CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani Discussion 2W

1. Triangle Inequality

Recall the triangle inequality, which states that for real numbers x_1 and x_2,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

Use induction to prove the generalized triangle inequality:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

2. Power Inequality

Use induction to prove that for all integers $n \geq 1$, $2^n + 3^n \leq 5^n$.

3. Convergence of Series

Use induction to prove that for all integers $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2.$$

Hint: Strengthen the induction hypothesis to $\displaystyle\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2 - \frac{1}{\sqrt{n}}$.

4. Grid Induction

A bug lives on the grid $\mathbb{N}^2$. He starts at some location $(i, j) \in \mathbb{N}^2$, and every second he does one of the following (if possible):

(i) Jump one inch down to $(i, j-1)$, as long as $(i, j-1) \in \mathbb{N}^2$.

(ii) Jump one inch left to $(i-1, j)$, as long as $(i-1, j) \in \mathbb{N}^2$.

For example, if the bug is at $(5,0)$, then his only option is to jump left to $(4,0)$. Prove that no matter where the bug starts and how the bug jumps, he will always reach $(0,0)$ in finite time.

Supplemental Problems

5. Divergence of Harmonic Series

You may have seen the *harmonic series* $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ in calculus. We will prove that the harmonic series diverges, i.e., the sum approaches infinity.

Let $H_j = \sum_{k=1}^{j} \frac{1}{k}$. Use induction to show that for all integers $n \geq 0$, $H_{2^n} \geq 1 + \frac{n}{2}$, thus showing that H_j must grow unboundedly as $j \to \infty$.

6. Fibonacci Expansion

The *Fibonacci numbers* are defined recursively by $F_1 = F_2 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 3$.

Prove that every positive integer n has a binary expansion in the Fibonacci basis that does not use two consecutive Fibonacci numbers, i.e., we can write:

$$n = c_k \cdot F_k + c_{k-1} \cdot F_{k-1} + \cdots + c_2 \cdot F_2 + c_1 \cdot F_1$$

for some $k \in \mathbb{N}$ and $c_1, \ldots, c_k \in \{0,1\}$ with the property that $c_i \cdot c_{i+1} = 0$ for all $1 \leq i \leq k-1$.

For example, we could write $6 = F_1 + F_3 + F_4$, but this uses consecutive Fibonacci numbers. We can write it instead as $6 = F_1 + F_5$, which satisfies the desired property.