MAE 280B
Linear Control Design – Winter 2018

Final Exam

Instructions

- Due on 03/19/2018 in my office EBU I 1602 by 5:00 PM;
- Use Matlab or Mathematica;
- You get marks for clarity;
- You loose marks for obscurantism;
- This exam has 3 questions, 40 total points and 2 bonus points
- Good luck!

Questions

1. **LQG Design.**

   The LTI system

$$\dot{x}(t) = \begin{bmatrix} -.0558 & -.9968 & .0802 & .0415 \\ .598 & -.115 & -.0318 & 0 \\ -3.05 & .388 & -.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} .00729 & 0 \\ -0.475 & 0.00775 \\ 0.153 & 0.143 \\ 0 & 0 \end{bmatrix} u(t)$$

   is a simplified trim model of a Boeing 747 during cruise flight. The four states are

   x_1: side-slip angle,

   x_2: yaw rate,

   x_3: roll rate, and

   x_4: bank angle.

   The two inputs are

   u_1: rudder deflection, and

   u_2: aileron deflection.

   All angles and angular velocities are in radians and radians/sec. This is one of the demos of the Matlab Control Toolbox, so check it out before you answer the next questions! Look for *Yaw Damper Design for a 747 Jet Aircraft.*

   (a) (6 points) Design an LQG controller that measures the yaw rate and feeds back rudder deflections that achieves the following two objectives: damping coefficient $\zeta > 0.35$, with natural frequency $\omega_n < 1.0$ rad/sec for the most lightly damped closed loop pole. Here you will have to augment the system with noisy inputs and outputs in order to define a cost function and effectively solve the problem. You're free to do it in any way that you think can solve the proposed problem! On your answer show me the following:

      1. The final arrangement of your augmented system, with noisy inputs w and controlled output z and the corresponding system matrices;

2. Write down the Riccati equations that you used to compute the state feedback and state estimation gains and their computed values;
3. The controller transfer function that results from your calculation, both the state space realization as well as the controller transfer function;
4. The closed loop poles, indicating whether you were able to achieve the specifications or how close were you;
5. How does the controller affects the aileron to bank angle behavior? Compare the impulse response of the open-loop system with the closed-loop system in the first 20s. Compare also the open loop and closed loop impulse responses from rudder to yaw rate.

(b) (4 points) If you follow the discussion in the Matlab demo, you will see that the controller ends up being augmented with a *washout filter*. Here we will let LQG achieve the same effect for you. In order to do that you will have to show the following:

1. The important part of the washout filter is the zero at zero. So first show how you can modify the system you used to solve part (a) so that $\dot{x}_2$ is the output to be used by the controller without increasing the number of states of the system! Why is this equivalent to having a zero at zero at the controller?
2. The resulting system will not be strictly proper. So, like in one of your homework problems, you will have to modify your system in order to be able to design an LQG controller. Here is a refresher: let $y = C_y x + D_u u + D_{yw} w$ be the measured output you obtain using item b.1). In the homework you showed that $C(s) = [I + \tilde{C}(s)]^{-1}\tilde{C}(s)$ would be an admissible controller to the original system if $\tilde{C}(s)$ were an admissible controller for the system with output $\tilde{y} = C_y x + D_{yw} w = y - D_u u$. Show that if $\tilde{C}(s) = C_c(sI - A_c)^{-1}B_c$ then
$$C(s) = [I + \tilde{C}(s)]^{-1}\tilde{C}(s) = C_c[sI - (A_c - B_c D_u C_c)]^{-1}B_c.$$
3. The last step is to modify the controller you obtained at the end of part (b.2) by augmenting it with the zero that was placed at the plant in order to facilitate the design of the controller in item (b.1). Compute the state-space realization of the controller $K(s) = s\,C(s)$, where $C(s) = C_c[sI - (A_c - B_c D_u C_c)]^{-1}B_c$. If you now substitute the values of (A_c, B_c, C_c) from your LQG controller, $K(s)$ will be your final controller.

(c) (6 points) Use what you learned in part (b) to redo part (a), that is to design an LQG controller that has a zero at zero and satisfy the desired closed-loop specifications.

(d) (2 points (bonus)) What is the impact of the zero on the controller on the overall closed loop performance? Explain what is the role zero of the zero in this problem.

2. **Multiobjective LQR design**

Consider the LTI system with N controlled outputs

$$\begin{aligned}
\dot{x} &= Ax + B_u u + B_w w, \quad x(0) = 0, \quad x \in \mathbb{R}^n \\
z_0 &= C_{z0} x + D_{zu0} u, \\
z_1 &= C_{z1} x + D_{zu1} u, \\
&\vdots \\
z_N &= C_{zN} x + D_{zuN} u,
\end{aligned}$$

where $w(t)$ is zero mean Gaussian white-noise vector with variance $W \succ 0$. Derive a semidefinite program (SDP) that solves the problem of computing the state feedback control

$$u = Kx,$$

which minimizes the cost function

$$J_0 = \lim_{t \to \infty} E\{z_0(t)^T z_0(t)\}$$

subject to the constraints

$$J_1 < \mu_1, \qquad \cdots \qquad J_N < \mu_N,$$

where μ_i are given and

$$J_i = \lim_{t \to \infty} E\left\{z_i(t)^T z_i(t)\right\}.$$

for $i = 1, \ldots, N$.

(a) (4 points) Show how the cost function and the constraints can be computed in terms of the observability and controllability Gramians.

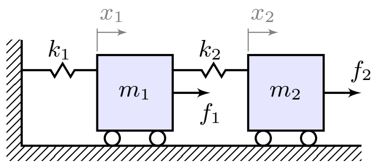
(b) (4 points) Show that only one of the Gramians leads to LMIs which include

$$\begin{bmatrix} Z_i & C_{zi}X + D_{zui}L \\ XC_{zi}^T + L^T D_{zui}^T & X \end{bmatrix} \succ 0, \qquad i = 1, \ldots, N,$$
$$AX + XA^T + B_u L + L^T B_u^T + B_w W B_w^T \prec 0,$$

after relaxing equations into inequalities and performing a change-of-variables. Explain why the other Gramian cannot produce similar LMIs.

3. **Mass-Spring-Damper Design.**

Consider the following mass-and-spring system with two masses:



where $m_1 = m_2 = 1$ and $k_1 = k_2 = 1$, x_1 and x_2 represent the displacement of masses m_1 and m_2, and f_1 and f_2 are forces applied at the center of each mass. Assume zero friction at the rollers.

(a) (2 points) Write state-space dynamic equations for this system.

(b) (6 points) Suppose that you can use both f_1 and f_2 as control inputs, i.e.

$$u(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix},$$

and that a zero-mean Gaussian white-noise disturbance with unitary variance is applied at the mass m_2. Show how the semidefinite program that you derived in Question 2 can be used to compute the state feedback control which minimizes $\lim_{t\to\infty} E\{u(t)^T u(t)\}$ subject to the performance constraints

$$\lim_{t\to\infty} E\{x_1(t)^2\} < \mu^2 \qquad \text{and} \qquad \lim_{t\to\infty} E\{x_2(t)^2\} < \mu^2$$

where $\mu = 1/2$ is given. Solve the SDP and compute the resulting controller.

(c) (2 points) What is the structure of the control law you obtained as a result to part (b)? Can you implement it using only springs and dampers?

(d) (6 points) Explain how the SDP you derived in question b) can be modified to provide a control law that could be implemented as a spring-damper system between mass m_1 and the wall and another spring-damper system between mass m_1 and m_2. Solve the resulting SDP problem and find a state feedback controller that satisfies this structural constraint.