

Nodal Discontinuous Galerkin Method for the Euler Equations of Fluid Dynamics

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1. Discontinuous Galerkin Scheme

We consider the system of conservation laws with sources

$$\partial_t (\sqrt{\gamma} U) + \sum_{i=1}^d \partial_{x^i} (\sqrt{\gamma} F^i(U)) = \sqrt{\gamma} (G(U) + Q(U, M)), \quad (1)$$

where we have split the source term into two contributions; G is due to gravity and the use of curvilinear spatial coordinates, while Q is due to coupling with an external field M (e.g., radiation).

The semi-discrete DG scheme is given by (letting the test function be given by ℓ_j)

$$\begin{aligned} \partial_t \int_K \sqrt{\gamma} U_{\text{DG}} \ell_j d\mathbf{x} + \sum_{i=1}^d \int_{\tilde{K}^i} (\sqrt{\gamma} \widehat{F}^i \ell_j|_{x_{\text{H}}^i} - \sqrt{\gamma} \widehat{F}^i \ell_j|_{x_{\text{L}}^i}) d\tilde{\mathbf{x}}^i \\ - \sum_{i=1}^d \int_K \sqrt{\gamma} F^i (\partial_{x^i} \ell_j) d\mathbf{x} = \int_K \sqrt{\gamma} (G + Q) \ell_j d\mathbf{x}, \end{aligned} \quad (2)$$

where \widehat{F}^i are numerical fluxes; computed by solving Riemann problems on element interfaces. In each element, the DG solution is given by the polynomial expansion

$$U_{\text{DG}}(\mathbf{x}, t) = \sum_{k=1}^N U_k(t) \ell_k(\mathbf{x}), \quad (3)$$

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where the basis $\{\ell_k\}_{k=1}^N$ is constructed from a tensor product of one-dimensional Lagrange polynomials based on a set of interpolation points $S = \{\mathbf{x}_k\}_{k=1}^N$. We may use Gaussian points, but leave room for flexibility in the specification of these points. We use a polynomial representation of the metric functions as well; e.g.,

$$\sqrt{\gamma}(\mathbf{x}) = \sum_{k=1}^N \widehat{\sqrt{\gamma}}_k \hat{\ell}_k(\mathbf{x}), \quad (4)$$

where the basis $\{\hat{\ell}_k\}_{k=1}^N$ is also constructed from a tensor product of one-dimensional Lagrange polynomials, but is based on the Lobatto points $\hat{S} = \{\hat{\mathbf{x}}_k\}_{k=1}^N$. (The use of Lobatto points will ensure continuity of the metric functions across element interfaces.) We assume that the metric functions are time-independent. Then, once the metric functions are computed analytically in the Lobatto points $\hat{\mathbf{x}}_k \in \hat{S}$, they can be interpolated to the points $\mathbf{x}_i \in S$; e.g.,

$$\sqrt{\gamma}_i = \sqrt{\gamma}(\mathbf{x}_i) = \sum_{k=1}^N \hat{\ell}_k(\mathbf{x}_i) \widehat{\sqrt{\gamma}}_k. \quad (5)$$

An example of the interpolation matrix (using Gaussian points in S), $L_{ik}^{\text{L2G}} = \hat{\ell}_k(\mathbf{x}_i) \in \mathbb{R}^{N \times N}$, is shown in Figure 1.

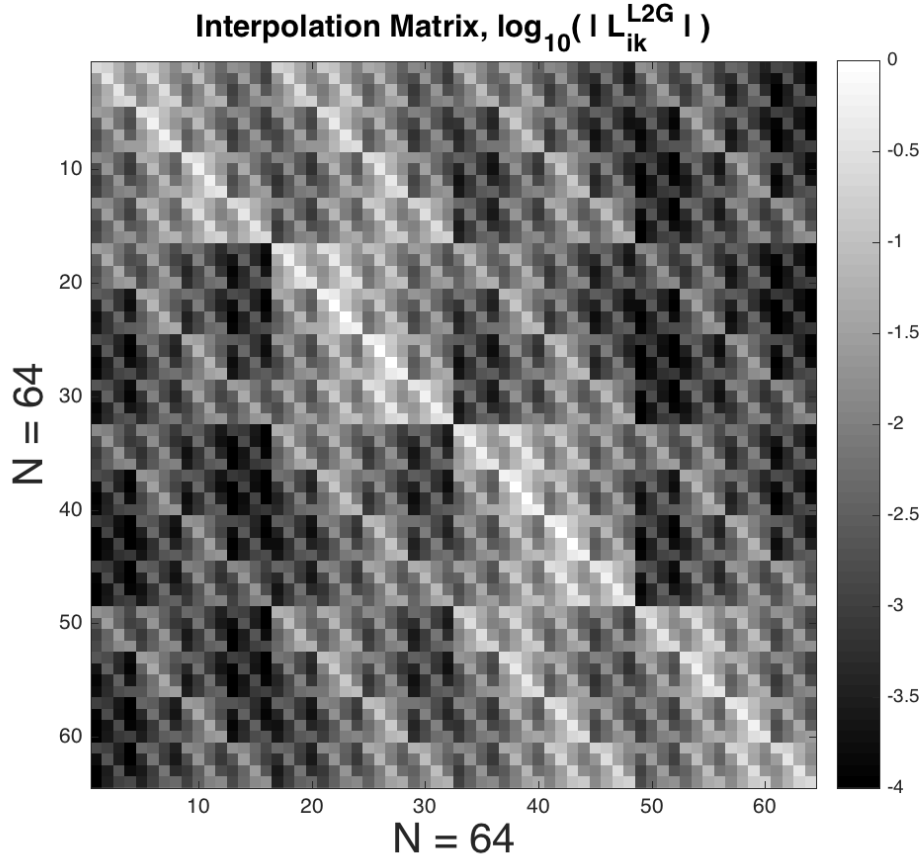


Fig. 1.—

We proceed by using the polynomial representation

$$\sqrt{\gamma}(\mathbf{x}) = \sum_{i=1}^N \sqrt{\gamma}_i \ell_i(\mathbf{x}) \quad (6)$$

for the metric functions as well, where the basis functions are constructed from points in S , and $\sqrt{\gamma}_i$ given by Eq. (5). Note that these representations are equivalent, and

$$\sqrt{\gamma}(\hat{\mathbf{x}}_q) = \sum_{i=1}^N \sqrt{\gamma}_i \ell_i(\hat{\mathbf{x}}_q) = \sum_{k=1}^N \left(\sum_{i=1}^N \hat{\ell}_k(\mathbf{x}_i) \ell_i(\hat{\mathbf{x}}_q) \right) \widehat{\sqrt{\gamma}}_k = \widehat{\sqrt{\gamma}}_q. \quad (7)$$

Inserting the polynomial representations, we can write the time-derivative term in Eq. (2) as

$$\partial_t \int_{\mathbf{K}} \sqrt{\gamma} \mathbf{U}_{\text{DG}} \ell_j d\mathbf{x} = \sum_{k=1}^N M_{jk}^{(0)} \frac{dU_k}{dt}, \quad (8)$$

where the volume mass matrix is given by

$$M_{jk}^{(0)} = \sum_{m=1}^N \int_{\mathbf{K}} \ell_m(\mathbf{x}) \ell_j(\mathbf{x}) \ell_k(\mathbf{x}) d\mathbf{x} \sqrt{\gamma}_m. \quad (9)$$

For the flux terms, we define a polynomial representation on the surface $\tilde{\mathbf{K}}^i$

$$[\sqrt{\gamma} \widehat{\mathbf{F}}^i](x^i, \tilde{\mathbf{x}}^i) = \sum_{k=1}^{\tilde{N}^i} [\sqrt{\gamma} \widehat{\mathbf{F}}^i]_k(x^i) \tilde{\ell}_k(\tilde{\mathbf{x}}^i), \quad (10)$$

where the basis functions are based on a set of interpolation points $\tilde{S}^i = \{\tilde{\mathbf{x}}_k^i\}_{k=1}^{\tilde{N}^i}$ on the element surface $\tilde{\mathbf{K}}^i$. Then, the surface integrals are evaluated as

$$\int_{\tilde{\mathbf{K}}^i} \sqrt{\gamma} \widehat{\mathbf{F}}^i \ell_j|_{x^i} d\tilde{\mathbf{x}}^i = \sum_{k=1}^{\tilde{N}^i} \widetilde{M}_{jk}^i(x^i) [\sqrt{\gamma} \widehat{\mathbf{F}}^i]_k(x^i), \quad (11)$$

where the surface mass matrices $\widetilde{M}_{jk}^i \in \mathbb{R}^{N \times \tilde{N}^i}$ are given by

$$\widetilde{M}_{jk}^i(x^i) = \int_{\tilde{\mathbf{K}}^i} \ell_j(x^i, \tilde{\mathbf{x}}^i) \tilde{\ell}_k(\tilde{\mathbf{x}}^i) d\tilde{\mathbf{x}}^i, \quad (12)$$

and plotted in Figure 2.

In order to compute numerical fluxes, the evolved quantities and metric functions are interpolated to the relevant surface. For example, to interpolate to $x_L^{i,+} = \lim_{\delta \rightarrow 0} x_L^i + \delta$ ($\delta > 0$) we evaluate

$$U_k(x_L^{i,+}) = U_{\text{DG}}(x_L^{i,+}, \tilde{\mathbf{x}}_k^i) = \sum_{l=1}^N L_{kl}^i(x_L^i) U_l, \quad (13)$$

where we have defined the interpolation matrix $L_{kl}^i(x_L^i) = \ell_l(x_L^i, \tilde{\mathbf{x}}_k^i) \in \mathbb{R}^{\tilde{N}^i \times N}$. Interpolation matrices are shown in Figure 3. The metric functions are interpolated similarly.

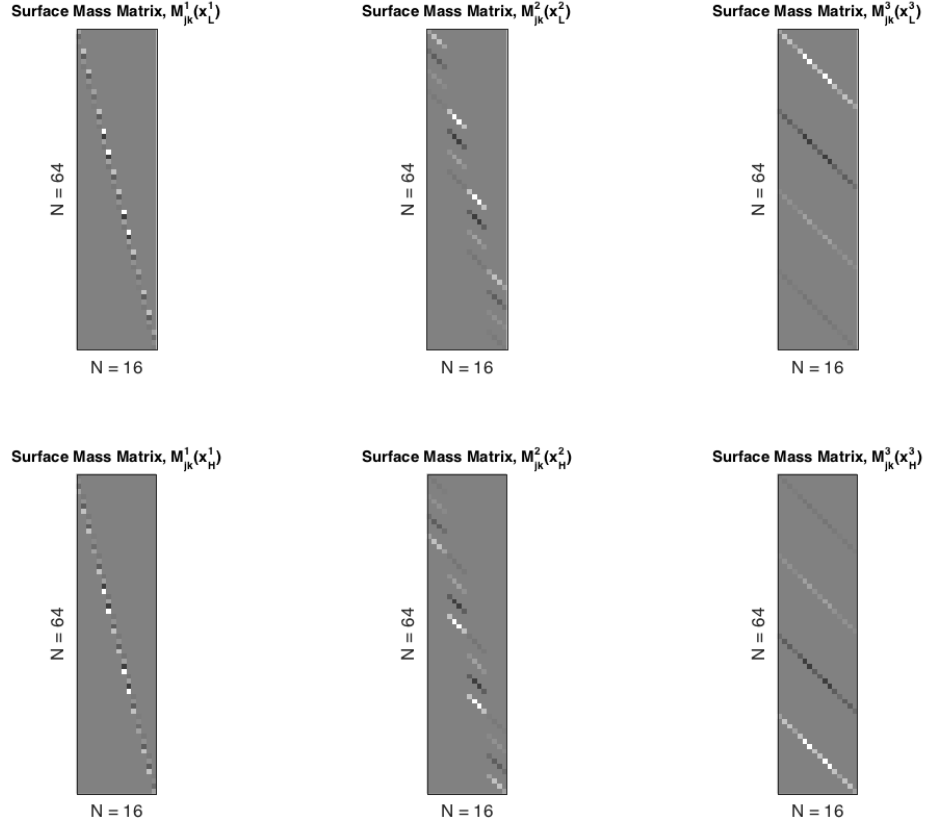


Fig. 2.—

For the volume term and source terms due to gravity and curvilinear coordinates we use the expansions

$$\sqrt{\gamma} \mathbf{F}^i = \sum_{k=1}^N [\sqrt{\gamma} \mathbf{F}^i]_k \ell_k(\mathbf{x}) \quad \text{and} \quad \sqrt{\gamma} \mathbf{G} = \sum_{k=1}^N [\sqrt{\gamma} \mathbf{G}]_k \ell_k(\mathbf{x}) \quad (14)$$

where the basis functions are based on the points S . Then,

$$\int_{\mathbf{K}} \sqrt{\gamma} \mathbf{F}^i (\partial_{x^i} \ell_j) d\mathbf{x} = \sum_{k=1}^N S_{jk}^i [\sqrt{\gamma} \mathbf{F}^i]_k \quad \text{and} \quad \int_{\mathbf{K}} \sqrt{\gamma} \mathbf{G} \ell_j d\mathbf{x} = \sum_{k=1}^N M_{jk} [\sqrt{\gamma} \mathbf{G}]_k, \quad (15)$$

where

$$S_{jk}^i = \int_{\mathbf{K}} \frac{\partial \ell_j}{\partial x^i}(\mathbf{x}) \ell_k(\mathbf{x}) d\mathbf{x} \quad \text{and} \quad M_{jk} = \int_{\mathbf{K}} \ell_j(\mathbf{x}) \ell_k(\mathbf{x}) d\mathbf{x}. \quad (16)$$

These matrices are displayed on Figure 4. Note again the sparsity patterns.

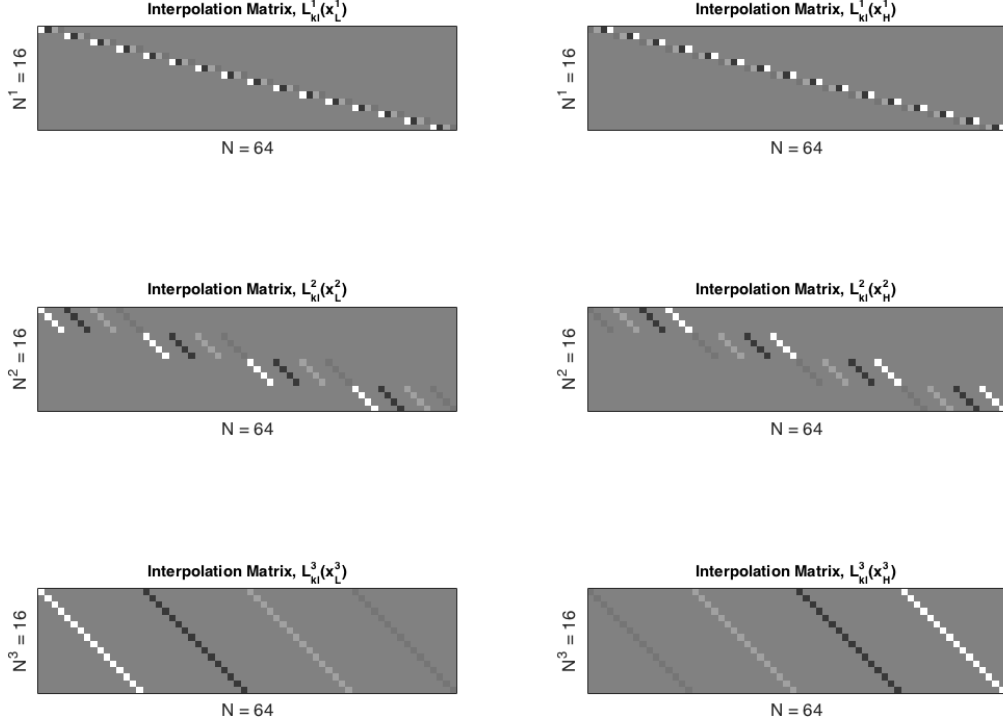


Fig. 3.—

For the source terms due to coupling with the external field, we write

$$\mathbf{Q} = \sum_{k=1}^N \mathbf{Q}_k \ell_k(\mathbf{x}), \quad (17)$$

where the basis is constructed from the points in S . Then

$$\int_{\mathbf{K}} \sqrt{\gamma} \mathbf{Q} \ell_j(\mathbf{x}) d\mathbf{x} = \sum_{k=1}^N M_{jk}^{(0)} \mathbf{Q}_k. \quad (18)$$

We can then write the DG scheme in the compact form

$$M^{(0)} \frac{d\mathbf{U}}{dt} = - \sum_{i=1}^d \left(\widetilde{M}^i(x_{\text{H}}^{i,-}) [\sqrt{\gamma} \widehat{\mathbf{F}}^i](x_{\text{H}}^i) - \widetilde{M}^i(x_{\text{L}}^{i,+}) [\sqrt{\gamma} \widehat{\mathbf{F}}^i](x_{\text{L}}^i) - S^i [\sqrt{\gamma} \mathbf{F}^i] \right) + M [\sqrt{\gamma} \mathbf{G}] + M^{(0)} \mathbf{Q}, \quad (19)$$

where $M^{(0)}, M, S^i \in \mathbb{R}^{N \times N}$ and $\widetilde{M}^i \in \mathbb{R}^{N \times \widetilde{N}^i}$. In Eq (19), $\mathbf{U} = (\mathbf{U}_1, \dots, \mathbf{U}_N)^\top$, $[\sqrt{\gamma} \mathbf{F}^i] = ([\sqrt{\gamma} \mathbf{F}^i]_1, \dots, [\sqrt{\gamma} \mathbf{F}^i]_N)^\top$,

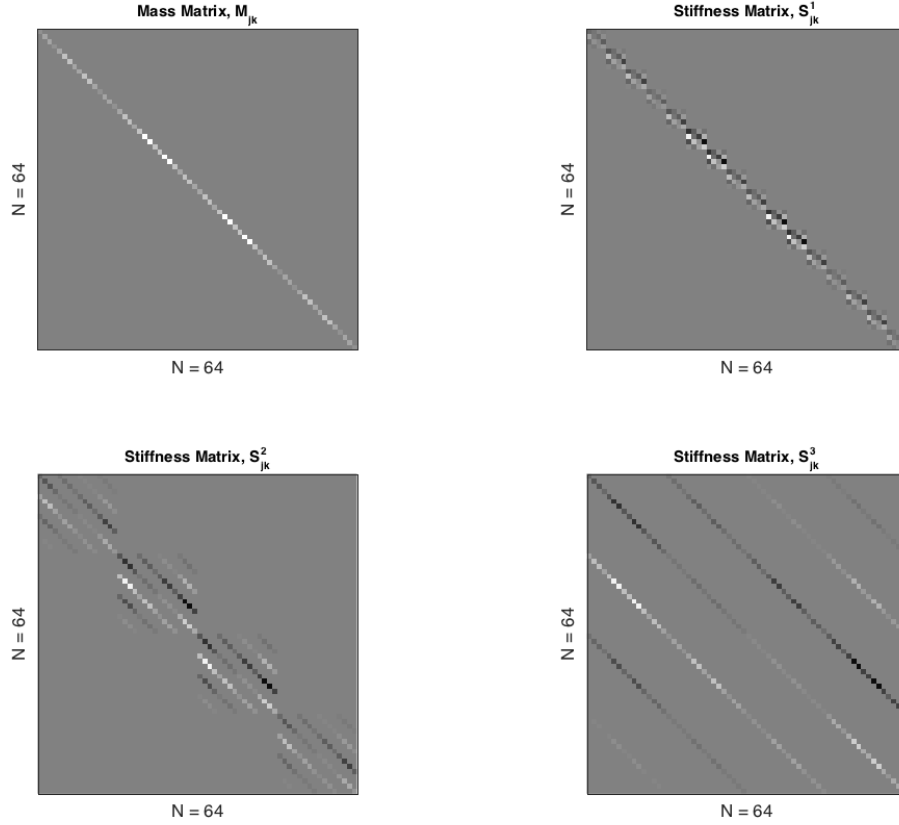


Fig. 4.—

$[\sqrt{\gamma}\mathbf{G}] = ([\sqrt{\gamma}\mathbf{G}]_1, \dots, [\sqrt{\gamma}\mathbf{G}]_N)^\top$, $\mathbf{Q} = (\mathbf{Q}_1, \dots, \mathbf{Q}_N)^\top$, and $[\sqrt{\gamma}\widehat{\mathbf{F}}]^i = ([\sqrt{\gamma}\widehat{\mathbf{F}}^i]_1, \dots, [\sqrt{\gamma}\widehat{\mathbf{F}}^i]_{\tilde{N}^i})^\top$; i.e., point values within \mathbf{K} or on the element faces $\tilde{\mathbf{K}}^i$ — not to be confused with the continuous fields in Eq. (1).

In multi-dimensional problems, the evaluation of $d_t\mathbf{U}$ in Eq. (19) can become costly for several reasons. Assume $d = 3$, and that the basis functions are constructed from polynomials of degree $p = 3$. Then, $N = (p+1)^d = 64$. If we use the same polynomial degree to construct the numerical flux on the element interfaces we have $\tilde{N}^i = (p+1)^{d-1} = 16$. The matrices involve the metric function $\sqrt{\gamma}$, which depends on spatial coordinate \mathbf{x} . We can precompute matrices of the form

$$\int_{\mathbf{K}} \ell_m(\mathbf{x}) \ell_j(\mathbf{x}) \ell_k(\mathbf{x}) d\mathbf{x}, \quad (20)$$

which contains 1024 entries, and contract with $\sqrt{\gamma}_m$ in each element to avoid storing matrices for each element. The contraction operation involves N^3 operations. Additional (similar) cost is incurred by inverting the mass matrix. We achieve significant computational savings from lumping the mass matrix; i.e., when computing the mass matrix, we use a quadrature where the quadrature points coincide with the interpolation points S . Then, the mass matrix takes the

following simple (diagonal) form

$$M_{jk}^{(0)} \approx \sum_{m=1}^N \sum_{q=1}^N w_q \ell_m(\mathbf{x}_q) \ell_j(\mathbf{x}_q) \ell_k(\mathbf{x}_q) \sqrt{\gamma}_m = \delta_{jk} w_k \sqrt{\gamma}_k, \quad (21)$$

where $\sqrt{\gamma}_k = \sum_{m=1}^N \widehat{\sqrt{\gamma}}_m \hat{\ell}_m(\mathbf{x}_k)$. In our example with $p = 3$ we are using 4 points to integrate a polynomial of degree $3 \times p = 9$, while a 4-point Gaussian quadrature only integrates polynomials of degree $\leq 2 \times N - 1 = 7$ exactly.

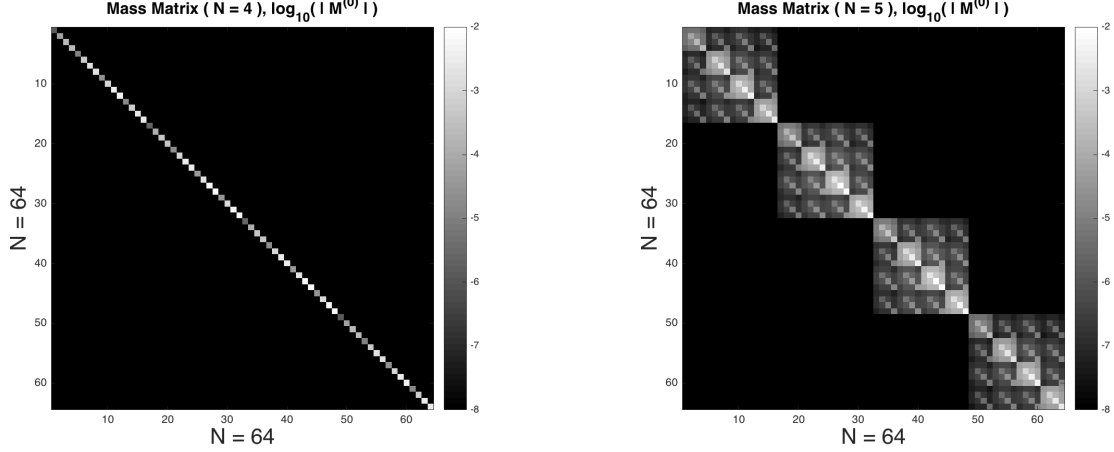


Fig. 5.—

2. Cell Averages and Conservation

The DG method evolves a high-order approximation the solution \mathbf{U} . Each element contains N degrees of freedom $\{\mathbf{U}_i\}_{i=1}^N$. Let the cell-average in element \mathbf{K} be given by

$$\mathbf{U}_{\mathbf{K}} = \frac{1}{V_{\mathbf{K}}} \int_{\mathbf{K}} \mathbf{U}_{\text{DG}} \sqrt{\gamma} d\mathbf{x}, \quad \text{where} \quad V_{\mathbf{K}} = \int_{\mathbf{K}} \sqrt{\gamma} d\mathbf{x}. \quad (22)$$

The cell volume is computed as

$$V_{\mathbf{K}} = \sum_{k=1}^N w_k \sqrt{\gamma}_k, \quad \text{where} \quad w_k = \int_{\mathbf{K}} \ell_k(\mathbf{x}) d\mathbf{x}, \quad (23)$$

while the cell average becomes

$$\mathbf{U}_{\mathbf{K}} = \frac{1}{V_{\mathbf{K}}} \sum_{j=1}^N \sum_{k=1}^N M_{jk} \sqrt{\gamma}_j \mathbf{U}_k. \quad (24)$$

Note that M_{jk} is diagonal (cf. Figure 4), which simplifies the computation of $\mathbf{U}_{\mathbf{K}}$.

Define the vector of ones $\mathbf{e}^T = (1, \dots, 1)$. Then we see that

$$\mathbf{e}^T M^{(0)} \frac{d\mathbf{U}}{dt} = V_{\mathbf{K}} \frac{d\mathbf{U}_{\mathbf{K}}}{dt}, \quad (25)$$

since $\sum_{j=1}^N \ell_j(\mathbf{x}) = 1$. The evolution equation for the cell average is then given by

$$\frac{d\mathbf{U}_K}{dt} = -\frac{1}{V_K} \left[\sum_{i=1}^d \left((\tilde{w}^i)^\top [\sqrt{\gamma} \widehat{\mathbf{F}}^i](x_{\text{H}}^i) - (\tilde{w}^i)^\top [\sqrt{\gamma} \widehat{\mathbf{F}}^i](x_{\text{L}}^i) \right) - w^\top [\sqrt{\gamma} \mathbf{G}] \right] + \mathbf{Q}_K, \quad (26)$$

where we have defined the surface integral weights

$$\tilde{w}_k^i = \int_{\tilde{K}^i} \tilde{\ell}_k(\tilde{\mathbf{x}}^i) d\tilde{\mathbf{x}}^i. \quad (27)$$

Equation (26) is familiar from the literature on finite volume methods for hyperbolic conservation laws. In the absence of sources, it expresses exact conservation of the cell average quantities \mathbf{U}_K .

3. Source Terms from Geometry Fields

The source terms from geometry fields (i.e., gravity and curvilinear coordinates) involve spatial derivatives. One could use the polynomial expansions of these fields and simply take derivatives. This approach reduces the order of accuracy in the representation of these fields, and leads to vanishing source terms in the case with polynomial degree $p = 0$. To avoid this, we define expansions of derivatives of the geometry fields. To this end, let f represent a geometry field (e.g., $f = \sqrt{\gamma}$) and define $g^i = \frac{\partial f}{\partial x^i}$. Then

$$g_i(\mathbf{x}) = \sum_{k=1}^N (g_i)_k \ell_k(\mathbf{x}), \quad (28)$$

where the basis is constructed from points $\mathbf{x}_k \in S$, and

$$\begin{aligned} (g_i)_j &= M_{jj}^{-1} \int_K g_i \ell_j d\mathbf{x} \\ &= M_{jj}^{-1} \left\{ \int_{\tilde{K}^i} [\hat{f} \ell_j|_{x_{\text{H}}^i} - \hat{f} \ell_j|_{x_{\text{L}}^i}] d\tilde{\mathbf{x}}^i - \int_K f \frac{\partial \ell_j}{\partial x^i} d\mathbf{x} \right\}. \end{aligned} \quad (29)$$

(Note that $M_{jk} = M_{jj} \delta_{jk}$.) By construction, the geometry fields are continuous across element interfaces. (In the $p = 0$ case, the geometry fields on the faces are computed by averaging values from elements abutting the face.) Introducing the expansions

$$\hat{f}(x^i, \tilde{\mathbf{x}}^i) = \sum_{k=1}^{\tilde{N}^i} \hat{f}_k(x^i) \tilde{\ell}_k(\tilde{\mathbf{x}}^i) \quad \text{and} \quad f(\mathbf{x}) = \sum_{k=1}^N f_k \ell_k(\mathbf{x}) \quad (30)$$

on the faces and the interior of K , we have

$$(g_i)_j = M_{jj}^{-1} \left\{ \sum_{k=1}^{\tilde{N}^i} \left(\widetilde{M}_{jk}^i(x_{\text{H}}^{i,-}) \hat{f}_k(x_{\text{H}}^i) - \widetilde{M}_{jk}^i(x_{\text{L}}^{i,+}) \hat{f}_k(x_{\text{L}}^i) \right) - \sum_{k=1}^N S_{jk}^i f_k \right\}. \quad (31)$$

Writing this in matrix form, letting $g_i = ((g_i)_1, \dots, (g_i)_N)^\top$, $f = (f_1, \dots, f_N)^\top$, and $\hat{f} = (\hat{f}_1, \dots, \hat{f}_{\tilde{N}^i})^\top$, we have

$$g_i = M^{-1} \left\{ \widetilde{M}^i(x_{\text{H}}^{i,-}) \hat{f}(x_{\text{H}}^i) - \widetilde{M}^i(x_{\text{L}}^{i,+}) \hat{f}(x_{\text{L}}^i) - S^i f \right\}. \quad (32)$$

For a continuous field approximated by $f(\mathbf{x}) = \sum_{i=1}^N f_k \ell_k(\mathbf{x})$, these representations are equivalent in the sense that

$$M_{jj}^{-1} \int_{\mathbf{K}} \left(g_i - \frac{\partial f}{\partial x^i} \right) \ell_j = 0. \quad (33)$$

We approximate metric factors $\sqrt{\gamma}$, $\sqrt{\gamma_{11}}$, $\sqrt{\gamma_{22}}$, and $\sqrt{\gamma_{33}}$, and the gravitational potential Φ with polynomial representations that are continuous across element interfaces, as discussed for $\sqrt{\gamma}$ in Section 1.

4. Source Terms from Curvilinear Coordinates

In the absence of gravity ($\Phi = 0$), the geometry sources are purely due to curvilinear coordinates and contribute only to the right-hand side of the momentum equation. We consider the case where γ_{ij} is diagonal, and write

$$\frac{1}{2} P^{ik} \frac{\partial \gamma_{ik}}{\partial x^j} = \sum_{i=1}^d P^i \frac{1}{\sqrt{\gamma_{ii}}} \frac{\partial \sqrt{\gamma_{ii}}}{\partial x^j}. \quad (34)$$

Next we investigate the balance between the curvilinear sources and the geometry terms in the spatial divergence of the momentum equation. For a fluid associated with an isotropic and spatially homogeneous stress tensor, i.e., $P^i_k = P \delta^i_k$, the balance should be exact so as to not induce artificial flows. The cell average of the j -th component of the momentum equation is denoted $(S_j)_{\mathbf{K}}$. Under the stated conditions, the evolution equation is given by

$$\frac{d(S_j)_{\mathbf{K}}}{dt} = -\frac{1}{V_{\mathbf{K}}} \left[\left((\tilde{w}^j)^T [\sqrt{\gamma}] (x_{\mathbf{H}}^j) - (\tilde{w}^j)^T [\sqrt{\gamma}] (x_{\mathbf{L}}^j) \right) - \sum_{i=1}^d w^T \left[\frac{\sqrt{\gamma}}{\sqrt{\gamma_{ii}}} \frac{\partial \sqrt{\gamma_{ii}}}{\partial x^j} \right] \right], \quad (35)$$

where, without loss of generality, we have set $P = 1$. In Cartesian coordinates, the balance is trivial. Next, we consider the balance for the case of spherical polar and cylindrical coordinates.

4.1. Spherical Polar Coordinates

In spherical polar coordinates, we have $\sqrt{\gamma_{11}} = 1$, $\sqrt{\gamma_{22}} = r$, $\sqrt{\gamma_{33}} = r \sin \theta$, and $\sqrt{\gamma} = r^2 \sin \theta$.

For $j = 1$ (the radial component of the momentum equation), we have

$$\sum_{i=1}^d w^T \left[\frac{\sqrt{\gamma}}{\sqrt{\gamma_{ii}}} \frac{\partial \sqrt{\gamma_{ii}}}{\partial x^j} \right] = w^T [2r \sin \theta] = (\tilde{w}^1)^T [r_{\mathbf{H}}^2 \sin \theta] - (\tilde{w}^1)^T [r_{\mathbf{L}}^2 \sin \theta] = (\tilde{w}^1)^T [\sqrt{\gamma}] (r_{\mathbf{H}}) - (\tilde{w}^1)^T [\sqrt{\gamma}] (r_{\mathbf{L}}). \quad (36)$$

Hence the balance is exact for a quadrature that integrates polynomials of degree 1 in the radial direction.

For $j = 2$ (the latitudinal component of the momentum equation), we have

$$\sum_{i=1}^d w^T \left[\frac{\sqrt{\gamma}}{\sqrt{\gamma_{ii}}} \frac{\partial \sqrt{\gamma_{ii}}}{\partial x^j} \right] = w^T \left[\sqrt{\gamma_{22}} \frac{\partial \sqrt{\gamma_{33}}}{\partial x^2} \right]. \quad (37)$$

Since $\sqrt{\gamma_{22}}$ and $\sqrt{\gamma_{33}}$ are approximated by polynomials of degree 1 in the radial direction and $\sqrt{\gamma_{33}}$ is approximated by a degree p polynomial in the latitudinal direction, the balance is exact if the quadrature exactly integrates polynomials of degree 2 in the radial direction and degree $p - 1$ in the latitudinal direction (e.g., the Gaussian quadrature with $N = 2$). By using the approximation in Eq. (31), we also achieve exact balance in the case with $p = 0$ polynomials.

The balance is trivial for $j = 3$ (the azimuthal component of the momentum equation).

4.2. Cylindrical Coordinates

In spherical polar coordinates, we have $\sqrt{\gamma_{11}} = 1$, $\sqrt{\gamma_{22}} = 1$, $\sqrt{\gamma_{33}} = R$, and $\sqrt{\gamma} = R$.

5. Source Terms from Gravitational Fields

For the source terms due to gravitational fields, we investigate conservation of energy.

5.1. Time-Independent Gravitational Fields

Multiplying the mass conservation equation with a time-independent gravitational field $\Phi = \Phi(\mathbf{x})$, we obtain

$$\partial_t(\sqrt{\gamma} \rho \Phi) + \sum_{i=1}^d \partial_i(\sqrt{\gamma}(\rho \Phi v^i)) = \sqrt{\gamma} \sum_{i=1}^d \rho v^i \partial_i \Phi. \quad (38)$$

Adding this equation to the fluid energy equation

$$\partial_t(\sqrt{\gamma} E) + \sum_{i=1}^d \partial_i(\sqrt{\gamma}(E + p) v^i) = -\sqrt{\gamma} \sum_{i=1}^d \rho v^i \partial_i \Phi, \quad (39)$$

where we ignore the sources \mathbf{Q} due to energy exchange with the external field \mathbf{M} , gives the conservation law

$$\partial_t(\sqrt{\gamma}(E + \rho \Phi)) + \sum_{i=1}^d (\sqrt{\gamma}(E + p + \rho \Phi) v^i) = 0. \quad (40)$$

In the semi-discrete DG representation, the approximation to the mass density ρ_{DG} satisfies

$$\partial_t \int_{\mathbf{K}} \sqrt{\gamma} \rho_{\text{DG}} \ell_j d\mathbf{x} + \sum_{i=1}^d \int_{\tilde{\mathbf{K}}^i} (\sqrt{\gamma} \widehat{\rho v^i} \ell_j|_{x_{\text{H}}^i} - \sqrt{\gamma} \widehat{\rho v^i} \ell_j|_{x_{\text{L}}^i}) = \sum_{i=1}^d \int_{\mathbf{K}} \sqrt{\gamma}(\rho v^i) \partial_i \ell_j. \quad (41)$$

Contracting this with Φ_j , noting that $\Phi(\mathbf{x}) = \sum_{j=1}^N \Phi_j \ell_j(\mathbf{x})$, gives

$$\partial_t(\rho \Phi)_{\mathbf{K}} + \frac{1}{V_{\mathbf{K}}} \sum_{i=1}^d \int_{\tilde{\mathbf{K}}^i} (\sqrt{\gamma} \widehat{F_g^i}|_{x_{\text{H}}^i} - \sqrt{\gamma} \widehat{F_g^i}|_{x_{\text{L}}^i}) = \frac{1}{V_{\mathbf{K}}} \sum_{i=1}^d \int_{\mathbf{K}} \sqrt{\gamma}(\rho v^i) \partial_i \Phi, \quad (42)$$

where the cell-averaged gravitational energy density is

$$(\rho \Phi)_{\mathbf{K}} = \frac{1}{V_{\mathbf{K}}} \int_{\mathbf{K}} \rho_{\text{DG}} \Phi \sqrt{\gamma} d\mathbf{x}, \quad (43)$$

and $\widehat{F_g^i} = \widehat{\rho \Phi v^i} = \widehat{\rho v^i} \Phi$.

Remark. The requirement that Φ is continuous across element interfaces is needed for conservation.

The cell-averaged energy density satisfies

$$\partial_t E_{\mathbf{K}} + \frac{1}{V_{\mathbf{K}}} \sum_{i=1}^d \int_{\tilde{\mathbf{K}}^i} (\sqrt{\gamma} \widehat{F_E^i}|_{x_{\text{H}}^i} - \sqrt{\gamma} \widehat{F_E^i}|_{x_{\text{L}}^i}) = -\frac{1}{V_{\mathbf{K}}} \sum_{i=1}^d \int_{\mathbf{K}} \sqrt{\gamma}(\rho v^i) \partial_i \Phi. \quad (44)$$

Thus, adding the equation for the cell-averaged gravitational energy density and the equation for the total fluid energy density results in an exact conservation law for the total (gravitational plus fluid) energy. (Complications may arise when limiters are invoked.)