$$
\begin{aligned}
f_{15} = \frac{\partial \delta \alpha_{b_{k+1}}}{\partial \delta b_k^g} &= \frac{1}{4} \frac{\partial q_{b_i b_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}(\omega - \delta b_k^g)\delta t \end{bmatrix} (a^{b_{k+1}} - b_k^g) \cdot \delta t^2}{\partial \delta b_k^g} \\
&= \frac{1}{4} \frac{\partial R_{b_i b_k} \cdot exp\{[(\omega - \delta b_k^g)\delta t]_\times\}(a^{b_{k+1}} - b_k^g)\, \delta t^2}{\partial \delta b_k^g} \\
&= \frac{1}{4} \frac{\partial R_{b_i b_{k+1}} \left(exp([\omega \delta t]_\times)\, exp([-J_r(\omega \delta t)\delta b_k^g \delta t]_\times)(a^{b_{k+1}} - b_k^g)\, \delta t^2 \right.}{\partial \delta b_k^g} \\
&= \frac{1}{4} \frac{\partial R_{b_i b_{k+1}} \cdot [-J_r(\omega \delta t)\, \delta b_k^g \delta t]_\times (a^{b_{k+1}} - b_k^g)\, \delta t^2}{\partial \delta b_k^g} \\
&= -\frac{1}{4} \frac{\partial R_{b_i b_{k+1}} \left([(a^{b_{k+1}} - b_k^g)\delta t^2]_\times\right) \cdot \left(-J_r(\omega \delta t) \cdot \delta b_k^g \delta t\right)}{\partial \delta b_k^g} \\
&= -\frac{1}{4} \left(R_{b_i b_{k+1}} [(a^{b_{k+1}} - b_k^g)_\times \delta t^2] \right) \left(-J_r(\omega \delta t) \cdot \delta t\right) \\
&= -\frac{1}{4} \left(R_{b_i b_{k+1}} [(a^{b_{k+1}} - b_k^g)_\times \delta t^2] (-\delta t) \right)
\end{aligned}
$$

$$
\begin{aligned}
g_{12} = \frac{\partial \delta \alpha_{b_{k+1}}}{\partial \delta n_k^g} &= \frac{1}{4} \frac{\partial q_{b_i b_k} \begin{bmatrix} 1 \\ \frac{1}{2}(\omega + \frac{1}{2}\delta n_k^g)\delta t \end{bmatrix} (a^{b_{k+1}} - b_k^g)\, \delta t^2}{\partial \delta n_k^g} \\
&= \frac{1}{4} \frac{\partial R_{b_i b_k} \cdot exp\left([(\omega + \frac{1}{2}\delta n_k^g)\delta t]_\times (a^{b_{k+1}} - b_k^g)\, \delta t^2 \right.}{\partial \delta n_k^g} \\
&= \frac{1}{4} \frac{\partial R_{b_i b_{k+1}} \left(I + [J_r(\omega \delta t) \cdot \frac{1}{2}\delta n_k^g \delta t]_\times\right) \cdot (a^{b_{k+1}} - b_k^a)\, \delta t^2}{\partial \delta n_k^g} \\
&= \frac{1}{4} \frac{\partial \left(-R_{b_i b_{k+1}} [(a^{b_{k+1}} - b_k^a)\delta t^2]_\times \left(J_r(\omega \delta t) \cdot \frac{1}{2}\delta n_k^g \delta t\right)\right)}{\partial \delta n_k^g} \\
&= -\frac{1}{4} \left(R_{b_i b_{k+1}} [(a^{b_{k+1}} - b_k^a)]_\times \delta t^2 \right) \cdot \left(J_r(\omega \delta t) \cdot \frac{1}{2}\delta t\right) \\
&= -\frac{1}{4} \left(R_{b_i b_{k+1}} [(a^{b_{k+1}} - b_k^a)]_\times \delta t^2 \right) \left(\frac{1}{2}\delta t\right)
\end{aligned}
$$

$$(J^TJ+\mu I)\Delta x_{lm} = -J^Tf \quad , \text{when } \mu \geq 0$$

由于半正定矩阵 J^TJ 是实对称矩阵，其特征值为 $\{\lambda_i\}$，对应特征向量为 $\{v_i\}$，则有 $J^TJ = V\Lambda V^T$，V 为 J^TJ 特征向量组成的特征矩阵，且各向量之间相互正交。Λ 为 J^TJ 的特征值组成的对角矩阵，另，$F'(x) = (J^Tf)^T$

所以：

$$(V\Lambda V^T + \mu I)\cdot \Delta x_{lm} = -J^Tf$$

$$(V\Lambda V^T + \mu VV^T)\Delta x_{lm} = -J^Tf$$

$$(V\Lambda V^T + V\mu I V^T)\Delta x_{lm} = -F'^T$$

$$[V(\Lambda + \mu I)V^T]\Delta x_{lm} = -F'^T$$

其中：

$$V = [v_1 \ v_2 \ \cdots \ v_n] \qquad V^T = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$\Lambda + \mu I = \begin{bmatrix} \lambda_1+\mu & & & \\ & \lambda_2+\mu & & \\ & & \ddots & \\ & & & \lambda_n+\mu \end{bmatrix}$$

所以：

$$[v_1 \ v_2 \cdots v_n]\begin{bmatrix} \lambda_1+\mu & & & \\ & \lambda_2+\mu & & \\ & & \ddots & \\ & & & \lambda_n+\mu \end{bmatrix}\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}\Delta x_{lm} = -F'^T$$

由于 V，$\Lambda+\mu I$，V^T 均可逆，且 $V^{-1} = V^T$，有：

$$\Delta x_{lm} = -\sum_{i=1}^{n} \frac{v_i^T F'^T}{\lambda_i+\mu} v_i \quad , \text{证毕}$$