THE PERVERSE FILTRATION FOR $M_{4,1}$

Recall the following description of the Chow and cohomology rings of $M_{4,1}$:

Theorem 0.1 ([1, Theorem 6.5]). *The Chow ring of $M_{4,1}$ is given by*

$$\begin{aligned}
A^\bullet(M_{4,1}) \simeq \mathbb{Q}[\alpha,\beta,x,y,z]/\langle & xz-yz, \beta^2z-3yz-9z^2, 3\alpha^2z-\alpha\beta z+yz, \beta^2y-3y^2-9yz,\\
& \beta^2x-xy-3y^2-3\alpha\beta z-9yz+9z^2, \beta^4+3x^2-9xy-3y^2-54yz-81z^2,\\
& \beta yz+9\alpha z^2-3\beta z^2, 2\beta xy-3\beta y^2-9\alpha yz-27\alpha z^2+9\beta z^2, 3\beta x^2-7\beta y^2-36\alpha yz\\
& -108\alpha z^2+36\beta z^2, \alpha^{12}+3\alpha^{11}\beta+3\alpha^{10}(\beta^2+2x-y)+\alpha^9(-\beta^3+12\beta x+2\beta y)\\
& +3\alpha^8(9x^2-16xy+17y^2)+28\alpha^7\beta y^2+56\alpha^6y^3+201\alpha\beta z^5-19yz^5-613z^6,\\
& 6\alpha^{10}xy-12\alpha^{10}y^2-10\alpha^9\beta y^2-45\alpha^8y^3-104\alpha\beta z^6+2yz^6+310z^7\rangle,
\end{aligned}$$

where α,β are of algebraic degree 1 and x,y,z of degree 2. This also gives the cohomology ring $H^\bullet(M_{4,1},\mathbb{Q})$ with the degrees of the generators doubled.

In this note we describe the perverse filtration on $H^\bullet(M_{4,1},\mathbb{Q})$ explicitly in these generators. Denote by $\mathrm{Gr}_i^P H^\bullet(M_{4,1})$ the i-th graded piece of the perverse filtration on the total cohomology ring. We separate the terms of different cohomological degrees in a common graded piece by a semicolon, so one easily counts $n_4^{i,j} := \dim \mathrm{Gr}_i^P H^{i+j}(M_{4,1})$.

$$\begin{aligned}
\mathrm{Gr}_0^P H^\bullet(M_{4,1}) = \langle & 1;\alpha;\alpha^2;\ldots;\alpha^{11}; 3\alpha^{11}\beta+3\alpha^{10}\beta^2-\alpha^9\beta^3+6\alpha^{10}x+12\alpha^9\beta x+27\alpha^8x^2-3\alpha^{10}y+2\alpha^9\beta y\\
& -48\alpha^8xy+51\alpha^8y^2+28\alpha^7\beta y^2+56\alpha^6y^3+201\alpha\beta z^5-19yz^5-613z^6; 6\alpha^{11}\beta^2+10\alpha^{10}\beta^3\\
& -6\alpha^{11}x+6\alpha^{10}\beta x-18\alpha^9x^2+3\alpha^{11}y-11\alpha^{10}\beta y+57\alpha^9xy+66\alpha^9y^2+98\alpha^8\beta y^2+196\alpha^7y^3\\
& +1357\alpha yz^5+2674\alpha z^6-999\beta z^6; 12\alpha^{11}\beta^3-36\alpha^{11}\beta x-108\alpha^{10}x^2+30\alpha^{11}\beta y\\
& +486\alpha^{10}y^2+543\alpha^9\beta y^2+2136\alpha^8y^3+5851\alpha\beta z^6+374yz^6-17799z^7\rangle
\end{aligned}$$

$$\begin{aligned}
\mathrm{Gr}_1^P H^\bullet(M_{4,1}) = \langle & \alpha\beta+4x-4y-4z; \alpha^2\beta+4\alpha x-4\alpha y-4\alpha z; 3\alpha^3\beta+12\alpha^2x-12\alpha^2y-4\alpha\beta z+4yz;\\
& 3\alpha^4\beta+12\alpha^3x-12\alpha^3y-24\alpha z^2+4\beta z^2; 3\alpha^5\beta+12\alpha^4x-12\alpha^4y-4\alpha\beta z^2+8yz^2;\\
& 3\alpha^6\beta+12\alpha^5x-12\alpha^5y+4\alpha yz^2-24\alpha z^3+4\beta z^3; \alpha^7\beta+4\alpha^6x-4\alpha^6y+4yz^3-4z^4;\\
& \alpha^8\beta+4\alpha^7x-4\alpha^7y+4\alpha yz^3-4\alpha z^4; 3\alpha^9\beta+12\alpha^8x-12\alpha^8y+8\alpha\beta z^4+16yz^4-36z^5;\\
& 3\alpha^{10}\beta+12\alpha^9x-12\alpha^9y+24\alpha yz^4+12\alpha z^5-8\beta z^5; -3\alpha^{10}\beta^2+\alpha^9\beta^3+6\alpha^{10}x-12\alpha^9\beta x\\
& -27\alpha^8x^2-9\alpha^{10}y-2\alpha^9\beta y+48\alpha^8xy-51\alpha^8y^2-28\alpha^7\beta y^2-56\alpha^6y^3-181\alpha\beta z^5+39yz^5\\
& +541z^6; -60\alpha^{10}\beta^3-30\alpha^{11}x+90\alpha^{10}\beta x+360\alpha^9x^2+75\alpha^{11}y+75\alpha^{10}\beta y-765\alpha^9xy\\
& +180\alpha^9y^2-210\alpha^8\beta y^2-420\alpha^7y^3-5365\alpha yz^5-7920\alpha z^6+3185\beta z^6\rangle
\end{aligned}$$

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\begin{aligned}
\mathrm{Gr}_2^P H^\bullet(M_{4,1}) = \langle & \beta; 3\beta^2-8y-40z, -4x+4y+4z; 2\beta^3+3\beta x-\alpha y-11\beta y-17\alpha z-11\beta z,\\
& 3\alpha\beta^2-8\alpha y-40\alpha z, -4\alpha x+4\alpha y+4\alpha z; 9\alpha\beta x+18x^2+3\alpha^2 y-9\alpha\beta y-36xy+18y^2\\
& -4\alpha\beta z-5yz+18z^2, 3\alpha\beta^3-9x^2-3\alpha^2 y-12\alpha\beta y+18xy-9y^2-23\alpha\beta z+11yz-9z^2,\\
& 9\alpha^2\beta^2-24\alpha^2 y-40\alpha\beta z+40yz, -4\alpha^2 x+4\alpha^2 y+\frac{4}{3}\alpha\beta z-\frac{4}{3}yz;\\
& 9\alpha^2\beta x+18\alpha x^2+3\alpha^3 y-9\alpha^2\beta y-36\alpha xy+18\alpha y^2-9\alpha yz-6\alpha z^2+4\beta z^2,\\
& 3\alpha^2\beta^3-9\alpha x^2-3\alpha^3 y-12\alpha^2\beta y+18\alpha xy-9\alpha y^2-12\alpha yz-147\alpha z^2+23\beta z^2,\\
& 9\alpha^3\beta^2-24\alpha^3 y-240\alpha z^2+40\beta z^2, -4\alpha^3 x+4\alpha^3 y+8\alpha z^2-\frac{4}{3}\beta z^2;\\
& 9\alpha^3\beta x+18\alpha^2 x^2+3\alpha^4 y-9\alpha^3\beta y-36\alpha^2 xy+18\alpha^2 y^2-7\alpha\beta z^2-7yz^2+27z^3,\\
& 3\alpha^3\beta^3-9\alpha^2 x^2-3\alpha^4 y-12\alpha^3\beta y+18\alpha^2 xy-9\alpha^2 y^2-38\alpha\beta z^2+37yz^2+36z^3,\\
& 9\alpha^4\beta^2-24\alpha^4 y-40\alpha\beta z^2+80yz^2, -4\alpha^4 x+4\alpha^4 y+\frac{4}{3}\alpha\beta z^2-\frac{8}{3}yz^2;\\
& 9\alpha^4\beta x+18\alpha^3 x^2+3\alpha^5 y-9\alpha^4\beta y-36\alpha^3 xy+18\alpha^3 y^2-14\alpha yz^2-15\alpha z^3+7\beta z^3,\\
& 3\alpha^4\beta^3-9\alpha^3 x^2-3\alpha^5 y-12\alpha^4\beta y+18\alpha^3 xy-9\alpha^3 y^2-\alpha yz^2-192\alpha z^3+38\beta z^3,\\
& 9\alpha^5\beta^2-24\alpha^5 y+40\alpha yz^2-240\alpha z^3+40\beta z^3, -4\alpha^5 x+4\alpha^5 y-\frac{4}{3}\alpha yz^2+8\alpha z^3-\frac{4}{3}\beta z^3;\\
& 3\alpha^5\beta x+6\alpha^4 x^2+\alpha^6 y-3\alpha^5\beta y-12\alpha^4 xy+6\alpha^4 y^2-4\alpha\beta z^3-3yz^3+14z^4,\\
& \alpha^5\beta^3-3\alpha^4 x^2-\alpha^6 y-4\alpha^5\beta y+6\alpha^4 xy-3\alpha^4 y^2-9\alpha\beta z^3+21yz^3+z^4,\\
& 3\alpha^6\beta^2-8\alpha^6 y+40yz^3-40z^4, -4\alpha^6 x+4\alpha^6 y-4yz^3+4z^4;\\
& 3\alpha^6\beta x+6\alpha^5 x^2+\alpha^7 y-3\alpha^6\beta y-12\alpha^5 xy+6\alpha^5 y^2-7\alpha yz^3-10\alpha z^4+4\beta z^4,\\
& \alpha^6\beta^3-3\alpha^5 x^2-\alpha^7 y-4\alpha^6\beta y+6\alpha^5 xy-3\alpha^5 y^2+12\alpha yz^3-53\alpha z^4+9\beta z^4,\\
& 3\alpha^7\beta^2-8\alpha^7 y+40\alpha yz^3-40\alpha z^4, -4\alpha^7 x+4\alpha^7 y-4\alpha yz^3+4\alpha z^4;\\
& 9\alpha^7\beta x+18\alpha^6 x^2+3\alpha^8 y-9\alpha^7\beta y-36\alpha^6 xy+18\alpha^6 y^2-19\alpha\beta z^4-11yz^4+63z^5,\\
& 3\alpha^7\beta^3-9\alpha^6 x^2-3\alpha^8 y-12\alpha^7\beta y+18\alpha^6 xy-9\alpha^6 y^2+10\alpha\beta z^4+89yz^4-108z^5,\\
& 9\alpha^8\beta^2-24\alpha^8 y+80\alpha\beta z^4+160yz^4-360z^5, -4\alpha^8 x+4\alpha^8 y-\frac{8}{3}\alpha\beta z^4-\frac{16}{3}yz^4+12z^5;\\
& 9\alpha^8\beta x+18\alpha^7 x^2+3\alpha^9 y-9\alpha^8\beta y-36\alpha^7 xy+18\alpha^7 y^2-30\alpha yz^4-51\alpha z^5+19\beta z^5,\\
& 3\alpha^8\beta^3-9\alpha^7 x^2-3\alpha^9 y-12\alpha^8\beta y+18\alpha^7 xy-9\alpha^7 y^2+99\alpha yz^4-48\alpha z^5-10\beta z^5,\\
& 9\alpha^9\beta^2-24\alpha^9 y+240\alpha yz^4+120\alpha z^5-80\beta z^5, -4\alpha^9 x+4\alpha^9 y-8\alpha yz^4-4\alpha z^5+\frac{8}{3}\beta z^5;\\
& 3\alpha^{10}x+5\alpha^9\beta y+3\alpha^8 xy+12\alpha^8 y^2+14\alpha^7\beta y^2+28\alpha^6 y^3+98\alpha\beta z^5-15yz^5-296z^6,\\
& 3\alpha^9\beta^3-9\alpha^8 x^2-7\alpha^9\beta y+21\alpha^8 xy+3\alpha^8 y^2+14\alpha^7\beta y^2+28\alpha^6 y^3+166\alpha\beta z^5+95yz^5-575z^6,\\
& 9\alpha^{10}\beta^2+40\alpha^9\beta y+24\alpha^8 xy+96\alpha^8 y^2+112\alpha^7\beta y^2+224\alpha^6 y^3+944\alpha\beta z^5+40yz^5-2944z^6,\\
& \alpha^{11}\beta; 3\alpha^{11}x+5\alpha^{10}\beta y+3\alpha^9 xy+12\alpha^9 y^2+14\alpha^8\beta y^2+28\alpha^7 y^3+83\alpha yz^5+292\alpha z^6-98\beta z^6,\\
& 3\alpha^{10}\beta^3-9\alpha^9 x^2-7\alpha^{10}\beta y+21\alpha^9 xy+3\alpha^9 y^2+14\alpha^8\beta y^2+28\alpha^7 y^3+261\alpha yz^5+421\alpha z^6-166\beta z^6,\\
& 9\alpha^{11}\beta^2+40\alpha^{10}\beta y+24\alpha^9 xy+96\alpha^9 y^2+112\alpha^8\beta y^2+224\alpha^7 y^3+984\alpha yz^5+2720\alpha z^6-944\beta z^6;\\
& 6\alpha^{11}\beta x+12\alpha^{10}x^2+36\alpha^{10}y^2+92\alpha^9\beta y^2+309\alpha^8 y^3+876\alpha\beta z^6-22yz^6-2624z^7,\\
& 6\alpha^{11}\beta^3-18\alpha^{10}x^2+216\alpha^{10}y^2+315\alpha^9\beta y^2+1155\alpha^8 y^3+3301\alpha\beta z^6+176yz^6-10029z^7;\\
& 36\alpha^{11}x^2-162\alpha^{11}y^2+1371\alpha^9 y^3-1266\alpha yz^6-2454\alpha z^7+968\beta z^7\rangle
\end{aligned}
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$$\begin{aligned}
\mathrm{Gr}^P_3 H^\bullet(M_{4,1}) = \langle & \frac{8}{3}y - \frac{56}{3}z; \beta^3 - 4\beta y - 24\alpha z - 4\beta z, \frac{8}{3}\alpha y - \frac{56}{3}\alpha z; 3\alpha\beta x - 15xy + 24y^2 + 29\alpha\beta z - 29yz - 42z^2, \\
& \alpha\beta^3 - 20xy + 32y^2 + 20\alpha\beta z - 20yz - 56z^2, 8\alpha^2 y - \frac{56}{3}\alpha\beta z + \frac{56}{3}yz; \\
& 3\alpha^2\beta x - 15\alpha xy + 24\alpha y^2 + 132\alpha z^2 - 29\beta z^2, \alpha^2\beta^3 - 20\alpha xy + 32\alpha y^2 + 64\alpha z^2 - 20\beta z^2, \\
& 8\alpha^3 y - 112\alpha z^2 + \frac{56}{3}\beta z^2; 3\alpha^2 x^2 - 3\alpha^2 y^2 + 3\alpha\beta y^2 + 6y^3 + 28\alpha\beta z^2 + 7yz^2 - 93z^3, \\
& 6\alpha^3\beta x + 18\alpha^2 y^2 + 15\alpha\beta y^2 + 30y^3 + 175\alpha\beta z^2 - 68yz^2 - 465z^3, \\
& \alpha^3\beta^3 + 12\alpha^2 y^2 + 10\alpha\beta y^2 + 20y^3 + 98\alpha\beta z^2 - 8yz^2 - 310z^3, \\
& 8\alpha^4 y - \frac{56}{3}\alpha\beta z^2 + \frac{112}{3}yz^2; 3\alpha^3 x^2 - 3\alpha^3 y^2 + 3\alpha^2\beta y^2 + 6\alpha y^3 + 35\alpha yz^2 + 75\alpha z^3 - 28\beta z^3, \\
& 6\alpha^4\beta x + 18\alpha^3 y^2 + 15\alpha^2\beta y^2 + 30\alpha y^3 + 107\alpha yz^2 + 585\alpha z^3 - 175\beta z^3, \\
& \alpha^4\beta^3 + 12\alpha^3 y^2 + 10\alpha^2\beta y^2 + 20\alpha y^3 + 90\alpha yz^2 + 278\alpha z^3 - 98\beta z^3, \\
& 8\alpha^5 y + \frac{56}{3}\alpha yz^2 - 112\alpha z^3 + \frac{56}{3}\beta z^3; 3\alpha^4 x^2 - 3\alpha^4 y^2 + 3\alpha^3\beta y^2 + 6\alpha^2 y^3 + 32\alpha\beta z^3 + 10yz^3 - 105z^4, \\
& 6\alpha^5\beta x + 18\alpha^4 y^2 + 15\alpha^3\beta y^2 + 30\alpha^2 y^3 + 127\alpha\beta z^3 - 88yz^3 - 321z^4, \\
& 3\alpha^5\beta^3 + 36\alpha^4 y^2 + 30\alpha^3\beta y^2 + 60\alpha^2 y^3 + 254\alpha\beta z^3 - 8yz^3 - 810z^4, \\
& \frac{8}{3}\alpha^6 y + \frac{56}{3}yz^3 - \frac{56}{3}z^4; 3\alpha^5 x^2 - 3\alpha^5 y^2 + 3\alpha^4\beta y^2 + 6\alpha^3 y^3 + 42\alpha yz^3 + 87\alpha z^4 - 32\beta z^4, \\
& 6\alpha^6\beta x + 18\alpha^5 y^2 + 15\alpha^4\beta y^2 + 30\alpha^3 y^3 + 39\alpha yz^3 + 441\alpha z^4 - 127\beta z^4, \\
& 3\alpha^6\beta^3 + 36\alpha^5 y^2 + 30\alpha^4\beta y^2 + 60\alpha^3 y^3 + 246\alpha yz^3 + 714\alpha z^4 - 254\beta z^4, \\
& \frac{8}{3}\alpha^7 y + \frac{56}{3}\alpha yz^3 - \frac{56}{3}\alpha z^4; 3\alpha^6 x^2 - 3\alpha^6 y^2 + 3\alpha^5\beta y^2 + 6\alpha^4 y^3 + 39\alpha\beta z^4 + 13yz^4 - 126z^5, \\
& 6\alpha^7\beta x + 18\alpha^6 y^2 + 15\alpha^5\beta y^2 + 30\alpha^4 y^3 + 59\alpha\beta z^4 - 108yz^4 - 117z^5, \\
& 3\alpha^7\beta^3 + 36\alpha^6 y^2 + 30\alpha^5\beta y^2 + 60\alpha^4 y^3 + 230\alpha\beta z^4 + 8yz^4 - 738z^5, 8\alpha^8 y + \frac{112}{3}\alpha\beta z^4 \\
& + \frac{224}{3}yz^4 - 168z^5; 3\alpha^7 x^2 - 3\alpha^7 y^2 + 3\alpha^6\beta y^2 + 6\alpha^5 y^3 + 52\alpha yz^4 + 108\alpha z^5 - 39\beta z^5, \\
& 6\alpha^8\beta x + 18\alpha^7 y^2 + 15\alpha^6\beta y^2 + 30\alpha^5 y^3 - 49\alpha yz^4 + 237\alpha z^5 - 59\beta z^5, \\
& 3\alpha^8\beta^3 + 36\alpha^7 y^2 + 30\alpha^6\beta y^2 + 60\alpha^5 y^3 + 238\alpha yz^4 + 642\alpha z^5 - 230\beta z^5, \\
& 8\alpha^9 y + 112\alpha yz^4 + 56\alpha z^5 - \frac{112}{3}\beta z^5; \\
& 6\alpha^9\beta x + 18\alpha^8 y^2 + 15\alpha^7\beta y^2 + 30\alpha^6 y^3 - 29\alpha\beta z^5 - 128yz^5 + 147z^6, \\
& \alpha^9\beta^3 + 12\alpha^8 y^2 + 10\alpha^7\beta y^2 + 20\alpha^6 y^3 + 74\alpha\beta z^5 + 8yz^5 - 238z^6, \\
& 8\alpha^{10} y + \frac{280}{3}\alpha\beta z^5 + \frac{280}{3}yz^5 - 336z^6; \\
& 6\alpha^{10}\beta x + 18\alpha^9 y^2 + 15\alpha^8\beta y^2 + 30\alpha^7 y^3 - 157\alpha yz^5 - 27\alpha z^6 + 29\beta z^6, \\
& \alpha^{10}\beta^3 + 12\alpha^9 y^2 + 10\alpha^8\beta y^2 + 20\alpha^7 y^3 + 82\alpha yz^5 + 206\alpha z^6 - 74\beta z^6, \\
& 8\alpha^{11} y + \frac{560}{3}\alpha yz^5 + 224\alpha z^6 - \frac{280}{3}\beta z^6; \\
& 6\alpha^{11}\beta x - 24\alpha^9\beta y^2 - 123\alpha^8 y^3 - 596\alpha\beta z^6 - 178yz^6 + 1860z^7, \\
& 3\alpha^{11}\beta^3 - 48\alpha^9\beta y^2 - 246\alpha^8 y^3 - 688\alpha\beta z^6 - 20yz^6 + 2040z^7; \\
& 81\alpha^{11} y^2 - \frac{2133}{2}\alpha^9 y^3 + 1323\alpha yz^6 + 2457\alpha z^7 - 954\beta z^7\rangle
\end{aligned}$$

$$\begin{aligned}
\mathrm{Gr}_4^P H^\bullet(M_{4,1}) = \langle & 32z; \beta^3 - 32\beta z, 32\alpha z; x^2 - yz + 22z^2, \alpha\beta^3 - 32yz, \frac{32}{3}\alpha\beta z - \frac{32}{3}yz; 2\beta y^2 + 30\alpha yz - 19\beta z^2, \\
& 3\alpha x^2 - 3\alpha yz + 11\beta z^2, \alpha^2\beta^3 - 32\alpha yz, 64\alpha z^2 - \frac{32}{3}\beta z^2; \alpha\beta y^2 + 26yz^2 - 45z^3, \\
& 3\alpha^2 x^2 + 13yz^2 + 9z^3, \alpha^3\beta^3 - 96yz^2 + 96z^3, \frac{32}{3}\alpha\beta z^2 - \frac{64}{3}yz^2; \alpha^2\beta y^2 + 111\alpha z^3 - 26\beta z^3, \\
& 3\alpha^3 x^2 + 87\alpha z^3 - 13\beta z^3, \alpha^4\beta^3 - 480\alpha z^3 + 96\beta z^3, -\frac{32}{3}\alpha yz^2 + 64\alpha z^3 - \frac{32}{3}\beta z^3; \\
& \alpha^3\beta y^2 + 11\alpha\beta z^3 - 37z^4, 3\alpha^4 x^2 + 16\alpha\beta z^3 - 29z^4, \alpha^5\beta^3 - 64\alpha\beta z^3 + 160z^4, -32yz^3 + 32z^4; \\
& \alpha^4\beta y^2 + 40\alpha z^4 - 11\beta z^4, 3\alpha^5 x^2 + 83\alpha z^4 - 16\beta z^4, \alpha^6\beta^3 - 288\alpha z^4 + 64\beta z^4, \\
& -32\alpha yz^3 + 32\alpha z^4; 2\alpha^5\beta y^2 - 36yz^4 + 21z^5, 2\alpha^6 x^2 - 34yz^4 + 35z^5, \alpha^7\beta^3 + 160yz^4 - 144z^5, \\
& -\frac{64}{3}\alpha\beta z^4 - \frac{128}{3}yz^4 + 96z^5; 2\alpha^6\beta y^2 + 39\alpha z^5 - 12\beta z^5, 3\alpha^7 x^2 + 78\alpha z^5 - 17\beta z^5, \\
& 3\alpha^8\beta^3 - 672\alpha z^5 + 160\beta z^5, -64\alpha yz^4 - 32\alpha z^5 + \frac{64}{3}\beta z^5; 5\alpha^7\beta y^2 - 35yz^5 + 9z^6, \\
& 15\alpha^8 x^2 - 175yz^5 + 162z^6, 5\alpha^9\beta^3 + 480yz^5 - 384z^6, -\frac{160}{3}\alpha\beta z^5 - \frac{160}{3}yz^5 + 192z^6; \\
& 10\alpha^8\beta y^2 + 102\alpha z^6 - 35\beta z^6, 30\alpha^9 x^2 + 744\alpha z^6 - 175\beta z^6, \alpha^{10}\beta^3 - 192\alpha z^6 + 48\beta z^6, \\
& -\frac{320}{3}\alpha yz^5 - 128\alpha z^6 + \frac{160}{3}\beta z^6; 9\alpha^{10}x^2 - 89yz^6 + 73z^7, 3\alpha^{11}\beta x + 56yz^6 - 40z^7, \\
& 3\alpha^{11}\beta^3 + 224yz^6 - 160z^7; 15\alpha^{11}y^2 + 221\alpha z^7 - 56\beta z^7, 15\alpha^{11}x^2 + 359\alpha z^7 - 89\beta z^7; z^8\rangle
\end{aligned}$$

$$\begin{aligned}
\mathrm{Gr}_5^P H^\bullet(M_{4,1}) = \langle & 32yz - 128z^2; 96\alpha yz - 64\beta z^2; 480yz^2 - 864z^3; 3360\alpha z^3 - 800\beta z^3; 64\alpha\beta z^3 - 224z^4; 2016\alpha z^4 - 576\beta z^4; \\
& -160yz^4 + 80z^5; 672\alpha z^5 - 224\beta z^5; -480yz^5; 192\alpha z^6 - 80\beta z^6; -\frac{224}{3}yz^6 - \frac{224}{3}z^7; -\frac{359}{15}\alpha z^7 + \frac{359}{15}\beta z^7\rangle
\end{aligned}$$

$$\mathrm{Gr}_6^P H^\bullet(M_{4,1}) = \langle 32\beta z; 128z^2; \tfrac{64}{3}\beta z^2; \tfrac{384}{5}z^3; \tfrac{128}{7}\beta z^3; 64z^4; \tfrac{128}{7}\beta z^4; 64z^5; \tfrac{64}{3}\beta z^5; \tfrac{384}{5}z^6; 32\beta z^6; 128z^7; -18\beta z^7; yz^7; \beta z^8\rangle$$

References

[1] K. Chung, H.-B. Moon, *Chow ring of the moduli space of stable sheaves supported on quartic curves,* Q. J. Math. 68 (2017), no. 3, 851–887.