MAE 280B Final Weilun Zhang A53239629

Problem 1. LQG Design.

(a)

1. $\dot{x} = Ax + B_u u + B_w w$

$y = C_y x + D_{yw} w$

$z = C_z x + D_{zu} u$

where $A = \begin{bmatrix} -.0558 & -.9968 & .0802 & .0415 \\ .598 & -.115 & -.0318 & 0 \\ -3.05 & .388 & -.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix}$ $B_u = \begin{bmatrix} .00729 & 0 \\ -0.475 & 0.00775 \\ 0.153 & 0.143 \\ 0 & 0 \end{bmatrix}$

$B_w = \begin{bmatrix} 0.0075 & 0.0026 \\ 0.0081 & 0.0066 \\ 0.0050 & 0.0075 \\ 0.0040 & 0.0046 \end{bmatrix}$ $C_y = [0\ 1\ 0\ 0]$ % measure x_2

$C_y = [0\ 0\ 0\ 1]$ % measure x_4

$D_{yw} = [0.0067\ \ 0.0089]$ $D_{zu} = [0\ 0\ 0\ 0\ 1]'$

$C_z = [\,eye(4);\ zeros(1,4)]$ $W = eye(2)$.

2. feedback gain $K^* = -(D_{zu}^T D_{zu})^{-1} B_u^T X^*$

where X^* satisfies the ARE: $A^T X^* + X^* A - X^* B_u (D_{zu}^T D_{zu})^{-1} B_u^T X^* + C_z^T C_z = 0$

estimation gain $F^* = -Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1}$

where Y^* satisfies the ARE: $AY^* + Y^* A^T - Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1} C_y Y^* + B_w W B_w^T = 0$

$k = \begin{bmatrix} -4.3274 & 4.7375 & 1.1920 & 0.8319 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $F = \begin{bmatrix} -0.2161 & -0.9703 & 0.4150 & -1.9495 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T$

3.

controller transfer function:

```
tfctrl =

  From input 1 to output...
           4.788 s^3 + 6.835 s^2 + 7.641 s + 3.457
   1:  ---------------------------------------------
        s^4 + 3.706 s^3 + 4.763 s^2 + 4.567 s + 1.788

   2:  0

  From input 2 to output...
   1:  0

   2:  0
```

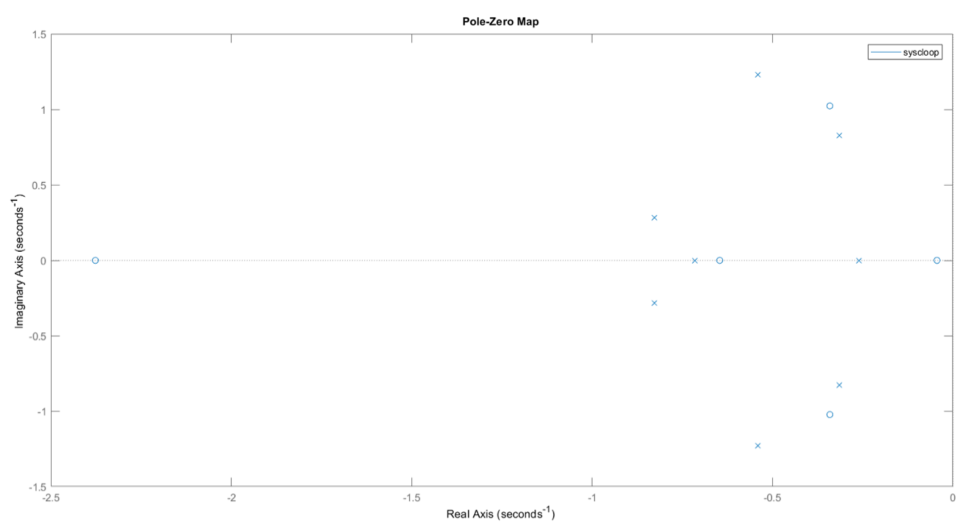
state-space realization →

```
sysctrl =

  A =
             x1        x2        x3        x4
   x1  -0.08735    -1.178   0.08889   0.04756
   x2     2.654    -3.336    -0.598   -0.3952
   x3    -3.712     1.528   -0.2826    0.1273
   x4         0    -1.869         1         0

  B =
           u1      u2
   x1  0.2161       0
   x2  0.9703       0
   x3  -0.415       0
   x4   1.949       0

  C =
           x1      x2      x3      x4
   y1  -4.327   4.738   1.192  0.8319
   y2       0       0       0       0

  D =
       u1  u2
   y1   0   0
   y2   0   0
```

$\dot{x} = A_c x + B_c y$

$u = C_c x$

4. pole-zero map is showed below.



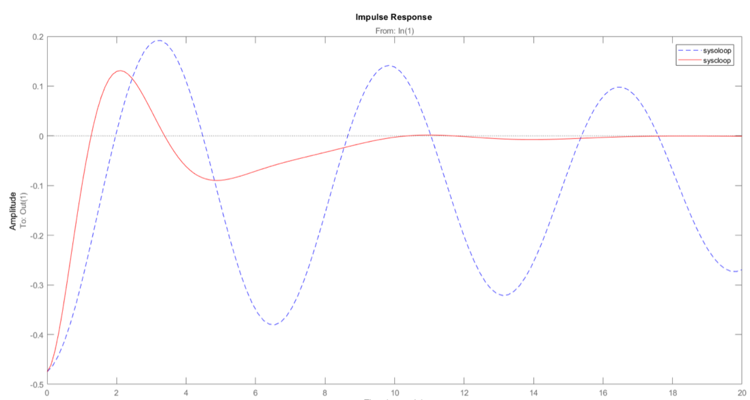
see next page for detail values.

frequency	damping coefficient
Wn =	zeta =
0.2606	1.0000
0.7161	1.0000
0.8732	0.9468
0.8732	0.9468
0.8862	0.3551
0.8862	0.3551
1.3423	0.4030
1.3423	0.4030

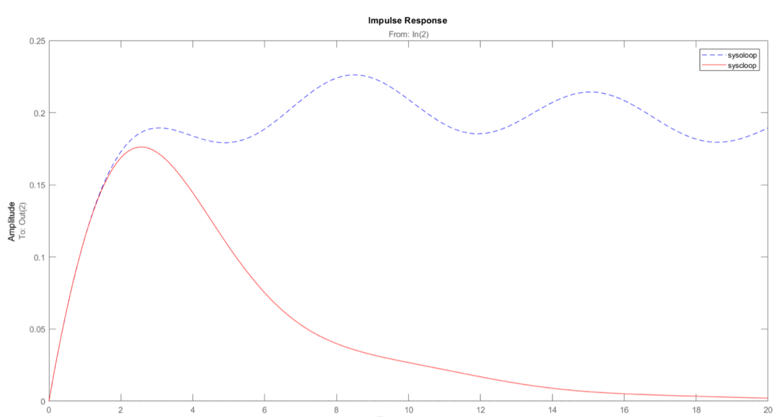
min

Apparently, I was able to achieve the two specifications.

5.



rudder to yaw rate



airelon to bank angle

△ How does controller affect the rudder to yaw rate behavior?
The open-loop system comes with an impulse and goes to zero at 600s.
The closed-loop system comes with an impulse and goes to zero at 10 s which is very fast.

△ How does the controller affect the aileron to bank angle behavior?
The open-loop system keeps unsteady in the first 100s and goes to zero at 800 s.
The closed-loop system comes with an impulse and goes to zero at 20s.

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b) 1. define $m = \dot{x} - B_u u$. $\ddot{x} = A\dot{x} + B_u\dot{u}$

$$\Rightarrow \dot{m} = \ddot{x} - B_u\dot{u} = A\dot{x} + B_u\dot{u} - B_u\dot{u} = A\dot{x}$$

plug into $\dot{x} = m + B_u u$ we get:

$$\dot{m} = Am + AB_u u$$

$$\tilde{y} = C_y\dot{x} = C_y m + C_y B_u u$$

Thus, apply Laplace transform to $\dot{x}$ we will get an "s" on the numerator of system transfer function. At the same time, "s" can be freely moved between system transfer function $S(s)$ and controller transfer function $C(s)$ as long as $S(s)$ and $C(s)$ are strictly proper. This is equivalent to having a zero at zero.

2. $\tilde{C}(s) = C_c(sI - A_c)^{-1}B_c$ ---②

$\tilde{y} = y - D_u u.$ ---①

First prove $C(s) = [I + \tilde{C}(s) \cdot D_u]^{-1} \tilde{C}(s)$:

proof: take Laplace tranform of ① we get

$$\tilde{Y}(s) = Y(s) - D_u U(s)$$

$$\Rightarrow U(s) = \tilde{C}(s) \tilde{Y}(s)$$

$$= \tilde{C}(s) (Y(s) - D_u U(s))$$

rearranging the equations:

$$(I + \tilde{C}(s) D_u) U(s) = \tilde{C}(s) Y(s)$$

$$\Rightarrow C(s) = \frac{U(s)}{Y(s)} = [I + \tilde{C}(s) D_u]^{-1} \tilde{C}(s) \quad \text{---③}$$

Then, we plug ② into the inverse of the equation ③

$$C^{-1}(s) = \tilde{C}^{-1}(s) [I + \tilde{C}(s) D_u]$$

$$= \tilde{C}^{-1}(s) + D_u$$

$$= B_c^{-1}(sI - A_c) C_c^{-1} + B_c^{-1} B_c D_u C_c C_c^{-1}$$

$$= B_c^{-1} (sI - (A_c - B_c D_u C_c)) C_c^{-1}$$

$$\xrightarrow{\text{inverse}} C(s) = C_c[sI - (A_c - B_c D_u C_c)]^{-1} B_c = [I + \tilde{C}(s) D_u]^{-1} \tilde{C}(s)$$

3. By using the same zero-adding method, like b) 1:

$C(s) = C_c[sI - (A_c - B_c D_u C_c)]^{-1} B_c$

define $n = \tilde{\dot{x}} - B_c y$

$$K(s) = sC(s) \Rightarrow \begin{cases} \dot{n} = (A_c - B_c D_u C_c)n + (A_c - B_c D_u C_c) B_c y \\ \tilde{u} = C_c n + C_c B_c y \end{cases}$$

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c)

1. define $m = \dot{x} - B_u u$

$\dot{m} = Am + AB_u u + B_w w$

$\tilde{y} = C_y m + C_y B_u u + D_{yw} w$

$\tilde{z} = C_z m + D_{zu} u$

$$A = \begin{bmatrix} -0.0558 & -0.9968 & -0.0802 & 0.0415 \\ 0.598 & -0.115 & -0.0318 & 0 \\ -3.05 & 0.388 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix} \quad B_{uk} = AB_u = \begin{bmatrix} 0.4853 & 0.0037 \\ 0.0541 & -0.0054 \\ -0.2777 & -0.0635 \\ 0.1148 & 0.1436 \end{bmatrix}$$

$$B_w = \begin{bmatrix} 0.0066 & 0.0038 \\ 0.0025 & 0.0089 \\ 0.0076 & 0.0046 \\ 0.0082 & 0.0038 \end{bmatrix} \quad C_y = [0\ 1\ 0\ 0] \quad C_z = [\mathrm{eye}(4); \mathrm{zeros}(1,4)];$$

$$C_{yt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{yu} = C_{yt} \cdot B_u = \begin{bmatrix} -0.4750 & 0.0077 \\ 0 & 0 \end{bmatrix} \quad D_{zu} = [0\ 0\ 0\ 0\ 1]' \quad W = \mathrm{eye}(2).$$

2. feedback gain: $K^* = -(D_{zu}^T D_{zu})^{-1} (A B_u)^T X^*$

where X^* satisfies the ARE: $A^T X^* + X^* A - X^*(AB_u)(D_{zu}^T D_{zu})^{-1}(AB_u)^T X^* + C_z^T C_z = 0$

estimation gain $F^* = -Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1}$

where Y^* satisfies the ARE: $AY^* + Y^* A^T - Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1} C_y Y^* + B_w W B_w^T = 0$

$$K = \begin{bmatrix} -2.9532 & 0.0086 & 0.9362 & 0.6838 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} -0.2102 & 0 \\ -0.9333 & 0 \\ 0.6949 & 0 \\ -2.2154 & 0 \end{bmatrix}$$

3.

$C(s)$

```
tfCs =

  From input 1 to output...
           0.2513 s^3 + 2.453 s^2 + 4.339 s + 2.18
   1:  ---------------------------------------------
       s^4 + 3.064 s^3 + 4.093 s^2 + 2.07 s + 0.1554

   2:  0

  From input 2 to output...
   1:  0

   2:  0

Cs =

  A =
            x1      x2      x3      x4
   x1   -1.784  -1.202  0.6281  0.4416
   x2   -0.871  -1.044  0.4339  0.3401
   x3   -1.255   1.078  -1.034 -0.4156
   x4   -3.447  -2.125   2.093   0.798

  B =
            u1      u2
   x1   0.2102       0
   x2   0.9333       0
   x3  -0.6949       0
   x4    2.215       0

  C =
             x1        x2        x3        x4
   y1    -2.953  0.008605    0.9362    0.6838
   y2         0         0         0         0

  D =
        u1  u2
   y1    0   0
   y2    0   0
```

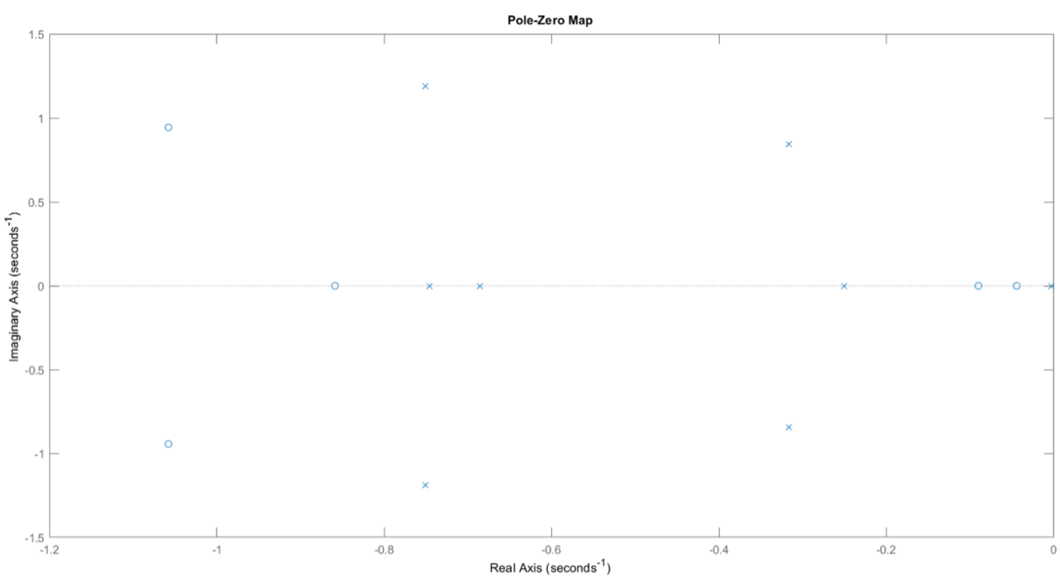
$K(s) = sC(s)$

```
tfKs =

  From input 1 to output...
        0.2513 s^4 + 2.453 s^3 + 4.339 s^2 + 2.18 s + 1.232e-15
   1:  ---------------------------------------------------------
             s^4 + 3.064 s^3 + 4.093 s^2 + 2.07 s + 0.1554

   2:  0

  From input 2 to output...
   1:  0

   2:  0

Ks =

  A =
            x1       x2       x3       x4
   x1   -1.784   -1.202   0.6281   0.4416
   x2   -0.871   -1.044   0.4339   0.3401
   x3   -1.255    1.078   -1.034  -0.4156
   x4   -3.447   -2.125    2.093    0.798

  B =
            u1       u2
   x1   -0.955        0
   x2  -0.7055        0
   x3   0.5399        0
   x4   -2.394        0

  C =
              x1        x2        x3        x4
   y1     -2.953  0.008605    0.9362    0.6838
   y2          0         0         0         0

  D =
            u1       u2
   y1   0.2513        0
   y2        0        0
```

where $C(s)$: $\begin{cases} \dot{\hat{x}} = (A_c - B_c D_u C_c)\hat{x} + B_c y \\ u = C_c \hat{x} \end{cases}$

$K(s) = sC(s) \Rightarrow \begin{cases} \dot{n} = (A_c - B_c D_u C_c)n + (A_c - B_c D_u C_c) B_c y \\ \tilde{u} = C_c n + C_c B_c y \end{cases}$

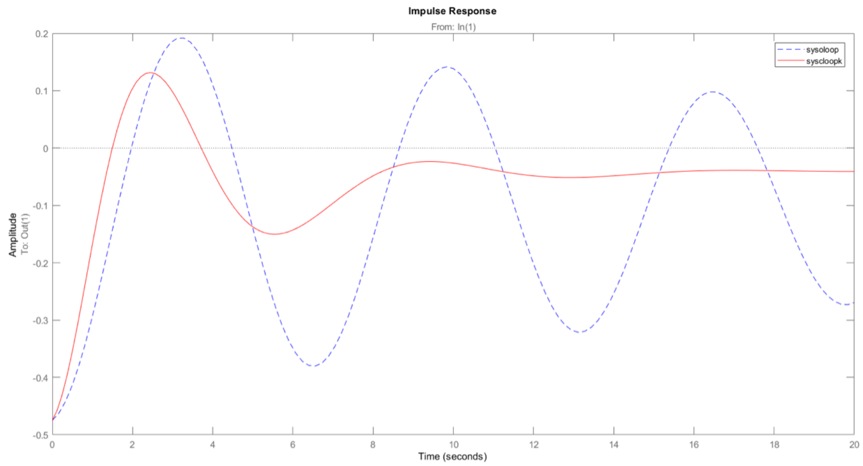
4. The pole-zero map is showed below.



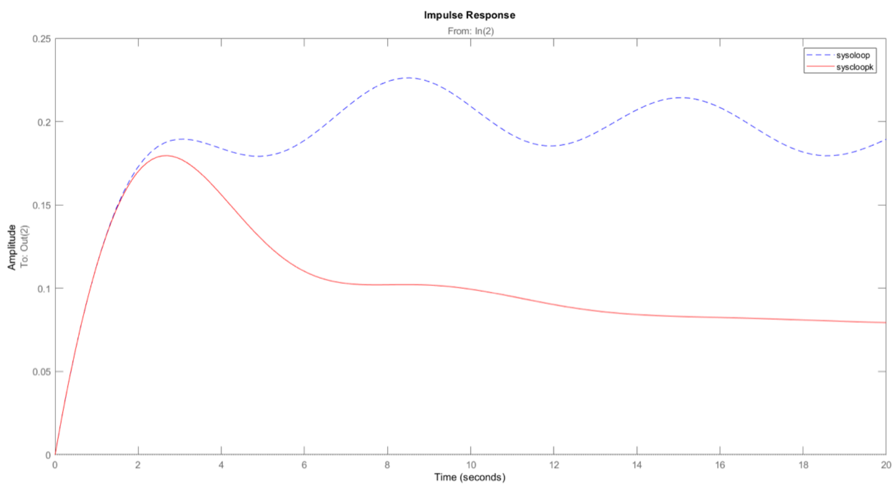
Wn =	zeta =
0.0028	1.0000
0.2508	1.0000
0.6857	1.0000
0.7460	1.0000
0.8991	0.3517
0.8991	0.3517 min
1.4063	0.5338
1.4063	0.5338

Apparently, it satisfies the two specifications.

5.



rudder to yaw rate



aileron to bank angle

△ How does controller affect the rudder to yaw rate behavior?
The open-loop system oscillates at the first 200 s and goes to zero at 1000 s.

The closed-loop system tends to keep decreasing steadily at 10 s and can hold a pretty long time which is good for controlling the plane.

△ How does the controller affect the aileron to bank angle behavior?
The open-loop system comes with an impulse and begins to decrease.
The closed-loop system comes with an impulse and oscillates a lot near 0.2 (amplitude) for a long time and goes to zero finally.

d) Zero on the controller will let the system keep steady first and then maintain its amplitude around a non-zero value for a long time then goes to zero finally.

This behavior of the system ensures the plane to maintain a small but steady yaw rate and bank angle when changing its flying direction.

In addition, I consider this "zero" as a differentiator and has similar influence as a low-pass filter which allows low frequency to pass. Since low frequency means small gain, the system will go to zero slowly.

Problem 2. a)

$$J = \lim_{t\to\infty} E\left[z_i(t)^T z_i(t) \right]$$

$$= \lim_{t\to\infty} E\left[\left((C_{zi} X + D_{zui} u \right)^T \left((C_{zi} X + D_{zui} u \right) \right]$$

$$= \lim_{t\to\infty} E\left[x^T C_{zi}^T C_{zi} X + u^T D_{zui}^T D_{zui} u \right] \qquad \leftarrow u = Kx$$

$$= \lim_{t\to\infty} E\left[x^T \left(C_{zi}^T C_{zi} + K^T D_{zui}^T D_{zui} K \right) x \right]$$

set $Q = C_{zi}^T C_{zi} + K^T D_{zui}^T D_{zui} K$

Then $J = \lim_{t\to\infty} E(x^T Q x) = \lim_{t\to\infty} E\left[\mathrm{trace}(Q x x^T) \right]$

$\because$ $x = \int_0^t e^{\hat{A}(t-\tau)} B_w w(\tau) d\tau$ where $\hat{A} = A + B_u K$.

$$\Rightarrow J = \lim_{t\to\infty} E\left[\mathrm{trace}\left(Q \int_0^t e^{\hat{A}(t-\tau)} B_w w(\tau) d\tau \cdot \int_0^t w^T(\sigma) B_w^T e^{\hat{A}^T(t-\sigma)} d\sigma \right) \right]$$

$$= \lim_{t\to\infty} \mathrm{trace}\left(Q \int_0^t e^{\hat{A}(t-\tau)} B_w \cdot \int_0^t E[w(\tau) w^T(\sigma)] e^{\hat{A}^T(t-\sigma)} d\sigma \, d\tau \right)$$

Integrate the equation above with respect to σ

$$J = \lim_{t\to\infty} \mathrm{trace}\left(Q \int_0^t e^{\hat{A}(t-\tau)} B_w W B_w^T e^{\hat{A}(t-\tau)} d\tau \right)$$

$$= \lim_{t\to\infty} \mathrm{trace}\left(Q \int_0^t e^{\hat{A}\xi} B_w W B_w^T e^{\hat{A}\xi} d\xi \right. \quad \text{where } \xi = t - \tau$$

$$= \lim_{t\to\infty} \mathrm{trace}\left[\left(\int_0^t e^{\hat{A}^T\xi} Q e^{\hat{A}\xi} d\xi \right) B_w W B_w^T \right]$$

$$= \mathrm{trace}\left(\left[\int_0^{\xi} e^{\hat{A}^T\xi} Q e^{\hat{A}\xi} d\xi \right] B_w W B_w^T \right)$$

$$= \mathrm{trace}\left(\bar{X} B_w W B_w^T \right)$$

where $\bar{X}$ is the solution to the Lyapunov equation:

$$\hat{A}^T \bar{X} + \bar{X} \hat{A} + Q = 0 \iff (A + B_u K)^T \bar{X} + \bar{X}(A + B_u K) + C_{zi}^T C_{zi} + K^T D_{zui}^T D_{zui} K = 0$$

in which $\bar{X} = \int_0^\infty e^{\hat{A}^T\xi} Q e^{\hat{A}\xi} d\xi$ is the observability Gramian.

$$= \int_0^\infty e^{(A+B_u K)^T \xi} \left(C_{zi}^T C_{zi} + K^T D_{zui}^T D_{zui} K \right) e^{(A+B_u K)\xi} d\xi$$

$$J = \mathrm{trace}\left(\left[\int_0^\infty e^{\hat{A}^T t} Q e^{\hat{A} t} dt\right] B_w w B_w^T\right)$$

$$= \mathrm{trace}\left(\left[\int_0^\infty e^{\hat{A} t} B_w w B_w^T e^{\hat{A}^T t} dt\right] Q\right)$$

$$= \mathrm{trace}\ (Y Q)$$

where Y is the solution to Lyapunov equation

$$(A + B_u K) Y + Y (A + B_u K)^T + B_w W B_w^T = 0$$

in which $Y = \int_0^\infty e^{(A+B_u K)t} B_w W B_w^T e^{(A+B_u K)^T t} dt$ is the controllability Gramian

b) Consider Controllability Gramian

First apply comparison Lemma : for $Y^* \geq Y$ $\mathrm{trace}\left(\left(C_z + D_{zu} K\right) Y^* \left(\left(C_z + D_{zu} K\right)^T\right) \geq J\right.$

Assuming $Y^* = X$, $L = KX$

Then we get $(AX + B_u L)^T + AX + B_u L + B_w W B_w^T \leq 0$ ... ①

Then introduce a new variable Z : $\Rightarrow Z \geq (C_z + D_{zu} K) X (C_z + D_{zu} K)^T$

$$= (C_z X + D_{zu} L) X^{-1} (C_z X + D_{zu} L)^T$$

Apply Schur- Complement to the inequality above

$$\Rightarrow \begin{bmatrix} Z & C_z X + D_{zu} L \\ (C_z X + D_{zu} L)^T & X \end{bmatrix} \geq 0$$

consider z_i instead of z $\Rightarrow$ $\begin{bmatrix} z_i & C_{zi} X + D_{zui} L \\ (C_{zi} X + D_{zui} L)^T & X \end{bmatrix} \geq 0$ ... ②

① and ② construct a LMI which can be used.

Consider Observability Gramian

First apply comparison Lemma : $X^* \geq \bar{X} \Rightarrow \mathrm{trace}\ (X^* B_w W B_w^T) \geq J$

X^* satisfies $(A + B_u K)^T X^* + X^* (A + B_u K) + (C_z + D_{zu} K)^T (C_z + D_{zu} K) \leq 0$

introduce a new variable Z^* $\Rightarrow Z^* \geq B_w^T X^* B_w$

$\mathrm{trace}\ (Z^* W) = \mathrm{trace}\ (B_w^T X^* B_w W) \geq J$

apply Schur complement:

$$Z^* \geq B_w^T X^* B_w \iff \begin{bmatrix} Z^* & B_w^T \\ B_w & (X^*)^{-1} \end{bmatrix} \geq 0 \quad \cdots (3)$$

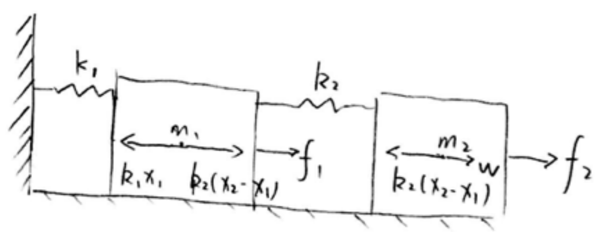
consider z_i instead of z: $\begin{bmatrix} Z_i^* & B_w^T \\ B_w & (X^*)^{-1} \end{bmatrix} \geq 0$

B_w is not a multi-valued matrix, which is not enough to constrain Z_i^*, i.e the multiobjective problem

While Controllability Gramian yields (1) and (2), which use C_{zi} and D_{zwi} to constrain Z_i. This satisfies the constrain conditions of LMI.

Thus, we can use Controllability Gramian to solve this LMI problem while Observability Gramian can not be used.

3. (a)



for m_1: $\quad m_1\ddot{x}_1 = k_2(x_2-x_1) + f_1 - k_1x_1$

for m_2: $\quad m_2\ddot{x}_2 = f_2 - k_2(x_2-x_1) + w$

choose state $x = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + B_w \cdot w$

plug $m_1 = m_2 = 1$ $k_1 = k_2 = 1$ into the state space we get:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} w$$

$$z = C_z x + D_{zu} u$$

where $C_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $D_{zu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ as computed in b)

b) $Z_0 = u = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = C_z(1{:}2, :) X + D_{zu}(1{:}2, :) u$

$$= \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{C_{z0}} X + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{D_{zu0}} u$$

$$z_1 = x_1 = C_z(3, :) X + D_{z_u}(3, :) u = \underbrace{[1 \; 0 \; 0 \; 0]}_{C_{z1}} x + \underbrace{[0 \; 0]}_{D_{zu1}} u$$

$$z_2 = x_2 = C_z(4, :) X + D_{zu}(4, :) u = \underbrace{[0 \; 1 \; 0 \; 0]}_{C_{z2}} X + \underbrace{[0 \; 0]}_{D_{zu2}} u$$

LMI satisfies min trace(Z_0)

LMI 1: $AX + XA^T + B_u L + L^T B_u^T + B_w W B_w^T \le 0$

LMI 2: $\begin{bmatrix} Z_0 & C_{z0} X + D_{zu0} L \\ X C_{z0}^T + L D_{zu0}^T & X \end{bmatrix} \ge 0$

LMI 3: $\begin{bmatrix} Z_1 & C_{z1} X + D_{zu1} L \\ (C_{z1} X + D_{zu1} L)^T & X \end{bmatrix} \ge 0$

LMI 4: $\begin{bmatrix} Z_2 & C_{z2} X + D_{zu2} L \\ (C_{z2} X + D_{zu2} L)^T & X \end{bmatrix} \ge 0$

LMI 5: $\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \text{trace}(z_1) \\ \text{trace}(z_2) \end{bmatrix} \ge \begin{bmatrix} 0 \\ -\mu^2 \\ 0 \\ -\mu^2 \end{bmatrix}$ LMI 6 $X \ge 0$

code is attached.

$$K = \begin{bmatrix} -0.1686 & -0.0418 & -0.4943 & -0.3047 \\ -0.4548 & -0.6912 & -0.3047 & -1.1356 \end{bmatrix}$$

controller: $\dot{x} = Ax + B_u u$

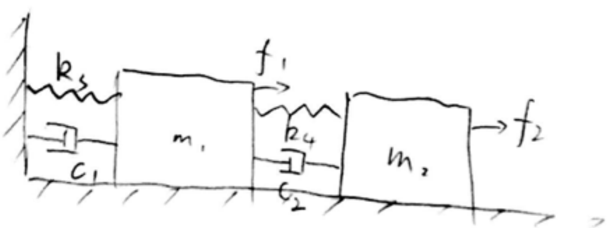
$u = Kx$

corresponding state-space is $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$C = \begin{bmatrix} -0.1686 & -0.0418 & -0.4943 & -0.3047 \\ -0.4548 & -0.6912 & -0.3047 & -1.1356 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and closed-loop system is computed in code, too.

c). We apply new springs paraller to k_1 and k_2 which will not increase the number of states but will change k_1 and k_2 to k_3 and k_4.
Again, add dampers paraller to k_3 and k_4 springs



for m_1: $\ddot{x}_1 m_1 = k_4(x_2 - x_1) + f_1 + c_2(\dot{x}_2 - \dot{x}_1) - c_1\dot{x}_1 - k_3 x_1$ ... ①

for m_2: $\ddot{x}_2 m_2 = f_2 + w - k_4(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1)$ ... ②

Control law obtained in b): the 4^{th} row, 1^{st} column element equals to the element locates in the 4^{th} row, 2^{nd} column.

~~we obtain K~~

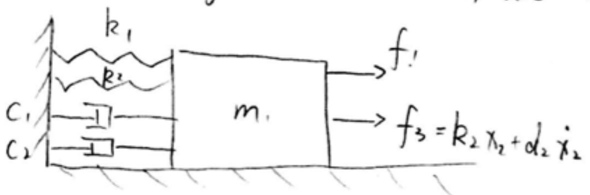
write state-space using ① and ②:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_3+k_4}{m_1} & \frac{k_4}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_4}{m_2} & -\frac{k_4}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} w$$

from $\hat{A}$ matrix we can conclude 4th row, 1st column element still equals to the element in - 4th row, 2nd column element.

The control seems to be the same, i.e. the control law is not implemented.

d) Consider a system between m_1 and wall



$$m_1 \ddot{x}_1 = f_1 + f_3 - (k_1+k_2) x_1 - (c_1+c_2)\dot{x}_1$$

choose state $x = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix}$, we can obtain state space:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} w$$

$$z_0 = f_1 = [0 \quad 0]\, x + [1 \quad 0] \begin{bmatrix} f_1 \\ f_3 \end{bmatrix}$$

$$z_1 = [1 \quad 0] \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} + [0 \quad 0]\, u$$

construct LMI which satisfies min trace(z_0)

LMI 1: $\begin{bmatrix} Z_0 & C_{z0}X + D_{zw0}L \\ X C_{z0}^T + L^T D_{zw0}^T & X \end{bmatrix} \geq 0$

LMI 3: $AX + XA^T + B_u L + L^T B_u^T + B_w W B_w^T \leq 0$

LMI 2: $\begin{bmatrix} Z_1 & C_{z1}X + D_{zw1}L \\ (X C_{z1}^T + L^T D_{zw1})^T & X \end{bmatrix} \geq 0$

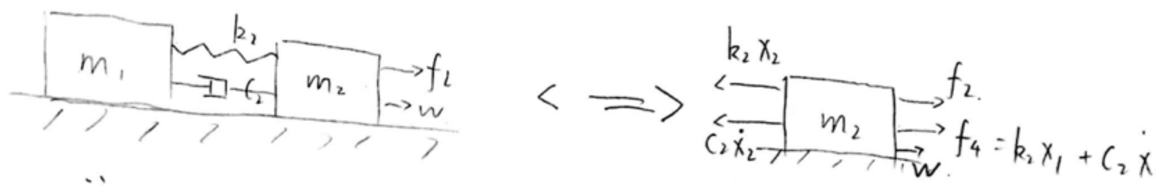
LMI 4: trace$(Z_1) \leq \mu^2$

we obtain $K = \begin{bmatrix} 0.0001 & 0.0001 \\ -0.6170 & -1.1739 \end{bmatrix}$

compute closed-loop matrix $\hat{A} = \begin{bmatrix} 0 & 1 \\ -2.6169 & -1.1738 \end{bmatrix}$

$$\Rightarrow \quad k_1^* + k_2^* = 2.6169$$
$$c_1 + c_2 = 1.1738$$

Consider system between m_1 and m_2



$m_2 \ddot{x}_2 = f_2 + f_4 + w - k_2 x_2 - c_2 \dot{x}_2$ choose state $x = \begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f_2 \\ f_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} w$$

$$z_0 = f_2 = [0 \;\; 0]\, x + [1 \;\; 0] \begin{bmatrix} f_2 \\ f_4 \end{bmatrix}$$

$$z_1 = [1 \;\; 0] \begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix} + [0 \;\; 0]\, u$$

formulate the same LMI as the last wall - m_1 system

$$K = \begin{bmatrix} 0 & 0 \\ -1.6162 & -3.2267 \end{bmatrix}$$

compute closed-loop matrix $\hat{A} = \begin{bmatrix} 0 & 1 \\ -2.6161 & -3.2266 \end{bmatrix}$

$$\Rightarrow \quad k_2^* = 0.0008$$

$c_1 = -1.0528$ $\rightarrow$ trivial

$c_2 = 3.2266$

Thus, $k_1^* = 2.6161$ $k_2^* = 0.0008$, $C_1 = -1.0578$ $C_2 = 3.2266$

↳ trivial! No matter how I choose tol this C_1 value still remains below zero. I wrote a loop to compute but cannot yield an answer.

If we ignore the noise added on m_2 and set tol $= 1.e^{-6}$, we can get the system below.

The closed-loop system matrix $\bar{A}$:

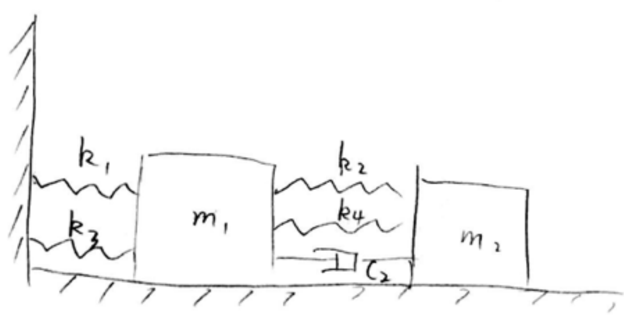
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1^*+k_2^*}{m_1} & \frac{k_2^*}{m_1} & -\frac{(C_1+C_2)}{m_1} & \frac{C_2}{m_1} \\ \frac{k_2^*}{m_2} & -\frac{k_2^*}{m_2} & \frac{C_2}{m_2} & -\frac{C_2}{m_2} \end{bmatrix}$$

$$A + BK^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.6172 & 1.4625 & -1.1742 & 1.1742 \\ 1.4625 & -1.4625 & 1.1742 & -1.1742 \end{bmatrix}$$

A. B is known.

$\Rightarrow$ we can obtain $K^* = \begin{bmatrix} -0.6172 & 0.4625 & -1.1742 & 1.1742 \\ 0.4625 & -0.4625 & 1.1742 & -1.1742 \end{bmatrix}$

Thus.



where $k_1 = 1$

$k_3 = k_1^* - k_1 = 0.1547$

$k_2 = 1$

$k_4 = k_2^* - k_2 = 0.4625$

$C_2 = 1.1742$

Therefore, with noise w added on m_2, we can only yield to trivial solution i.e $C_2 < 0$.