left but magically lost his gun. Somehow you both agree to put the two bullets into your six-chambered revolver in successive order, spin the revolver, and then take turn shooting. Your first shot was a blank. You want your friend to shoot a blank too, should you spin the revolver again before you hand it to your friend?

Answer: No, you shouldn't.

If you don't spin the revolver, the probability that a bullet will be fired in the next round is

$$\Pr[\text{next shot is a bullet}|\text{this shot was blank}] = \frac{\Pr[\text{this shot was a blank and next shot is a bullet}]}{\Pr[\text{this shot was blank}]} = \frac{1/6}{4/6} = \frac{1}{4}.$$

If you spin the revolver, the probability that a bullet will be fired is simply 2/6, or 1/3.

5. Combinatorial proof

Prove the following identity by a combinatorial argument:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Answer:

The left hand side is the number of ways to choose k elements out of n. Looking at this another way, we look at the first element and decide whether we are going to choose it or not. If we choose it, then we need to choose $k-1$ more elements from the remaining $n-1$. If we don't choose it, then we need to choose all our k elements from the remaining $n-1$. We are not double counting, since in one of our cases we chose the first element and in the other, we did not.