(*i*) $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

If $f(a) = 0$ then $\frac{1}{f(D)} \cdot e^{ax} = x \cdot \frac{1}{f'(a)} \cdot e^{ax}$

If $f'(a) = 0$ then $\frac{1}{f(D)} \cdot e^{ax} = x^2 \frac{1}{f''(a)} \cdot e^{ax}$

(*ii*) $\frac{1}{f(D)}x^n = [f(D)]^{-1}x^n$ Expand $[f(D)]^{-1}$ and then operate.

(*iii*) $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$ and $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$

If $f(-a^2) = 0$ then $\frac{1}{f(D^2)}\sin ax = x \cdot \frac{1}{f'(-a^2)} \cdot \sin ax$

(*iv*) $\frac{1}{f(D)}e^{ax} \cdot \phi(x) = e^{ax} \cdot \frac{1}{f(D+a)}\phi(x)$

(*v*) $\frac{1}{D+a}\phi(x) = e^{-ax}\int e^{ax} \cdot \phi(x)\,dx$

Solvable for p

Example 1. *Solve* : $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$. **(P.T.U**

Sol. $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$

$$p-\frac{1}{p} = \frac{x^2-y^2}{xy}, \text{ where } p=\frac{dy}{dx}$$

$$xyp^2-(x^2-y^2)p-xy = 0$$

$$p = \frac{(x^2-y^2)\pm\sqrt{(x^2-y^2)^2+4x^2y^2}}{2xy}$$

$$= \frac{(x^2-y^2)\pm(x^2+y^2)}{2xy}$$

$$\therefore \quad p = \frac{2x^2}{2xy} \quad ; \quad p=-\frac{2y^2}{2xy}$$

or $$p = \frac{x}{y} \quad ; \quad p=-\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x}{y} \quad ; \quad \frac{dy}{dx}=-\frac{y}{x}$$

$$ydy = xdx \quad ; \quad \frac{1}{y}dy = -\frac{1}{x}dx$$

Integrating both sides

$$\frac{y^2}{2} = \frac{x^2}{2}+c \quad ; \quad \log y = -\log x + c$$

or $$y^2-x^2-2c = 0 \quad \text{or} \quad \log xy = c \quad \text{or} \quad xy=e^c$$

$\therefore$ The general solution of given equation is $(y^2-x^2-2c)\,(xy-e^c)=0$.

Example 2. *Solve*: $p\,(p+y)=x\,(x+y)$.

Sol. $$p^2+py = x^2+xy$$

or $$p^2+py-(x^2+xy) = 0, \text{ which is quadratic in } p$$

$\therefore \quad p = \dfrac{-y \pm \sqrt{y^2 + 4\left(x^2 + xy\right)}}{2} = \dfrac{-y \pm \sqrt{\left(y + 2x\right)^2}}{2}$

$\therefore \quad p = \dfrac{-y + y + 2x}{2}$

or $\quad p = x$

or $\quad \dfrac{dy}{dx} = x$

Integrating both sides,

$$y = \frac{x^2}{2} + c$$

or $\quad y - \dfrac{x^2}{2} - c = 0 \qquad$ ...(2)

and $\quad p = \dfrac{-y - y - 2x}{2}$

or $\quad p = -y - x$

or $\quad \dfrac{dy}{dx} = -y - x$

or $\quad \dfrac{dy}{dx} + y = -x$

which is linear equation in y

Its $\quad$ I.F. $= e^{\int 1 \cdot dx} = e^x$

$\therefore$ Its solution is

$$y\, e^x = \int e^x \left(-x\right) dx + c = -\int x e^x dx + c$$

or $\quad y\, e^x = -(x-1)\, e^x + c \qquad$ [Integrating by parts]

or $\quad y = -(x-1) + ce^{-x}.$

or $\quad y + x - 1 - ce^{-x} = 0 \qquad$ ...(3)

Combining (2) and (3), general solution is

$$\left(y - \frac{x^2}{2} - c\right)\left(y + x - 1 - ce^{-x}\right) = 0.$$

Solvable for y

Example 1. *Solve :* $y + px = x^4 p^2$.

Sol. Given equation is $y = -px + x^4 p^2$ ...(1)

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = p = -p - x\frac{dp}{dx} + 4x^3 p^2 + 2x^4 p\frac{dp}{dx}$$

or $$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0 \quad \text{or} \quad \left(2p + x\frac{dp}{dx}\right)\left(1 - 2px^3\right) = 0$$

Discarding the factor $(1 - 2px^3)$, we have $2p + x\,\frac{dp}{dx} = 0$ or $\frac{dp}{p} + 2\frac{dx}{x} = 0$

Integrating $\log p + 2\log x = \log c$ or $\log px^2 = \log c$ or $px^2 = c$

or $$p = \frac{c}{x^2}.$$

Putting this value of p in (1), we have $y = -\frac{c}{x} + c^2$, which is the required solution.

Example 2. *Solve :* $y = 2px - p^2$.

Sol. The given equation is $y = 2px - p^2$ ...(1)

Differentiating both sides w.r.t. x, $\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$

or $$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

or $$p + (2x - 2p)\,\frac{dp}{dx} = 0 \quad \text{or} \quad p\frac{dx}{dp} + 2x - 2p = 0$$

or $$\frac{dx}{dp} + \frac{2}{p}x = 2 \qquad \text{...(2)}$$

which is a linear equation.

$$\text{I.F.} = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$

$\therefore$ The solution of (2) is $x\,(\text{I.F.}) = \int 2(\text{I.F.})\,dp + c$ or $xp^2 = \int 2p^2\,dp + c$

or $$xp^2 = \frac{2}{3}p^3 + c \quad \text{or} \quad x = \frac{2}{3}p + cp^{-2} \qquad \text{...(3)}$$

p cannot be easily eliminated from (1) and (3)

$\therefore$ Putting the value of x in (1), we have $y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2$

Solvable for x

Example 1. *Solve:* $y = 2px + p^2 y$

Sol. Given differential equation is

$$y = 2px + p^2 y$$

Solving for x, we have

$$x = \frac{y}{2p} - \frac{py}{2}$$

Differentiate w.r.t., y $\quad \dfrac{dx}{dy} = \dfrac{1}{2}\,\dfrac{p\cdot 1 - y\dfrac{dp}{dy}}{p^2} - \dfrac{1}{2}\left(p\cdot 1 + y\dfrac{dp}{dy}\right)$

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\frac{1}{2p} + \frac{p}{2} + \frac{y}{2p^2}\frac{dp}{dy} + \frac{y}{2}\frac{dp}{dy} = 0$$

$$\frac{1+p^2}{p} + \left(\frac{1}{p^2} + 1\right) y\frac{dp}{dy} = 0$$

$$\frac{1+p^2}{p} + \frac{1+p^2}{p^2}\, y\frac{dp}{dy} = 0$$

or $$\frac{1+p^2}{p}\left\{1 + \frac{y}{p}\frac{dp}{dy}\right\} = 0$$

Discarding the factor $\dfrac{1+p^2}{p}$, we have

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

or $$\frac{dp}{p} = -\frac{dy}{y}$$

Integrating both sides

$$\log p = -\log y + \log c$$

or $$py = c$$

Eliminate p from (1) and (2)

$$y = 2x\cdot\frac{c}{y} + \frac{c^2}{y^2}\, y$$

or $$y = \frac{2cx}{y} + \frac{c^2}{y}$$

or $$y^2 = 2cx + c^2 \text{; required solution.}$$

Example 2. *Solve :* $p = tan\left(x - \dfrac{p}{1+p^2}\right)$.

Sol. Solving for x, we have $x = \tan^{-1} p + \dfrac{p}{1+p^2}$

Differentiating both sides w.r.t. y, $\dfrac{dx}{dy} = \dfrac{1}{p} = \dfrac{1}{1+p^2}\cdot\dfrac{dp}{dy} + \dfrac{(1+p^2)-2p^2}{(1+p^2)^2}\cdot\dfrac{dp}{dy}$

or $$\frac{1}{p} = \frac{2(1+p^2)-2p^2}{(1+p^2)^2}\frac{dp}{dy} \quad\text{or}\quad dy = \frac{2p}{(1+p^2)^2}\, dp$$

Integrating $\quad y = c - \dfrac{1}{1+p^2}$

Equations (1) and (2) together constitute the general solution.

Variation of parameters

Example 1. *Find the general solution of the equation $y'' + 16y = 32 \sec 2x$; using method of variation of parameters.* **(P.T.U., May 2008, 2010)**

Sol. Given equation in symbolic form is $(D^2 + 16)\, y = 32 \sec 2x$

$$\text{A.E. is } D^2 + 16 = 0 \quad \therefore \quad D = \pm 4i$$

$$\text{C.F. is } y = c_1 \cos 4x + c_2 \sin 4x$$

Here $\quad y_1 = \cos 4x,\ y_2 = \sin 4x,\ X = 32 \sec 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4 \sin 4x & 4 \cos 4x \end{vmatrix} = 4$$

$$\text{P.I.} = uy_1 + vy_2 \text{ where } u = -\int \frac{y_2 X}{W}\, dx \text{ and } v = \int \frac{y_1 X}{W}$$

$\therefore$
$$\text{P.I.} = -\cos 4x \int \frac{\sin 4x \cdot 32 \sec 2x}{4}\, dx + \sin 4x \int \frac{\cos 4x \cdot 32 \sec 2x}{4}\, dx$$

$$= -8 \cos 4x \int 2 \sin 2x \cos 2x \cdot \frac{1}{\cos 2x}\, dx + 8 \sin 4x \int \frac{2\cos^2 2x - 1}{\cos 2x}\, dx$$

$$= -16 \cos 4x \int \sin 2x\, dx + 8 \sin 4x \int (2 \cos 2x - \sec 2x)\, dx$$

$$= -16 \cos 4x \left[-\frac{\cos 2x}{2} \right] + 8 \sin 4x \left[\frac{2 \sin 2x}{2} - \frac{\log(\sec 2x + \tan 2x)}{2} \right]$$

$$= 8 \cos 4x \cos 2x + 8 \sin 4x \sin 2x - 4 \sin 4x \log(\sec 2x + \tan 2x)$$

$$= 8 \cos(4x - 2x) - 4 \sin 4x \log(\sec 2x + \tan 2x)$$

$$= 8 \cos 2x - 4 \sin 4x \log(\sec 2x + \tan 2x)$$

$\therefore$
$$\text{C.S. is } y = \text{C.F.} + \text{P.I.}$$

$$= c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x - 4 \sin 4x \log(\sec 2x + \tan 2x).$$

Example 2. *Solve: $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$ by variation of parameter method.*

(P.T.U., May 2010, 2012, Dec. 2012, 2013)

Sol. Equation in the symbolic form is

$$(D^2 - 6D + 9)\, y = \frac{e^{3x}}{x^2}$$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.*, $(D - 3)^2 = 0$ *i.e.*, $D = 3, 3$

C.F. $= (c_1 + c_2 x)\, e^{3x} = c_1 e^{3x} + c_2 x e^{3x} = c_1 y_1 + c_2 y_2$, where $y_1 = e^{3x}$, $y_2 = x e^{3x}$

and
$$X = \frac{e^{3x}}{x^2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & (1 + 3x)\, e^{3x} \end{vmatrix} = e^{6x}$$

P.I. $= uy_1 + vy_2$, where $u = -\displaystyle\int \frac{y_2 X}{W}\, dx$ and $v = \displaystyle\int \frac{y_1 X}{W}\, dx$

$\therefore$
$$\text{P.I.} = -e^{3x} \int \frac{x e^{3x} \cdot e^{3x}}{e^{6x} \cdot x^2}\, dx + x e^{3x} \int \frac{e^{3x} \cdot e^{3x}}{e^{6x} \cdot x^2}\, dx = -e^{3x} \int \frac{1}{x}\, dx + x e^{3x} \int \frac{1}{x^2}\, dx$$

$$= -e^{3x} \log x + x e^{3x} \left(-\frac{1}{x} \right) = -e^{3x} (1 + \log x)$$

$$\text{C.S. is } y = (c_1 + c_2 x)_2\, e^{3x} - e^{3x} (1 + \log x)$$

$$= e^{3x} [c_1 + c_2 x - 1 - \log x] = e^{3x} [(c_1 - 1) + c_2 x - \log x]$$

$$= e^{3x} \left[c_1' + c_2 x - \log x \right], \text{ where } c_1' = c_1 - 1 \text{ is the required solution.}$$

Cauchy Euler Equation

Example 1. *Obtain the general solution of the equation* $2x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 6y = 0$. **(P.T.U., Dec. 2013)**

Sol. $2x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 6y = 0$ ...(1)

It is Cauchy's homogeneous linear differential equation

Put $\quad x = e^z \quad \therefore \quad z = \log x$

$$x\frac{dy}{dx} = \text{D}y; \quad x^2\frac{d^2y}{dx^2} = \text{D}(\text{D}-1)\,y, \text{ where D} = \frac{d}{dz}$$

Equation (1) becomes

$$2\text{D}(\text{D}-1)\,y + \text{D}y - 6y = 0$$

or

$$(2\text{D}^2 - \text{D} - 6)\,y = 0$$

$$\text{A.E. is } 2\text{D}^2 - \text{D} - 6 = 0$$

$$(\text{D}-2)(2\text{D}+3) = 0$$

i.e.,

$$\text{D} = 2, \text{D} = -\frac{3}{2}$$

Solution is

$$y = c_1 e^{2z} + c_2 e^{-\frac{3}{2}z} = c_1x^2 + c_2x^{-3/2}.$$

Example 2. *Solve* $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$. **(P.T.U., June 2003, May 2009, Dec. 2012)**

Sol. Given equation is Cauchy's homogeneous linear equation

so that

$$x\frac{dy}{dx} = \text{D}y, \quad x^2\frac{d^2y}{dx^2} = \text{D}(\text{D}-1)\,y$$

$$x^3\frac{d^3y}{dx^3} = \text{D}(\text{D}-1)(\text{D}-2)y, \text{ where D} = \frac{d}{dz}$$

Substituting these values in the given equation, it reduces to

$$[\text{D}(\text{D}-1)(\text{D}-2) + 2\text{D}(\text{D}-1) + 2]y = 10(e^z + e^{-z})$$

or

$$(\text{D}^3 - \text{D}^2 + 2)\,y = 10(e^z + e^{-z})$$

which is a linear equation with constant coefficients.

Its A.E. is $\text{D}^3 - \text{D}^2 + 2 = 0$ or $(\text{D}+1)(\text{D}^2 - 2\text{D} + 2) = 0$

$$\therefore \quad \text{D} = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1e^{-z} + e^z(c_2\cos z + c_3\sin z) = \frac{c_1}{x} + x\,[c_2\cos(\log x) + c_3\sin(\log x)]$$

$$\text{P.I.} = 10\frac{1}{\text{D}^3 - \text{D}^2 + 2}(e^z + e^{-z}) = 10\left(\frac{1}{\text{D}^3 - \text{D}^2 + 2}e^z + \frac{1}{\text{D}^3 - \text{D}^2 + 2}e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}e^z + z.\frac{1}{3\text{D}^2 - 2\text{D}}e^{-z}\right) = 10\left(\frac{1}{2}e^z + z.\frac{1}{3(-1)^2 - 2(-1)}e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x}\log x$$

Hence the C.S. is $y = \dfrac{c_1}{x} + x\left[c_2\cos(\log x) + c_3\sin(\log x)\right] + 5x + \dfrac{2}{x}\log x.$