Output Feedback Nonlinear Control of An Underactuated Flying Inverted Pendulum: Supplementary Material

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Abstract

This is a complementary document to the paper presented in [1]. Here we fully disclose all the detailed derivations and equations that are essential to implement the controller reported therein.

I. The transition matrix and the observability Gramian calculations

$$\mathbf{\Phi}(t,t_0) := \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 \cdots \quad \text{(SM 1)}$$

According to the transition matrix definition in (SM 1), the transition matrix's second and third terms are calculated as

$$\int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 = \begin{bmatrix} \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{m_Q \ell}{m_T}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top & \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and} \quad \text{(SM 2)}$$

$$\int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{n_{t2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & \mathbf{0} & \mathbf{0} & \boldsymbol{n_{t2}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ respectively,} \quad \text{(SM 3)}$$

where $\boldsymbol{n_{t1}} := (t-t_0)\mathbf{I}$, $\boldsymbol{n_{t2}} := \frac{1}{2}(t-t_0)^2\mathbf{I}$, $\boldsymbol{n_{q1}} := \frac{m_Q\ell}{m_T}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top$ and $\boldsymbol{n_{q2}} := \frac{m_Q\ell}{m_T}\int_{t_0}^{t}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top d\tau d\tau$.

It is noticed that the matrix of $\mathbf{A}$ is nilpotent of index 3, i.e. $\mathbf{A}^n = \mathbf{0}$ for $n \geq 3$ where n is a positive integer. Based on the definition of the observability Gramian , using (SM 1) and the nilpotent property of $\mathbf{A}$, it is given by

$$\boldsymbol{W}(t_0,t_f) = \int_{t_0}^{t_f} \begin{bmatrix} 1 & t-t_0 & n_t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ t-t_0 & (t-t_0)^2 & n_t(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ n_t & n_t(t-t_0) & {n_t}^2 + {\boldsymbol{n_{q2}}}^\top\boldsymbol{n_{q2}} & {\boldsymbol{n_{q2}}}^\top & (t-t_0){\boldsymbol{n_{q2}}}^\top & n_t{\boldsymbol{n_{q2}}}^\top \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & 1 & t-t_0 & n_t \\ \mathbf{0} & \mathbf{0} & (t-t_0)\boldsymbol{n_{q2}} & t-t_0 & (t-t_0)^2 & (t-t_0)n_t \\ \mathbf{0} & \mathbf{0} & n_t\boldsymbol{n_{q2}} & n_t & (t-t_0)n_t & {n_t}^2 \end{bmatrix} \otimes \mathbf{I}\ dt \quad \text{(SM 4)}$$

where $n_t := \frac{1}{2}(t-t_0)^2$.

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Setting $t_0 = t - \delta$ and $t_f = t$, $\boldsymbol{W}(t_0, t_f)$ is obtained as

$$\boldsymbol{W}(t-\delta,t) = \begin{bmatrix} \delta & \frac{\delta^2}{2} & \frac{\delta^3}{6} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\delta^2}{2} & \frac{\delta^3}{3} & \frac{\delta^4}{8} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\delta^3}{6} & \frac{\delta^4}{8} & \frac{\delta^5}{20} + \int_{t-\delta}^{t} \boldsymbol{n_{q2}}^\top \boldsymbol{n_{q2}} & \boldsymbol{n_{q2}}^\top & (t-t_0)\boldsymbol{n_{q2}}^\top & n_t \boldsymbol{n_{q2}}^\top \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & \delta & \frac{\delta^2}{2} & \frac{\delta^3}{6} \\ \mathbf{0} & \mathbf{0} & (t-t_0)\boldsymbol{n_{q2}} & \frac{\delta^2}{2} & \frac{\delta^3}{3} & \frac{\delta^4}{8} \\ \mathbf{0} & \mathbf{0} & n_t \boldsymbol{n_{q2}} & \frac{\delta^3}{6} & \frac{\delta^4}{8} & \frac{\delta^5}{20} \end{bmatrix} \otimes \mathbf{I} \tag{SM 5}$$

II. Computation of auxiliary variables

We start by noting that (4a) may be rewritten as

$$\dot{\mathbf{v}}_L = \dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{m_T}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel}, \tag{SM 6}$$

where

$$\dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = g\mathbf{e}_3 - \frac{1}{m_T}\widehat{\boldsymbol{\xi}}^{\parallel}. \tag{SM 7}$$

All the time derivatives that explicitly feature $\dot{\mathbf{v}}_L$ can be divided into three terms: one related to $\mathbf{f}_d^{\parallel}$, one related to the error $\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}$, and another related to $\widetilde{\mathbf{f}}_v$ and $\widetilde{\mathbf{f}}_q$.

Similarly, we can rearrange (8) as

$$\dot{\mathbf{z}}_v = \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{m_T}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \frac{1}{m_T}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L}(\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel}), \tag{SM 8}$$

where

$$\dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = \dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \ddot{\mathbf{p}}_d \tag{SM 9}$$

and

$$\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L} = \mathbf{I}. \tag{SM 10}$$

Then

$$\dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\frac{1}{m_T}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel}\right), \tag{SM 11}$$

where

$$\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K_p}\mathbf{z}_v + \mathbf{K_v}\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \dddot{\mathbf{p}}_d\right), \tag{SM 12}$$

its corresponding estimation

$$\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K_p}\widehat{\mathbf{z}}_v + \mathbf{K_v}\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \dddot{\mathbf{p}}_d\right) \tag{SM 13}$$

and $\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} := m_T \mathbf{K_v}$.

We start by presenting the expression for $\widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$, first shown in (29). We have

$$\begin{aligned} \widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} =& \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)\left(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{est} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right) \\ & - \frac{k_\mathbf{q}}{h_\mathbf{q}}\left(\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\widehat{\mathbf{q}}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}^2(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right), \end{aligned} \tag{SM 14}$$

where

$$\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}, \tag{SM 15}$$

$$\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = -\frac{1}{m_T}\left[(\widehat{\boldsymbol{\varpi}}\mathbf{q}^\top + \mathbf{q}\widehat{\boldsymbol{\varpi}}^\top)\widehat{\boldsymbol{\xi}} + \mathbf{q}\mathbf{q}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right] - \dddot{\mathbf{p}}_d, \tag{SM 16}$$

and, finally,

$$\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K_p}\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{K_v}\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \ddddot{\mathbf{p}}_d\right). \tag{SM 17}$$

The expression for $\partial \mathbf{z}_\varpi / \partial \mathbf{v}_L$, also first shown in (29), is given by

$$\left.\frac{\partial \mathbf{z}_\varpi}{\partial \mathbf{v}_L}\right|_{\text{est}} = -\frac{k_q}{h_q}\mathbf{S}^2(\mathbf{q})\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} - \mathbf{S}(\mathbf{q})\mathbf{S}\Big(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\widehat{\boldsymbol{\xi}}\|}\Big)\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} - \frac{\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)}{\|\widehat{\boldsymbol{\xi}}\|^2}\left(\|\widehat{\boldsymbol{\xi}}\|\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} + \frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}}{\|\widehat{\boldsymbol{\xi}}\|}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right), \tag{SM 18}$$

where

$$\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}, \tag{SM 19}$$

with

$$\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} = m_T \mathbf{K}_{\mathbf{v}}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L}, \tag{SM 20}$$

and where

$$\frac{\partial \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = -\frac{1}{m_T}\mathbf{q}\mathbf{q}^{\mathsf{T}}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}, \tag{SM 21}$$

$$\frac{\partial \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = m_T\left(\mathbf{K}_{\mathbf{p}}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L} + \mathbf{K}_{\mathbf{v}}\frac{\partial \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}\right), \tag{SM 22}$$

and, finally,

$$\frac{\partial \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = \frac{1}{\|\boldsymbol{\xi}\|}\left(\mathbf{q}_d\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}^{\mathsf{T}} - \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\mathbf{q}_d^{\mathsf{T}}\right)\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L} - \frac{\mathbf{S}^2(\mathbf{q}_d)\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^{\mathsf{T}}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|^3} + \frac{\mathbf{S}^2(\mathbf{q}_d)\frac{\partial \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|}. \tag{SM 23}$$

In the following we present some useful partial derivatives of the Lyapunov function candidates with respect to $\mathbf{v}_L$ and $\boldsymbol{\omega}$. They are given by

$$\frac{\partial V_3}{\partial \mathbf{v}_L} = \mathbf{e}^{\mathsf{T}}\frac{\partial \mathbf{e}}{\partial \mathbf{v}_L} - h_q\mathbf{z}_q^{\mathsf{T}}\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}, \tag{SM 24}$$

$$\frac{\partial V_4}{\partial \mathbf{v}_L} = \frac{\partial V_3}{\partial \mathbf{v}_L} + h_\omega\mathbf{z}_\omega^{\mathsf{T}}\frac{\partial \mathbf{z}_\omega}{\partial \mathbf{v}_L}, \tag{SM 25}$$

$$\frac{\partial V_5}{\partial \mathbf{v}_L} = \frac{\partial V_4}{\partial \mathbf{v}_L} - h_r\mathbf{r}_{3d}^{\mathsf{T}}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}\mathbf{S}(\mathbf{r}_{3d})\frac{\partial \mathbf{r}_{3d}}{\partial \mathbf{v}_L}, \tag{SM 26}$$

and

$$\frac{\partial V_5}{\partial \boldsymbol{\omega}} = h_\omega\mathbf{z}_\omega^{\mathsf{T}}\frac{\partial \mathbf{z}_\omega}{\partial \boldsymbol{\omega}} - h_r\mathbf{r}_{3d}^{\mathsf{T}}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}\mathbf{S}(\mathbf{r}_{3d})\frac{\partial \mathbf{r}_{3d}}{\partial \boldsymbol{\omega}}, \tag{SM 27}$$

where

$$\frac{\partial \mathbf{z}_\omega}{\partial \boldsymbol{\omega}} = \mathbf{S}(\mathbf{q}), \tag{SM 28}$$

$$\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}} = -\mathbf{S}(\mathbf{q}), \tag{SM 29}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = -\frac{1}{m_T}\left(\mathbf{q}^{\mathsf{T}}\boldsymbol{\xi} + \mathbf{q}\boldsymbol{\xi}^{\mathsf{T}}\right)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}, \tag{SM 30}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_L} = -\frac{(\dot{\mathbf{q}}\mathbf{q}^{\mathsf{T}} + \mathbf{q}\dot{\mathbf{q}}^{\mathsf{T}})\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} + \mathbf{q}\mathbf{q}^{\mathsf{T}}\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}}{m_T}, \tag{SM 31}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = \epsilon(\mathbf{K}_{\mathbf{p}} + \mathbf{K}_{\mathbf{v}})\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}}, \tag{SM 32}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_L} = \epsilon^2(\mathbf{I} + \mathbf{K}_{\mathbf{v}}\mathbf{K}_{\mathbf{p}})\frac{\partial \widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} + \epsilon(\mathbf{K}_{\mathbf{p}} + \mathbf{K}_{\mathbf{v}})\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_L}, \tag{SM 33}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = m_T\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}}, \tag{SM 34}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_L} = m_T\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_L}, \tag{SM 35}$$

$$
\begin{aligned}
\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = &-\frac{k_q}{h_q}\left(\left[\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})+\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})\right]\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}+\mathbf{S}^2(\mathbf{q})\frac{\partial \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}\right)-\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\boldsymbol{\xi}\|})\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\\
&+\frac{\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}_d)+\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}})}{\|\boldsymbol{\xi}\|}\left(\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}-\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|^2}\right)-\mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\boldsymbol{\xi}\|})\frac{\partial \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}\\
&-\mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\|\boldsymbol{\xi}\|^2\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}-\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\boldsymbol{\xi}\|^3})\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}+\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\left(-\frac{\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}-\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|^3}\right.\\
&\left.\frac{\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_L}\|\boldsymbol{\xi}\|-\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\boldsymbol{\xi}^\mathsf{T}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|^2}+\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\left(\|\boldsymbol{\xi}\|^3\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}^\mathsf{T}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}-3\|\boldsymbol{\xi}\|\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right)}{\|\boldsymbol{\xi}\|^6}\right), \qquad \text{(SM 36)}
\end{aligned}
$$

and, at last,

$$
\begin{aligned}
\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \boldsymbol{\omega}} = &-\mathbf{S}(\boldsymbol{\omega})\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}+\mathbf{S}(\dot{\mathbf{q}})-\frac{k_q}{h_q}(\mathbf{q}\mathbf{q}_d^\mathsf{T}-\mathbf{q}_d\mathbf{q}^\mathsf{T})\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}+\frac{k_q}{h_q}\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}+\frac{\|\boldsymbol{\xi}\|\epsilon}{h_q m_T}\left((\mathbf{q}\mathbf{e}^\mathsf{T}-\mathbf{e}\mathbf{q}^\mathsf{T})\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}\right.\\
&\left.-\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{e})\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}\right)+\frac{\mathbf{q}_d\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}^\mathsf{T}-\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\mathbf{q}_d^\mathsf{T}}{\|\boldsymbol{\xi}\|}\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}+\frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}}, \qquad \text{(SM 37)}
\end{aligned}
$$

Other important partial derivatives used in the derivations are the following:

$$
\frac{\partial \mathbf{f}_d}{\partial \mathbf{v}_L} = -\mathbf{q}\mathbf{q}^\mathsf{T}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}+m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}+\frac{h_q}{h_\omega}\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right), \qquad \text{(SM 38)}
$$

$$
\frac{\partial \mathbf{f}_d}{\partial \boldsymbol{\omega}} = 2m_Q\ell\mathbf{q}\boldsymbol{\omega}^\mathsf{T}\mathbf{I}+m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \boldsymbol{\omega}}+\frac{k_\omega}{h_\omega}\mathbf{S}(\mathbf{q})\right), \qquad \text{(SM 39)}
$$

$$
\frac{\partial \mathbf{r}_{3d}}{\partial \boldsymbol{\omega}} = -\frac{\|\mathbf{f}_d\|\mathbf{I}-\frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\mathsf{T}}{\|\mathbf{f}_d\|^2}\frac{\partial \mathbf{f}_d}{\partial \boldsymbol{\omega}}, \qquad \text{(SM 40)}
$$

and

$$
\frac{\partial \mathbf{r}_{3d}}{\partial \mathbf{v}_L} = -\frac{\|\mathbf{f}_d\|\mathbf{I}-\frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\mathsf{T}}{\|\mathbf{f}_d\|^2}\frac{\partial \mathbf{f}_d}{\partial \mathbf{v}_L}. \qquad \text{(SM 41)}
$$

We also need to compute the pseudo-estimate of $\dot{\mathbf{r}}_{3d}$, which is given by

$$
\widehat{\dot{\mathbf{r}}_{3d}} = -\frac{1}{\|\mathbf{f}_d\|^2}\left(\|\mathbf{f}_d\|\widehat{\dot{\mathbf{f}}}_d-\frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\mathsf{T}\widehat{\dot{\mathbf{f}}}_d\right), \qquad \text{(SM 42)}
$$

where

$$
\widehat{\dot{\mathbf{f}}}_d = \widehat{\dot{\mathbf{f}}_d^{\parallel}}+\widehat{\dot{\mathbf{f}}_d^{\perp}}, \qquad \text{(SM 43)}
$$

with

$$
\widehat{\dot{\mathbf{f}}_d^{\parallel}} = -\widehat{\boldsymbol{\varpi}}\mathbf{q}^\mathsf{T}\widehat{\boldsymbol{\xi}}-\mathbf{q}\widehat{\boldsymbol{\varpi}}^\mathsf{T}\widehat{\boldsymbol{\xi}}-\mathbf{q}\mathbf{q}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\text{est}}-m_T\left(\widehat{\boldsymbol{\varpi}}\mathbf{q}^\mathsf{T}\widehat{\mathbf{f}}_v-\mathbf{q}\widehat{\boldsymbol{\varpi}}^\mathsf{T}\widehat{\mathbf{f}}_v-\mathbf{q}\mathbf{q}^\mathsf{T}\widehat{\dot{\mathbf{f}}}_v\right)-m_Q\ell\left(\widehat{\boldsymbol{\varpi}}\mathbf{q}^\mathsf{T}\widehat{\mathbf{f}}_q-\mathbf{q}\widehat{\boldsymbol{\varpi}}^\mathsf{T}\widehat{\mathbf{f}}_q-\mathbf{q}\mathbf{q}^\mathsf{T}\widehat{\dot{\mathbf{f}}}_q\right) \qquad \text{(SM 44)}
$$

and

$$
\begin{aligned}
\widehat{\dot{\mathbf{f}}_d^{\perp}} = &\; m_Q\ell\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\left(\widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}+\frac{h_\mathbf{q}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{q}}_d-\frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{z}}_\varpi-\widehat{\mathbf{f}}_q\right)+m_Q\ell\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\left(\widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}+\frac{h_\mathbf{q}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{q}}_d-\frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{z}}_\varpi-\widehat{\mathbf{f}}_q\right)\\
&+m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}+\frac{h_\mathbf{q}}{h_{\boldsymbol{\varpi}}}\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}}-\frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\frac{d}{dt}\left(\widehat{\mathbf{z}}_\varpi\right)_{\text{est}}-\widehat{\dot{\mathbf{f}}}_q\right),
\end{aligned}
$$

where,

$$
\hat{\dot{\mathbf{v}}}_L = \frac{1}{m_T}\mathbf{f}^{\parallel}+\widehat{\mathbf{f}}_v+\frac{m_Q\ell}{m_T}\widehat{\mathbf{f}}_q^{\parallel}+g\mathbf{e}_3, \qquad \text{(SM 45)}
$$

$$\hat{\dot{\mathbf{z}}}_v = \hat{\dot{\mathbf{v}}}_L - \ddot{\mathbf{p}}_d, \tag{SM 46}$$

$$\hat{\dot{\boldsymbol{\xi}}} = m_T \left(\mathbf{K_p} \hat{\mathbf{z}}_v + \mathbf{K_v} \hat{\dot{\mathbf{z}}}_v - \dddot{\mathbf{p}}_d \right), \tag{SM 47}$$

$$\frac{d}{dt} (\hat{\mathbf{q}}_d)_{\text{est}} = \frac{1}{\|\hat{\boldsymbol{\xi}}\|} \mathbf{S}^2(\hat{\mathbf{q}}_d) \hat{\dot{\boldsymbol{\xi}}} \tag{SM 48}$$

$$\frac{d}{dt} \left(\dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = -\frac{(\hat{\boldsymbol{\varpi}} \mathbf{q}^{\mathsf{T}} + \mathbf{q} \hat{\boldsymbol{\varpi}}^{\mathsf{T}}) \hat{\boldsymbol{\xi}} + \mathbf{q}\mathbf{q}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}}}{m_T} - \dddot{\mathbf{p}}_d, \tag{SM 49}$$

$$\frac{d}{dt} \left(\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = m_T \left(\mathbf{K_p} \hat{\dot{\mathbf{z}}}_v + \mathbf{K_v} \frac{d}{dt} \left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \ddddot{\mathbf{p}}_d \right), \tag{SM 50}$$

$$\frac{d}{dt} \left(\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = \frac{1}{\|\boldsymbol{\xi}\|^2} \left(\|\boldsymbol{\xi}\| \left(\mathbf{S}(\hat{\dot{\mathbf{q}}}_d)\mathbf{S}(\mathbf{q}_d) + \mathbf{S}(\mathbf{q}_d)\mathbf{S}(\hat{\dot{\mathbf{q}}}_d) \right) - \frac{1}{\|\boldsymbol{\xi}\|} \mathbf{S}^2(\mathbf{q}_d) \boldsymbol{\xi}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}} \right) \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{\|\boldsymbol{\xi}\|} \mathbf{S}^2(\mathbf{q}_d) \frac{d}{dt} \left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}}, \tag{SM 51}$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{d}{dt} \left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} |_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = & -\frac{1}{m_T} \left(\hat{\dot{\boldsymbol{\varpi}}} \mathbf{q}^{\mathsf{T}} \hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\varpi}} \hat{\boldsymbol{\varpi}}^{\mathsf{T}} \hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\varpi}} \mathbf{q}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}} + \hat{\boldsymbol{\varpi}} \hat{\boldsymbol{\varpi}}^{\mathsf{T}} \hat{\boldsymbol{\xi}} + \mathbf{q} \hat{\dot{\boldsymbol{\varpi}}}^{\mathsf{T}} \hat{\boldsymbol{\xi}} + \mathbf{q} \hat{\boldsymbol{\varpi}}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}} + \hat{\boldsymbol{\varpi}} \mathbf{q}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right. \\ & \left. + \mathbf{q} \hat{\boldsymbol{\varpi}}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{q}\mathbf{q}^{\mathsf{T}} \frac{d}{dt} \left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \right) - \ddddot{\mathbf{p}}_d, \end{aligned}$$

$$\frac{d}{dt} \left(\frac{d}{dt} \left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} |_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = m_T \left(\mathbf{K_p} \frac{d}{dt} \left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} + \mathbf{K_v} \frac{d}{dt} \left(\frac{d}{dt} \left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} |_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \mathbf{p}_d^{(5)} \right), \tag{SM 52}$$

and, finally,

$$\begin{aligned} \frac{d}{dt} \left(\hat{\dot{\boldsymbol{\varpi}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = & -\frac{k_{\mathbf{q}}}{h_{\mathbf{q}}} \left(\mathbf{S}(\hat{\dot{\boldsymbol{\varpi}}})\mathbf{S}(\mathbf{q})\hat{\mathbf{q}}_d + \mathbf{S}(\hat{\boldsymbol{\varpi}})\mathbf{S}(\hat{\boldsymbol{\varpi}})\hat{\mathbf{q}}_d + \mathbf{S}(\hat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q}) \frac{d}{dt} (\hat{\mathbf{q}}_d)_{\text{est}} + \mathbf{S}(\hat{\boldsymbol{\varpi}})\mathbf{S}(\hat{\boldsymbol{\varpi}})\hat{\mathbf{q}}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\hat{\dot{\boldsymbol{\varpi}}})\hat{\mathbf{q}}_d \right. \\ & + \mathbf{S}(\mathbf{q})\mathbf{S}(\hat{\boldsymbol{\varpi}}) \frac{d}{dt} (\hat{\mathbf{q}}_d)_{\text{est}} + \mathbf{S}(\hat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\hat{\boldsymbol{\varpi}})\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}^2(\mathbf{q}) \frac{d}{dt} \left(\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \Bigg) - \Bigg(\frac{1}{\|\hat{\boldsymbol{\xi}}\|} \mathbf{S}(\hat{\dot{\boldsymbol{\varpi}}})\mathbf{S}(\hat{\mathbf{q}}_d) \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\ & - \frac{1}{\|\hat{\boldsymbol{\xi}}\|} \mathbf{S}(\hat{\boldsymbol{\varpi}})\mathbf{S}(\frac{d}{dt} (\hat{\mathbf{q}}_d)_{\text{est}}) \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\hat{\boldsymbol{\xi}}\|^2} \mathbf{S}(\hat{\boldsymbol{\varpi}})\mathbf{S}(\hat{\mathbf{q}}_d) \left(\|\hat{\boldsymbol{\xi}}\| \frac{d}{dt} \left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \frac{1}{\|\hat{\boldsymbol{\xi}}\|} \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \hat{\boldsymbol{\xi}}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}} \right) + \mathbf{S}(\hat{\boldsymbol{\varpi}})\mathbf{S}(\frac{\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\hat{\boldsymbol{\xi}}\|}) \hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\ & + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\|\hat{\boldsymbol{\xi}}\| \frac{d}{dt} \left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \frac{1}{\|\hat{\boldsymbol{\xi}}\|} \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \hat{\boldsymbol{\xi}}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}}}{\|\hat{\boldsymbol{\xi}}\|^2}) \hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\hat{\boldsymbol{\xi}}\|}) \frac{d}{dt} \left(\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \left(\mathbf{S}(\hat{\boldsymbol{\varpi}})\mathbf{S}(\hat{\mathbf{q}}_d) + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{d}{dt} (\hat{\mathbf{q}}_d)_{\text{est}}) \right) \\ & - \mathbf{S}(\mathbf{q})\mathbf{S}(\hat{\mathbf{q}}_d) \Bigg[\frac{1}{\|\hat{\boldsymbol{\xi}}\|^2} \left(\|\hat{\boldsymbol{\xi}}\| \frac{d}{dt} \left(\frac{d}{dt} \left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} |_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \right. \\ & \left. - \frac{1}{\|\hat{\boldsymbol{\xi}}\|} \frac{d}{dt} \left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} |_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \hat{\boldsymbol{\xi}}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}} \right) - \frac{1}{\|\hat{\boldsymbol{\xi}}\|^3} \left(\frac{d}{dt} \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \hat{\boldsymbol{\xi}}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\hat{\boldsymbol{\xi}}\|^6} \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \left(\|\hat{\boldsymbol{\xi}}\|^3 \hat{\dot{\boldsymbol{\xi}}}^{\mathsf{T}} - 3\|\hat{\boldsymbol{\xi}}\| \hat{\boldsymbol{\xi}}^{\mathsf{T}} (\hat{\boldsymbol{\xi}}^{\mathsf{T}} \hat{\dot{\boldsymbol{\xi}}}) \right) \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\ & + \frac{1}{\|\hat{\boldsymbol{\xi}}\|^3} \hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \hat{\boldsymbol{\xi}}^{\mathsf{T}} \frac{d}{dt} \left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \Bigg] \Bigg). \end{aligned} \tag{SM 53}$$

III. Estimation Errors Associated term

Baesd on auxiliary calculation in Section II, the estimation related terms $\widetilde{W}_1$ in (25), $\widetilde{W}_2$ in (27), $\widetilde{W}_3$ in (28) and $\widetilde{W}_4$ in (34) are expressed by

$$\widetilde{W}_1 := \frac{\partial V_1}{\partial \mathbf{v}_L} \left(\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q \ell}{m_T} \widetilde{\mathbf{f}}_q^{\parallel} + \frac{1}{m_T} \left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{z}_v} \right)^{\parallel} \widetilde{\mathbf{v}}_L \right) \in \mathbb{R}, \tag{SM 54}$$

$$\widetilde{W}_2 := \widetilde{W}_1 - \frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|} \mathbf{q}^{\mathsf{T}} \mathbf{S}^2(\mathbf{q}_d) \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} \left(\widetilde{\mathbf{f}}_v + \frac{m_Q \ell}{m_T} \widetilde{\mathbf{f}}_q \right)^{\parallel} \in \mathbb{R}, \tag{SM 55}$$

$$\widetilde{W}_3 := \widetilde{W}_2 + h_{\boldsymbol{\varpi}} \mathbf{z}_{\varpi}^{\mathsf{T}} \left(\widetilde{\mathbf{f}}_q - \left(\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \hat{\dot{\boldsymbol{\varpi}}}_{\boldsymbol{d}} \right) - \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}} (\mathbf{q}_d - \hat{\mathbf{q}}_d) + \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}} (\mathbf{z}_{\varpi} - \hat{\mathbf{z}}_{\varpi}) \right) \in \mathbb{R}, \text{ and} \tag{SM 56}$$

$$\widetilde{W}_4 := \widetilde{W}_3 + h_{\mathbf{r}}\mathbf{r}_{3d}^{\mathsf{T}}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\left(-\mathbf{R}^{\mathsf{T}}\mathbf{S}(\mathbf{r}_{3d})(\dot{\mathbf{r}}_{3d} - \widehat{\dot{\mathbf{r}}_{3d}}) - \frac{T_d}{h_{\mathbf{r}}m_T}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}(\boldsymbol{\delta}_2^{\mathsf{T}} - \widehat{\boldsymbol{\delta}_2}^{\mathsf{T}})^{\parallel} + \frac{h_{\boldsymbol{\varpi}}}{h_{\mathbf{r}}}\frac{T_d}{m_Q\ell}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}(\mathbf{z}_{\varpi} - \widehat{\mathbf{z}_{\varpi}})^{\perp}\right) \tag{SM 57}$$

, respectively.

References

[1] Yang, W., Yu, G., Reis, J. and Silvestre, C. Robust Nonlinear 3D Control of an Inverted Pendulum Balanced on a Quadrotor. *Automatica*. pp. 1-12 (2023)