Nodal Discontinuous Galerkin Method for the Euler Equations in GR

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Contents

1. Discontinuous Galerkin Scheme

We assume a spacetime metric

$$ds^2 = -\alpha^2\,dt^2 + \gamma_{ij}\,dx^i\,dx^j\,, \tag{1}$$

and consider the system of conservation laws with sources

$$\partial_t\big(\sqrt{\gamma}\,\boldsymbol{U}\big) + \sum_{i=1}^{d}\partial_i\big(\alpha\,\sqrt{\gamma}\,\boldsymbol{F}^i(\boldsymbol{U})\big) = \alpha\,\sqrt{\gamma}\,\boldsymbol{G}(\boldsymbol{U}), \tag{2}$$

where

$$\boldsymbol{U} = \big(D,\,S_j,\,\tau\big)^{\mathrm{T}} = \big(\rho\,W,\,\rho\,h\,W^2\,v_j,\,\rho\,h\,W^2 - p - D\big)^{\mathrm{T}}, \tag{3}$$

$$\boldsymbol{F}^i(\boldsymbol{U}) = \big(D\,v^i,\,\big)^{\mathrm{T}} \tag{4}$$

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2. Bound-Preserving Methods Using First-Order DG Scheme

2.1. Cartesian Coordinates

This section closely follows Qin et al. (2016).

2.1.1. *Set of Admissible States*

We consider a one-dimensional system of conservation laws:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}\left(\boldsymbol{U}\right) = \boldsymbol{0},$$

where $\boldsymbol{U}$ is a vector of conserved variables, defined as:

$$\boldsymbol{U} \longrightarrow \begin{pmatrix} D \\ S \\ \tau \end{pmatrix} = \begin{pmatrix} \rho\, W \\ \rho\, h\, W^2\, v \\ \rho\, W\left(h\, W - 1\right) - p \end{pmatrix}, \tag{5}$$

and $\boldsymbol{F}\left(\boldsymbol{U}\right)$ are the fluxes of those conserved quantities:

$$\boldsymbol{F}\left(\boldsymbol{U}\right) \longrightarrow \begin{pmatrix} \rho\, W\, v \\ \rho\, h\, W^2\, v^2 + p \\ \rho\, h\, W^2\, v - D\, v \end{pmatrix}. \tag{6}$$

The physics leads us to define a set of admissible states, $\mathcal{G}_p$ (the subscript p stands for primitive), as:

$$\mathcal{G}_p \equiv \left\{\boldsymbol{U}\middle|\rho > 0,\, p > 0,\, v^2 < 1\right\}.$$

It is shown in Mignone & Bodo (2005) that $\mathcal{G}$ is a convex set[3] and can equivalently be written as:

$$\mathcal{G} \equiv \left\{\boldsymbol{U}\middle|D > 0,\, \tau + D > \sqrt{D^2 + S^2}\right\}. \tag{7}$$

2.1.2. *Time-Step Derivation/CFL Condition*

For the first-order DG method using forward-Euler time-stepping, we evolve the vector of conserved variables as:

$$\overline{\boldsymbol{U}}_i^{n+1} = \overline{\boldsymbol{U}}_i^{n} - \eta\left[\hat{\boldsymbol{F}}\left(\overline{\boldsymbol{U}}_i^{n}, \overline{\boldsymbol{U}}_{i+1}^{n}\right) - \hat{\boldsymbol{F}}\left(\overline{\boldsymbol{U}}_{i-1}^{n}, \overline{\boldsymbol{U}}_i^{n}\right)\right], \tag{8}$$

where

$$\overline{\boldsymbol{U}} \equiv \frac{1}{\Delta x}\int_k \boldsymbol{U}\, dx,$$

$\eta \equiv \Delta t/\Delta x$, and $\hat{\boldsymbol{F}}$ is the numerical flux. In this document we use the local Lax-Friedrichs flux, defined as:

$$\hat{\boldsymbol{F}}\left(a, b\right) = \frac{1}{2}\left[\boldsymbol{F}\left(a\right) + \boldsymbol{F}\left(b\right) - \alpha_{ab}\left(b - a\right)\right], \tag{9}$$

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[3] Convex in the sense that if $\boldsymbol{U}_1 \in \mathcal{G}$ and $\boldsymbol{U}_2 \in \mathcal{G}$, then $\alpha_1\, \boldsymbol{U}_1 + \alpha_2\, \boldsymbol{U}_2 \in \mathcal{G}$, where $\alpha_1,\ \alpha_2 \in [0, 1]$ and $\alpha_1 + \alpha_2 = 1$.

where a and b represent the state of the fluid in two different elements, α_{ab} is an estimate for the wave-speed:

$$\alpha_{ab} = \max\left[\alpha\left(a\right), \alpha\left(b\right)\right],$$

and α is the largest (in absolute value) eigenvalue of the flux-Jacobian:

$$\alpha = \left\|\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}\right\|.$$

Using this we define the following variables:

$$\alpha_{i+\frac{1}{2}} = \max\left[\alpha\left(\overline{\boldsymbol{U}}_i\right), \alpha\left(\overline{\boldsymbol{U}}_{i+1}\right)\right], \qquad \alpha_{i-\frac{1}{2}} = \max\left[\alpha\left(\overline{\boldsymbol{U}}_{i-1}\right), \alpha\left(\overline{\boldsymbol{U}}_i\right)\right]. \tag{10}$$

Substituting (9) with (10) into (8):

$$\begin{aligned}
\overline{\boldsymbol{U}}_i^{n+1} &= \overline{\boldsymbol{U}}_i^n - \frac{\eta}{2}\Big[\boldsymbol{F}\left(\overline{\boldsymbol{U}}_i^n\right) + \boldsymbol{F}\left(\overline{\boldsymbol{U}}_{i+1}^n\right) - \alpha_{i+\frac{1}{2}}\left(\overline{\boldsymbol{U}}_{i+1}^n - \overline{\boldsymbol{U}}_i^n\right) \\
&\qquad\qquad -\boldsymbol{F}\left(\overline{\boldsymbol{U}}_i^n\right) - \boldsymbol{F}\left(\overline{\boldsymbol{U}}_{i-1}^n\right) + \alpha_{i-\frac{1}{2}}\left(\overline{\boldsymbol{U}}_i^n - \overline{\boldsymbol{U}}_{i-1}^n\right)\Big] \\
&= \left[1 - \frac{\eta}{2}\left(\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}\right)\right]\overline{\boldsymbol{U}}_i^n + \frac{\eta}{2}\,\alpha_{i+\frac{1}{2}}\left[\overline{\boldsymbol{U}}_{i+1}^n - \frac{1}{\alpha_{i+\frac{1}{2}}}\boldsymbol{F}\left(\overline{\boldsymbol{U}}_{i+1}^n\right)\right] \\
&\qquad\qquad + \frac{\eta}{2}\,\alpha_{i-\frac{1}{2}}\left[\overline{\boldsymbol{U}}_{i-1}^n + \frac{1}{\alpha_{i-\frac{1}{2}}}\boldsymbol{F}\left(\overline{\boldsymbol{U}}_{i-1}^n\right)\right] \\
&= \left[1 - \frac{\eta}{2}\left(\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}\right)\right]\overline{\boldsymbol{U}}_i^n + \frac{\eta}{2}\,\alpha_{i+\frac{1}{2}}\,\boldsymbol{H}^-\left(\overline{\boldsymbol{U}}_{i+1}^n, \alpha_{i+\frac{1}{2}}\right) + \frac{\eta}{2}\,\alpha_{i+\frac{1}{2}}\,\boldsymbol{H}^+\left(\overline{\boldsymbol{U}}_{i-1}^n, \alpha_{i-\frac{1}{2}}\right),
\end{aligned} \tag{11}$$

where

$$\boldsymbol{H}^{\pm}\left(\overline{\boldsymbol{U}}, \alpha\right) \equiv \overline{\boldsymbol{U}} \pm \frac{1}{\alpha}\,\boldsymbol{F}\left(\overline{\boldsymbol{U}}\right). \tag{12}$$

The proof that $\boldsymbol{H}^{\pm} \in \mathcal{G}$ is given in Qin et al. (2016). Therefore, we see that with a restrictions on $\alpha_{i\pm\frac{1}{2}}$ that (11) is a convex combination. The restriction is (recalling that $\eta = \Delta t/\Delta x$):

$$1 - \frac{\eta}{2}\left(\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}\right) > 0 \implies \frac{\eta}{2}\left(\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}\right) < 1 \implies \Delta t < \frac{2\,\Delta x}{\alpha_{i+\frac{1}{2}} + \alpha_{i-\frac{1}{2}}}.$$

We want a time-step that is the same for all elements at a given time, so we tighten the restriction slightly to:

$$\Delta t < \frac{\Delta x}{\max_i\left(\alpha_{i-\frac{1}{2}}\right)}.$$

2.2. Curvilinear Coordinates

NOTE: We assume a conformally-flat, time-independent spatial three-metric.

2.2.1. *Set of Admissible States*

We again consider a one-dimensional system of conservation laws, but this time with a curvilinear metric:

$$\partial_t(\sqrt{\gamma}\,\boldsymbol{U}) + \partial_1(\sqrt{\gamma}\,\boldsymbol{F}) = \sqrt{\gamma}\,\boldsymbol{S},$$

where $\boldsymbol{U}$ is given by:

$$\boldsymbol{U} \longrightarrow \begin{pmatrix} D \\ S_1 \\ \tau \end{pmatrix} = \begin{pmatrix} \rho\,W \\ \rho\,h\,W^2\,v_1 \\ \rho\,W\,(h\,W-1)-p \end{pmatrix} = \begin{pmatrix} \rho\,W \\ \rho\,h\,W^2\,\gamma_{1j}\,v^j \\ \rho\,W\,(h\,W-1)-p \end{pmatrix},$$

$\boldsymbol{F}\,(\boldsymbol{U})$ are the fluxes of those conserved quantities:

$$\boldsymbol{F}\,(\boldsymbol{U}) \longrightarrow \begin{pmatrix} \rho\,W\,v^1 \\ \rho\,h\,W^2\,v^1\,v_1 + p\,\delta^1{}_1 \\ \rho\,h\,W^2\,v^1 - D\,v^1 \end{pmatrix} = \begin{pmatrix} \rho\,W\,v^1 \\ \rho\,h\,W^2\,\gamma_{1j}\,v^1\,v^j + p \\ \rho\,h\,W^2\,v^1 - D\,v^1 \end{pmatrix},$$

and $\boldsymbol{S}$ is a source term:

$$\boldsymbol{S} \longrightarrow \begin{pmatrix} 0 \\ \frac{1}{2}\,P^{1k}\,\partial_1\gamma_{1k} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\,P^{11}\,\partial_1\gamma_{11} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\left[S^1\,v^1 + p\,\gamma^{11}\right]\,\partial_1\gamma_{11} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\gamma^{11}\left[S_1\,v^1 + p\right]\partial_1\gamma_{11} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2}\left[S_1\,v^1 + p\right]\partial_1 \ln\gamma_{11} \\ 0 \end{pmatrix},$$

where the last equality follows because for a conformally-flat spatial three-metric, $\gamma^{11} = 1/\gamma_{11}$.

These definitions lead us to define the same set of admissible states as before, namely:

$$\mathcal{G}_p \equiv \left\{\boldsymbol{U}\middle|\rho > 0,\, p > 0,\, v^2 < 1\right\},$$

the difference being that v^2 now involves the metric:

$$v^2 = v^i\,v_i = v^1\,v_1 = \gamma_{1j}\,v^1\,v^j.$$

Before continuing, we show that the introduction of the metric doesn't affect the translation between $\mathcal{G}_p$ and $\mathcal{G}$...(SD: Need to do this)

2.2.2. *Time-Step Derivation/CFL Condition*

We start by integrating both sides over the element and dividing by the volume, V:

$$\frac{1}{V}\int_k \partial_t(\sqrt{\gamma}\,\boldsymbol{U})dx + \frac{1}{V}\int_k \partial_x(\sqrt{\gamma}\,\boldsymbol{F}\,(\boldsymbol{U}))dx = \frac{1}{V}\int_k \sqrt{\gamma}\,\boldsymbol{S}\,dx,$$

where:

$$V = \int_k dV = \int_k \sqrt{\gamma}\, dx.$$

By defining the cell-average as:

$$\overline{\boldsymbol{W}} \equiv \frac{1}{V} \int_k \boldsymbol{W}\, dV,$$

we have:

$$\frac{d\overline{\boldsymbol{U}}}{dt} + \frac{1}{V} \left(\sqrt{\gamma}\, \hat{\boldsymbol{F}} \left(\overline{\boldsymbol{U}} \right) \right) \Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \overline{\boldsymbol{S}},$$

or, using the common notation of the time step being represented as a superscript and the spatial element represented by a subscript:

$$\overline{\boldsymbol{U}}_i^{n+1} = \overline{\boldsymbol{U}}_i^n - \frac{\Delta t}{V} \left[\sqrt{\gamma}_{i+\frac{1}{2}}\, \hat{\boldsymbol{F}}_{i+\frac{1}{2}}^n - \sqrt{\gamma}_{i-\frac{1}{2}}\, \hat{\boldsymbol{F}}_{i-\frac{1}{2}}^n \right] + \Delta t\, \overline{\boldsymbol{S}}_i^n.$$

Now we define a parameter a la Zhang & Shu (2011): $\varepsilon \in (0,1)$, such that (NOTE: Zhang & Shu (2011) set $\varepsilon = 1/2$):

$$\overline{\boldsymbol{U}}_i^n = \varepsilon\, \overline{\boldsymbol{U}}_i^n + (1-\varepsilon)\, \overline{\boldsymbol{U}}_i^n.$$

We can use the first term to balance out the term in the square brackets and the second term to balance out the source term.

So, we get:

$$\begin{aligned}
\overline{\boldsymbol{U}}_i^{n+1} &= \varepsilon \left\{ \overline{\boldsymbol{U}}_i^n - \frac{\Delta t}{\varepsilon\, V} \left[\sqrt{\gamma}_{i+\frac{1}{2}}\, \hat{\boldsymbol{F}}_{i+\frac{1}{2}}^n - \sqrt{\gamma}_{i-\frac{1}{2}}\, \hat{\boldsymbol{F}}_{i-\frac{1}{2}}^n \right] \right\} + (1-\varepsilon)\, \overline{\boldsymbol{U}}_i^n + \Delta t\, \overline{\boldsymbol{S}}_i^n \\
&= \varepsilon \left\{ \overline{\boldsymbol{U}}_i^n - \eta_i \left[\sqrt{\gamma}_{i+\frac{1}{2}}\, \hat{\boldsymbol{F}} \left(\overline{\boldsymbol{U}}_{i+1}^n, \overline{\boldsymbol{U}}_i^n \right) - \sqrt{\gamma}_{i-\frac{1}{2}}\, \hat{\boldsymbol{F}} \left(\overline{\boldsymbol{U}}_i^n, \overline{\boldsymbol{U}}_{i-1}^n \right) \right] \right\} + (1-\varepsilon)\, \overline{\boldsymbol{U}}_i^n + \Delta t\, \overline{\boldsymbol{S}}_i^n, \quad \eta_i \equiv \frac{\Delta t}{\varepsilon\, V_i}.
\end{aligned}$$

We proceed by focusing on each term individually.

2.2.3. *Numerical flux term*

We start with the term in the curly brackets and again we use the Local-Lax-Friedrichs flux, (9), yielding:

$$\begin{aligned}
&\overline{\boldsymbol{U}}_i^n - \frac{\eta_i}{2} \Big\{ \sqrt{\gamma}_{i+\frac{1}{2}} \left[\boldsymbol{F} \left(\overline{\boldsymbol{U}}_{i+1}^n \right) + \boldsymbol{F} \left(\overline{\boldsymbol{U}}_i^n \right) - \alpha_{i+\frac{1}{2}} \left(\overline{\boldsymbol{U}}_{i+1}^n - \overline{\boldsymbol{U}}_i^n \right) \right] \\
&\qquad - \sqrt{\gamma}_{i-\frac{1}{2}} \left[\boldsymbol{F} \left(\overline{\boldsymbol{U}}_i^n \right) + \boldsymbol{F} \left(\overline{\boldsymbol{U}}_{i-1}^n \right) - \alpha_{i-\frac{1}{2}} \left(\overline{\boldsymbol{U}}_i^n - \overline{\boldsymbol{U}}_{i-1}^n \right) \right] \Big\} \\
= &\left(1 - \frac{1}{2}\, \eta_i\, \sqrt{\gamma}_{i+\frac{1}{2}}\, \alpha_{i+\frac{1}{2}} - \frac{1}{2}\, \eta_i\, \sqrt{\gamma}_{i-\frac{1}{2}}\, \alpha_{i-\frac{1}{2}} \right) \overline{\boldsymbol{U}}_i^n \\
&- \frac{1}{2}\, \eta_i\, \sqrt{\gamma}_{i+\frac{1}{2}}\, \boldsymbol{F} \left(\overline{\boldsymbol{U}}_i^n \right) + \frac{1}{2}\, \eta_i\, \sqrt{\gamma}_{i-\frac{1}{2}}\, \boldsymbol{F} \left(\overline{\boldsymbol{U}}_i^n \right) \\
&+ \frac{1}{2}\, \eta_i\, \sqrt{\gamma}_{i-\frac{1}{2}}\, \alpha_{i-\frac{1}{2}} \left[\overline{\boldsymbol{U}}_{i-1}^n + \frac{1}{\alpha_{i-\frac{1}{2}}} \boldsymbol{F} \left(\overline{\boldsymbol{U}}_{i-1}^n \right) \right] + \frac{1}{2}\, \eta_i\, \sqrt{\gamma}_{i+\frac{1}{2}}\, \alpha_{i+\frac{1}{2}} \left[\overline{\boldsymbol{U}}_{i+1}^n - \frac{1}{\alpha_{i+\frac{1}{2}}} \boldsymbol{F} \left(\overline{\boldsymbol{U}}_{i+1}^n \right) \right].
\end{aligned}$$

Now we add and subtract $\frac{1}{2}\,\eta_i\,\sqrt{\gamma}_{i+\frac{1}{2}}\,\alpha_{i+\frac{1}{2}}\,\overline{\boldsymbol{U}}_i^n$ and $\frac{1}{2}\,\eta_i\,\sqrt{\gamma}_{i-\frac{1}{2}}\,\alpha_{i-\frac{1}{2}}\,\overline{\boldsymbol{U}}_i^n$, yielding:

$$\begin{aligned}
&\left(1-\eta_i\,\sqrt{\gamma}_{i+\frac{1}{2}}\,\alpha_{i+\frac{1}{2}}-\eta_i\,\sqrt{\gamma}_{i-\frac{1}{2}}\,\alpha_{i-\frac{1}{2}}\right)\overline{\boldsymbol{U}}_i^n\\
&+\frac{1}{2}\,\eta_i\,\sqrt{\gamma}_{i+\frac{1}{2}}\,\alpha_{i+\frac{1}{2}}\left[\overline{\boldsymbol{U}}_i^n-\frac{1}{\alpha_{i+\frac{1}{2}}}\boldsymbol{F}\left(\overline{\boldsymbol{U}}_i^n\right)\right]+\frac{1}{2}\,\eta_i\,\sqrt{\gamma}_{i-\frac{1}{2}}\,\alpha_{i-\frac{1}{2}}\left[\overline{\boldsymbol{U}}_i^n+\frac{1}{\alpha_{i-\frac{1}{2}}}\boldsymbol{F}\left(\overline{\boldsymbol{U}}_i^n\right)\right]\\
&+\frac{1}{2}\,\eta_i\,\sqrt{\gamma}_{i-\frac{1}{2}}\,\alpha_{i-\frac{1}{2}}\left[\overline{\boldsymbol{U}}_{i-1}^n+\frac{1}{\alpha_{i-\frac{1}{2}}}\boldsymbol{F}\left(\overline{\boldsymbol{U}}_{i-1}^n\right)\right]+\frac{1}{2}\,\eta_i\,\sqrt{\gamma}_{i+\frac{1}{2}}\,\alpha_{i+\frac{1}{2}}\left[\overline{\boldsymbol{U}}_{i+1}^n-\frac{1}{\alpha_{i+\frac{1}{2}}}\boldsymbol{F}\left(\overline{\boldsymbol{U}}_{i+1}^n\right)\right].
\end{aligned}$$

All of the terms in square brackets are similar to the $\boldsymbol{H}$ quantities in Qin et al. (2016), and are therefore in $\mathcal{G}$. It can easily be seen that the sum of the coefficients is unity. The final condition is that the coefficient of $\overline{\boldsymbol{U}}_i^n > 0$, or (recalling that $\eta = \Delta t/\Delta x$):

$$\begin{aligned}
1-\eta_i\,\sqrt{\gamma}_{i+\frac{1}{2}}\,\alpha_{i+\frac{1}{2}}-\eta_i\,\sqrt{\gamma}_{i-\frac{1}{2}}\,\alpha_{i-\frac{1}{2}}>0 &\implies \eta_i\left(\sqrt{\gamma}_{i+\frac{1}{2}}\,\alpha_{i+\frac{1}{2}}+\sqrt{\gamma}_{i-\frac{1}{2}}\,\alpha_{i-\frac{1}{2}}\right)<1\\
&\implies \Delta t<\frac{\Delta x}{\sqrt{\gamma}_{i+\frac{1}{2}}\,\alpha_{i+\frac{1}{2}}+\sqrt{\gamma}_{i-\frac{1}{2}}\,\alpha_{i-\frac{1}{2}}}.
\end{aligned}$$

Again we want a time-step that is the same for all elements at a given time, so:

$$\Delta t<\frac{\Delta x}{2\max_i\left(\sqrt{\gamma}_{i+\frac{1}{2}}\,\alpha_{i+\frac{1}{2}}\right)}.$$

We see that this differs from the case of Cartesian coordinates by a factor of two, as well as a factor of the metric determinant at the boundaries of the element. (SD: Why doesn't this reduce to the Cartesian result when $\sqrt{\gamma}_{i+\frac{1}{2}}=1$?)

Next we handle the source term.

2.2.4. Source term

We have to show that $(1-\varepsilon)\,\overline{\boldsymbol{U}}_i^n+\Delta t\,\overline{\boldsymbol{S}}_i^n\in\mathcal{G}$. Following Zhang & Shu (2011), we write out explicitly the cell-averages:

$$(1-\varepsilon)\,\overline{\boldsymbol{U}}_i^n+\Delta t\,\overline{\boldsymbol{S}}_i^n=\frac{1-\varepsilon}{V}\int\left[\boldsymbol{U}_i^n+\frac{\Delta t}{1-\varepsilon}\boldsymbol{S}_i^n\right]dV.$$

Here we note that for a one-dimensional first-order DG scheme we have:

$$\int\boldsymbol{W}_i^n\left(x\right)\,dV=\Delta x\,w_1\,\boldsymbol{W}^n\left(\eta_1\right)\,\sqrt{\gamma}\left(\eta_1\right),$$

and

$$\int dV=\Delta x\,w_1\,\sqrt{\gamma}\left(\eta_1\right),$$

so,

$$\frac{1}{V}\int\boldsymbol{W}_i^n\left(x\right)\,dV=\boldsymbol{W}^n\left(\eta_1\right).$$

So, we have:

$$\frac{1-\varepsilon}{V}\int\left[\boldsymbol{U}_i^n+\frac{\Delta t}{1-\varepsilon}\boldsymbol{S}_i^n\right]dV=(1-\varepsilon)\left[\boldsymbol{U}^n\left(\eta_1\right)+\frac{\Delta t}{1-\varepsilon}\boldsymbol{S}^n\left(\eta_1\right)\right].$$

So we need to show that if $\boldsymbol{U}_i^n \in \mathcal{G}$, then (omitting the arguments η_1 and superscripts n):

$$\begin{pmatrix} D \\ S_1 + \frac{\Delta t}{2(1-\varepsilon)} \left[S_1\, v^1 + p\right] \partial_1 \ln \gamma_{11} \\ \tau \end{pmatrix} \in \mathcal{G}.$$

The first condition of $\mathcal{G}$ is immediately obvious, i.e. that $D > 0$. The second condition reduces to the requirement that the argument of the square root in the second condition of $\mathcal{G}$ be non-negative:

$$D^2 + \left\{ S_1 + \frac{\Delta t}{2\,(1-\varepsilon)} \left[S_1\, v^1 + p\right] \partial_1 \ln \gamma_{11} \right\}^2 \geq 0.$$

We define the (scalar) term in the square brackets as K, yielding (noting that for the second quantity in curly brackets we convert all the contravariant indices to covariant indices and vice-versa):

$$D^2 + \left\{ S_1 + \frac{\Delta t}{2\,(1-\varepsilon)} K\, \partial_1 \ln \gamma_{11} \right\} \times \left\{ S^1 + \frac{\Delta t}{2\,(1-\varepsilon)} K\, \partial^1 \ln \gamma^{11} \right\} \geq 0.$$

Here we note that:

$$\partial^1 \ln \gamma^{11} = \gamma^{11}\, \partial_1 \ln \gamma^{11} = \gamma^{11}\, \partial_1 \ln \frac{1}{\gamma_{11}} = -\gamma^{11}\, \partial_1 \ln \gamma_{11}.$$

Using this, we have:

$$D^2 + \left\{ S_1 + \frac{\Delta t}{2\,(1-\varepsilon)} K\, \partial_1 \ln \gamma_{11} \right\} \times \left\{ S^1 - \frac{\Delta t}{2\,(1-\varepsilon)} K\, \gamma^{11}\, \partial_1 \ln \gamma_{11} \right\} \geq 0,$$

which gives:

$$D^2 + S_1\, S^1 - \left(\frac{K\, \partial_1 \ln \gamma_{11}}{2\,(1-\varepsilon)} \right)^2 \gamma^{11}\; (\Delta t)^2 - S_1 \frac{\Delta t}{2\,(1-\varepsilon)} K\, \gamma^{11}\, \partial_1 \ln \gamma_{11} + S^1 \frac{\Delta t}{2\,(1-\varepsilon)} K\, \partial_1 \ln \gamma_{11} \geq 0.$$

Since $S^1 = \gamma^{11}\, S_1$, the last two terms cancel, yielding (recalling that $\gamma^{11} = 1/\gamma_{11}$):

$$\begin{aligned} & D^2 + S_1\, S^1 - \left(\frac{K\, \partial_1 \ln \gamma_{11}}{2\,(1-\varepsilon)} \right)^2 \gamma^{11}\, (\Delta t)^2 \geq 0 \\ \Longrightarrow\;\; & D^2 + S_1\, S^1 \geq \left(\frac{K\, \partial_1 \gamma_{11}}{2\,(1-\varepsilon)} \right)^2 \frac{1}{\gamma_{11}^3}\, (\Delta t)^2 \\ \Longrightarrow\;\; & \Delta t \leq \frac{2\,(1-\varepsilon)\, \sqrt{D^2 + S_1\, S^1}}{\left|\left[S_1\, v^1 + p\right] \partial_1 \gamma_{11}\right|} \gamma_{11}^{3/2}. \end{aligned}$$

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