Throughout this document, lowercase latin indices (i, j, etc.) run from 1 to 3, lowercase Greek indices (μ, ν, etc.) run from 0 to 3, and uppercase latin indices (J) run from 1 to 5.

1. Equations

We are solving the general relativistic hydrodynamics (GRH) equations (Rezzolla & Zanotti 2013):

$$\partial_t(\sqrt{\gamma}\,\boldsymbol{U}) + \partial_i\big(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}^i\big) = \sqrt{\gamma}\,\boldsymbol{S}, \tag{1}$$

where $\boldsymbol{U}$ is the vector of conserved variables, defined as

$$\boldsymbol{U} \longrightarrow U^J = \begin{pmatrix} D \\ S_j \\ E \end{pmatrix} = \begin{pmatrix} \rho\,W \\ \rho\,h\,W^2\,v_j \\ \rho\,h\,W^2 - p \end{pmatrix}, \tag{2}$$

$\boldsymbol{F}^i$ is the vector of fluxes of the conserved variables in the x^i-direction, defined as

$$\boldsymbol{F}^i \longrightarrow \left(F^i\right)^J = \begin{pmatrix} D\left(v^i - \eta^i\right) \\ S_j\left(v^i - \eta^i\right) + p\,\delta^i{}_j \\ S^i - \eta^i\,E \end{pmatrix}, \tag{3}$$

$\boldsymbol{S}$ is the vector of sources, defined as

$$\boldsymbol{S} \longrightarrow S^J = \begin{pmatrix} 0 \\ \frac{1}{2}\,\alpha\,P^{ik}\,\partial_j\gamma_{ik} + S_i\,\partial_j\beta^i - E\,\partial_j\alpha \\ \alpha\,P^{ij}\,K_{ij} - S^j\,\partial_j\alpha \end{pmatrix}, \tag{4}$$

where ρ is the mass-density of the fluid, v^i are the contravariant components of the fluid three-velocity, and p is the pressure of the fluid. The quantity h is the specific enthalpy of the fluid, defined as

$$h \equiv 1 + \frac{e+p}{\rho}, \tag{5}$$

where e is the internal energy-density of the fluid. The quantity W is the Lorentz factor of the fluid, defined as

$$W \equiv \left(1 - \overrightarrow{v}\cdot\overrightarrow{v}\right)^{-1/2}. \tag{6}$$

The quantity α is the lapse function, η^i are the ratio of the components shift-vector, β^i, to the lapse function:

$$\overrightarrow{\eta} \longrightarrow \eta^i \equiv \alpha^{-1}\,\beta^i, \tag{7}$$

and γ_{ij} are the covariant components of the spatial three-metric tensor, i.e. $\gamma^{ij}\,\gamma_{jk} = \delta^i{}_k$, where $\delta^i{}_k$ are the usual components of the Kronecker delta tensor. The P^{ij} are the components of the stress-tensor, defined as

$$\boldsymbol{P} \longrightarrow P^{ij} \equiv \rho\,h\,W^2\,v^i\,v^j + p\,\gamma^{ij}, \tag{8}$$

and the K_{ij} are the components of the extrinsic curvature, defined as

$$\text{FILL IN HERE.} \tag{9}$$

2. Derivation of Middle Wave-Speed Estimate for General Relativistic HLLC Numerical Flux

For a given Riemann problem we assume that a discontinuity breaks up into four distinct regions (see Figure 1).

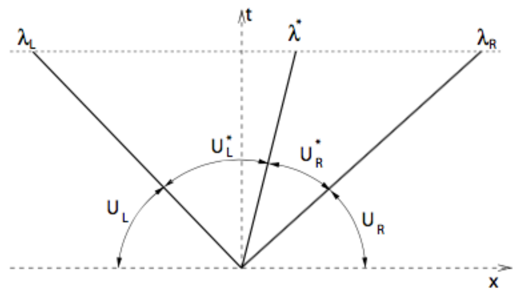

Fig. 1.— HLLC Riemann fan from Mignone & Bodo (2005).

2.1. Derivation of Rankine-Hugoniot Jump Conditions

This derivation closely follows Rezzolla & Zanotti (2013).

We start the derivation by integrating the one-dimensional version of (1) in space from a point x_L to a point x_R, that contains a shock, which we define to be at a time-dependent point $x_L < \lambda(t) < x_R$:

$$\int_{x_L}^{x_R} \partial_t(\sqrt{\gamma}\,\boldsymbol{U})\,dx + \int_{x_L}^{x_R} \partial_x(\alpha\sqrt{\gamma}\,\boldsymbol{F}^x)\,dx = \int_{x_L}^{x_R} \sqrt{\gamma}\,\boldsymbol{S}\,dx. \tag{10}$$

First, we note that in the first integral on the LHS we can pull out the partial derivative with respect to time, which converts it into a total derivative. We also perform an integration-by-parts on the second integral on the LHS, yielding

$$\frac{d}{dt}\int_{x_L}^{x_R} \sqrt{\gamma}\,\boldsymbol{U}\,dx + [\alpha\sqrt{\gamma}\,\boldsymbol{F}^x]_{x_L}^{x_R} = \int_{x_L}^{x_R} \sqrt{\gamma}\,\boldsymbol{S}\,dx \tag{11}$$

$$\implies \frac{d}{dt}\int_{x_L}^{x_R} \sqrt{\gamma}\,\boldsymbol{U}\,dx + \alpha_R\sqrt{\gamma_R}\,\boldsymbol{F}^x_R - \alpha_L\sqrt{\gamma_L}\,\boldsymbol{F}^x_L = \int_{x_L}^{x_R} \sqrt{\gamma}\,\boldsymbol{S}\,dx, \tag{12}$$

where $\alpha_L = \alpha(x_L)$, etc. The integral over the vector of conserved variables contains the discontinuity, and therefore it's derivative is not well-defined. To overcome this we split the integral into one integral from x_L to the location of the shock as approached from below, $s^-(t)$, and another integral from the location of the shock as approached from above, $s^+(t)$, to x_R:

$$\int_{x_L}^{x_R} \sqrt{\gamma}\,\boldsymbol{U}\,dx = \int_{x_L}^{s^-(t)} \sqrt{\gamma}\,\boldsymbol{U}\,dx + \int_{s^+(t)}^{x_R} \sqrt{\gamma}\,\boldsymbol{U}\,dx. \tag{13}$$

Both of these integrals are smooth. Next we use the rule for differentiation of an integral depending on a parameter

(Rezzolla & Zanotti 2013):

$$\frac{d}{dt}\int\limits_{x_1(t)}^{x_2(t)} Q\left(x,t\right)\,dx = \int\limits_{x_1(t)}^{x_2(t)} \partial_t Q\left(x,t\right)dx + Q\left(x_2\left(t\right),t\right)\,\frac{dx_2\left(t\right)}{dt} - Q\left(x_1\left(t\right),t\right)\,\frac{dx_1\left(t\right)}{dt}. \tag{14}$$

This yields, since x_L and x_R are constant,

$$\int\limits_{x_L}^{s^-(t)} \partial_t(\sqrt{\gamma}\boldsymbol{U})\,dx + \sqrt{\gamma\left(s^-\left(t\right),t\right)}\,\boldsymbol{U}\left(s^-\left(t\right),t\right)\,\frac{ds^-}{dt} \tag{15}$$

$$+\int\limits_{s^+(t)}^{x_R} \partial_t(\sqrt{\gamma}\boldsymbol{U})\,dx - \sqrt{\gamma\left(s^+\left(t\right),t\right)}\,\boldsymbol{U}\left(s^+\left(t\right),t\right)\,\frac{ds^+}{dt} \tag{16}$$

$$+\,\alpha_R\sqrt{\gamma_R}\,\boldsymbol{F}_R - \alpha_L\sqrt{\gamma_L}\,\boldsymbol{F}_L = \int\limits_{x_L}^{x_R}\sqrt{\gamma}\,\boldsymbol{S}\,dx. \tag{17}$$

We will now take the limit that $x_L \longrightarrow s^-$ and $x_R \longrightarrow s^+$. When we do this, we see that the integrals vanish, because the integrands are all smooth in the regions considered. This yields (defining $\boldsymbol{U}_L \equiv \boldsymbol{U}\left(s^-\left(t\right),t\right)$, etc.)

$$\sqrt{\gamma_L}\,\boldsymbol{U}_L\,\frac{ds_L}{dt} - \sqrt{\gamma_R}\,\boldsymbol{U}_R\,\frac{ds_R}{dt} = \alpha_L\sqrt{\gamma_L}\,\boldsymbol{F}_L - \alpha_R\sqrt{\gamma_R}\,\boldsymbol{F}_R. \tag{18}$$

Now, we note that $s_L = s_R$, and that geometry fields are continuous across discontinuities, which means that the metric determinant cancels. This yields, defining the middle wave-speed λ as

$$\lambda \equiv \frac{ds}{dt}, \tag{19}$$

$$\lambda\left(\boldsymbol{U}_L - \boldsymbol{U}_R\right) = \alpha\left(\boldsymbol{F}_L - \boldsymbol{F}_R\right). \tag{20}$$

These are the *Rankine-Hugoniot jump conditions* and they describe how the fluid variables change across a discontinuity. Next we derive an expression for an estimate of the value of the middle wave-speed, λ.

2.2. Derivation of Estimate of Middle Wave-Speed λ

Now we apply the jump conditions to the shocked and un-shocked fluid (either the left or the right state):

$$\lambda^i\left(\boldsymbol{U}^* - \boldsymbol{U}\right) = \alpha\left(\left(\boldsymbol{F}^*\right)^i - \boldsymbol{F}^i\right). \tag{21}$$

We will use the shorthand notation:

$$\boldsymbol{F}^{*i} = \left(\boldsymbol{F}^*\right)^i. \tag{22}$$

Recalling that the geometry fields are continuous, we assume that the flux in the shocked region can be written as

$$\boldsymbol{F}^{*i} \longrightarrow \left(F^{*i}\right)^J = \begin{pmatrix} D^*\left(\lambda^{*i} - \eta^i\right) \\ S^*_j\left(\lambda^{*i} - \eta^i\right) + p^*\,\delta^i{}_j \\ S^{*i} - \eta^i\,E^* \end{pmatrix}. \tag{23}$$

Applying the jump conditions to the momentum-density equation we get

$$\lambda^i\left(S^*_j - S_j\right) = \alpha\left[S^*_j\left(\lambda^{*i} - \eta^i\right) + p^*\,\delta^i{}_j - S_j\left(v^i - \eta^i\right) - p\,\delta^i{}_j\right] \tag{24}$$

$$\implies S^*_j\left[\lambda^i - \alpha\left(\lambda^{*i} - \eta^i\right)\right] = S_j\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] + \alpha\left(p^* - p\right)\delta^i{}_j. \tag{25}$$

Now we apply the jump conditions to the energy-density equation, which gives

$$\lambda^i\left(E^* - E\right) = \alpha\left[S^{*i} - \eta^i\,E^* - S^i + \eta^i\,E\right]. \tag{26}$$

Now, we note that

$$S^i = \left(E + p\right)v^i \tag{27}$$

$$S^{*i} = \left(E^* + p^*\right)\lambda^{*i}. \tag{28}$$

Substituting this into the energy-density equation gives

$$\lambda^i\left(E^* - E\right) = \alpha\left[\left(E^* + p^*\right)\lambda^{*i} - \eta^i\,E^* - \left(E + p\right)v^i + \eta^i\,E\right] \tag{29}$$

$$\implies E^*\left[\lambda^i - \alpha\left(\lambda^{*i} - \eta^i\right)\right] = E\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] + \alpha\left(p^*\,\lambda^{*i} - p\,v^i\right). \tag{30}$$

Now we substitute (27) into the momentum-density equation which yields

$$\left(E^* + p^*\right)\lambda^*_j\left[\lambda^i - \alpha\left(\lambda^{*i} - \eta^i\right)\right] = S_j\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] + \alpha\left(p^* - p\right)\delta^i{}_j \tag{31}$$

$$\implies E^*\left[\lambda^i - \alpha\left(\lambda^{*i} - \eta^i\right)\right]\lambda^*_j = -p^*\left[\lambda^i - \alpha\left(\lambda^{*i} - \eta^i\right)\right]\lambda^*_j + S_j\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] + \alpha\left(p^* - p\right)\delta^i{}_j. \tag{32}$$

Equating this with (30) gives

$$\left\{E\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] + \alpha\left(p^*\,\lambda^{*i} - p\,v^i\right)\right\}\lambda^*_j \tag{33}$$

$$= -p^*\left[\lambda^i - \alpha\left(\lambda^{*i} - \eta^i\right)\right]\lambda^*_j + S_j\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] + \alpha\left(p^* - p\right)\delta^i{}_j. \tag{34}$$

We see that the term $\alpha\,p^*\,\lambda^{*i}\,\lambda^*_j$ cancels from both sides. We now isolate p^* to get

$$p^* = \frac{E\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right]\lambda^*_j - \alpha\,p\,v^i\,\lambda^*_j - S_j\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] + \alpha\,p\,\delta^i{}_j}{\alpha\,\delta^i{}_j - \left(\lambda^i + \alpha\,\eta^i\right)\lambda^*_j} \tag{35}$$

$$= \frac{\left\{E\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] - \alpha\,p\,v^i\right\}\lambda^*_j - \left\{S_j\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] - \alpha\,p\,\delta^i{}_j\right\}}{\alpha\,\delta^i{}_j - \left(\lambda^i + \alpha\,\eta^i\right)\lambda^*_j}. \tag{36}$$

Now we note that

$$E\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] - \alpha\,p\,v^i = \lambda^i\,E - \alpha\left[\left(E + p\right)v^i - \eta^i\,E\right] = \lambda^i\,E - \alpha\left(S^i - \eta^i\,E\right) = \lambda^i\,E - \alpha\,F^i_E, \tag{37}$$

and

$$S_j\left[\lambda^i - \alpha\left(v^i - \eta^i\right)\right] - \alpha\,p\,\delta^i{}_j = \lambda^i\,S_j - \alpha\left[S_j\left(v^i - \eta^i\right) + p\,\delta^i{}_j\right] = \lambda^i\,S_j - \alpha\,F^i_{S_j}. \tag{38}$$

With this, we have

$$p^* = \frac{\left(\lambda^i\,E - \alpha\,F^i_E\right)\lambda^*_j - \left(\lambda^i\,S_j - \alpha\,F^i_{S_j}\right)}{\alpha\,\delta^i{}_j - \left(\lambda^i + \alpha\,\eta^i\right)\lambda^*_j}. \tag{39}$$

Now, since the pressure is continuous across contact discontinuities we can equate the left and right states, giving

$$\frac{\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right)\lambda^*_j - \left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right)}{\alpha\,\delta^i{}_j - \left(-\lambda^i_L + \alpha\,\eta^i\right)\lambda^*_j} = \frac{\left(\lambda^i_R\,E_R - \alpha\,F^i_{E,R}\right)\lambda^*_j - \left(\lambda^i_R\,S_{j,R} - \alpha\,F^i_{S_j,R}\right)}{\alpha\,\delta^i{}_j - \left(\lambda^i_R + \alpha\,\eta^i\right)\lambda^*_j}. \tag{40}$$

Now, cross-multiplying and focusing on the LHS:

$$\left[\alpha\,\delta^i{}_j - \left(\lambda^i_R + \alpha\,\eta^i\right)\lambda^*_j\right]\left[\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right)\lambda^*_j - \left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right)\right] \tag{41}$$

$$= \left[\alpha\,\delta^i{}_j - \left(\lambda^i_R + \beta^i\right)\lambda^*_j\right]\left[\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right)\lambda^*_j - \left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right)\right] \tag{42}$$

$$= \alpha\,\delta^i{}_j\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right)\lambda^*_j - \alpha\,\delta^i{}_j\left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right) \tag{43}$$

$$- \left(\lambda^i_R + \beta^i\right)\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right)\left(\lambda^*_j\right)^2 + \left(\lambda^i_R + \beta^i\right)\left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right)\lambda^*_j \tag{44}$$

$$= -\left(\lambda^i_R + \beta^i\right)\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right)\left(\lambda^*_j\right)^2 \tag{45}$$

$$+ \left[\alpha\,\delta^i{}_j\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right) + \left(\lambda^i_R + \beta^i\right)\left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right)\right]\lambda^*_j \tag{46}$$

$$- \alpha\,\delta^i{}_j\left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right). \tag{47}$$

Expanding out the terms gives

$$\left[\lambda^i_R\,\lambda^i_L\,E_L + \lambda^i_R\,\alpha\,F^i_{E,L} - \beta^i\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right)\right]\left(\lambda^*_j\right)^2 \tag{48}$$

$$+ \left[\alpha\,\delta^i{}_j\left(-\lambda^i_L\,E_L - \alpha\,F^i_{E,L}\right) - \lambda^i_R\,\lambda^i_L\,S_{j,L} - \lambda^i_R\,\alpha\,F^i_{S_j,L} + \beta^i\left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right)\right]\lambda^*_j \tag{49}$$

$$- \alpha\,\delta^i{}_j\left(-\lambda^i_L\,S_{j,L} - \alpha\,F^i_{S_j,L}\right). \tag{50}$$

From symmetry, by letting $L \leftrightarrow R$ and $\lambda^i_L \leftrightarrow -\lambda^i_R$, we can immediately write down the RHS as

$$\left[\lambda^i_R\,\lambda^i_L\,E_R - \lambda^i_L\,\alpha\,F^i_{E,R} - \beta^i\left(\lambda^i_R\,E_R - \alpha\,F^i_{E,R}\right)\right]\left(\lambda^*_j\right)^2 \tag{51}$$

$$+ \left[\alpha\,\delta^i{}_j\left(\lambda^i_R\,E_R - \alpha\,F^i_{E,R}\right) - \lambda^i_R\,\lambda^i_L\,S_{j,R} + \lambda^i_L\,\alpha\,F^i_{S_j,R} + \beta^i\left(\lambda^i_R\,S_{j,R} - \alpha\,F^i_{S_j,R}\right)\right]\lambda^*_j \tag{52}$$

$$- \alpha\,\delta^i{}_j\left(\lambda^i_R\,S_{j,R} - \alpha\,F^i_{S_j,R}\right). \tag{53}$$

Now we subtract the LHS from the RHS, giving

$$\left[\lambda^i_R\,\lambda^i_L\left(E_R - E_L\right) - \left(\lambda^i_R\,\alpha\,F^i_{E,L} + \lambda^i_L\,\alpha\,F^i_{E,R}\right) - \beta^i\left(\lambda^i_R\,E_R + \lambda^i_L\,E_L + \alpha\,F^i_{E,L} - \alpha\,F^i_{E,R}\right)\right]\left(\lambda^*_j\right)^2 \tag{54}$$

$$+ \Big[\alpha\,\delta^i{}_j\left(\lambda^i_R\,E_R + \lambda^i_L\,E_L + \alpha\,F^i_{E,L} - \alpha\,F^i_{E,R}\right) - \lambda^i_R\,\lambda^i_L\left(S_{j,R} - S_{j,L}\right) + \lambda^i_R\,\alpha\,F^i_{S_j,L} + \lambda^i_L\,\alpha\,F^i_{S_j,R} \tag{55}$$

$$+\beta^i\left(\lambda^i_R\,S_{j,R} + \lambda^i_L\,S_{j,L} + \alpha\,F^i_{S_j,L} - \alpha\,F^i_{S_j,R}\right)\Big] \tag{56}$$

$$- \alpha\,\delta^i{}_j\left(\lambda^i_R\,S_{j,R} + \lambda^i_L\,S_{j,L} + \alpha\,F^i_{S_j,L} - \alpha\,F^i_{S_j,R}\right). \tag{57}$$

By making use of the HLL conserved variables and fluxes,

$$\left(\lambda^i_R + \lambda^i_L\right)\boldsymbol{U}_{HLL} = \lambda^i_R\,\boldsymbol{U}_R + \lambda^i_L\,\boldsymbol{U}_L + \alpha\,\boldsymbol{F}^i_L - \alpha\,\boldsymbol{F}^i_R \tag{58}$$

$$\left(\lambda^i_R + \lambda^i_L\right)\boldsymbol{F}^i_{HLL} = \lambda^i_R\,\alpha\,\boldsymbol{F}^i_L + \lambda^i_L\,\alpha\,\boldsymbol{F}_R - \lambda^i_R\,\lambda^i_L\left(\boldsymbol{U}_R - \boldsymbol{U}_L\right), \tag{59}$$

we can simplify the quadratic equation to (assuming that $\lambda^i_L \neq 0$ **and** $\lambda^i_R \neq 0$)

$$\left(-F^i_{E,HLL} - \beta^i\,E_{HLL}\right)\left(\lambda^*_j\right)^2 + \left[\alpha\,\delta^i{}_j\,E_{HLL} + F^i_{S_j,HLL} + \beta^i\,S_{j,HLL}\right]\lambda^*_j - \alpha\,\delta^i{}_j\,S_{j,HLL} = 0. \tag{60}$$

Now we just multiply the equation by -1, which gives

$$\left(F^i_{E,HLL} + \beta^i\,E_{HLL}\right)\left(\lambda^*_j\right)^2 - \left[\alpha\,\delta^i{}_j\,E_{HLL} + F^i_{S_j,HLL} + \beta^i\,S_{j,HLL}\right]\lambda^*_j + \alpha\,\delta^i{}_j\,S_{j,HLL} = 0. \tag{61}$$

Finally, we use the metric to raise the index on the contact wave-speed, which gives

$$(\gamma_{jk})^2 \left(F^i_{E,HLL} + \beta^i\, E_{HLL}\right) \left(\lambda^{*k}\right)^2 - \gamma_{jk} \left[\alpha\, \delta^i{}_j\, E_{HLL} + F^i_{S_j,HLL} + \beta^i\, S_{j,HLL}\right] \lambda^{*k} + \alpha\, \delta^i{}_j\, S_{j,HLL} = 0. \quad (62)$$

We see that in the special relativistic, Cartesian coordinate limit of $\alpha \to 1$, $\beta^i \to 0^i$, and $\gamma_{ij} = \delta_{ij}$, we recover the result of Mignone & Bodo (2005).

REFERENCES

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