Nonlinear Output Feedback Control of an Underactuated Flying Inverted Pendulum: Supplementary Material

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Abstract

This is a complementary document to the paper presented in [1]. Here we fully disclose all the detailed derivations and equations that are essential to implement the controller reported therein.

I. The transition matrix and the observability Gramian calculations

The most general transition matrix is given by the Peano–Baker series

$$\mathbf{\Phi}(t,t_0) := \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)\int_{t_0}^{\sigma_2} \mathbf{A}(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 \cdots \tag{SM 1}$$

It is noticed that the matrix of $\mathbf{A}$ for the proposed LTV system as shown in (12) [1] is nilpotent of index 3, meaning $\mathbf{A}^n = \mathbf{0}$ for $n \geq 3$ where n is a positive integer. Hence, the proposed LTV system's transition matrix is given by

$$\mathbf{\Phi}(t,t_0) := \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 \tag{SM 2}$$

where

$$\int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 = \begin{bmatrix} \mathbf{0} & \boldsymbol{n_t} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_t} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{n_t} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_q} & \mathbf{0} & \mathbf{0} & \boldsymbol{n_t} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and} \tag{SM 3}$$

$$\int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \frac{1}{2}\boldsymbol{n_t}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & \mathbf{0} & \mathbf{0} & \frac{1}{2}\boldsymbol{n_t}^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{SM 4}$$

where $\boldsymbol{n_t} := (t-t_0)\mathbf{I}$, $\boldsymbol{n_q} := \frac{m_Q \ell}{m_T}\int_{t_0}^{t} \mathbf{q}\mathbf{q}^\intercal d\tau$ and $\boldsymbol{n_{q2}} := \int_{t_0}^{t} \boldsymbol{n_q} d\tau$.

Based on the definition of the observability Gramian and using (SM 2) along with the nilpotent property of $\mathbf{A}$, it is given by

$$\boldsymbol{W}(t_0,t_f) = \int_{t_0}^{t_f} \begin{bmatrix} \mathbf{I} & \boldsymbol{n_t} & \frac{1}{2}\boldsymbol{n_t}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{n_t} & \boldsymbol{n_t}^2 & \frac{1}{2}\boldsymbol{n_t}^3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{1}{2}\boldsymbol{n_t}^2 & \frac{1}{2}\boldsymbol{n_t}^3 & \frac{1}{4}\boldsymbol{n_t}^4 + \boldsymbol{n_{q2}}^\intercal \boldsymbol{n_{q2}} & \boldsymbol{n_{q2}}^\intercal & \boldsymbol{n_t}\boldsymbol{n_{q2}}^\intercal & \frac{1}{2}\boldsymbol{n_t}^2\boldsymbol{n_{q2}}^\intercal \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & \mathbf{I} & \boldsymbol{n_t} & \frac{1}{2}\boldsymbol{n_t}^2 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_t}\boldsymbol{n_{q2}} & \boldsymbol{n_t} & \boldsymbol{n_t}^2 & \frac{1}{2}\boldsymbol{n_t}^3 \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\boldsymbol{n_t}^2\boldsymbol{n_{q2}} & \frac{1}{2}\boldsymbol{n_t}^2 & \frac{1}{2}\boldsymbol{n_t}^3 & \frac{1}{4}\boldsymbol{n_t}^2 \end{bmatrix} dt \tag{SM 5}$$

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Setting $t_0 = t - \delta$ and $t_f = t$, $\boldsymbol{W}(t_0, t_f)$ is obtained as follows

$$\boldsymbol{W}(t-\delta,t) = \begin{bmatrix} \delta\mathbf{I} & \frac{\delta^2}{2}\mathbf{I} & \frac{\delta^3}{6}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\delta^2}{2}\mathbf{I} & \frac{\delta^3}{3}\mathbf{I} & \frac{\delta^4}{8}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\delta^3}{6}\mathbf{I} & \frac{\delta^4}{8}\mathbf{I} & \frac{\delta^5}{20}\mathbf{I} + \int_{t-\delta}^{t} \boldsymbol{n_{q2}}^\top \boldsymbol{n_{q2}} d\tau & \int_{t-\delta}^{t} \boldsymbol{n_{q2}}^\top d\tau & \boldsymbol{\varphi}_1^\top & \boldsymbol{\varphi}_2^\top \\ \mathbf{0} & \mathbf{0} & \int_{t-\delta}^{t} \boldsymbol{n_{q2}} d\tau & \delta\mathbf{I} & \frac{\delta^2}{2}\mathbf{I} & \frac{\delta^3}{6}\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_1 & \frac{\delta^2}{2}\mathbf{I} & \frac{\delta^3}{3}\mathbf{I} & \frac{\delta^4}{8}\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_2 & \frac{\delta^3}{6}\mathbf{I} & \frac{\delta^4}{8}\mathbf{I} & \frac{\delta^5}{20}\mathbf{I} \end{bmatrix} \quad \text{(SM 6)}$$

where $\boldsymbol{\varphi}_1 := \frac{\delta^2}{2}\boldsymbol{n_{q2}} - \frac{\delta^2}{2}\int_{t-\delta}^{t} \boldsymbol{n_q} d\tau$ and $\boldsymbol{\varphi}_2 := \frac{\delta^3}{6}\boldsymbol{n_{q2}} - \frac{\delta^3}{6}\int_{t-\delta}^{t} \boldsymbol{n_q} d\tau$.

Since $\mathbf{q}$ is a unit vector, each element in (SM 6) is bounded for some δ. Consequently, $\boldsymbol{W}(t-\delta,t)$ is bounded as well.

Remark 1. *When $\mathbf{q}$ remains constant, such as when it equals $-\mathbf{e}_3$, $\boldsymbol{n_q}$ tends to zero as t approaches infinity. However, in reality, it is impossible to maintain $\mathbf{q}$ constantly at $-\mathbf{e}_3$ throughout all FIP maneuvers, even for a constant position tracking.*

II. Proof of Theorem 3

Based on $\boldsymbol{x}(t)$ (10) and $\widehat{\boldsymbol{x}}(t)$ (19) in [1], the estimation error is defined as $\widetilde{\boldsymbol{x}}(t) := \boldsymbol{x}(t) - \widehat{\boldsymbol{x}}(t)$. Using $\dot{\boldsymbol{x}}(t)$ in (11) and $\dot{\widehat{\boldsymbol{x}}}(t)$ in (18) from [1], we obtain the time derivative of the estimation error $\widetilde{\boldsymbol{x}}(t)$ as:

$$\dot{\widetilde{\boldsymbol{x}}}(t) = (\mathbf{A}(t) - \boldsymbol{\mathcal{K}}(t)\mathbf{C})\widetilde{\boldsymbol{x}}(t) + \boldsymbol{h}(t). \quad \text{(SM 7)}$$

Considering the Lyapunov function $V_{\text{KF}} := \widetilde{\boldsymbol{x}}^\top \boldsymbol{\mathcal{P}}^{-1}\widetilde{\boldsymbol{x}}$, where $\boldsymbol{\mathcal{P}}(t)$ denotes the covariance matrix of the state estimate, and Assumption made in [1], we can derive the following inequality for the time derivative of V_{KF}:

$$\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})\|\widetilde{\boldsymbol{x}}\|^2 \le V_{\text{KF}} \le \lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})\|\widetilde{\boldsymbol{x}}\|^2 \quad \text{(SM 8)}$$

where $\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1}) \ge \lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1}) > 0$.

Using (18), the time derivative of V_{KF} can be written as

$$\dot{V}_{\text{KF}} \le -\lambda_{\min}(\boldsymbol{\Upsilon}(t))\|\widetilde{\boldsymbol{x}}(t)\|^2 + 2\widetilde{\boldsymbol{x}}^\top(t)\boldsymbol{\mathcal{P}}^{-1}(t)\boldsymbol{h}(t) \quad \text{(SM 9)}$$

where $\boldsymbol{\Upsilon}(t) := \mathbf{C}^\top\boldsymbol{\mathcal{R}}^{-1}(t)\mathbf{C} + \boldsymbol{\mathcal{P}}^{-1}(t)\boldsymbol{\mathcal{Q}}(t)\boldsymbol{\mathcal{P}}^{-1}(t) \in \mathbb{R}^{18\times18}_{\succ 0}$. Notice that $\|\partial V_{\text{KF}}/\partial\widetilde{\boldsymbol{x}}\| \le 2\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})\|\widetilde{\boldsymbol{x}}\|$.

Based on Assumption 1, suppose the perturbation term $\boldsymbol{h}(t)$ satisfies $\|\boldsymbol{h}(t)\| \le \varepsilon < \frac{\lambda_{\min}(\boldsymbol{\Upsilon})}{2\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}\sqrt{\frac{\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}}\epsilon r$, for all $t \ge 0$, for all $\widetilde{\boldsymbol{x}} \in D$ where $D = \{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{18} | \|\widetilde{\boldsymbol{x}}\| < r\}$, and for some positive constant $\epsilon < 1$.

According to Lemma 9.2 in [2], for all $\widetilde{\boldsymbol{x}}(t_0) < \sqrt{\frac{\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}}r$, and some finite T, the solution of $\widetilde{\boldsymbol{x}}(t)$ satisfies

$$\|\widetilde{\boldsymbol{x}}(t)\| \le \sqrt{\frac{\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}} \exp\left(-\frac{(1-\epsilon)\lambda_{\min}(\boldsymbol{\Upsilon})}{2\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}(t-t_0)\right)\|\widetilde{\boldsymbol{x}}(t_0)\|, \quad \forall t_0 \le t < t_0 + T \quad \text{(SM 10)}$$

and

$$\|\widetilde{\boldsymbol{x}}(t)\| \le \frac{2\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\min}(\boldsymbol{\Upsilon})}\sqrt{\frac{\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}}\frac{\varepsilon}{\epsilon} \quad \text{(SM 11)}$$

III. Computation of auxiliary variables

We start by noting that (4a) may be rewritten as

$$\dot{\mathbf{v}}_L = \dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{m_T}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel}, \quad \text{(SM 12)}$$

where

$$\dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = g\mathbf{e}_3 - \frac{1}{m_T}\widehat{\boldsymbol{\xi}}^{\parallel}. \quad \text{(SM 13)}$$

All the time derivatives that explicitly feature $\dot{\mathbf{v}}_L$ can be divided into three terms: one related to $\mathbf{f}_d^{\parallel}$, one related to the error $\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}$, and another related to $\widetilde{\mathbf{f}}_v$ and $\widetilde{\mathbf{f}}_q$.

Similarly, we can rearrange (8) as

$$\dot{\mathbf{z}}_v = \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{m_T}\frac{\partial\mathbf{z}_v}{\partial\mathbf{v}_L}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \frac{1}{m_T}\frac{\partial\mathbf{z}_v}{\partial\mathbf{v}_L}(\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel}), \quad \text{(SM 14)}$$

where

$$\dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = \dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \ddot{\mathbf{p}}_d \quad \text{(SM 15)}$$

and
$$\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L} = \mathbf{I}. \tag{SM 16}$$

Then
$$\dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\frac{1}{m_T}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel}\right), \tag{SM 17}$$

where
$$\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K}_{\mathbf{p}}\mathbf{z}_v + \mathbf{K}_{\mathbf{v}}\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \dddot{\mathbf{p}}_d\right), \tag{SM 18}$$

its corresponding estimation
$$\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K}_{\mathbf{p}}\widehat{\mathbf{z}}_v + \mathbf{K}_{\mathbf{v}}\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \dddot{\mathbf{p}}_d\right) \tag{SM 19}$$

and $\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} := m_T\mathbf{K}_{\mathbf{v}}$.

We start by presenting the expression for $\widehat{\boldsymbol{\varpi}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$, first shown in (29). We have

$$\begin{aligned}\widehat{\boldsymbol{\varpi}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} =& \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)\left(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{est} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)\\ &- \frac{k_{\mathbf{q}}}{h_{\mathbf{q}}}\left(\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\widehat{\mathbf{q}}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}^2(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right),\end{aligned} \tag{SM 20}$$

where
$$\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}, \tag{SM 21}$$

$$\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = -\frac{1}{m_T}\left[(\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}} + \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}})\widehat{\boldsymbol{\xi}} + \mathbf{q}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right] - \ddddot{\mathbf{p}}_d, \tag{SM 22}$$

and, finally,
$$\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K}_{\mathbf{p}}\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{K}_{\mathbf{v}}\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \ddddot{\mathbf{p}}_d\right). \tag{SM 23}$$

The expression for $\partial \mathbf{z}_{\varpi}/\partial \mathbf{v}_L$, also first shown in (29), is given by

$$\left.\frac{\partial \mathbf{z}_{\varpi}}{\partial \mathbf{v}_L}\right|_{\text{est}} = -\frac{k_q}{h_q}\mathbf{S}^2(\mathbf{q})\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\widehat{\boldsymbol{\xi}}\|})\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} - \frac{\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)}{\|\widehat{\boldsymbol{\xi}}\|^2}\left(\|\widehat{\boldsymbol{\xi}}\|\frac{\partial \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} + \frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}}{\|\widehat{\boldsymbol{\xi}}\|}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right), \tag{SM 24}$$

where
$$\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}, \tag{SM 25}$$

with
$$\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} = m_T\mathbf{K}_{\mathbf{v}}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L}, \tag{SM 26}$$

and where
$$\frac{\partial \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = -\frac{1}{m_T}\mathbf{q}\mathbf{q}^{\mathsf{T}}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}, \tag{SM 27}$$

$$\frac{\partial \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = m_T\left(\frac{1}{m_T}\mathbf{K}_{\mathbf{p}}\mathbf{I}^{\parallel} + \mathbf{K}_{\mathbf{v}}\frac{\partial \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}\right). \tag{SM 28}$$

We also need to compute the pseudo-estimate of $\dot{\mathbf{r}}_{3d}$, which is given by
$$\widehat{\dot{\mathbf{r}}_{3d}} = -\frac{1}{\|\mathbf{f}_d\|^2}\left(\|\mathbf{f}_d\|\widehat{\dot{\mathbf{f}}}_d - \frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^{\mathsf{T}}\widehat{\dot{\mathbf{f}}}_d\right), \tag{SM 29}$$

where
$$\widehat{\dot{\mathbf{f}}}_d = \widehat{\dot{\mathbf{f}}_d^{\parallel}} + \widehat{\dot{\mathbf{f}}_d^{\perp}}, \tag{SM 30}$$

with
$$\widehat{\dot{\mathbf{f}}_d^{\parallel}} = -\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} - \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} - \mathbf{q}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\text{est}} - m_T\left(\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\mathbf{f}}_v - \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\mathbf{f}}_v - \mathbf{q}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\mathbf{f}}}_v\right) - m_Q\ell\left(\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\mathbf{f}}_q - \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\mathbf{f}}_q - \mathbf{q}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\mathbf{f}}}_q\right) \tag{SM 31}$$

and

$$\widehat{\dot{\mathbf{f}}_d^{\perp}} = m_Q\ell\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{q}}_d - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{z}}_{\varpi} - \widehat{\mathbf{f}}_q\right) + m_Q\ell\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{q}}_d - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{z}}_{\varpi} - \widehat{\mathbf{f}}_q\right)$$
$$+ m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\frac{d}{dt}\left(\widehat{\mathbf{z}}_{\varpi}\right)_{\text{est}} - \widehat{\dot{\mathbf{f}}}_q\right),$$

where,

$$\widehat{\dot{\mathbf{v}}}_L = \frac{1}{m_T}\mathbf{f}^{\parallel} + \widehat{\mathbf{f}}_v + \frac{m_Q\ell}{m_T}\widehat{\mathbf{f}}_q^{\parallel} + g\mathbf{e}_3, \tag{SM 32}$$

$$\widehat{\dot{\mathbf{z}}}_v = \widehat{\dot{\mathbf{v}}}_L - \ddot{\mathbf{p}}_d, \tag{SM 33}$$

$$\widehat{\dot{\boldsymbol{\xi}}} = m_T\left(\mathbf{K}_{\mathbf{p}}\widehat{\mathbf{z}}_v + \mathbf{K}_{\mathbf{v}}\widehat{\dot{\mathbf{z}}}_v - \dddot{\mathbf{p}}_d\right), \tag{SM 34}$$

$$\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}} \tag{SM 35}$$

$$\frac{d}{dt}\left(\dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = -\frac{(\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}} + \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}})\widehat{\boldsymbol{\xi}} + \mathbf{q}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}}{m_T} - \ddddot{\mathbf{p}}_d, \tag{SM 36}$$

$$\frac{d}{dt}\left(\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = m_T\left(\mathbf{K}_{\mathbf{p}}\widehat{\dot{\mathbf{z}}}_v + \mathbf{K}_{\mathbf{v}}\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \ddddot{\mathbf{p}}_d\right), \tag{SM 37}$$

$$\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = \frac{1}{\|\boldsymbol{\xi}\|^2}\left(\|\boldsymbol{\xi}\|\left(\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d)\mathbf{S}(\mathbf{q}_d) + \mathbf{S}(\mathbf{q}_d)\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d)\right) - \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\boldsymbol{\xi}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}\right)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}, \tag{SM 38}$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = -\frac{1}{m_T}\left(\widehat{\dot{\boldsymbol{\varpi}}}\mathbf{q}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} + \widehat{\boldsymbol{\varpi}}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} + \widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}} + \widehat{\boldsymbol{\varpi}}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} + \mathbf{q}\widehat{\dot{\boldsymbol{\varpi}}}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} + \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}} + \widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right.$$
$$\left. + \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{q}\mathbf{q}^{\mathsf{T}}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\right) - \dddot{\mathbf{p}}_d,$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = m_T\left(\mathbf{K}_{\mathbf{p}}\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} + \mathbf{K}_{\mathbf{v}}\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \mathbf{p}_d^{(5)}\right), \tag{SM 39}$$

and, finally,

$$\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = -\frac{k_{\mathbf{q}}}{h_{\mathbf{q}}}\left(\mathbf{S}(\widehat{\dot{\boldsymbol{\varpi}}})\mathbf{S}(\mathbf{q})\widehat{\mathbf{q}}_d + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\dot{\boldsymbol{\varpi}}})\widehat{\mathbf{q}}_d\right.$$
$$+\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}^2(\mathbf{q})\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\Bigg) - \Bigg(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\dot{\boldsymbol{\varpi}}})\mathbf{S}(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$$
$$-\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}})\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^2}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d)\left(\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}\right) + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\widehat{\boldsymbol{\xi}}\|})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$$
$$+\mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}}{\|\widehat{\boldsymbol{\xi}}\|^2})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\widehat{\boldsymbol{\xi}}\|})\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \left(\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d) + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}})\right)$$
$$-\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)\left[\frac{1}{\|\widehat{\boldsymbol{\xi}}\|^2}\left(\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\right.\right.$$
$$\left.-\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}\right) - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\left(\frac{d}{dt}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^6}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\left(\|\widehat{\boldsymbol{\xi}}\|^3\widehat{\dot{\boldsymbol{\xi}}}^{\mathsf{T}} - 3\|\widehat{\boldsymbol{\xi}}\|\widehat{\boldsymbol{\xi}}^{\mathsf{T}}(\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}})\right)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$$
$$\left.\left.+\frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\right]\right). \tag{SM 40}$$

IV. Estimation Errors Associated term

Based on auxiliary calculation in Section III, the estimation related terms Ψ_1 in (25), Ψ_2 in (27), Ψ_3 in (28) and Ψ_4 in (34) are expressed by

$$\Psi_1 := \frac{\partial V_1}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel} + \frac{1}{m_T}\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right)^{\parallel}\widetilde{\mathbf{v}}_L\right) \in \mathbb{R}, \tag{SM 41}$$

$$\Psi_2 := \Psi_1 - \frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|}\mathbf{q}^{\mathsf{T}}\mathbf{S}^2(\mathbf{q}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q\right)^{\parallel} \in \mathbb{R}, \tag{SM 42}$$

$$\Psi_3 := \Psi_2 + h_{\boldsymbol{\varpi}}\mathbf{z}_{\varpi}^{\mathsf{T}}\left(\widetilde{\mathbf{f}}_q - \left(\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right) - \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}(\mathbf{q}_d - \widehat{\mathbf{q}}_d) + \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}(\mathbf{z}_{\varpi} - \widehat{\mathbf{z}}_{\varpi})\right) \in \mathbb{R}, \text{ and} \tag{SM 43}$$

$$\Psi_4 := \Psi_3 + h_{\mathbf{r}}\mathbf{r}_{3d}^{\mathsf{T}}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\left(-\mathbf{R}^{\mathsf{T}}\mathbf{S}(\mathbf{r}_{3d})(\dot{\mathbf{r}}_{3d} - \widehat{\dot{\mathbf{r}}_{3d}}) - \frac{T_d}{h_{\mathbf{r}}m_T}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}(\boldsymbol{\delta}_2^{\mathsf{T}} - \widehat{\boldsymbol{\delta}_2}^{\mathsf{T}})^{\parallel} + \frac{h_{\boldsymbol{\varpi}}}{h_{\mathbf{r}}}\frac{T_d}{m_Q\ell}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}(\mathbf{z}_{\varpi} - \widehat{\mathbf{z}_{\varpi}})^{\perp}\right) \tag{SM 44}$$

, respectively.

Using Young's inequality, Ψ_1 can be rewritten as

$$\begin{aligned}\Psi_1 = \frac{\partial V_1}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel} + \frac{1}{m_T}\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right)^{\parallel}\widetilde{\mathbf{v}}_L\right) &\leq \frac{\gamma_1}{2}(\beta\mathbf{z}_p^{\mathsf{T}} + \mathbf{z}_v^{\mathsf{T}})(\beta\mathbf{z}_p + \mathbf{z}_v) + \frac{1}{2\gamma_1}\delta_{\Psi_1} \\ &\leq \frac{\gamma_1}{2}\beta^2\|\mathbf{z}_p\|^2 + \frac{\gamma_1}{2}\|\mathbf{z}_v\|^2 + \frac{1}{2\gamma_1}\delta_{\Psi_1}\end{aligned} \tag{SM 45}$$

where $\gamma_1 > 0$ and $\delta_{\Psi_1} := \left\|\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel} + \frac{1}{m_T}\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right)^{\parallel}\widetilde{\mathbf{v}}_L\right\|^2$. According to Section II, $\widetilde{\boldsymbol{x}}$ is bounded, hence $\widetilde{\mathbf{v}}_L$, $\widetilde{\boldsymbol{\varpi}}$, $\widetilde{\mathbf{f}}_v$ and $\widetilde{\mathbf{f}}_q$ are also bounded. In this case, δ_{Ψ_1} is bounded as well.

Then using Young's inequality, Ψ_2 can be rewritten as

$$\Psi_2 = \Psi_1 - \frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|}\mathbf{q}^{\mathsf{T}}\mathbf{S}^2(\mathbf{q}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q\right)^{\parallel} \leq \Psi_1 + \frac{\gamma_2}{2}\frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|}\|\mathbf{q}^{\mathsf{T}}\mathbf{S}^2(\mathbf{q}_d)\|^2 + \frac{1}{2\gamma_2}\delta_{\Psi_2} \tag{SM 46}$$

where $\gamma_2 > 0$ and $\delta_{\Psi_2} := \left\|\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q\right)^{\parallel}\right\|^2$. According to $\widetilde{\mathbf{f}}_v$ and $\widetilde{\mathbf{f}}_q$ are bounded, δ_{Ψ_2} is bounded.

Using Young's inequality for Ψ_3, it can be formulated as

$$\Psi_3 = \Psi_2 + h_{\boldsymbol{\varpi}}\mathbf{z}_{\varpi}^{\mathsf{T}}\left(\widetilde{\mathbf{f}}_q - \left(\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right) - \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}(\mathbf{q}_d - \widehat{\mathbf{q}}_d) + \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}(\mathbf{z}_{\varpi} - \widehat{\mathbf{z}}_{\varpi})\right) \leq \Psi_2 + \frac{\gamma_3}{2}\|\mathbf{z}_{\varpi}\|^2 + \frac{1}{2\gamma_3}\delta_{\Psi_3} \tag{SM 47}$$

where $\gamma_3 > 0$ and $\delta_{\Psi_3} := \left\|\widetilde{\mathbf{f}}_q - \left(\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right) - \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}(\mathbf{q}_d - \widehat{\mathbf{q}}_d) + \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}(\mathbf{z}_{\varpi} - \widehat{\mathbf{z}}_{\varpi})\right\|^2$. Due to $\widetilde{\mathbf{v}}_L$ is bounded, then $\boldsymbol{\xi} - \widehat{\boldsymbol{\xi}}$ and $\mathbf{q}_d - \widehat{\mathbf{q}}_d$ are bounded. Then based on the reference trajectory described in Section IV [1] are bounded by construction, $\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$, $\dot{\mathbf{q}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$ and $\frac{d}{dt}\left(\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{d}{dt}\left(\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$ are bounded. Moreover, on account of $\widetilde{\boldsymbol{x}}$ is bounded, $\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$ and $\mathbf{z}_{\varpi} - \widehat{\mathbf{z}}_{\varpi}$ are bounded. In this case, δ_{Ψ_3} is bounded.

Similarly, $\dot{\mathbf{r}}_{3d} - \widehat{\dot{\mathbf{r}}_{3d}}$ and $\boldsymbol{\delta}_2^{\mathsf{T}} - \widehat{\boldsymbol{\delta}_2}^{\mathsf{T}}$ are also bounded. Due to Rotation matrix $\mathbf{R}$ property and, $\mathbf{r}_3$ and $\mathbf{r}_{3d}$ are unit vector, $\delta_{\Psi_4} := h_{\mathbf{r}}\mathbf{r}_{3d}^{\mathsf{T}}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\left(-\mathbf{R}^{\mathsf{T}}\mathbf{S}(\mathbf{r}_{3d})(\dot{\mathbf{r}}_{3d} - \widehat{\dot{\mathbf{r}}_{3d}}) - \frac{T_d}{h_{\mathbf{r}}m_T}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}(\boldsymbol{\delta}_2^{\mathsf{T}} - \widehat{\boldsymbol{\delta}_2}^{\mathsf{T}})^{\parallel} + \frac{h_{\boldsymbol{\varpi}}}{h_{\mathbf{r}}}\frac{T_d}{m_Q\ell}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}(\mathbf{z}_{\varpi} - \widehat{\mathbf{z}_{\varpi}})^{\perp}\right)$ is bounded as well. And all of the estimation error related term that remained in time derivative of Lyapunov function V_4 is introduced as

$$\delta_{V_4} := \delta_{\Psi_1} + \delta_{\Psi_2} + \delta_{\Psi_3} + \delta_{\Psi_4}. \tag{SM 48}$$

In turn, the boundedness of δ_{V_4} is confirmed. Then the time derivative of the fourth Lyapunov function candidate V_4 can be expressed as

$$\dot{V}_4 = -(\mathbf{P}\mathbf{z})^{\mathsf{T}}\mathbf{J}\overline{\mathbf{Q}}^{-1}\overline{\mathbf{Q}}(\mathbf{P}\mathbf{z}) + \Psi_4 \leq -(\mathbf{P}\mathbf{z})^{\mathsf{T}}\mathbf{J}^{\star}\overline{\mathbf{Q}}^{-1}\overline{\mathbf{Q}}(\mathbf{P}\mathbf{z}) + \delta_{V_4} \leq -\frac{\lambda_{\min}(\mathbf{J}^{\star})}{\lambda_{\max}(\overline{\mathbf{Q}})}V_4 + \delta_{V_4}. \tag{SM 49}$$

where

$$\mathbf{J}^{\star} := \begin{bmatrix} \beta\mathbf{K_p} - \frac{\gamma_1}{2}\beta^2\mathbf{I} & \frac{\beta}{2}\mathbf{K_v} - \frac{\gamma_1}{4}\mathbf{I} & -\frac{\beta\|\boldsymbol{\xi}\|}{2m_T}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \frac{\beta}{2}\mathbf{K_v} - \frac{\gamma_1}{4}\mathbf{I} & -(\beta - \mathbf{K_v}) & -\frac{\|\boldsymbol{\xi}\|}{2m_T}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\frac{\beta\|\boldsymbol{\xi}\|}{2m_T}\mathbf{I} & -\frac{\|\boldsymbol{\xi}\|}{2m_T}\mathbf{I} & (k_{\mathbf{q}} - \frac{\gamma_2}{2}\frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|})\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (k_{\boldsymbol{\varpi}} - \frac{\gamma_3}{2})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k_{\mathbf{r}}\mathbf{I} \end{bmatrix}$$

Remark 2. *γ_1, γ_2 and γ_3 are simple analysis parameters that can always be adjusted to render $\mathbf{J}^{\star}$ positive definite and they do not play any role whatsoever in the controller performance.*

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