CS 70	Discrete Mathematics and Probability Theory
Spring 2015	Vazirani Discussion 3W

1. Stable Marriage

Consider the following list of preferences:

Men	Preferences
A	$4 > 2 > 1 > 3$
B	$2 > 4 > 3 > 1$
C	$4 > 3 > 1 > 2$
D	$3 > 1 > 4 > 2$

Women	Preferences
1	$A > D > B > C$
2	$D > C > A > B$
3	$C > D > B > A$
4	$B > C > A > D$

1. Is $\{(A,4),(B,2),(C,1),(D,3)\}$ a stable pairing?

2. Find a stable matching by running the Traditional Propose & Reject algorithm.

3. Show that there exist a stable matching where women 1 is matched to men A.

2. Objective Preferences

Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking.

1. Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.

2. Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.

3. Continuing this way, assume that we have determined the pairs for the first $k-1$ women in the ranking. Who should the k-th woman be paired with?

4. Prove that there is a unique stable pairing.

3. **Least Preferred Marriage**

   Is there any instance of stable marriage with more than 3 men and 3 women, where a man m and woman w end up being paired to each other, despite having each other at the end of their preference lists? Provide an example, or disprove.

4. **Large Number of Stable Pairings**

   How many different stable pairings can there be for an instance with n men and n women? In this question we will see how to construct instances with a very large number of stable pairings.

   The overall plan is as follows: imagine we have already constructed an instance of stable marriage with m men and m women which admits X different stable pairings. We will show how to use this to construct a new instance with $2m$ men and $2m$ women which has at least X^2 stable pairings.

   1. Construct an instance with $n = 2$ and 2 different stable pairings. Now assuming construction in the overall plan above, show that there is an instance with $n = 4$ and 4 different stable pairings. What does that tell you about $n = 8$? In general if we continue this for $n = 2^k$, how many different stable pairings do we get? Express that as a function of n.

   2. Implement the overall plan: square the number of stable pairings at the cost of doubling the size of the instance. Start with an instance of stable marriage with m men and m women which admits X different stable pairings, and create a new instance with $2m$ men and $2m$ women. To do this, create two copies of each person in the instance you start with. Call one of them the original, and the other the alternate. Let original people prefer original people above alternate people, and alternate people prefer alternate people above originals, but in every other way let preferences remain as they were in the instance you started with. Thus, for example, the preference list of an alternate man would look like the preference list repeated twice of the man he was cloned from in the original instance, with the first half consisting of alternate women and the second half consisting of original women. Prove that the number of stable pairings in this new instance is at least X^2. To do so, it suffices to exhibit that for any pair of stable pairings of the instance you started with there is a unique stable pairing in the new instance.

5. **Calculator Induction**

   Suppose that your calculator is malfunctioning and only the keys $3, 6, 9$ and $+, -, *, (,)$ are working. You can construct any valid expression out of these. Can you construct the number 7? Either provide a way, or prove that this cannot be done.