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CSc 301 HW#2

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Q1. $P = \dfrac{P_R(x-x_1) - P_L(x-x_3)}{x_3 - x_1}$

First order divide differences:

$f[x_1, x_2] = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Second order divide differences:

$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1 - x_2]}{x_3 - x_1}$

Divide differences table:

x_i	0th Order	1th Order	2nd Order
x_1	$f(x_1)$		
		$\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$	
x_2	$f(x_2)$		
		$\dfrac{f(x_3)-f(x_2)}{x_3-x_2}$	$\dfrac{f[x_2, x_3]-f[x_1 - x_2]}{x_3-x_1}$
x_3	$f(x_3)$		
		$\dfrac{f(x_4)-f(x_3)}{x_4-x_3}$	$\dfrac{f[x_4, x_3]-f[x_2, x_3]}{x_4-x_2}$
x_4	$f(x_4)$		
⋮	⋮	⋮	⋮

The geometric meaning of first order differences:

Assuming the <u>linear interpolant</u> is given by

$f(x) = c_0 + c_1(x - x_1)$

$c_0 = f(x_1)$

$c_1 = f[x_1, x_2]$

$f(x) = f(x_1) + \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$

The geometric meaning of second order difference:

Assuming the <u>quadratic interpolant</u> is given by

$f(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2)$

$c_0 = f(x_1)$

$c_2 = f[x_1, x_2]$

$c_3 = f[x_1, x_2, x_3]$

$f(x) = f(x_1) + \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + \dfrac{\dfrac{f(x_3) - f(x_2)}{x_3 - x_2} - \dfrac{f(x_2) - f(x_1)}{x_3 - x_1}}{x_3 - x_1}(x - x_1)(x - x_2)$

Lagrangian interpolation

Q2.

x_i	$f(x_i)$	$f(x_i, x_{i+1})$	$f(x_{i-1}, x_i, x_{i+1})$				
-5	$\frac{1}{1+(-5)^2} = \frac{1}{26}$						
-4	$\frac{1}{1+(-4)^2} = \frac{1}{17}$	$\frac{\frac{1}{17} - \frac{1}{26}}{-4-(-5)} = \frac{9}{442}$					
-3	$\frac{1}{1+(-3)^2} = \frac{1}{10}$	$\frac{7}{170}$	$\frac{391}{37570}$				
-2	$\frac{1}{1+(-2)^2} = \frac{1}{5}$	$\frac{5}{50}$	$\frac{1}{34}$	$\frac{357}{56355}$			
-1	$\frac{1}{1+(-1)^2} = \frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{2}{85}$	$\frac{323}{75140}$		
0	1	$\frac{1}{2}$	$\frac{1}{10}$	0	$\frac{1}{170}$	$\frac{9}{4420}$	
1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{30}$	$-\frac{1}{120}$	$-\frac{1}{408}$	$-\frac{11}{159120}$
2	$\frac{1}{5}$	$-\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{30}$	0	$\frac{1}{720}$	$\frac{47}{12240}$
3	$\frac{1}{10}$	$-\frac{5}{50}$	$\frac{1}{10}$	0	$\frac{1}{120}$	$-\frac{1}{720}$	
4	$\frac{1}{17}$	$-\frac{7}{170}$	$\frac{1}{34}$	$-\frac{2}{85}$	$\frac{1}{170}$	$\frac{1}{408}$	
5	$\frac{1}{26}$	$-\frac{9}{442}$	$\frac{391}{37570}$	$\frac{357}{56355}$	$-\frac{323}{75140}$	$\frac{9}{4420}$	

$C_0 = \frac{1}{26} = 0.0385$

$C_1 = \frac{9}{442} = 0.0204$

$C_2 = \frac{391}{37570} = 0.0104$

$C_3 = \frac{357}{56355} = 0.0063$

$C_4 = \frac{323}{75140} = 0.0043$

$C_5 = -\frac{9}{4420} = -0.0020$

$C_6 = -\frac{11}{159120} = -0.0000691$

$C_7 = 0.0039 = 0.0005584$

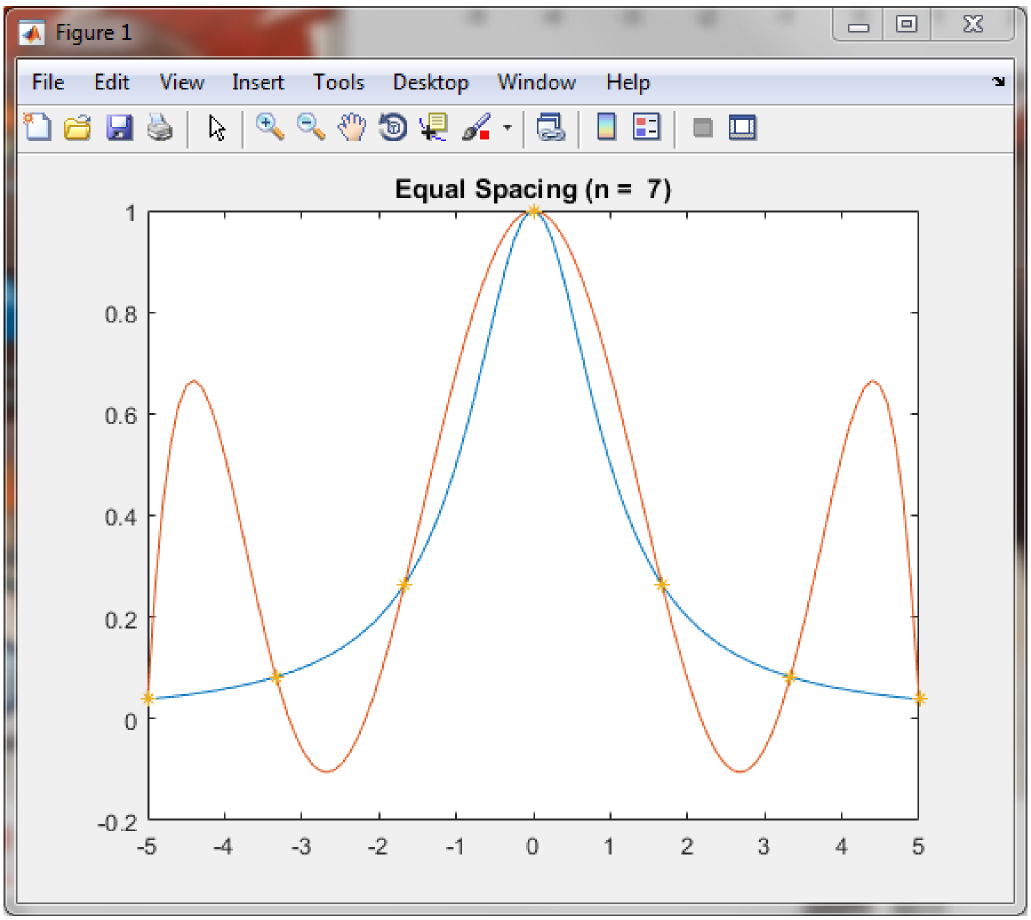
Its coefficients are decreasing from C_0 to C_6, but at C_7, its coefficients are increasing.

According to the Runge phenomenon, if a function has ill-behaved higher derivatives (as this example), then the quality of polynomial interpolants may actually decrease as the degree increases.

If we see the graph, while the interplant "captures" the trend of the function in the middle part of the interval, it blows up near the endpoints.

```
% Script File: RungeEg2
% For n = 7:2:15, the equal-spacing interpolants of f(x) = 1/(1+x^2) on
[-5,5]'
% are of plotted.

close all
x = linspace(-5,5,100)';
y = ones(100,1)./(1 + x.^2);
for n=7:2:15
   figure
   xEqual = linspace(-5,5,n)';
   yEqual = ones(n,1)./(1+xEqual.^2);
   cEqual=InterpN(xEqual,yEqual);
   pValsEqual = HornerN(cEqual,xEqual,x);
   plot(x,y,x,pValsEqual,xEqual,yEqual,'*')
   title(sprintf('Equal Spacing (n = %2.0f)',n))
end
```



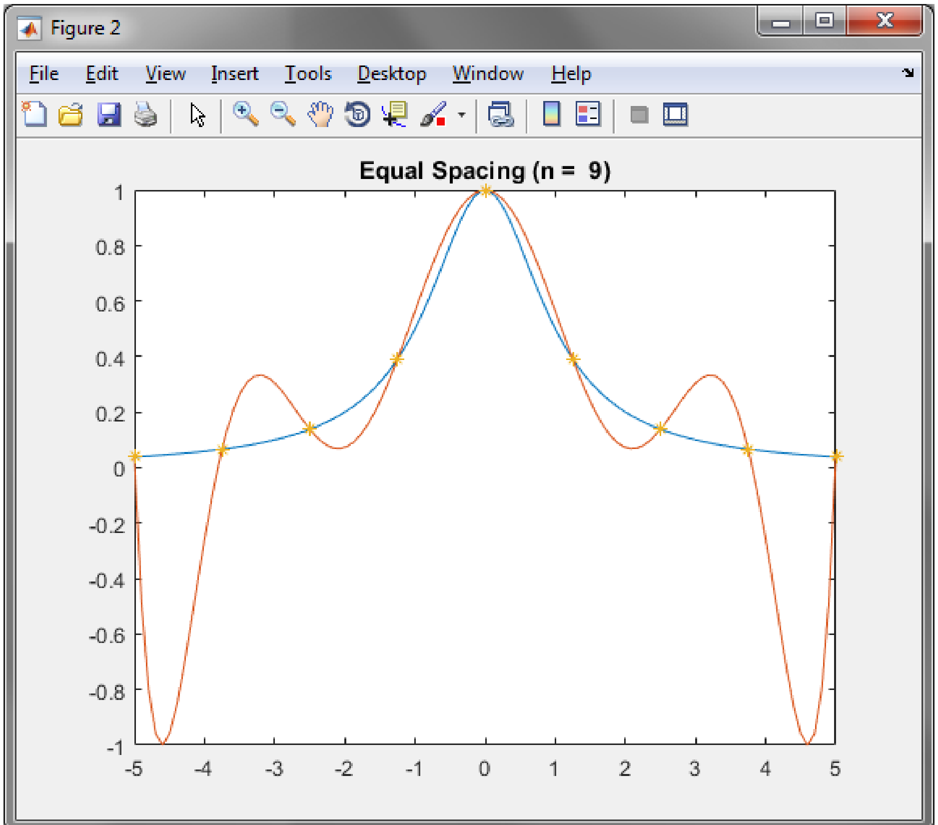


Figure 2
File Edit View Insert Tools Desktop Window Help
Equal Spacing (n = 9)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-5 -4 -3 -2 -1 0 1 2 3 4 5

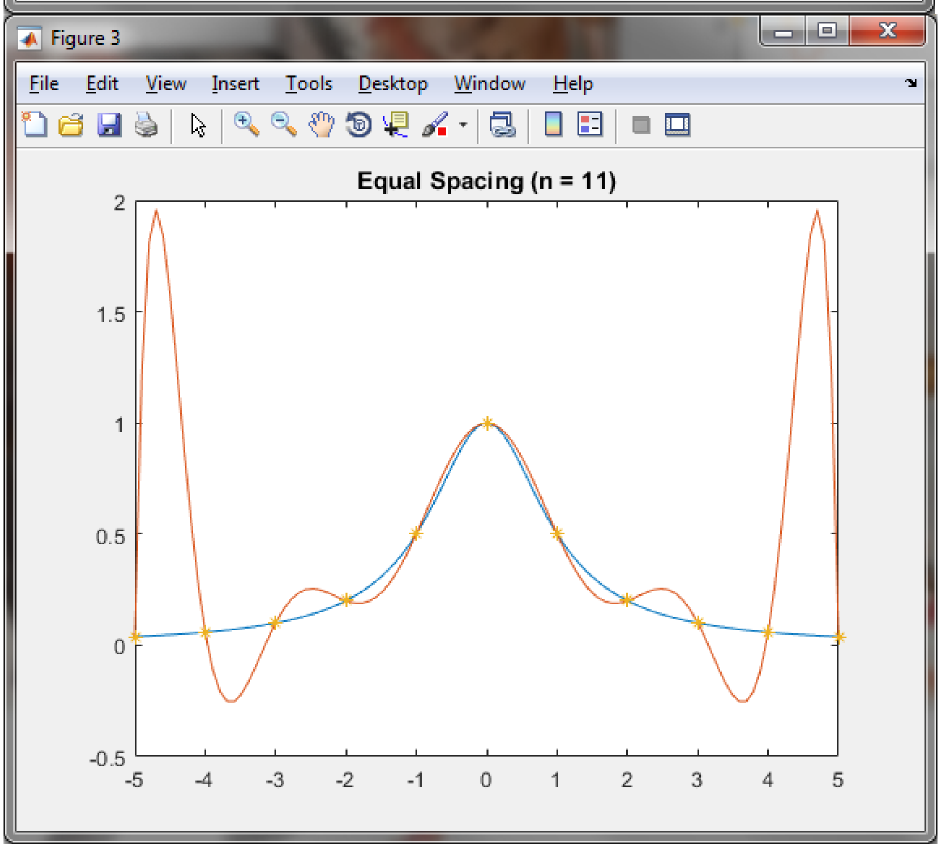


Figure 3
File Edit View Insert Tools Desktop Window Help
Equal Spacing (n = 11)
2
1.5
1
0.5
0
-0.5
-5 -4 -3 -2 -1 0 1 2 3 4 5

Q3. Determine (analytically) the spacing h in a table of

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$

a) $f(x) = \underbrace{f(x_i) + f'(x_i)(x-x_i)}_{\text{1st degree polynomial }(P_1)} + \underbrace{\frac{f''(x_i)}{2!}(x-x_i)(x-x_{i+1})}_{\text{Error term of 1st degree polynomial}}$

So, $f(x) - P_1(x) = \frac{f''(x_i)}{2}(x-x_i)(x-x_{i+1}) = E_1$

$|E_1| = |-\frac{2}{8}x^{-\frac{3}{2}}||(x-x_i)(x-x_{i+1})|$

where $\Delta x = h = (x_{i+1}, x_i)$

$\max(x-x_i)(x-x_{i+1}) = \left(\frac{h}{2}\right)^2$

$x \in [x_i, x_{i+1}]$

$x \in [1,2]$ So, $|-\frac{1}{8}x^{-\frac{3}{2}}||(x-x_i)(x-x_{i+1})| \le |-\frac{1}{8}| \cdot \frac{h^2}{4}$

$\le \frac{1}{8} \cdot \frac{h^2}{4}$

$\Rightarrow \boxed{|E_1| \le \frac{h^2}{8}}$

Since this table will yield an accuracy "N".

$|E_1| \le \frac{h^2}{8} \le N$

$\Rightarrow \boxed{h \le \sqrt{N \cdot 8}}$

b) $f(x) = \underbrace{f(x_i) + f'(x_i)(x-x_i) + \frac{f''(x_i)}{2!}(x-x_i)(x-x_{i+1})}_{\text{2nd degree polynomial }(P_2)} + \underbrace{\frac{f'''(x_i)}{3!}(x-x_{i-1})(x-x_i)(x-x_{i+1})}_{\text{Error term of 2nd degree polynomial}}$

So, $f(x) - P_2(x) = E_2$

$|E_2| = |\frac{f'''(x_i)}{3!}||(x-x_{i-1})(x-x_i)(x-x_{i+1})|$ where $\Delta x = h = (x_{i+1}, x_i)$

$x \in [1,2]$ So, $|\frac{\frac{3}{8}x^{-\frac{5}{2}}}{3!}||(x-x_{i-1})(x-x_i)(x-x_{i+1})| \le \frac{3}{8} \cdot \frac{1}{3!} \cdot \frac{2h^3}{3\sqrt{3}}$

$\max(x-x_{i-1})(x-x_i)(x-x_{i+1}) = \left(\frac{2h^3}{3\sqrt{3}}\right)$

$x \in [x_{i-1}, x_i, x_{i+1}]$

$\boxed{|E_2| \le \frac{h^3}{24\sqrt{3}}}$

Since this table will yield an accuracy "N".

$|E_2| \le \frac{h^3}{24\sqrt{3}} \le N$

$\Rightarrow \boxed{h \le \sqrt[3]{24\sqrt{3}N}}$

Q4

1) Calling interp2 once:

Interpolate Over a Grid Using Bilinear Method:

```
[X,Y] = meshgrid(-3:3);
V = peaks(7);
figure
surf(X,Y,V)
title('Original Sampling');

[Xq,Yq] = meshgrid(-3:0.25:3);
Vq = interp2(X,Y,V,Xq,Yq,'bilinear');
figure
surf(Xq,Yq,Vq);
title('Bilinear Interpolation Over Finer Grid');
```

Reference: https://www.mathworks.com/help/matlab/ref/interp2.html

2) N times iteratively doubling the size:

```
[X,Y] = meshgrid(-3:3);
V = peaks(7);
figure
surf(X,Y,V)
title('Original Sampling');

[Xq,Yq] = meshgrid(-3:3);
Xq=2*Xq;
Yq=2*Yq;
Vq=2*V;
figure
surf(Xq,Yq,Vq)
title('Doubling the image size');
```

- If we are using first way, calling interp2, it not just magnifies image, it also refines image, let the image becomes clear.
- If we iteratively doubling the size, it resizes image and makes image bigger, but it doesn’t change the grid or change distance between each point. That cannot make the image becomes clear.