| CS 70 | Discrete Mathematics and Probability Theory |
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| Spring 2015 | Vazirani Discussion 3W |

## 1. Stable Marriage

Consider the following list of preferences:

| Men | Preferences |
|---|---|
| $A$ | $4 > 2 > 1 > 3$ |
| $B$ | $2 > 4 > 3 > 1$ |
| $C$ | $4 > 3 > 1 > 2$ |
| $D$ | $3 > 1 > 4 > 2$ |

| Women | Preferences |
|---|---|
| 1 | $A > D > B > C$ |
| 2 | $D > C > A > B$ |
| 3 | $C > D > B > A$ |
| 4 | $B > C > A > D$ |

1. Is $\{(A,4),(B,2),(C,1),(D,3)\}$ a stable pairing?

   **Answer:** No. Rogue pair: $(C,3)$.

2. Find a stable matching by running the Traditional Propose & Reject algorithm.

   **Answer:** A men-optimal pairing: $\{(A,2),(B,4),(C,3),(D,1)\}$.

3. Show that there exist a stable matching where women 1 is matched to men A.

   **Answer:** A pairing can be: $\{(A,1),(B,4),(C,3),(D,2)\}$. This is stable because each woman gets their first preference.

## 2. Objective Preferences

Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking.

1. Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.

   **Answer:** If the first woman is not paired with her first choice, then she and her first choice would form a rogue couple, because her first choice prefers her over any other woman, and vice versa.

2. Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.

   **Answer:** If the first and second women have different first choices, then the second woman must be matched to her first choice. Otherwise she and her first choice would form a rogue couple (since her first choice is not matched to the first woman, he would prefer the second woman over his current match).

   If the first choices are the same, then the second woman must be paired with her second choice, otherwise she and her second choice would form a rogue couple (neither of them are matched to their first choices, and they are each other's second choice).

3. Continuing this way, assume that we have determined the pairs for the first $k-1$ women in the ranking. Who should the $k$-th woman be paired with?

   **Answer:** The $k$-th woman should be paired with the first man on her list who has not been matched yet (with the first $k-1$ women). If she's not matched to him, they would form a rogue couple. This is because the man would have to be matched to a woman ranked worse than $k$, so she would prefer the $k$-th woman over his current partner, and the $k$-th woman obviously prefers him to whoever she's matched with.

4. Prove that there is a unique stable pairing.

   **Answer:** In the previous parts, we saw that for each woman, given the pairs for the lower-ranked women, her pair would be determined uniquely. So there is only one stable pairing.

   This can be stated and proved more rigorously using induction. Namely that there is a unique pairing for the first $k$ women, assuming stability. An induction on $k$ would prove this.

## 3. Least Preferred Marriage

Is there any instance of stable marriage with more than 3 men and 3 women, where a man $m$ and woman $w$ end up being paired to each other, despite having each other at the end of their preference lists? Provide an example, or disprove.

**Answer:** A very simple example is when we have global preference lists. Imagine that men are sorted the same way by all women, and women are sorted the same way by all men. Then there is exactly one stable pairing, and it is the one where the $i$-th man is paired with the $i$-th woman. So the last man does get paired with the last woman.

## 4. Large Number of Stable Pairings

How many different stable pairings can there be for an instance with $n$ men and $n$ women? In this question we will see how to construct instances with a very large number of stable pairings.

The overall plan is as follows: imagine we have already constructed an instance of stable marriage with $m$ men and $m$ women which admits $X$ different stable pairings. We will show how to use this to construct a new instance with $2m$ men and $2m$ women which has at least $X^2$ stable pairings.

1. Construct an instance with $n = 2$ and 2 different stable pairings. Now assuming construction in the overall plan above, show that there is an instance with $n = 4$ and 4 different stable pairings. What does that tell you about $n = 8$? In general if we continue this for $n = 2^k$, how many different stable pairings do we get? Express that as a function of $n$.

   **Answer:** The $n = 2$ case can be the following.

   | Men | Preferences |
   |---|---|
   | $A$ | $1 > 2$ |
   | $B$ | $2 > 1$ |

   | Women | Preferences |
   |---|---|
   | 1 | $B > A$ |
   | 2 | $A > B$ |

   In this instance both $\{(A,1),(B,2)\}$ and $\{(A,2),(B,1)\}$ are stable.

   Now using the previous part for $n = 4$ we get an instance with $2^2 = 4$ stable pairings. For $n = 8$ we get an instance with $4^2 = 16$ stable pairings. In general it seems like for $n = 2^k$ we get $2^{2^{k-1}} = 2^{n/2}$ different pairings. So let us prove that using induction on $k$.

   The base case of $k = 1$ is correct because for $n = 2$ we have $2^{2^{1-1}} = 2$ different stable pairings. Now if the statement is true for $k$, we know that for $k+1$ we get the square of what we had for $k$. So for $k+1$ we have $(2^{2^{k-1}})^2 = 2^{2\times 2^{k-1}} = 2^{2^k}$ different stable pairings.

2. Implement the overall plan: square the number of stable pairings at the cost of doubling the size of the instance. Start with an instance of stable marriage with $m$ men and $m$ women which admits $X$ different stable pairings, and create a new instance with $2m$ men and $2m$ women. To do this, create two copies of each person in the instance you start with. Call one of them the original, and the other the alternate.

Let original people prefer original people above alternate people, and alternate people prefer alternate people above originals, but in every other way let preferences remain as they were in the instance you started with. Thus, for example, the preference list of an alternate man would look like the preference list repeated twice of the man he was cloned from in the original instance, with the first half consisting of alternate women and the second half consisting of original women. Prove that the number of stable pairings in this new instance is at least $X^2$. To do so, it suffices to exhibit that for any pair of stable pairings of the instance you started with there is a unique stable pairing in the new instance.

**Answer:**

Given two stable pairings, we use one of them to pair the original people, and one of them to pair the alternate people. Note that we end up with no original-alternate couple. Now it is easy to see that there are no rogue couples; if $(M,W)$ is a rogue couple, then they can't both be original or both alternate (because the two pairings we used were stable). And they can't be original-alternate, because then they would prefer their current partners over each other.

Note that for any pair of starting stable pairings, we end up with a unique pairing (because we can get back the original pairings in the smaller instance, by just narrowing our view to original/alternate people). So by doing this we produce $X^2$ different pairings.

## 5. Calculator Induction

Suppose that your calculator is malfunctioning and only the keys $3,6,9$ and $+,-,*,(,)$ are working. You can construct any valid expression out of these. Can you construct the number 7? Either provide a way, or prove that this cannot be done.

**Answer:**

We will prove that all of the expressions that appear on the calculator have to be multiples of 3. Therefore 7 which is not a multiple of 3 cannot be constructed.

To prove this let us use the well-ordering principle and consider (for the sake of contradiction) the smallest expression (in terms of the number of key presses) that is not a multiple of 3. If there are parenthesis around the whole expression, clearly we can remove them and get a smaller expression. So that expression is either a single number or is the sum, difference, or product of two smaller expressions. If it is a single number, then all of its digits are multiples of 3 and so the whole number is a multiple of 3 (when dividing by 3 we can just divide each digit by 3, and so we get a whole number). But if it's the sum, difference or the product of two smaller expressions, we know that they are multiples of 3; but the sum, difference, or product of two multiples of 3 is also a multiple of 3. So we get a contradiction and the statement is proved.