Example 2: Find the Fourier series of $f(x) = x^2$ in the interval $(0,2\pi)$ and hence, deduce that $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dots$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x^2 \mathrm{d}x = \frac{1}{2\pi}\left|\frac{x^3}{3}\right|_0^{2\pi} = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right) = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_0^{2\pi} = \frac{1}{\pi}\left(-\frac{4\pi^2}{n}\right) = -\frac{4\pi}{n}$$

Hence, $f(x) = \dfrac{4\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$ (1) ... (1)

Putting $x = \pi$ in Eq. (1),

$$f(\pi) = \pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\pi^2 = \frac{4\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2}\dots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Example 3: Find the Fourier series of $f(x) = \dfrac{1}{2}(\pi - x)$ in the interval $(0, 2\pi)$. Hence, deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dots$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\mathrm{d}x = \frac{1}{4\pi}\left|\pi x - \frac{x^2}{2}\right|_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{2\pi} = \frac{1}{2\pi}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) = \frac{1}{n}$$

Hence, $$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx \qquad \ldots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Example 6: Find the Fourier series of $f(x) = x \sin x$ in the interval $(0, 2\pi)$ and hence, deduce that $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$.

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x\sin x\,\mathrm{d}x = \frac{1}{2\pi}\left|x(-\cos x) - (-\sin x)\right|_0^{2\pi} = -1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right|_0^{2\pi}, \qquad n \neq 1$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos(n+1)2\pi}{n+1} + \frac{\cos(n-1)2\pi}{n-1}\right\}\right], \quad n \neq 1$$

$$= -\frac{1}{n+1} + \frac{1}{n-1}, \quad n \neq 1$$

$$= \frac{2}{n^2 - 1}, \quad n \neq 1$$

For $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left|x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right|_0^{2\pi} = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]\mathrm{d}x$$

$$= \frac{1}{2\pi}\left| x\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right] - (1)\left[-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]\right|_0^{2\pi}, \qquad n \neq 1$$

$$= \frac{1}{2\pi}\left[\frac{\cos(n-1)2\pi}{(n-1)^2} - \frac{\cos(n+1)2\pi}{n+1} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right], \qquad n \neq 1$$

$$= 0, \qquad n \neq 1$$

For $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\mathrm{d}x$$

$$= \frac{1}{2\pi}\left| x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right|_0^{2\pi} = \frac{1}{2\pi}(2\pi^2) = \pi$$

Hence, $f(x) = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x \qquad \ldots (1)$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = -1 - \frac{1}{2} + \sum_{n=2}^{\infty}\frac{2}{n^2-1}$$

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \frac{3}{4}$$

Example 8: Find the Fourier series of

$$f(x) = -1 \qquad 0 < x < \pi$$
$$= 2 \qquad \pi < x < 2\pi.$$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\left[\int_0^{\pi}(-1)\mathrm{d}x + \int_{\pi}^{2\pi} 2\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[\left|-x\right|_0^{\pi} + \left|2x\right|_{\pi}^{2\pi}\right] = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi}(-1)\cos nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[-\left|\frac{\sin nx}{n}\right|_0^{\pi} + 2\left|\frac{\sin nx}{n}\right|_{\pi}^{2\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_0^{\pi}(-1)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 2\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\cos nx}{n}\right|_0^{\pi} + \left|-\frac{2\cos nx}{n}\right|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n} - \frac{2\cos 2n\pi}{n} + \frac{2\cos n\pi}{n}\right]$$

$$= \frac{3}{n\pi}[(-1)^n - 1]$$

Hence, $f(x) = \frac{1}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\sin nx$

Example 11: Find the Fourier series of $f(x) = -\pi \quad -\pi < x < 0$

$\qquad = x \quad 0 < x < \pi$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Solution: The Fourier series of $f(x)$ with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-\pi)\mathrm{d}x + \int_0^{\pi} x\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[\left|-\pi x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_0^{\pi}\right] = -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,\mathrm{d}x + \int_0^{\pi} x\cos nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_0^{\pi}\right] = \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,\mathrm{d}x + \int_{0}^{\pi} x\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\pi\left|\frac{\cos nx}{n}\right|_{-\pi}^{0} + \left|x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right|_{0}^{\pi}\right] = \frac{1}{n}[1 - 2\cos n\pi]$$

$$= \frac{1}{n}[1 - 2(-1)^n]$$

Hence, $$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx + \sum_{n=1}^{\infty}\left[\frac{1 - 2(-1)^n}{n}\right]\sin nx \qquad \text{... (1)}$$

At $x = 0$, $f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = \frac{-\pi + 0}{2} = -\frac{\pi}{2}$

Putting $x = 0$ in Eq. (1),

$$f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 20: Find the Fourier series of $f(x) = 4 - x \qquad 3 < x < 4$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = x - 4 \qquad 4 < x < 5.$

Solution: The Fourier series of $f(x)$ with period $2l = 5 - 3 = 2$ is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\mathrm{d}x = \frac{1}{2}\int_3^5 f(x)\mathrm{d}x = \frac{1}{2}\left[\int_3^4 (4-x)\mathrm{d}x + \int_4^5 (x-4)\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[\left|4x - \frac{x^2}{2}\right|_3^4 + \left|\frac{x^2}{2} - 4x\right|_4^5\right] = \frac{1}{2}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \int_3^4 (4-x)\cos n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\cos n\pi x\,\mathrm{d}x$$

$$= \left[\left|(4-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_3^4 + \left|(x-4)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_4^5\right]$$

$$= -\frac{1}{n^2\pi^2}(\cos 4n\pi - \cos 3n\pi) + \frac{1}{n^2\pi^2}(\cos 5n\pi - \cos 4n\pi)$$

$$= -\frac{1}{n^2\pi^2}[(-1)^{4n} - (-1)^{3n} - (-1)^{5n} + (-1)^{4n}]$$

$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\mathrm{d}x = \int_3^4 (4-x)\sin n\pi x\,\mathrm{d}x + \int_4^5 (x-4)\sin n\pi x\,\mathrm{d}x$$

$$= \left[\left|(4-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_3^4\right.$$

$$\left. + \left|(x-4)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_4^5\right]$$

$$= -\frac{1}{n\pi}\cos 3n\pi - \frac{1}{n\pi}\cos 5n\pi$$

$$= 0$$

Hence, $f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos n\pi x$

Example 21: Find the Fourier series of $f(x) = 0 \qquad -5 < x < 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 3 \qquad 0 < x < 5.$

Solution: The Fourier series of $f(x)$ with period $2l = 10$ is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{5}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{5}$$

$$a_0=\frac{1}{2l}\int_{-l}^{l}f(x)\mathrm{d}x=\frac{1}{10}\left(\int_{-5}^{0}0\,\mathrm{d}x+\int_{0}^{5}3\mathrm{d}x\right)=\frac{1}{10}\left|3x\right|_0^5=\frac{3}{2}$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$$

$$=\frac{1}{5}\left(\int_{-5}^{0}0\cdot\cos\frac{n\pi x}{5}\mathrm{d}x+\int_{0}^{5}3\cos\frac{n\pi x}{5}\mathrm{d}x\right)=\frac{3}{5}\left|\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right|_0^5$$

$$=0$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$=\frac{1}{5}\left(\int_{-5}^{0}0\cdot\sin\frac{n\pi x}{5}\mathrm{d}x+\int_{0}^{5}3\sin\frac{n\pi x}{5}\mathrm{d}x\right)=\frac{3}{5}\left|\frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right)\right|_0^5$$

$$=\frac{3}{n\pi}[1-(-1)^n]$$

Hence, $f(x)=\frac{3}{2}+\frac{3}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin\frac{n\pi x}{5}$

Example 22: Find the Fourier series of $f(x)=x \quad -1<x<0$
$\quad = x+2 \quad 0<x<1$.

Solution: The Fourier series of $f(x)$ with period $2l=2$ is given by,

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$=a_0+\sum_{n=1}^{\infty}a_n\cos n\pi x+\sum_{n=1}^{\infty}b_n\sin n\pi x$$

$$a_0=\frac{1}{2l}\int_{-l}^{l}f(x)\,\mathrm{d}x=\frac{1}{2}\left[\int_{-1}^{0}x\mathrm{d}x+\int_{0}^{1}(x+2)\mathrm{d}x\right]=\frac{1}{2}\left[\left|\frac{x^2}{2}\right|_{-1}^{0}+\left|\frac{x^2}{2}+2x\right|_0^1\right]=1$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x=\left[\int_{-1}^{0}x\cos n\pi x\,\mathrm{d}x+\int_{0}^{1}(x+2)\cos n\pi x\,\mathrm{d}x\right]$$

$$=\left[\left|x\left(\frac{\sin n\pi x}{n\pi}\right)-(1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0}+\left|(x+2)\left(\frac{\sin n\pi x}{n\pi}\right)-(1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1\right]=0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x = \left[\int_{-1}^{0} x\sin n\pi x\,\mathrm{d}x + \int_{0}^{1}(x+2)\sin n\pi x\,\mathrm{d}x\right]$$

$$= \left[\left| x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{-1}^{0} + \left|(x+2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{0}^{1}\right]$$

$$= \left[\frac{-(-1)^n}{n\pi} - \frac{3(-1)^n}{n\pi} + \frac{2}{n\pi}\right]$$

$$= \frac{2}{n\pi}[1-2(-1)^n]$$

Hence, $f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-2(-1)^n}{n}\right]\sin n\pi x$

Example 1: Find the Fourier series of $f(x) = x^2$ in the interval $(-\pi, \pi)$. Hence, deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

Solution: $f(x) = x^2$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x^2\mathrm{d}x = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi} = \frac{4}{n^2}\cos n\pi$$

$$= \frac{4}{n^2}(-1)^n$$

Hence, $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \qquad \dots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Example 2: Find the Fourier series of $f(x) = x^3$ in the interval $(-\pi, \pi)$.

Solution: $f(x) = x^3$ is an odd function.

Hence, $a_0 = 0$ and $a_n = 0$

The Fourier series of an odd function with period 2π is given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx$$

$$= \frac{2}{\pi}\left| x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\pi^3\frac{\cos n\pi}{n} + 6\pi\frac{\cos n\pi}{n^3}\right) = 2(-1)^n\left(\frac{-\pi^2}{n} + \frac{6}{n^3}\right)$$

Hence, $f(x) = 2\sum_{n=1}^{\infty}(-1)^n\left(-\frac{\pi^2}{n} + \frac{6}{n^3}\right)\sin nx$

Example 4: Find the Fourier series of $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in the interval $[-\pi, \pi]$ and deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$.

Solution: $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)dx = \frac{1}{\pi}\left|\frac{\pi^2 x}{12} - \frac{x^3}{12}\right|_0^{\pi}$$

$$= 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left|\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{x}{2}\right)\left(-\frac{\cos nx}{n^2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sin nx}{n^3}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{2n^2}\cos n\pi\right)$$

$$= \frac{-(-1)^n}{n^2}$$

Hence, $f(x) = \sum_{n=1}^{\infty}\frac{-(-1)^n}{n^2}\cos nx$

$$= \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

Example 5: Find the Fourier series of $f(x) = |x|$ in the interval $[-\pi, \pi]$.

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$.

Solution: $f(x) = |x| \quad -\pi < x < \pi$

i.e. $\quad f(x) = -x \quad -\pi < x \le 0$

$\quad\quad = x \quad 0 \le x < \pi$

$f(x) = |x|$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x\,\mathrm{d}x = \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi}$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left|x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right|_0^{\pi} = \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

Hence, $$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right) \qquad \ldots (1)$$

Putting $x = 0$ in Eq. (1),

$$f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 6: Find the Fourier series of $f(x) = \sin ax$ in the interval $(-\pi, \pi)$.

Solution: $f(-x) = \sin a(-x) = -\sin ax$

$f(-x) = -f(x)$

$f(x) = \sin ax$ is an odd function.

Hence, $\quad a_0 = 0 \quad$ and $\quad a_n = 0$

The Fourier series of an odd function with period 2π is given by,

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\sin ax\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi}[\cos(n-a)x-\cos(n+a)x]\,\mathrm{d}x \\
&= \frac{1}{\pi}\left|\frac{\sin(n-a)x}{n-a}-\frac{\sin(n+a)x}{n+a}\right|_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a}-\frac{\sin(n+a)\pi}{n+a}\right] \\
&= \frac{1}{\pi}\left(\frac{\sin n\pi\cos a\pi-\sin a\pi\cos n\pi}{n-a}-\frac{\sin n\pi\cos a\pi+\sin a\pi\cos n\pi}{n+a}\right) \\
&= \frac{1}{\pi}\left[\frac{-(-1)^n\sin a\pi}{n-a}-\frac{(-1)^n\sin a\pi}{n+a}\right] = \frac{-(-1)^n\sin a\pi}{\pi}\left(\frac{1}{n-a}+\frac{1}{n+a}\right) \\
&= \frac{2n(-1)^n\sin a\pi}{\pi(a^2-n^2)}
\end{aligned}$$

Hence, $f(x)=\dfrac{2\sin a\pi}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2-n^2}\sin nx$

Example 7: Find the Fourier series of $f(x) = x \sin x$ in the interval $(-\pi, \pi)$.

Hence, deduce that $\frac{\pi - 1}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots$.

Solution: $f(-x) = -x \sin(-x)$

$= x \sin x$

$= f(x)$

$f(x) = x \sin x$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period 2π is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x \sin x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left| x(-\cos x) - (-\sin x)\right|_0^{\pi}$$

$$= 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\mathrm{d}x$$

$$= \frac{1}{\pi}\left| x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right|_0^{\pi}, n \neq 1$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos(n+1)\pi}{n+1} + \pi\frac{\cos(n-1)\pi}{n-1}\right], \qquad n \neq 1$$

$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1},\ n \neq 1 \quad [\because\ (-1)^{n+1} = (-1)^{n-1} = -(-1)^n]$$

For $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left| -x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right|_0^{\pi}$$

$$= -\frac{1}{2}$$

Hence, $$f(x) = 1 - \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$= \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx \qquad \dots (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1),

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{1}{2} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\frac{\pi}{2} = \frac{1}{2} - \frac{2}{3}\cos\pi - \frac{2}{15}\cos 2\pi - \frac{2}{35}\cos 3\pi - \dots$$

$$\frac{\pi-1}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots$$

Example 15: Find the Fourier series of $f(x) = 0 \quad -2 < x < -1$

$$\begin{aligned} f(x) &= 0 & -2 < x < -1 \\ &= 1 + x & -1 < x < 0 \\ &= 1 - x & 0 < x < 1 \\ &= 0 & 1 < x < 2. \end{aligned}$$

Solution:

$$\begin{aligned} f(-x) &= 0 & -2 < -x < -1 \quad &\text{or} \quad 1 < x < 2 \\ &= 1 - x & -1 < -x < 0 \quad &\text{or} \quad 0 < x < 1 \\ &= 1 + x & 0 < -x < 1 \quad &\text{or} \quad -1 < x < 0 \\ &= 0 & 1 < -x < 2 \quad &\text{or} \quad -2 < x < -1 \end{aligned}$$

$$f(-x) = f(x)$$

$f(x)$ is an even function.

Hence, $b_n = 0$

The Fourier series of an even function with period $2l = 4$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\mathrm{d}x = \frac{1}{2}\left[\int_0^1 (1-x)\mathrm{d}x + \int_1^2 0\cdot\mathrm{d}x\right] = \frac{1}{2}\left|x - \frac{x^2}{2}\right|_0^1 = \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x = \frac{2}{2}\int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)\mathrm{d}x + \int_1^2 0\cdot\mathrm{d}x$$

$$= \left|(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right|_0^1 = -\cos\left(\frac{n\pi}{2}\right)\frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$$

Hence, $$f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]\cos\frac{n\pi x}{2}$$