EECS 70 Discrete Mathematics and Probability Theory
Spring 2013 Anant Sahai

HW 2

Due Feb 4

Administrative details of preparing and submitting homework submissions: In CS 70 this semester, you will be submitting all homework solutions online. (You might recognize the submission system from CS 61ABC.) Use your instructional account and follow the same approach you were instructed to use for Homework 1. See your GSI or Piazza for instructions on how to do this.

You are welcome to form small groups (up to four people) to work through the homework, but you **must** write up all your solutions on your own.

Although your final submission must be neat (and so typesetting with LaTeXis not a bad option), you are strongly encouraged **not** to directly try to solve the problems in any sort of typesetting environment. Use paper and pencil and scratch paper. Only typeset your solutions after you already pretty-much know exactly what you want to write. Otherwise, you risk wasting a lot of time.

1. **(2 pts.) Proof by induction**
For $n \in \mathbb{N}$ with $n \geq 2$, define s_n by

$$s_n = (1 - \frac{1}{2}) \times (1 - \frac{1}{3}) \times \cdots \times (1 - \frac{1}{n}).$$

Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

2. **(2 pts.) Another induction proof**
Let $a_n = 3^{n+2} + 4^{2n+1}$. Prove that 13 divides a_n for every $n \in \mathbb{N}$.

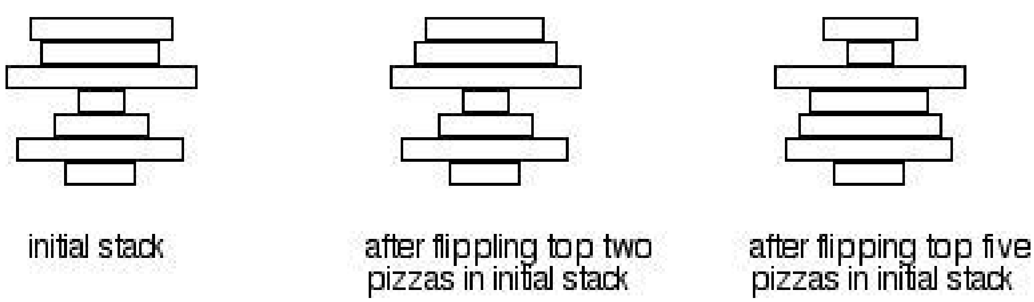
(Hint: What can you say about $a_{n+1} - 3a_n$?)

3. **(2 pts.) Tower of Brahma**
This puzzle was invented by the French mathematician, Edouard Lucas, in 1883. Accompanying the puzzle is a story:

In the great temple at Benares beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at creation, God placed sixty-four disks of pure gold, the largest disk resting on the brass plate and the others getting smaller and smaller up to the top one. This is the Tower of Brahma. Day and Night unceasingly, the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk below it. When all the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple and priests alike will crumble into dust, and with a thunderclap the world will vanish.

Prove by induction the exact number of moves required to carry out this task if we assume that the priests are trying to hasten the end of the world. If there are n disks on the original needle. Assuming that the priests can move a disk each second, roughly how many centuries does the prophecy predict before the destruction of the World?

4. **(2 pts.) The proof of the π is in the eating**
Philip J. Fry has a job at the local pizza parlor, where he tends to be a bit distractable. One day, he has a stack of unbaked pizza doughs and for some unknown reason, he decides to arrange them in order of size, with the largest pizza on the bottom, the next largest pizza just above that, and so on. He has learned how to place his spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their order). The figure below shows two sample flips.



This is the only move Fry can do to change the order of the stack; however, he is willing to keep repeating this move until he gets the stack in order. Is it always possible for him to get the pizzas in order via some sequence of moves, no matter how many pizzas he starts with and what order they are originally in? Prove your answer.

5. (2 pts.) Boring Birds choosing the base case is tricky?

In a casual game that didn't quite sweep the nation, there is a bucket that contains some number of yellow birds, red birds, and black birds. When it is a player's turn, the player may either: (i) slingshot away one yellow bird from the bucket, and add up to 3 red birds into the bucket; (ii) slingshot away two red birds from the bucket, and add up to 7 black birds into the bucket; or, (iii) explode a single black bird in the bucket. These are the only legal moves. The last player that can make a legal move wins.

Prove by induction that, if the bucket initially contains a finite number of birds at the start of the game, then the game will end after a finite number of moves.

6. (8 pts.) You be the grader

Assign a grade of A (correct) or F (failure) to the following proofs. If you give a F, please explain clearly where the logical error in the proof lies. Saying that the claim is false is *not* a valid explanation of what is wrong with the proof. If you give an A, you do not need to explain your grade.

1. **Claim:** For every $n \in \mathbb{N}$, $n^2 + 3n$ is odd.

   Proof: The proof will be by induction on n.
   Base case: The number $n = 1$ is odd.
   Induction step: Suppose $k \in \mathbb{N}$ and $k^2 + 3k$ is odd. Then,

   $$(k+1)^2 + 3(k+1) = (k^2 + 2k + 1) + (3k + 3) = (k^2 + 3k) + (2k + 4)$$

   is the sum of an odd and an even integer. Therefore, $(k+1)^2 + 3(k+1)$ is odd. Therefore, by the principle of mathematical induction, $n^2 + 3n$ is odd for all natural numbers n. $\Box$

2. **Claim:** For every real number x, if x is irrational, then $2008x$ is irrational.

   Proof: Suppose $2008x$ is rational. Then $2008x = p/q$ for some integers p, q with $q \neq 0$. Therefore $x = p/(2008q)$ where p and $2008q$ are integers with $2008q \neq 0$, so x is rational. Therefore, if $2008x$ is rational, then x is rational. By the contrapositive, if x is irrational, then $2008x$ is irrational. $\Box$

3. **Claim:** For every $n \in \mathbb{N}$, if $n \geq 4$, then $2^n < n!$.

   Proof: The proof will be by induction on n.
   Base case: $2^4 = 16$ and $4! = 24$ and $16 < 24$, so the statement is true for $n = 4$.
   Induction step: Suppose $k \in \mathbb{N}$ and $2^k < k!$. Then

   $$2^{k+1} = 2 \times 2^k < 2 \times k! < (k+1) \times k! = (k+1)!,$$

   so $2^{k+1} < (k+1)!$. By the principle of mathematical induction, the statement is true for all $n \geq 4$. $\Box$

4. **Claim:** For all $x, y, n \in \mathbb{N}$, if $\max(x, y) = n$, then $x \leq y$.

   Proof: The proof will be by induction on n.
   Base case: Suppose that $n = 0$. If $\max(x, y) = 0$ and $x, y \in \mathbb{N}$, then $x = 0$ and $y = 0$, hence $x \leq y$.
   Inductive hypothesis: Assume that, whenever we have $\max(x, y) = k$, then $x \leq y$ must follow.
   Inductive step: We must prove that if $\max(x, y) = k + 1$, then $x \leq y$. Suppose x, y are such that $\max(x, y) = k + 1$. Then it follows that $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 \leq y - 1$. In this case, we have $x \leq y$, completing the induction step. $\Box$