Building a computer package for manipulating 2-associahedra

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Geboren am 10. November 2002 in Köln

01. August 2023

Bachelorarbeit Mathematik

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Zusammenfassung

Das Hauptziel dieser Arbeit ist die Entwicklung eines Computerpakets, das in der Lage ist, die Poset-Struktur der von Bottman in [Bot19] eingeführten 2-Assoziahydra zu berechnen. Die 2-Assoziahydra ist ein *relativer 2-Operad*, der die Informationen hinter den $(\mathcal{A}_\infty, 2)$-Operationen in `Symp`, der $(\mathcal{A}_\infty, 2)$-Kategorie mit symplektischen Mannigfaltigkeiten als Objekten, verwaltet. Das Computerpaket eröffnet Möglichkeiten zur Erforschung verschiedener kombinatorischer Aspekte, die mit den 2-Assoziahydra verbunden sind. In dieser Arbeit wird der Leser in die symplektische Geometrie und ihre Verbindungen zu algebraischen Strukturen eingeführt.

Das Feld der symplektischen Geometrie stammt aus der klassischen Mechanik, insbesondere der Hamiltonschen Mechanik. Diese beschreibt die Bewegung von Teilchen in einem n-dimensionalen Raum, die eine Energie-Funktion optimieren. Dieses Phänomen wird durch das Konzept einer *symplektischen Mannigfaltigkeit* verkörpert, dem zentralen Objekt der symplektischen Geometrie, das in Kapitel 2 eingeführt wird.

Diesen symplektischen Mannigfaltigkeiten kann man eine algebraische Invariante namens *Fukaya-Kategorie* zuordnen. Die Objekte der Fukaya-Kategorie sind Lagrange-Untermannigfaltigkeiten, die in der symplektischen Geometrie von großer Bedeutung sind. Eine Anwendung besteht darin, folgende Frage zu beantworten: Gibt es für zwei gegebene Konfigurationen von Teilchen, von deren eine die Startkonfiguration ist, einen Startimpuls, sodass das System nach einer Sekunde die andere Konfiguration erreicht hat? Diese Frage wird durch die Betrachtung des Schnitts zweier solcher Lagrange-Untermannigfaltigkeiten beantwortet.

Die Fukaya-Kategorien erweisen sich als $\mathcal{A}_\infty$-Kategorien. Wir werden sehen, wie sie mithilfe des Operads namens Assoziahydra konstruiert werden. Die Assoziahydra erfasst dabei die Operationen innerhalb der Fukaya-Kategorie. Da wir die Fukaya-Kategorien als algebraische Invarianten für symplektische Mannigfaltigkeiten konstruieren, wollen wir, dass bestimmte Abbildungen zwischen ihnen Funktoren zwischen ihren Fukaya-Kategorien induzieren. In Kapitel 3 konstruieren wir Fukaya-Kategorien und untersuchen ihre Funktorialität.

Wie sich herausstellt, ist der richtige Rahmen, um die Funktorialitätseigenschaften der Fukaya-Kategorien zu verstehen, die symplektische $(\mathcal{A}_\infty, 2)$-Kategorie `Symp`, die in Kapitel 4 eingeführt wird. Ein weiterer Vorteil von `Symp` besteht darin, dass sie in einem allgemeineren Rahmen definiert ist. Allerdings ist die Definition von `Symp` bisher nicht vollständig abgeschlossen. `Symp` wird mithilfe des relativen 2-Operad namens 2-Assoziahydra konstruiert, welcher wiederum alle Operationen innerhalb von `Symp` verwaltet.

In Kapitel 5 werden wir einige kombinatorische Darstellungen der 2-Assoziahydra sehen, die wir dann zur Konstruktion des Computerpakets verwenden. Die Konstruktion des Computerpakets in Kapitel 6 ist originale Arbeit und existiert meines Wissens nach noch nicht. Sobald die Konstruktion von `Symp` abgeschlossen sein wird, wird das Computerpaket hilfreich sein, um einige Berechnungen darin durchzuführen.

Als Anwendung des Computerpakets, berechnen wir in Kapitel 7 eine kombinatorische Eigenschaft namens CD-Index für die 2-Assoziahydra.

Contents

1 Introduction

The primary objective of this thesis is to develop a computer package capable of computing the poset structure of the 2-associahedra introduced by Bottman in [Bot19]. The 2-associahedra form a *relative 2-operad* which stores the information behind the $(\mathcal{A}_\infty, 2)$-operations in `Symp`, the $(\mathcal{A}_\infty, 2)$-category whose objects are symplectic manifolds. The computer package will open up possibilities for exploring various combinatorial aspects associated with these structures. Through this work, the reader is introduced to symplectic geometry and its connections to algebraic structures.

The field of symplectic geometry emerges from classical mechanics, particularly Hamiltonian mechanics, which describes the motion of particles in an n-dimensional space, optimizing an energy function. This phenomenon is encapsulated by the concept of a *symplectic manifold*, the central object of study in symplectic geometry, introduced in Chapter 2.

To this symplectic manifold one can associate an algebraic invariant called the *Fukaya Category*. The objects of this Fukaya category are Lagrangian submanifolds which have significant importance in symplectic geometry. One application is that they can answer the following question: For two given configurations of particles, starting with one configuration, is there a start momentum, such that the particles end up in the other configuration? This question is answered by considering the intersection of two such Lagrangians.

The Fukaya categories turn out to be $\mathcal{A}_\infty$-categories. We will see how they are constructed using the operad called associahedra, which keeps track of the operations within the Fukaya category. Since we construct the Fukaya categories as algebraic invariants for symplectic manifolds we hope that certain maps between them induce functors between their Fukaya categories. We construct Fukaya categories in Chapter 3 and examine their functoriality.

As it turns out, the right framework for understanding the functoriality properties of the Fukaya categories is the symplectic $(\mathcal{A}_\infty, 2)$-category `Symp` which is introduced in Chapter 4. Another benefit of `Symp` is that it is defined in a more general setting. However, the definition of `Symp` has not been completed until now. `Symp` is built over the relative 2-operad called 2-associahedra, which again keeps track of all the operations within `Symp`.

In the following Chapter 5, we are going to see some combinatorial representations of the 2-associahedra which we then use to construct the computer package. The construction of the computer package in Chapter 6 is original work and, to the best of my knowledge, does not yet exist. Once the construction of `Symp` will be completed, it will be helpful to make some computations in there.

As an application of the computer package, we calculate a combinatorial property of the 2-associahedra, called the CD-Index, in Chapter 7.

2 Symplectic Geometry and the Floer complex

In this section, we will first give an introduction to some fundamental concepts in symplectic geometry. Afterward, we will construct the so-called *Floer chain complex*. This construction lays the groundwork for defining the Fukaya categories as algebraic invariants for symplectic manifolds. More details can be found in [MS17] and [AD14].

2.1 Basics in symplectic geometry

Symplectic geometry finds its roots in classical mechanics, where it serves as a natural framework for describing Hamiltonian systems. Consider k particles moving in an m-dimensional space, minimizing an energy function. To model such a system, we associate them with a $2km$-dimensional manifold. For each particle, we take m coordinates representing the particle's location, and m coordinates representing its momentum. This idea is formalized in the definition of symplectic manifolds, which emerge as the central objects of interest in symplectic geometry.

Definition 2.1. A *symplectic manifold* (M, ω) is a $2n-$dimensional, smooth manifold M equipped with a 2-form $\omega \in \Omega^2(M)$, which is non-degenerate, i.e. $\omega^{\wedge n} \neq 0$ and closed, i.e. $d\omega = 0$.

Remark 2.2. The assumptions on ω already imply that the manifold has to be even dimensional since there are no skew-symmetric, non-degenerate forms in space of odd dimension. One can see this by constructing a basis $\{e_1, ..., e_n, f_1, ..., f_n\}$, where $\omega(e_i, f_j) = \delta_{i,j}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$. The part of math that deals with the odd-dimensional manifolds is called *contact geometry* and is a whole other branch but with a lot of connections to symplectic geometry.

Example 2.3. The most straightforward example of a symplectic manifold is $\mathbb{R}^{2n}$ equipped with the standard symplectic form. For coordinates $(p, q) \in \mathbb{R}^n \times \mathbb{R}^n$, it is defined as:

$$\omega_0 := \sum_{i=0}^{n} dp_i \wedge dq_i.$$

However, interpreting $\mathbb{R}^{2n}$ as the cotangent space $T^*(\mathbb{R}^n) = \mathbb{R}^{2n}$ provides a more intuitive understanding of the location and momenta of particles. In this view, the coordinates of the cotangent space at a fixed point describe the momenta of the particles. This leads to a multitude of examples by considering:

$$\left(T^*Q, \sum_i dp_i \wedge dq^i \right)$$

for any smooth manifold Q, where p_i are the coordinates on Q and q^i are the coordinates on the fiber T^*_pQ.

Moving forward, given the non-degeneracy of ω we want to have a look at a so-called *Hamiltonian* $H \in C^\infty(M)$. The Hamiltonian is closely related to the energy function of the physical system. The Hamiltonian induces a vector field, the so-called *Hamiltonian vector field* (sometimes called the *symplectic gradient*) given by the following equation:

$$\omega(\cdot, X_H) = dH.$$

Moreover, we can associate with this vector field its flow, which results from integrating the vector field. The flow describes the time evolution of the physical system and is given by the following equation:

$$\frac{d}{dt}\varphi_H^t = X_H \circ \varphi_H^t$$

Since these are very natural definitions we want them of course to be somehow compatible with ω. For this, we define what in this setting is a notion of an isomorphism.

Definition 2.4. A diffeomorphism $\varphi : M_1 \to M_2$ satisfying $\varphi^*\omega_2 = \omega_1$ is called a *symplectomorphism*. The group of symplectomorphisms of a symplectic manifold M is denoted by $\mathrm{Symp}(M, \omega)$. A *symplectic isotopy* of (M, ω) is a smooth map $\psi : [0, 1] \times M \to M, (t, q) \mapsto \psi_t(q)$ such that ψ_t is a symplectomorphism for all t and $\psi_0 = \mathrm{id}$.

Understanding $\mathrm{Symp}(M, \omega)$ is crucial to understand the nature of a symplectic manifold, as this is the group of structure-preserving transformations. The flow of a Hamiltonian vector field is indeed a symplectomorphism. We call a symplectomorphism that is induced by a Hamiltonian a *Hamiltonian symplectomorphism* and analogous we call a symplectic isotopy a *Hamiltonian isotopy* if it is induced by a time-dependent Hamiltonian $H \in C^\infty([0, 1] \times M)$.

Proposition 2.5. *For any $H \in \mathcal{C}^\infty(M)$, the flow φ_H^t of X_H is a symplectomorphism for any time t.*

Proof. We want to show that φ_H^t is a symplectomorphism for every t. By Cartan's formula

$$\frac{d}{dt}((\varphi_{X_H}^t)^*\omega) = \mathcal{L}_{X_H}\omega = d(i_{X_H}\omega) + i_{X_H}d\omega = d(-dH) = 0.$$

Therefore $(\varphi_H^t)^*\omega$ does not depend on t and thus $(\varphi_H^t)^*\omega = (\varphi_H^0)^*\omega = \mathrm{id}^*\,\omega = \omega$ for all $t \in [0, 1]$. $\square$

This shows that symplectic geometry generalizes the notion of a Hamiltonian system. Even though symplectic geometry originated from classical mechanics it nowadays has its rights as its own branch in mathematics.
A very fundamental result in symplectic geometry is the following theorem.

Theorem 2.6 (Darboux)**.** *Let (M, ω) be a symplectic manifold and let $p \in M$. Then there is a chart (U, φ) around p such that*

$$\omega\,|_U = \sum_{i=1}^n x_i \wedge y_i$$

where $(x_1, x_2, \ldots, y_{n-1}, y_n)$ are local coordinates on U.

This theorem shows that there are no local invariants for symplectic manifolds. Its original proof is credited to Darboux, although a more modern approach was presented by Moser in [Mos65]. Since we want to assign an algebraic invariant to a symplectic manifold, we seek some global geometric properties. One fundamental goal is to understand the geometry of Lagrangian submanifolds.

Definition 2.7. A *Langrangian submanifold* $L^n \subset M^{2n}$ of a symplectic Manifold M^{2n} is a submanifold of M^{2n} with half the dimension of M^{2n}, satisfying $\omega\mid_{L^n} = 0$.

Remark 2.8. Lagrangians naturally occur as the 0-section of the cotangent bundle T^*Q, where Q is a smooth manifold. Hence, every smooth manifold can be regarded as the Lagrangian 0-section of its cotangent bundle.
Another reason why we should care about Lagrangians is the following. Given two points $p, q \in Q$, describing the configuration of k particles, it is a natural question to ask whether there is an initial momentum at p such that after 1 second the particles moved to q. This question is tied to the position and momenta described by the cotangent bundle T^*Q. The fibers T_p^*Q and T_q^*Q are Lagrangians and also $\varphi_H^1(T_p^*Q)$ is a Lagrangian, even though it is not a fiber of the cotangent bundle anymore. Thus the question we are asking here is if the two Lagrangians T_q^*Q and $\varphi_H^1(T_p^*Q)$ intersect. This gives also a reason why we should care about the intersection of Lagrangians.

The goal of the next chapters is to define an algebraic invariant for symplectic manifolds called the *Fukaya category*. We start by providing an overview of a simplified version of this category, to see what we need to construct.

Definition 2.9. The Fukaya category $\mathrm{Fuk}(M, \omega)$ of a symplectic manifold (M, ω), satisfying certain assumptions, consists of the following data.

- The objects of the Fukaya category are compact, closed, oriented Lagrangian submanifolds, such that $\pi_2(M, L) = 0$.
- Let $L, K \in \mathrm{ob}(\mathrm{Fuk}(M, \omega))$. Then we define the class of morphisms between L and K to be the Floer chain complex $CF(L, K)$ of these two Lagrangians.
- Additionally, there is a whole bunch of other data, which will not be discussed at this moment.

One might initially consider defining the class of morphisms to be the set of intersection points between the two Lagrangians. As we mentioned earlier, the intersections of Lagrangians hold significant importance. However, this approach runs mainly into two problems. Firstly it is not invariant under a Hamiltonian symplectomorphism $\psi \in \mathrm{Symp}(M, \omega)$, and secondly, we don't have composition maps for intersection points. Instead, the right way to do this is to use the Floer chain complex which we will construct in the following.

2.2 The Floer chain complex

For this section, we assume every manifold and Lagrangian submanifold to be closed. Furthermore, we assume $\pi_2(M, L) = 0$ for all Lagrangians involved in the construction. We follow [Aur13] for the introduction to Lagrangian Floer Homology and the Fukaya category. The original motivation for defining Floer Homology was the *Arnold conjecture*, which gives a lower bound for the number of 1-periodic orbits of a Hamiltonian system.

Theorem 2.10 (Arnold's conjecture)**.** *Let $\Psi \in Symp(M)$ be a Hamiltonian symplectomorphism and assume that $L \pitchfork \Psi(L)$. Then,*

$$|L \cap \Psi(L)| \geq \sum_i \dim H^i(L, \mathbb{Z}/2).$$

Floer's approach is to associate to a pair of Lagrangians (L, K) a chain complex which is freely generated by the intersection points of the two Lagrangians. To achieve this, we need suitable coefficients.

Definition 2.11. The *Novikov field* is defined as

$$\Lambda = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in K, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty\}$$

where K is some field.

For our purposes, we will pick $K = \mathbb{Z}/2$. With this coefficient field, we can define

$$CF(L, K) := \bigoplus_{p \in L \cap K} \Lambda \cdot p$$

which will turn out to be a graded vector space.

The next step is to define a differential $\partial : CF(L, K) \to CF(L, K)$ with the following properties:

1. $\partial^2 = 0$ to obtain a well-defined chain complex.
2. If $K \simeq K'$ are Hamiltonian isotopic we want $HF^*(L, K) \cong HF^*(L, K')$.
3. $HF(L, L) \cong H^*_{\text{sing}}(L)$ with suitable coefficients.

Using this properties the Arnold conjecture 2.10 follows immediately:

$$\begin{aligned} |L \cap \Psi(L)| = \sum_i \dim CF^i(L, \psi(L)) &\geq \sum_i \dim HF^i(L, \psi(L)) \\ &= \sum_i \dim HF^i(L, L) = \sum_i \dim H^i(L, \mathbb{Z}/2) \end{aligned}$$

Next, we proceed to construct the Floer differential. In this thesis, we will only construct the differential for Lagrangians L, K with $L \pitchfork K$. In general, if they do not intersect transversely we would need to introduce so-called *Hamiltonian pertubation* terms to achieve transversality. These are, for $H \in C^\infty(M)$ we look at a shift by φ^t_H.

The main idea behind constructing the differential is to count certain maps that connect two intersection points of two Lagrangians. These maps are called $J-$holomorphic strips. They map a "strip" $\mathbb{R} \times [0, 1]$ into our manifold, as illustrated in Figure 1.

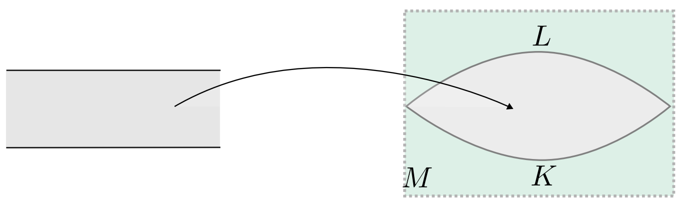


Figure 1: J-holomorphic strip $u : \mathbb{R} \times [0, 1] \to M$

To define these maps, we require the notion of an ω-compatible almost complex structure.

Definition 2.12. An ω-*compatible almost complex structure* on M is an endomorphism $J : TM \to TM$ of the tangent bundle with $J^2 = -\mathrm{id}$ such that $\omega(\cdot, J\cdot)$ forms a Riemannian metric on M. The space of all such J is denoted by $\mathcal{J}(M, \omega)$.

Now fix a *time dependent* almost complex structure $J_t : [0,1] \to \mathcal{J}(M, \omega)$ compatible with ω in a way that it ensures transversality of the moduli space of J-holomorphic strips. The existence of such structures is guaranteed by the fact that $\mathcal{J}(M, \omega)$ is non-empty and contractible ([MS17], Proposition 4.1.1). The Floer differential is then defined by counting maps $u : \mathbb{R} \times [0,1] \to M$ that satisfy the following three conditions:

1. *Floer equation*

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = 0 \tag{2.1}$$

2. *Boundary condition*

$$\begin{cases} u(s,0) \in L_0, \quad u(s,1) \in L_1 \\ \lim\limits_{s\to-\infty} u(s,t) = q, \ \lim\limits_{s\to\infty} u(s,t) = p \end{cases} \tag{2.2}$$

3. *Finite energy*

$$E(u) := \int u^*\omega = \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial u}{\partial s}\right|^2 dsdt < \infty \tag{2.3}$$

The Floer equation is analogous to the Cauchy-Riemann differential equation for J-holomorphic strips. The boundary condition ensures that the image of the map is bounded by the two Lagrangians L_0 and L_1. The third condition is necessary for the well-definedness of the differential.
To count the number of maps solving this boundary value problem, we consider the space of solutions $\widehat{\mathcal{M}}(p, q, [\alpha], J)$ representing a certain homotopy class $[\alpha] \in \pi_2(M, L \cup K)$. There is an action by $\mathbb{R}$ on $\widehat{\mathcal{M}}(p, q, [\alpha], J)$ given by $a \in \mathbb{R}$ acting on $u(s,t) \mapsto u_a(s,t) = u(s-a, t)$. So we define

$$\mathcal{M}(p, q, [\alpha], J) := \widehat{\mathcal{M}}(p, q, [\alpha], J)\big/\mathbb{R}$$

The boundary value problem (2.1)-(2.3) is a Fredholm problem, which means the following. We can view the solution set of (2.1)-(2.3) as the zero set of a certain section of a bundle of Banach manifolds, and the linearizations of this section at points in the zero set are Fredholm operators, as defined below. For more information about this Fredhom problem, we refer the reader to [Mor19], chapter 4 or [AD14], chapter 8.

Definition 2.13. A linear map $f : E \to F$ between Banach spaces is a *Fredholm operator* if its kernel is finite dimensional and its image has finite codimension. In this case, its index is given by

$$\mathrm{Ind}(f) = \dim\ker f - \dim\operatorname{coker} f$$

If $\mathcal{F} : E \to F$ is a smooth map between Banach manifolds, we say that it is *Fredholm* if $d\mathcal{F}_x$ if Fredholm for every $x \in E$.

The following theorem now shows that $\mathcal{M}(p,q,[\alpha],J)$ forms a smooth manifold of dimension $\mathrm{ind}([\alpha])-1$, given that the Fredholm operator is surjective. This is due to the transversality. Here, $\mathrm{ind}([\alpha])$ denotes the *Maslov index* of $[\alpha]$. The Maslov index will be discussed briefly at the end of this section.

Theorem 2.14 ([Mor19], Theorem 4.7)**.** *Let $\mathcal{F}: E \to F$ be a Fredholm map between Banach manifolds, and let $y \in F$ such that $d\mathcal{F}_x$ is surjective $\forall x \in \mathcal{F}^{-1}(y)$. Then $\mathcal{F}^{-1}(y)$ is a smooth manifold of dimension Ind($\mathcal{F}$).*

If we now pick a homotopy class $[\alpha]$ of index 1, the moduli space $\mathcal{M}(p,q,[\alpha],J)$ becomes a 0-dimensional smooth manifold and as we will see, it is also compact. Hence it is a finite collection of points. With this understanding, we can now define the differential of the Floer complex.

Definition 2.15. The differential $\partial : CF(L,K) \to CF(L,K)$ is defined as follows.

$$\partial(p) = \sum_{q \in L \cap K} \left(\sum_{\substack{[\alpha] \in \pi_2(M, L \cup K) \\ ind([\alpha])=1}} \#_2 \mathcal{M}(p,q,[\alpha],J) T^{\omega([\alpha])} \right) q \tag{2.4}$$

where $\omega([\alpha])$ denotes the symplectic area of $[\alpha]$.

In the following we are going to show that this indeed defines a differential.
Well-definedness: To ensure the well-definedness of the differential, we need to examine two aspects. Firstly, the inner sum only sums over countably many summands due to condition (3). By Gromov's compactness theorem, for a given finite energy E_0, there are only finitely many homotopy classes with energy less then E_0 and non-empty solutions ([Gro85], Theorem 1.5B). Secondly, we have to show that the number of points in $\mathcal{M}(p,q,[\alpha],J)$ is finite. To establish this, we show that this manifold is compact by taking a closer look at J-holomorphic curves. Again by Gromov's compactness theorem there are only three different limit strips that can occur for J-holomorphic curves. They are obtained as reparametrizations, where a non-zero amount of energy concentrates in different regions of the strip.

(STRIP BREAKING) Energy concentrates at either $\pm\infty$ (see Figure 2a).

(DISK BUBBLING) Energy concentrates at a boundary point of the strip (see Figure 2b).

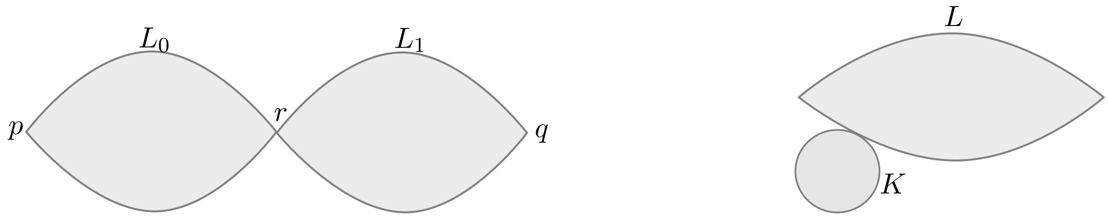
(SPHERE BUBBLING) Energy concentrates at an interior point of the strip.

However, due to the condition $\pi_2(M,L) = 0$, the cases of disk and sphere bubbling cannot occur, leaving us only with the case of strip breaking.

Definition 2.16. The *Gromov compactification* is given by

$$\overline{\mathcal{M}}(p,q,[\alpha],J) := \mathcal{M}(p,q,[\alpha],J) \cup \{\text{broken strips}\}$$

Using that the Maslov index is additive in this case and the fact that each strip has an index of at least 1 due to transversality, it follows that our 0-dimensional manifold is already compact. This concludes the well-definedness of the differential.



(a) Strip breaking (b) Disk bubbling

Figure 2: Limits of J-holomorphic curves

$\partial^2 = 0$: Next, we aim to show that the differential squares to zero. For this fix a homotopy class $[\alpha]$ of index 2. Then $\overline{\mathcal{M}}(p, q, [\alpha], J)$ is a smooth, compact 1-manifold with boundary given by

$$\partial\overline{\mathcal{M}}(p, q, [\alpha], J) = \bigsqcup_{\substack{r \in L \cap K \\ [\beta]+[\gamma]=[\alpha] \\ ind([\beta])=ind([\gamma])=1}} \mathcal{M}(p, r, [\beta], J) \times \mathcal{M}(r, q, [\gamma], J)$$

consisting of the broken strips connecting p to q. Counting the modulo 2 number of points in this space gives us precisely the coefficient of q in $\partial^2(p)$. Now since the modulo 2 number of boundary points of a 1-manifold is 0 we conclude that $\partial^2 = 0$.

Remark 2.17. The third technical issue in defining the Floer chain complex, besides the compactness and transversality of the moduli space, is the issue of its orientability. We sidestep this issue here by working with a $\mathbb{Z}/2$ base field.

2.2.1 Maslov Index

For our purposes, we will focus on a special case of the Maslov index that suits our needs. In the following denote by $LGr(n)$ the Grassmannian of Lagragian n-planes of $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic form. There is a homotopy equivalence $LGr(n) \simeq {}^{U(n)}\!/_{O(n)}$ ([MS17], chapter 2.3), which can be constructed by associating Lagrangians to matrices $X, Y \in GL(n, \mathbb{R})$ as follows

$$\Lambda = \operatorname{im} Z, \quad Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad U_\Lambda := X + iY.$$

As it turns out U_Λ forms an orthogonal basis of Λ if and only if it is unitary. Moreover, $U(n)$ is determined by Λ up to multiplication by $O(n)$. Define the map

$$\rho : LGr(n) \to S^1, \rho(\Lambda) = \det(U_\Lambda^2)$$

The Maslov index of a loop in $\Lambda(t) \subset LGr(n)$ can be defined as the degree of the composition map $\rho \circ \Lambda : \mathbb{R}/\mathbb{Z} \to S^1$.

Now, consider transverse Lagrangians $\Lambda_0, \Lambda_1 \in LGr(n)$ and identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. There exists a symplectic matrix $A \in Sp(\mathbb{R}^{2n}, \omega_0)$ that maps Λ_0 to $\mathbb{R}^n \subset \mathbb{C}^n$ and Λ_1 to $i\mathbb{R}^n \subset \mathbb{C}^n$. Then

$$\Lambda_t := A^{-1}(e^{-i\pi t/2}\mathbb{R})$$

forms a homotopy class of paths connecting Λ_0 to Λ_1 and is called the *canonical short path*.

Definition 2.18. Given $p, q \in L \cap K$. Denote by Λ_p the canonical short path connecting T_pL to T_pK and by Λ_q the canonical short path connecting T_qK to T_qL. Let $u : \mathbb{R} \times [0,1] \to M$ be a map connecting p to q. Then let L_i be the path $u^* \mid_{\mathbb{R}\times\{i\}}$ oriented with s going from $+\infty$ to $-\infty$. View all these as paths in $LGr(n)$ by fixing a trivialization of u^*TM. Then the *Maslov index* of u is the Maslov index of the loop consisting of the concatenation of $-L_0, \Lambda_p, l_1$ and finally $-\Lambda_q$.

Example 2.19. Take a look at the strip $u : \mathbb{R} \to \mathbb{R}^2$, illustrated in Figure 1. The Grassmanian $LGr(1)$ is canonically isomorphic to $\mathbb{R}P^1$. The path we get from putting together all the sub-paths is now homotopic to walking around twice in $\mathbb{R}P^1$. Since S^1 double covers $\mathbb{R}P^1$ the mapping degree of the map $\rho \circ \Lambda$ is 1. Therefore $\text{ind}(u) = 1$.

2.3 Product operations on the Floer complex

In the preceding section, we constructed the Floer chain complex, which is intended to represent the morphism class between two Lagrangians in the Fukaya Category. In this section, we aim to introduce a product operation on the Floer complex, which will enable us to compose morphisms in the Fukaya Category. More precisely for Lagrangians $L_0, L_1, L_2 \subset M$ we want to have a map

$$CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2).$$

The idea for this is again to count certain maps. For the differential, we counted maps $u : \mathbb{R} \times [0,1] \to M$. Now note that $\mathbb{R} \times [0,1]$ is homeomorphic to a disc with two boundary points missing. In a very similar way, we can now count maps of discs with three boundary points missing into the area bounded by three Lagrangians as indicated in figure 3.

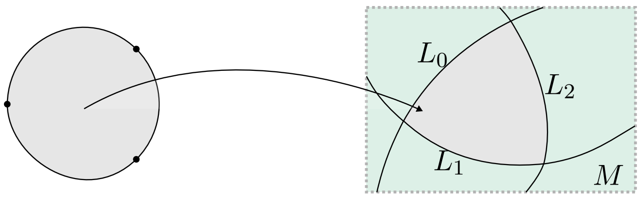


Figure 3: Product operation μ^2

Let $p_1 \in L_0 \cap L_1$, $p_2 \in L_1 \cap L_2$, and $q \in L_0 \cap L_2$. We denote the moduli space of solutions to the boundary value problem by $\mathcal{M}(p_1, p_2, q, [\alpha], \mathcal{J})$. As before this defines a smooth manifold.

Definition 2.20. We define the *Floer product* to be the Λ-linear map $\mu^2 : CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2)$ with

$$\mu^2(p_1, p_2) = \sum_{q \in L_0 \cap L_2} \sum_{\substack{[\alpha] \in \pi_2(M, L \cup K), \\ [\alpha] : ind([\alpha]) = 0}} (\#_2 \mathcal{M}(p_1, p_2, q, [\alpha], J)) T^{\omega([\alpha])} q.$$

Proposition 2.21 ([Aur13], Proposition 2.3). *If $[\omega] \cdot \pi_2(M, L_i) = 0$ for all i, then the Floer product satisfies the Leibniz rule:*

$$\partial(p_1 \cdot p_2) = \partial(p_1) \cdot p_2 + p_1 \cdot \partial(p_2).$$

Consequently, it induces a well-defined product $HF(L_0, L_1) \otimes HF(L_1, L_2) \to HF(L_0, L_2)$ that is associative and independent of the chosen almost complex structure (and Hamiltonian perturbations).

Proof sketch. To prove this result, we examine the index 1 moduli space of J-holomorphic curves. This space admits a compactification using Gromov compactness. As our assumptions exclude the presence of disk or sphere bubbling, we are only left with strip breaking. There are only three limit configurations, all consisting of an index 0-strip bounded by L_0, L_1, L_2 and an index 1-strip bounded by two of them. The three configuration exactly contribute to the coefficients of $T^{\omega([\alpha])}$ in $\partial(p_1 \cdot p_2), (\partial p_1) \cdot p_2, p_1 \cdot (\partial p_2)$ and all arise as an end of $\mathcal{M}(p_1, p_2, q, [\alpha], J)$. Thus we are counting the modulo 2 number of boundary points of a compact 1-manifold, which is equal to zero. $\square$

However, the product operation on the chain level is not associative. To handle this non-associativity, we introduce additional structure to the chain complex. For this generalize the Floer product as follows: Similar to definition 2.20, we can define $\mathcal{M}(p_1, ..., p_r, q, [\alpha], \mathcal{J})$ to be the space of solutions for maps from disks with k boundary points missing into our manifold, such that each missing boundary point gets mapped to an intersection point $p_i \in L_i \cap L_{i+1}$.

Definition 2.22. The operation $\mu^k : CF(L_0, L_1) \otimes \cdots \otimes CF(L_{k-1}, L_k) \to CF(L_0, L_k)$ is a Λ-linear map defined by

$$\mu^k(p_1, ..., p_k) = \sum_{\substack{q \in L_0 \cap L_k, \\ [\alpha]: ind([\alpha]) = 2-k}} (\#\mathcal{M}(p_1, ..., p_k, q, [\alpha], J)) T^{\omega([u])} q$$

The higher product operations somehow capture the non-associativity of μ^2 and we will utilize these operations for the definition of the Fukaya categories in the next chapter.

3 Fukaya categories

The goal of this section is to complete the definition of the Fukaya Category $\mathrm{Fuk}(M,\omega)$ of a symplectic manifold (M,ω). In the last section, we constructed the Floer chain complex and defined the product operations on it, to get a composition function in the Fukaya Category. However, as mentioned earlier, the operation μ^2 defined above is not associative, which is necessary for a composition function of a category. To resolve this, we introduce the concept of $\mathcal{A}_\infty$-categories, which relaxes the strict associativity condition and allows for higher homotopy associativity.

3.1 $\mathcal{A}_\infty$-categories and operads

An $\mathcal{A}_\infty$-category somehow relaxes the condition of associativity in the definition of a category up to higher homotopies. The symbol $\mathcal{A}$ is related to the associativity, and the ∞ signifies that we have homotopies μ^k for every $k \in \mathbb{N}$.

Definition 3.1. The linear $\mathcal{A}_\infty$-relations over $\mathbb{Z}/2$ are given by

$$\sum_{i+j+k=n} \mu^{i+1+k}(\mathbf{1}^{\otimes i}, \mu^j, \mathbf{1}^{\otimes k}) = 0, \quad \text{for all } n \geq 1.$$

Definition 3.2. A linear, $\mathbb{Z}$-graded $\mathcal{A}_\infty-$category $\mathcal{C}$ consists of the following data

- For all $A, B \in \mathrm{ob}(\mathcal{C})$ the class $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is a finite dimensional chain complex of $\mathbb{Z}-$graded modules.
- For all $k \geq 1$, $X_0, ..., X_k \in \mathrm{ob}(\mathcal{C})$, there is a family of composition maps

  $$\mu^k : \mathrm{Hom}_{\mathcal{C}}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}(X_{k-1}, X_k) \to \mathrm{Hom}_{\mathcal{C}}(X_0, X_k)$$

  of degree $k-2$, which satisfy the $\mathcal{A}_\infty$-relations.

Remark 3.3. One can also construct the Fukaya category using other gradings. For example [MWW18] works with $\mathbb{Z}/N$ coefficients.

A linear $\mathcal{A}_\infty$-category can be described in a much more general way, using the notion of an operad. This will help us to show that the product operations of the Floer chain complex indeed satisfy the $\mathcal{A}_\infty$-relations.

Definition 3.4. A *non-Σ operad* in a symmetric, monoidal category $(\mathcal{C}, \otimes, 1)$ is a collection $(P_r)_{r\geq 1} \subseteq \mathcal{C}$ together with a family of morphisms:

$$\gamma_{r,(s_i)} : P_r \otimes \bigotimes_{1\leq i\leq r} P_i \to P_{\Sigma_i s_i}, \quad r, s_1, ..., s_r \geq 1$$

satisfying the following additional axioms:

(ASSOCIATIVITY) The following diagram commutes:

$$\begin{tikzcd}
P_r \otimes \bigotimes_{1\le i\le r} P_{s_i} \otimes \bigotimes_{1\le i\le r}\bigotimes_{1\le j\le s_i} P_{t_{ij}} \arrow[r, "\cong"] \arrow[dd, "\gamma_{r,(s_i)}\times id"'] & P_r \otimes \bigotimes_{1\le i\le r} (P_{s_i} \otimes \bigotimes_{1\le j\le s_i} P_{t_{ij}}) \arrow[d, "id\otimes_{1\le i\le r}\gamma_{s_i,(t_{ij})j}"] \\
 & P_r \otimes \prod_{1\le i\le r} P\Sigma_j t_{ij} \arrow[d, "\gamma_{r,\Sigma_j t_{ij}}"] \\
P_{\Sigma_i s_i} \otimes \bigotimes_{1\le i\le r}\bigotimes_{1\le j\le s_i} P_{t_{ij}} \arrow[r, "\gamma_{\Sigma_i s_i,(t_{1,1},\dots,t_{1,s_1},\dots,t_{r,1},\dots,t_{r,s_r})}"'] & P_{\Sigma_{i,j} t_{ij}}
\end{tikzcd}$$

(UNIT) There is a unit map $\eta : 1 \to P_1$ such that the compositions

$$P_r \otimes 1^{\otimes r} \overset{id\otimes\eta^{\otimes r}}{\to} P_r \otimes P_1^{\otimes r} \overset{\gamma_{r,(1,\dots,1)}}{\to} P_r, \quad 1 \otimes P_s \overset{\eta\otimes id}{\to} P_1 \otimes P_s \overset{\gamma_{1,(s)}}{\to} P_s$$

are the iterated right respectively left unit morphisms in $\mathcal{C}$.

The operadic structure can be thought of as keeping track of different operations. The next step is to form these operations into the operations of a category.

Definition 3.5. Let $\mathcal{O}$ be an operad in $\mathscr{C}$. Then a *category* $\mathcal{A}$ *over* $\mathcal{O}$ consists of the following data:

- A set of objects $\mathrm{ob}(\mathcal{A})$.
- For every $X, Y \in \mathrm{ob}(\mathcal{A})$ a class of morphisms $\mathrm{Hom}(X, Y) \in \mathscr{C}$.
- For every $r \geq 1$, sequence $X_0, ..., X_r \in \mathrm{ob}(\mathcal{A})$ a *composition operation*

$$\mathcal{O}(r) \times \mathrm{Hom}(X_0, X_1) \times \cdots \times \mathrm{Hom}(X_{r-1}, X_r) \to \mathrm{Hom}(X_0, X_r)$$

compatible with the composition maps in $\mathcal{O}$ and with the unit $1 \in \mathcal{O}(1)$.

We proceed with defining the operad that we use in the context of the definition of the Fukaya Category. This operad is constructed based on the domains of the product operations on the Floer chain complex.

Definition 3.6. For $r \geq 2$ define the space M_r to be the moduli space of discs with $(r+1)$ distinct, unlabeled boundary marked points, one of which is distinguished, modulo Möbius transformations that preserve the unit disc.

This space is not compact since the limits in which points collide do not lie in the space. There is a compactification of this space, following the compactification proposed by Fulton-MacPherson in [FM94]. In cases where points in the space collide, we "zoom in" at this point to "remember" the configuration of the space right before the collision. We denote the compactified space by $\overline{M_r}$. It can be realized as a convex polytope. We give an example in Figure 4.

Definition 3.7. The poset of strata of the geometric realization as a convex polytope of $\overline{M_r}$ is called the *associahedron* and is denoted by K_r.

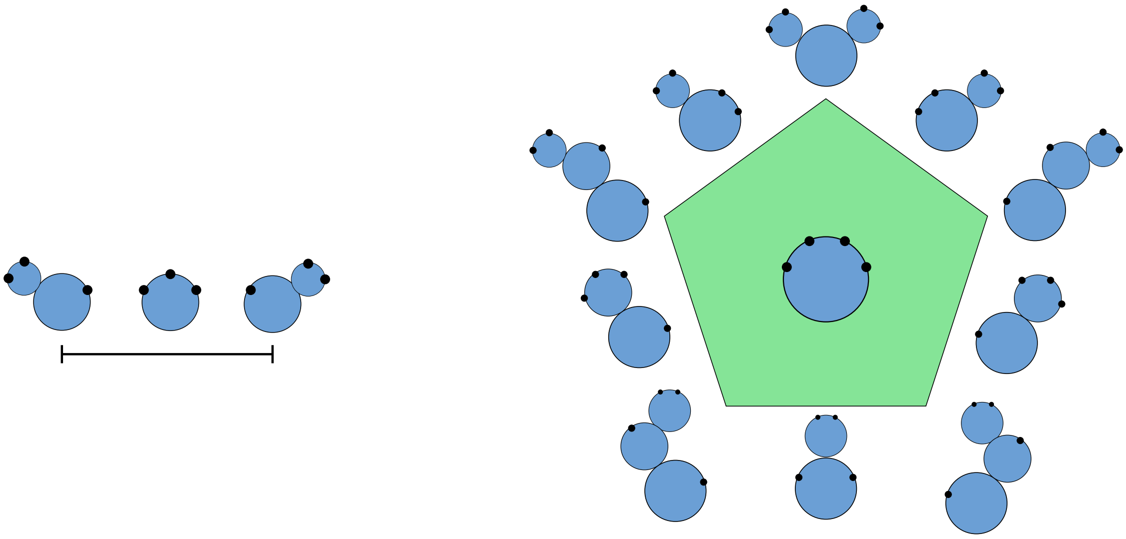


Figure 4: K_3 and K_4 realized as a convex polytope

Remark 3.8. The collection $(K_r)_{r\geq 1}$ forms an operad in the category of topological spaces `TOP`. The operations $K_r \times K_{s_1} \times \cdots \times K_{s_r} \to K_{\sum s_i}$ are given by replacing a boundary marked point of the disk in K_r by another disk. By considering the cellular homology of the operad, $(C_*^{\text{cell}}(K_r))_{r\geq 1}$ forms an operad in the category of chain complexes `Ch`.

The Fukaya category $\text{Fuk}(M, \omega)$ is a category over $(C_*^{\text{cell}}(K_r))$, as we can associate to a cell $C \subseteq K_r$ the r-ary operation defined by counting pseudo-holomorphic maps whose domains are in C. The following proposition now shows that the Fukaya category is an $\mathcal{A}_\infty$-category.

Proposition 3.9 ([AB22], Proposition 2.21)**.** *The categories over $(C_*^{\text{cell}}(K_r))$ are precisely the linear $\mathcal{A}_\infty$-categories.*

Proof sketch. We only prove how to construct an $\mathcal{A}_\infty$-category out of a category over $(C_*^{\text{cell}}(K_r))$. Denote by φ_r the composition operations. We define the following operations: Define the unary operation φ_1 to be the differential of the chain complex μ^1. For $r \geq 1$ we define μ^r to be the result of feeding the chain corresponding to the codimension 0 face of K_r into φ_r.

We first notice that the maps φ_r are chain maps as shown for the special case of the product operation on the Floer complex in the Leibniz rule. This gives us the following equation:

$$\varphi_r(\partial K_r, x_1, ..., x_r) + \sum_{1\leq i\leq r} \varphi_r(K_r, x_1, ..., x_{i-1}, \partial x_i, x_{i+1}, ..., x_r) \\ = \partial\varphi_r(K_r, x_1, ..., x_r)$$

Now the operadic structure of the associahedra gives us the following result when performing the boundary operator on K_r:

$$\sum_{\substack{1\le i\le r-1\\ 2\le a\le r-i+1}} \mu_{r-a+1}(x_1,...,x_{i-1},\mu_a(x_i,...,x_{i+a-1},)x_{i+a},...,x_r)$$
$$+\sum_{1\le i\le r}\mu_r(x_1,...,x_{i-1},\mu_1(x_i),x_{i+1},...,x_r)=\mu_1\mu_r(x_1,...,x_r)$$
$$\Leftrightarrow \sum_{\substack{1\le i\le r\\ 1\le a\le r-i+1}} \mu_{r-a+1}(x_1,...,x_{i-1},\mu_a(x_i,...,x_{i+a-1}),x_{i+a},...,x_r)=0,$$

where the last term represents exactly the $\mathcal{A}_\infty$-relations of the category. □

3.2 Definition of the Fukaya category

As we now have seen, the product operations on the Floer complex satisfy the $\mathcal{A}_\infty$-relations. We can now finish the definition of the Fukaya $\mathcal{A}_\infty$-category. The following definition is only a sketch definition. The complete definition can be found in [Aur13], Definition 2.9. Furthermore, notice that there are many different versions of the Fukaya category.

Definition 3.10. The *Fukaya Category* is built in the following way. Let (M,ω) be a symplectic manifold, which is compact, and $2c_1(TM)=0$.

- The objects of the Fukaya category are compact, closed Lagrangians $L\subset M$ that satisfy $[\omega]\cdot\pi_2(M,L)=0$, equipped with a spin structure and some more auxiliary data.
- For $L,K\in ob(\mathrm{Fuk}(M,\omega))$ we have $\mathrm{Hom}(L,K)=CF^*(L,K)$ ($\mathbb{Z}-$graded) for some perturbation data and an almost complex structure to ensure transversality.
- For $r\ge 2$ we have r-ary morphisms

$$\mu^r: CF^*(L_0,L_1)\otimes\cdots\otimes CF^*(L_{r-1},L_r)\to CF^{*-2+r}(L_0,L_r)$$

  given by counting maps whose domains are pseudo-holomorphic discs.

3.3 Functoriality of the Fukaya category

Our next objective is to define a functor between the Fukaya categories $\mathrm{Fuk}(M_1,\omega_1)$ and $\mathrm{Fuk}(M_2,\omega_2)$. Because we constructed the Fukaya category as an algebraic invariant for a symplectic manifold, we hope that a certain map between symplectic manifolds induces a functor between their Fukaya categories.
Let us begin by examining the maps between symplectic manifolds. For sure, these maps should preserve the symplectic structure, which leads us to consider symplectomorphisms. However, they are not very exciting themselves, which brings us to the notion of the more general Lagrangian correspondences.

Definition 3.11. A *Lagrangian correspondence* of two symplectic manifolds (M_1,ω_1) and (M_2,ω_2) is a Lagrangian $L_{12}\subset M_1^-\times M_2:=(M_1\times M_2,-\omega_1\oplus\omega_2)$.

Remark 3.12. Let $\varphi_{12} : M_1 \to M_2$ be a symplectomorphism. Then the graph Γ_{12} of φ_{12} is a Lagrangian correspondence $\Gamma_{12} \subset M_1^- \times M_2$. This shows that Lagrangian correspondences generalize the notion of a symplectomorphism.

Ma'u, Wehrheim, and Woodward constructed functors from such Lagrangian correspondences. The results of their construction of the $\mathcal{A}_\infty$-functor are given by the following theorem.

Theorem 3.13 (Ma'u, Wehrheim, Woodward, [MWW18], Theorem 1.1)**.** *Suppose that M_1 and M_2 are compact, monotone symplectic manifolds with the same monotonicity constant. Given an admissible Lagrangian correspondence $L_{12} \subset M_1^- \times M_2$ equipped with a brane structure, there exists an $\mathcal{A}_\infty$-functor*

$$\Phi^{\#}_{L_{12}} : \mathit{Fuk}^{\#}(M_1) \to \mathit{Fuk}^{\#}(M_2)$$

which acts on objects by appending L_{12} to a generalized Lagrangian $\underline{L_1} \in \mathit{Fuk}^{\#}(M_1)$ and on morphism by counting quilted disks with two patches and boundary marked points.

In the context of the above theorem, a symplectic manifold is said to be *monotone* if $c_1(M) = \lambda \cdot [\omega]$ for some $\lambda \in \mathbb{R}^+$, known as the *monotonicity constant.* The brane structure allows for a richer collection of objects in the Fukaya category. The theorem is stated in terms of the *extended Fukaya categories* $\mathrm{Fuk}^{\#}(M_i)$. In the following, we briefly introduce these extended Fukaya categories. We refer the reader to [MWW18], chapter 4 for more details.

3.3.1 Quilted Floer cohomology and the extended Fukaya category

The definitions introduced in the following are analogous to the definitions made for the Lagrangian Floer chain complex.

Definition 3.14. A *generalized Lagrangian correspondence* from M_0 to M_1 is a sequence

$$\underline{L} := (M_0 = N_0 \overset{L_{01}}{\to} N_1 \overset{L_{12}}{\to} \cdots \overset{L_{(r-1)r}}{\to} N_r = M_1)$$

A *cyclic generalized Lagrangian correspondence* is a generalized correspondence with $M_0 = M_1$. A *generalized Lagrangian* is a generalized correspondence from *pt* to M_0. The *generalized intersection points* of a cyclic correspondence $\underline{L}$ are defined as

$$\mathcal{I}(\underline{L}) := \{(p_{01}, ..., p_{(r-1)r})) \in L_{01} \times \cdots \times L_{(r-1)r} \mid \pi_j(p_{(j-1)j})) = \pi_j(p_{j(j+1)}\}.$$

As we will see, we are primarily interested in cyclic Lagrangian correspondences. In the following, we set $L_{r(r+1)} = L_{01}$ and assume moreover that $\mathcal{I}(\underline{L})$ is cut out transversely. We now define the following moduli space of trajectories for points $p, q \in \mathcal{I}(\underline{L})$ and some width $\underline{\delta} \in (0, \infty)^r$.

$$\mathcal{M}_{\underline{\delta}}(\underline{p}, \underline{q}) := \{\underline{u} := (u_j : \mathbb{R} \times [0,1] \to N_j)_{1 \le j \le r} \mid (1), ..., (4)\}\big/\sim$$

The quotient indicates that we identify two *quilted Floer trajectories* if they differ by a translation in the s-coordinate. The four conditions in the definition are the following.

$$(1) \quad \frac{\partial u_j}{\partial s} + J_j(u_j)\frac{\partial u_j}{\partial t} = 0 \forall j, (2) \quad (u_j(s, \delta_j), u_{j+1}(s, 0)) \in L_{j(j+1)} \forall j, \forall s \in \mathbb{R}$$

$$(3) \quad E(\underline{u}) := \sum_{i=1}^{r} \int_{[0,\delta_j]} u_j^* \omega < \infty, \quad (4) \quad \lim_{s \to \pm\infty} u(s, \cdot) = p^{\mp} \forall j$$

It turns out that $\mathcal{M}_{\underline{\delta}}(p, q)$ is again a smooth manifold that admits a compactification in the same way as the moduli space of unquilted Floer trajectories. We view the maps $\underline{u}$ as quilted maps whose domains are quilted cylinders.

Definition 3.15. For every cyclic generalized correspondence $\underline{L}$ and tuple of width $\underline{\delta}$ we define the complex

$$CF_{\underline{\delta}}(\underline{L}) := (\mathbb{K} \langle \underline{p} \rangle_{\underline{p} \in \mathcal{I}(\underline{L})}, d_{\underline{\delta}}).$$

We then define *Quilted Floer cohomology* of $\underline{L}$ to be

$$HF^*(\underline{L}) := H_*(\mathcal{I}(\underline{L}), d_{\underline{\delta}})$$

where $d_{\underline{\delta}}$ is the differential resulting from counting quilted Floer trajectories.

Remark 3.16. Definition 3.15 generalizes the Floer chain complex defined in section 2.2. Given two Lagrangians L, K we can realize them as a cyclic correspondence $\underline{L}$ in the following way:

$$\begin{tikzcd} pt \arrow[r, bend left, "L"] & M \arrow[l, bend left, "K"] \end{tikzcd}$$

Then $CF_{\underline{\delta}}(\underline{L}) = CF(L, K)$.

With this, we can now define the extended Fukaya category. The exact construction can be found in [MWW18], chapter 4.

Definition 3.17. Let M be a monotone and compact symplectic manifold. The *extended Fukaya category* $\mathrm{Fuk}^{\#}(M)$ is the $\mathcal{A}_\infty$-category constructed as follows:

- The objects of $\mathrm{Fuk}^{\#}(M)$ are generalized Lagranians $pt \xrightarrow{\underline{L}} M$.
- For $\underline{L}, \underline{K} \in \mathrm{ob}(\mathrm{Fuk}^{\#}(M))$, the homomorphism class is defined by the quilted Floer chain complex $\mathrm{Hom}(\underline{L}, \underline{K}) := CF^*(\underline{L}, \underline{K})$
- The $\mathcal{A}_\infty$-operations

$$\mu^d : CF^*(\underline{L}_0, \underline{L}_1) \otimes \cdots \otimes CF^*(\underline{L}_{d-1}, \underline{L}_d) \to CF^*(\underline{L}_0, \underline{L}_d)$$

  defined by counting quilted disks, as depicted in Figure 5.

The key feature of these quilted disks is that the seams approach the output point transversely. This approach allows us to obtain strip-like coordinates near the output point.

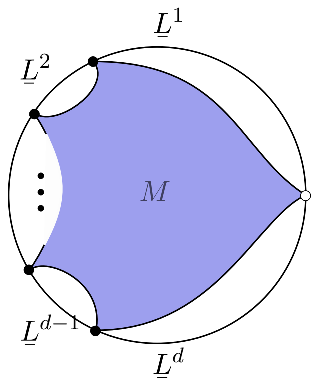


Figure 5: Quilted disk

3.3.2 Construction of Ma'u-Wehrheim-Woodward's functor

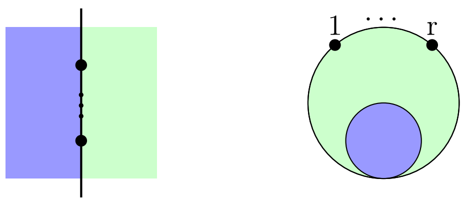
To construct Ma'u-Wehrheim-Woodward's $\mathcal{A}_\infty$-functor, we start with the definition on the level of objects. Using the notion of extended Fukaya categories there is an obvious definition:

$$\Phi^{\#}_{L_{12}}(pt = N_0 \xrightarrow{K_{01}} N_1 \xrightarrow{K_{12}} \cdots \xrightarrow{K_{(r-1)r}} N_r = M_1)$$
$$\mapsto (pt = N_0 \xrightarrow{K_{01}} N_1 \xrightarrow{K_{12}} \cdots \xrightarrow{K_{(r-1)r}} N_r = M_1 \xrightarrow{L_{12}} M_2)$$

We denote $\Phi^{\#}_{L_{12}}(\underline{L}_1) =: \underline{L}_1 \circ L_{12}$.

As before in this thesis, for the construction of the functor on the level of morphisms we want to use the operadic principle. To do so we first need to define a suitable operad.

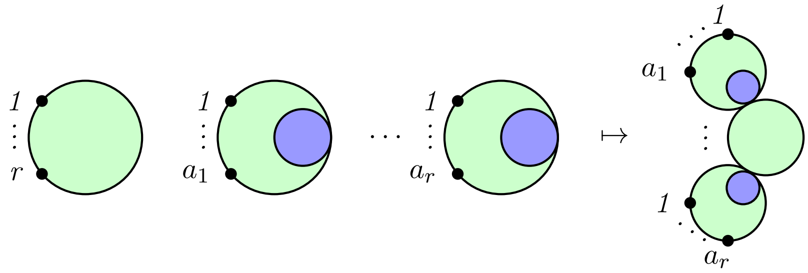
Definition 3.18. For $r \geq 1$, the *multiplihedron* J_r is defined as the moduli space of configurations of a vertical line in the half right plane $\mathbb{H}_0$ and r marked points on the imaginary axis. Alternatively, it can be visualized as a disk with r boundary marked points and an additional stratum.



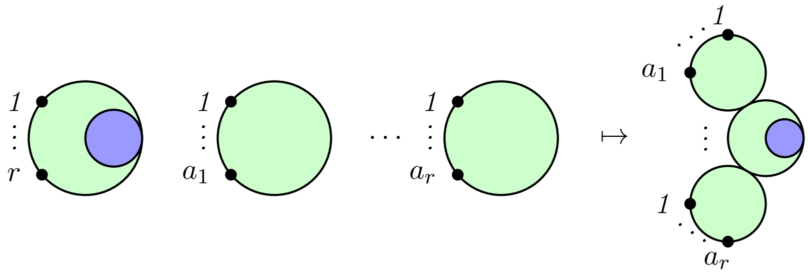
Remark 3.19. This mulitplehedron has a geometric realization as a $(r-1)$-dimensional manifold convex polytope, as shown in [LM23], chapter 1.

Lemma 3.20. *The multipehedra (J_r) form a bimodule structure over the operad (K_r) of associahedra with the following operations and certain compatibility conditions:*

(Left Module) *A map $K_r \times J_{a_1} \times \cdots \times J_{a_r} \to J_{a_1+\cdots+a_r}$*



(Right Module) *A map* $J_a \times K_{r_1} \times \cdots \times K_{r_a} \to J_{r_1+\cdots+r_a}$



Applying C_*^{cell} to J_r in the same way as to the associahedra, we see that $(C_*^{\text{cell}}(J_r))$ is a bimodule over $(C_*^{\text{cell}}(K_r))$. The central result we need for the construction of the functor is the following:

Proposition 3.21 ([AB22], chapter 3.4)**.** *An $\mathcal{A}_\infty$-functor is equivalent to a functor over $(C_*^{cell}(J_r))$. Such an $\mathcal{A}_\infty$-functor $F : C \to D$ consists of the following data:*

- *A map $F : ob(C) \to ob(D)$*
- *Maps of morphisms*

$$C_*^{cell}(J_r) \otimes Hom(O_{r-1}, O_r) \otimes \cdots \otimes \mathrm{Hom}(O_1, O_2) \to \mathrm{Hom}(O_1, O_r),$$

that satisfy the same relations as the one given by the composition maps in $(C_*^{cell}(J_r))$.

Remark 3.22. By analyzing the codimension 1-degenerations of the multiplehedra one can see that they provide the $\mathcal{A}_\infty$-relations of the extended Fukaya category. This analysis relies on the assumption of the monotonicity of M, which excludes so-called *figure eight bubbles* shown in Figure 6. Figure eight bubbling occurs from energy concentrating at the interior seam of the multiplehedra.

With this construction, we can now define the functor as follows:

Definition 3.23. Given a correspondence $L_{12} \subset M_1^- \times M_2$ we define the functor $\Phi_{L_{12}}^{\#} : \mathrm{Fuk}(M_1) \to \mathrm{Fuk}(M_2)$ as follows:

- On the level of objects $\Phi_{L_{12}}^{\#}$ acts by

$$\Phi_{L_{12}}^{\#}(pt = N_0 \overset{K_{01}}{\to} N_1 \overset{K_{12}}{\to} \cdots \overset{K_{(r-1)r}}{\to} N_r = M_1)$$
$$\mapsto (pt = N_0 \overset{K_{01}}{\to} N_1 \overset{K_{12}}{\to} \cdots \overset{K_{(r-1)r}}{\to} N_r = M_1 \overset{L_{12}}{\to} M_2)$$

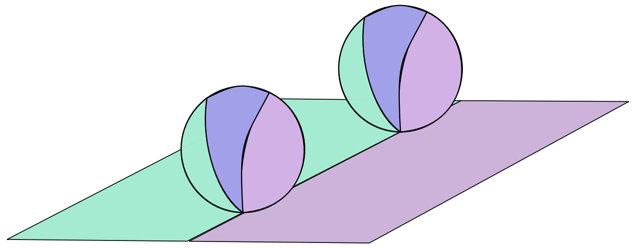


Figure 6: Figure eight bubbles

- On the level of morphisms we have a map

$$\Phi^{\#}_{L_{12}} : CF(\underline{L}_1^0, \underline{L}_1^1) \otimes \cdots \otimes CF(\underline{L}_1^{r-1}, \underline{L}_1^r) \to CF((\underline{L}_1^0, L_{12}), (\underline{L}_1^r, L_{12}))$$

  by counting pseudo holomorphic quilts of the form in Figure 7.

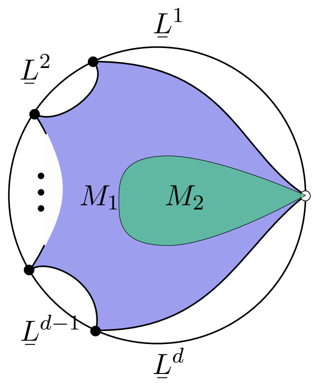


Figure 7: Quilts to define the $\mathcal{A}_\infty$-operations of Ma'u-Wehrheim-Woodward's functor.

Notice that, once again, the strata approach the output point transversely rather than tangentially.
Ma'u, Wehrheim, and Woodward continued with this procedure and constructed homotopies between two $\mathcal{A}_\infty$-functors in a similar way.

Theorem 3.24 (Ma'u, Wehrheim, Woodward, [MWW18], Theorem 1.2)**.** *Let M_0, M_1, M_2 be monotone symplectic manifolds with the same monotonicity constant. Let $L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2$, such that $L_{01} \circ L_{12}$ is smooth. Then there exists a homotopy of $\mathcal{A}_\infty-$functors*

$$\Phi^{\#}_{L_{01}} \circ \Phi^{\#}_{L_{12}} \simeq \Phi^{\#}_{L_{01} \circ L_{12}}$$

The idea here is to count maps whose domain is a quilted disk with two interior strata.
There is a more general category called the symplectic $(\mathcal{A}_\infty, 2)$-category. It incorporates all the operations constructed above and requires fewer assumptions. We will again follow the operadic principle to construct this category. This will lead us to the definition of the *2-associahedra*.

4 The symplectic category Symp

In the previous sections, we constructed an $\mathcal{A}_\infty$-functor between Fukaya categories and stated that one can define a homotopy between such $\mathcal{A}_\infty$-functors in a similar fashion. However, this approach excluded the consideration of figure eight bubbling. The natural follow-up question now is, if there is an algebraic structure that works in a more general setting and incorporates all the operations above. This is indeed possible and due to the work of Bottman and his collaborators, surveyed in [AB22].

Following the operadic principle as in the previous chapter, we start with the construction of an operad or in this case a *relative 2-operad*. This leads us to define `Symp` as an $(\mathcal{A}_\infty, 2)$-category. The construction of `Symp` is particularly valuable since it provides a chain-level form of the 2-category established by Wehrheim and Woodward in [WW10]. This chain-level form carries more information than the homology version, making it a powerful tool for the study of symplectic manifolds.

To define the relative 2-operad, we will take so-called *witch balls* which generalize the notion of the associahedra and the multiplehedra, to give us an overall composition function.

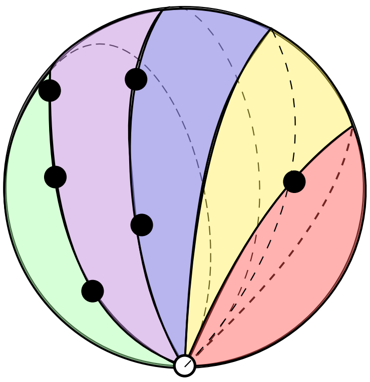


Figure 8: Witch ball

We can view a witch ball as describing r vertical lines in $\mathbb{R}^2$ with coordinates $x_1, ..., x_r$, along with n_i marked points on the i-th line with coordinates $(x_i, y_1), ..., (x_i, y_{n_i})$. By compactifying $\mathbb{R}^2$ to S^2 we get the view of a witch ball as in Figure 8.

$$2\mathcal{M}_{\mathbf{n}} := \left\{ \begin{array}{c|c} (x_1, ...x_r) \in \mathbb{R}^r & x_1 < \cdots < x_r \\ (y_{11}, ..., y_{1n_1}) \in \mathbb{R}^{n_1} & y_{11} < \cdots < y_{1n_1} \\ \vdots & \vdots \\ (y_{r1}, ..., y_{rn_r}) \in \mathbb{R}^{n_r} & y_{r1} < \cdots < y_{rn_r} \end{array} \right\} \Big/ \mathbb{R}^2 \rtimes \mathbb{R}_{>0}$$

The quotient can be regarded as the action of the group, generated by affine linear transformations of the plane and positive dilations. There is a compactification of this space that works, as before, by "zooming in" at points or seams that collide. It is called *2-associahedron* and denoted by $\overline{2\mathcal{M}_{\mathbf{n}}}$ if we refer to the stratified space or by $W_{\mathbf{n}}$ if we refer to the poset of strata. The definition of the allowed degeneration of the space will be made precise in the next chapter.

As shown in [BC21] the 2-associahedra $(\overline{2\mathcal{M}_{\mathbf{n}}})$ together with the forgetful maps $\pi : (\overline{2\mathcal{M}_{\mathbf{n}}}) \to (\overline{\mathcal{M}_r})$ and certain structure maps form a relative 2-operad to the associahedra $(\overline{\mathcal{M}_r})$.

4.1 Relative 2-operads

Relative 2-operads have been introduced by Bottman and Carmeli in [BC21]. The motivation behind this concept is to extend the associahedra by incorporating additional operations.

Definition 4.1 ([BC21], Def 2.3)**.** A *(nonsymmetric) relative 2-operad* in a category $\mathcal{C}$ with finite limits is a pair

$$\Big((P_r)_{r\geq 1}, (Q_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^r_{\geq 0}\backslash\{0\}, r\geq 1}\Big)$$

where (P_r) is a nonsymmetric operad in $\mathcal{C}$ and $(Q_{\mathbf{m}}) \subseteq \mathcal{C}$ is a collection of objects together with maps

$$\Gamma_{\mathbf{m},(\mathbf{n}_i^a)} : Q_{\mathbf{m}} \times \prod_{1\leq i\leq r} \prod_{1\leq a\leq m_i}^{P_{s_i}} Q_{\mathbf{n}_i^a} \to Q_{\sum_a \mathbf{n}_1^a,\dots,\sum_a \mathbf{n}_r^a}$$

where $r, s_1, ..., s_r \geq 1, \mathbf{m} \in \mathbb{Z}^r_{\geq 0}\backslash\{0\}, \mathbf{n}_i^a \in \mathbb{Z}^{s_i}_{\geq 0}\backslash\{0\}$. Here the P_{s_i} on top of the second product denotes the fiber product with respect to the projection maps. The maps have to satisfy the following assumptions:

(PROJECTIONS) $((P_r), (Q_{\mathbf{m}}))$ is equipped with projections

$$\pi_{\mathbf{m}} : Q_{\mathbf{m}} \to P_r, \quad r \geq 1, \mathbb{Z}^r_{\geq 0}\backslash\{0\}$$

that make the following diagram commute

$$\begin{tikzcd}
Q_{\mathbf{m}} \times \prod\limits_{1\leq i\leq r} \prod\limits_{1\leq a\leq m_i}^{P_{s_i}} Q^a_{\mathbf{n}_i} \arrow[r, "\Gamma_{\mathbf{m},(\mathbf{n}_i)^a}"] \arrow[d, "\left(\pi_{\mathbf{m}}, \prod\limits_{1\leq i\leq r} \pi\right)"'] & Q_{\sum_a \mathbf{n}_1^a,\dots,\sum_a \mathbf{n}_r^a} \arrow[d, "\pi_{\sum_a \mathbf{n}_1^a,\dots,\sum_a \mathbf{n}_r^a}"] \\
P_r \times \prod\limits_{1\leq i\leq r} P_{s_i} \arrow[r, "\gamma_{r,(s_i)}"'] & P_{\sum_i s_i}
\end{tikzcd}$$

(ASSOCIATIVE) The structure maps satisfy an associativity condition, similar to the one of normal operads.

(UNIT) There is a unit map $1 \to Q_{\mathbf{m}}$ satisfying the expected conditions.

Remark 4.2. Notice that the collection $C_*(Q_{\mathbf{m}}); R)$ does not define a relative 2-operad in the category of chain complexes.

To still get a definition of a category over a relative 2-operad where the 2-morphisms are chain complexes, we use the following definition.

Definition 4.3 ([BC21], Definition 3.7)**.** Let R be a ring. A *R-linear category* over a relative 2-operad $((P_r),(Q_{\mathbf{m}}))$ in **Top** consists of the following data:

- A category with objects Ob and morphisms Mor.
- For each pair of morphisms $L, K : M \to N$, a complex of free R-modules $2\mathrm{Mor}(L, K)$.
- Composition maps: For $r \geq 1$ and $\mathbf{m} \in \mathbb{Z}^r_{\geq 0}\backslash\{\mathbf{0}\}$ and sequence of objects $M_0, ..., M_r \in \mathrm{Ob}$, and for each collection of sequences $L_1^0, ..., L_1^{m_1}, ..., L_r^0, ..., L_r^{m_r}$ with L_i^j a morphism from M_{i-1} to M_i a composition map

$$C_*(Q_{\mathbf{m}}) \otimes \bigotimes_{\substack{1\leq i\leq r,\\ 1\leq j\leq m_i}} 2\mathrm{Mor}(L_i^{j-1}, L_i^j) \to 2\mathrm{Mor}(L_1^0 \circ \cdots \circ L_r^0, L_1^{m_1} \circ \cdots \circ L_r^{m_r})$$

where $C_*(Q_{\mathbf{m}})$ denotes the complex of singular chains in $Q_{\mathbf{m}}$ with coefficients in R.

Definition 4.4 ([BC21], Definition 3.9)**.** An *R-linear* $(\mathcal{A}_\infty, 2)$*-category* is an R-linear category over the relative 2-operad $((\overline{M_r}), (\overline{2M_{\mathbf{n}}}))$.

We can now define `Symp` as an $(\mathcal{A}_\infty, 2)$-category, with the structure of its operations coming from counting maps with domains in the 2-associahedra.

4.2 `Symp`

In the following, we want to use the structure that is given to us by the constructed relative 2-operad of 2-associahedra. The symplectic $(\mathcal{A}_\infty, 2)$-category incorporates all this structure.

Definition 4.5 ([AB22], Chapter 4)**.** The category `Symp` consists of the following data:

- The objects of `Symp` are symplectic manifolds (M, ω).
- The class $\mathrm{Hom}(M_0, M_1)$ is given by Lagrangian correspondences $M_0 \overset{L_{01}}{\to} M_1$ and are called the *1-morphisms.*
- For each pair of Lagrangian correspondences $M_0 \overset{L_{01}, L'_{01}}{\longrightarrow} M_1$ we have the class of 2-morphisms $CF^*(L_{01}, L'_{01})$.
- For each $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}^r_{\geq 0}\backslash\{\mathbf{0}\}$ and sequence $M_0, ..., M_r$ and collections

$$L_{01}^0, ..., L_{01}^{n_1} \subset M_0^- \times M_1, \quad ... \quad , L_{(r-1)r}^0, ..., L_{(r-1)r}^{n_r} \subset M_{r-1}^- \times M_r$$

we have a compositions map

$$2c_{\mathbf{m}} : C_*^{\mathrm{Sing}}(\overline{2M_{\mathbf{n}}}) \otimes \bigotimes_{\substack{1\leq i\leq r,\\ 1\leq j\leq m_i}} CF^*(L_{(i-1)i}^{j-1}, L_{(i-1)i}^{j-1})$$
$$\to CF^*(L_{01}^0 \circ \cdots \circ L_{(r-1)r}^0, L_{01}^{m_1} \circ \cdots \circ L_{(r-1)r}^{m_r})$$

given by counting maps of the form as in Figure 9.

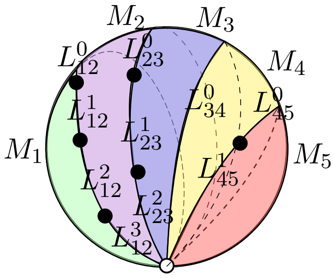


Figure 9: Example of the domain of maps to count for the composition maps.

Remark 4.6. The definition of `Symp` has not been completed until now. The major step that is still missing is the construction of a regularization theory of moduli spaces of witch balls.

5 Combinatorics of the 2-associahedra

In this section, we will define a combinatorial representation of the 2-associahedra. This allows us to see some combinatorial properties which will be useful to calculate the poset structure of the 2-associahedra. In addition, it will make the definition of the compactification of the space $2\mathcal{M}_{\mathbf{n}}$ precise. To do so we start with the analysis of the associahedra.

5.1 Combinatorics of the associahedra

So far we have viewed the associahedra as a poset of strata of the space of disks with boundary marked points. To find a suitable representation of the associahedra, we introduce the concept of *1-brackets* and *1-bracketings*.

Definition 5.1 ([Bot19], Definition 2.11)**.** A *1-bracket* of $r \geq 2$ is a subset $B \subseteq \{1, ..., r\}$. A *1-bracketing* $\mathscr{B}$ is a set of 1-brackets with the following properties:

(BRACKETING) If $B, B' \in \mathscr{B}$ with $B \cap B' \neq \emptyset$ then $B \subseteq B'$ or $B' \subseteq B$.

(ROOT AND LEAVES) $\mathscr{B}$ contains $\{1, ..., r\}$ and $\{i\}$ for every $i \in \{1, ..., r\}$.

We define the collection K_r^{br} to be the set of all 1-bracketings of r and equip it with a poset structure by declaring $\mathscr{B}' < \mathscr{B}$ if $\mathscr{B} \subset \mathscr{B}'$. We denote $K_r := K_r^{\mathrm{br}}$.

Remark 5.2. To be precise, one needs to define the combinatorial version of the associahedra before defining the topological one.

Another combinatorial representation of the associahedra, denoted by K_r^{tree}, is provided by Bottman in [Bot19]. This representation uses rooted ribbon trees. However, for our purposes, the representation using rooted ribbon trees is not as useful as the one described above.

Proposition 5.3 ([Bot19], Proposition 2.14)**.** *The following are some key combinatorial properties of the 2-associahedra.*

(ABSTRACT POLYTOPE) *For $r \geq 2, K_r \cup F_{-1}$ is an abstract polytope of dimension $r-2$. Here F_{-1} denotes the unique smallest element which we simply add to K_r.*

(RECURSIVE) *For every $\mathscr{B} \in K_r^{br}$ there is an inclusion of posets:*

$$\prod_{B \in \mathscr{B}} K_{\#in(B)}^{br} \hookrightarrow K_r^{br}$$

which restrict to a poset isomorphism onto $cl(\mathscr{B}) := (F_{-1}, \mathscr{B}]$.

The (RECURSIVE) property endows the operadic structure of the associahedra. To facilitate later discussions, we introduce some technical definitions.

Definition 5.4. Let $\mathscr{B}$ be a 1-bracketing in K_r^{br}. For $B \in \mathscr{B}$ we define $in(B)$ to be the set

$$in(B) := \{B' \in \mathscr{B} \mid B' \subsetneq B, \nexists B'' : B' \subsetneq B'' \subsetneq B\}$$

Now, we define *moves* on a face of a 2-associahedron, which correspond to codimension 1 degenerations of this face. For this, we can identify a 1-bracketing naturally with making brackets around r elements. From this point of view, a move on the face looks as follows:

$$((...), ..., (...), ..., (...), ..., (...)) \mapsto ((...), ..., ((...), ..., (...)), ..., (...))$$

The definition of a move will be used in the analysis of the computer package.

5.2 2-associahedra

We now seek a representation of the 2-associahedra similar to that of the associahedra. For this purpose, we build upon the concept of a 1-bracketing. Throughout, let $\mathbf{n} \in \mathbb{Z}^r_{\geq 0} \backslash \{\mathbf{0}\}$, where $\mathbf{n} = (n_1, ..., n_r)$ and $r \in \mathbb{Z}_{\geq 1}$.

Definition 5.5 ([Bot19], Definition 3.11)**.** A *2-bracket* of $\mathbf{n}$ is a pair $\mathbf{2B} = (B, (2B_i))$ consisting of a 1-bracket $B \subseteq \{1, ..., r\}$ and subsets $2B_i \subseteq \{1, ..., n_i\}$ for every $i \in B$ such that at least one $2B_i$ is not empty.
We write $\mathbf{2B'} \subset \mathbf{2B}$ if $B' \subset B$ and $2B'_i \subset 2B_i$ for every $i \in B'$. Furthermore we define $\pi(\mathbf{2B}) = \pi(B, (2B_i)) := B$.

Definition 5.6 ([Bot19], Definition 3.12)**.** A *2-bracketing* of $\mathbf{n}$ is a pair $(\mathscr{B}, 2\mathscr{B})$ where $\mathscr{B}$ is a 1-bracketing of r and $2\mathscr{B}$ is a collection of 2-brackets of $\mathbf{n}$ satisfying following properties:

(1-BRACKETING) For every $\mathbf{2B} \in 2\mathscr{B}, \pi(\mathbf{2B})$ is contained in $\mathscr{B}$.

(2-BRACKETING) $\mathbf{2B}, \mathbf{2B'} \in 2\mathscr{B}$ and $2B_{i_0} \cap 2B'_{i_0} \neq \emptyset$. Then either $\mathbf{2B} \subset \mathbf{2B'}$ or $\mathbf{2B'} \subset \mathbf{2B}$.

(ROOT AND MARKED POINTS) $2\mathscr{B}$ contains $(\{1, ..., r\}, (\{1, ..., n_1\}, ..., \{1, ..., n_r\})$ and every 2-bracket of the form $(i, \{j\})$.

For any $B_0 \in \mathscr{B}$ write $2\mathscr{B}_{B_0} := \{(B, (2B)_i) \in 2\mathscr{B} \mid B = B_0\}$

(MARKED SEAMS ARE UNFUSED)

- For any $B_0 \in \mathscr{B}, i \in B_0$, we have $\bigcup_{\mathbf{2B} \in 2\mathscr{B}_{B_0}} 2B_i = \{1, ..., n_i\}$
- For every $\mathbf{2B} \in 2\mathscr{B}_{B_0}$ for which there exists a $\mathbf{2B'} \subsetneq \mathbf{2B}$, $i \in B_0, j \in 2B_i$, there exists $\mathbf{2B''} \subsetneq \mathbf{2B}$ with $2B''_i \ni j$.

(PARTIAL ORDER) For every $B_0 \in \mathscr{B}, 2\mathscr{B}_{B_0}$ has a partial order with the following properties:

- $\mathbf{2B}, \mathbf{2B'} \in 2\mathscr{B}_{B_0}$ are comparable if and only if $2B_i \cap 2B'_i = \emptyset$.
- For any i and $j < j'$, we have $(\{i\}, (\{j\})) < (\{i\}, (\{j'\}))$.
- For any 2-brackets $\mathbf{2B^j} \in 2\mathscr{B}_{B_0}, \tilde{\mathbf{2B^j}} \in 2\mathscr{B}_{\tilde{B}_0}$ with $\tilde{\mathbf{2B^j}} \subset \mathbf{2B^j}$ we have:

$$\mathbf{2B^1} < \mathbf{2B^2} \Rightarrow \tilde{\mathbf{2B^1}} < \tilde{\mathbf{2B^2}}.$$

We define $W_{\mathbf{n}}^{br}$ to be the set of 2-bracketings of $\mathbf{n}$, with the poset structure defined by declaring $(\mathscr{B}', 2\mathscr{B}') < (\mathscr{B}, 2\mathscr{B})$ if the containments $\mathscr{B} \subset \mathscr{B}', 2\mathscr{B} \subset 2\mathscr{B}'$ hold and at least one of the containments is proper.

Remark 5.7. It is convenient to depict 2-bracketings in the format illustrated in Figure 10. The 1-brackets are shown in the bottom row and the 2-brackets are shown above their respective 1-brackets. The partial order is given by the ordering according to their height. One doesn't depict the 1- & 2-brackets in (ROOTS AND LEAVES).

As for the associahedra, there is another combinatorial representation $W_{\mathbf{n}}^{\text{tree}}$ of the 2-associahedra, provided by Bottman in [Bot19]. This representation utilizes stable tree pairs.

Theorem 5.8 ([Bot19], Theorem 4.1)**.** *For any $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}_{\geq 0}^{r} \backslash \{0\}$, the 2-associahedron $W_{\mathbf{n}}$ is a poset, with the following properties:*

(ABSTRACT POLYTOPE) *For $\mathbf{n} \neq \mathbf{1}, \widehat{W_{\mathbf{n}}} := W_{\mathbf{n}} \cup F_{-1}$ is an abstract polytope of dimension $|n| + r - 3$.*

(FORGETFUL) *$W_{\mathbf{n}}$ is equipped with forgetful maps $\pi : W_{\mathbf{n}} \to K_r$ which are surjective maps of posets.*

(RECURSIVE) *Each closed face of $W_{\mathbf{n}}$ decomposes in a canonical way as a product of fiber products of lower-dimensional 2-associahedra, where the fiber products are with respect to the forgetful maps π.*

The fact that each face of the 2-associahedra decomposes into a product of fiber product and not a product of products will play a very important role in the construction of the computer package.
We will now again define moves on the 2-associahedra which correspond precisely to the codimension 1 degenerations of the face. These moves fall into three distinct types:

Type-1 move: Fix $B \in \mathscr{B} \in W_{\mathbf{n}}$. Let $2B \in 2\mathscr{B}_B$. Let $in(2B) = (2B_1, ..., 2B_k)$ ($in(2B) \subseteq 2\mathscr{B}_B$ is just defined as for 1-brackets in). A type 1-move consists of adding a new 2-bracket of the form

$$\mathbf{2B}_{\text{new}} = (B, \bigcup_{n \leq i \leq m} 2B_i)$$

to $2\mathscr{B}_B$, where $n, m \in \{1, ..., k\}$ and the union is not empty.

Type-2 move: Fix $B \notin \mathscr{B} \in W_{\mathbf{n}}$, but B comes from a degeneration in $\mathscr{B}$. Let

$$S_B = \bigcup_{B' \in in(B)} 2\mathscr{B}_{B'} = \{\mathbf{2B_1}, ..., \mathbf{2B_k}\}$$

where $in(B)$ is defined as one would expect, eventhough $B \notin \mathscr{B}$. A type-2 move consists of adding 2-brackets $(B, 2B)_i = \bigcup_{j \in J_i} \mathbf{2B_j}$ to the 2-bracketing, such that the partial order is preserved, $2B_i \cap 2B_j = \emptyset, \forall i \neq j$ and $\bigcup_i J_i = \{1, ..., k\}$.

Type-3-move: Let $\mathbf{2B} \in 2\mathscr{B}_B$ be minimal, where $B \in \mathscr{B}$ is not a singleton. Let

$$S_{\mathbf{2B}} = \bigcup_{\mathbf{2B}' \in in(\mathbf{2B})} 2B'$$

A type-3 move consists of adding 2-brackets $(B, 2B)_i$ to $2\mathscr{B}_B$, consisting of the union of $2B'$s in the union above, such that the partial order is preserved, $2B_i \cap 2B_j = \emptyset, \forall i \neq j$ and $\bigcup_i 2B_i = S_{\mathbf{2B}}$.

There is a major difference between type-1&3 and type-2 moves, being that we only change $\mathscr{B}$ in type-2 moves. This is important to construct the computer package.

Example 5.9. Different types of moves in W_{40}

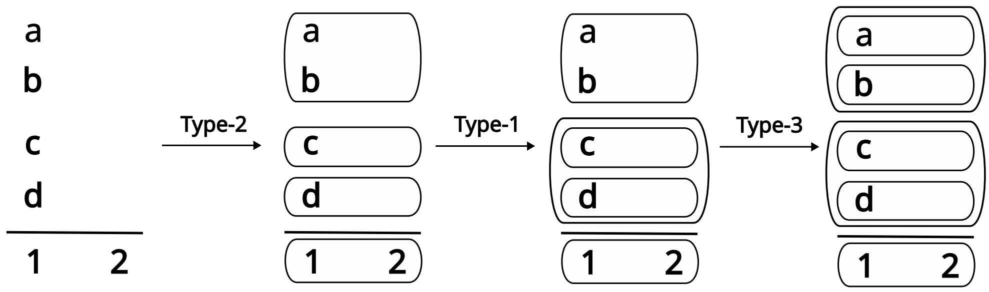


Figure 10: Example of moves on faces of W_{40}

Definition 5.10. Let $F \in W_{\mathbf{n}}$. We call $G \in W_{\mathbf{n}}$ a *child* of F if G is the result of making a move on F.

6 Building the computer package

In this section, we are going to develop a computer package capable of computing the poset structure of the 2-associahedra. This package will enable computations and analysis of the 2-associahedra, which will be valuable for studying and understanding its combinatorial properties. In Chapter 7 we will calculate one specific combinatorial property called the *CD-Index*.

Before constructing the computer package for the 2-associahedra, we will first build a similar package for the associahedra. This initial step will serve as a foundation since the basic ideas behind the packages are pretty similar.

6.1 Associahedra

The idea behind the construction algorithm is to start with the codimension 0 face of K_r. To build the entire poset structure, we recursively calculate all the children and store them in a dictionary. This storage mechanism allows for efficient retrieval and avoids redundant calculations by checking if a particular child has already been computed. Additionally, a list of children is maintained for each node, representing the poset structure.
To calculate the poset structure of the associahedra we are going to use the representation K_r^{br} of the associahedra. In this way, there is an easy way to store the data of a face. We will use the following methods to build the poset structure.

Methods for building associahedra		
Name	Symbol	Description
Build	B_A	Build the poset structure of K_r
Make all children	C_A	Makes all children of a given face of the poset

6.1.1 Storage

To store a face f of K_r^{br}, which is represented as a 1-bracketing $\mathscr{B}$ of r, we use a list of lists approach. Specifically, we create a list that contains lists representing the individual 1-brackets B in $\mathscr{B}$. This is possible due to (BRACKETING). Each B is enclosed in a list unless it is a singleton, or if B equals $\{1, 2, \ldots, r\}$, as these two cases are common to all bracketings due to (ROOTS AND LEAVES).

Example 6.1. Let $r = 5$ and $\mathscr{B} = \{\{1,2,3\},\{1,2\},\{4,5\}\}$. Then the corresponding list representation is $[[[1,2],3],[4,5]]$.

In the context of this work, we will use the list representations to identify and store the faces of K_r, along with their respective codimensions. For the storage of the entire associahedra, we use dictionaries that organizes all the faces based on their codimensions. Each face in the dictionary stores all its children.

6.1.2 Building the poset

The algorithm for calculating the poset structure of an associahedron is provided in Algorithm 1.

The algorithm begins by adding the codimension 0 face to the dictionaries of calculated faces. It then recursively calculates the children of a given face using the function

---

Algorithm 1 Building associahedra: B_A

```
Input
    r ∈ ℤ
Output
    P    Poset structure of K_r
F_r = [{}, ..., {}]                                   ▷ Storage for faces
Queue Q ← codim 0 face of K_r
while not Q.empty() do
    f ← Q.pop()
    C_f ← C_A(f)
    for c ∈ C_f do
        if s(c) in F_r[codim(f) + 1] then
            f.children ← c
        else
            F_r[codim(f) + 1] ← c
            f.children ← c
            Q.put(c)
```

C_A (Algorithm 2). In the following, it checks if the calculated children are already in the dictionaries $\mathcal{F}_r$ of faces. If not we can add them. The way it checks whether a face is already in a dictionary is to associate to the face f a key in the dictionary. For this, we take the string of the list representation $s(f)$. The reason why this works here is, that the string we use as a key is unambiguous, i.e. no other faces can have this string. The average runtime for dictionary inserts and lookups are $\mathcal{O}(1)$. The time to obtain the string representation is $O(r)$.

6.1.3 Calculation of children

To get all the children of a given face f, we need to find all the possible 1-brackets, that can be added to f. The algorithm for this is provided in Algorithm 2.

Algorithm 2 Make all children: C_A

```
Input
    f    face of K_r
Output
    C_f  List of children of the face
P ← (f choose 2) \ {(f[0], f[−1])}
for (i, j) ∈ P do
    C_f ← f ∪ (i, j)
for g ∈ f do
    if g is a list then
        C_f ← C_A(g)
```

Lemma 6.2. *Algorithm 2 calculates all the children of a given face in runtime* $\mathcal{O}(r^3)$.

Proof. The algorithm works as follows: First, it makes brackets, i.e. lists, by combining elements of f itself. These can be lists or integers. We denote by $\binom{f}{2}$ all the possibilities to select 2 elements in f and by $f \cup (i, j)$ that we add a 1-bracket around the elements between i and j. After that, the algorithm recursively calls C_A on the lists in f to make brackets inside of them.
Since we only make new 1-brackets by combining consecutive 1-brackets in $in(B)$, for some 1-bracket B of the face, (BRACKETING) is satisfied, and thus we indeed produce children. We also go over all possible combinations of how one can make new 1-brackets, which is why we produce all children.
To prove the stated running time, we first take a look at how many children we are producing. We can show that we produce less than $\binom{r}{2}$ children by making induction over the codimension. For codimension 0, we have that there are exactly $\binom{r}{2} - 1$ many children because there are no brackets inside yet. So let's assume that the codimension is greater than 0. Let $f = [B_1, ..., B_k]$. By induction we know, that for each B_i we produce less than $\binom{r_i}{2}$ children if r_i is the length of B_i. Together with the children in the first part of the algorithm, we get

$$\sum_{i=1}^{k} \binom{r_i}{2} + \binom{k}{2} \leq \binom{r}{2}.$$

So we have $\mathcal{O}(\binom{r}{2}) = \mathcal{O}(r^2)$ loop iterations. In each loop, we make a new child by adding a bracket to the previous face. Since making the new bracket is in $\mathcal{O}(r)$ this shows the overall run time is $\mathcal{O}(r^3)$. As one can see it is not possible to improve the runtime a lot since the storage of a face can not be made with capacity $\leq r$ and we have to go over all children of the face. □

6.1.4 Overall runtime

Theorem 6.3. *The algorithms above calculate the poset structure of K_r in $\mathcal{O}(6^r r^3)$.*

Proof. Based on the previous runtime analysis in lemma 6.2, we observe that, given a face, we require $\mathcal{O}(r^2(r + r)) = \mathcal{O}(r^3)$ time to calculate all its children and insert them into the dictionary of faces. Since we need to perform this process for every face once, the total runtime becomes $\mathcal{O}(\#K_r \cdot r^3)$, where $\#K_r$ represents the number of faces in the associahedron K_r. The number of faces of K_r is given by the *Schröder–Hipparchus numbers* or *small Schröder numbers* s_r. They satisfy the recursion

$$s_n = \frac{1}{n}((6n - 9)s_{n-1} - (n - 3)s_{n-2}), \quad s_1 = s_2 = 1.$$

It is straightforward to see that this sequence has a growth rate of $\mathcal{O}(6^r)$. This proves the stated runtime. Moreover [Au19], Proposition 4 provides an asymptotic growth of

$$s_n \sim Cn^{-\frac{3}{2}}(3 + 2\sqrt{2})^n$$

where $C \in \mathbb{R}$ is a constant. This shows that the given bound can not be improved by a lot. □

In the appendix, we provide a list of calculated f-vectors of the associahedra, along with their corresponding runtimes.

6.2 2-Associahedra

The approach to calculating the poset structure of the 2-associahedra follows a similar idea as the one used for the associahedra. We begin with the root, i.e. the codimension 0 face, and recursively calculate all the children of a given face.
Let's provide an overview of all the methods used for building the 2-associahedra and their corresponding notations.

Methods for building 2-associahedra		
Name	Symbol	Description
Build	B_{2A}	Builds the poset structure of $W_{\mathbf{n}}$
Make all children	C_{2A}	Makes all children of a given face of the poset
Make 2-br over 1-br	C_{B_0}	Makes possible 2-bracketings in $\pi^{-1}(B_0), B_0 \in \mathscr{B}$
Partition	$\mathcal{P}$	Creates possible partitions of 2-brackets for new ones

6.2.1 Storage

Also here, we use the combinatorial representation $W_{\mathbf{n}}^{\mathrm{br}}$ of the 2-associahedra. A face of the 2-associahedra has the form $(\mathscr{B}, 2\mathscr{B})$. We store the 1-bracketing $\mathscr{B}$ in the same way as for the associahedra. However, for the 2-bracketing we use an additional dictionary to store all the 2-brackets over some 1-bracket. Denote by d a dictionary, then

$$d(B) \leftarrow \pi^{-1}(B) \subseteq 2\mathscr{B}.$$

However, we do not simply store the 2-brackets over a specific 1-bracket as a list. Each time we create a new 2-bracket out of smaller 2-brackets $[B_1, ..., B_k]$ we add a copy of this list to the dictionary, where we replace the brackets that are in the new 2-bracket with the new 2-bracket. After this, we sort them after their length. We will see later why we do that.

Example 6.4. Let's consider the faces of W_{300} in Figure 11 and see how they are stored:

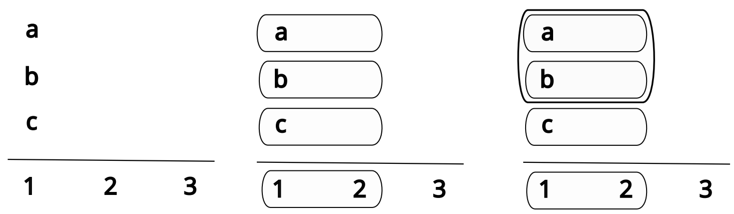


Figure 11: Faces of W_{300}

They are stored in the following way (from left to right):

- $\{'[1]' : [[[a], [b], [c]]], '[2]' : [[]], '[3]' : [[]]\}$
- $\{'[1]' : [[[a], [b], [c]]], '[2]' : [[]], '[3]' : [[]], '[1, 2]' : [[[a], [b], [c]]]\}$
- $\{'[1]' : [[[a], [b], [c)]]], '[2]' : [[]], '[3]' : [[]], '[1, 2]' : [[[a, b], [c]], [[a], [b], [c]]]\}$

As with the associahedra, we will have dictionaries of all the faces of the 2-associahedra, sorted by their codimension, for the overall storage. The children of each face will also be stored within their respective faces.

6.2.2 Building the poset

The algorithm to build the 2-associahedra is quite similar to the one for associahedra and is described in Algorithm 3.

Algorithm 3 Building 2-associahedra: B_{2A}

Input
  $\mathbf{n} = (n_1, ...n_r)$
Output
  P Poset structure of $W_{\mathbf{n}}$
$\mathcal{F}_{\mathbf{n}} = [\{\}, ..., \{\}]$
Queue $Q \leftarrow$ codim 0 face of $W_{\mathbf{n}}$
while not Q.empty() **do**
  $F \leftarrow Q$.pop()
  $C_F \leftarrow C_{2A}(F)$
  for $c \in C_F$ **do**
    if $s(c)$ in $\mathcal{F}_{\mathbf{n}}[\mathrm{codim}(F) + 1]$ **then**
      F.children $\leftarrow c$
    else
      $\mathcal{F}_{\mathbf{n}}[\mathrm{codim}(F) + 1] \leftarrow c$
      F.children $\leftarrow c$
      Q.put(c)

The main difference is, that we need another way to associate a key to a face of the 2-associahedra which is unique. Another issue arises due to the storage method for faces, which stores each collision separately. Therefore the way to store one face of the 2-associahedra is ambiguous. For instance, consider the following two dictionaries representing the same face of W_{32}:

a) $\{'[0]' : [[[(0,0)], [(0,1)], [(0,2)]]], '[1]' : [[[(1,0)], [(1,1)]]],$
$'[0,1]' : [[[(0,0),(0,1),(1,0)], [(0,2),(1,1)]], [[(0,0),(1,0)], [(0,1)], [(0,2),(1,1)]],$
$[[(1,0)], [(0,0)], [(0,1)], [(0,2)], [(1,1)]]]\}$
b) $\{'[0]' : [[[(0,0)], [(0,1)], [(0,2)]]], '[1]' : [[[(1,0)], [(1,1)]]],$
$'[0,1]' : [[[(0,0),(0,1),(1,0)], [(0,2),(1,1)]], [[(0,0),(1,0)], [(0,1)], [(0,2)], [(1,1)]],$
$[[(1,0)], [(0,0)], [(0,1)], [(0,2)], [(1,1)]]]\}$

Both dictionaries represent the same face of W_{32}, but due to the different chains of degenerations, the face is stored in two different ways. Therefore we need to find a string representation of a face that is independent of the specific way of storage. To achieve this, we define the function

$$\beta : d(B_0) \to \mathscr{B}_{B_0}$$

which extracts all the 2-brackets from the separated lists of 2-brackets resulting from collisions (We will also write $\beta(f) := \mathscr{B}_f$ for a face f of an associahedron). The function can be implemented by putting all the 2-brackets out of $d(B_0)$ in a set. Doing this, every 2-bracket is only left once. We then define the string representation of a face $F \in W_{\mathbf{n}}$

to be

$$s(F) := s(\pi(F)) + \sum_{k \in \text{sorted(d)}} (\beta(d(k)) + d(k)[-1]).$$

where the sum is over all keys in the dictionary, sorted with the Unicode table, and "adding" to strings is concatenating them.

Lemma 6.5. *The string representation of a face F of the 2-associahedra is unique and independent of the specific way of storage.*

Proof. Let F, F' be two calculated faces that represent the same face, but which are stored differently. We first notice that $\pi(F) = \pi(F') = \mathscr{B}$. Also since they represent the same face, they have the same 2-brackets which implies $\beta(d(B)) = \beta(d'(B))$ for every $B \in \mathscr{B}$. The last thing though is to check whether $d(B)[-1] = d'(B)[-1]$. This is due to the following reason: The 2-brackets we store in the list on the right are the ones, which have no other 2-brackets as a subset. This is due to sorting the lists in $d(k)$ after their length and how Algorithm 5 works (Lemma 6.8). If now $d(B)[-1] \neq d'(B)[-1]$, one of them would have a smaller 2-bracket in it than the other one. However, since they both have the same set of 2-brackets, this leads to a contradiction. $\square$

This approach guarantees a reliable key association for each face of the 2-associahedra, which we use to store the faces in a dictionary. Creating this string representation $s(F)$ takes time $\mathcal{O}(\dim(F) \cdot |\mathbf{n}|)$ since the size of one collision is bounded by $\mathcal{O}(|\mathbf{n}|)$. Here $|\mathbf{n}| = n_1 + \cdots + n_r$.

6.2.3 Calculation of children

The Algorithm that calculates all possible collisions of a face is given in Algorithm 4.

Algorithm 4 Make all children: C_{2A}

Input
F Face of the 2-associahedra
Output
$\mathcal{C}_F$ List of all children of F
$F = (\mathscr{B}, 2\mathscr{B})$
$\mathcal{B} \leftarrow \bigcup_{c \in C_A(\pi(F))} \beta(c)$
for $B_0 \in \mathcal{B}$ **do**
$\mathcal{C}_F \leftarrow C_B(F, B_0)$

In this algorithm, we use a subroutine C_{B_0}, which generates all the children of the node F resulting from a collision over a specific 1-bracket B_0. We see, that if C_{B_0} calculates all the right children over B_0, we will obtain all the children of the face F.

This subroutine is probably the greatest difference to the calculation of the associahedra and is what makes the calculation much harder. The algorithm C_{B_0} is given in Algorithm 5.

Theorem 6.6. *Algorithm 5 works correctly. This means for an input (F, B_0) it produces all correct children of F over the 1-bracket B_0.*

Algorithm 5 Make 2-bracketings over 1-bracket: C_{B_0}

Input

F Poset Node, $F = (\mathscr{B}_F, 2\mathscr{B}_F)$

B_0 1-bracket of the Node F

Output

$\mathcal{C}_F^{B_0}$ Children of F coming from a collision over B_0

Step 1: Collect all relevant marked points and 2-brackets.

$B_{B_0} \leftarrow \min\{BB \in \mathscr{B}_F \mid B_0 \subsetneq BB\}$

$B_{B_0}M \leftarrow d(BB_0)[-1]$

$C_{B_0}M \leftarrow d(B_0),$ ▷ $C_{B_0}M = [CB_1, ..., CB_k]$

$S_{B_0}M \leftarrow \bigcup_{b \in in(B_0)} d(b)[0]$

$M \leftarrow [BB_0M, CB_1, ..., CB_k, SB_0M]$

Step 2: Make all possible collisions

for $i = 0, ..., \#M - 2$ **do**

  for Brackets $2B \in M[i]$ **do**

    Collect Brackets $[2b_1, ..., 2b_k]$ in $M[i+1]$, that are in $2B$

    $BP[i] \leftarrow \mathcal{P}([2b_1, ..., 2b_k])$

if $CB_0M \neq \emptyset$ **then** ▷ type 1&3 moves

  for $i = 0, ..., \#M - 2$ **do**

    for $L2b \in \text{flatten}BP[i])$ **do** ▷ List of 2-brackets

      if $i < \#M - 2$ AND there is more then 1 new 2-bracket **then**

        Continue ▷ For type-1 moves

      $P \leftarrow (M[i] \backslash \{2B = \bigcup L2b\}) \cup L2b$

else ▷ type 2 moves

  for $b_{\mathcal{C}} \in BP$ **do**

    for Tuples $(b_1, ..., b_k) \in b_{\mathcal{C}}$ **do**

      $P \leftarrow (b_1, ..., b_k)$

Step 3: Make Poset Nodes

for $C \in P$ **do**

  if $B_0 \notin \mathscr{B}_F$ **then**

    $\mathscr{B}_C \leftarrow \mathscr{B}_F \cup B_0$

  $2\mathscr{B}_C \leftarrow 2\mathscr{B}_F \cup C$

  $\mathcal{C}_F^{B_0} \leftarrow (\mathscr{B}_C, 2\mathscr{B}_C)$

Proof. Let's first describe how Algorithm 5 works. The algorithm takes a 2-associahedra face F and a 1-bracket B_0 as input. First, it collects all the 2-brackets that are above B_0 and stores them in $C_{B_0}M$. Additionally, it takes the smallest 2-brackets $d(B_{B_0})[-1]$, above the smallest 1-bracket B_{B_0} that contains B_0 and the greatest 2-brackets $d(b)[0]$ over all the greatest 1-brackets b that are contained in B_0, i.e. $b \in in(B_0)$.
Now the idea is the following. The algorithm combines these lists in a specific order, starting with $B_{B_0}M$, followed by all lists in $C_{B_0}M$, and then $S_{B_0}M$. Crucially, it ensures that if we consider two consecutive lists L and R in this order, the smaller list (the right list R) exactly contains the biggest 2-brackets which are contained in the 2-brackets of the bigger list (the left list L). This gives us the following important property:

$$\big(B \in L, B' \in R : B' \subseteq B\big) \Rightarrow \Big(\nexists \tilde{B} \in \mathscr{B} : B' \subsetneq \tilde{B} \subsetneq B\Big)$$

Using this fact, we can take for every 2-bracket $2B \in L$ all the 2-brackets in R, that are subsets of $2B$ and let them collide. The property guarantees that any 2-bracket constructed from L and R will be valid and not contain any overlapping brackets. This is done at the beginning of step 2. The algorithm in the following checks if it needs to do a type-1 or -3 or a type-2 move. This can be seen easily by checking if there are any 2-brackets above B_0. Afterwards, it generates 2-brackets depending on the type of move.
In step 3 the algorithm produces the children by adding 2-brackets to the input face.
In the following, we prove that the algorithm works correctly.
Step 1: The children that are produced are correct.
To show this, we check that the created children satisfy the axioms of a valid 2−Bracketing.

- (1-BRACKETING) We only need to check, whether $B_0 \in \mathscr{B}_c$ for every $c \in \mathcal{C}_F^{B_0}$. If $B_0 \in \mathscr{B}_F$ we are done. Otherwise we get that $C_{B_0}M = \emptyset$. For this reason, we make a type-2 move in the algorithm. Making a type-2 move we add the bracket B_0 to $\mathscr{B}_F$, so $B_0 \in \mathscr{B}_c$.

- (2-BRACKETING) The algorithm guarantees that the resulting faces satisfy the 2-bracketing property because, as explained above, all the collisions we make in this algorithm are partitions of previous brackets, that are contained in one other 2-bracket. By induction the axiom is satisfied.

- (ROOT AND MARKED POINTS) We use here, that for all $c \in \mathcal{C}_F^{B_0}$ we have, that
$$(\mathscr{B}_F, 2\mathscr{B}_F) \subsetneq (\mathscr{B}_c, 2\mathscr{B}_c)$$
Now since
$$(\{1, ..., r\}, (\{1, ..., n_1\}, ..., \{1, ..., n_r\})) \cup \bigcup_{\substack{i \in \{1,...,r\} \\ j \in \{1,...,n_i\}}} (\{i\}, \{j\}) \subseteq (\mathscr{B}_{\text{root}}, 2\mathscr{B}_{\text{root}})$$
the axiom is satisfied.

- (MARKED SEAMS ARE UNFUSED) Let $B_0 \in \mathscr{B}_c$. The only way, that we added B_0 to $\mathscr{B}_c$ is, that we made a type-2 move. Since we pick tuples $(b_1, ..., b_k) \in b_{\mathcal{C}}$ we have, that
$$\forall p \in M_{B_0}, \exists 2B \in 2\mathscr{B}_{B_0,c} : p \in 2B$$

where M_{B_0} consists of all marked points above B_0. Thus we get

$$\bigcup_{2B \in d_c(B_0)} 2B_i = \{1, ..., n_i\}, \quad \forall i \in B_0$$

For the second condition in (MARKED SEAMS ARE UNFUSED), we need to take a look at type-3 moves made in the algorithm. The condition is satisfied, if we replace one minimal 2-bracket $2B$ with 2-brackets whose union is $2B$. As one can see this is exactly the case in the algorithm. We take a minimal bracket since we look at the last entrance of the current 2-brackets and replace one whole 2-bracket with smaller ones. This shows the second condition.

- (PARTIAL ORDER) We use here that the partial order of $2\mathscr{B}_{B_0}$ is stored in the last entrance of $d(B_0)$. The last entrance of $d(B_0)$ always stores the smallest 2-brackets in there. We need to look separately at type-1&3 moves and type-2 moves.
  Type-1&3 moves: The only way to get a new relation is to make a new 2-bracket inside a 2-bracket of the last entrance. In this case, we will just add the new relation to the current ones, and the other ones are preserved.
  Type-2 move: Here we are in the case, that $B_0 \notin \mathscr{B}_F$. By analyzing the Partition function in Lemma 6.7, we will see that the Partial order is preserved and also that we create all the allowed orders.

Step 2: Every child of F with additional brackets over B_0 is created as a result of the algorithm.
The algorithm starts by constructing all possible sub-2-brackets over B_0 using the Partition function. The Partition function produces all possible collisions as shown in Lemma 6.7. By iterating through all the possible combinations of 2-brackets and making different types of moves (Type 1, Type 2, and Type 3), it ensures that all the children with additional brackets over B_0 are generated. □

We have now shown how the 2-bracketings are generated. However, we are still missing the method to make partitions out of the 2-brackets in Algorithm 5. This is described in Algorithm 6.

Algorithm 6 Partition

Input
  $\mathcal{B}$ Array of 2-Brackets over different 1-brackets, that are not in bigger 1-brackets
Output
  $\mathcal{P}(\mathcal{B})$ Partitions of the 2-brackets, that can occur in a collision in the face
Make list $L \leftarrow [[0, ..., \#B_0], ..., [0, ..., \#B_k]]$ where $\#B_0$ is the number of 2-brackets over B_0 in the input.
for Tuple $(b_0, ..., b_k) \in L \backslash \{(0, ..., 0)\}$ **do**
  for $P_{\text{sub}} \in \mathcal{P}(\mathcal{B}[b_0 :, ..., b_k :])$ **do**
    $P = \mathcal{B}[: b_0, ..., : b_k] \cup P_{\text{sub}}$
    $\mathcal{P}(\mathcal{B}) \leftarrow P$

Lemma 6.7. *The algorithm calculates all allowed partitions of the given 2-brackets.*

Proof. The algorithm selects tuples of indices, where the elements under the indices form the first set of the partition. It then recursively calls the function for the remaining nodes. This is illustrated in figure 12. There we take all the nodes under the line as our first set and continue with the rest. One can see in figure 12 that the partial order is preserved and moreover, that we make all possible partitions. □

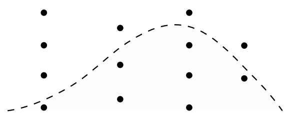


Figure 12: Partitioning of given marked nodes and 2-brackets

To analyze the running time, we observe that each partition is created in linear time. Let $a_{\mathbf{n}}$ denote the number of possible partitions. Then the running time for the Partition function is $\mathcal{O}(a_{\mathbf{n}}|\mathbf{n}|)$, where $\mathbf{n}$ is the multi-index, describing the size of the input. However, determining the precise number $a_{\mathbf{n}}$ is pretty hard; to the best of my knowledge, even finding an asymptotic growth rate is an open problem in number theory. The numbers $a_{\mathbf{n}}$ are equivalent to the number of partitions of multipartite numbers. Hardy and Ramanujan showed in their famous paper [HR18] from 1918, in the special case of $r = 2$, that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

We will find the generating function for these numbers in Lemma 6.10.

After we discussed how the algorithm that produces the children works we can now prove what was missing for Lemma 6.5.

Lemma 6.8. *Let $B \in \mathscr{B}$. The 2-brackets stored in the rightmost list of $d(B)$ satisfy the property that no other 2-brackets over B are subsets of them.*

Proof. We prove this statement using induction based on the number of collisions over B. If we have no collisions over B, we have that $d(B) = \emptyset$, so the statement is satisfied. Also if we had only one collision over B, there is only one list in $d(B)$ and the statement is satisfied. Now let's assume that there has been at least one collision over B. By induction hypothesis the right list stores 2-brackets, which have no other 2-brackets as subsets.
To introduce a new 2-bracket that is a subset of an existing one in the rightmost list, we must be at $i = \#M - 2$ in the first for loop of Algorithm 5. However, at this step, we create additional 2-brackets and hence have more 2-brackets in total. As a result, the newly created list is going to be the new rightmost list. Thus, the statement remains valid through induction. □

6.2.4 Overall runtime

Estimating the runtime to build the poset structure of a 2-associahedra is more difficult than for the associahedra and we will not give a sharp bound here. For example, there is no obvious upper bound for the number of faces of the 2-associahedra.

Theorem 6.9. *The overall runtime to build the poset structure of* $W_{\mathbf{n}}$ *can be bounded by*

$$\mathcal{O}(\#W_{\mathbf{n}} \cdot a_{\mathbf{n}} \cdot \dim(W_{\mathbf{n}}) \cdot |n|).$$

Proof. To analyze the overall runtime, let's examine Algorithm 3 where we iterate over every face of the 2-associahedra once. For each face, we calculate all its children. The time to create one child, given the partition, is $\mathcal{O}(\dim(W_{\mathbf{n}}) \cdot |n|)$. Thus we require $\mathcal{O}(2\dim(W_{\mathbf{n}} \cdot |n|) = \mathcal{O}(\dim(W_{\mathbf{n}}) \cdot |n|)$ time to calculate a child and insert it into the dictionary, as the string representation of a face F takes time $\mathcal{O}(\dim(F) \cdot |n|)$. So the key factor that remains to be determined is how many children one face can have. For this let $\mathcal{B} = \bigcup_{c \in C_A(\pi(F))} \beta(c)$. We notice, that the number $p(F)$ of possible children of a face F is bounded by

$$p(F) \leq \sum_{B \in \mathcal{B}} a_{\mathbf{n}_B}$$

where $\mathbf{n}_B$ is the multi-index associated to the number of 2-brackets above B, as in Algorithm 6. We show in Lemma 6.10 that

$$a_{\mathbf{n}} = \sum_{\substack{\mathbf{m} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\} \\ m_i \leq n_i, \forall i \\ \mathbf{m} \neq \mathbf{n}}} a_{\mathbf{m}}.$$

Thus we can conclude that $p(F) \leq 2a_{\mathbf{n}}$, which proves the overall runtime bound. $\square$

In the following, we analyze the numbers $a_{\mathbf{n}}$ and find the generating function of them.

Lemma 6.10. *Denote by* $a_{\mathbf{n}}$ *the number of partitions of multipartite numbers. Denote the generating function by*

$$F(x_1, ..., x_r) = \sum_{\mathbf{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}} a_{\mathbf{n}} x^{\mathbf{n}}.$$

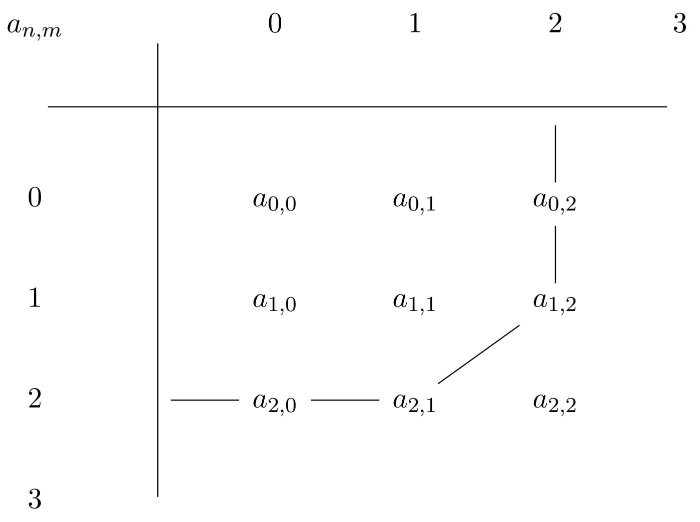
Then we have that

$$F(x_1, ..., x_r) = \frac{x_1 + \cdots + x_r}{\left(2 \sum_{k=1}^r (-1)^k \sum_{\substack{\mathbf{i} \in \{0,1\}^r \\ |\mathbf{i}| = k}} x^{\mathbf{i}}\right) + 1}$$

Proof. We first see, that the $a_{\mathbf{n}}$ satisfy the following recursive relations:

$$a_{\mathbf{n}} = \sum_{\substack{\mathbf{m} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\} \\ m_i \leq n_i, \forall i \\ \mathbf{m} \neq \mathbf{n}}} a_{\mathbf{m}}$$

This recursion naturally comes from Algorithm 6. We can think of the summand $a_{\mathbf{m}}$ as taking $a_{\mathbf{n}-\mathbf{m}}$ as the first set of our partition. Now let's first take a look at the special case where $r = 2$. For this, we take a look at the following table:



As one can see we get $a_{2,2}$ by summing over all the number, that are on the other side or on the line. This gives the expression $a_{2,2} = 2(a_{2,1} + a_{1,2} - a_{1,1})$. So in general for $r = 2$ we get that

$$a_{i,j} = 2(a_{i-1,j} + a_{i,j-1} - a_{i-1,j-1}) + [(i,j) = (0,1)] + [(i,j) = (1,0)]$$

In the same way, as for $r = 2$, we get for $r \geq 2$ the following formula by using the inclusion, exclusion principle.

$$a_{\mathbf{n}} = \sum_{k=1}^{r} (-1)^{k-1} \sum_{\substack{\mathbf{i} \in \{0,1\}^r \\ |i| = k}} a_{\mathbf{n}-\mathbf{i}} + \sum_{i=1}^{r} [\mathbf{n} = e_i]$$

Now we can use this recursive formula to get

$$\begin{aligned} F(x_1, ..., x_r) &= \sum_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\mathbf{0}\}} a_{\mathbf{n}} x^{\mathbf{n}} \\ &= \sum_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{\mathbf{0}\}} \left(\sum_{k=1}^{r} (-1)^{k-1} \sum_{\substack{\mathbf{i} \in \{0,1\}^r \\ |i| = k}} a_{\mathbf{n}-\mathbf{i}} + \sum_{i=1}^{r} [\mathbf{n} = e_i] \right) x^{\mathbf{n}} \\ &= \left(2 \sum_{k=1}^{r} (-1)^{k-1} \sum_{\substack{\mathbf{i} \in \{0,1\}^r \\ |i| = k}} x^{\mathbf{i}} \right) F(x_1, ..., x_r) + (x_1 + \cdots + x_r) \end{aligned}$$

Now by simply rearranging the terms, we get that

$$F(x_1, ..., x_r) = \frac{x_1 + \cdots + x_r}{\left(2 \sum_{k=1}^{r} (-1)^k \sum_{\substack{\mathbf{i} \in \{0,1\}^r \\ |\mathbf{i}| = k}} x^{\mathbf{i}} \right) + 1}$$

□

6.3 Some observations

We here give some examples of patterns of the 2-associahedra, which were observed while looking at the results of the computer program. There are a lot more observations to be made by looking at the results of the computer program.

Lemma 6.11. *For* $\mathbf{n} = 2, 0, ..., 0$ *with* $|n| = r$ *we have an isomorphism of posets*

$$W_{\mathbf{n}}^{br} \cong K_{r+1}^{br}$$

Proof. Let K_r^{br} be indexed by $\{0, 1, ..., r\}$ and the 1-bracketing $\mathscr{B}$ by $\{1, ..., r\}$. Take $(\mathscr{B}, 2\mathscr{B}) \in W_{\mathbf{n}}$. We want to send this element to a 1-bracketing.

$$(\mathscr{B}, 2\mathscr{B}) \mapsto \bigcup_{\substack{2B \in 2\mathscr{B}_B \\ 1 \in B \in \mathscr{B}}} \{f_B(2B)\} \cup \bigcup_{\substack{B \in \mathscr{B} \\ 1 \notin B}} \{B\}$$

where $f_B : 2\mathscr{B}_B \to \{\{0\}, B, \{0\} \cup B\}$,

$$f_B(2B) = \begin{cases} \{0\} \cup B, & \text{if } 2B = \{\{(0,0),(0,1)\}\} \\ B, & \text{if } 2B = \{\{(0,0)\},\{(0,1)\}\} \\ \{0\} & \text{else} \end{cases}$$

Well-defindness: Denote the map defined above by φ. We have to check that the axioms of a 1-bracketing are satisfied. (ROOT AND LEAVES) follows directly from the definition of the map. For (BRACKETING) assume there are 1-brackets $B, B' \in \varphi((\mathscr{B}, 2\mathscr{B}))$ with $B \cap B' \neq \emptyset$ but non is contained in the other. We know then, that neither of them has only one element. If $1 \notin B \cap B'$ every element in the intersection is ≥ 2 which shows that (BRACKETING) was already violated in $\mathscr{B}$. So let now $1 \in B \cap B'$ and let w.l.o.g. $B = \{0, ..., k\}, B' = \{1, ..., l\}, k < l$. Otherwise (BRACKETING) would not be violated. Then

$$\varphi^{-1}(B) = (\{1, ..., k\}, \{\{(0,0),(0,1)\}\}), \quad \varphi^{-1}(B') = (\{1, ..., l\}, \{\{(0,0)\},\{(0,1)\}\}$$

This violates the (2-BRACKETING) axiom of a 2-bracketing. So we showed well-definedness. *Bijection*: Injectivity of φ is clear since every 2-bracket gets sent to a unique 1-bracket. So it suffices to show that every 1-Bracketing $\mathscr{B}_K \in K_r^{\mathrm{br}}$ has a preimage. The preimage is given by taking as $\mathscr{B}$ all the 1-brackets in $\mathscr{B}_K$ and as 2-brackets the following. For $\{0, ..., k\} \in \mathscr{B}_K$ let $(\{1, ..., k\}, \{(0,0),(0,1)\}) \in (\mathscr{B}, 2\mathscr{B})$ and for $\{1, ..., k\}$ let $(\{1, ..., k\}, \{\{(0,0)\},\{(0,1)\}\}) \in (\mathscr{B}, 2\mathscr{B})$. It is easy to check that this satisfies (1-BRACKETING) and (2-BRACKETING). For the rest, there is nothing to check.
The poset structure is preserved since the poset structure on both posets is given by $\subset$. This ordering is preserved under φ.

□

Lemma 6.12. *For* $\mathbf{n} = 020...0$, $l(\mathbf{n}) = r + 1$, $\mathbf{m} = 30...0, l(\mathbf{n}) = r$, $\mathbf{l} = 110...0, l(\mathbf{l})) = r + 1$ *there are isomorphisms of posets*

$$W_{\mathbf{n}} \cong W_{\mathbf{m}} \cong W_{\mathbf{l}}$$

Proof. The proof of this lemma works similarly to the proof of Lemma 6.11. We just need to construct an isomorphism of the 2-bracketings.
For $W_{\mathbf{n}} \cong W_{\mathbf{m}}$ we do

$$\begin{aligned}
(\{0,1,...\},(\{(1,0),(1,1)\})) &\mapsto (\{1,2,...\},(\{(1,0),(1,1),(1,2)\}))\\
(\{0,1,...\},(\{(1,0)\},\{(1,1)\})) &\mapsto (\{1,2,...\},(\{(1,0),(1,1)\},\{(1,2)\}))\\
(\{1,2,...\},(\{(1,0),(1,1)\})) &\mapsto (\{1,2,...\},(\{(1,0)\},\{(1,1),(1,2)\}))\\
(\{1,2,...\},(\{(1,0)\},\{(1,1)\})) &\mapsto (\{1,2,...\},(\{(1,0)\},\{(1,1)\},\{(1,2)\}))
\end{aligned}$$

For $W_{\mathbf{n}} \cong W_{\mathbf{l}}$ we do

$$\begin{aligned}
(\{0,1,...\},(\{(1,0),(1,1)\})) &\mapsto (\{0,1,...\},(\{(0,0),(1,0)\}))\\
\{0,1,...\},(\{(1,0)\},\{(1,1)\})) &\mapsto (\{0,1,...\},(\{(0,0)\},\{(1,0)\}))\\
(\{1,2,...\},(\{(1,0),(1,1)\})) &\mapsto (\{0,1,...\},(\{(1,0)\},\{(0,0)\}))\\
(\{1,2,...\},(\{(1,0)\},\{(1,1)\})) &\mapsto (\{1,2,...\},(\{(1,0)\}))
\end{aligned}$$

On the other 2-Brackets, we just pick the identity. Checking that the axioms of a 2-bracketing are satisfied and that this defines an isomorphism is straightforward and similar to 6.11. □

7 Calculation of the CD Index

In this section, we aim to associate a noncommutative polynomial, the CD-Index to the 2-associahedra, which efficiently encodes its flag f-vector. The flag f-vector counts the number of chains in a poset. The CD-Index is introduced for example in [Sta94].

Definition 7.1. Let P be a finite graded poset of rank $n+1$ with $\hat{0}$ and $\hat{1}$, representing the unique minimal and maximal elements, respectively. Denote the rank function of P by ρ. Given a subset $S \subseteq \{1, ..., n\} =: [n]$, we define

$$P_S := \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}.$$

The *flag f-vector* is the function $\alpha : 2^{[n]} \to \mathbb{Z}$ which associates to each subset $S \subseteq [n]$ the number $\alpha(S)$ of maximal chains in P_S. Additionally, we introduce the *flag h-vector* $\beta : 2^{[n]} \to \mathbb{Z}$, defined by the expression

$$\beta(S) = \sum_{T \subseteq S} (-1)^{\#S-T} \alpha(T).$$

Remark 7.2. Equivalently, by the inclusion, exclusion formula, we obtain the expression

$$\sum_{T \subseteq S} \beta(T) = \alpha(S).$$

We now proceed to define the CD-Index $\Phi_P(c, d)$, which is a noncommutative polynomial in c and d. For this define for $S \subseteq [n]$ the noncommutative monomial $u_S = u_1 \cdots u_n$ where

$$u_i = \begin{cases} a & i \notin S \\ b & i \in S \end{cases}.$$

Furthermore, we define

$$\Upsilon_P(a, b) = \sum_{S \subseteq [n]} \alpha_P(S) u_S,$$
$$\Psi_P(a, b) = \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

Definition 7.3. The *CD-Index $\Phi_P(c, d)$* is the noncommutative polynomial defined by expressing $\Psi_P(a, b)$ in the variables $c = a + b$ and $d = ab + ba$.

A priori it is not clear that the CD-Index of a Poset exists and in fact, it doesn't exist for all posets. It is shown in [Bay91], Theorem 4 that the CD-Index of a poset exists and has integer coefficients if and only if the poset satisfies the so-called *generalized Dehn-Sommerville equations*. We can see that the CD-Index for the 2-associahedra exist, by introducing the following definition.

Definition 7.4. Let P be a finite graded poset of rank $n + 1$ with $\hat{0}$ and $\hat{1}$. We say that P is Eulerian if and only if every interval of rank at least one contains as many elements of even as of odd rank, i.e.

$$\sum_{u \in [s,t]} (-1)^{\rho(u)} = 0, \text{ if } s < t \text{ in } P.$$

The following theorem shows that the CD-Index exists for Eulerian Posets. Since the 2-associahedra are Eulerian, as proved in [BM19], Theorem 1.1, this shows that the CD-Index for 2-associahedra exists. The theorem is stated in terms of the *incidence algebras*. The elements of the incidence algebra are functions f assigning to every nonempty interval $[a,b] \subseteq P$ a scalar $f(a,b) =: f_{ab}$ in some base field. The multiplication is defined as follows:

$$(f * g)(a,b) := \sum_{a \leq x \leq b} f(a,x)g(x,b)$$

With this, we can now state the theorem.

Theorem 7.5 ([Sta94], Theorem 1.1)**.** *Let P be Eulerian. Then*

$$\begin{aligned} 2\Psi_P = 2\Psi_{\hat{0}\hat{1}} = & \sum_{\substack{\hat{0}<x<\hat{1} \\ \rho(x,\hat{1})=2j-1}} \Psi_{\hat{0}x} c(c^2-2d)^{j-1} \\ & - \sum_{\substack{\hat{0}<x<\hat{1} \\ \rho(x,\hat{1})=2j}} \Psi_{\hat{0}x}(c^2-2d)^j + \begin{cases} 2(c^2-2d)^{k-1} & \rho(\hat{0},\hat{1}) = 2k-1 \\ 0 & \rho(\hat{0},\hat{1}) = 2k \end{cases} \end{aligned}$$

Hence $\Phi_P(c,d)$ exists by induction of the rank of the poset.

By applying this theorem multiple times, we can conclude the following formula, which we can use to calculate the CD-Index of the 2-associahedra.

Corollary 7.6 ([Sta94], Corollary 1.1)**.** *For $j \geq 1$ define*

$$\begin{aligned} \omega(2j-1) &:= \frac{1}{2}c(c^2-2d)^{j-1} \\ \omega(2j) &:= -\frac{1}{2}(c^2-2d)^j \end{aligned}$$

If P is an Eulerian poset, then we can use the following formula for the calculation of the CD Index:

$$\begin{aligned} \Phi_P(c,d) = \sum_S (c^2-2d)^{\frac{1}{2}(a_1-1)} \omega(a_2-a_1)\omega(a_3-a_2)\cdots\omega(n+1-a_S)\alpha_P(S) \\ + \begin{cases} (c^2-2d)^{n/2} & \textit{if } n \textit{ is even} \\ 0 & \textit{else} \end{cases} \end{aligned}$$

where S ranges over all subsets $\{a_1,...,a_S\} \subseteq [n]$ such that $a_1 < \cdots < a_S$ and a_1 is odd.

Remark 7.7. Notice that Stanley forgot the last summand in the formula as shown in the following proof.

Proof. By applying Theorem 7.5 multiple times and expanding the resulting product, we obtain a sum where each term corresponds to one descending chain

$$\hat{1} > x_1 > \cdots > x_m > \hat{0}$$

in the poset. The last element of the chain indicates the index at which the last summand of the theorem was selected in the product. Notice that each chain with the same ranks of the elements gives the same summand. The summand of a chain with ranks $S \subseteq [n]$ is given by

$$\omega(n+1-a_S)\cdots\omega(a_2-a_1)\cdot\begin{cases}(c^2-2d)^{\frac{1}{2}(a_1-1)} & \text{if } a_1 \text{ is odd}\\ 0 & \text{else}\end{cases}$$

Thus, we only need to consider all subsets $S \subseteq [n]$ with a_1 odd. Using this we get the stated formula, by adding the last summand of Theorem 7.5 for the whole poset as it was not involved in the above considerations. □

With the derived formula for the CD-Index of Eulerian posets, we have an easy way to calculate the CD-Index of a 2-associahedra. This is also part of the computer package. However, to do this, we need to compute the flag f-vector of the poset, which can be done using the following algorithm:

Algorithm 7 Element Flag f-vector: α_E

Require: Poset Node p, $S \subseteq [n]$
Ensure: Number of maximal Chains in P_S, starting with P.

```
if #S ≤ 1 then return 1
d = S[1] − S[0], s = 0
N ← Elements that are d smaller then p
for n ∈ N do
    s+ = α_E(n, S[1 :])
Return s
```

The algorithm starts by finding the elements that are d smaller than the current element, where d is the difference between the second and first element in S. These elements are found by iterating recursively through the lists of children. For each of these smaller elements, the algorithm then calls itself recursively to build a chain in S. This process continues until S contains only one element.

By calling this algorithm on the root of the poset, one can obtain the flag f-vector α. Subsequently, we can calculate the CD-Index by plugging it into the formula.
In the appendix 8.2 there is a list of the CD-Indices of 4-dimensional 2-associahedra.

8 Appendix

8.1 Runtimes

We give here lists of runtimes on an average laptop in the year 2023.

Runtime for associahedra			
r	Seconds	# faces	face vector
3	0.0	3	$(2, 1)$
4	0.0	11	$(5, 5, 1)$
5	0.001	45	$(14, 21, 9, 1)$
6	0.006	197	$(42, 84, 56, 14, 1)$
7	0.033	903	$(132, 330, 300, 120, 20, 1)$
8	0.172	4279	$(429, 1287, 1485, 825, 225, 27, 1)$
9	1.120	20793	$(1430, 5005, 7007, 5005, 1925, 385, 35, 1)$
10	7.135	103049	$(4862, 19448, 32032, 28028, 14014, 4004, 616, 44, 1)$
11	43.269	518859	$(16796, 75582, 143208, 148512, 91728, 34398, 7644, 936, 54, 1)$
12	271.343	2646723	$(58786, 293930, 629850, 755820, 556920,$ $259896, 76440, 13650, 1365, 65, 1)$

The following are some sequences of 2-associahedra and their runtime.

Runtime for 2-associahedra			
n	Seconds	# faces	dimension
11	0.0	3	1
111	0.04	99	3
1111	6.89	7845	5
11111	1538.560	1064469	7
101	0.003	9	2
1001	0.013	33	3
10001	0.062	135	4
100001	0.401	591	5
1000001	2.263	2709	6
10000001	13.226	12837	7
100000001	78.195	62379	8
1000000001	460.419	309147	9
10	0.0	1	0
20	0.0	3	1
30	0.003	13	2
40	0.027	67	3
50	0.225	381	4
60	1.986	2311	5
70	17.911	17681	6

8.2 Lists of 2-associahedra

The following table consists of of all the face vectors and CD-Indices of 4-dimensional 2-associahedra.

4-dim 2-associahedra		
n	face vector	CD-Index
50	$(80, 165, 110, 25, 1)$	$128d^2 + 23c^2d + 85cdc + 78dc^2 + c^4$
41	$(170, 361, 244, 53, 1)$	$300d^2 + 51c^2d + 191cdc + 168dc^2 + c^4$
32	$(248, 534, 365, 79, 1)$	$458d^2 + 77c^2d + 286cdc + 246dc^2 + c^4$
400	$(75, 152, 99, 22, 1)$	$114d^2 + 20c^2d + 77cdc + 73dc^2 + c^4$
040	$(108, 219, 140, 29, 1)$	$168d^2 + 27c^2d + 111cdc + 106dc^2 + c^4$
310	$(164, 336, 215, 43, 1)$	$262d^2 + 41c^2d + 172cdc + 162dc^2 + c^4$
301	$(92, 188, 123, 27, 1)$	$142d^2 + 25c^2d + 96cdc + 90dc^2 + c^4$
130	$(184, 376, 238, 46, 1)$	$296d^2 + 44c^2d + 192cdc + 182dc^2 + c^4$
220	$(218, 448, 285, 55, 1)$	$354d^2 + 53c^2d + 230cdc + 216dc^2 + c^4$
202	$(104, 214, 141, 31, 1)$	$162d^2 + 29c^2d + 110cdc + 102dc^2 + c^4$
211	$(212, 436, 279, 55, 1)$	$342d^2 + 53c^2d + 224cdc + 210dc^2 + c^4$
121	$(248, 508, 320, 60, 1)$	$404d^2 + 58c^2d + 260cdc + 246dc^2 + c^4$
3000	$(56, 112, 73, 17, 1)$	$82d^2 + 15c^2d + 56cdc + 54dc^2 + c^4$
0300	$(84, 168, 107, 23, 1)$	$126d^2 + 21c^2d + 84cdc + 82dc^2 + c^4$
2100	$(102, 204, 129, 27, 1)$	$154d^2 + 25c^2d + 102cdc + 100dc^2 + c^4$
2010	$(74, 148, 95, 21, 1)$	$110d^2 + 19c^2d + 74cdc + 72dc^2 + c^4$
2001	$(48, 96, 63, 15, 1)$	$70d^2 + 13c^2d + 48cdc + 46dc^2 + c^4$
1200	$(112, 224, 141, 29, 1)$	$170d^2 + 27c^2d + 112cdc + 110dc^2 + c^4$
1020	$(82, 164, 105, 23, 1)$	$122d^2 + 21c^2d + 82cdc + 80dc^2 + c^4$
0120	$(142, 284, 177, 35, 1)$	$218d^2 + 33c^2d + 142cdc + 140dc^2 + c^4$
1110	$(140, 280, 175, 35, 1)$	$214d^2 + 33c^2d + 140cdc + 138dc^2 + c^4$
1101	$(84, 168, 107, 23, 1)$	$126d^2 + 21c^2d + 84cdc + 82dc^2 + c^4$
20000	$(42, 84, 56, 14, 1)$	$60d^2 + 12c^2d + 42cdc + 40dc^2 + c^4$
02000	$(56, 112, 73, 17, 1)$	$82d^2 + 15c^2d + 56cdc + 54dc^2 + c^4$
00200	$(60, 120, 78, 18, 1)$	$88d^2 + 16c^2d + 60cdc + 58dc^2 + c^4$
11000	$(56, 112, 73, 17, 1)$	$82d^2 + 15c^2d + 56cdc + 54dc^2 + c^4$
10100	$(46, 92, 61, 15, 1)$	$66d^2 + 13c^2d + 46cdc + 44dc^2 + c^4$
10010	$(38, 76, 51, 13, 1)$	$54d^2 + 11c^2d + 38cdc + 36dc^2 + c^4$
10001	$(28, 56, 39, 11, 1)$	$38d^2 + 9c^2d + 28cdc + 26dc^2 + c^4$
01100	$(74, 148, 95, 21, 1)$	$110d^2 + 19c^2d + 74cdc + 72dc^2 + c^4$
01010	$(56, 112, 73, 17, 1)$	$82d^2 + 15c^2d + 56cdc + 54dc^2 + c^4$

The following table consists of a few face vectors of 5- and 6-dimensional 2-associahedra.

5-dim 2-associahedra		6-dim 2-associahedra	
n	face vector	**n**	face vector
7	$(132, 330, 300, 120, 20, 1)$	8	$(429, 1287, 1485, 825, 225, 27, 1)$
06	$(322, 841, 788, 313, 46, 1)$	07	$(1348, 4272, 5183, 2984, 809, 84, 1)$
15	$(824, 2222, 2139, 860, 121, 1)$	16	$(4060, 13286, 16619, 9818, 2695, 270, 1)$
24	$(1400, 3836, 3750, 1526, 214, 1)$	25	$(7880, 26214, 33332, 19994, 5550, 554, 1)$
33	$(1656, 4560, 4480, 1831, 257, 1)$	34	$(10740, 35992, 46106, 27852, 7774, 776, 1)$
005	$(340, 875, 800, 305, 42, 1)$	006	$(1632, 5119, 6090, 3383, 859, 79, 1)$
014	$(932, 2439, 2240, 833, 102, 1)$	015	$(5420, 17332, 20901, 11628, 2873, 234, 1)$
023	$(1500, 3954, 3642, 1343, 157, 1)$	024	$(10282, 33196, 40327, 22486, 5497, 424, 1)$
113	$(1424, 3748, 3458, 1289, 157, 1)$	033	$(12864, 41632, 50636, 28196, 6841, 513, 1)$
122	$(2084, 5492, 5042, 1843, 211, 1)$	114	$(9580, 30822, 37365, 20876, 5177, 424, 1)$
0004	$(260, 657, 591, 224, 32, 1)$	123	$(16888, 54584, 66268, 36844, 8963, 691, 1)$
0013	$(628, 1598, 1426, 519, 65, 1)$	222	$(19952, 64620, 78700, 44014, 10854, 872, 1)$
0022	$(868, 2212, 1966, 704, 84, 1)$	0005	$(1320, 4060, 4725, 2565, 640, 60, 1)$
0112	$(1076, 2744, 2438, 871, 103, 1)$	0014	$(4022, 12510, 14609, 7839, 1870, 152, 1)$
		0023	$(6784, 21196, 24780, 13222, 3090, 236, 1)$
		0113	$(8328, 26012, 30400, 16224, 3801, 293, 1)$
		0122	$(12608, 39384, 45844, 24178, 5506, 396, 1)$
		1112	$(10528, 32976, 38688, 20764, 4903, 379, 1)$
		00004	$(900, 2724, 3128, 1688, 427, 43, 1)$
		00022	$(2296, 6984, 7988, 4224, 1012, 88, 1)$
		00013	$(3246, 9882, 11272, 5906, 1384, 114, 1)$
		00112	$(4436, 13512, 15408, 8060, 1880, 152, 1)$
		01111	$(6496, 19776, 22456, 11616, 2641, 201, 1)$
		000003	$(594, 1782, 2040, 1110, 290, 32, 1)$
		000012	$(1170, 3510, 3980, 2110, 520, 50, 1)$
		000111	$(1888, 5664, 6396, 3352, 804, 72, 1)$
		0000002	$(429, 1287, 1485, 825, 225, 27, 1)$
		0000011	$(594, 1782, 2040, 1110, 290, 32, 1)$

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