Nonlinear Output Feedback Control of an Underactuated Flying Inverted Pendulum: Supplementary Material

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Abstract

This is a complementary document to the paper presented in [1]. Here we fully disclose all the detailed derivations and equations that are essential to implement the controller reported therein.

I. The transition matrix and the observability Gramian calculations

$$\mathbf{\Phi}(t,t_0) := \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 \cdots \tag{SM 1}$$

According to the transition matrix definition in (SM 1), the transition matrix's second and third terms are calculated as

$$\int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 = \begin{bmatrix} \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{m_Q \ell}{m_T}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top & \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and} \tag{SM 2}$$

$$\int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{n_{t2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & \mathbf{0} & \mathbf{0} & \boldsymbol{n_{t2}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ respectively,} \tag{SM 3}$$

where $\boldsymbol{n_{t1}} := (t-t_0)\mathbf{I}$, $\boldsymbol{n_{t2}} := \frac{1}{2}(t-t_0)^2\mathbf{I}$, $\boldsymbol{n_{q1}} := \frac{m_Q\ell}{m_T}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top$ and $\boldsymbol{n_{q2}} := \frac{m_Q\ell}{m_T}\int_{t_0}^{t}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top d\tau d\tau$.

It is noticed that the matrix of $\mathbf{A}$ is nilpotent of index 3, i.e. $\mathbf{A}^n = \mathbf{0}$ for $n \geq 3$ where n is a positive integer. Based on the definition of the observability Gramian, using (SM 1) and the nilpotent property of $\mathbf{A}$, it is given by

$$\boldsymbol{W}(t_0,t_f) = \int_{t_0}^{t_f} \begin{bmatrix} 1 & t-t_0 & n_t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ t-t_0 & (t-t_0)^2 & n_t(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ n_t & n_t(t-t_0) & {n_t}^2 + {\boldsymbol{n_{q2}}}^\top\boldsymbol{n_{q2}} & {\boldsymbol{n_{q2}}}^\top & (t-t_0){\boldsymbol{n_{q2}}}^\top & n_t{\boldsymbol{n_{q2}}}^\top \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & 1 & t-t_0 & n_t \\ \mathbf{0} & \mathbf{0} & (t-t_0)\boldsymbol{n_{q2}} & t-t_0 & (t-t_0)^2 & (t-t_0)n_t \\ \mathbf{0} & \mathbf{0} & n_t\boldsymbol{n_{q2}} & n_t & (t-t_0)n_t & {n_t}^2 \end{bmatrix} \otimes \mathbf{I}\, dt \tag{SM 4}$$

where $n_t := \frac{1}{2}(t-t_0)^2$.

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Setting $t_0 = t - \delta$ and $t_f = t$, $\boldsymbol{W}(t_0, t_f)$ is obtained as

$$\boldsymbol{W}(t-\delta,t) = \begin{bmatrix} \delta & \frac{\delta^2}{2} & \frac{\delta^3}{6} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\delta^2}{2} & \frac{\delta^3}{3} & \frac{\delta^4}{8} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\delta^3}{6} & \frac{\delta^4}{8} & \frac{\delta^5}{20} + \int_{t-\delta}^{t} \boldsymbol{n_{q2}}^\top \boldsymbol{n_{q2}} & \int_{t-\delta}^{t} \boldsymbol{n_{q2}}^\top & \boldsymbol{\varphi}_1^\top & \boldsymbol{\varphi}_2^\top \\ \mathbf{0} & \mathbf{0} & \int_{t-\delta}^{t} \boldsymbol{n_{q2}} & \delta & \frac{\delta^2}{2} & \frac{\delta^3}{6} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_1 & \frac{\delta^2}{2} & \frac{\delta^3}{3} & \frac{\delta^4}{8} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\varphi}_2 & \frac{\delta^3}{6} & \frac{\delta^4}{8} & \frac{\delta^5}{20} \end{bmatrix} \otimes \mathbf{I} \tag{SM 5}$$

where $\boldsymbol{\varphi}_1 := \frac{\delta^2}{2}\boldsymbol{n_{q2}} - \frac{\delta^2}{2}\int_{t-\delta}^{t}\boldsymbol{n_{q1}}$ and $\boldsymbol{\varphi}_2 := \frac{\delta^3}{6}\boldsymbol{n_{q2}} - \frac{\delta^3}{6}\int_{t-\delta}^{t}\boldsymbol{n_{q1}}$.

II. Proof of Theorem 3

Based on $\boldsymbol{x}(t)$ (10) and $\widehat{\boldsymbol{x}}(t)$ (19), the estimation error is defined as $\widetilde{\boldsymbol{x}}(t) := \boldsymbol{x}(t) - \widehat{\boldsymbol{x}}(t)$. Using $\dot{\boldsymbol{x}}(t)$ in (11) and $\dot{\widehat{\boldsymbol{x}}}(t)$ in (18), it follows that

$$\dot{\widetilde{\boldsymbol{x}}}(t) = (\mathbf{A}(t) - \boldsymbol{\mathcal{K}}(t)\mathbf{C})\widetilde{\boldsymbol{x}}(t) + \boldsymbol{h}(t). \tag{SM 6}$$

The Lyapunov function for KF is chosen as $V_{\text{KF}} := \widetilde{\boldsymbol{x}}^\top \boldsymbol{\mathcal{P}}^{-1}\widetilde{\boldsymbol{x}}$. Notice that, rely on Remark 1, the Lyapunov function V_{KF} satisfies

$$\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})\|\widetilde{\boldsymbol{x}}\|^2 \leq V_{\text{KF}} \leq \lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})\|\widetilde{\boldsymbol{x}}\|^2 \tag{SM 7}$$

where $\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1}) \geq \lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1}) > 0$.

Using (18), the time derivative of V_{KF} can be written as

$$\dot{V}_{\text{KF}} \leq -\lambda_{\min}(\boldsymbol{\Upsilon}(t))\|\widetilde{\boldsymbol{x}}(t)\|^2 + 2\widetilde{\boldsymbol{x}}^\top(t)\boldsymbol{\mathcal{P}}^{-1}(t)\boldsymbol{h}(t) \tag{SM 8}$$

where $\boldsymbol{\Upsilon}(t) := \mathbf{C}^\top\boldsymbol{\mathcal{R}}^{-1}(t)\mathbf{C} + \boldsymbol{\mathcal{P}}^{-1}(t)\boldsymbol{\mathcal{Q}}(t)\boldsymbol{\mathcal{P}}^{-1}(t) \in \mathbb{R}^{18\times18}_{\succ 0}$. Notice that $\|\partial V_{\text{KF}}/\partial\widetilde{\boldsymbol{x}}\| \leq 2\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})\|\widetilde{\boldsymbol{x}}\|$.

Based on Assumption 1, suppose the perturbation term $\boldsymbol{h}(t)$ satisfies $\|\boldsymbol{h}(t)\| \leq \varepsilon < \frac{\lambda_{\min}(\boldsymbol{\Upsilon})}{2\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}\sqrt{\frac{\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}}\epsilon r$, for all $t \geq 0$, for all $\widetilde{\boldsymbol{x}} \in D$ where $D = \{\widetilde{\boldsymbol{x}} \in \mathbb{R}^{18} | \|\widetilde{\boldsymbol{x}}\| < r\}$, and for some positive constant $\epsilon < 1$.

According to Lemma 9.2 in [2], for all $\widetilde{\boldsymbol{x}}(t_0) < \sqrt{\frac{\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}}r$, and some finite T, the solution of $\widetilde{\boldsymbol{x}}(t)$ satisfies

$$\|\widetilde{\boldsymbol{x}}(t)\| \leq \sqrt{\frac{\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}}\exp\left(-\frac{(1-\epsilon)\lambda_{\min}(\boldsymbol{\Upsilon})}{2\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}(t-t_0)\right)\|\widetilde{\boldsymbol{x}}(t_0), \quad \forall t_0 \leq t < t_0 + T \tag{SM 9}$$

and

$$\|\widetilde{\boldsymbol{x}}(t)\| \leq \frac{2\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\min}(\boldsymbol{\Upsilon})}\sqrt{\frac{\lambda_{\max}(\boldsymbol{\mathcal{P}}^{-1})}{\lambda_{\min}(\boldsymbol{\mathcal{P}}^{-1})}}\frac{\varepsilon}{\epsilon} \tag{SM 10}$$

III. Computation of auxiliary variables

We start by noting that (4a) may be rewritten as

$$\dot{\mathbf{v}}_L = \dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\|}} + \frac{1}{m_T}(\mathbf{f}^{\|} - \mathbf{f}_d^{\|}) + \widetilde{\mathbf{f}}_v^{\|} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q^{\|}, \tag{SM 11}$$

where

$$\dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\|}} = g\mathbf{e}_3 - \frac{1}{m_T}\widehat{\boldsymbol{\xi}}^{\|}. \tag{SM 12}$$

All the time derivatives that explicitly feature $\dot{\mathbf{v}}_L$ can be divided into three terms: one related to $\mathbf{f}_d^{\|}$, one related to the error $\mathbf{f}^{\|} - \mathbf{f}_d^{\|}$, and another related to $\widetilde{\mathbf{f}}_v$ and $\widetilde{\mathbf{f}}_q$.

Similarly, we can rearrange (8) as

$$\dot{\mathbf{z}}_v = \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}} + \frac{1}{m_T}\frac{\partial\mathbf{z}_v}{\partial\mathbf{v}_L}(\mathbf{f}^{\|} - \mathbf{f}_d^{\|}) + \frac{1}{m_T}\frac{\partial\mathbf{z}_v}{\partial\mathbf{v}_L}(\widetilde{\mathbf{f}}_v^{\|} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}}_q^{\|}), \tag{SM 13}$$

where

$$\dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}} = \dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\|}} - \ddot{\mathbf{p}}_d \tag{SM 14}$$

and

$$\frac{\partial\mathbf{z}_v}{\partial\mathbf{v}_L} = \mathbf{I}. \tag{SM 15}$$

Then

$$\dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} + \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\frac{1}{m_T}(\mathbf{f}^{\|} - \mathbf{f}_d^{\|}) + \widetilde{\mathbf{f}}_v^{\ \|} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q^{\ \|}\right), \tag{SM 16}$$

where

$$\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} = m_T\left(\mathbf{K_p}\mathbf{z}_v + \mathbf{K_v}\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}} - \ddot{\mathbf{p}}_d\right), \tag{SM 17}$$

its corresponding estimation

$$\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} = m_T\left(\mathbf{K_p}\widehat{\mathbf{z}}_v + \mathbf{K_v}\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}} - \ddot{\mathbf{p}}_d\right) \tag{SM 18}$$

and $\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} := m_T \mathbf{K_v}$.

We start by presenting the expression for $\widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\|}}$, first shown in (29). We have

$$\begin{aligned}\widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\|}} =& \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d)\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}})\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)\left(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\frac{d}{dt}\left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{est} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right) \\ &- \frac{k_{\mathbf{q}}}{h_{\mathbf{q}}}\left(\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\widehat{\mathbf{q}}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}^2(\mathbf{q})\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right),\end{aligned} \tag{SM 19}$$

where

$$\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}, \tag{SM 20}$$

$$\frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} = -\frac{1}{m_T}\left[(\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}} + \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}})\widehat{\boldsymbol{\xi}} + \mathbf{q}\mathbf{q}^{\mathsf{T}}\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right] - \dddot{\mathbf{p}}_d, \tag{SM 21}$$

and, finally,

$$\frac{d}{dt}\left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} = m_T\left(\mathbf{K_p}\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}} + \mathbf{K_v}\frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} - \ddddot{\mathbf{p}}_d\right). \tag{SM 22}$$

The expression for $\partial \mathbf{z}_\varpi / \partial \mathbf{v}_L$, also first shown in (29), is given by

$$\left.\frac{\partial \mathbf{z}_\varpi}{\partial \mathbf{v}_L}\right|_{\text{est}} = -\frac{k_q}{h_q}\mathbf{S}^2(\mathbf{q})\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\|\widehat{\boldsymbol{\xi}}\|})\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} - \frac{\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)}{\|\widehat{\boldsymbol{\xi}}\|^2}\left(\|\widehat{\boldsymbol{\xi}}\|\frac{\partial \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} + \frac{\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}}{\|\widehat{\boldsymbol{\xi}}\|}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right), \tag{SM 23}$$

where

$$\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}, \tag{SM 24}$$

with

$$\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} = m_T \mathbf{K_v}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L}, \tag{SM 25}$$

and where

$$\frac{\partial \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} = -\frac{1}{m_T}\mathbf{q}\mathbf{q}^{\mathsf{T}}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}, \tag{SM 26}$$

$$\frac{\partial \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} = m_T\left(\frac{1}{m_T}\mathbf{K_p}\mathbf{I}^{\|} + \mathbf{K_v}\frac{\partial \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L}\right). \tag{SM 27}$$

We also need to compute the pseudo-estimate of $\dot{\mathbf{r}}_{3d}$, which is given by

$$\widehat{\dot{\mathbf{r}}_{3d}} = -\frac{1}{\|\mathbf{f}_d\|^2}\left(\|\mathbf{f}_d\|\hat{\dot{\mathbf{f}}}_d - \frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^{\mathsf{T}}\hat{\dot{\mathbf{f}}}_d\right), \tag{SM 28}$$

where

$$\hat{\dot{\mathbf{f}}}_d = \widehat{\dot{\mathbf{f}}_d^{\|}} + \widehat{\dot{\mathbf{f}}_d^{\perp}}, \tag{SM 29}$$

with

$$\widehat{\dot{\mathbf{f}}_d^{\|}} = -\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} - \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} - \mathbf{q}\mathbf{q}^{\mathsf{T}}\hat{\dot{\boldsymbol{\xi}}}|_{\text{est}} - m_T\left(\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\mathbf{f}}_v - \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\mathbf{f}}_v - \mathbf{q}\mathbf{q}^{\mathsf{T}}\hat{\dot{\mathbf{f}}}_v\right) - m_Q\ell\left(\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\mathbf{f}}_q - \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\mathbf{f}}_q - \mathbf{q}\mathbf{q}^{\mathsf{T}}\hat{\dot{\mathbf{f}}}_q\right) \tag{SM 30}$$

and

$$
\begin{aligned}
\widehat{\dot{\mathbf{f}}_d^{\perp}} &= m_Q \ell \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q}) \left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{q}}_d - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{z}}_{\varpi} - \widehat{\mathbf{f}}_q \right) + m_Q \ell \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}}) \left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{q}}_d - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{z}}_{\varpi} - \widehat{\mathbf{f}}_q \right) \\
&\quad + m_Q \ell \mathbf{S}^2(\mathbf{q}) \left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\frac{d}{dt}\left(\widehat{\mathbf{z}}_{\varpi}\right)_{\text{est}} - \dot{\widehat{\mathbf{f}}}_q \right),
\end{aligned}
$$

where,

$$
\widehat{\dot{\mathbf{v}}}_L = \frac{1}{m_T}\mathbf{f}^{\parallel} + \widehat{\mathbf{f}}_v + \frac{m_Q \ell}{m_T}\widehat{\mathbf{f}}_q^{\parallel} + g\mathbf{e}_3, \tag{SM 31}
$$

$$
\widehat{\dot{\mathbf{z}}}_v = \widehat{\dot{\mathbf{v}}}_L - \ddot{\mathbf{p}}_d, \tag{SM 32}
$$

$$
\widehat{\dot{\boldsymbol{\xi}}} = m_T \left(\mathbf{K}_{\mathbf{p}}\widehat{\mathbf{z}}_v + \mathbf{K}_{\mathbf{v}}\widehat{\dot{\mathbf{z}}}_v - \dddot{\mathbf{p}}_d \right), \tag{SM 33}
$$

$$
\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}} \tag{SM 34}
$$

$$
\frac{d}{dt}\left(\dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = -\frac{(\widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}} + \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}})\widehat{\boldsymbol{\xi}} + \mathbf{q}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}}{m_T} - \dddot{\mathbf{p}}_d, \tag{SM 35}
$$

$$
\frac{d}{dt}\left(\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = m_T \left(\mathbf{K}_{\mathbf{p}}\widehat{\dot{\mathbf{z}}}_v + \mathbf{K}_{\mathbf{v}}\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \ddddot{\mathbf{p}}_d \right), \tag{SM 36}
$$

$$
\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = \frac{1}{\|\boldsymbol{\xi}\|^2}\left(\|\boldsymbol{\xi}\| \left(\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d)\mathbf{S}(\mathbf{q}_d) + \mathbf{S}(\mathbf{q}_d)\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d) \right) - \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\boldsymbol{\xi}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}} \right)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}}, \tag{SM 37}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = -\frac{1}{m_T}\Big(& \widehat{\dot{\boldsymbol{\varpi}}}\mathbf{q}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} + \widehat{\boldsymbol{\varpi}}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} + \widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}} + \widehat{\boldsymbol{\varpi}}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} + \mathbf{q}\widehat{\dot{\boldsymbol{\varpi}}}^{\mathsf{T}}\widehat{\boldsymbol{\xi}} + \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}} + \widehat{\boldsymbol{\varpi}}\mathbf{q}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\
& + \mathbf{q}\widehat{\boldsymbol{\varpi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{q}\mathbf{q}^{\mathsf{T}}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \Big) - \ddddot{\mathbf{p}}_d,
\end{aligned}
$$

$$
\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = m_T \left(\mathbf{K}_{\mathbf{p}}\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} + \mathbf{K}_{\mathbf{v}}\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \mathbf{p}_d^{(5)} \right), \tag{SM 38}
$$

and, finally,

$$
\begin{aligned}
&\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\varpi}}_d}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} = -\frac{k_{\mathbf{q}}}{h_{\mathbf{q}}}\Bigg(\mathbf{S}(\widehat{\dot{\boldsymbol{\varpi}}})\mathbf{S}(\mathbf{q})\widehat{\mathbf{q}}_d + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\dot{\boldsymbol{\varpi}}})\widehat{\mathbf{q}}_d \\
&+ \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}^2(\mathbf{q})\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \Bigg) - \Bigg(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\dot{\boldsymbol{\varpi}}})\mathbf{S}(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\
&- \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}})\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^2}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d)\left(\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}} \right) + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\widehat{\boldsymbol{\xi}}\|})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\
&+ \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}}{\|\widehat{\boldsymbol{\xi}}\|^2})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\widehat{\boldsymbol{\xi}}\|})\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} - \left(\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d) + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}}) \right) \\
&- \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)\Bigg[\frac{1}{\|\widehat{\boldsymbol{\xi}}\|^2}\Bigg(\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \\
&- \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}} \Bigg) - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\left(\frac{d}{dt}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^6}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\left(\|\widehat{\boldsymbol{\xi}}\|^3\widehat{\dot{\boldsymbol{\xi}}}^{\mathsf{T}} - 3\|\widehat{\boldsymbol{\xi}}\|\widehat{\boldsymbol{\xi}}^{\mathsf{T}}(\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}) \right)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\
&+ \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \right)_{\text{est}} \Bigg]\Bigg).
\end{aligned}
\tag{SM 39}
$$

IV. Estimation Errors Associated term

Based on auxiliary calculation in Section III, the estimation related terms Ψ_1 in (25), Ψ_2 in (27), Ψ_3 in (28) and Ψ_4 in (34) are expressed by

$$\Psi_1 := \frac{\partial V_1}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel} + \frac{1}{m_T}\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right)^{\parallel}\widetilde{\mathbf{v}}_L\right) \in \mathbb{R}, \tag{SM 40}$$

$$\Psi_2 := \Psi_1 - \frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|}\mathbf{q}^\top \mathbf{S}^2(\mathbf{q}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q\right)^{\parallel} \in \mathbb{R}, \tag{SM 41}$$

$$\Psi_3 := \Psi_2 + h_{\boldsymbol{\varpi}}\mathbf{z}_\varpi^\top\left(\widetilde{\mathbf{f}}_q - \left(\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right) - \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}(\mathbf{q}_d - \widehat{\mathbf{q}}_d) + \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}(\mathbf{z}_\varpi - \widehat{\mathbf{z}}_\varpi)\right) \in \mathbb{R}, \text{ and} \tag{SM 42}$$

$$\Psi_4 := \Psi_3 + h_{\mathbf{r}}\mathbf{r}_{3d}^\top \mathbf{R}\mathbf{S}(\mathbf{e}_3)\left(- \mathbf{R}^\top \mathbf{S}(\mathbf{r}_{3d})(\dot{\mathbf{r}}_{3d} - \widehat{\dot{\mathbf{r}}_{3d}}) - \frac{T_d}{h_{\mathbf{r}} m_T}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^\top(\boldsymbol{\delta}_2^\top - \widehat{\boldsymbol{\delta}_2}^\top)^{\parallel} + \frac{h_{\boldsymbol{\varpi}}}{h_{\mathbf{r}}}\frac{T_d}{m_Q \ell}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^\top(\mathbf{z}_\varpi - \widehat{\mathbf{z}_\varpi})^{\perp}\right) \tag{SM 43}$$

, respectively.

Using Young's inequality, Ψ_1 can be rewritten as

$$\begin{aligned}\Psi_1 = \frac{\partial V_1}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel} + \frac{1}{m_T}\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right)^{\parallel}\widetilde{\mathbf{v}}_L\right) &\leq \frac{\gamma_1}{2}(\beta\mathbf{z}_p^\top + \mathbf{z}_v^\top)(\beta\mathbf{z}_p + \mathbf{z}_v) + \frac{1}{2\gamma_1}\delta_{\Psi_1} \\ &\leq \frac{\gamma_1}{2}\beta^2\|\mathbf{z}_p\|^2 + \frac{\gamma_1}{2}\|\mathbf{z}_v\|^2 + \frac{1}{2\gamma_1}\delta_{\Psi_1}\end{aligned} \tag{SM 44}$$

where $\gamma_1 > 0$ and $\delta_{\Psi_1} := \left\|\widetilde{\mathbf{f}}_v^{\parallel} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q^{\parallel} + \frac{1}{m_T}\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right)^{\parallel}\widetilde{\mathbf{v}}_L\right\|^2$. According to Section II, $\widetilde{\boldsymbol{x}}$ is bounded, hence $\widetilde{\mathbf{v}}_L$, $\widetilde{\boldsymbol{\varpi}}$, $\widetilde{\mathbf{f}}_v$ and $\widetilde{\mathbf{f}}_q$ are also bounded. In this case, δ_{Ψ_1} is bounded as well.

Then using Young's inequality, Ψ_2 can be rewritten as

$$\begin{aligned}\Psi_2 = \Psi_1 - \frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|}\mathbf{q}^\top \mathbf{S}^2(\mathbf{q}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q\right)^{\parallel} &\leq \Psi_1 + \frac{\gamma_2}{2}\left(\frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|}\mathbf{q}^\top \mathbf{S}^2(\mathbf{q}_d)\right)^\top\left(\frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|}\mathbf{q}^\top \mathbf{S}^2(\mathbf{q}_d)\right) + \frac{1}{2\gamma_2}\delta_{\Psi_2} \\ &\leq \Psi_1 + \frac{\gamma_2}{2}\frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|^2} + \frac{\gamma_2}{2}\|\mathbf{q}^\top \mathbf{S}^2(\mathbf{q}_d)\|^2 + \frac{1}{2\gamma_2}\delta_{\Psi_2}\end{aligned} \tag{SM 45}$$

where $\gamma_2 > 0$ and $\delta_{\Psi_2} := \left\|\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q\right)^{\parallel}\right\|^2$. According to $\widetilde{\mathbf{f}}_v$ and $\widetilde{\mathbf{f}}_q$ are bounded, δ_{Ψ_2} is bounded.

Using Young's inequality for Ψ_3, it can be formulated as

$$\Psi_3 = \Psi_2 + h_{\boldsymbol{\varpi}}\mathbf{z}_\varpi^\top\left(\widetilde{\mathbf{f}}_q - \left(\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right) - \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}(\mathbf{q}_d - \widehat{\mathbf{q}}_d) + \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}(\mathbf{z}_\varpi - \widehat{\mathbf{z}}_\varpi)\right) \leq \Psi_2 + \frac{\gamma_3}{2}\|\mathbf{z}_\varpi\|^2 + \frac{1}{2\gamma_3}\delta_{\Psi_3} \tag{SM 46}$$

where $\gamma_3 > 0$ and $\delta_{\Psi_3} := \left\|\widetilde{\mathbf{f}}_q - \left(\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right) - \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}(\mathbf{q}_d - \widehat{\mathbf{q}}_d) + \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}(\mathbf{z}_\varpi - \widehat{\mathbf{z}}_\varpi)\right\|^2$. Due to $\widetilde{\mathbf{v}}_L$ is bounded, then $\boldsymbol{\xi} - \widehat{\boldsymbol{\xi}}$ and $\mathbf{q}_d - \widehat{\mathbf{q}}_d$ are bounded. Then based on the reference trajectory described in Section IV [1] are bounded by construction, $\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$, $\dot{\mathbf{q}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$ and $\frac{d}{dt}\left(\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{d}{dt}\left(\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$ are bounded. Moreover, on account of $\widetilde{\boldsymbol{x}}$ is bounded, $\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$ and $\mathbf{z}_\varpi - \widehat{\mathbf{z}}_\varpi$ are bounded. In this case, δ_{Ψ_3} is bounded.

Similarly, $\dot{\mathbf{r}}_{3d} - \widehat{\dot{\mathbf{r}}_{3d}}$ and $\boldsymbol{\delta}_2^\top - \widehat{\boldsymbol{\delta}_2}^\top$ are also bounded. Due to Rotation matrix $\mathbf{R}$ property and, $\mathbf{r}_3$ and $\mathbf{r}_{3d}$ are unit vector, $\delta_{\Psi_4} := h_{\mathbf{r}}\mathbf{r}_{3d}^\top \mathbf{R}\mathbf{S}(\mathbf{e}_3)\left(- \mathbf{R}^\top \mathbf{S}(\mathbf{r}_{3d})(\dot{\mathbf{r}}_{3d} - \widehat{\dot{\mathbf{r}}_{3d}}) - \frac{T_d}{h_{\mathbf{r}} m_T}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^\top(\boldsymbol{\delta}_2^\top - \widehat{\boldsymbol{\delta}_2}^\top)^{\parallel} + \frac{h_{\boldsymbol{\varpi}}}{h_{\mathbf{r}}}\frac{T_d}{m_Q \ell}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^\top(\mathbf{z}_\varpi - \widehat{\mathbf{z}_\varpi})^{\perp}\right)$ is bounded as well. And all of the estimation error related term that remained in time derivative of Lyapunov function V_4 is introduced as

$$\delta_{V_4} := \delta_{\Psi_1} + \delta_{\Psi_2} + \delta_{\Psi_3} + \delta_{\Psi_4}. \tag{SM 47}$$

In turn, the boundedness of δ_{V_4} is confirmed.

References

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[2] Khalil, H. K.. Pearson education, Nonlinear systems (3rd ed.). Upper Saddle River, New Jersey: Prentice Hall. (2002)