Mathematical background

- Sets and De Morgan's laws
- Sequences and ther limits
- Infinite series
  - The geometric series
- Sums with multiple indices
- Countable and uncountable sets

Sets

- A collection of distinct elements

Unions and intersections

Set properties

$$
\begin{aligned}
S \cup T &= T \cup S, & S \cup (T \cup U) &= (S \cup T) \cup U, \\
S \cap (T \cup U) &= (S \cap T) \cup (S \cap U), & S \cup (T \cap U) &= (S \cup T) \cap (S \cup U), \\
(S^c)^c &= S, & S \cap S^c &= \emptyset, \\
S \cup \Omega &= \Omega, & S \cap \Omega &= S.
\end{aligned}
$$

De Morgan's laws

$$\Big(\bigcap_n S_n\Big)^c = \bigcup_n S_n^c$$

$$\Big(\bigcup_n S_n\Big)^c = \bigcap_n S_n^c$$

Mathematical background: Sequences and their limits

Mathematical background: When does a sequence converge?

- If $a_i \leq a_{i+1}$, for all i, then either:
  - the sequence "converges to ∞"
  - the sequence converges to some real number a

- If $|a_i - a| \leq b_i$, for all i, and $b_i \rightarrow 0$, then $a_i \rightarrow a$

Mathematical background: Infinite series

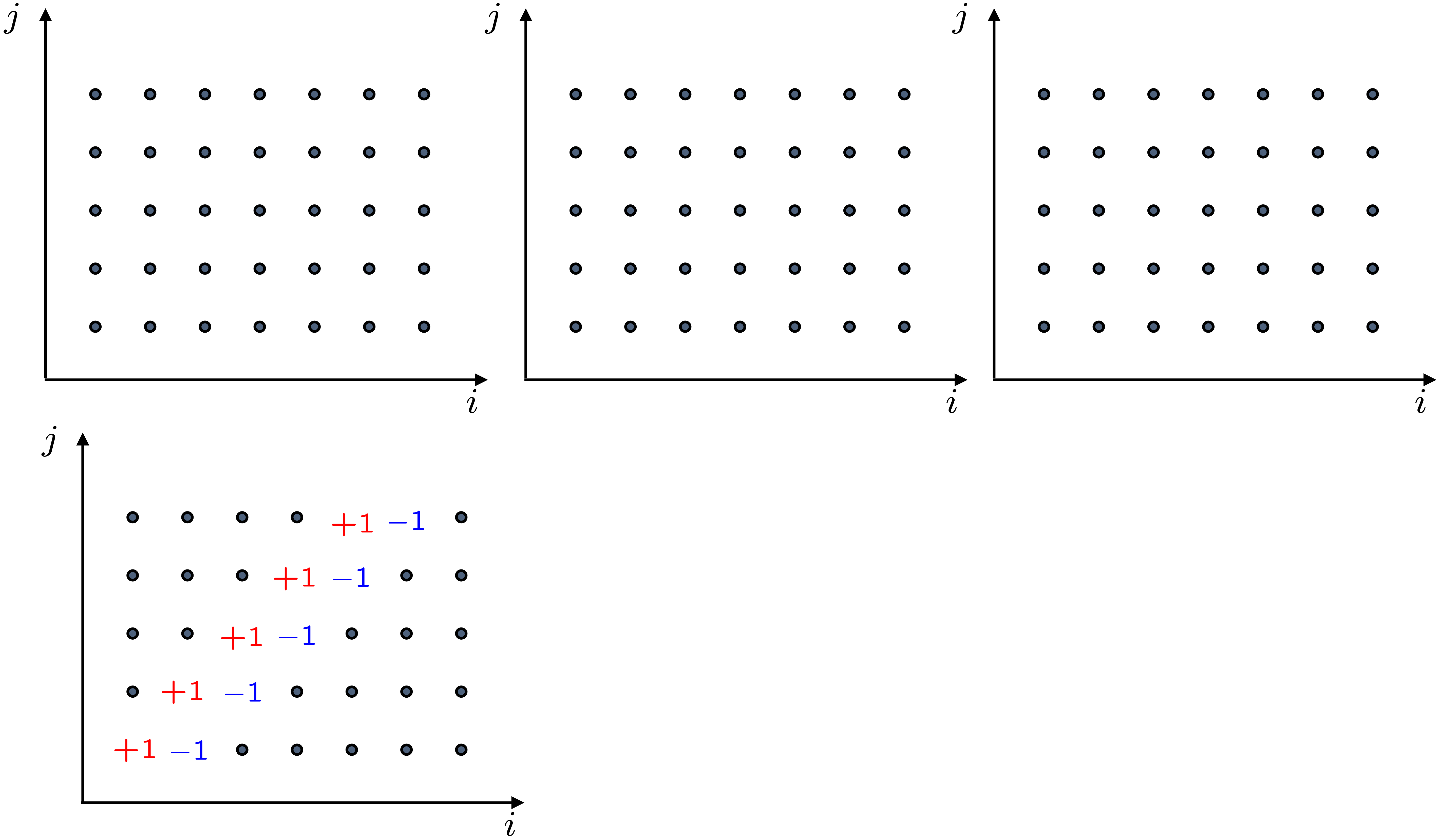
$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

- If $a_i \geq 0$: limit exists

- if terms a_i do not all have the same sign:
  - limit need not exist
  - limit may exist but be different if we sum in a different order
  - **Fact:** limit exists and independent of order of summation if $\sum_{i=1}^{\infty} |a_i| < \infty$

About the order of summation in series with multiple indices

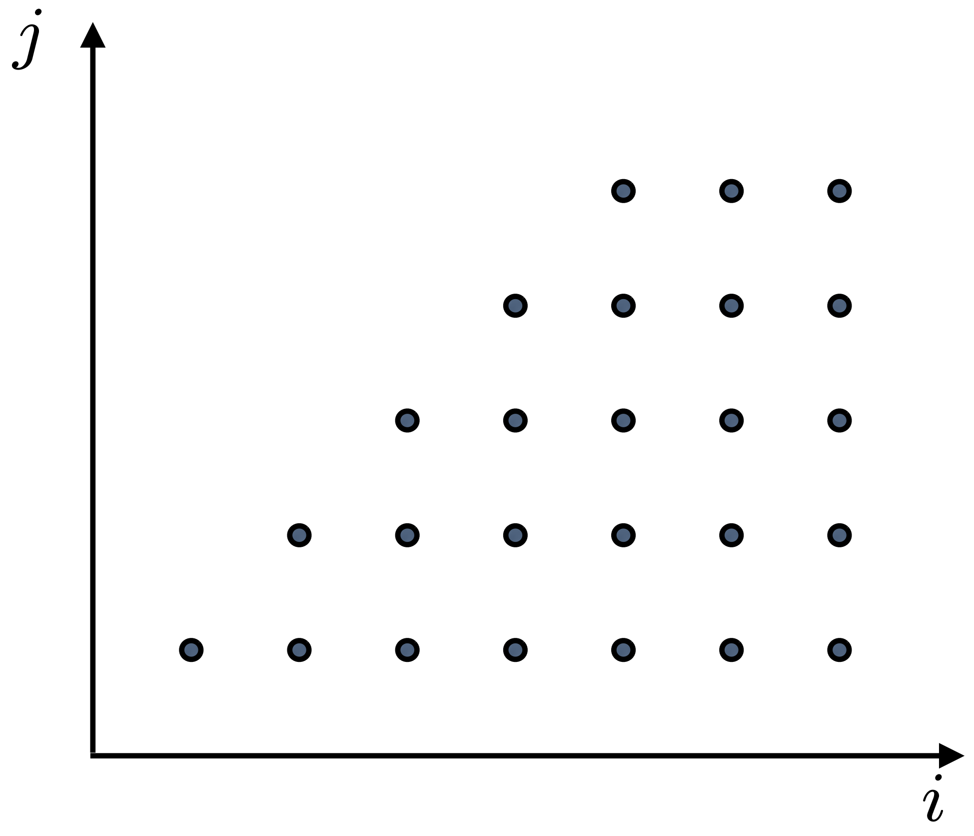
$$\sum_{i\geq 1, j\geq 1} a_{ij}$$

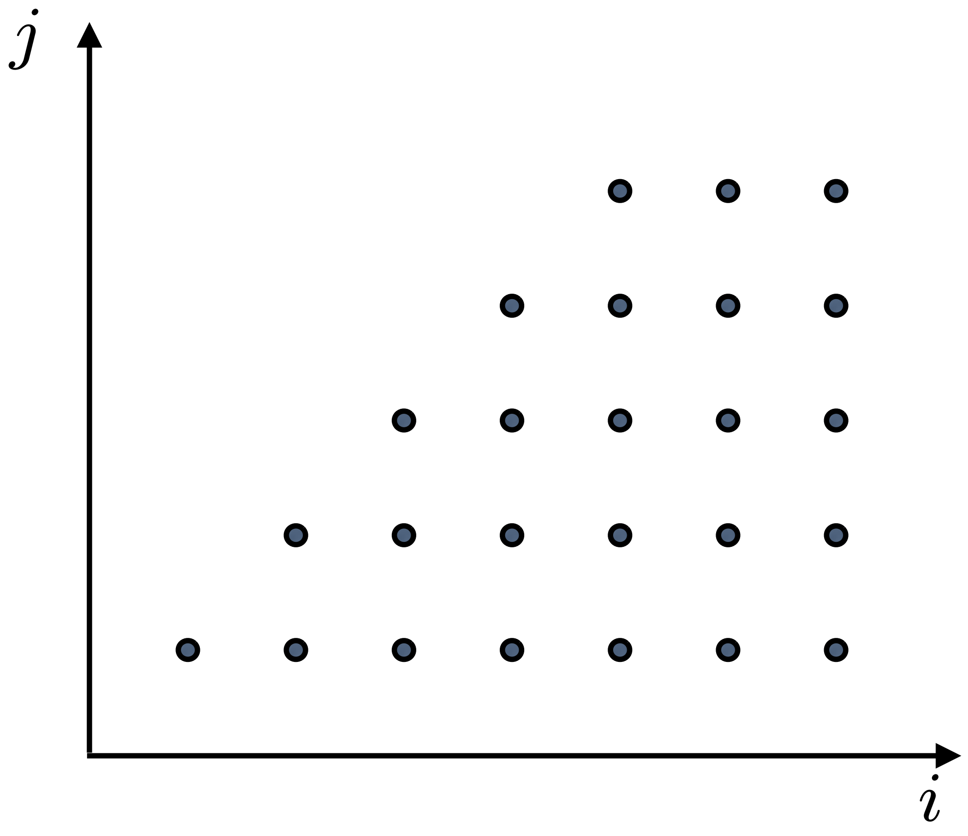


About the order of summation in series with multiple indices

$$\sum_{(i,j):\ j\le i} a_{ij}$$

$$\sum |a_{ij}| < \infty$$



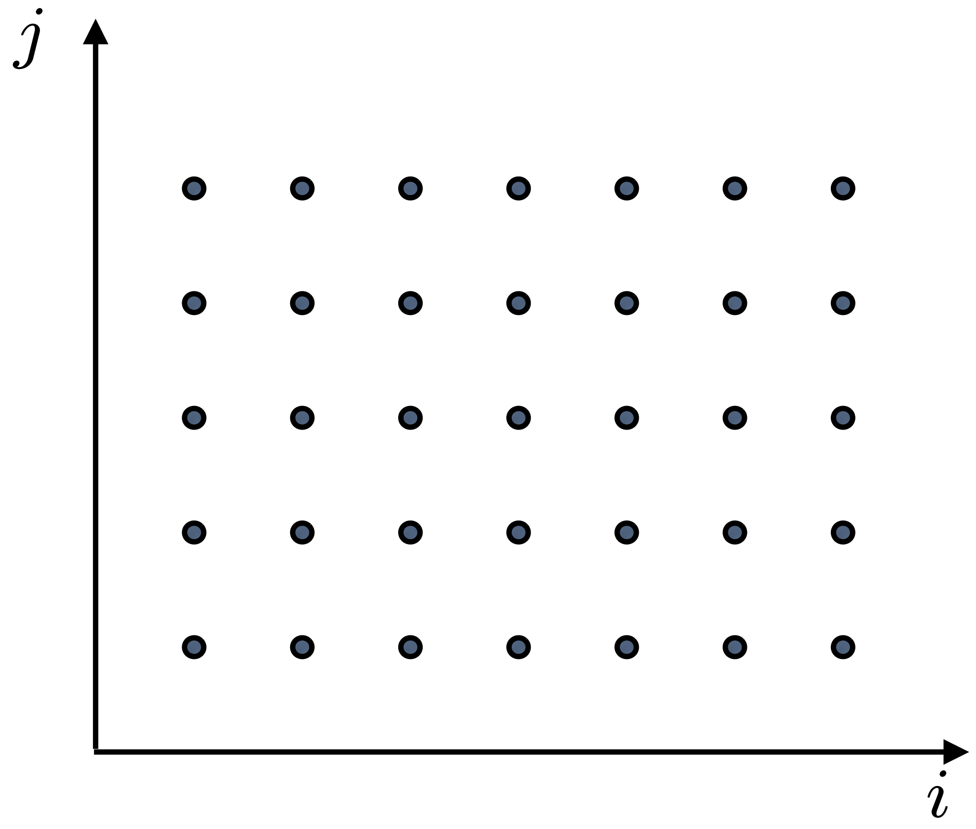


Mathematical background: Geometric series

$$\sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + \cdots \quad = \frac{1}{1-\alpha} \qquad\qquad |\alpha| < 1$$

Countable versus uncountable infinite sets

- Countable: can be put in 1-1 correspondence with positive integers
  - positive integers
  - integers
  - pairs of positive integers
  - rational numbers q, with $0 < q < 1$



- Uncountable: not countable
  - the interval $[0, 1]$
  - the reals, the plane,...

The reals are uncountable

- Cantor's diagonalization argument

Interpreting the union bound and the Bonferroni inequality

- Suppose that:
  - very few of the students are smart
  - very few students are beautiful
- Then: very few students are smart or beautiful

- Suppose that:
  - most of the students are smart
  - most students are beautiful
- Then: most students are smart and beautiful

$$\mathbf{P}(A_1 \cap A_2) \geq \mathbf{P}(A_1) + \mathbf{P}(A_2) - 1$$

The Bonferroni inequality

$$\mathbf{P}(A_1 \cap A_2) \geq \mathbf{P}(A_1) + \mathbf{P}(A_2) - 1$$

$$\mathbf{P}(A_1 \cap \cdots \cap A_n) \geq \mathbf{P}(A_1) + \cdots + \mathbf{P}(A_n) - (n - 1)$$