(iv) $L\left\{(\sin 2t - \cos 2t)^2\right\} = L\left\{\sin^2 2t + \cos^2 2t - 2\cos 2t \sin 2t\right\}$

$$= L\left\{1 - \sin 4t\right\} = L\left\{1\right\} - L\left\{\sin 4t\right\}$$

$$= \frac{1}{s} - \frac{4}{s^2 + 16}$$

(v) $L\left\{\cos(\omega t + b)\right\} = L\left\{\cos \omega t \cos b - \sin \omega t \sin b\right\}$

$$= \cos b\, L\left\{\cos \omega t\right\} - \sin b\, L\left\{\sin \omega t\right\}$$

$$= \cos b \cdot \frac{s}{s^2 + \omega^2} - \sin b \cdot \frac{\omega}{s^2 + \omega^2}$$

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$\left[\textbf{Ans. :}\ \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^4+9}\right]$$

2. $e^{2t} + 4t^3 - \sin 2t \cos 3t$

$$\left[\textbf{Ans. :}\ \frac{1}{s-2} + \frac{24}{s^4} - \frac{5}{2} \cdot \frac{1}{s^2+25} + \frac{1}{2(s^2+1)}\right]$$

3. $3t^2 + e^{-t} + \sin^3 2t$

$$\left[\textbf{Ans. :}\ \frac{6}{s^3} + \frac{1}{s+1} + \frac{3}{2} \cdot \frac{1}{s^2+4} - \frac{3}{2} \cdot \frac{1}{s^2+36}\right]$$

4. $(t^2 + a)^2$

$$\left[\textbf{Ans. :}\ \frac{a^2 s^4 + 4as^2 + 24}{s^5}\right]$$

(i) $L\left\{4t^2 + \sin 3t + e^{2t}\right\} = 4L\left\{t^2\right\} + L\left\{\sin 3t\right\} + L\left\{e^{2t}\right\}$

$$= 4 \cdot \frac{2}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$$

$$= \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$$

Change of Scale

$$\text{If } L\left\{f(t)\right\} = F(s), \text{ then } L\left\{f(at)\right\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Example 1: If $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$, find $L\{f(2t)\}$.

Solution: $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$

By change of scale property,

$$L\{f(2t)\} = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right) = \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Example 2: If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$, find $L\{\sin 2\sqrt{t}\}$.

Solution: $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$

By change of scale property,

$$L\{\sin 2\sqrt{t}\} = L\{\sin\sqrt{4t}\} = \frac{1}{4}\frac{\sqrt{\pi}}{2\cdot\frac{s}{4}\sqrt{\frac{s}{4}}}e^{-\frac{1}{4\left(\frac{s}{4}\right)}} = \frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{s}}$$

1. If $L\{f(t)\} = \frac{8(s-3)}{(s^2-6s+25)^2}$, find $L\{f(2t)\}$.

$$\left[\textbf{Ans. :}\ \frac{1}{4}\frac{(s-6)}{(s^2-12s+100)^2}\right]$$

2. If $L\{f(t)\} = \frac{2}{s^3}e^{-s}$, find $L\{f(3t)\}$.

$$\left[\textbf{Ans. :}\ \frac{18}{s^3}e^{-\frac{s}{3}}\right]$$

3. If $L\{f(t)\} = \frac{s^2-s-1}{\cdots s + \;^{2}\; s -}$, find $L\{f(2t)\}$.

$$\left[\textbf{Ans. :}\ \frac{s^2-2s+4}{4(s+1)^2(s-2)}\right]$$

First Shifting Theorem

$$\text{If } L\{f(t)\} = F(s), \text{ then } L\{e^{-at}f(t)\} = F(s+a)$$

(i) $$L\left\{t^4\right\}=\frac{4!}{s^5}$$

By first shifting theorem,

$$L\left\{e^{-3t}t^4\right\}=\frac{4!}{(s+3)^5}$$

(ii) $$L\left\{(t+1)^2\right\}=L\left\{t^2+2t+1\right\}=\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}$$

By first shifting theorem,

$$L\left\{(t+1)^2e^t\right\}=\frac{2}{(s-1)^3}+\frac{2}{(s-1)^2}+\frac{1}{s-1}$$

(iii) $$L\left\{\left(1+\sqrt{t}\right)^4\right\}=L\left\{1+4\sqrt{t}+6\left(\sqrt{t}\right)^2+4\left(\sqrt{t}\right)^3+\left(\sqrt{t}\right)^4\right\}$$

$$=L\left\{1+4t^{\frac{1}{2}}+6t+4t^{\frac{3}{2}}+t^2\right\}=\frac{1}{s}+\frac{4\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}+\frac{6\Gamma(2)}{s^2}+\frac{4\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}}+\frac{\Gamma(3)}{s^3}$$

$$=\frac{1}{s}+\frac{4\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}+\frac{6}{s^2}+\frac{4\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}}+\frac{2}{s^3}=\frac{1}{s}+\frac{2\sqrt{\pi}}{s^{\frac{3}{2}}}+\frac{6}{s^2}+\frac{3\sqrt{\pi}}{s^{\frac{5}{2}}}+\frac{2}{s^3}$$

By first shifting theorem,

$$L\left\{e^t\left(1+\sqrt{t}\right)^4\right\}=\frac{1}{s-1}+\frac{2\sqrt{\pi}}{(s-1)^{\frac{3}{2}}}+\frac{6}{(s-1)^2}+\frac{3\sqrt{\pi}}{(s-1)^{\frac{5}{2}}}+\frac{2}{(s-1)^3}$$

(iv) $$L\left\{\sin^3 t\right\}=\frac{1}{4}L\left\{3\sin t-\sin 3t\right\}=\frac{3}{4(s^2+1)}-\frac{3}{4(s^2+9)}$$

By first shifting theorem,

$$L\left\{e^{4t}\sin^3 t\right\}=\frac{3}{4\left[(s-4)^2+1\right]}-\frac{3}{4\left[(s-4)^2+9\right]}$$

$$=\frac{3}{4(s^2-8s+17)}-\frac{3}{4(s^2-8s+25)}=\frac{6}{(s^2-8s+7)(s^2-8s+25)}$$

Multiplication by 't'

$$\text{If } L\{f(t)\} = F(s), \text{ then } L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

(i) $L\{\sin at\} = \dfrac{a}{s^2 + a^2}$

$$L\{t \sin at\} = -\frac{d}{ds} L\{\sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

(ii) $L\{\cos^2 t\} = L\left\{\dfrac{1+\cos 2t}{2}\right\} = \dfrac{1}{2} L\{1 + \cos 2t\} = \dfrac{1}{2}\left(\dfrac{1}{s} + \dfrac{s}{s^2+4}\right)$

$$L\{t \cos^2 t\} = -\frac{d}{ds} L\{\cos^2 t\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{(s^2+4)\cdot 1 - s \cdot 2s}{(s^2+4)^2}\right] = \frac{1}{2s^2} + \frac{s^2-4}{2(s^2+4)^2}$$

(iii) $L\{\sin^3 t\} = L\left\{\dfrac{3\sin t - \sin 3t}{4}\right\} = \dfrac{1}{4}\left(\dfrac{3}{s^2+1} - \dfrac{3}{s^2+9}\right) = \dfrac{3}{4}\left(\dfrac{1}{s^2+1} - \dfrac{1}{s^2+9}\right)$

$$L\{t \sin^3 t\} = -\frac{d}{ds} L\{\sin^3 t\} = -\frac{3}{4}\frac{d}{ds}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$= -\frac{3}{4}\left[\frac{-2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{(s^2+9)^2 - (s^2+1)^2}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4 + 18s^2 + 81 - s^4 - 2s^2 - 1}{(s^2+1)^2(s^2+9)^2}\right] = \frac{3s}{2} \cdot \frac{16(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

$$= \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2}$$

(v) $L\{\sqrt{1+\sin t}\} = L\left\{\sin\dfrac{t}{2} + \cos\dfrac{t}{2}\right\} = \dfrac{\frac{1}{2}}{s^2 + \frac{1}{4}} + \dfrac{s}{s^2 + \frac{1}{4}}$

$$= \frac{1}{2} \cdot \frac{4}{4s^2+1} + \frac{4s}{4s^2+1} = \frac{4s+2}{4s^2+1}$$

$$L\{t\sqrt{1+\sin t}\} = -\frac{d}{ds} L\{\sqrt{1+\sin t}\} = -\frac{d}{ds}\left(\frac{4s+2}{4s^2+1}\right)$$

$$= -\left[\frac{(4s^2+1)4 - (4s+2)8s}{(4s^2+1)^2}\right] = \frac{-16s^2 - 4 + 32s^2 + 16s}{(4s^2+1)^2}$$

$$= \frac{16s^2 + 16s - 4}{(4s^2+1)^2} = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

(vi) $$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{t\sin t\} = -\frac{d}{ds}L\{\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

By first shifting theorem,

$$L\{e^{3t}\,t\sin t\} = \frac{2(s-3)}{\left[(s-3)^2+1\right]^2} = \frac{2(s-3)}{(s^2-6s+10)^2}$$

(vii) $$f(t) = t\left(\frac{\sin t}{e^t}\right)^2 = t\,e^{-2t}\sin^2 t = t\,e^{-2t}\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}\,t\,e^{-2t}\left(1-\cos 2t\right)$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\{t(1-\cos 2t)\} = -\frac{d}{ds}L(1-\cos 2t) = -\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= -\left[-\frac{1}{s^2} - \frac{(s^2+4)\cdot 1 - s\cdot 2s}{(s^2+4)^2}\right] = \frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}$$

By first shifting theorem,

$$L\left\{\frac{1}{2}t\cdot e^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\left[\frac{1}{(s+2)^2} + \frac{4-(s+2)^2}{\left\{(s+2)^2+4\right\}^2}\right]$$

(viii) $$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{t^2\cos at\} = (-1)^2\frac{d^2}{ds^2}L\{\cos at\}$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{d}{ds}\left[\frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2}\right] = \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2(-2s) - (a^2-s^2)\cdot 2(s^2+a^2)(2s)}{(s^2+a^2)^4}$$

$$= \frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

(ix) $$L\{\sin 4t\} = \frac{4}{s^2+16}$$

$$L\{t^2\sin 4t\} = (-1)^2\frac{d^2}{ds^2}L\{\sin 4t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{4}{s^2+16}\right) = -\frac{d}{ds}\left[\frac{4(2s)}{(s^2+16)^2}\right] = -\frac{d}{ds}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$= -\left[\frac{(s^2+16)^2\cdot 8 - 8s\cdot 2(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$= \frac{-8s^2-128+32s^2}{(s^2+16)^3} = \frac{24s^2-128}{(s^2+16)^3} = \frac{8(3s^2-16)}{(s^2+16)^3}$$

By first shifting theorem,

$$L\left\{t^2 e^t \sin 4t\right\} = \frac{8\left[3(s-1)^2-16\right]}{\left[(s-1)^2+16\right]^3} = \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

Division by 't'

$$L\left\{f(t)\right\} = F(s), \text{ then } L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\mathrm{d}s$$

(i) $L\left\{1-e^{-t}\right\} = \frac{1}{s} - \frac{1}{s+1}$

$$L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^\infty L\left\{1-e^{-t}\right\}\mathrm{d}s = \int_s^\infty \left(\frac{1}{s}-\frac{1}{s+1}\right)\mathrm{d}s = \left|\log s - \log(s+1)\right|_s^\infty$$

$$= \left|\log\frac{s}{s+1}\right|_s^\infty = \log\left|\frac{1}{1+\frac{1}{s}}\right|_s^\infty = \log 1 - \log\left(\frac{1}{1+\frac{1}{s}}\right)$$

$$= -\log\frac{s}{s+1} = \log\frac{s+1}{s}$$

(ii) $L\left\{e^{-at}-e^{-bt}\right\} = \frac{1}{s+a} - \frac{1}{s+b}$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty L\{e^{-at}-e^{-bt}\}\mathrm{d}s = \int_s^\infty \left(\frac{1}{s+a}-\frac{1}{s+b}\right)\mathrm{d}s$$

$$= \left|\log(s+a)-\log(s+b)\right|_s^\infty = \left|\log\frac{s+a}{s+b}\right|_s^\infty = \left|\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right|_s^\infty$$

$$= \log 1 - \log\frac{1+\frac{a}{s}}{1+\frac{b}{s}} = -\log\frac{s+a}{s+b} = \log\frac{s+b}{s+a}$$

(v) $L\left\{1-\cos t\right\} = \frac{1}{s} - \frac{s}{s^2+1}$

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty L\left\{1-\cos t\right\}\mathrm{d}s = \int_s^\infty \left(\frac{1}{s}-\frac{s}{s^2+1}\right)\mathrm{d}s$$

$$= \left|\log s - \frac{1}{2}\log\left(s^2+1\right)\right|_s^\infty = -\frac{1}{2}\left|\log(s^2+1)-\log s^2\right|_s^\infty$$

$$= -\frac{1}{2}\left|\log\frac{s^2+1}{s^2}\right|_s^\infty = -\frac{1}{2}\left|\log\left(1+\frac{1}{s^2}\right)\right|_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}\log\left(1+\frac{1}{s^2}\right) = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

(vi) $L\{\cos at - \cos bt\} = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty L\{\cos at - \cos bt\}\,\mathrm{d}s = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)\mathrm{d}s$$

$$= \left|\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log\frac{s^2+a^2}{s^2+b^2}\right|_s^\infty = \frac{1}{2}\left|\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right|_s^\infty$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}} = -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

(vii) $L\{\sin t\} = \dfrac{1}{s^2+1}$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty L\{e^{-t}\sin t\}\,\mathrm{d}s = \int_s^\infty \frac{1}{(s+1)^2+1}\,\mathrm{d}s$$

$$= \left|\tan^{-1}(s+1)\right|_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

Find the Laplace transforms of the following functions:

1. $\dfrac{\sin t}{t}$

$$\left[\textbf{Ans.:} \cot^{-1} s\right]$$

2. $\dfrac{\sin^2 t}{t}$

$$\left[\textbf{Ans.:} \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)\right]$$

3. $\left(\dfrac{\sin 2t}{\sqrt{t}}\right)^2$

$$\left[\textbf{Ans.:} \frac{1}{4}\log\left(\frac{s^2+16}{s^2}\right)\right]$$

4. $\dfrac{\sin^3 t}{t}$

$$\left[\textbf{Ans.:} \frac{1}{4}\left(3\cot^{-1} s - \cot^{-1}\frac{s}{3}\right)\right]$$

5. $\dfrac{1-\cos at}{t}$

$$\left[\textbf{Ans.:} \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)\right]$$

6. $\dfrac{\sin t \sin 5t}{t}$

$$\left[\textbf{Ans.:} \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

7. $\dfrac{2\sin t \sin 2t}{t}$

$$\left[\textbf{Ans.:} \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)\right]$$

8. $\dfrac{e^{2t}\sin t}{t}$

$$\left[\textbf{Ans.:} \cot^{-1}(s-2)\right]$$

9. $\dfrac{e^{2t}\sin^3 t}{t}$

$$\left[\begin{aligned}\textbf{Ans.:} & \frac{3}{4}\cot^{-1}(s-2) \\ & -\frac{1}{4}\cot^{-1}\left(\frac{s-2}{3}\right)\end{aligned}\right]$$

10. $\dfrac{1-\cos t}{t^2}$

$$\left[\textbf{Ans.:} s\log\frac{s}{\sqrt{s^2+1}} + \cot^{-1} s\right]$$

Laplace Transform of Derivatives

If $L\{f(t)\} = F(s)$, then

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)$$

In general

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{(n-1)}(0)$$

(i) $L\{f(t)\} = F(s) = L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\,\mathrm{d}s = \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s$

$$= \left|\tan^{-1} s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

(iii) $L\{f(t)\} = F(s) = L\{e^{-5t}\sin t\} = \frac{1}{(s+5)^2+1}$

$$L\{f'(t)\} = sF(s) - f(0) = s\cdot\frac{1}{s^2+10s+26} - e^0\sin 0 = \frac{s}{s^2+10s+26}$$

Laplace Transform of Integrals

$$L\{f(t)\} = F(s), \text{ then } L\left\{\int_0^t f(t)\,\mathrm{d}t\right\} = \frac{F(s)}{s}$$

(i) $L\{e^{-2t}t^3\} = \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$

$$L\left\{\int_0^t e^{-2t}t^3\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-2t}t^3\} = \frac{6}{s(s+2)^4}$$

(iii) $L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$

$$L\{t\,e^{-4t}\sin 3t\} = \frac{6(s+4)}{\left[(s+4)^2+9\right]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$L\left\{\int_0^t t\,e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\,e^{-4t}\sin 3t\} = \frac{6(s+4)}{s(s^2+8s+25)^2}$$

(iv) $L\{t\sin 3t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\}$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\} = \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{\left[(s+4)^2+9\right]^2} = \frac{6}{(s^2+8s+25)^2}$$

(v) $$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\left\{e^{-4t}\sin 3t\right\} = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$$

$$L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{e^{-4t}\sin 3t\right\} = \frac{3}{s^3+8s^2+25s}$$

$$L\left\{t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^3+8s^2+25s}\right)$$
$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

(vi) $$L\left\{\sin^2 t\right\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$L\left\{t\sin^2 t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\left\{\sin^2 t\right\} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$
$$= -\frac{1}{2}\left[-\frac{1}{s^2}-\left\{\frac{s^2+4-s\cdot 2s}{(s^2+4)^2}\right\}\right] = \frac{1}{2}\left[\frac{1}{s^2}-\frac{s^2-4}{(s^2+4)^2}\right]$$

$$L\left\{t\,e^{-3t}\sin^2 t\right\} = \frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{(s+3)^2-4}{\left\{(s+3)^2+4\right\}^2}\right] = \frac{1}{2}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

$$L\left\{\int_0^t t\,e^{-3t}\sin^2 t\,\mathrm{d}t\right\} = \frac{1}{s}\,L\left\{t\,e^{-3t}\sin^2 t\right\} = \frac{1}{2s}\left[\frac{1}{(s+3)^2}-\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$$

(viii) $$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\,\mathrm{d}s = \int_s^\infty \frac{1}{s^2+1}\,\mathrm{d}s = \left|\tan^{-1}s\right|_s^\infty = \frac{\pi}{2}-\tan^{-1}s = \cot^{-1}s$$

$$L\left\{\int_0^t \frac{\sin t}{t}\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{\frac{\sin t}{t}\right\} = \frac{1}{s}\cot^{-1}s$$

$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,\mathrm{d}t\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

Find the Laplace transforms of the following functions:

1. $\int_0^t e^{-t} t^4 \, dt$

$$\left[\textbf{Ans.}: \frac{4!}{s(s+1)^5} \right]$$

2. $\int_0^t \frac{1+e^{-t}}{t} \, dt$

$$\left[\textbf{Ans.}: \frac{1}{s} \log[s(s+1)] \right]$$

3. $\int_0^t \frac{e^t \sin t}{t} \, dt$

$$\left[\textbf{Ans.}: \frac{1}{s} \cot^{-1}(s-1) \right]$$

4. $\int_0^t t e^{-2t} \sin 3t \, dt$

$$\left[\textbf{Ans.}: \frac{1}{s} \cdot \frac{3(2s+4)}{(s^2+4s+13)^2} \right]$$

5. $e^{-3t} \int_0^t t \sin 3t \, dt$

$$\left[\textbf{Ans.}: -\frac{6}{(s^2+6s+18)^2} \right]$$

6. $\int_0^t t^2 \sin t \, dt$

$$\left[\textbf{Ans.}: -\frac{2}{s} \frac{(1-3s^2)}{(s^2+1)^3} \right]$$

7. $\int_0^t t \cos^2 t \, dt$

$$\left[\textbf{Ans.}: \frac{1}{2s^3} + \frac{1}{2} \cdot \frac{s^2-4}{s(s^2+4)^2} \right]$$

8. $\int_0^t t e^{-3t} \cos^2 2t \, dt$

$$\left[\textbf{Ans.}: \frac{1}{2s(s+3)^2} + \frac{1}{2} \cdot \frac{s^2+6s-7}{s(s^2+6s+25)^2} \right]$$