Find Laplace Inverse

$$\text{(ii)}\ \ F(s)=\frac{3s+4}{s^2+9}=\frac{3s}{s^2+9}+\frac{4}{s^2+9}$$

$$L^{-1}\{F(s)\}=3\cos 3t+\frac{4}{3}\sin 3t$$

$$\text{(iv)}\ \ F(s)=\frac{2s+2}{s^2+2s+10}=\frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\}=2\,e^{-t}\,L^{-1}\left\{\frac{s}{s^2+9}\right\}=2\,e^{-t}\cos 3t$$

$$\text{(v)}\ \ F(s)=\frac{2s+3}{s^2+2s+2}=\frac{2s+2+1}{(s+1)^2+1}=\frac{2(s+1)+1}{(s+1)^2+1}=2\frac{(s+1)}{(s+1)^2+1}+\frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\}=2\,e^{-t}\,L^{-1}\left\{\frac{s}{s^2+1}\right\}+e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\}=2\,e^{-t}\cos t+e^{-t}\sin t$$

$$\text{(vii)}\ \ F(s)=\frac{s}{(2s+1)^2}=\frac{1}{4}\,\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2}=\frac{1}{4}\left[\frac{1}{s+\frac{1}{2}}-\frac{1}{2}\cdot\frac{1}{\left(s+\frac{1}{2}\right)^2}\right]$$

$$L^{-1}\{F(s)\}=\frac{1}{4}L^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}-\frac{1}{8}\,e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2}\right\}=\frac{1}{4}e^{-\frac{t}{2}}-\frac{1}{8}\,e^{-\frac{t}{2}}t$$

$$\text{(ix)}\ \ F(s)=\frac{3s+1}{(s+1)^4}=\frac{3(s-1)-2}{(s+1)^4}=\frac{3}{(s+1)^3}-\frac{2}{(s+1)^4}$$

$$L^{-1}\{F(s)\}=3e^{-t}\,L\left\{\frac{1}{s^3}\right\}-2e^{-t}\left\{\frac{1}{s^4}\right\}=3e^{-t}\,\frac{t^2}{2!}-2e^{-t}\,\frac{t^3}{3!}=\frac{3}{2}e^{-t}\,t^2-\frac{1}{3}\ e^{-t}\,t^3$$

Partial Fractions Cases

Case I: Factors are linear and distinct,

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II: Factors are linear and repeated,

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \ldots + \frac{B_n}{(s+b)^n}$$

Case III: Factors are quadratic and distinct,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV: Factors are quadratic and repeated,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1 s+D_1}{s^2+cs+d} + \frac{C_2 s+D_2}{(s^2+cs+d)^2} + \ldots + \frac{C_n s+D_n}{(s^2+cs+d)^n}$$

(i) $F(s) = \dfrac{s+2}{s(s+1)(s+3)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$s + 2 = A(s + 1)(s + 3) + Bs(s + 3) + Cs(s + 1) \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3A$$

$$A = \frac{2}{3}$$

Putting $s = -1$ in Eq. (1),

$$1 = B(-1)(2)$$

$$B = -\frac{1}{2}$$

Putting $s = -3$ in Eq. (1),

$$-1 = C(-3)(-2)$$

$$C = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{\mathrm{F}(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

(ii) $F(s) = \dfrac{s+2}{s^2(s+3)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s + 2 = As(s + 3) + B(s + 3) + Cs^2 \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1),

$$2 = 3B$$

$$B = \frac{2}{3}$$

Putting $s = -3$ in Eq. (1),

$$-1 = 9C$$

$$C = -\frac{1}{9}$$

Equating the coefficients of s^2,

$$0 = A + C$$

$$A = \frac{1}{9}$$

$$F(s) = \frac{1}{9}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s^2} - \frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

(iii) $F(s) = \dfrac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$5s^2 - 15s - 11 = A(s - 2)^2 + B(s + 1)(s - 2) + C(s + 1) \qquad \dots (1)$$

Putting $s = -1$ in Eq. (1),

$$9 = 9A$$

$$A = 1$$

Putting $s = 2$ in Eq. (1),

$$-21 = 3C$$

$$C = -7$$

Equating the coefficients of s^2,

$$5 = A + B$$

$$B = 4$$

$$F(s) = \frac{1}{s+1} + \frac{4}{s-2} - \frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} + 4\,L^{-1}\left\{\frac{1}{s-2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= e^{-t} + 4e^{2t} - 7\,te^{2t}$$

(iv) $F(s) = \dfrac{s+2}{(s+3)(s+1)^3}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$s + 2 = A(s + 1)^3 + B(s + 3)\,(s + 1)^2 + C(s + 3)\,(s + 1) + D(s + 3) \qquad \ldots (1)$$

Putting $s = -3$ in Eq. (1),

$$-1 = -8A$$

$$A = \frac{1}{8}$$

Putting $s = -1$ in Eq. (1),

$$1 = 2D$$

$$D = \frac{1}{2}$$

Equating the coefficients of s^3,

$$0 = A + B$$

$$B = -\frac{1}{8}$$

Equating the coefficients of s^2,

$$0 = 3A + 5B + C$$

$$C = -\frac{3}{8} + \frac{5}{8} = \frac{1}{4}$$

$$F(s) = \frac{1}{8}\cdot\frac{1}{s+3} - \frac{1}{8}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

$$L^{-1}\{F(s)\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{8}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= \frac{1}{8}e^{-3t} - \frac{1}{8}e^{-t} + \frac{1}{4}t\,e^{-t} + \frac{1}{2}\cdot\frac{t^2}{2}\cdot e^{-t}$$

$$= \frac{1}{8}[e^{-3t} + (2t^2 + 2t - 1)e^{-t}]$$

(v) $F(s) = \dfrac{s^3 + 6s^2 + 14s}{(s+2)^4}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}$$

$$s^3 + 6s^2 + 14s = A(s+2)^3 + B(s+2)^2 + C(s+2) + D$$

$$= As^3 + (6A + B)s^2 + (12A + 4B + C)s + (8A + 4B + 2C + D) \quad \text{... (1)}$$

Equating the coefficients of s^3,

$$A = 1$$

Equating the coefficients of s^2,

$$6 = 6A + B$$

$$B = 0$$

Equating the coefficients of s,

$$14 = 12A + 4B + C$$

$$C = 14 - 12 - 0 = 2$$

Equating the coefficients of s^0,

$$0 = 8A + 4B + 2C + D$$

$$D = -8 - 0 - 4 = -12$$

$$F(s) = \frac{1}{s+2} + \frac{2}{(s+2)^3} - \frac{12}{(s+2)^4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^3}\right\} - 12L^{-1}\left\{\frac{1}{(s+2)^4}\right\}$$

$$= e^{-2t} + 2\cdot\frac{t^2}{2}\cdot e^{-2t} - 12\cdot\frac{t^3}{6}\cdot e^{-2t} = e^{-2t}(1 + t^2 - 2t^3)$$

(vi) $F(s) = \dfrac{3s+1}{(s+1)(s^2+2)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$3s + 1 = A(s^2 + 2) + (Bs + C)(s + 1) \quad \text{... (1)}$$

Putting $s = -1$ in Eq. (1),

$$-2 = 3A$$

$$A = -\frac{2}{3}$$

Equating the coefficients of s^2,

$$0 = A + B$$

$$B = \frac{2}{3}$$

Equating the coefficients of s°,

$$1 = 2A + C$$

$$C = 1 + \frac{4}{3} = \frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\}$$

$$= -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$$

(vii) $F(s) = \dfrac{s+4}{s(s-1)(s^2+4)}$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$s + 4 = A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)\,s(s-1) \qquad \ldots (1)$$

Putting $s = 0$ in Eq. (1),

$$4 = -4A$$

$$A = -1$$

Putting $s = 1$ in Eq. (1),

$$5 = 5B$$

$$B = 1$$

Equating the coefficients of s^3,

$$0 = A + B + C$$

$$C = 1 - 1 = 0$$

Equating the coefficients of s,

$$1 = 4A + 4B - D$$

$$D = -4 + 4 - 1 = -1$$

$$F(s) = -\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}$$

$$L^{-1}\{F(s)\} = -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s^2+4}\right\} = -1 + e^t - \frac{1}{2}\sin 2t$$

(viii) $$F(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right] = \frac{1}{3}[\cos t - \cos 2t]$$

(ix) $F(s) = \dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $s^2 = x$

$$G(x) = \frac{x}{(x+a^2)(x+b^2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+a^2} + \frac{B}{x+b^2}$$

$$x = A(x+b^2) + B(x+a^2) \qquad \dots (1)$$

Putting $x = -a^2$ in Eq. (1),

$$-a^2 = A(-a^2+b^2)$$

$$A = \frac{a^2}{a^2-b^2}$$

Putting $x = -b^2$ in Eq. (1),

$$-b^2 = B(-b^2+a^2)$$

$$B = -\frac{b^2}{a^2-b^2}$$

$$G(x) = \frac{a^2}{a^2-b^2}\cdot\frac{1}{x+a^2} - \frac{b^2}{a^2-b^2}\cdot\frac{1}{x+b^2}$$

$$F(s) = \frac{a^2}{a^2-b^2}\cdot\frac{1}{s^2+a^2} - \frac{b^2}{a^2-b^2}\cdot\frac{1}{s^2+b^2}$$

$$L^{-1}\{F(s)\} = \frac{a^2}{a^2-b^2}\,L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - \frac{b^2}{a^2-b^2}\,L^{-1}\left\{\frac{1}{s^2+b^2}\right\}$$

$$= \frac{a^2}{a^2-b^2}\,\frac{1}{a}\sin at - \frac{b^2}{a^2-b^2}\,\frac{1}{b}\sin bt$$

$$= \frac{1}{a^2-b^2}\,(a\sin at - b\sin bt)$$

(x) $F(s) = \dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

Let $s^2 + 2s = x$

$$G(x) = \frac{x+3}{(x+5)(x+2)}$$

By partial fraction expansion,

$$G(x) = \frac{A}{x+5} + \frac{B}{x+2}$$

$$x + 3 = A(x+2) + B(x+5) \qquad \dots (1)$$

Putting $x = -5$ in Eq. (1),

$$-2 = -3A$$

$$A = \frac{2}{3}$$

Putting $x = -2$ in Eq. (1),

$$1 = 3B$$

$$B = \frac{1}{3}$$

$$G(x) = \frac{2}{3} \cdot \frac{1}{x+5} + \frac{1}{3} \cdot \frac{1}{x+2}$$

$$F(s) = \frac{2}{3} \cdot \frac{1}{(s^2+2s+5)} + \frac{1}{3} \cdot \frac{1}{(s^2+2s+2)} = \frac{2}{3} \cdot \frac{1}{(s+1)^2+4} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\} = \frac{2}{3} L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t + \frac{1}{3} e^{-t} \sin t = \frac{1}{3} e^{-t} (\sin 2t + \sin t)$$

(xi)
$$F(s) = \frac{s+2}{(s^2+4s+8)(s^2+4s+13)} = \frac{s+2}{5}\left[\frac{s^2+4s+13-s^2-4s-8}{(s^2+4s+8)(s^2+4s+13)}\right]$$

$$= \frac{1}{5}\left[\frac{s+2}{s^2+4s+8} - \frac{s+2}{s^2+4s+13}\right] = \frac{1}{5}\left[\frac{s+2}{(s+2)^2+4} - \frac{s+2}{(s+2)^2+9}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{5}\left[L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\}\right]$$

$$= \frac{1}{5}\left[e^{-2t} L^{-1}\left\{\frac{s}{s^2+4}\right\} - e^{-2t} L^{-1}\left\{\frac{s}{s^2+9}\right\}\right]$$

$$= \frac{1}{5}\left(e^{-2t} \cos 2t - e^{-2t} \cos 3t\right) = \frac{e^{-2t}}{5}(\cos 2t - \cos 3t)$$