# CS70–Spring 2013 — Homework 7

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## 1. Error-Detecting Codes

1. We need $k$ extra messages. Since $n$ points determine 1 unique polynomial of deg $n-1$. We send $n+k$ points, there are at most k errors, so we can always use a subset of $n$ points out of $n+k$ points to construct a polynomial, and every time there are some number of points $e \leq k$ not on the polynomial, we pick $e$ to be the smallest, then we know there are $e$ number of errors there.

2. Use checksum to detect. Say the extra message is $s$, we make it this way.

$$\sum_{i=1}^{n} P(i) + s = 0 \text{ (mod q)}$$

   So if any one of the $n$ points changes, the checksum is not 0, the error must be detected. And if multiple points have error, the error is not detected only when the values they changed offset each other, the probability is low for that.

## 2. Spies

We need $n + 2k = 10 + 2 * 5 = 20$ generals to guarantee we can launch the missiles. First we need $n > 9$, then we pick minimum 10, and since there are at most 5 spies, which means the number of errors $k$ is at most 5. So in order to reconstruct the correct polynomial and get the secret, we can need $n + 2k = 20$ messages.

## 3. Magic!

1. The degree 1 polynomial is

$$y = -2x + 6$$

   So the other 4 messages we sent are 0, $-2 \ mod \ 7 = 5$, $-4 \ mod \ 7 = 3$, $-6 \ mod \ 7 = 1$. The four are 0, 5, 3, 1.

2. Given $n = 2$, $k = 2$, $deg(Q(x)) = n + k - 1 = 3$, $deg(E(x)) = k = 2$, $deg(P(x)) = n - 1 = 1$

$$\begin{aligned} Q(x) &= a_3x^3 + a_2x^2 + a_1x + a0 \\ E(x) &= x^2 + b_1x + b0 \end{aligned}$$

And given

$$Q(x) = P(x)E(x) = r_xE(x)$$

We have the following 6 linear equations:

$$\begin{aligned} a_3 + a_2 + a_1 + a_0 &= 5 + 5b_1 + 5b_0 \\ a_3 + 4a_2 + 2a_1 + a_0 &= 6 + 3b_1 + 5b_0 \\ 6a_3 + 2a_2 + 3a_1 + a_0 &= 1 + 5b_1 + 4b_0 \\ a_4 + 2a_2 + 4a_1 + a_0 &= 0 \\ 6a_4 + 4a_2 + 5a_1 + a_0 &= 5 + b_1 + 3b_0 \\ 6a_4 + a_2 + 6a_1 + a_0 &= 6 + b_1 + 6b_0 \end{aligned}$$

3. solving this linear equations, we got 2 solutions.

   (a)

$$Q(x) = 3x^3 + 2x^2 + 2x + 6E(x) = x^2 + 3P(x) = 3x + 2$$

   (b)

$$Q(x) = 3x^3 + 5x^2 + 5x + 2E(x) = x^2 + x + 1P(x) = 3x + 2$$

   The polynomials are the same for both solutions

4. (a)

$$E(x) = (x - 2)(x - 5)$$

   (b)

$$E(x) = (x - 2)(x - 4)$$

   And we put all the points to the polynomial $P(x)$, we found only $P(2)$ is incorrect. So it means there is only one error, which lies in point 2. And the terms for $E(x)$ can be arbitrary anything. $(x - 5)$ and $(x - 4)$ just happened to be there to make $E(x) = 0$.

## 4. Graphs

1. Without loss of generosity, if there exists a cycle, we pick the start vertex and end vertex $u$ to be in $L$, we know, there are no edges between edges in $L$ or $R$. So every time we goes out from $L$, we must reach $R$, and if we need to go back to the starting vertex $u$ in $L$, we need to go out from $R$, so this procedure takes at least 2 edges. So the length of the cycle finally must be multiple of 2, which is not odd.

2. Since edges only connects vertices between $L$ and $R$. So $\sum_{v \in L} degree(v) = |E|$, the same for $\sum_{v \in R} degree(v) = |E|$. So

$$\sum_{v \in V} degree(v) = 2|E|$$
$$\sum_{v \in L} degree(v) = \sum_{v \in R} degree(v)$$

## 5. Count it

1.

$$\binom{10}{3} = \frac{10 * 9 * 8}{3 * 2} = 120$$

2.

$$10^3 = 1000$$

3.

$$\binom{52}{4} = 270725$$

4.

$$\binom{16}{4} = 1820$$

5.

$$\binom{52}{4} - \binom{48}{4} = 76145$$

6.

$$\binom{4}{2} \times \binom{13}{2} \times \binom{13}{2} = 36504$$

7.

$$13^4 = 28561$$

8.

$$11 \times 4^4 = 2816$$

9. we have 7 letters, the permutations is 7!, and 3 'z', 2 'a' are repeated, so we need to divide by their order, which are 3! and 2!.

$$\frac{7!}{3! \times 2!} = 420$$

10.

$$\binom{10}{4} = 210$$

11. The smallest one has $n-1$ choices to be put in, the second smallest has $n-2$ choices to be put in, all the way up to the second largest has 1 choice to put in.

$$(n-1)!$$

## 6. Prove it

1. Think of it in this way. We have $n$ eggs, we want to choose $k$ eggs to cook, and within these $k$ eggs, we want $i$ of them to be boiled and the rest of them to fried. So the left $\binom{n}{i}\binom{n-i}{k-i}$ is to first pick $i$ eggs to be boiled, them from the rest $n-i$ we pick $k-i$ to be fried. The right $\binom{k}{i}\binom{n}{k}$ is that we first pick $k$ eggs to cook from $n$, then we decide which $i$ of them to be boiled

2. Say we have $m$ undergrads and $p$ graduates, we want to choose $n$ people from them to form a committee. The left side is saying we don't care about how these $n$ students are constructed (what proportion from undergrads or what proportion from grads).
The right side is saying, we pick these members separately from undergrads and grads, so let $k$ be the number of students in the committee from undergrads, then there are $m+1$ possible of proportion constructions, $k \in [0, m]$, and for each value $k$, there are $\binom{m}{k}\binom{p}{n-k}$ ways of choosing the members, and we sum all the $k$ up from 0 to $m$, we will get the same thing with the left.

## 7. Fibonacci and combinatorics

Use induction to prove.
Base case: $n = 0$, $F(0) = \binom{0}{0} = 1$; $F(1) = \binom{1}{0} = \sum_{k=0}^{1} \binom{1}{0} = \binom{1}{0} + \binom{0}{1} = 1 + 0 = 1$.
Induction hypothesis: we use strong induction we assume $F(n) = \sum_{k=0}^{n} \binom{n-k}{k}$ holds for all 0 to $n-1$.
Induction step:

$$\begin{aligned}
&F(n) = F(n-1) = F(n-2) \\
&F(n) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} + \sum_{k=0}^{n-2} \binom{n-2-k}{k} \\
&F(n) = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \ldots + \binom{m+1}{m} + \binom{m}{m+1} \\
&\qquad \binom{n-2}{0} + \binom{n-3}{1} + \ldots + \binom{m+1}{m-1} + \binom{m}{m}
\end{aligned}$$

And using the given property

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

and $\binom{n}{0} = 1, \forall n \in \mathbb{N}$, so that $\binom{n-1}{0} = \binom{n}{0} = 1$ We have:

$$
\begin{aligned}
F(n) &= \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \ldots + \binom{m+1}{m} + \binom{m}{m+1} \\
&\binom{n-2}{0} + \binom{n-3}{1} + \ldots + \binom{m+1}{m-1} + \binom{m}{m} \\
&= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots + \binom{m}{m} \\
&= \sum_{k=0}^{n} \binom{n-k}{k}
\end{aligned}
$$