CS70–Spring 2013 — Homework 8

Felix He, SID 303308****

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Collaborators: None

1. Deja Vu

1. 4. FFFBBB, FFBFBB for going forward first. The same for going backward first.

2. 0. cannot return when the number of moves are odd

3. 10. The first 2 moves must be the same with $t = 6$, then the 3^{rd} move and 4^{th} move can be FF, BF, FB. For FF, there is only one way for the rest 4 moves, for BF, FB, the rest moves would be the same situation with $t = 6$, so it's $(1 + 2 + 2) \times 2 = 10$

4. 0

5. Since for returning to the starting point, the first move and the last move must be (Forward, Backward) or (Backward, Forward). So the rest is $2n$ steps. It's the same thing with going up or right not crossing the diagonal in a $n \times n$ grid where total steps is $2n$. Here is not returning to the starting point before last step. So the possible paths for $t = 2n + 2$ if going forward first is

$$\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

And we multiply by 2 we get

$$\frac{2}{n+1}\binom{2n}{n}$$

2. The Monty Hall Problem

1. The Monty Hall problem is saying behind 3 doors, there is a prize, once the contestant choose 1 door, then the host opens a door revealing a goat which is not the prize. Under this circumstance, should the contestant switch his choice to get the prize?

2. The sample space is as below

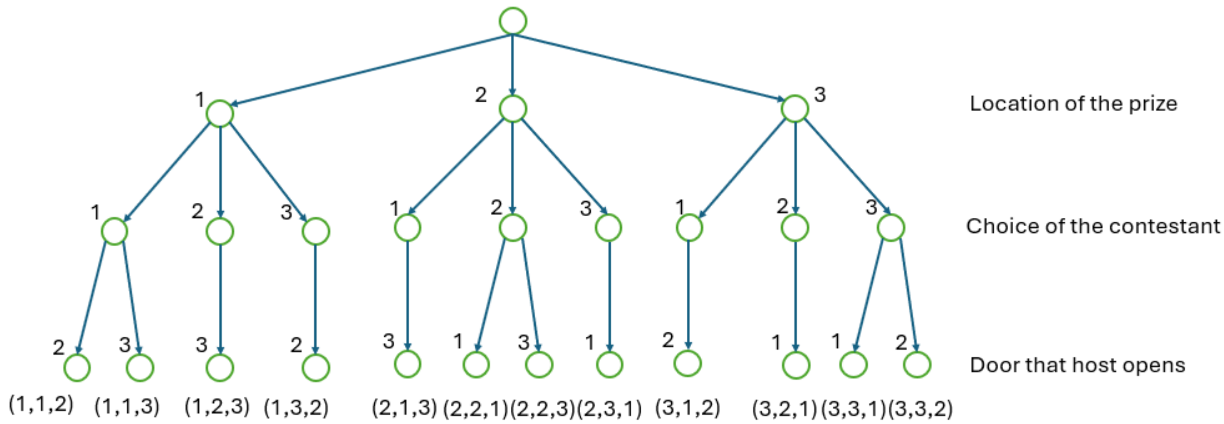

Figure 1: Sample space of Monty Hall problem

3. The probability of the prize behind each door is uniform $\frac{1}{3}$, the choice that contestant makes is uniform $\frac{1}{3}$, the door that host opens is also uniform, either $\frac{1}{2}$ or 1.
   So $Pr[(1,1,2)] = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{18}$, $Pr[(1,2,3)] = \frac{1}{3} \times \frac{1}{3} \times 1 = \frac{1}{9}$. The others are calculated in the same way.

4. The event A is Use Switching Strategy to get the Prize. Suppose event B is host opening a door revealing a goat. Then we want to calculate
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
   And since we know the host must open 1 door, so $P(B) = 1$
$$P(A|B) = P(A \cap B)$$

5. The probability $P(\bar{A}|B) = P(\bar{A} \cap B) = P(1,1,2) + P(1,1,3) + P(2,2,1) + P(2,2,3) + P(3,3,1) + P(3,3,2) = \frac{6}{18} = \frac{1}{3}$

6. The probability $P(A|B) = P(A \cap B) = P(1,2,3) + P(1,3,2) + P(2,1,3) + P(2,3,1) + P(3,1,2) + P(3,2,1) = \frac{6}{9} = \frac{2}{3}$

3. Error Correction Codes

1. $m = 2$, tolerate 1 error. So $P = \binom{7}{1}0.1^1 0.9^6 + \binom{7}{7}0.1^0 0.9^7 \approx 0.85$

2. $m = 3$, tolerate $3/2 = 1$ error. So $P = \binom{8}{1}0.1^1 0.9^7 + \binom{8}{8}0.1^0 0.9^8 \approx 0.81$

3. Not always helpful

4. $m = 4$. $P = \binom{9}{0}0.9^9 + \binom{9}{1}0.1^1 0.9^8 + \binom{9}{2}0.1^2 0.9^7 \approx 0.95 \geq 0.9$

5. False.
   For example, $n = 1, m = 0, p = 0.9$
$$P[success, m] = 0.1$$
$$P[success, m+2] = 0.1^3 + \binom{3}{1}0.9^1 0.1^2 = 0.028$$

Apparently, sending 1 messages is better than sending 3 messages.

4. Best-of-Five Series

1. 3 Wins and 2 Losses, the last win must be in 5_{th} position

$$\binom{4}{2} = 6$$

2. the last win at position 5, 4, 3, then we have

$$\binom{4}{2} + \binom{3}{2} + \binom{2}{2} = 6 + 3 + 1 = 10$$

3.

$$\sum_{i=0}^{n} \binom{n+i}{i} = \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \ldots + \binom{2n}{n}$$

Use the property

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

We can sum all the terms up

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n+i}{i} &= \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \ldots + \binom{2n}{n} \\
&= \binom{n+2}{1} + \binom{n+2}{2} + \ldots + \binom{2n}{n} \\
&= \binom{2n}{n-1} + \binom{2n}{n} \\
&= \binom{2n+1}{n} \\
&= \binom{2n+1}{n+1}
\end{aligned}$$

4. The possible outcomes for Bears to win is the sum of the below, where n represents the n wins before the last 1 win, i represents i losses for the first $n + i$ games

$$\sum_{i=0}^{n} \binom{n+i}{i}$$

So it's $\binom{2n+1}{n+1}$

5. Intuitively, for a best-of-$(2n + 1)$ series, there are $(2n + 1)$ total games choose $(n + 1)$ wins ways for a team to win. Any losses after $n + 1$ wins doesn't matter

6. It's the same with above, since the locations of the losses doesn't matter. e.g. WWW is the same with WWWLL for a best-of-5 series. So for best-of-$(n_1 + n_2 - 1)$, the possible outcomes for Bears to win is $(n_1 + n_2 - 1)$ choose n_1, so it's $\binom{n_1+n_2-1}{n_1}$

7. $$Pr[3W, 2L] = \binom{4}{2} 0.7^2 0.3^2 \times 0.7 = 0.185$$

8. the last win at position 5, 4, 3, then we have

$$\begin{aligned} P &= P[3W, 0L] + P[3W, 1L] + P[3W, 2L] \\ &= 0.7^3 + \binom{3}{1} 0.7^2 0.3^1 \times 0.7 + \binom{4}{2} 0.7^2 0.3^2 \times 0.7 \\ &= 0.837 \end{aligned}$$

9. If change to best-of-3, probability that Bears win is:

$$\begin{aligned} P &= P[2W, 0L] + P[2W, 1L] \\ &= 0.7^2 + \binom{2}{1} 0.7^1 0.3^1 \times 0.7 \\ &= 0.784 \end{aligned}$$

It's advantageous to Cardinal.

10.
Pr[(Bears win series) | (Bears win first game)]
= Pr[(Bears win series) ∩ (Bears win first game)] / Pr[(Bears win first game)]
$= (0.7 \times 0.7^2 + 0.7 \times \binom{2}{1} 0.7^2 0.3 + 0.7^2 \times \binom{3}{1} 0.7^1 0.3^2) \ / \ 0.7$
= 0.641 / 0.7
= 0.916

11.
Pr[(Bears win first game) | (Bears win series)]
= Pr[(Bears win first game) ∩ (Bears win series)] / Pr[(Bears win series))]
= 0.641 / 0.837
= 0.766

5. Stirling's Approximation

1. The plot of $f(x) = lnx$ is as below

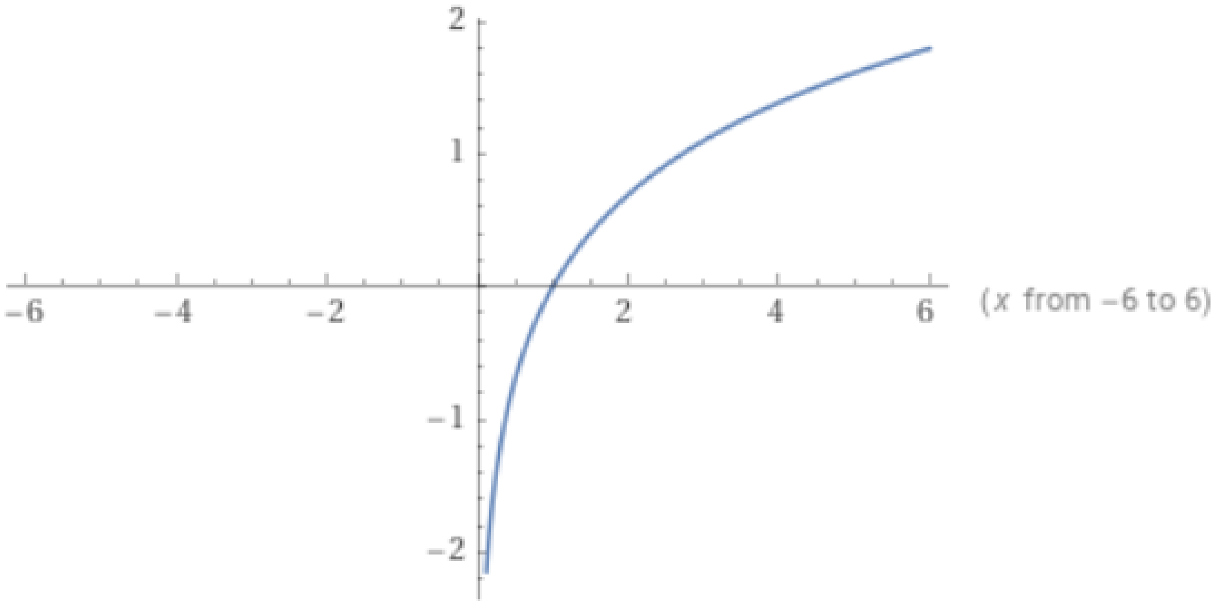

Figure 2: ln x

2. We can think of it of comparing area.
   LHS is summing the area of the little rectangle, while the RHS is summing the area bonded by the lnx curve and the x axis. Apparently, LHS has those little extra areas than the RHS

   The plot of $f(x) = lnx$ is as below

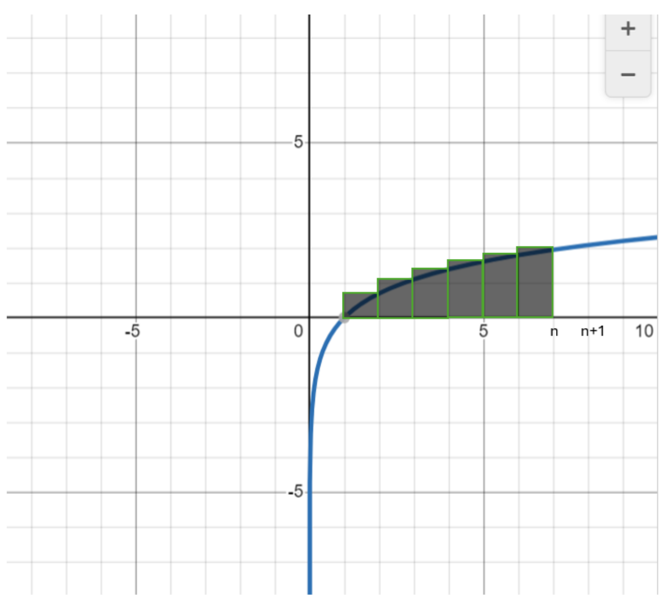

Figure 3: integral of ln x from 1 to n

3. Similar to part2, the LHS is the same, we shift right by 1 unit. The RHS is doing integral from 1 to $n+1$ instead of 1 to n. Apparently, the summing rectangle regions is less than the area bounded by the curve and x axis from 1 to $n+1$

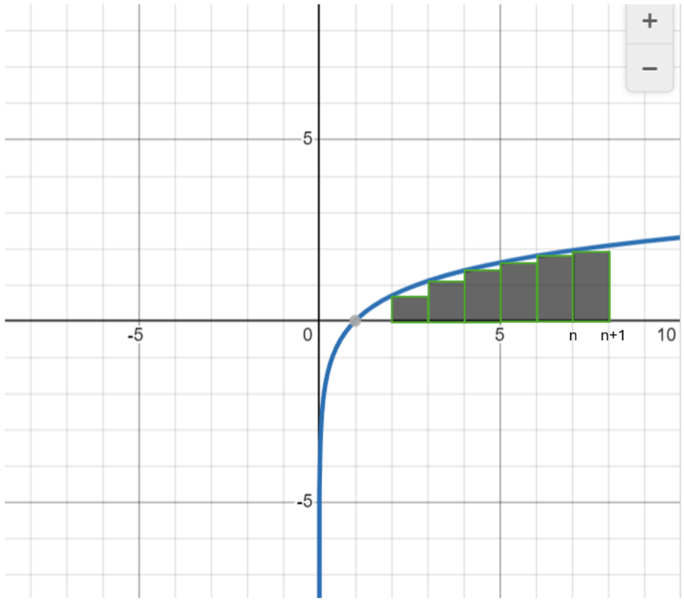

Figure 4: integral of ln x from 1 to n+1

4. LHS is calculating the purple trapezoid area, while the RHS is calculating the area bounded by lnx, $x = a$, $x = a + 1$ and $y = 0$

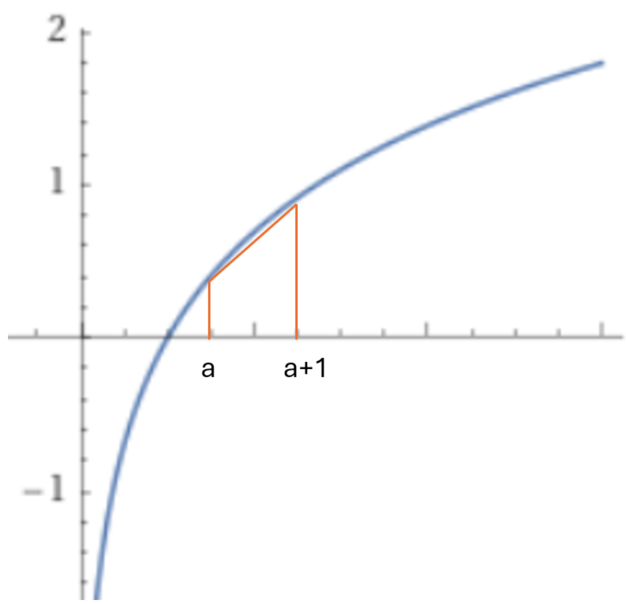

Figure 5: q4

5.

$$LHS : \ln 1 + \ln 2 + \ldots + \ln n = lnn!$$
$$RHS : \int_1^n \ln x \, dx = [x \ln x - x]_1^n = (n \ln n - n) - (1 \ln 1 - 1)$$

We take exponential by e to both side, we get

$$n! \geq e^{nlnn-n+1} = e\frac{n^n}{e^n} = e(\frac{n}{e})^n$$

6.

$$\begin{aligned}\ln 1 + \ln 2 + \ldots + \ln n &= \ln 1 + \ln 2 + \ldots + \ln n - 1 + \ln n \\ &< \int_1^n \ln x + \ln n \\ &< n \ln n - n + 1 + \ln n\end{aligned}$$

Take exponential of e to both side we have

$$\begin{aligned}e^{\ln 1 + \ln 2 + \ldots + \ln n} &< e^{n \ln n - n + 1 + \ln n} \\ n! &< en(\frac{n}{e})^n\end{aligned}$$

7.

$$\begin{aligned}\frac{\ln 1 + \ln 2}{2} + \frac{\ln 2 + \ln 3}{2} + \ldots + \frac{\ln n - 1 + \ln n}{2} &< \int_1^2 \ln x + \int_2^3 \ln x + \ldots + \int_{n-1}^n \ln x \\ \frac{\ln 1 + 2\ln 2 + 2\ln 3 + \ldots + \ln n}{2} &< \int_1^n \ln x\end{aligned}$$

Add $(\frac{\ln 1}{2} + \frac{\ln n}{2})$ to both sides, we have

$$\ln n! \leq n \ln n - n + 1 + \frac{\ln n}{2} + 0$$

Take exponential of e to both side we have

$$n! \leq e\sqrt{n}(\frac{n}{e})^n$$

8. We can see when n goes larger, it approximate to 1

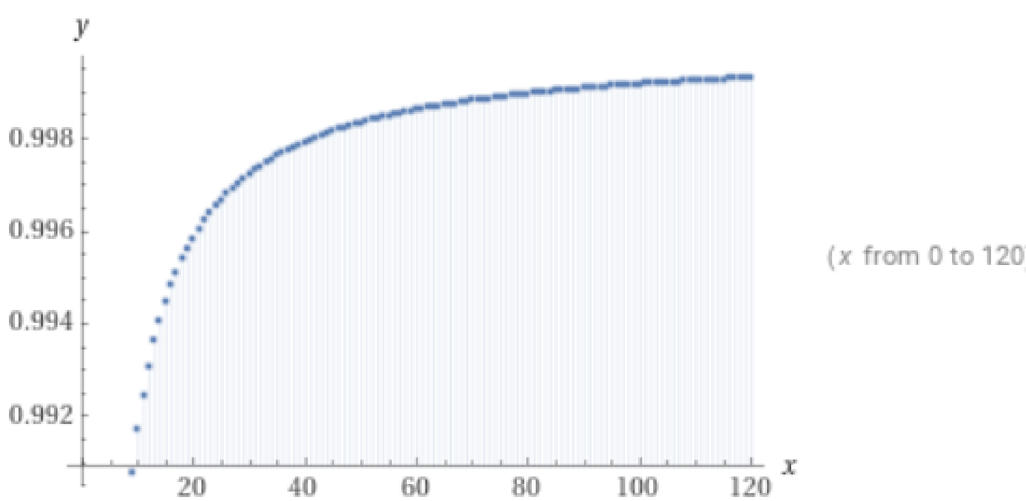

Figure 6: Plot $f(n) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!}$

9.

$$\begin{aligned}
\frac{n!}{(n-k)!k!} &= \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{n-k}{e}\right)^{n-k} \cdot \left(\frac{k}{e}\right)^k} \\
&= \frac{n^n}{(n-k)^{n-k}k^k} \\
&= \left(\frac{n}{n-k}\right)^{n-k} \left(\frac{n}{k}\right)^k \\
&= \left(\frac{1}{1-m}\right)^{(1-m)n} \left(\frac{1}{m}\right)^{mn}
\end{aligned}$$

10. We can see these functions are linear on log-scaled of $f(n)$, and $f(n)$ in $m_1 = 0.25$, $m3 = 0.75$ overlap

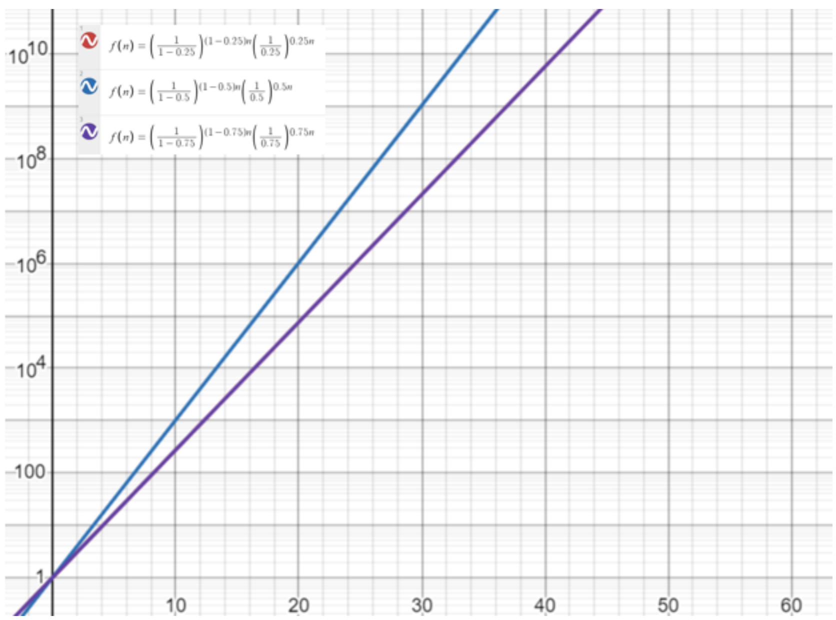

Figure 7: Plot log-scaled of $f(n) = \left(\frac{1}{1-m}\right)^{(1-m)n} \left(\frac{1}{m}\right)^{mn}$

11. We take the derivative of $log(f(n))$ by n, we get the slope is

$$(1-m)\log_{10}(\frac{1}{1-m}) + m\log_{10}(\frac{1}{m})$$

$m = 0.25$ or 0.75, the slope is 0.244, $m = 0.5$, the slope is 0.301