

Predicting spatial curvature Ω_K in globally CPT -symmetric universes

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Boyle and Turok’s CPT -symmetric universe model posits that the universe was symmetric at the Big Bang, addressing numerous problems in both cosmology and the Standard Model of particle physics. We extend this model by incorporating the symmetric conditions at the end of the Universe, which impose constraints on the allowed perturbation modes. These constrained modes conflict with the integer wave vectors required by the global spatial geometry in a closed universe. To resolve this conflict, only specific values of curvature are permissible, and in particular the curvature density is constrained to be $\Omega_K \in \{-0.014, -0.009, -0.003, \dots\}$, consistent with *Planck* observations.

INTRODUCTION

The concordance Λ CDM model and the Standard Model successfully explain a range of observed and theoretical phenomena in cosmology [1] and particle physics [2]. However, several issues remain unresolved: For cosmology, there are the horizon [3] and flatness [4] problems, as well as the composition and nature of dark matter and energy. For the standard model of particle physics, there is the cosmological constant problem [5], Weyl anomaly [6] and the strong CP problem [7]. Attempts to address these challenges often involve introducing new degrees of freedom, such as an era of inflation [8], a wide variety of dark matter particles [9], axions [10], or modifying General Relativity [11]. Despite these efforts, no single model can resolve all these issues simultaneously, and such modifications compromise the apparent simplicity of our universe as suggested by recent observations [1, 12].

To address these challenges, Boyle et al. [13, 14] propose a novel cosmological model known as the CPT -symmetric Universe. Rather than introducing new degrees of freedom, they posit that the universe is CPT -symmetric at the Big Bang singularity. Given this initial condition, the aforementioned issues could be solved without adding extra complexity [15–18]. For example, the CPT -symmetric universe exhibits a periodic global structure, which allows for the calculation of its gravitational entropy [19]. As noted by Turok and Boyle [20], maximizing this entropy resolves the flatness, homogeneity, and isotropy problems without requiring inflation.

In this letter, we further investigate the impact of this periodic global structure on cosmological perturbations. Previous work by Lasenby et al. [21] has explored this concept within the context of flat universes. Their findings suggest that the periodic conditions impose constraints on perturbations within the universe, which have observable effects on the Cosmic Microwave Background (CMB) power spectrum [22, 23].

Building on this work, we extend the analysis of Lasenby et al. [21] to include curved universes. Our findings indicate that the constraints imposed by the future

conformal boundary conflict with the spatial boundary conditions in a closed universe. To reconcile this conflict, only specific values of curvature are permissible, which are consistent with the observations from *Planck* 2018 [1], adding to the recent conversation in the literature surrounding closed universes [24–27].

BACKGROUND AND PERTURBATIONS

We build on the derivation presented in Lasenby et al. [21] and extend it to a curved universe. The perturbed FLRW metric can be written as:

$$ds^2 = a^2 \left[-(1 + 2\Phi)d\eta^2 + (1 - 2\Phi) \left(\frac{1}{1 - \kappa r^2} dr^2 + r^2 d\Omega^2 \right) \right], \quad (1)$$

where Φ is the perturbation in the Newtonian gauge, and κ controls the spatial curvature of the universe.

We analyse the evolution of the background in conformal time η instead of physical time to render explicit the periodic behaviour of the universe. Following Lasenby [21], only three components are considered: dark energy λ , radiation r , and spatial curvature κ , omitting matter. The zero-order Friedmann equations are:

$$\dot{a}^2 = \frac{1}{3}\lambda a^4 - \kappa a^2 + \frac{1}{3}r, \quad \ddot{a} = \frac{2}{3}\lambda a^3 - \kappa a, \quad (2)$$

where $\hbar = c = 8\pi G = 1$ and dots denote $d/d\eta$.

The solution to eq. (2) can be expressed as a Jacobi elliptic function [28], which exhibits periodic global structure in conformal time. The universe begins at the Big Bang $a = 0$, evolves to the end of the universe at $a = \infty$ at a finite conformal time η_∞ termed the future conformal boundary (FCB), and then returns from the future boundary back to a future Big Bang, starting the cycle over again. However, this cyclic behaviour does not by default hold for perturbations. To maintain the behaviour, the perturbations should be symmetric not just at the Bang, but also at the FCB. The symmetry constrains the allowed modes of perturbations (see fig. 1).

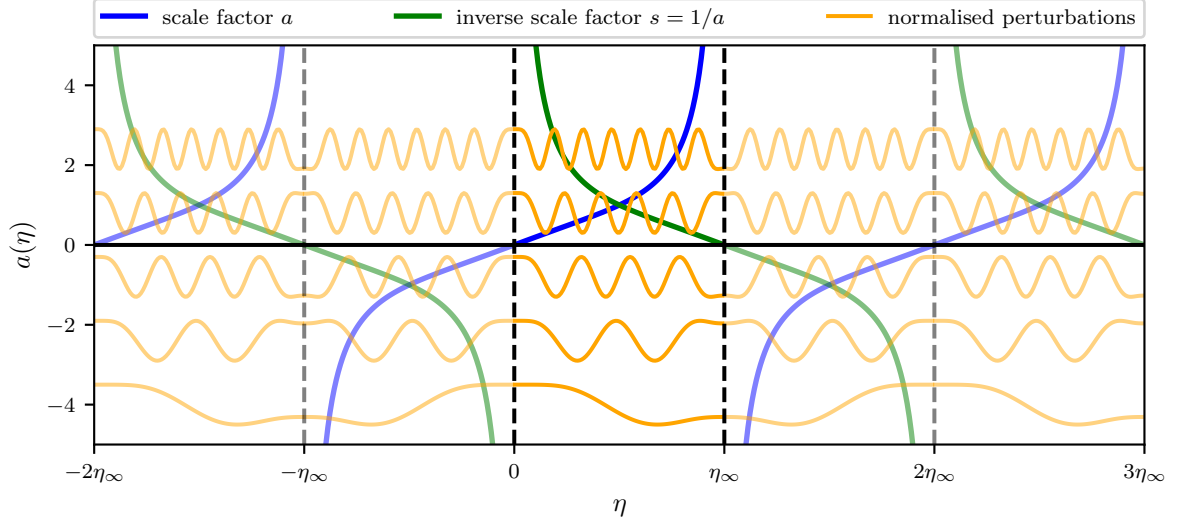


FIG. 1. Evolution of the scale factor a , its reciprocal $s \equiv 1/a$, and the first five allowed perturbations which are (anti)-symmetric about the future conformal boundary. The perturbations have been normalised to have constant oscillatory amplitude across cosmic time for clarity.

To analyse the perturbations, we write the first-order Einstein equation as:

$$\ddot{\Phi} + 4aH\dot{\Phi} + \frac{1}{3}(k^2 + 4a^2\lambda - 12\kappa)\Phi = 0. \quad (3)$$

where k is the comoving wave vector of the perturbation. Switching the independent variable from conformal time η to scale factor a , one finds

$$(\lambda a^2 - 3\kappa + \frac{r}{a^2})\Phi'' + (6\lambda a - \frac{15\kappa}{a} + \frac{4r}{a^3})\Phi' + (4\lambda + \frac{k^2 - 12\kappa}{a^2})\Phi = 0, \quad (4)$$

where a prime denotes d/da .

One degree of freedom between r and λ can be removed from eqs. (2) and (4) by fixing the scaling of a . Following [21], $r = \lambda$ is chosen, equivalent to demanding $a = 1$ when the energy density of matter radiation and dark energy are equal, after which eq. (4) becomes

$$a(a^4 - 2a^2\tilde{\kappa} + 1)\Phi'' + (6a^4 - 10a^2\tilde{\kappa} + 4)\Phi' + a(4a^2 + \tilde{k}^2 - 8\tilde{\kappa})\Phi = 0,$$

$$\text{where } \tilde{k} \equiv k/\sqrt{\lambda}, \quad \tilde{\kappa} \equiv \frac{3}{2\lambda}\kappa$$

are the dimensionless wave vector and curvature respectively. This equation reduces to eq. (23) in Lasenby et al. [21] when $\tilde{\kappa} = 0$. Note that if we replace $\Phi(a)$ with $\varphi(a) \equiv a^2\Phi(a)$, the equation remains invariant under the transformation $a \leftrightarrow 1/a$. This symmetry is important in the next section when considering perturbations passing through the FCB as $a \rightarrow \infty$.

This equation can be solved analytically, yielding two solutions. The solution can be expressed in terms of Heun

functions:

$$\Phi(a) = \text{HeunG} \left(-1 + 2\tilde{\kappa}\tilde{a}_+^2, \frac{\tilde{k}^2 - 8\tilde{\kappa}}{-4\tilde{a}_-^2}, \frac{1}{2}, 2, \frac{5}{2}, \frac{1}{2}, a^2\tilde{a}_+^2 \right) \quad (5)$$

$$\tilde{a}_\pm = \sqrt{\tilde{\kappa} \pm \sqrt{\tilde{\kappa}^2 - 1}}$$

where \tilde{a}_\pm are the extremum values of the scale factor, corresponding to $\dot{a} = 0$ in eq. (2). Since the perturbation is frozen at the beginning and starts to oscillate after entering the horizon $k = aH$, a specific linear combination of these solutions is selected to ensure that $\Phi(a = 0) = 1$ and $\Phi'(a = 0) = 0$.

CONSTRAINTS FROM THE FUTURE CONFORMAL BOUNDARY

Next, we investigate the behaviour of perturbations near the FCB. For universes with curvature $\tilde{\kappa} \equiv \frac{3}{2\lambda}\kappa > 1$, the scale factor a reaches a maximum before they collapse in a “big crunch”, which are not physically relevant, since we observe an accelerating Universe today. This implies that the Universe cannot possess excessive curvature, establishing an upper limit of $\tilde{\kappa} < 1$. In this case, the scale factor a approaches infinity at the FCB. Consequently, we should consider the reciprocal scale factor $s \equiv 1/a$. Because of the symmetry in eq. (5), the solution can be expressed as $\Phi(s) = s^4\Phi(a(s))$. As a result, the solution would continue oscillating through the FCB until the future Big Bang. Lasenby et al. [21] found that most perturbations diverge before reaching the future Big Bang. Only those solutions that are finite and symmetric or an-

tisymmetric at the FCB remain non-singular (see fig. 1). This symmetry requirement means that only a discrete spectrum of wavevectors are allowed.

To calculate the discrete spectrum of wave vectors, we re-write the perturbation solution into an integral form eq. (6), where the sine component captures the solution's oscillatory nature:

$$\Phi(a) = \frac{3\sqrt{a^4 + (\tilde{k}^2 - 2\tilde{\kappa})a^2 + 1}}{a^3\tilde{k}\sqrt{(\tilde{k}^2 - 2\tilde{\kappa})^2 - 4}} \sin\theta(\tilde{k}, \tilde{\kappa}, a),$$

$$\theta = \int_0^a \frac{\tilde{k}a'^2\sqrt{(\tilde{k}^2 - 2\tilde{\kappa})^2 - 4}}{\sqrt{1 + a'^4 - 2\tilde{\kappa}a'^2(a'^4 + (\tilde{k}^2 - 2\tilde{\kappa})a'^2 + 1)}} da', \quad (6)$$

The integration in $\theta(a)$ can be expressed as an incomplete elliptic integral of the third kind Π :

$$\theta(a) = \tilde{k}\tilde{a}_- \times \left[\Pi\left(\frac{a}{\tilde{a}_-}, -\frac{1}{2}\tilde{a}_-^2\tilde{K}_-^2, \tilde{a}_-^2\right) - \Pi\left(\frac{a}{\tilde{a}_+}, -\frac{1}{2}\tilde{a}_-^2\tilde{K}_+^2, \tilde{a}_-^2\right) \right], \quad (7)$$

$$\Pi(z, \nu, k) = \int_0^z \frac{1}{(1 - \nu t^2)\sqrt{1 - t^2}\sqrt{1 - k^2 t^2}} dt,$$

$$\tilde{K}_{\pm} = \sqrt{(\tilde{k}^2 - 2\tilde{\kappa}) \pm \sqrt{(\tilde{k}^2 - 2\tilde{\kappa})^2 - 4}}.$$

For the flat case ($\tilde{\kappa} = 0$), the solution eqs. (6) and (7) reduces to eq. (26-28) in Lasenby et al. [21].

If we require the solution to be (anti)symmetric at the FCB, the cycles of oscillation should be complete at the boundary. This implies that the angle θ in the sine function should equal $n\frac{\pi}{2}$ at $a \rightarrow \infty$.

$$\theta(\tilde{k}, \tilde{\kappa}, a \rightarrow \infty) \stackrel{!}{=} \frac{n\pi}{2}. \quad (8)$$

The corresponding \tilde{k} are the allowed values of discrete wave vectors. The function $\theta(\tilde{k}, a \rightarrow \infty)$ with different $\tilde{\kappa}$ is illustrated in fig. 2. While increasing the curvature value $\tilde{\kappa}$, θ becomes steeper, and the corresponding discrete wave vectors \tilde{k} have smaller spacing.

IMPLICATIONS FOR SPATIAL CURVATURE Ω_K

These observations have significant implications for closed universes, where the compactness of spatial slices necessitates integer wave vectors (Sec. 2.4 in [29]). This requirement is generally in conflict with the discrete spectrum of wave vectors arising from FCB considerations, prompting the question of whether this conflict can be resolved by matching the two discrete sets.

In fig. 2 $\theta(\tilde{k})$ approaches a straight line for high- k modes due to perturbations oscillating as $\sim \cos(\tilde{k}\eta/\sqrt{3})$.

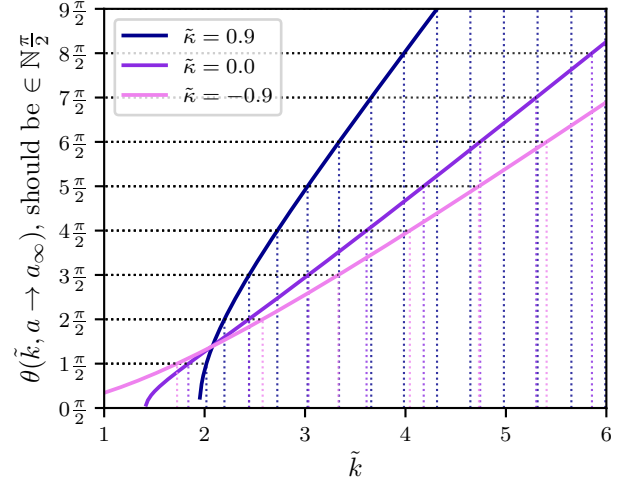


FIG. 2. Phase of perturbations $\theta(\tilde{k}, a \rightarrow \infty)$ with different curvature $\tilde{\kappa}$. Larger $\tilde{\kappa}$ results in a steeper curve. The value of θ should be $\frac{\pi}{2}n, n \in \mathbb{N}$, where the corresponding \tilde{k} are the allowed wave vectors forming a discrete spectrum.

This implies that the allowed wave vectors are equally spaced for high- k modes. By adjusting $\tilde{\kappa}$, we can align the allowed wavevectors to be integers. Note that the unit of \tilde{k} differs from that of the integer wave vectors k_{int} . The transformation between them can be expressed as: $\tilde{k} \equiv \sqrt{\frac{2\tilde{\kappa}}{3}} \sqrt{k_{\text{int}}(k_{\text{int}} + 2)} \approx \sqrt{\frac{2\tilde{\kappa}}{3}} k_{\text{int}}$ in high- k modes. Consequently, the actual gradient is given by:

$$\frac{d\theta}{dk_{\text{int}}}(\tilde{\kappa}, \tilde{k} \gg 1, a \rightarrow \infty) \approx \sqrt{\frac{2\tilde{\kappa}}{3}} \frac{\eta_{\infty}(\tilde{\kappa})}{\sqrt{3}} \stackrel{!}{=} N \pm \frac{\pi}{2}, \quad (9)$$

where to match the integer wave vectors, the gradient should be $N\frac{\pi}{2}$ or $\frac{1}{N}\frac{\pi}{2}$ (where $N \in \mathbb{N}$), corresponding to either sparser discrete θ or sparser k_{int} (see fig. 3). Since the gradient depends on $\tilde{\kappa}$, this constrains the allowed curvature values. Some allowed $\tilde{\kappa}$ values are shown in fig. 4. The gradient increases with $\tilde{\kappa}$ and approaches infinity as $\tilde{\kappa}$ approaches unity, the maximum allowed curvature value. Beyond this value, the universe would collapse, resulting in a crunch.

The predicted values of $\tilde{\kappa}$ are then transformed to the current curvature density $\Omega_K = -2\tilde{\kappa}\sqrt{\Omega_{\Lambda}\Omega_r}$ and compared with the Planck 2018 data (fig. 5). The $\tilde{\kappa}$, for unit gradient, corresponds to $\Omega_K \sim -0.009$. The maximum curvature, $\tilde{\kappa} = 1$, corresponds to $\Omega_K \sim -0.014$. Although this is closer to flat than the observational value of $-0.044^{+0.018}_{-0.015}$ [1, 24, 25], incorporating matter into our model would likely yield more precise predictions. Despite its simplicity, our model recovers curvature values consistent with observational data.

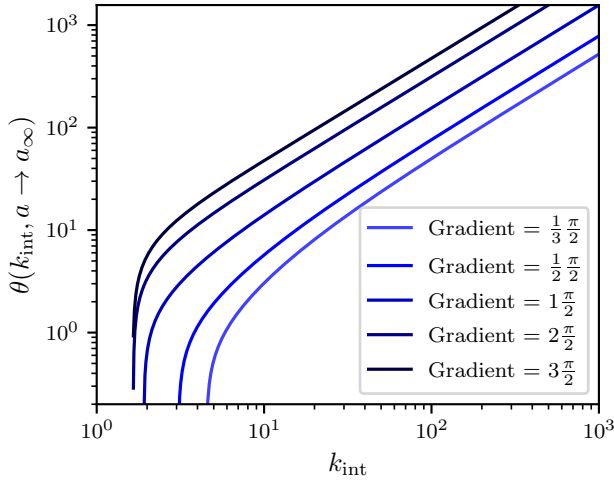


FIG. 3. Phase of perturbations θ , in log-log plot. In the high- k mode, the gradient of θ approximates a straight line, indicating that the discrete wave vectors are equally spaced. In a closed universe, these wave vectors must be integers, constraining the gradient to be $N \frac{\pi}{2}$ or $\frac{1}{N} \frac{\pi}{2}$ (where $N \in \mathbb{N}$)..

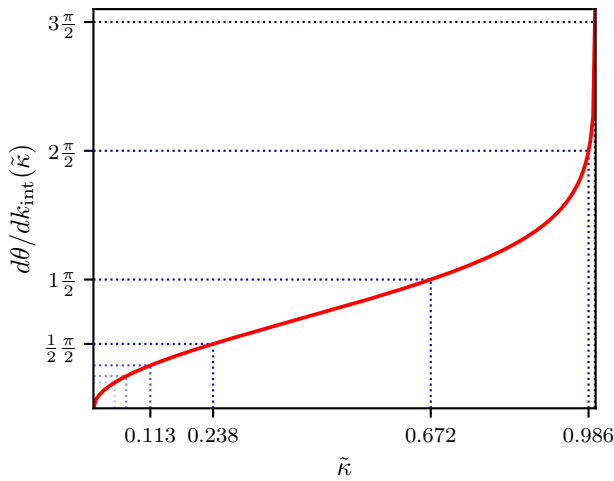


FIG. 4. The gradient of the phase of perturbations θ , as a function of $\tilde{\kappa}$. The gradient constraint determines the preferred curvature values.

CONCLUSION

In this letter, we extended the work of Lasenby et al. [21] and Boyle and Turok [28] by solving the first-order perturbation equation to derive analytic solutions. These solutions remain physical if they exhibit symmetry or antisymmetry at the FCB. This temporal boundary condition imposes a discrete spectrum on the wave vectors, which conflicts with the integer wave vectors derived from the spatial compactness of closed universes. Since both sets of discrete k values exhibit equal spacing in the high k regime, we can reconcile them by constraining the spa-

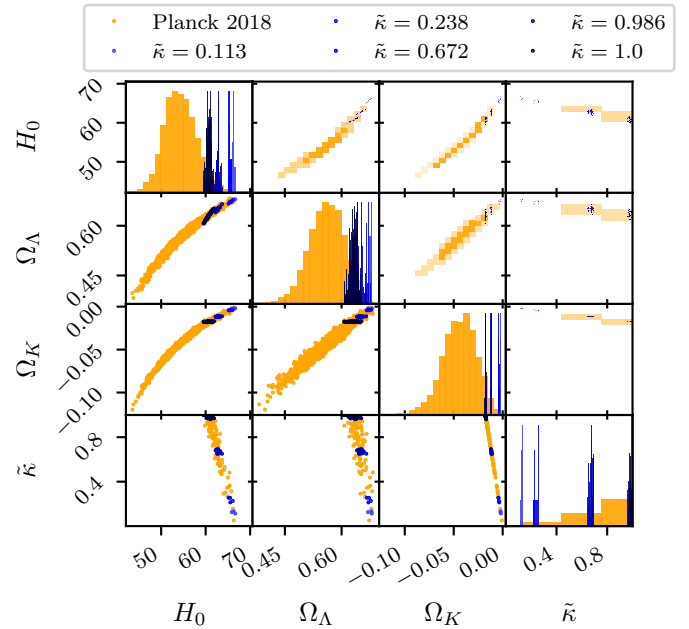


FIG. 5. Comparison of predicted curvature values with Planck 2018 data (base_omgak_plikHM.TTTEEE_lowl_lowE from [1]). Predicted curvature values from fig. 4 are shown.

tial curvature to specific values..

To refine the predictions of this allowed curvature spectrum further, we plan to solve the perturbations numerically in a more detailed universe model that includes matter and radiation anisotropy. The resulting solutions will be used to generate the CMB power spectra, which will then be compared with observational data, following the methodologies of [22] and [23].

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