3.1 Basic Operations

1) Matrix convention

Example of matrix in daily life

Sets of numerical information can often be presented as a rectangular **"array"** of numbers. For example, a football results table might look like this:

TEAM	PLAYED	WON	DRAWN	LOST
Blackburn	22	11	6	5
Burnley	22	9	6	7
Chelsea	21	7	8	6
Leicester	21	6	8	7
Stoke	21	6	6	9

If the headings are taken for granted, we write simply

$$\begin{bmatrix} 22 & 11 & 6 & 5 \\ 22 & 9 & 6 & 7 \\ 21 & 7 & 8 & 6 \\ 21 & 6 & 8 & 7 \\ 21 & 6 & 6 & 9 \end{bmatrix}$$

and this symbol is called a **"matrix"**.

2) Terminology

a. **Matrix**

- Rectangular array of numbers written within brackets

b. **Element**

- Each number in a matrix

c. **Size**

- No. of rows x no. of column

Example 1: Please give the size of the following matrices.

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 5 & 12 \\ 0 & 1 & 8 \\ -3 & 10 & 9 \\ -6 & 0 & -1 \end{bmatrix}$$

d. **Square matrix of order n**
   - Matrix with n rows and n columns

e. **Column and row matrices**
   i. Column matrix - Matrix with only 1 column
   ii. Row matrix - Matrix with only 1 row

Example 2:

$$\underset{\text{Square matrix of order 3}}{\overset{3 \times 3}{\begin{bmatrix} 0.5 & 0.2 & 1.0 \\ 0.0 & 0.3 & 0.5 \\ 0.7 & 0.0 & 0.2 \end{bmatrix}}} \qquad \underset{\text{Column matrix}}{\overset{4 \times 1}{\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}}} \qquad \underset{\text{Row matrix}}{\overset{1 \times 4}{\begin{bmatrix} 2 & \frac{1}{2} & 0 & -\frac{2}{3} \end{bmatrix}}}$$

f. **Position of an element in a matrix**
   i. Depend on the row and column of the element
   ii. Use **double subscript notation** a_{ij}

   Where i = row no. (from left to right)
   j = column no. (top down)

Example 3:

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad \begin{matrix} a_{11} = 1, & a_{12} = -4, & a_{13} = 5 \\ a_{21} = 7, & a_{22} = 0, & a_{23} = -2 \end{matrix}$$

3) **Addition and Subtraction**- Only applicable to matrices of the same size.

<u>Addition</u>

- Add the corresponding elements of 2 matrices.

Example 4:

a. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} (a+w) & (b+x) \\ (c+y) & (d+z) \end{bmatrix}$

b. $\begin{bmatrix} 2 & -3 & 0 \\ 1 & -3 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ -3 & 2 & 5 \end{bmatrix} =$

c. $\begin{bmatrix} 2 & -3 & 0 \\ 1 & -3 & 8 \end{bmatrix} + \begin{bmatrix} -1 & 7 \\ 0 & 6 \\ -2 & 8 \end{bmatrix}$

<u>Subtraction</u>

- ~~Add~~ the corresponding elements of 2 matrices.

Example 5:

a. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} (a-w) & (b-x) \\ (c-y) & (d-z) \end{bmatrix}$

b. $\begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 3 & 4 \end{bmatrix} =$

4) Multiplication

- 2 types of matrix multiplications

a. Scalar multiplication - Product of a number and a matrix

b. Matrix multiplication - Product of 2 matrices

Scalar Multiplication

- Multiply each entries by the number

Example 10:

a. $k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$, k = real number

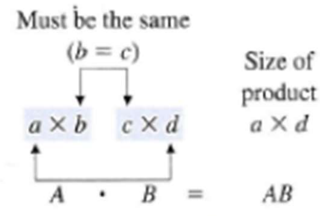
b. $-2\begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix} =$

Matrix Multiplication

Sequence and size of the 2 matrices are important.

a. Matrix A x B is defined only if the **inner pair matched**.

b. **Outer pair** is the size of the product, matrix AB.



Element of AB, $\mathbf{ab_{ij}}$

a. Takes info from
   i. i-th row of matrix A
   ii. j-th column of matrix B.

b. Multiply the corresponding entries from the A's i-th row and B's jth column together and then add up the resulting products.

$AB \neq BA$ because of

a. Inner pair may not match.

b. Order is important in matrix multiplication.

Example 6:

a. $\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} =$

b. $\begin{bmatrix} 1 & -1 & 2 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix} =$

3.2 Determinant of A Matrix

- The **determinant of a matrix** is a single number.
- We obtain this value by multiplying and adding its elements in a special way.
- We can use the c[illegible]nt of a matrix to solve a system of simultaneous equations.
- Determinant of matrix M is written: $|M|$ or det M.
- Determinant of a 2 x 2 matrix,

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then det($A$) = $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3.2.1 Determinant 2 x 2

For example, if we have the (square) 2 × 2 matrix:

$$\begin{pmatrix} 5 & 7 \\ 2 & -3 \end{pmatrix}$$

The determinant of this matrix is written within vertical lines as follows:

$$\begin{vmatrix} 5 & 7 \\ 2 & -3 \end{vmatrix} =$$

Ans: -29

Example 7:

Find the determinant of the following matrix:

$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. $\qquad B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

Example 8:

Given that $|M| = 3$. Find the values of x for the matrix:
$\begin{bmatrix} x-5 & 1 \\ 2-x & x+6 \end{bmatrix}$.

3.3 Inverse Matrix 2 x 2

The procedure to find the inverse matrix, M^{-1} for a 2 x 2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$$

1. Find the determinant, $|M| = ad - bc$
2. Rewrite the matrix M as $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
3. Inverse matrix is: $M^{-1} = \dfrac{1}{|M|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 9:

Given that $A = \begin{bmatrix} 3 & 6 \\ 3 & 8 \end{bmatrix}$

Find the inverse matrix, A^{-1}.

Example 10:

Find the inverse of the following matrices.

(a) $A=\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (b) $B=\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

3.3.1 Singular Matrix

A matrix is singular if its inverse does not exist.

$\Rightarrow A$ is a singular matrix if $|A| = 0$

$\Rightarrow A$ is a non-singular matrix if $|A| \neq 0$

3.4 Solving Linear Equations (2 variables) using inverse matrix

Example 11:

Use matrices to solve the following simultaneous equations:

(a) $2x + 5y = 8$

$3x - 2y = -7$

Ans : $x = -1$, $y = 2$

Example 12:

Solve the following simultaneous equations:

(b) $3x + 2y = 7$

$2x - 3y = -4$

Ans : $x = 1$, $y = 2$

Example 13:

Solve the following simultaneous equations:

(c) $x + 2y = 1$

$3x + 7y = 8$

Ans : $x = -9$, $y = 5$

3.5 Solving Linear Equations (3 variables) using Cramer's Rule

Cramer's rule:

Cramer's Rules for the Systems of Linear Equations with Three V	
Given a linear system	x-column, z-column $\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$ y-column, constant-col

Labeling each of the four matrices	**coefficient matrix**	$D = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$
	X - matrix	$D_x = \begin{bmatrix} d_1 & b_1 \\ d_2 & b_2 \\ d_3 & b_3 \end{bmatrix}$
	Y - matrix	$D_y = \begin{bmatrix} a_1 & d_1 \\ a_2 & d_2 \\ a_3 & d_3 \end{bmatrix}$
	Z - matrix	$D_z = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$

To solve for x:	$x=\dfrac{\lvert D_x\rvert}{\lvert D\rvert}=\dfrac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$
To solve for y:	$y=\dfrac{\lvert D_y\rvert}{\lvert D\rvert}=\dfrac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$
To solve for z:	$z=\dfrac{\lvert D_z\rvert}{\lvert D\rvert}=\dfrac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$

$$\begin{aligned} ax + by + cz &= k_1 \\ dx + ey + fz &= k_2 \\ gx + hy + mz &= k_3 \end{aligned}$$

1. Express in the form of $AX = B$, where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix};\quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix};\quad B = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

Example 14:

$$\begin{aligned} 2x + 1y + 1z &= 3 \\ 1x - 1y - 1z &= 0 \\ 1x + 2y + 1z &= 0 \end{aligned}$$

Example 15:

$$\begin{aligned} 2x + y + 4z &= 5 \\ 3x + 2y + 5z &= 3 \\ -y + z &= 8 \end{aligned}$$