Fourier Transformation

Fourier series decomposes a periodic function into a discrete set of contributions of various frequencies. The Fourier transform provides a continuous frequency resolution of a function.

Fourier transform is useful in study of frequency response, solution of PDE, discrete Fourier transform and fast fourier transform in signal analysis

Fourier transform of $f(x)$

is $$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

Here $F(\alpha)$ is the Fourier transform of $f(x)$.

Fourier Cosine transform of $f(x)$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(\alpha x)\, dx$$

Fourier sine transform of $f(x)$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin(\alpha x)\, dx$$

Linearity Property

Fourier transform, Fourier cosine transform and Fourier sine transform are all linear operations. For any two functions $f(x)$ and $g(x)$; and choosing a and b to be any constants, the fourier transform of $af(x)+bg(x)$ is given by

$$F(af(x)+bg(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af(x)+bg(x))\, e^{-i\alpha x}\, dx$$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx + b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{-i\alpha x}\, dx$$

$$= a \cdot F(f(x)) + b\, F(g(x))$$

IIly $$F_c(af(x)+bg(x)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} [af(x)+bg(x)] \cos(\alpha x)\, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\cos(\alpha x)\, dx + b\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(x)\cos(\alpha x)\, dx.$$

$$= a\, F_c(f(x)) + b\, F_c(g(x))$$

$$F_s(af(x)+bg(x)) = a F_s(f(x)) + b F_s(g(x)).$$

(2) <u>Shifting Property</u> :-

$$F(f(x-a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{-i\alpha x}\, dx$$

$$= e^{-i\alpha a}\, F(f(x)).$$

(3) <u>Scaling Property</u> :-

$$F(g(ct)) = \frac{1}{c} F\left(\frac{\alpha}{c}\right). \qquad \because F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du$$

Q Find the Fourier transform of

$$f(x) = \frac{1}{2a}, \quad \text{if } |x| \le a$$

$$= 0 \quad \text{if } |x| > a$$

<u>Sol</u>

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x) e^{-i\alpha x} dx + \int_{-a}^{a} f(x) e^{-i\alpha x} dx + \int_{a}^{\infty} f(x) e^{-i\alpha x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{1}{2a} e^{-i\alpha x}\, dx$$

$$= \frac{1}{2\sqrt{2\pi}\,a}\left[\frac{e^{-i\alpha x}}{-i\alpha}\right]_{-a}^{a} = \frac{-1}{2\sqrt{2\pi}\,i\alpha}\left(e^{-i\alpha a} - e^{i\alpha a}\right)$$

$$= \frac{e^{i\alpha a} - e^{-i\alpha a}}{2\sqrt{2\pi}\,a i\alpha} = \frac{1}{\sqrt{2\pi}\,a\alpha}\left(\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i}\right)$$

$$= \frac{\sin(\alpha a)}{\sqrt{2\pi}\,a\alpha}.$$

Q Find the Fourier transform of $f(x)$

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$$

Sol $$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} 0\, dx + \int_{0}^{\pi} \sin x\, e^{-i\alpha x}\, dx + \int_{\pi}^{\infty} 0\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{\pi} \sin x\, e^{-i\alpha x}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\pi}\left(\frac{e^{ix} - e^{-ix}}{2i}\right) e^{-i\alpha x}\, dx.$$

$$= \frac{1}{2\sqrt{2\pi}\, i}\int_{0}^{\pi}\left(e^{i(1-\alpha)x} - e^{-i(1+\alpha)x}\right) dx$$

$$= \frac{1}{2\sqrt{2\pi}\, i}\left[\frac{e^{i(1-\alpha)x}}{i(1-\alpha)} - \frac{e^{-i(1+\alpha)x}}{-i(1+\alpha)}\right]_0^{\pi}$$

$$= \frac{1}{2\sqrt{2\pi}\, i}\left[\frac{e^{i(1-\alpha)\pi}}{i(1-\alpha)} + \frac{e^{-i(1+\alpha)\pi}}{i(1+\alpha)} - \frac{1}{i(1-\alpha)} - \frac{1}{i(1+\alpha)}\right]$$

$$= \frac{-1}{2\sqrt{2\pi}}\left(\frac{e^{i\pi}\cdot e^{-i\alpha\pi}}{1-\alpha} + \frac{e^{-i\pi}\cdot e^{-i\alpha\pi}}{1+\alpha} - \frac{1}{1-\alpha} - \frac{1}{1+\alpha}\right)$$

$$= \frac{-1}{2\sqrt{2\pi}}\left(\frac{-1\cdot e^{-i\alpha\pi}}{1-\alpha} + \frac{(-1)e^{-i\alpha\pi}}{1+\alpha} - \frac{1}{1-\alpha} - \frac{1}{1+\alpha}\right)$$

$$= \frac{1}{2\sqrt{2\pi}}\left[e^{-i\alpha\pi}\left(\frac{1}{1-\alpha} + \frac{1}{1+\alpha}\right) + \left(\frac{1}{1-\alpha} + \frac{1}{1+\alpha}\right).\right.$$

$$= \frac{1}{2\sqrt{2\pi}}\left[e^{-i\alpha\pi}\cdot\frac{2}{1-\alpha^2} + \frac{2}{1-\alpha^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-i\alpha\pi}+1}{1-\alpha^2}\right)$$

Q Find the a) Fourier Cosine & b) Sine Transform of $f(x) = e^{-ax}$ for $x \geq 0$ and $a > 0$

<u>Sol</u> $F_c(f(x)) = F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos\alpha x\, dx.$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+\alpha^2}\left(-a\cos\alpha x + \alpha\sin\alpha x\right)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{e^0}{a^2+\alpha^2}(-a) \right] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+\alpha^2}$$

Fourier sine Transform

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int^{\infty} e^{-ax} \sin\alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+\alpha^2} (-a\sin\alpha x - \alpha\cos\alpha x) \right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left(0 - \frac{e^0}{a^2+\alpha^2} (-\alpha) \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\alpha}{a^2+\alpha^2} \right)$$

Parseval's Identity For Fourier Transforms

Let $F(\alpha)$ and $G(\alpha)$ be the Fourier transforms of $f(x)$ and $g(x)$ then

$$\int_{-\infty}^{\infty} F(\alpha)\overline{G(\alpha)}\, d\alpha = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$$

where bar implies the complex conjugate

For $g(x) = f(x)$

$$\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Parseval's Identity for Fourier Cosine transform

$$\int_0^\infty F_c(\alpha)\, G_c(\alpha)\, d\alpha = \int_0^\infty f(x) g(x)\, dx.$$

$$\int_0^\infty |F_c(\alpha)|^2 d\alpha = \int_0^\infty |f(x)|^2 dx.$$

Parseval's Identity for Fourier Sine Transform

- $$\int_0^\infty F_s(\alpha)\, G_s(\alpha)\, d\alpha = \int_0^\infty f(x) g(x)\, dx.$$

$$\int_0^\infty |F_s(\alpha)|^2 d\alpha = \int_0^\infty |f(x)|^2 dx$$

Q Using Parseval's Identity for cosine

Show that

- $$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

Sol As $f(x) = e^{-x}$. have Fourier Cosine Transform

$$F[f(x)] = \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\alpha^2}\right).$$

Using Parseval's Identity for Cosine transform

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\alpha^2}\right)\right)^2 d\alpha = \int_0^\infty |e^{-x}|^2 dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{1}{(1+\alpha^2)^2}\,d\alpha = \int_0^\infty e^{-2x}\,dx$$

$$= \left|\frac{e^{-2x}}{-2}\right|_0^\infty = \frac{-1}{2}\left(0-e^0\right)$$

$$\int_0^\infty \frac{1}{(1+\alpha^2)^2}\,d\alpha = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}.$$

- Q Using Parseval's Identity, prove that

$$\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

Sol We know that e^{-ax} & e^{-bx} have Fourier cosine transformations $\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\alpha^2}\right)$,

- $\sqrt{\frac{2}{\pi}}\;\frac{b}{b^2+\alpha^2}$ resp.

So Using $\int_0^\infty F_c(\alpha)\,G_c(\alpha)\,d\alpha = \int_0^\infty f(x)\,g(x)\,dx$.

$$\frac{2}{\pi}\int_0^\infty \frac{a}{a^2+\alpha^2}\cdot\frac{b}{b^2+\alpha^2}\,d\alpha = \int_0^\infty e^{-ax}\cdot e^{-bx}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+\alpha^2)(b^2+\alpha^2)}\,d\alpha = \int_0^\infty e^{-x(a+b)}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+\alpha^2)}\,d\alpha = \left[\frac{e^{-x(a+b)}}{-(a+b)}\right]_0^\infty$$

$$= \frac{1}{a+b}\left(0+e^0\right) = \frac{1}{a+b}.$$

$$\int_0^\infty \frac{1}{(a^2+\alpha^2)(b^2+\alpha^2)}\,d\alpha = \frac{\pi}{2ab(a+b)}.$$