Solved Problems :

1. Using Laplace transform, solve $y' - y = t, y(0) = 0$.

Solution :

Given $y' - y = t, y(0) = 0$

Taking Laplace transfrom on both sides,

$$L(y') - L(y) = L(t)$$

$$sL(y) - y(0) - L(y) = \frac{1}{s^2}$$

$$L(y)\left[S - 1\right] = \frac{1}{s^2}$$

$$L(y) = \frac{1}{s^2(s-1)}$$

$$\therefore y = L^{-1}\left[\frac{1}{s^2(s-1)}\right]$$

$$y = \int_0^t \int_0^t L^{-1}\left(\frac{1}{s-1}\right) dt\ dt$$

$$y = \int_0^t \int_0^t e^t\ dt\ dt$$

$$= \int_0^t \left[e^t\right]_0^t dt$$

$$= \int_0^t \left[e^t - 1\right]_0^t dt$$

$$= \left(e^t - t\right)_0^t$$

$$= e^t - t - 1$$

2. Solve $y'' - 4y' + 8y = e^{2t}, y(0) = 2$ and $y'(0) = -2$.

Solution :

Taking Laplace transforms on the sides of the equation, we get

$$L(y'') - 4L(y') + 8L(y) = L(e^{2t})$$

$$\left[s^2L(y) - sy(0) - y^1(0)\right] - 4\left[sL(y) - y(0)\right] + 8L(y) = \frac{1}{s-2}$$

$$i.e., \left[s^2 - 4s + 8\right]L(y) = \frac{1}{s-2} + 2s - 10$$

$$L(y) = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8}$$

$$= \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8} + \frac{2s-10}{s^2-4s+8}$$

Solving we get $A=\frac{1}{4}, \quad B=\frac{-1}{4}, \quad C=\frac{1}{2}$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{-1}{4}s+\frac{1}{2}}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}s-\frac{19}{2}}{s^2-4s+8}$$

$$=\frac{\frac{1}{4}}{S-2}+\frac{\frac{7}{4}(S-2)-6}{(S-2)^2+4}$$

$$y=\frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right)+e^{2t}\left(\frac{\frac{7}{4}s-6}{s^2+4}\right)$$

$$=\frac{1}{4}e^{2t}+e^{2t}\left(\frac{7}{4}\cos 2t-3\sin 2t\right)$$

$$=\frac{1}{4}e^{2t}\left(1+7\cos 2t-12\sin 2t\right)$$

3. Use Laplace transform to solve $y'-y=e^t$ given that $y(0)=1$

Solution:

$$y'-y=e^t$$

Taking Laplace transform on both sides of the equation, we get $y'-y=t, y(0)=0$

$$\left[sL(y)-y(0)\right]-L(y)=\frac{1}{s-1}$$

$$L(y)[s-1]=\frac{1}{s-1}+1$$

$$L(y)=\frac{s}{(s-1)^2}$$

$$y=L^{-1}\left[\frac{s}{(s-1)^2}\right]$$

$$=L^{-1}\left[\frac{(s-1)+1}{(s-1)^2}\right]$$

$$=L^{-1}\left[\frac{1}{s-1}\right]+L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$=e^t+te^t$$

$$=e^t(1+t)$$

Exercise :

1. Solve $y'' - 4y' + 8y = e^{2t}$, $y(0) = 2$ and $y'(0) = -2$

2. Solve $y'' + 4y = \sin wt$, $y(0) = 0$ and $y'(0) = 0$

3. Solve $y'' + y' - 2y = 3\cos 3t - 11\sin 3t$, $y(0) = 0$ and $y'(0) = 6$

4. Solve $(D^2 + 4D + 13)y = e^{-t}\sin t$, $y = 0$ and $Dy = 0$ at $t = 0$ where $D = \dfrac{d}{dt}$

5. Solve $(D^2 + 6D + 9)x = 6t^2 e^{-3t}$, $x = 0$ and $Dx = 0$ at $t = 0$.

6. Solve $x'' + 3x' + 2x = 2(t^2 + t + 1)$, $x(0) = 2$, $x'(0) = 0$.

7. Solve $y'' - 3y' - 4y = 2e^{t}$, $y(0) = y'(0) = 1$.

1. $y = \dfrac{1}{4}e^{2t}(1 + 7\cos 2t - 12\sin 2t)$
2. $y = \dfrac{1}{8}(\sin 2t - 2t\cos 2t)$
3. $y = \sin 3t - e^{-2t} + e^{t}$
4. $y = \dfrac{1}{85}\left[e^{-t}\{-2\cos t + 9\sin t\}\right] + e^{-2t}\left\{2\cos 3t - \dfrac{7}{3}\sin 3t\right\}$
5. $x = \dfrac{1}{2}t^4 e^{-3t}$
6. $x = t^2 - 2t + 3 - e^{-2t}$
7. $y = \dfrac{1}{25}(13e^{-t} - 10te^{-t} + 12e^{4t})$