Inverse Laplace Transform

If $\bar{f}(s)$ is the laplace transform of any function $f(t)$ then $f(t)$ is called the inverse laplace transform of $\bar{f}(s)$

$$L\{f(t)\} = \bar{f}(s)$$

$$f(t) = L^{-1}\{\bar{f}(s)\}$$

Note:→ The inverse laplace transfor is mainly done with the help of partial fractions.

Standard form of the inverse Laplace transformation

1. $L^{-1}\left[\frac{1}{s}\right] = 1.$

2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

4. $L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}.t^{n-1}}{(n-1)!}$ where n is positive intiger

5. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

6. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

7. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

Types 2 :→ Adjusment

[1] $L^{-1}\left[\frac{s}{s^2+4s+5}\right] = L^{-1}\left[\frac{s}{s^2+4s+4+5-4}\right]$

$= L^{-1}\left[\frac{(s+2)-2}{(s+2)^2+1^2}\right]$

$= e^{-2t}.\cos t - 2e^{-2t}\sin t$ Ans

Types 3 :→ Partial fraction :→

[1] $\frac{s}{(s+1)\ (s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$

[2] $\frac{s}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$

[3] $\frac{s}{(s+1)(s^2+2)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+2)}$

[4] $\frac{s}{(s^2+1)(s^2+2)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+2)}$

EXERSIZE

Find the inverse Laplace tranform:→

1. $\dfrac{3(s^2-2)^2}{2s^5}$

Solution:→ $\dfrac{3(s^2-2)^2}{2s^5} = \dfrac{3(s^4+4-4s^2)}{2s^5}$

$$= \frac{3}{2s} + \frac{6}{s^5} - \frac{6}{s^3}$$

$$L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2}L^{-1}\left(\frac{1}{s}\right) + 6L^{-1}\left[\frac{1}{s^5}\right] - 6L^{-1}\left[\frac{1}{s^3}\right]$$

$$= \frac{3}{2} + 6\,\frac{t^{(5-1)}}{(5-1)!} - 6\,\frac{t^{(3-1)}}{(3-1)!}$$

$$= \frac{3}{2} + \frac{t^{5-1}}{4} - 3t^{3-1}$$

$$= \frac{3}{2} + \frac{t^4}{4} - 3t^2 \text{ Ans}$$

2. $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}$

Solution:→

$$L^{-1}\left[\frac{2s-5}{4s^2+25} + \frac{4s-18}{(-s^2+9)}\right] = 2L^{-1}\left[\frac{s}{(2s)^2+5^2}\right] - 5L^{-1}\left[\frac{1}{(2s)^2+5^2}\right]$$

$$-4L^{-1}\left(\frac{s}{s^2-3^2}\right] + 18L^{-1}\left[\frac{1}{s^2-3^2}\right]$$

$$= \frac{2}{4}\cos\frac{5}{2}t - \frac{5}{4}\cdot\frac{2}{5}\sin\frac{5}{2}t$$

$$- 4\cosh 3t + 18\cdot\frac{1}{3}\sinh 3t$$

$$= \frac{2}{4}\cos\frac{5}{2}t - \frac{2}{4}\sin\frac{5}{2}t - 4\cosh 3t + 6\sinh 3t$$

Ans

3. $\frac{3s}{s^2+2s-8}$

Solution:→ $L^{-1}\left[\frac{3s}{s^2+2s-8}\right] = L^{-1}\left[\frac{3s}{s^2+2s+1-8-1}\right]$

$$= L^{-1}\left[\frac{3s}{(s+1)^2-9}\right]$$

$$= L^{-1}\left[\frac{3s+3-3}{(s+1)^2-9}\right]$$

$$= 3L^{-1}\left[\frac{s+1}{(s+1)^2-3^2}\right] - 3L^{-1}\left[\frac{1}{(s+1)^2-3^2}\right]$$

$$= 3e^{-t}\cosh 3t - \frac{3}{3}e^{-t}\sinh 3t$$

$$= e^{-t}(3\cosh 3t - \sinh 3t)$$

$$= e^{-t}\left[\frac{3(e^{3t}+e^{-3t})}{2} - \frac{(e^{3t}-e^{-3t})}{2}\right]$$

$$= \frac{e^{-t}}{2}(3e^{3t}+3e^{-3t}-e^{3t}+e^{-3t})$$

$$= e^{-t}(e^{3t}+2e^{-3t})$$

$$= e^{2t}+2e^{-4t}$$ **Ans**

4. $\frac{3s+7}{s^2-2s-3}$

Solution:- $L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = L^{-1}\left[\frac{3s+7}{s^2-2s+1-4}\right]$

$$= L^{-1}\left[\frac{3s+7}{(s-1)^2-2^2}\right]$$

$$= L^{-1}\left[\frac{3s-3+10}{(s-1)^2-2^2}\right]$$

$$= 3L^{-1}\left[\frac{(s-1)}{(s-1)^2-2^2}\right] + 10L^{-1}\left[\frac{1}{(s-1)^2-2^2}\right]$$

$$= 3e^t\cosh t + \frac{10e^t}{2}\sinh 2t$$

$$= e^t(3\cosh 2t + 5\sinh 2t)$$

$$= \frac{e^t}{2}\left[3(e^{2t}+e^{-2t}) + 5(e^{2t}-e^{-2t})\right]$$

$$= 4e^{3t} - e^{-t}$$ Ans

5. $\left[\frac{(s^2+s-2)}{s(s+3)(s-2)}\right]$

Solution:→ Let $\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$

$$s^2+s-2 = A(s+3)(s-2) + Bs(s-2) + C[s(s+3)]$$

Put $s=0$ $\quad -2 = [A(0)+3A](0-2) \Rightarrow A = \frac{1}{3}$

Put $s=2$ $\quad 4 = C(2)(5) \quad \Rightarrow C = \frac{2}{5}$

Put $s=-3$ $\quad 4 = B(-3)(-5) \quad \Rightarrow B = \frac{4}{15}$

$$\therefore L^{-1}\left[\frac{(s^2+s-2)}{s(s+3)(s-2)}\right] = L^{-1}\left[\frac{1}{3s}\right] + L^{-1}\left[\frac{4}{15(s+3)}\right] + L^{-1}\left[\frac{2}{5(s-2)}\right]$$

$$= \frac{1}{3} + \frac{4}{15}\cdot e^{-3t} + \frac{2}{5}\cdot e^{2t}$$

$$= \frac{5+4e^{-3t}+6\cdot e^{2t}}{15} \quad \text{Ans}$$

6. $\frac{s}{(s^2-1)^2}$

Solution:→ $L^{-1}\left[\frac{s}{(s^2-1)^2}\right]$

Let:→

$$\frac{s}{(s^2-1)^2} = \frac{s}{(s+1)^2(s-1)^2}$$

$$\frac{s}{(s^2-1)^2} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$$

$s = A(s+1)(s-1)^2 + B(s-1)^2 + C(s-1)(s+1)^2 + D(s+1)^2$ ①

Put $s=1$ in eq ①

$4D = 1$

or $D = \frac{1}{4}$

Put $s=-1$ in (1)

$4B = -1$

or $B = \frac{-1}{4}$

Compare the coff. of s^3

$A + C = 0$

Compare the constent terms :-

$A + D + B - C = 0$

or $A + \left(-\frac{1}{4}\right) - C + \left(\frac{1}{4}\right) = 0$

$A = C$

$\Rightarrow \quad 2A = 0$

$A = C = 0$

$\therefore \quad L^{-1} \frac{s}{(s^2-1)^2} = L^{-1}\left[-\frac{1}{4}\frac{1}{(s+1)^2} + \frac{1}{4}\frac{1}{(s-1)^2}\right]$

$= -\frac{1}{4} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{4} L^{-1}\left[\frac{1}{(s-1)^2}\right]$

$= -\frac{1}{4} e^{-t} t + \frac{1}{4} e^{t} t$

$= \frac{t}{4}\left(e^{t} - e^{-t}\right)$

$= \frac{t}{2} \sinh t$ Ans

26

7. $\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]$

Let $\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$

$$1+2s = A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2$$

Put $s=-2$, $-3 = B(-2-1)^2$ $B = -\frac{1}{3}$

Put $s=1$ $3 = D(1+2)^2$ $D = \frac{1}{3}$

$$1+2s = A[(s+2)(s^2+1-2s)] + B[s^2+1-2s] + C[(s-1)(s^2+4+4s)] + D[s^2+4+4s]$$

Compare coff. of s^3 :→

$A + C = 0$

compare the cofficent of constent→

$2A + B - 4C + 4D = 1$

$2A - \frac{1}{3} - 4C + \frac{4}{3} = 1$

$A - 2C = 0$

$A = 2C$

∴ $A = 0$ $C = 0$

∴ $\frac{1+2s}{(s+2)^2(s-1)^2} = -\frac{1}{3}\frac{1}{(s+2)^2} + \frac{1}{3}\frac{1}{(s-1)^2}$

∴ $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = L^{-1}\left[\frac{-1}{3(s+2)^2}\right] + L^{-1}\left[\frac{1}{3(s-1)^2}\right]$

$= -\frac{1}{3}\cdot e^{-2t}\cdot\frac{t}{1!} + \frac{1}{3}\cdot e^{t}\cdot\frac{t}{1!}$

$= \frac{1}{3}(e^{t} - e^{2t})\cdot t$ Ans

8. $\dfrac{s}{(s-3)(s^2+4)}$

Let $\dfrac{s}{(s-3)(s^2+4)} = \dfrac{A}{(s-3)} + \dfrac{Bs+D}{(s^2+4)}$

$$s = A(s^2+4) + Bs(s-3) + D(s-3)$$

Put $s=3 \Rightarrow 3 = 13A$ or $A = \dfrac{3}{13}$

$A + B = 0$ $\qquad$ $4A - 3D = 0$

$B = -A$ $\qquad$ $D = \dfrac{4}{3}A$

or $B = -\dfrac{3}{13}$ $\qquad$ $D = \dfrac{4}{13}$

$$\therefore L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right] = L^{-1}\left[\frac{3}{13}\frac{1}{(s-3)} - \frac{3}{13}\frac{s}{(s^2+4)} + \frac{4}{13}\frac{1}{(s^2+4)}\right]$$

$$= \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{4}{2\cdot 13}\sin 2t$$

$$= \frac{1}{13}\left[3e^{3t} - 3\cos 2t + 2\sin 2t\right]$$

Ans

9. $\left[\frac{s^3}{s^4-a^4}\right]$

Solution:→ let $\left[\frac{s^3}{s^4-a^4}\right] = \left[\frac{s^3}{(s^2+a^2)(s+a)(s-a)}\right]$

$$\left[\frac{s^3}{s^4-a^4}\right] = \frac{A}{s-a} + \frac{B}{s+a} + \frac{Cs+D}{s^2+a^2}$$

$$s^3 = A(s+a)(s^2+a^2) + B(s-a)(s^2+a^2) + (Cs+D)(s^2-a^2)$$

Put $s=-a$ $\quad -a^3 = B(-a-a)(a^2+a^2)$

$$B = \frac{1}{4}$$

Put $s=a$ $\quad a^3 = A(a+a)(a^2+a^2)$

$$A = \frac{1}{4}$$

Compare the cofficent of s^3:→

$$A+B+C=1$$

$$\frac{1}{4}+\frac{1}{4}+C=1$$

$$C=\frac{1}{2}$$

compare the constant term:→

$$Aa^3 - Ba^3 - Da^2 = 0$$

$$a^3\left(\frac{1}{4}-\frac{1}{4}\right) - Da^2 = 0$$

$$D=0$$

$$\therefore \quad L^{-1}\left[\frac{s^3}{s^4-a^4}\right] = L^{-1}\left[\frac{1}{4(s-a)}\right] + L^{-1}\left[\frac{1}{4(s+a)}\right] + L^{-1}\left[\frac{s}{2(s^2+a^2)}\right]$$

$$= \frac{1}{4}\cdot e^{at} + \frac{1}{4}\cdot e^{-at} + \frac{1}{2}\cdot \cos at$$

$$= \frac{1}{4}(e^{at}+e^{-at}) + \frac{1}{2}\cos at$$

$$= \frac{1}{2}(\cosh at + \cos at)$$ Ans

10. $\dfrac{1}{s^3-a^3}$

Solution:-

$$L^{-1}\left[\frac{1}{s^3-a^3}\right] = L^{-1}\left[\frac{1}{(s-a)(s^2+as+a^2)}\right]$$

Let $$\frac{1}{s^3-a^3} = \frac{A}{s-a} + \frac{Bs+C}{s^2+a^2+as}$$

$$1 = A(s^2+a^2+as) + (Bs+C)(s-a)$$

Put $s=a$, $$1 = A(a^2+a^2+a^2)$$

$$A = \frac{1}{3a^2}$$

Compare the coffic. of s^2:-

$$A+B=0$$

$$B = -\frac{1}{3a^2}$$

Compare the constant terms

$$Aa^2 - aC = 1$$

$$\frac{1}{3} - aC = 1$$

$$C = -\frac{2}{3a}$$

$$\therefore L^{-1}\left[\frac{1}{s^3-a^3}\right] = L^{-1}\left[\frac{1}{3a^2(s-a)}\right] + L^{-1}\left[\frac{-\frac{s}{a}-2}{3a(s^2+a^2+as)}\right]$$

$$= \frac{1}{3a^2}\cdot e^{at} - \frac{1}{3a^2}L^{-1}\left[\frac{s+2a}{\left(s+\frac{a}{2}\right)^2+\left(\sqrt{3}a/2\right)^2}\right]$$

$$= \frac{1}{3a^2}\cdot e^{at} - \frac{1}{3a^2}L^{-1}\left[\frac{s+\frac{a}{2}+\frac{3a}{2}}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right]$$

$$= \frac{1}{3a^2}\left[e^{at} - e^{-\frac{a}{2}t}\cos\frac{\sqrt{3}}{2}at + \frac{3a}{2}\times\frac{2}{\sqrt{3}a}\sin\frac{\sqrt{3}}{2}at\cdot e^{-a/2t}\right]$$

$$= \frac{1}{3a^2}\left[e^{at} - e^{-a/2t}\left(\cos\frac{\sqrt{3}}{2}at + \sqrt{3}\sin\frac{\sqrt{3}}{2}at\right)\right]$$ Ans

11:→ $\left[\frac{a(s^2-2a^2)}{s^4+4a^4}\right]$

Solution:→ $\left[\frac{a(s^2-2a^2)}{s^4+4a^4}\right] = \left[\frac{a(s^2-2a^2)}{s^4+4a^4-4a^2s^2+4a^2s^2}\right]$

$$= \left[\frac{a(s^2-2a^2)}{(s^2+2a^2)^2-4a^2s^2}\right] = \left(\frac{a(s^2-2a^2)}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right)$$

Let $\frac{a(s^2-2a^2)}{(s^2+2a^2-2as)(s^2+2a^2+2as)} = \frac{As+B}{(s^2+2a^2-2as)} + \frac{Cs+D}{(s^2+2a^2+2as)}$

$$a(s^2-2a^2) = (As+B)(s^2+2a^2+2as)+(Cs+D)(s^2+2a^2-2as)$$

compare coeff. of s^3 $\quad A+C=0$

compare coeff. of s^2 $\quad B+2aA+D-2aC=a$

compare coeff. of s $\quad 2a^2A+2aB+2a^2C-2aD=0$

$$2a^2(A+C)+2a(B-D)=0$$

$$B-D=0 \Rightarrow B=D$$

Compare constant terms:→ $\quad 2a^2B+2a^2D=-2a^3$

$$B+D=-a \text{ or } 2D=-a \Rightarrow D=-\frac{a}{2} \therefore B=\frac{-a}{2}$$

$\therefore \quad B+D+2a(A-C)=a$

or $\quad -\frac{a}{2}-\frac{a}{2}+2a(A-C)=a$

$$2a(A-C)=2a \Rightarrow A-C=1$$

$$A+C=1$$

$\therefore$ we get $\quad A=\frac{1}{2}\ ;\ C=-1/2$

$$L^{-1}\left(\frac{a(s^2-2a^2)}{s^4+4a^4}\right) = L^{-1}\left(\frac{\frac{1}{2}s-\frac{a}{2}}{s^2+2a^2-2as}\right) + L^{-1}\left(\frac{-\frac{1}{2}s-\frac{a}{2}}{s^2+2a^2+2as}\right)$$

$$= \frac{1}{2}L^{-1}\left[\frac{s-a}{(s-a)^2+a^2}\right] + \left(\frac{-1}{2}\right)L^{-1}\left[\frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}\cdot e^{at}\cdot \cos at - \frac{1}{2}e^{-at}\cos at$$

$$= \cos at \sinh at$$

Question 12:→ $\frac{s+2}{(s^2+4s+5)^2}$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = L^{-1}\left[\frac{s+2}{((s+2)^2+1)^2}\right]$$

$$= e^{-2t}\cdot\frac{t\sin t}{2}$$ Ans

13. $\left[\frac{s^2+6}{(s^2+1)(s^2+4)}\right]$

Solution:→ Let $\left[\frac{s^2+6}{(s^2+1)(s^2+4)}\right] = \frac{A}{(s^2+1)} + \frac{B}{(s^2+4)}$

$$s^2+6 = A(s^2+4) + B(s^2+1)$$

Compare coff. of s^2 — Compare constant term

$A+B = 1$ —① $\quad 4A+B=6$ —②

$$\begin{array}{r} 4A+4B=4 \\ \underline{4A+B=6} \\ 3B=-2 \end{array}$$

$B = -2/3 \quad . \quad A = \frac{5}{3}$

$$\therefore \quad L^{-1}\left[\frac{s^2+6}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[\frac{5}{3(s^2+1)}\right] + L^{-1}\left[\frac{-2}{3(s^2+4)}\right]$$

$$= \frac{5}{3}\cdot \sin t - \frac{1}{3}\cdot \sin 2t = \left[\frac{5\sin t - \sin 2t}{3}\right]_{Ans}$$

14. $\left[\frac{s+3}{(s^2+6s+13)^2}\right] = \left[\frac{s+3}{(s^2+6s+9+4)^2}\right]$

$$L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = \left[\frac{s+3}{((s+3)^2+4)^2}\right] = \frac{e^{-3t}\cdot t\sin 2t}{2\cdot 2}$$

$$= \frac{1}{4}\cdot e^{-3t}\, t\sin 2t$$

93

15. $L^{-1}\left[\frac{2s-3}{s^2+4s+13}\right]$

Solutions:- $L^{-1}\left[\frac{2s-3}{s^2+4s+13}\right] = L^{-1}\left[\frac{2s-3}{s^2+4s+4+9}\right]$

$= L^{-1}\left[\frac{2s+4-7}{(s+2)^2+3^2}\right]$

$= 2L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - 7L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$

$= 2e^{-2t}\cos 3t - 7e^{-2t}\cdot\frac{1}{3}\cdot\sin 3t$

$= \frac{1}{3}\cdot e^{-2t}[6\cos 3t - 7\sin 3t]$ Ans