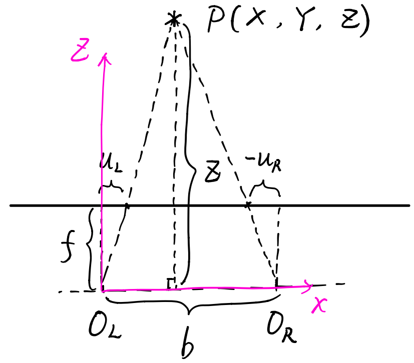
4. 双目视差的使用

理论部分推导：

(1) $\underbrace{u、v}_{\text{像素坐标}}、\underbrace{d}_{\text{视差}} \longleftrightarrow \underbrace{X、Y、Z}_{\text{相机坐标系下坐标}}$

左目相机下：



$$\boxed{d = u_L - u_R}$$

由相似关系，很容易有 $\frac{Z-f}{Z} = \frac{b-u_L-(-u_R)}{b} = \frac{b-d}{b}$

$$\Rightarrow 1-\frac{f}{Z} = 1-\frac{d}{b} \quad \Rightarrow Z = f\frac{b}{d} = f_x\frac{b}{d}$$

注：当 d 表示像素坐标之差（单位为 pixel）时 f 取 f_x

当 d 表示物理成像平面坐标之差（单位为 mm）时 f 即取 f

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{Z}\begin{bmatrix} f_x & 0 & C_x \\ 0 & f_y & C_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

代入 $Z = f_x\frac{b}{d}$，得：

$$X = \frac{(u-C_x)f_x\frac{b}{d}}{f_x} = (u-C_x)\frac{b}{d}$$

$$Y = \frac{(v-C_y)f_x\frac{b}{d}}{f_y} = \frac{f_x}{f_y}(v-C_y)\frac{b}{d}$$

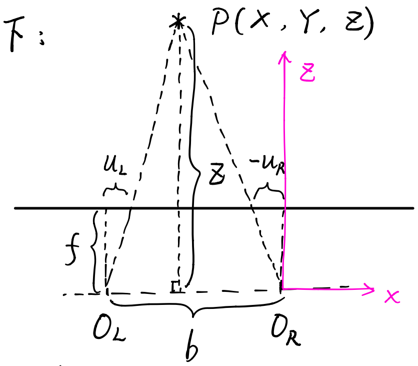
由 X, Y, Z 推导 u, v, d

$$d = f_x\frac{b}{Z}$$

$$u = \frac{Xd}{b} + C_x = \frac{X}{Z}\cdot f_x + C_x$$

$$v = \frac{Yd}{b}\cdot\frac{f_y}{f_x} + C_y = \frac{Y}{Z}f_y + C_y$$

(2). 右目相机下：



$$d = -[u_L + (-u_R)]$$
$$= u_R - u_L$$

由相似关系，很容易有：$$\frac{Z-f}{Z} = \frac{b - u_L - (-u_R)}{b} = \frac{b+d}{b}$$

$$\Rightarrow Z = -\frac{fb}{d} = -\frac{f_x b}{d}$$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{Z}\begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

代入 $Z = -f_x \frac{b}{d}$，得：

$$X = \frac{-(u-c_x) f_x \frac{b}{d}}{f_x} = -(u-c_x)\frac{b}{d}$$

$$Y = \frac{-(v-c_y) f_x \frac{b}{d}}{f_y} = -\frac{f_x}{f_y}(v-c_y)\frac{b}{d}$$

由 X, Y, Z 推导 u, v, d

$$d = -f_x \frac{b}{Z}$$

$$u = -\frac{Xd}{b} + c_x = \frac{X}{Z} \cdot f_x + c_x$$

$$v = -\frac{Yd}{b} \cdot \frac{f_y}{f_x} + c_y = \frac{Y}{Z} f_y + c_y$$

5. 矩阵微分运算

(1) $$\frac{d(Ax)}{dx} = A$$

(2). $$\frac{d(x^T A x)}{dx} = (A + A^T) x$$

(3). 证明 $x^T A x = tr(A x x^T)$

左边 $= [x_1 \ x_2 \ \cdots \ x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= x_1 \sum_{i=1}^{n} a_{1i} x_i + x_2 \sum_{i=1}^{n} a_{2i} x_i + \cdots + x_n \sum_{i=1}^{n} a_{ni} x_i$

右边 $= tr(A x x^T)$

$= tr\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [x_1 \ x_2 \ \cdots \ x_n] \right)$

$= tr\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^2 & x_1 x_2 & \cdots & x_1 x_n \\ x_2 x_1 & x_2^2 & \cdots & x_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n x_1 & x_n x_2 & \cdots & x_n^2 \end{bmatrix} \right)$

$= x_1 \sum_{i=1}^{n} a_{1i} x_i + x_2 \sum_{i=1}^{n} a_{2i} x_i + \cdots \ x_n \sum_{i=1}^{n} a_{ni} x_i$

左边 = 右边，证毕

7. 批量最大似然估计

$$\begin{cases} x_1 = x_0 + v_1 + w_1 \\ x_2 = x_1 + v_2 + w_2 \\ x_3 = x_2 + v_3 + w_3 \\ y_1 = x_1 + n_1 \\ y_2 = x_2 + n_2 \\ y_3 = x_3 + n_3 \end{cases} \Longrightarrow \begin{cases} w_1 = -v_1 + x_1 - x_0 \cong v_1 + x_0 - x_1 \\ w_2 = -v_2 + x_2 - x_1 \cong v_2 + x_1 - x_2 \\ w_3 = -v_3 + x_3 - x_2 \cong v_3 + x_2 - x_3 \\ n_1 = y_1 - x_1 \\ n_2 = y_2 - x_2 \\ n_3 = y_3 - x_3 \end{cases}$$

即 $$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ 即 $\vec{e} = \vec{z} - H\vec{x}$

(1). $$H = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2).

由于 $\vec{e} = \vec{z} - H\vec{x} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}$ 故 $W = \begin{bmatrix} Q & & & & & \\ & Q & & & 0 & \\ & & Q & & & \\ & & & R & & \\ & 0 & & & R & \\ & & & & & R \end{bmatrix}$

(3).

$$(z - Hx)^T W^{-1} (z - Hx)$$

$$= z^T W^{-1} z - z^T W^{-1} Hx - x^T H^T W^{-1} z + x^T H^T W^{-1} Hx$$

对上式求导，令其对 x 导数为 0：

$$-H^T W^{-1} z - H^T W^{-1} z + \left[H^T W^{-1} H + (H^T W^{-1} H)^T\right] x = 0$$

不难得出：

$$x = (H^T W^{-1} H)^{-1} H^T W^{-1} z$$

只要 $H^T W^{-1} H$ 可逆，x 存在唯一解 $x^* = (H^T W^{-1} H)^{-1} H^T W^{-1} z$