Output Feedback Nonlinear Control of An Underactuated Flying Inverted Pendulum: Supplementary Material

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Abstract

This is a complementary document to the paper presented in [1]. Here we fully disclose all the detailed derivations and equations that are essential to implement the controller reported therein. The main paper addresses the nonlinear control problem of balancing an inverted pendulum on a flying underactuated unmanned aerial vehicle. To simultaneously tackle the system's slow and fast transients, a novel error transformation approach is considered where no linearization method whatsoever is used, making this controller suitable beyond the scope of trim maneuvers. Furthermore, the controller features an adaptive bounded law that is used to compensate for unknown external disturbances. Our Lyapunov-based control design is rooted in an integral backstepping process, wherein the origin of the closed-loop total system error is shown to be almost globally asymptotically stable.

I. TITLE

$$\mathbf{\Phi}(t,t_0) := \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 \cdots \tag{SM 1}$$

According to the transition matrix definition in (SM 1), the transition matrix's second and third terms are calculated as

$$\int_{t_0}^{t} \mathbf{A}(\sigma_1)d\sigma_1 = \begin{bmatrix} \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{m_Q \ell}{m_T}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top & \mathbf{0} & \mathbf{0} & \mathbf{I}(t-t_0) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{SM 2}$$

$$\int_{t_0}^{t} \mathbf{A}(\sigma_1)\int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)d\sigma_2 d\sigma_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{n_{t2}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & \mathbf{0} & \mathbf{0} & \boldsymbol{n_{t2}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{SM 3}$$

where $\boldsymbol{n_{t1}} := (t-t_0)\mathbf{I}$, $\boldsymbol{n_{t2}} := \frac{1}{2}(t-t_0)^2\mathbf{I}$, $\boldsymbol{n_{q1}} := \frac{m_Q\ell}{m_T}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top$ and $\boldsymbol{n_{q2}} := \frac{m_Q\ell}{m_T}\int_{t_0}^{t}\int_{t_0}^{t}\mathbf{q}\mathbf{q}^\top d\tau d\tau$.

$$\boldsymbol{W}(t_0,t_f) = \int_{t_0}^{t_f} \begin{bmatrix} 1 & t-t_0 & n_t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ t-t_0 & (t-t_0)^2 & n_t(t-t_0) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ n_t & n_t(t-t_0) & {n_t}^2 + {\boldsymbol{n_{q2}}}^\top\boldsymbol{n_{q2}} & {\boldsymbol{n_{q2}}}^\top & (t-t_0){\boldsymbol{n_{q2}}}^\top & n_t{\boldsymbol{n_{q2}}}^\top \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & 1 & t-t_0 & n_t \\ \mathbf{0} & \mathbf{0} & (t-t_0)\boldsymbol{n_{q2}} & t-t_0 & (t-t_0)^2 & (t-t_0)n_t \\ \mathbf{0} & \mathbf{0} & n_t\boldsymbol{n_{q2}} & n_t & (t-t_0)n_t & {n_t}^2 \end{bmatrix} \otimes \mathbf{I}\ dt \tag{SM 4}$$

$$\boldsymbol{W}(t-\delta,t) = \begin{bmatrix} \delta & \frac{\delta^2}{2} & \frac{\delta^3}{6} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\delta^2}{2} & \frac{\delta^3}{3} & \frac{\delta^4}{8} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\delta^3}{6} & \frac{\delta^4}{8} & \frac{\delta^5}{20} + \int_{t-\delta}^{t}{\boldsymbol{n_{q2}}}^\top\boldsymbol{n_{q2}} & {\boldsymbol{n_{q2}}}^\top & (t-t_0){\boldsymbol{n_{q2}}}^\top & n_t{\boldsymbol{n_{q2}}}^\top \\ \mathbf{0} & \mathbf{0} & \boldsymbol{n_{q2}} & \delta & \frac{\delta^2}{2} & \frac{\delta^3}{6} \\ \mathbf{0} & \mathbf{0} & (t-t_0)\boldsymbol{n_{q2}} & \frac{\delta^2}{2} & \frac{\delta^3}{3} & \frac{\delta^4}{8} \\ \mathbf{0} & \mathbf{0} & n_t\boldsymbol{n_{q2}} & \frac{\delta^3}{6} & \frac{\delta^4}{8} & \frac{\delta^5}{20} \end{bmatrix} \otimes \mathbf{I} \tag{SM 5}$$

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II. Estimation related terms

$$\widetilde{W_1} := \frac{\partial V_1}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}_v}^{\parallel} + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}_q}^{\parallel} + \frac{1}{m_T}\left(\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{z}_v}\right)^{\parallel}\widetilde{\mathbf{v}}_L\right) \in \mathbb{R} \tag{SM 6}$$

By exploiting the proposed KF, exchange the unmeasured linear velocity $\mathbf{v}_L$ in (**??**) with estimation value which is obtained from (**??**) as $\mathbf{z}_v = \widehat{\mathbf{z}}_v + \widetilde{\mathbf{v}}_L$, where $\widehat{\mathbf{z}}_v := \widehat{\mathbf{v}}_L - \dot{\mathbf{p}}_d \in \mathbb{R}^3$. Then $\boldsymbol{\xi} = \widehat{\boldsymbol{\xi}} + \partial\boldsymbol{\xi}/\partial\mathbf{z}_v\widetilde{\mathbf{v}}_L$, where $\widehat{\boldsymbol{\xi}} := m_T\left(\mathbf{K_p}\mathbf{z}_p + \mathbf{K_v}\widehat{\mathbf{z}}_v + g\mathbf{e}_3 - \ddot{\mathbf{p}}_d\right)$ and $\partial\boldsymbol{\xi}/\partial\mathbf{z}_v := m_T\mathbf{K_v}$.

$$\widetilde{W_2} := \widetilde{W_1} - \frac{h_{\mathbf{q}}}{\|\boldsymbol{\xi}\|}\mathbf{q}^{\mathsf{T}}\mathbf{S}^2(\mathbf{q}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\widetilde{\mathbf{f}}_v + \frac{m_Q \ell}{m_T}\widetilde{\mathbf{f}}_q\right)^{\parallel} \in \mathbb{R} \tag{SM 7}$$

$$\widetilde{W_3} := \widetilde{W_2} + h_{\boldsymbol{\varpi}}\mathbf{z}_{\varpi}^{\mathsf{T}}\left(\widetilde{\mathbf{f}}_q - \left(\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}} - \widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}\right) - \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\left(\mathbf{q}_d - \widehat{\mathbf{q}}_d\right) + \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\left(\mathbf{z}_{\varpi} - \widehat{\mathbf{z}}_{\varpi}\right)\right) \in \mathbb{R} \tag{SM 8}$$

$$\widetilde{W_4} := \widetilde{W_3} + h_{\mathbf{r}}\mathbf{r}_{3d}^{\mathsf{T}}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\left(-\mathbf{R}^{\mathsf{T}}\mathbf{S}(\mathbf{r}_{3d})(\dot{\mathbf{r}}_{3d} - \widehat{\dot{\mathbf{r}}_{3d}}) - \frac{T_d}{h_{\mathbf{r}}m_T}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}(\boldsymbol{\delta}_2^{\mathsf{T}} - \widehat{\boldsymbol{\delta}_2}^{\mathsf{T}})^{\parallel} + \frac{h_{\boldsymbol{\varpi}}}{h_{\mathbf{r}}}\frac{T_d}{m_Q\ell}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\mathsf{T}}(\mathbf{z}_{\varpi} - \widehat{\mathbf{z}_{\varpi}})^{\perp}\right) \tag{SM 9}$$

III. Computation of auxiliary variables

We start by noting that (4a) may be rewritten as

$$\dot{\mathbf{v}}_L = \dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{m_T}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \widetilde{\mathbf{f}_v}^{\parallel} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}_q}^{\parallel}, \tag{SM 10}$$

where

$$\dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = g\mathbf{e}_3 - \frac{1}{m_T}\widehat{\boldsymbol{\xi}}^{\parallel}. \tag{SM 11}$$

All the time derivatives that explicitly feature $\dot{\mathbf{v}}_L$ can be divided into three terms: one related to $\mathbf{f}_d^{\parallel}$ and $\widehat{\boldsymbol{\xi}}$, one related to the error $\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}$, and another related to $\widetilde{\mathbf{f}}_v$ and $\widetilde{\mathbf{f}}_q$.

Similarly, we can rearrange (8) as

$$\dot{\mathbf{z}}_v = \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{m_T}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \frac{1}{m_T}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L}(\widetilde{\mathbf{f}_v}^{\parallel} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}_q}^{\parallel}), \tag{SM 12}$$

where

$$\dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = \dot{\mathbf{v}}_L|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \ddot{\mathbf{p}}_d \tag{SM 13}$$

and

$$\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L} = \mathbf{I}. \tag{SM 14}$$

Then

$$\dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\left(\frac{1}{m_T}(\mathbf{f}^{\parallel} - \mathbf{f}_d^{\parallel}) + \widetilde{\mathbf{f}_v}^{\parallel} + \frac{m_Q\ell}{m_T}\widetilde{\mathbf{f}_q}^{\parallel}\right), \tag{SM 15}$$

where

$$\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K_p}\mathbf{z}_v + \mathbf{K_v}\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \dddot{\mathbf{p}}_d\right) \tag{SM 16}$$

$$\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K_p}\widehat{\mathbf{z}}_v + \mathbf{K_v}\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \dddot{\mathbf{p}}_d\right) \tag{SM 17}$$

and $\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} := m_T\mathbf{K_v}$.

We start by presenting the expression for $\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}$, first shown in (29). We have

$$\begin{aligned}\widehat{\dot{\boldsymbol{\varpi}}_{d}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} =& \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)\left(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{est} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right) \\ &- \frac{k_{\mathbf{q}}}{h_{\mathbf{q}}}\left(\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\widehat{\mathbf{q}}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}^2(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right),\end{aligned} \tag{SM 18}$$

where

$$\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}, \tag{SM 19}$$

$$\frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = -\frac{1}{m_T}\left[(\widehat{\boldsymbol{\varpi}}\mathbf{q}^\mathsf{T} + \mathbf{q}\widehat{\boldsymbol{\varpi}}^\mathsf{T})\widehat{\boldsymbol{\xi}} + \mathbf{q}\mathbf{q}^\mathsf{T}\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right] - \dddot{\mathbf{p}}_d, \tag{SM 20}$$

and, finally,

$$\frac{d}{dt}\left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = m_T\left(\mathbf{K}_\mathbf{p}\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{K}_\mathbf{v}\frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \ddddot{\mathbf{p}}_d\right). \tag{SM 21}$$

The expression for $\partial \mathbf{z}_\varpi/\partial \mathbf{v}_L$, also first shown in (29), is given by

$$\left.\frac{\partial \mathbf{z}_\varpi}{\partial \mathbf{v}_L}\right|_{\text{est}} = -\frac{k_q}{h_q}\mathbf{S}^2(\mathbf{q})\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\widehat{\boldsymbol{\xi}}\|})\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} - \frac{\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)}{\|\widehat{\boldsymbol{\xi}}\|^2}\left(\|\widehat{\boldsymbol{\xi}}\|\frac{\partial\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} + \frac{\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\widehat{\boldsymbol{\xi}}^\mathsf{T}}{\|\widehat{\boldsymbol{\xi}}\|}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right), \tag{SM 22}$$

where

$$\left.\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right|_{\text{est}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}, \tag{SM 23}$$

with

$$\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} = m_T\mathbf{K}_\mathbf{v}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L}, \tag{SM 24}$$

and where

$$\frac{\partial \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = -\frac{1}{m_T}\mathbf{q}\mathbf{q}^\mathsf{T}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}, \tag{SM 25}$$

$$\frac{\partial \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = m_T\left(\mathbf{K}_\mathbf{p}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_L} + \mathbf{K}_\mathbf{v}\frac{\partial \dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}\right), \tag{SM 26}$$

and, finally,

$$\frac{\partial \hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L} = \frac{1}{\|\boldsymbol{\xi}\|}\left(\mathbf{q}_d\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}^\mathsf{T} - \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\mathbf{q}_d^\mathsf{T}\right)\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L} - \frac{\mathbf{S}^2(\mathbf{q}_d)\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|^3} + \frac{\mathbf{S}^2(\mathbf{q}_d)\frac{\partial \dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|}. \tag{SM 27}$$

In the following we present some useful partial derivatives of the Lyapunov function candidates with respect to $\mathbf{v}_L$ and $\boldsymbol{\omega}$. They are given by

$$\frac{\partial V_3}{\partial \mathbf{v}_L} = \mathbf{e}^\mathsf{T}\frac{\partial \mathbf{e}}{\partial \mathbf{v}_L} - h_q\mathbf{z}_q^\mathsf{T}\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}, \tag{SM 28}$$

$$\frac{\partial V_4}{\partial \mathbf{v}_L} = \frac{\partial V_3}{\partial \mathbf{v}_L} + h_\omega\mathbf{z}_\omega^\mathsf{T}\frac{\partial \mathbf{z}_\omega}{\partial \mathbf{v}_L}, \tag{SM 29}$$

$$\frac{\partial V_5}{\partial \mathbf{v}_L} = \frac{\partial V_4}{\partial \mathbf{v}_L} - h_r\mathbf{r}_{3d}^\mathsf{T}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^\mathsf{T}\mathbf{S}(\mathbf{r}_{3d})\frac{\partial \mathbf{r}_{3d}}{\partial \mathbf{v}_L}, \tag{SM 30}$$

and

$$\frac{\partial V_5}{\partial \boldsymbol{\omega}} = h_\omega\mathbf{z}_\omega^\mathsf{T}\frac{\partial \mathbf{z}_\omega}{\partial \boldsymbol{\omega}} - h_r\mathbf{r}_{3d}^\mathsf{T}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^\mathsf{T}\mathbf{S}(\mathbf{r}_{3d})\frac{\partial \mathbf{r}_{3d}}{\partial \boldsymbol{\omega}}, \tag{SM 31}$$

where

$$\frac{\partial \mathbf{z}_\omega}{\partial \boldsymbol{\omega}} = \mathbf{S}(\mathbf{q}), \tag{SM 32}$$

$$\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}} = -\mathbf{S}(\mathbf{q}), \tag{SM 33}$$

$$\frac{\partial \frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = -\frac{1}{m_T}\left(\mathbf{q}^\mathsf{T}\boldsymbol{\xi} + \mathbf{q}\boldsymbol{\xi}^\mathsf{T}\right)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}, \tag{SM 34}$$

$$\frac{\partial \frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_L} = -\frac{(\dot{\mathbf{q}}\mathbf{q}^\mathsf{T} + \mathbf{q}\dot{\mathbf{q}}^\mathsf{T})\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} + \mathbf{q}\mathbf{q}^\mathsf{T}\frac{\partial\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_L}}{m_T}, \tag{SM 35}$$

$$\frac{\partial \frac{d}{dt}\left(\hat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = \epsilon(\mathbf{K}_\mathbf{p} + \mathbf{K}_\mathbf{v})\frac{\partial \frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}}, \tag{SM 36}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}}{\partial \mathbf{v}_L} = \epsilon^2(\mathbf{I}+\mathbf{K_v}\mathbf{K_p})\frac{\partial \widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} + \epsilon(\mathbf{K_p}+\mathbf{K_v})\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}}{\partial \mathbf{v}_L}, \tag{SM 37}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = m_T\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}}, \tag{SM 38}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}}{\partial \mathbf{v}_L} = m_T\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}}{\partial \mathbf{v}_L}, \tag{SM 39}$$

$$\begin{aligned}
\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} = &-\frac{k_q}{h_q}\left([\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})+\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})]\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L} + \mathbf{S}^2(\mathbf{q})\frac{\partial \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L}\right) + \frac{\epsilon}{h_q m_T}\frac{1}{\|\boldsymbol{\xi}\|}\Bigg[\mathbf{S}^2(\mathbf{q})(\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}})\frac{\partial \mathbf{e}}{\partial \mathbf{v}_L}\\
&+\frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q})\mathbf{e}\left(\|\boldsymbol{\xi}\|\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} - \frac{1}{\|\boldsymbol{\xi}\|}\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right) + \mathbf{S}^2(\mathbf{q})\mathbf{e}\boldsymbol{\xi}^\top\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} + \mathbf{S}^2(\mathbf{q})\widehat{\dot{\mathbf{e}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\\
&+[\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})+\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})]\left(\mathbf{e}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} + \|\boldsymbol{\xi}\|^2\frac{\partial \mathbf{e}}{\partial \mathbf{v}_L}\right) + \epsilon\|\boldsymbol{\xi}\|^2\mathbf{S}^2(\mathbf{q})\frac{\partial \widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L}\Bigg] - \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\|\boldsymbol{\xi}\|})\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\\
&+\frac{\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}_d)+\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}})}{\|\boldsymbol{\xi}\|}\left(\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} - \frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|^2}\right) - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\|\boldsymbol{\xi}\|})\frac{\partial \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L}\\
&-\mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\|\boldsymbol{\xi}\|^2\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}} - \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\|\boldsymbol{\xi}\|^3})\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L} + \mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\Bigg(-\frac{\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} - \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\boldsymbol{\xi}^\top\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|^3}\\
&\frac{\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}}{\partial \mathbf{v}_L}\|\boldsymbol{\xi}\| - \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}}{\|\boldsymbol{\xi}\|^2} + \frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\left(\|\boldsymbol{\xi}\|^3\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} - 3\|\boldsymbol{\xi}\|\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L}\right)}{\|\boldsymbol{\xi}\|^6}\Bigg),
\end{aligned} \tag{SM 40}$$

and, at last,

$$\begin{aligned}
\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \boldsymbol{\omega}} = &-\mathbf{S}(\boldsymbol{\omega})\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}} + \mathbf{S}(\dot{\mathbf{q}}) - \frac{k_q}{h_q}(\mathbf{q}\mathbf{q}_d^\top - \mathbf{q}_d\mathbf{q}^\top)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}} + \frac{k_q}{h_q}\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}} + \frac{\|\boldsymbol{\xi}\|\epsilon}{h_q m_T}\left((\mathbf{q}\mathbf{e}^\top - \mathbf{e}\mathbf{q}^\top)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}\right.\\
&\left.-\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{e})\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}\right) + \frac{\mathbf{q}_d\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}^\top - \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\mathbf{q}_d^\top}{\|\boldsymbol{\xi}\|}\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}} + \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}},
\end{aligned} \tag{SM 41}$$

Other important partial derivatives used in the derivations are the following:

$$\frac{\partial \mathbf{f}_d}{\partial \mathbf{v}_L} = -\mathbf{q}\mathbf{q}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_L} + m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \mathbf{v}_L} + \frac{h_q}{h_\omega}\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_L}\right), \tag{SM 42}$$

$$\frac{\partial \mathbf{f}_d}{\partial \boldsymbol{\omega}} = 2m_Q\ell\mathbf{q}\boldsymbol{\omega}^\top\mathbf{I} + m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\partial \boldsymbol{\omega}} + \frac{k_\omega}{h_\omega}\mathbf{S}(\mathbf{q})\right), \tag{SM 43}$$

$$\frac{\partial \mathbf{r}_{3d}}{\partial \boldsymbol{\omega}} = -\frac{\|\mathbf{f}_d\|\mathbf{I} - \frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\top}{\|\mathbf{f}_d\|^2}\frac{\partial \mathbf{f}_d}{\partial \boldsymbol{\omega}}, \tag{SM 44}$$

and

$$\frac{\partial \mathbf{r}_{3d}}{\partial \mathbf{v}_L} = -\frac{\|\mathbf{f}_d\|\mathbf{I} - \frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\top}{\|\mathbf{f}_d\|^2}\frac{\partial \mathbf{f}_d}{\partial \mathbf{v}_L}. \tag{SM 45}$$

We also need to compute the pseudo-estimate of $\dot{\mathbf{r}}_{3d}$, which is given by

$$\widehat{\dot{\mathbf{r}}_{3d}} = -\frac{1}{\|\mathbf{f}_d\|^2}\left(\|\mathbf{f}_d\|\widehat{\dot{\mathbf{f}}}_d - \frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\top\widehat{\dot{\mathbf{f}}}_d\right), \tag{SM 46}$$

where

$$\hat{\dot{\mathbf{f}}}_d = \widehat{\dot{\mathbf{f}}_d^{\parallel}} + \widehat{\dot{\mathbf{f}}_d^{\perp}}, \tag{SM 47}$$

with

$$\widehat{\dot{\mathbf{f}}_d^{\parallel}} = -\widehat{\boldsymbol{\varpi}}\mathbf{q}^\mathsf{T}\widehat{\boldsymbol{\xi}} - \mathbf{q}\widehat{\boldsymbol{\varpi}}^\mathsf{T}\widehat{\boldsymbol{\xi}} - \mathbf{q}\mathbf{q}^\mathsf{T}\hat{\dot{\boldsymbol{\xi}}}|_{\text{est}} - m_T\left(\widehat{\boldsymbol{\varpi}}\mathbf{q}^\mathsf{T}\widehat{\mathbf{f}}_v - \mathbf{q}\widehat{\boldsymbol{\varpi}}^\mathsf{T}\widehat{\mathbf{f}}_v - \mathbf{q}\mathbf{q}^\mathsf{T}\hat{\dot{\mathbf{f}}}_v\right) - m_Q\ell\left(\widehat{\boldsymbol{\varpi}}\mathbf{q}^\mathsf{T}\widehat{\mathbf{f}}_q - \mathbf{q}\widehat{\boldsymbol{\varpi}}^\mathsf{T}\widehat{\mathbf{f}}_q - \mathbf{q}\mathbf{q}^\mathsf{T}\hat{\dot{\mathbf{f}}}_q\right) \tag{SM 48}$$

and

$$\begin{aligned}\widehat{\dot{\mathbf{f}}_d^{\perp}} &= m_Q\ell\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{q}}_d - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{z}}_{\varpi} - \widehat{\mathbf{f}}_q\right) + m_Q\ell\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{q}}_d - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\widehat{\mathbf{z}}_{\varpi} - \widehat{\mathbf{f}}_q\right)\\ &\quad + m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\varpi}}_{\boldsymbol{d}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} + \frac{h_{\mathbf{q}}}{h_{\boldsymbol{\varpi}}}\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} - \frac{k_{\boldsymbol{\varpi}}}{h_{\boldsymbol{\varpi}}}\frac{d}{dt}\left(\widehat{\mathbf{z}}_{\varpi}\right)_{\text{est}} - \hat{\dot{\mathbf{f}}}_q\right),\end{aligned}$$

where,

$$\hat{\dot{\mathbf{v}}}_L = \frac{1}{m_T}\mathbf{f}^{\parallel} + \widehat{\mathbf{f}}_v + \frac{m_Q\ell}{m_T}\widehat{\mathbf{f}}_q^{\parallel} + g\mathbf{e}_3, \tag{SM 49}$$

$$\hat{\dot{\mathbf{z}}}_v = \hat{\dot{\mathbf{v}}}_L - \ddot{\mathbf{p}}_d, \tag{SM 50}$$

$$\hat{\dot{\boldsymbol{\xi}}} = m_T\left(\mathbf{K}_{\mathbf{p}}\widehat{\mathbf{z}}_v + \mathbf{K}_{\mathbf{v}}\hat{\dot{\mathbf{z}}}_v - \dddot{\mathbf{p}}_d\right), \tag{SM 51}$$

$$\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} = \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}^2(\widehat{\mathbf{q}}_d)\hat{\dot{\boldsymbol{\xi}}} \tag{SM 52}$$

$$\frac{d}{dt}\left(\dot{\mathbf{z}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = -\frac{(\widehat{\boldsymbol{\varpi}}\mathbf{q}^\mathsf{T} + \mathbf{q}\widehat{\boldsymbol{\varpi}}^\mathsf{T})\widehat{\boldsymbol{\xi}} + \mathbf{q}\mathbf{q}^\mathsf{T}\hat{\dot{\boldsymbol{\xi}}}}{m_T} - \dddot{\mathbf{p}}_d, \tag{SM 53}$$

$$\frac{d}{dt}\left(\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = m_T\left(\mathbf{K}_{\mathbf{p}}\hat{\dot{\mathbf{z}}}_v + \mathbf{K}_{\mathbf{v}}\frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \ddddot{\mathbf{p}}_d\right), \tag{SM 54}$$

$$\frac{d}{dt}\left(\hat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = \frac{1}{\|\boldsymbol{\xi}\|^2}\left(\|\boldsymbol{\xi}\|\left(\mathbf{S}(\hat{\dot{\mathbf{q}}}_d)\mathbf{S}(\mathbf{q}_d) + \mathbf{S}(\mathbf{q}_d)\mathbf{S}(\hat{\dot{\mathbf{q}}}_d)\right) - \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\boldsymbol{\xi}^\mathsf{T}\hat{\dot{\boldsymbol{\xi}}}\right)\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\frac{d}{dt}\left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}, \tag{SM 55}$$

$$\hat{\dot{\boldsymbol{\omega}}} = -\frac{1}{m_Q\ell}\mathbf{S}(\mathbf{q})(\mathbf{f} + \widehat{\mathbf{b}}), \tag{SM 56}$$

$$\hat{\ddot{\mathbf{q}}} = -\mathbf{S}(\mathbf{q})\hat{\dot{\boldsymbol{\omega}}} - \|\boldsymbol{\omega}\|^2\mathbf{q} \tag{SM 57}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = &-\frac{1}{m_T}\left(\hat{\ddot{\mathbf{q}}}\mathbf{q}^\mathsf{T}\boldsymbol{\xi} + \dot{\mathbf{q}}\dot{\mathbf{q}}^\mathsf{T}\boldsymbol{\xi} + \dot{\mathbf{q}}\mathbf{q}^\mathsf{T}\hat{\dot{\boldsymbol{\xi}}} + \dot{\mathbf{q}}\dot{\mathbf{q}}^\mathsf{T}\boldsymbol{\xi} + \mathbf{q}\hat{\ddot{\mathbf{q}}}^\mathsf{T}\boldsymbol{\xi} + \mathbf{q}\dot{\mathbf{q}}^\mathsf{T}\hat{\dot{\boldsymbol{\xi}}} + \dot{\mathbf{q}}\mathbf{q}^\mathsf{T}\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right.\\ &\left.+\mathbf{q}\dot{\mathbf{q}}^\mathsf{T}\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{q}\mathbf{q}^\mathsf{T}\frac{d}{dt}\left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\right) - \ddddot{\mathbf{p}}_d,\end{aligned}$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\hat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = \epsilon^2(\mathbf{I} + \mathbf{K}_{\mathbf{v}}\mathbf{K}_{\mathbf{p}})\frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} + \epsilon(\mathbf{K}_{\mathbf{p}} + \mathbf{K}_{\mathbf{v}})\frac{d}{dt}\left(\frac{d}{dt}\left(\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}, \tag{SM 58}$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\hat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = m_T\left(\frac{d}{dt}\left(\frac{d}{dt}\left(\hat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \mathbf{p}_d^{(5)}\right), \tag{SM 59}$$

and, finally,

$$
\begin{aligned}
&\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\varpi}}_d|}_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}} = -\frac{k_{\mathbf{q}}}{h_{\mathbf{q}}}\Bigg(\mathbf{S}(\dot{\widehat{\boldsymbol{\varpi}}})\mathbf{S}(\mathbf{q})\widehat{\mathbf{q}}_d + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\mathbf{q}}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\widehat{\boldsymbol{\varpi}}})\widehat{\mathbf{q}}_d \\
&+\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}} + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\boldsymbol{\varpi}})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}} + \mathbf{S}^2(\mathbf{q})\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}\Bigg) - \Bigg(\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\dot{\widehat{\boldsymbol{\varpi}}})\mathbf{S}(\widehat{\mathbf{q}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} \\
&-\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}})\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^2}\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d)\left(\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}\right) + \mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\|\widehat{\boldsymbol{\xi}}\|})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}} \\
&+\mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}}{\|\widehat{\boldsymbol{\xi}}\|^2})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}}{\|\widehat{\boldsymbol{\xi}}\|})\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}} - \left(\mathbf{S}(\widehat{\boldsymbol{\varpi}})\mathbf{S}(\widehat{\mathbf{q}}_d) + \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{d}{dt}\left(\widehat{\mathbf{q}}_d\right)_{\text{est}})\right) \\
&-\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\mathbf{q}}_d)\Bigg[\frac{1}{\|\widehat{\boldsymbol{\xi}}\|^2}\left(\|\widehat{\boldsymbol{\xi}}\|\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}\right. \\
&\left.-\frac{1}{\|\widehat{\boldsymbol{\xi}}\|}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}\right) - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\left(\frac{d}{dt}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} - \frac{1}{\|\widehat{\boldsymbol{\xi}}\|^6}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\left(\|\widehat{\boldsymbol{\xi}}\|^3\widehat{\dot{\boldsymbol{\xi}}}^{\mathsf{T}} - 3\|\widehat{\boldsymbol{\xi}}\|\widehat{\boldsymbol{\xi}}^{\mathsf{T}}(\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\widehat{\dot{\boldsymbol{\xi}}})\right)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}} \\
&+\frac{1}{\|\widehat{\boldsymbol{\xi}}\|^3}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\widehat{\boldsymbol{\xi}}^{\mathsf{T}}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\|}}\right)_{\text{est}}\Bigg]\Bigg). \qquad \text{(SM 60)}
\end{aligned}
$$

References

[1] Yang, W., Yu, G., Reis, J. and Silvestre, C. Robust Nonlinear 3D Control of an Inverted Pendulum Balanced on a Quadrotor. *Automatica*. pp. 1-12 (2023)