# Relations in degree d

In[ ]:=
```
(*The symbol c[k,j] stands for c_k(j)*)
(*Properties of generators*)
c[k_, j_] := 0 /; k < 0
c[0, 1] = d;
c[0, 0] = 0;
```

In[ ]:=
```
(*Normalization*)
c[1, 0] = 0;
c[1, 1] = 0;
```

## Truncated relations in C[T^+] with degree d - 1, d

In[ ]:=
```
(*A[s,n] and B[s,n] are A_s, B_s defined in Proposition 2.6*)
A[s_, n_] := (-1) ^ (s + 2) c[s + 1, 0] + (-1) ^ (s + 1) (3 - n - chi / d) c[s, 1] +
    (-1) ^ s / (2 d ^ 2) ((n - 7 / 2) d + chi) ((n - 5 / 2) d + chi) c[s - 1, 2];
B[s_, n_] := (-1) ^ (s + 2) c[s + 1, 0] + (-1) ^ (s + 1) (2 - n - chi / d) c[s, 1] +
    (-1) ^ s / (2 d ^ 2) ((n - 5 / 2) d + chi) ((n - 3 / 2) d + chi) c[s - 1, 2];
```

In[ ]:=
```
(*Given a partition m of l,
term9[m,n] is the corresponding term in the sum (9)*)
term9[m_, n_] :=
  (Times @@ Function[c, 1 / c[[2]] !] /@ Tally[m]) * Times @@ (Function[s,
        (s - 1) ! (A[s, n] - Sum[(-1) ^ i beta ^ i / i ! B[s - i, n], {i, 0, 2}])] /@ m)
```

In[ ]:=
```
(*rule for truncating equations*)
trunc = c[j_, k_] /; (j + k - d < -1) → 0;
```

In[ ]:=
```
(*rule to simplify factorials,
which mathematica does not do automatically*)
simpfactorials = {(d + 1) ! → (d + 1) d !, d ! → d * (d - 1) !,
    (d - 1) ! → (d - 2) ! (d - 1), (d - 3) ! → (d - 2) ! / (d - 2) };
```

In[ ]:=
```
(*relsproduct1 are the (truncated) relations in degree l=
 d+1 in the product M x P^2 produced by Proposition 2.6*)
relsproduct1[n_] =
   Simplify[1 / (d - 3) ! Plus @@ Function[m, term9[m, n]] /@ {{d + 1},
          {d, 1}, {d - 1, 1, 1}, {d - 1, 2}, {d - 2, 2, 1},
          {d - 2, 3}, {d - 2, 1, 1, 1}} /. trunc //. simpfactorials];
```

In[ ]:=
```
(*relsproduct2 are the (truncated) relations in degree l=
 d+2 in the product M x P^2 produced by Proposition 2.6*)
relsproduct2[n_] =
  Simplify[1 / (d - 3) ! Plus @@ Function[m, term9[m, n]] /@ {{d + 2},
        {d + 1, 1}, {d, 1, 1}, {d, 2}, {d - 1, 2, 1}, {d - 1, 3},
        {d - 1, 1, 1, 1}, {d - 2, 4}, {d - 2, 3, 1}, {d - 2, 2, 2},
        {d - 2, 2, 1, 1}, {d - 2, 1, 1, 1, 1}} /. trunc //. simpfactorials];
```

In[ ]:=
```
(*relations (a) are relations in degree d-
  1 obtained by pushing forward relations in M x P^2 in degree l=d+1*)
relationa[n_] := Coefficient[relsproduct1[n], beta, 2];
```

In[ ]:=
```
(*relations (b) are relations in degree d obtained from relations in M x
  P^2 in degree l=d+1 by multiplying by beta and pushing forward*)
relationb[n_] := Coefficient[relsproduct1[n], beta, 1];
```

In[ ]:=
```
(*relations (c) produces relations in degree d by
  pushing forward relations in M x P^2 in degree l=d+2*)
relationc[n_] := Coefficient[relsproduct2[n], beta, 2];
```

## Verifying linear independence

In[ ]:=
```
(*Determinant appearing when expressing generators in degree d-
 1 in terms of lower generators*)
det1 = Factor[
  Det[Table[Coefficient[relationa[n], c[d - j, j]], {n, 1, 3}, {j, 0, 2}]]]
```

Out[ ]=

$$\frac{1}{4} (-1)^{3 d} \text{chi} (-2+d)^4 (-1+d) (-2 \text{chi}+d) (-\text{chi}+d)$$

In[ ]:=
```
(*Set of monomials Mon_2*)
monomials2 =
 Join[{c[d + 1, 0], c[d, 1], c[d - 1, 2]}, Table[c[d - 1, 0] × c[3 - i, i], {i, 0, 2}]]
```

Out[ ]=
```
{c[1 + d, 0], c[d, 1], c[-1 + d, 2],
 c[3, 0] × c[-1 + d, 0], c[2, 1] × c[-1 + d, 0], c[1, 2] × c[-1 + d, 0]}
```

In[∘]:=
```
(*Determinant of 6x6 matrix of coefficients
 of monomials2 in relations R_(b), R_(c)*)
eqs = Join @@ Table[{relationb[n], relationc[n]}, {n, 1, 3}];
det2 = Factor[
  Det[Table[Coefficient[eqs⟦i⟧, monomials2⟦j⟧], {i, 1, 6}, {j, 1, 6}]] //.
   {d! → d * (d - 1) !, (d - 1) ! → (d - 2) ! (d - 2), (d - 3) ! → (d - 2) ! / (d - 2) }]
```

Out[∘]=

$$4\,(-1)^{6\,d}\,(-2+d)^6\,(-1+d)^3\,d^4$$

## Truncated relations in C[T] with degree d - 1, d

In[∘]:=
```
(*Writes c[d,0],c[d-1,1], c[d-2,2] in terms of lower degree generators*)
sol1 = Simplify[Solve[relationa[1] == 0 && relationa[2] == 0 && relationa[3] == 0,
      {c[d, 0], c[d - 1, 1], c[d - 2, 2]}] //.
     {d! → d * (d - 1) !, (d - 2) ! → (d - 1) ! / (d - 1)}]⟦1⟧;
```

In[∘]:=
```
(*Writes c[d+1,0],c[d,1], c[d-1,2] in terms of lower degree generators*)
sol2 = Simplify[Solve[relationb[1] == 0 && relationc[1] == 0 && relationc[2] == 0,
     {c[d + 1, 0], c[d, 1], c[d - 1, 2]}]]⟦1⟧;
```

In[∘]:=
```
(*Basis of C[T]^2 x T^(d-2) ordered by <*)
basis = Join[Join @@ Table[c[d - i, i - 1] × c[4 - j, j - 1], {i, 1, 3}, {j, 1, 3}],
  Join @@ Table[{c[d - i, i - 1] × c[2, 0] ^ 2,
     c[d - i, i - 1] × c[2, 0] × c[0, 2], c[d - i, i - 1] × c[0, 2] ^ 2}, {i, 1, 3}]]
```

Out[∘]=

```
{c[3, 0] × c[-1 + d, 0], c[2, 1] × c[-1 + d, 0], c[1, 2] × c[-1 + d, 0],
 c[3, 0] × c[-2 + d, 1], c[2, 1] × c[-2 + d, 1], c[1, 2] × c[-2 + d, 1],
 c[3, 0] × c[-3 + d, 2], c[2, 1] × c[-3 + d, 2], c[1, 2] × c[-3 + d, 2],
 c[2, 0]^2 c[-1 + d, 0], c[0, 2] × c[2, 0] × c[-1 + d, 0], c[0, 2]^2 c[-1 + d, 0],
 c[2, 0]^2 c[-2 + d, 1], c[0, 2] × c[2, 0] × c[-2 + d, 1], c[0, 2]^2 c[-2 + d, 1],
 c[2, 0]^2 c[-3 + d, 2], c[0, 2] × c[2, 0] × c[-3 + d, 2], c[0, 2]^2 c[-3 + d, 2]}
```

In[∘]:=
```
(*Matrix in row-echelon form with the coefficients of the 3 relations R_1,
R_2, R_3. Obtained by plugging sol1, sol2 into the remaining 3 equations*)
TruncRelations = Together[Simplify[
    Factor[RowReduce[Outer[Function[{rel, mon}, Coefficient[rel, mon]],
       {relationb[2], relationb[3], relationc[3]} /. sol2 /. sol1, basis]]]]];
```

In[∘]:=
```
MatrixForm[TruncRelations]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{\text{chi}\,(\text{chi}-d)\,(2\,\text{chi}-d)\,(-2+d)}{8\,d^3} & \frac{-4+d}{8\,(-2+d)} & -\frac{\text{chi}\,(\text{chi}-d)\,(2\,\text{chi}-d)}{2\,(-2+d)\,d^3} & \frac{-48\,\text{chi}^4+96\,\text{chi}^3\,d+24\,\text{chi}^4\,d-}{} \\ 0 & 1 & 0 & 1 & 0 & \frac{-12\,\text{chi}^2+12\,\text{chi}\,d+6\,\text{chi}^2\,d-d^2-6\,\text{chi}\,d^2}{8\,d^2} & 0 & \frac{24\,\text{chi}^2-24\,\text{chi}\,d+2\,d^2-d^3}{8\,(-2+d)\,d^2} & - \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{2}{-2+d} & 0 & \end{pmatrix}$$

```
(* Relations[d,chi] is same as the above TruncRelations
 but also keeps track of the dependence on d and chi. *)
```

In[∘]:=
```
Relations[d_, chi_] := {{1, 0, 0, 0, 0, -1/(8 d^3) chi (chi - d) (2 chi - d) (-2 + d),
   (-4 + d)/(8 (-2 + d)), -((chi (chi - d) (2 chi - d)) / (2 (-2 + d) d^3)),
   1/(32 (-2 + d) d^4) (-48 chi^4 + 96 chi^3 d + 24 chi^4 d - 48 chi^2 d^2 -
      48 chi^3 d^2 + 18 chi^2 d^3 + 3 d^4 + 6 chi d^4 + 6 chi^2 d^4 - d^5 - 6 chi d^5),
   (-6 chi^2 + 6 chi d - 2 d^2 + d^3) / (2 chi (2 chi^2 - 3 chi d + d^2)),
   (12 chi^4 - 6 chi^2 d - 24 chi^3 d - 12 chi^4 d + 6 chi d^2 + 24 chi^2 d^2 + 24 chi^3 d^2 -
      12 chi d^3 - 30 chi^2 d^3 - 4 d^4 + 18 chi d^4 + 6 chi^2 d^4 + 4 d^5 - 6 chi d^5 - d^6) /
    (8 chi (chi - d) (2 chi - d) (-2 + d) d), (64 chi^6 - 192 chi^5 d - 128 chi^6 d +
      208 chi^4 d^2 + 384 chi^5 d^2 + 128 chi^6 d^2 - 96 chi^3 d^3 - 392 chi^4 d^3 - 384 chi^5 d^3 -
      64 chi^6 d^3 - 8 chi^2 d^4 + 144 chi^3 d^4 + 344 chi^4 d^4 + 192 chi^5 d^4 + 24 chi d^5 +
      94 chi^2 d^5 - 48 chi^3 d^5 - 160 chi^4 d^5 - 4 d^6 - 102 chi d^6 - 196 chi^2 d^6 + 12 d^7 +
      156 chi d^7 + 140 chi^2 d^7 - 9 d^8 - 108 chi d^8 - 24 chi^2 d^8 + 2 d^9 + 24 chi d^9) /
    (128 chi (chi - d) (2 chi - d) (-2 + d) d^4), (4 - d)/(4 (-2 + d) d), (4 - 15 d + 10 d^2 - 2 d^3)/(16 (-2 + d) d),
   1/(256 (-2 + d) d^5) (192 chi^4 - 384 chi^3 d - 240 chi^4 d + 192 chi^2 d^2 + 480 chi^3 d^2 +
      96 chi^4 d^2 - 96 chi^2 d^3 - 192 chi^3 d^3 - 12 d^4 - 144 chi d^4 - 240 chi^2 d^4 + 31 d^5 +
      336 chi d^5 + 240 chi^2 d^5 - 20 d^6 - 240 chi d^6 - 48 chi^2 d^6 + 4 d^7 + 48 chi d^7),
   ((2 + d) (6 chi^2 - 6 chi d + 2 d^2 - d^3)) / (16 chi (chi - d) (2 chi - d) (-2 + d)),
   (-128 chi^6 + 384 chi^5 d + 160 chi^6 d - 464 chi^4 d^2 - 480 chi^5 d^2 - 80 chi^6 d^2 +
      288 chi^3 d^3 + 688 chi^4 d^3 + 240 chi^5 d^3 - 20 chi^2 d^4 - 576 chi^3 d^4 - 476 chi^4 d^4 -
      60 chi d^5 + 118 chi^2 d^5 + 552 chi^3 d^5 + 84 chi^4 d^5 + 16 d^6 + 90 chi d^6 -
      188 chi^2 d^6 - 168 chi^3 d^6 - 24 d^7 - 48 chi d^7 + 90 chi^2 d^7 + 16 d^8 - 6 chi d^8 -
      6 chi^2 d^8 - 6 d^9 + 6 chi d^9 + d^10) / (64 chi (chi - d) (2 chi - d) (-2 + d)^2 d^4),
   (-3072 chi^8 + 12 288 chi^7 d + 3840 chi^8 d - 18 688 chi^6 d^2 - 15 360 chi^7 d^2 -
      1152 chi^8 d^2 + 13 056 chi^5 d^3 + 21 312 chi^6 d^3 + 4608 chi^7 d^3 - 384 chi^8 d^3 -
      3904 chi^4 d^4 - 10 176 chi^5 d^4 - 1952 chi^6 d^4 + 1536 chi^7 d^4 + 384 chi^3 d^5 -
      816 chi^4 d^5 - 10 272 chi^5 d^5 - 6656 chi^6 d^5 - 208 chi^2 d^6 + 672 chi^3 d^6 +
      12 712 chi^4 d^6 + 14 592 chi^5 d^6 + 1280 chi^6 d^6 + 144 chi d^7 + 972 chi^2 d^7 -
      2928 chi^3 d^7 - 12 680 chi^4 d^7 - 3840 chi^5 d^7 - 24 d^8 - 444 chi d^8 - 1622 chi^2 d^8 +
      2832 chi^3 d^8 + 3680 chi^4 d^8 + 44 d^9 + 606 chi d^9 + 1180 chi^2 d^9 - 960 chi^3 d^9 - 34 d^10 -
```

```
    420 chi d^10 - 316 chi^2 d^10 + 13 d^11 + 156 chi d^11 + 24 chi^2 d^11 - 2 d^12 - 24 chi d^12) /
   (1024 chi (chi - d) (2 chi - d) (-2 + d)^2 d^6)},
{0, 1, 0, 1, 0, 1/(8 d^2) (-12 chi^2 + 12 chi d + 6 chi^2 d - d^2 - 6 chi d^2),
 0, (24 chi^2 - 24 chi d + 2 d^2 - d^3) / (8 (-2 + d) d^2),
 -((chi (chi - d) (2 chi - d) (-16 + 8 d + d^2)) / (8 (-2 + d) d^3)),
 (3 (-1 + d) d)/(chi (chi - d) (-2 + d)),
 (4 chi^2 - 4 chi d - 14 chi^2 d + 14 chi d^2 + 16 chi^2 d^2 + 3 d^3 -
    16 chi d^3 - 2 chi^2 d^3 - 3 d^4 + 2 chi d^4) / (4 chi (chi - d) (-2 + d) d),
 ((-1 + d) (48 chi^4 - 96 chi^3 d - 48 chi^4 d + 48 chi^2 d^2 + 96 chi^3 d^2 + 48 chi^4 d^2 -
      36 chi^2 d^3 - 96 chi^3 d^3 - 3 d^4 - 12 chi d^4 - 4 chi^2 d^4 + 6 d^5 + 52 chi d^5 +
      8 chi^2 d^5 - 8 chi d^6)) / (64 chi (chi - d) (-2 + d) d^3), 0, 0,
 (chi (chi - d) (2 chi - d) (-8 + 10 d - d^2 - 4 d^3 + d^4)) / (8 (-2 + d) d^4),
 (-8 chi^2 + 8 chi d + 8 chi^2 d + 6 d^2 - 8 chi d^2 + 2 chi^2 d^2 - 3 d^3 - 2 chi d^3 - 3 d^4) /
  (8 chi (chi - d) (-2 + d)^2 d),
 (96 chi^4 - 192 chi^3 d - 144 chi^4 d + 72 chi^2 d^2 + 288 chi^3 d^2 + 96 chi^4 d^2 + 24 chi d^3 -
    60 chi^2 d^3 - 192 chi^3 d^3 - 12 d^4 - 84 chi d^4 + 44 chi^2 d^4 + 18 d^5 + 52 chi d^5 - 16 chi^2
     d^5 - 9 d^6 + 16 chi d^6 + 2 chi^2 d^6 + 3 d^7 - 2 chi d^7) / (32 chi (chi - d) (-2 + d)^2 d^3),
 (1920 chi^6 - 5760 chi^5 d - 2304 chi^6 d + 6240 chi^4 d^2 + 6912 chi^5 d^2 - 288 chi^6 d^2 -
    2880 chi^3 d^3 - 8016 chi^4 d^3 + 864 chi^5 d^3 + 1536 chi^6 d^3 + 552 chi^2 d^4 +
    4512 chi^3 d^4 + 576 chi^4 d^4 - 4608 chi^5 d^4 - 384 chi^6 d^4 - 72 chi d^5 - 1368 chi^2 d^5 -
    2592 chi^3 d^5 + 3936 chi^4 d^5 + 1152 chi^5 d^5 + 18 d^6 + 264 chi d^6 + 1774 chi^2 d^6 -
    192 chi^3 d^6 - 1200 chi^4 d^6 - 33 d^7 - 334 chi d^7 - 864 chi^2 d^7 + 480 chi^3 d^7 +
    21 d^8 + 192 chi d^8 + 28 chi^2 d^8 - 6 d^9 - 76 chi d^9 - 8 chi^2 d^9 + 8 chi d^10) /
  (512 chi (chi - d) (-2 + d)^2 d^5)}, {0, 0, 1, 0, 0, 0, 2/(-2 + d), 0,
 (12 chi^2 - 12 chi d - d^2)/(8 (-2 + d) d), (4 d^2) / (chi (2 chi^2 - 3 chi d + d^2)),
 (-6 chi^2 + 6 chi d + 6 chi^2 d - 6 chi d^2 - d^3) / (chi (chi - d) (2 chi - d)),
 -(((-1 + 2 d) (-12 chi^2 + 12 chi d + 12 chi^2 d - d^2 - 12 chi d^2)) /
    (16 chi (chi - d) (2 chi - d))), 0, (3 - d)/(-2 + d),
 ((-3 + d) (12 chi^2 - 12 chi d - 12 chi^2 d + d^2 + 12 chi d^2)) / (8 (-2 + d) d^2),
 -((d^2 (2 + d)) / (2 chi (chi - d) (2 chi - d) (-2 + d))),
 (-48 chi^4 + 96 chi^3 d + 72 chi^4 d - 36 chi^2 d^2 - 144 chi^3 d^2 - 12 chi d^3 +
    66 chi^2 d^3 + 4 d^4 + 6 chi d^4 - 6 chi^2 d^4 - 2 d^5 + 6 chi d^5 + d^6) /
  (8 chi (chi - d) (2 chi - d) (-2 + d) d^2), (128 chi^6 - 384 chi^5 d - 1344 chi^6 d +
    224 chi^4 d^2 + 4032 chi^5 d^2 + 1216 chi^6 d^2 + 192 chi^3 d^3 - 3792 chi^4 d^3 - 3648 chi^5 d^3 -
    232 chi^2 d^4 + 864 chi^3 d^4 + 3520 chi^4 d^4 + 72 chi d^5 + 372 chi^2 d^5 - 960 chi^3 d^5 -
    6 d^6 - 132 chi d^6 - 212 chi^2 d^6 + 5 d^7 + 84 chi d^7 + 24 chi^2 d^7 - 2 d^8 - 24 chi d^8) /
  (128 chi (chi - d) (2 chi - d) (-2 + d) d^4)}}
```

```
(* Put the Relations[d,chi] into the matrix form. TopRelations
 for T^(d-2)xT^(2) and ExtRelations for T^(d-2)xSym^2(T^1). *)
```

In[∘]:=
```
TopRelations[d_, chi_] := {{Relations[d, chi]⟦1, 1 ;; 3⟧,
    Relations[d, chi]⟦1, 4 ;; 6⟧, Relations[d, chi]⟦1, 7 ;; 9⟧},
   {Relations[d, chi]⟦2, 1 ;; 3⟧,
    Relations[d, chi]⟦2, 4 ;; 6⟧, Relations[d, chi]⟦2, 7 ;; 9⟧},
   {Relations[d, chi]⟦3, 1 ;; 3⟧,
    Relations[d, chi]⟦3, 4 ;; 6⟧, Relations[d, chi]⟦3, 7 ;; 9⟧}}
```

In[∘]:=
```
ExtRelations[d_, chi_] := {{Relations[d, chi]⟦1, 10 ;; 12⟧,
    Relations[d, chi]⟦1, 13 ;; 15⟧, Relations[d, chi]⟦1, 16 ;; 18⟧},
   {Relations[d, chi]⟦2, 10 ;; 12⟧,
    Relations[d, chi]⟦2, 13 ;; 15⟧, Relations[d, chi]⟦2, 16 ;; 18⟧},
   {Relations[d, chi]⟦3, 10 ;; 12⟧,
    Relations[d, chi]⟦3, 13 ;; 15⟧, Relations[d, chi]⟦3, 16 ;; 18⟧}}
```

# Proof of main theorem

In[∘]:=
```
(* Automorphism matrices; AutA for SL(T^{d-2}), AutB for SL(T^2),
AutU for Hom(T^2, Sym^2(T^1)) and AutV for GL(Sym^2(T^1)). *)
AutA := Table[a[i, j], {i, 1, 3}, {j, 1, 3}]
AutB := Table[b[i, j], {i, 1, 3}, {j, 1, 3}]
AutU := Table[u[i, j], {i, 1, 3}, {j, 1, 3}]
AutV := Table[v[i, j], {i, 1, 3}, {j, 1, 3}]
```

In[∘]:=
```
(* Shuffle matrix S which is obtained by AutA^tr(-)AutB. *)
S := Table[s[i, j], {i, 1, 3}, {j, 1, 3}]
```

## Find the shuffle matrix

In[∘]:=
```
(* Determinant map defined by three
 dimensional span(TopRelations[d,chi]). *)
DET[d_, chi_] := Simplify[Det[Sum[TopRelations[d, chi]⟦i⟧ * x[i], {i, 1, 3}]]]
```

In[∘]:= 
```
DET[d, chi]
```
Out[∘]=

$$-\frac{1}{32\,(-2+d)\,d^6}\,\mathrm{chi}\,\left(2\,\mathrm{chi}^2-3\,\mathrm{chi}\,d+d^2\right)\,\Big(4\,\mathrm{chi}^3\,(-2+d)\,x[1]^3-6\,\mathrm{chi}^2\,(-2+d)\,d\,x[1]^2\,(x[1]+4\,x[2])+2\,\mathrm{chi}\,(-2+d)\,d^2\,x[1]\,\left(x[1]^2+12\,x[1]\times x[2]+24\,x[2]^2\right)+d^3\,x[2]\,\left(\left(8-5\,d+d^2\right)x[1]^2-4\left(-16+8\,d+d^2\right)x[2]^2+8\,x[1]\,(-3\,(-2+d)\,x[2]+d\,x[3])\right)\Big)$$

In[∘]:= 
```
(* Cubic curve defined by DET[d,chi]
 has a unique nodal singularity at [0:0:1]. *)
Solve[DET[d, chi] == 0 && D[DET[d, chi], x[1]] == 0 &&
  D[DET[d, chi], x[2]] == 0 && D[DET[d, chi], x[3]] == 0 &&
  (x[1] == 1 ∨ x[2] == 1 ∨ x[3] == 1), {x[1], x[2], x[3]}, Complexes]
```
Out[∘]=

```
{{x[1] → 0, x[2] → 0, x[3] → 1}}
```

In[∘]:= 
```
(* Solve the shuffle matrix S using the fact that it sends
 the determinant curve of (d,chi1) to that of (d,chi2). *)
f := CoefficientList[DET[d, chi1], {x[1], x[2], x[3]}, {4, 4, 4}]
g := CoefficientList[(DET[d, chi2] /.
    {x[1] → Sum[s[j, 1] * x[j], {j, 1, 3}], x[2] → Sum[s[j, 2] * x[j], {j, 1, 3}],
     x[3] → Sum[s[j, 3] * x[j], {j, 1, 3}]}), {x[1], x[2], x[3]}, {4, 4, 4}]
coeff[u_, v_, w_] := Simplify[(f - g)[[u + 1, v + 1, w + 1]]]
```

In[∘]:= 
```
(* By the singularity consideration, we have the following. *)
s[3, 1] := 0
s[3, 2] := 0
```

In[∘]:= 
```
(* The following equations allow us
 to divide the solutions into two cases. *)
coeff[0, 2, 1]
coeff[2, 0, 1]
coeff[1, 1, 1]
```
Out[∘]=

$$\frac{\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)s[2,1]\times s[2,2]\times s[3,3]}{4\,(-2+d)\,d^2}$$

Out[∘]=

$$\frac{\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)s[1,1]\times s[1,2]\times s[3,3]}{4\,(-2+d)\,d^2}$$

Out[∘]=

$$\frac{1}{4\,(-2+d)\,d^2}\Big(-2\,\mathrm{chi1}^3+3\,\mathrm{chi1}^2\,d-\mathrm{chi1}\,d^2+\mathrm{chi2}\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)(s[1,2]\times s[2,1]+s[1,1]\times s[2,2])\,s[3,3]\Big)$$

In[∘]:=
```
Solve[s[2, 1] × s[2, 2] × s[3, 3] == 0 && s[1, 1] × s[1, 2] × s[3, 3] == 0 &&
  (s[1, 2] × s[2, 1] + s[1, 1] × s[2, 2]) s[3, 3] ≠ 0,
 {s[1, 1], s[2, 2], s[3, 3], s[1, 2], s[2, 1]}]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=
```
{{s[1, 1] → 0, s[2, 2] → 0}, {s[1, 2] → 0, s[2, 1] → 0}}
```

## Type I solutions

In[∘]:=
```
(* We start with the first case,
namely s[1,1]=s[2,2]=0, which we call Type I. *)
s[1, 1] := 0
s[2, 2] := 0
```

In[∘]:=
```
coeff[0, 3, 0]
coeff[3, 0, 0]
coeff[1, 1, 1]
```

Out[∘]=

$$\frac{1}{16\,(-2+d)\,d^6}\left(4\,\text{chi1}^3\,d^3\,\left(-16+8\,d+d^2\right)-6\,\text{chi1}^2\,d^4\,\left(-16+8\,d+d^2\right)+2\,\text{chi1}\,d^5\,\left(-16+8\,d+d^2\right)+\text{chi2}^2\,(-2+d)\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2\,s[2,1]^3\right)$$

Out[∘]=

$$-\frac{1}{16\,d^6}\left(4\,\text{chi1}^6-12\,\text{chi1}^5\,d+13\,\text{chi1}^4\,d^2-6\,\text{chi1}^3\,d^3+\text{chi1}^2\,d^4+\frac{2\,\text{chi2}\,d^3\,\left(-16+8\,d+d^2\right)\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,s[1,2]^3}{-2+d}\right)$$

Out[∘]=

$$\frac{-2\,\text{chi1}^3+3\,\text{chi1}^2\,d-\text{chi1}\,d^2+\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,s[1,2]\times s[2,1]\times s[3,3]}{4\,(-2+d)\,d^2}$$

In[∘]:=
```
(* The above three equations imply that s[3,3]^3=
 1. By scaling the matrices using a cubic root of unity,
if necessary, we may assume that s[3,3]=1. *)
s[3, 3] := 1
```

In[∘]:=
```
(* From coeff[0,3,0]=0,
s[2,1] is given by a cubic root. From coeff[1,1,1]=0 and s[3,3]=1,
we also solve s[1,2]. *)
s[2, 1] := CubeRoot[-((2 chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2)) /
     (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2))] * d
```

In[∘]:=
```
Solve[coeff[1, 1, 1] == 0, s[1, 2]]
```

In[∘]:=

$$s[1,2] := -\left(\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big/\left(2^{1/3}\,\text{chi2}\,d\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)\right)$$

```
(* Solve the rest of S. We carefully check
 that the coefficients are nonzero before solving it. *)
```

In[∘]:=
```
Coefficient[coeff[1, 2, 0], s[2, 3], 1]
Coefficient[coeff[2, 1, 0], s[1, 3], 1]
```

Out[∘]=

$$\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{4\,(-2+d)\,d^2}$$

Out[∘]=

$$\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{4\,(-2+d)\,d^2}$$

In[∘]:=
```
Simplify[Solve[coeff[1, 2, 0] == 0 && coeff[2, 1, 0] == 0, {s[2, 3], s[1, 3]}]]
```

In[∘]:=

$$s[2,3] := \frac{1}{8\,(2-d)\,d^2}\,(-2+d)\left(-48\,\text{chi1}\,(-2+d)-48\,d+24\,d^2+2^{1/3}\left(24\,\text{chi2}^2\,(-2+d)-24\,\text{chi2}\,(-2+d)\,d-d^2\left(8-5\,d+d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)$$

$$s[1,3] := \frac{1}{8\,d^3}\,(-2+d)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+\frac{d^2\left(8-5\,d+d^2\right)}{-2+d}+\left(24\times 2^{2/3}\,\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big/\left(\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)-\left(12\times 2^{2/3}\,\text{chi1}\,d\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big/\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)\right)$$

In[∘]:= MatrixForm[S]

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & -\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{2^{1/3}\,\text{chi2}\,d\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,\sqrt[3]{\frac{\text{chi1}\,(-16+8\,d+d^2)\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}^2\,(-2+d)\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)^2}}} \\ -2^{1/3}\,d\,\sqrt[3]{\frac{\text{chi1}\,(-16+8\,d+d^2)\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}^2\,(-2+d)\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)^2}} & 0 \\ 0 & 0 \end{pmatrix}$$

## Type I - Solve for A, B

```
(* We have solved the shuffle matrix S uniquely
  up to a choice of a cubic root in the case of s[1,1]=
 s[2,2]=0. We claim that there is no compatible pair (AutA,AutB) such that
   AutA^t(-)AutB corresponds to this shuffle matrix. To linearize the problem,
we use the inverse matrix AutB^(-1). *)
```

In[∘]:=

```
At := Transpose[AutA]
Binv := Table[bb[i, j], {i, 1, 3}, {j, 1, 3}]
Diff[d_, chi1_, chi2_] := Table[At.TopRelations[d, chi1][[i]] -
    Sum[S[[i, j]] * TopRelations[d, chi2][[j]].Binv, {j, 1, 3}], {i, 1, 3}]
```

```
(* Use Diff[[3]] first. *)
(* From now on, we skip checking that we are solving
 linear equations with nonzero leading coefficients,
unless it is nontrivial, for simplicity of the code. The
 solutions have denominators which are evidently nonzero. *)
```

In[∘]:=

```
Solve[Diff[d, chi1, chi2][[3]] == 0,
 {bb[3, 1], bb[3, 2], bb[3, 3], bb[1, 1], bb[1, 2], bb[1, 3], a[3, 2], a[1, 2]}]
```

In[∘]:=
```
bb[3, 1] := 2 a[3, 1] / (-2 + d)
bb[3, 2] := 0
bb[3, 3] := - 1/(8 (-2 + d) d)
    (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1])
bb[1, 1] := -(((12 chi2^2 - 12 chi2 d - d^2) a[3, 1]) / (8 (-2 + d) d)) + a[3, 3]
bb[1, 2] := 0
bb[1, 3] := 1/(128 (-2 + d) d^2) (12 chi2^2 - 12 chi2 d - d^2)
     (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1]) -
   1/(16 d) (16 d a[1, 3] - 8 d^2 a[1, 3] - 12 chi1^2 a[3, 3] + 12 chi1 d a[3, 3] + d^2 a[3, 3])
a[3, 2] := 0
a[1, 2] := 0
```

In[∘]:=
```
Simplify[Diff[d, chi1, chi2][[3]]]
```

Out[∘]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* Solve b[2,j] using Diff[[2]][3,j]. *)
```

In[∘]:=
```
Simplify[
 Solve[Diff[d, chi1, chi2][[2]][[3, 1]] == 0 && Diff[d, chi1, chi2][[2]][[3, 2]] == 0 &&
   Diff[d, chi1, chi2][[2]][[3, 3]] == 0, {bb[2, 1], bb[2, 2], bb[2, 3]}]]
```

In[∘]:=
```
bb[2, 1] :=
 ((-2 + d) d^2 (-16 a[2, 3] + 1/((-2 + d)^2 d^3) 2^(1/3) (-24 chi2^4 (-2 + d) + 48 chi2^3 (-2 + d) d +
          (-3 + d) d^4 + 6 chi2 (-1 + d) d^4 - 6 chi2^2 d^2 (-8 + 3 d + d^2))
        a[3, 1] CubeRoot[(chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2)) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2)] -
      1/(2 × 2^(2/3) (-2 + d)^2) (-4 + d) (-12 chi2^2 a[3, 1] + 12 chi2 d a[3, 1] +
          d (-16 a[3, 3] + d (a[3, 1] + 8 a[3, 3])))
        CubeRoot[(chi1 (-16 + 8 d + d^2) (2 chi1^2 - 3 chi1 d + d^2)) / (chi2^2 (-2 + d) (2 chi2^2 - 3 chi2 d + d^2)^2)] -
      1/((2 - d) d^2) 4 a[3, 3] (48 chi1 (-2 + d) + 48 d - 24 d^2 -
          2^(1/3) (24 chi2^2 (-2 + d) - 24 chi2 (-2 + d) d - d^2 (8 - 5 d + d^2))
```

$$\sqrt[3]{\frac{\mathrm{chi1}\,\left(-16+8\,\mathrm{d}+\mathrm{d}^2\right)\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,\mathrm{d}+\mathrm{d}^2\right)}{\mathrm{chi2}^2\,(-2+\mathrm{d})\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,\mathrm{d}+\mathrm{d}^2\right)^2}}\Bigg)\Bigg)\Bigg)\Bigg/\Bigg(8\times 2^{1/3}\,\mathrm{chi2}$$

$$(\mathrm{chi2}-\mathrm{d})\,(-2\,\mathrm{chi2}+\mathrm{d})\,\sqrt[3]{\frac{\mathrm{chi1}\,\left(-16+8\,\mathrm{d}+\mathrm{d}^2\right)\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,\mathrm{d}+\mathrm{d}^2\right)}{\mathrm{chi2}^2\,(-2+\mathrm{d})\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,\mathrm{d}+\mathrm{d}^2\right)^2}}\Bigg)$$

$$\mathrm{bb}[2,2]:=\Big(\mathrm{d}^3\,(8\,\mathrm{a}[1,3]-\mathrm{a}[3,3])+$$

$$24\,\mathrm{chi1}^2\,\mathrm{a}[3,3]-24\,\mathrm{chi1}\,\mathrm{d}\,\mathrm{a}[3,3]+2\,\mathrm{d}^2\,(-8\,\mathrm{a}[1,3]+\mathrm{a}[3,3])\Big)\Big/$$

$$\Bigg(4\times 2^{1/3}\,\mathrm{chi2}\,(\mathrm{chi2}-\mathrm{d})\,(2\,\mathrm{chi2}-\mathrm{d})\,\sqrt[3]{\frac{\mathrm{chi1}\,\left(-16+8\,\mathrm{d}+\mathrm{d}^2\right)\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,\mathrm{d}+\mathrm{d}^2\right)}{\mathrm{chi2}^2\,(-2+\mathrm{d})\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,\mathrm{d}+\mathrm{d}^2\right)^2}}\Bigg)$$

$$\mathrm{bb}[2,3]:=\Bigg((-2+\mathrm{d})\,\Bigg(32\,\mathrm{d}^2\,\left(-6\,\mathrm{chi1}^2\,(-2+\mathrm{d})+6\,\mathrm{chi1}\,(-2+\mathrm{d})\,\mathrm{d}+\mathrm{d}^2\right)\,\mathrm{a}[2,3]+$$

$$\frac{1}{-2+\mathrm{d}}\,32\,\mathrm{chi1}\,(\mathrm{chi1}-\mathrm{d})\,(2\,\mathrm{chi1}-\mathrm{d})\,\mathrm{d}\,\left(-16+8\,\mathrm{d}+\mathrm{d}^2\right)\,\mathrm{a}[3,3]+$$

$$\frac{1}{(-2+\mathrm{d})^2}\,2^{1/3}\,\Big(-24\,\mathrm{chi2}^4\,(-2+\mathrm{d})+48\,\mathrm{chi2}^3\,(-2+\mathrm{d})\,\mathrm{d}+$$

$$(-3+\mathrm{d})\,\mathrm{d}^4+6\,\mathrm{chi2}\,(-1+\mathrm{d})\,\mathrm{d}^4-6\,\mathrm{chi2}^2\,\mathrm{d}^2\,\left(-8+3\,\mathrm{d}+\mathrm{d}^2\right)\Big)$$

$$\Big(\mathrm{d}^2\,(8\,\mathrm{a}[1,1]-\mathrm{a}[3,1])+12\,\mathrm{chi1}^2\,\mathrm{a}[3,1]-4\,\mathrm{d}\,(4\,\mathrm{a}[1,1]+3\,\mathrm{chi1}\,\mathrm{a}[3,1])\Big)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,\left(-16+8\,\mathrm{d}+\mathrm{d}^2\right)\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,\mathrm{d}+\mathrm{d}^2\right)}{\mathrm{chi2}^2\,(-2+\mathrm{d})\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,\mathrm{d}+\mathrm{d}^2\right)^2}}+\frac{1}{2\times 2^{2/3}\,(-2+\mathrm{d})^2}\,(4-\mathrm{d})\,\mathrm{d}^3$$

$$\Big(\left(12\,\mathrm{chi2}^2-12\,\mathrm{chi2}\,\mathrm{d}-\mathrm{d}^2\right)\,\Big(-12\,\mathrm{chi1}^2\,\mathrm{a}[3,1]+\mathrm{d}^2\,(-8\,\mathrm{a}[1,1]+\mathrm{a}[3,1])+$$

$$4\,\mathrm{d}\,(4\,\mathrm{a}[1,1]+3\,\mathrm{chi1}\,\mathrm{a}[3,1])\Big)+8\,(-2+\mathrm{d})\,\mathrm{d}\,\Big(\mathrm{d}^2\,(8\,\mathrm{a}[1,3]-\mathrm{a}[3,3])+$$

$$12\,\mathrm{chi1}^2\,\mathrm{a}[3,3]-4\,\mathrm{d}\,(4\,\mathrm{a}[1,3]+3\,\mathrm{chi1}\,\mathrm{a}[3,3])\Big)\Big)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,\left(-16+8\,\mathrm{d}+\mathrm{d}^2\right)\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,\mathrm{d}+\mathrm{d}^2\right)}{\mathrm{chi2}^2\,(-2+\mathrm{d})\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,\mathrm{d}+\mathrm{d}^2\right)^2}}+$$

$$\frac{1}{2-\mathrm{d}}\,4\,\mathrm{d}\,\Big(-12\,\mathrm{chi1}^2\,\mathrm{a}[3,3]+\mathrm{d}^2\,(-8\,\mathrm{a}[1,3]+\mathrm{a}[3,3])+$$

$$4\,\mathrm{d}\,(4\,\mathrm{a}[1,3]+3\,\mathrm{chi1}\,\mathrm{a}[3,3])\Big)\,\Bigg(48\,\mathrm{chi1}\,(-2+\mathrm{d})+48\,\mathrm{d}-24\,\mathrm{d}^2-$$

$$2^{1/3}\,\left(24\,\mathrm{chi2}^2\,(-2+\mathrm{d})-24\,\mathrm{chi2}\,(-2+\mathrm{d})\,\mathrm{d}-\mathrm{d}^2\,\left(8-5\,\mathrm{d}+\mathrm{d}^2\right)\right)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,\left(-16+8\,\mathrm{d}+\mathrm{d}^2\right)\,\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,\mathrm{d}+\mathrm{d}^2\right)}{\mathrm{chi2}^2\,(-2+\mathrm{d})\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,\mathrm{d}+\mathrm{d}^2\right)^2}}\Bigg)\Bigg)\Bigg)\Bigg/$$

$$\Big(128\times 2^{1/3}\,\mathrm{chi2}\,(\mathrm{chi2}-\mathrm{d})\,\mathrm{d}^2\,(-2\,\mathrm{chi2}+\mathrm{d})$$

$$\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\;\Bigg)$$

```
(* Use the rest of Diff⟦2⟧. *)
```

In[∘]:=
```
Simplify[
 Solve[Diff[d, chi1, chi2]⟦2⟧⟦1, 1⟧ == 0 && Diff[d, chi1, chi2]⟦2⟧⟦2, 1⟧ == 0 &&
   Diff[d, chi1, chi2]⟦2⟧⟦1, 2⟧ == 0, {a[2, 1], a[2, 2], a[1, 1]}]]
```

In[∘]:=

$$a[2,1]:=-2^{1/3}\,d\left(\left(\left(-12\,\text{chi2}^2+12\,\text{chi2}\,d+d^2\right)a[3,1]\right)/\left(8\,(-2+d)\,d\right)+a[3,3]\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}+\frac{1}{4\,(2-d)\,d^2}\,a[3,1]\Bigg(-48\,\text{chi1}\,(-2+d)-48\,d+24\,d^2+2^{1/3}\left(24\,\text{chi2}^2\,(-2+d)-24\,\text{chi2}\,(-2+d)\,d-d^2\left(8-5\,d+d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\Bigg)$$

$$a[2,2]:=\frac{1}{2\times 2^{2/3}\,d^2}\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,a[3,1]\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}$$

$$a[1,1]:=\left(\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)a[3,1]\right)/\left(8\,(-2+d)\,d^2\right)$$

```
Simplify[Solve[Diff[d, chi1, chi2]⟦2⟧⟦1, 3⟧ == 0, a[3, 1]]]
```

In[∘]:=

$$a[3,1]:=\Bigg((-2+d)\,d^2\left(d^3\,(8\,a[1,3]-a[3,3])+24\,\text{chi1}^2\,a[3,3]-24\,\text{chi1}\,d\,a[3,3]+2\,d^2\,(-8\,a[1,3]+a[3,3])\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\Bigg)\Big/\left(2^{2/3}\,\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)$$

In[∘]:=
```
Simplify[Diff[d, chi1, chi2]⟦2⟧]
```

Out[∘]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* We would like to use Diff⟦1⟧ to solve a[1,3] and a[2,3]. We first check
 that the leading coefficients of the linear equations are nonzero. *)
```

In[ ]:=
```
Simplify[Coefficient[Diff[d, chi1, chi2]⟦1⟧⟦2, 1⟧, a[1, 3], 1]]
Simplify[Coefficient[Diff[d, chi1, chi2]⟦1⟧⟦1, 1⟧, a[2, 3], 1]]
```

Out[ ]=

$$\frac{(-2+d)\, d\, \left(-24\, \text{chi2}^2 + 24\, \text{chi2}\, d + (-2+d)\, d^2\right)}{2\, \text{chi2}\, \left(-16 + 8\, d + d^2\right)\, \left(2\, \text{chi2}^2 - 3\, \text{chi2}\, d + d^2\right)}$$

Out[ ]=

$$\frac{2^{1/3}\, \text{chi1}\, (-2+d)\, d\, \left(2\, \text{chi1}^2 - 3\, \text{chi1}\, d + d^2\right)}{\text{chi2}^2\, (\text{chi2} - d)^2\, (-2\, \text{chi2} + d)^2\, \sqrt[3]{\frac{\text{chi1}\, \left(-16+8\, d+d^2\right)\, \left(2\, \text{chi1}^2-3\, \text{chi1}\, d+d^2\right)}{\text{chi2}^2\, (-2+d)\, \left(2\, \text{chi2}^2-3\, \text{chi2}\, d+d^2\right)^2}}^{\,2}}$$

In[ ]:=
```
(* Check that (-24 chi2^2+24 chi2 d+(-2+d) d^2) is nonzero
 if d is coprime to chi1 and chi2. Note that coprimality
 forces d to divide 24 for such solutions to exist. *)
Divlist := Divisors[24]

For[i = 1, i < Length[Divlist] + 1, i++,
 For[chi2 = 1, chi2 < Divlist⟦i⟧ / 2 , chi2++,
  If[(
     (-24 chi2^2 + 24 chi2 d + (-2 + d) d^2) /. d → Divlist⟦i⟧) == 0,
   Print[{Divlist⟦i⟧, chi1, chi2}]]
 ]
]

Clear[d, chi1, chi2]
```

```
(* Therefore we solve a[1,3] and a[2,3] using Diff⟦1⟧⟦2,1⟧ and Diff⟦1⟧⟦1,1⟧. *)
```

In[ ]:=
```
Simplify[Solve[Diff[d, chi1, chi2]⟦1⟧⟦2, 1⟧ == 0 &&
   Diff[d, chi1, chi2]⟦1⟧⟦1, 1⟧ == 0, {a[1, 3], a[2, 3]}]]
```

In[•]:=

```
a[1, 3] :=
```

$$\frac{1}{8\,(-2+d)\,d^2}\,a[3,3]\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2-\left(8\times 2^{2/3}\,\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big/\left(\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}\right)\right)$$

```
a[2, 3] :=
```

$$\left(a[3,3]\left(24\,\text{chi1}\,(\text{chi1}-d)\,(-2+d)\,(-2\,\text{chi1}+d)^2\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)+2^{1/3}\,\text{chi1}\,(-2+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(48\,\text{chi2}^4\,(4+d)-96\,\text{chi2}^3\,d\,(4+d)+d^4\left(8-6\,d+d^2\right)-6\,\text{chi2}\,d^3\left(8-2\,d+d^2\right)+6\,\text{chi2}^2\,d^2\left(40+6\,d+d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}+8\times 2^{2/3}\,\text{chi2}^2\,(-4+d)\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2\sqrt[3]{\frac{\text{chi1}\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}^2\,(-2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2}}^{\,2}\right)\right)\Big/\left(4\,\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,(-2+d)\,d^2\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\right)$$

```
(* Conclude a[3,3]=0 from Diff⟦1⟧[1,2]. *)
```

In[•]:=

```
Simplify[Diff[d, chi1, chi2]⟦1⟧⟦1, 2⟧]
```

Out[•]=

$$-\frac{2\,\text{chi1}\,(-4+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)a[3,3]}{d\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)}$$

In[•]:=

```
a[3, 3] := 0
```

```
(* This forces AutA and Binv to be zero,
hence a contradiction with Det[AutA]=Det[AutB]=1. *)
```

In[ ]:=
```
AutA
Binv
```

Out[ ]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

Out[ ]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

## Type II

In[ ]:=
```
Clear[s, a, bb]
```

In[ ]:=
```
(* Now we consider the type II case,
namely s[1,2]=s[2,1]=0. Recall that we also had s[3,1]=
   s[3,2]=0 from the singulariy consideration. *)
s[3, 1] := 0
s[3, 2] := 0
s[1, 2] := 0
s[2, 1] := 0
```

```
(* We solve the matrix S similar to the previous case. Consider
 the following three equations involving s[1,1],s[2,2],s[3,3].*)
```

In[ ]:=
```
coeff[3, 0, 0]
coeff[0, 3, 0]
coeff[1, 1, 1]
```

Out[ ]=

$$\frac{1}{16\,d^6}\left(-4\,\text{chi1}^6+12\,\text{chi1}^5\,d-13\,\text{chi1}^4\,d^2+6\,\text{chi1}^3\,d^3-\text{chi1}^2\,d^4+\text{chi2}^2\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)^2\,s[1,\,1]^3\right)$$

Out[ ]=

$$\frac{\left(-16+8\,d+d^2\right)\,\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)-\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,s[2,\,2]^3\right)}{8\,(-2+d)\,d^3}$$

Out[ ]=

$$\frac{-2\,\text{chi1}^3+3\,\text{chi1}^2\,d-\text{chi1}\,d^2+\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,s[1,\,1]\times s[2,\,2]\times s[3,\,3]}{4\,(-2+d)\,d^2}$$

```
(* The above three equations imply that s[3,3]^3=
 1. By scaling the matrices using a cubic root of unity,
if necessary, we may assume that s[3,3]=1. *)
```

In[ ]:=
```
s[3, 3] := 1
```

```
(* From coeff[0,3,0]=0,
s[2,2] is given by a cubic root. From coeff[1,1,1]=0 and s[3,3]=1,
we also solve s[1,1]=s[2,2]^2. *)
```

In[ ]:=
```
s[2, 2] := CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2)) / (chi2 (2 chi2^2 - 3 chi2 d + d^2))]
s[1, 1] := s[2, 2] ^2
```

```
(* Solve the rest of S. *)
```

In[ ]:=
```
Simplify[Solve[coeff[2, 1, 0] == 0 && coeff[1, 2, 0] == 0, {s[1, 3], s[2, 3]}]]
```

In[ ]:=
s[1, 3] :=

$$\frac{1}{8\,d^3}\left(-24\,\text{chi1}^2\,(-2+d)+24\,\text{chi1}\,(-2+d)\,d+d^2\left(8-5\,d+d^2\right)+\left(24\,\text{chi2}^2\,(-2+d)-24\,\text{chi2}\,(-2+d)\,d-d^2\left(8-5\,d+d^2\right)\right)\left(\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}\right)$$

$$s[2,3] := \frac{1}{d^2}\,3\,(-2+d)\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\left(\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{1/3}\right)$$

In[ ]:=
```
MatrixForm[S]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \left(\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\right)^2 & 0 & \frac{-24\,\text{chi1}^2\,(-2+d)+24\,\text{chi1}\,(-2+d)\,d+d^2\,(8-5\,d+d^2)+(24\,\text{chi2}}{} \\ 0 & \sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}} & \frac{3\,(-2+d)\,(2\,\text{chi1}-d+(-2}{} \\ 0 & 0 & \end{pmatrix}$$

## Type II - Solve for A, B

```
(* We have solved the shuffle matrix S uniquely up to a choice
 of a cubic root for s[2,2]. We claim that for each S there is
 a unique pair (AutA,AutB) up to scaling such that AutA^t(-)AutB
 corresponds to this shuffle matrix. To linearize the problem,
we again use the inverse matrix AutB^(-1). *)
```

In[ ]:=
```
At := Transpose[AutA]
Binv := Table[bb[i, j], {i, 1, 3}, {j, 1, 3}]
Diff'[d_, chi1_, chi2_] := Table[At.TopRelations[d, chi1][[i]] -
    Sum[S[[i, j]] * TopRelations[d, chi2][[j]].Binv, {j, 1, 3}], {i, 1, 3}]
```

In[ ]:=
```
(* Use Diff'[[3]] first. *)
```

In[ ]:=
```
Solve[Diff'[d, chi1, chi2][[3]] == 0,
 {bb[1, 1], bb[1, 2], bb[1, 3], bb[3, 1], bb[3, 2], bb[3, 3], a[3, 2], a[1, 2]}]
```

In[ ]:=
```
bb[1, 1] := -(((12 chi2^2 - 12 chi2 d - d^2) a[3, 1]) / (8 (-2 + d) d)) + a[3, 3]
bb[1, 2] := 0
bb[1, 3] := 1/(128 (-2 + d) d^2) (12 chi2^2 - 12 chi2 d - d^2)
    (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1]) -
   1/(16 d) (16 d a[1, 3] - 8 d^2 a[1, 3] - 12 chi1^2 a[3, 3] + 12 chi1 d a[3, 3] + d^2 a[3, 3])
bb[3, 1] := (2 a[3, 1])/(-2 + d)
bb[3, 2] := 0
bb[3, 3] := -1/(8 (-2 + d) d)
   (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1])
a[3, 2] := 0
a[1, 2] := 0
```

In[ ]:=
```
Simplify[Diff'[d, chi1, chi2][[3]]]
```

Out[ ]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* Solve b[2,j] using Diff'[[2]][1,j]. *)
```

In[ ]:=
```
Solve[Diff'[d, chi1, chi2][[2]][[1, 1]] == 0 && Diff'[d, chi1, chi2][[2]][[1, 2]] == 0 &&
  Diff'[d, chi1, chi2][[2]][[1, 3]] == 0, {bb[2, 1], bb[2, 2], bb[2, 3]}]
```

In[◦]:=
```
bb[2, 1] := (d^2 a[2, 1] - 12 chi1 a[3, 1] + 6 d a[3, 1] +
    12 chi2 a[3, 1] ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3) - 6 d a[3, 1]
     ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)) / (d^2 ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3))
bb[2, 2] :=
  (-16 d^2 a[1, 1] + 8 d^3 a[1, 1] + 24 chi1^2 a[3, 1] - 24 chi1 d a[3, 1] + 2 d^2 a[3, 1] -
    d^3 a[3, 1]) / (8 (-2 + d) d^2 ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3))
bb[2, 3] := -(((-1/(8 d^2)) (-12 chi1^2 + (12 chi1 + 6 chi1^2) d + (-1 - 6 chi1) d^2) a[2, 1] +
      (chi1 (-2 chi1 + d) (-chi1 + d) (-16 + 8 d + d^2) a[3, 1]) / (8 (-2 + d) d^3) - 1/(8 d^3)
       3 (16 d a[1, 1] - 8 d^2 a[1, 1] - 12 chi1^2 a[3, 1] + 12 chi1 d a[3, 1] + d^2 a[3, 1])
        (2 chi1 - d + (-2 chi2 + d) ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3))) /
    (((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)))
```

```
(* Use the rest of Diff'[[2]]. *)
```

In[◦]:=
```
Simplify[
 Solve[Diff'[d, chi1, chi2][[2]][[2, 1]] == 0 && Diff'[d, chi1, chi2][[2]][[3, 2]] == 0 &&
   Diff'[d, chi1, chi2][[2]][[3, 1]] == 0, {a[2, 2], a[1, 3], a[2, 3]}]]
```

In[∘]:=
```
a[2, 2] :=
 1/(8 (-2 + d) d^2) (-24 chi2^2 a[3, 1] + 24 chi2 d a[3, 1] + (-2 + d) d^2 (a[3, 1] + 8 a[3, 3]))
  ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)
a[1, 3] := 1/(64 (-2 + d)^2 d^4)
  ((24 chi2^2 - 24 chi2 d - (-2 + d) d^2) (d^3 (8 a[1, 1] - a[3, 1]) + 24 chi1^2 a[3, 1] -
      24 chi1 d a[3, 1] + 2 d^2 (-8 a[1, 1] + a[3, 1])) +
   8 (-2 + d) d^2 (-24 chi1^2 + 24 chi1 d + (-2 + d) d^2) a[3, 3])
a[2, 3] := 1/d^2 6 a[3, 3] (2 chi1 - d + (-2 chi2 + d) ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)) +
  (1/(8 (-2 + d)^2 d^4 ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)))
   ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3) (-2 chi2 (chi2 - d) (2 chi2 - d)
      d (-16 + 8 d + d^2) a[3, 1] ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3) -
      (-2 + d) (-24 chi2^2 + 24 chi2 d + (-2 + d) d^2) (d^2 a[2, 1] - 12 chi1 a[3, 1] +
         6 d a[3, 1] + 6 (2 chi2 - d) a[3, 1] ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)))
```

In[∘]:=
```
Simplify[Solve[Diff'[d, chi1, chi2][[2]][[3, 3]] == 0, {a[3, 3]}]]
```

In[∘]:=
```
a[3, 3] :=
 -(((6 (-24 chi2^2 + 24 chi2 d + (-2 + d) d^2) (d^3 (8 a[1, 1] - a[3, 1]) + 24 chi1^2 a[3, 1] -
           24 chi1 d a[3, 1] + 2 d^2 (-8 a[1, 1] + a[3, 1]))
        (2 chi1 - d + (-2 chi2 + d) ((chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2)))^(1/3)) -
       3 d (-24 chi2^2 + 24 chi2 d + (-2 + d) d^2) (d^3 (8 a[1, 1] - a[3, 1]) +
          24 chi1^2 a[3, 1] - 24 chi1 d a[3, 1] + 2 d^2 (-8 a[1, 1] + a[3, 1]))
```

$$\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\right)-$$

$$\left(6\,\text{chi1}^2\,(-2+d)-6\,\text{chi1}\,(-2+d)\,d-d^2\right)\left(-2\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\right.$$

$$d\,(-16+8\,d+d^2)\,a[3,1]\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}-$$

$$(-2+d)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\left(d^2\,a[2,1]-12\,\text{chi1}\,a[3,1]+\right.$$

$$\left.\left.6\,d\,a[3,1]+6\,(2\,\text{chi2}-d)\,a[3,1]\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\right)\right)+$$

$$d^2\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\left(\text{chi2}\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\right.$$

$$(-16+8\,d+d^2)\,\left(-12\,\text{chi1}^2\,a[3,1]+\right.$$

$$\left.d^2\,(-8\,a[1,1]+a[3,1])+4\,d\,(4\,a[1,1]+3\,\text{chi1}\,a[3,1])\right)-$$

$$\left((-2+d)\,\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2\right)\left(d\,\left(-6\,\text{chi1}^2\,(-2+d)+\right.\right.\right.$$

$$\left.6\,\text{chi1}\,(-2+d)\,d+d^2\right)\,a[2,1]+\frac{1}{-2+d}\,\text{chi1}\,(-2\,\text{chi1}+d)$$

$$(-\text{chi1}+d)\,(-16+8\,d+d^2)\,a[3,1]-3\,\left(-12\,\text{chi1}^2\,a[3,1]+\right.$$

$$\left.d^2\,(-8\,a[1,1]+a[3,1])+4\,d\,(4\,a[1,1]+3\,\text{chi1}\,a[3,1])\right)$$

$$\left.\left.\left.\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\right)\right)\right)\right)\Big/$$

$$\left.\left.\left(d\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\right)\right)\right)\Big/$$

$$\left.\left(8\,\text{chi1}\,(-2+d)\,d^3\,(-16+8\,d+d^2)\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)\right)\right)$$

In[∘]:=
```
Simplify[Diff'[d, chi1, chi2]〚2〛]
```

Out[∘]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* Use Diff'〚1〛 to solve the rest of AutA. *)
```

In[∘]:=
```
Solve[Diff'[d, chi1, chi2]〚1〛〚1, 2〛 == 0, a[3, 1]]
```

In[∘]:=
```
a[3, 1] := 0
```

In[∘]:=
```
Simplify[Solve[Diff'[d, chi1, chi2][[1]][[3, 1]] == 0, a[2, 1]]]
```

In[∘]:=
```
a[2, 1] :=
```

$$-\left(\left(\left((-4+d)\, a[1,1]\, \sqrt[3]{\frac{chi1\,(2\,chi1^2-3\,chi1\,d+d^2)}{chi2\,(2\,chi2^2-3\,chi2\,d+d^2)}}\,\left(chi1\,(2\,chi1^2-3\,chi1\,d+d^2)\,(-24\,chi2^2+24\,chi2\,d+(-4+d)\,d^2)+chi2\,(2\,chi2^2-3\,chi2\,d+d^2)\left(4\,(6\,chi1^2-6\,chi1\,d+d^2)-d^3\,\sqrt[3]{\frac{chi1\,(2\,chi1^2-3\,chi1\,d+d^2)}{chi2\,(2\,chi2^2-3\,chi2\,d+d^2)}}^{\,2}\right)\sqrt[3]{\frac{chi1\,(2\,chi1^2-3\,chi1\,d+d^2)}{chi2\,(2\,chi2^2-3\,chi2\,d+d^2)}}\right)\right)\right)\Big/\left(4\,chi1\,chi2\,(-2\,chi2+d)\,(-chi2+d)\,(2\,chi1^2-3\,chi1\,d+d^2)\,\sqrt[3]{\frac{chi1\,(2\,chi1^2-3\,chi1\,d+d^2)}{chi2\,(2\,chi2^2-3\,chi2\,d+d^2)}}^{\,2}\right)\right)$$

In[∘]:=
```
Simplify[Diff'[d, chi1, chi2][[1]]]
```

Out[∘]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
(* By rescaling AutA by a cubic root of unity and AutB by the inverse
 of it so that S stays unchanged, we may assume that a[1,1]=s[2,2]. *)
```

In[∘]:=
```
Simplify[Det[AutA]]
```

Out[∘]=

$$\frac{chi2\,(2\,chi2^2-3\,chi2\,d+d^2)\,a[1,1]^3}{chi1\,(2\,chi1^2-3\,chi1\,d+d^2)}$$

In[∘]:=
```
a[1, 1] := s[2, 2]
```

```
(* We record the unique solutions AutA,
AutB, S up to a choice of a cubic root. *)
```

In[∘]:=
```
Simplify[AutA]
Simplify[Adjugate[Binv]]
Simplify[S]
```

In[∘]:=

$$solA[d\_, chi1\_, chi2\_] := \left\{\left\{\sqrt[3]{\frac{chi1\,(2\,chi1^2-3\,chi1\,d+d^2)}{chi2\,(2\,chi2^2-3\,chi2\,d+d^2)}}\;, 0,\right.\right.$$

$$\frac{1}{8\,(-2+d)\,d^2}\ \sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ \Bigg(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+$$

$$\Bigg(\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Big/\ \left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)\Big\},$$

$$\Big\{-\Bigg(\Bigg((-4+d)\ \sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\Bigg(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+$$

$$\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\ \sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\Bigg(4\,\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\ \sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg)\Bigg)\Big/$$

$$\Bigg(4\,\text{chi1}\,\text{chi2}\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Bigg),\ 1,$$

$$\Bigg(\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ \Bigg(\frac{1}{-16+8\,d+d^2}\ 192\,\text{chi2}^2\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\left(32-32\,d+6\,d^2+d^3\right)\ \sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}$$

$$\Bigg(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\ \sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)+$$

$$\Bigg((-4+d)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\right/$$

$$\left.\left.((-2\,\text{chi2}+d)\,(-\text{chi2}+d))\right)\right)\Big/\left(32\,\text{chi1}\,\text{chi2}\,(-2+d)\,d^2\right.$$

$$\left.\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Bigg\},$$

$$\left\{0,\,0,\,\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right.\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Bigg/$$

$$\left.\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right\}\Big\}$$

In[ ]:=

```
solB[d_, chi1_, chi2_] :=
```

$$\left\{\left\{\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg/\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right),\right.\right.$$

$$0,\,-\left(\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right.\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-6\,\text{chi2}^2\,(-2+d)+6\,\text{chi2}\,(-2+d)\,d+d^2\right)-\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-6\,\text{chi1}^2\,(-2+d)+6\,\text{chi1}\,(-2+d)\,d+d^2\right)$$

$$\left.\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)\Bigg/$$

$$\left(8\,\text{chi1}\,(\text{chi1}-\text{d})\,(2\,\text{chi1}-\text{d})\,\text{d}^2\,\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}\right)\right)\Big\},$$

$$\Big\{\left((-4+\text{d})\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,\text{d}+(-4+\text{d})\,\text{d}^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}$$

$$\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,\text{d}+\text{d}^2\right)-\text{d}^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left(4\,\text{chi1}\,\text{chi2}\,(-2\,\text{chi2}+\text{d})\,(-\text{chi2}+\text{d})\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\right),\ 1,$$

$$\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\left(-\left(\left((-4+\text{d})\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)\right.\right.\right.\right.\right.$$

$$\left(-6\,\text{chi2}^2\,(-2+\text{d})+6\,\text{chi2}\,(-2+\text{d})\,\text{d}+\text{d}^2\right)-\text{chi2}$$

$$\left.\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)\left(-6\,\text{chi1}^2\,(-2+\text{d})+6\,\text{chi1}\,(-2+\text{d})\,\text{d}+\text{d}^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}\right)\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)\right.$$

$$\left(-24\,\text{chi2}^2+24\,\text{chi2}\,\text{d}+(-4+\text{d})\,\text{d}^2\right)+\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,\text{d}+\text{d}^2\right)-\right.$$

$$\left.\left.\left.\text{d}^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,\text{d}+\text{d}^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,\text{d}+\text{d}^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$
\left.(\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d))\right)+
$$

$$
\Bigg(\text{chi2}^2\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(32-32\,d+6\,d^2+d^3\right)
$$

$$
\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}
$$

$$
\Bigg(96\,(-2+d)\,\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{1/3}\right)+
$$

$$
\Bigg((-4+d)\,\left(6\,\text{chi1}^2\,(-2+d)-6\,\text{chi1}\,(-2+d)\,d-d^2\right)
$$

$$
\Bigg(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+
$$

$$
\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)
$$

$$
\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{1/3}\Bigg(4\,\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-
$$

$$
\left.\left.\left.d^3\,\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}\right)\right)\right)\Big/
$$

$$
\Bigg(\text{chi1}\,\text{chi2}\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)
$$

$$
\left.\left.\left.\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{1/3}\right)\right)\right)\Big/
$$

$$
\left.\left.\left((-2+d)\,\left(-16+8\,d+d^2\right)\right)\right)\right)\Big/\Bigg(32\,\text{chi1}\,\text{chi2}\,d^2
$$

$$
\left.\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\,\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}\right)\Big\},
$$

$$\left\{0,\ 0,\ \left(\text{chi2}\ \left(2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2\right)\ \left(\frac{\text{chi1}\ \left(2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2\right)}{\text{chi2}\ \left(2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2\right)}\right)^{2/3}\right)\Big/\left(\text{chi1}\ \left(2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2\right)\right)\right\}\right\}$$

In[∘]:=

$$\text{solS}[d\_,\ \text{chi1}\_,\ \text{chi2}\_]:=\left\{\left\{\left(\frac{\text{chi1}\ \left(2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2\right)}{\text{chi2}\ \left(2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2\right)}\right)^{2/3},\ 0,\ \frac{1}{8\ d^3}\left(-24\ \text{chi1}^2\ (-2+d)+24\ \text{chi1}\ (-2+d)\ d+d^2\ \left(8-5\ d+d^2\right)+\left(24\ \text{chi2}^2\ (-2+d)-24\ \text{chi2}\ (-2+d)\ d-d^2\ \left(8-5\ d+d^2\right)\right)\left(\frac{\text{chi1}\ \left(2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2\right)}{\text{chi2}\ \left(2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2\right)}\right)^{2/3}\right)\right\},\ \left\{0,\ \sqrt[3]{\frac{\text{chi1}\ \left(2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2\right)}{\text{chi2}\ \left(2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2\right)}},\ \frac{1}{d^2}\ 3\ (-2+d)\left(2\ \text{chi1}-d+(-2\ \text{chi2}+d)\ \sqrt[3]{\frac{\text{chi1}\ \left(2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2\right)}{\text{chi2}\ \left(2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2\right)}}\right)\right\},\ \{0,\ 0,\ 1\}\right\}$$

```
(* Solutions solA, solB,
solS are just sign matrices when chi1=chi2 and chi1+chi2=d. *)
```

In[∘]:=

```
Simplify[MatrixForm[solA[d, chi, chi]]]
Simplify[MatrixForm[solB[d, chi, chi]]]
Simplify[MatrixForm[solS[d, chi, chi]]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[∘]:= Simplify[MatrixForm[solA[d, chi, d - chi]]]
        Simplify[MatrixForm[solB[d, chi, d - chi]]]
        Simplify[MatrixForm[solS[d, chi, d - chi]]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Type II - Solve for U, V

```
(* Using the solutions above,
we find AutU and AutV. Existence of the solution,
which we assumed, will lead to a contradiction later. *)
```

```
In[∘]:= ExtDiff[d_, chi1_, chi2_] :=
         Table[Transpose[solA[d, chi1, chi2]].TopRelations[d, chi1][[i]].AutU +
           Transpose[solA[d, chi1, chi2]].ExtRelations[d, chi1][[i]].AutV - Sum[
            solS[d, chi1, chi2][[i, j]] * ExtRelations[d, chi2][[j]], {j, 1, 3}], {i, 1, 3}]
```

```
(* We solve the linear equation for variables u[i,1] and v[i,1] by
 looking at the first columns of ExtDiff[[1]],ExtDiff[[2]],ExtDiff[[3]]. *)
```

```
In[∘]:= MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2][[3]][[1 ;; 3, 1]], u[1, 1], 1]]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ \dfrac{2\,\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}\right)^{2/3}}{\text{chi1}\,(-2+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)} \end{pmatrix}$$

```
In[∘]:= Solve[ExtDiff[d, chi1, chi2][[3]][[3, 1]] == 0, {u[1, 1]}]
```

$$\text{In[∘]:= } u[1,1] := \left(\text{chi1}\,(-2+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right.$$

$$\left(-\left(\left(d^2\,(2+d)\right)/\left(2\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,(-2+d)\right)\right)-\right.$$

$$
\begin{aligned}
&\Bigg(\Bigg(\text{chi2}\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right) \\
&\qquad \left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)^2\Bigg)\Big/ \\
&\quad \left(8\,\text{chi1}\,(-2+d)\,d\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2} \\
&\quad \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\Bigg(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+ \\
&\qquad \Bigg(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right) \\
&\qquad\quad \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Big/ \\
&\qquad\quad \left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)\Bigg)\,u[3,1]- \\
&\Bigg(-\Bigg(\Bigg(\text{chi2}\,d^2\,(2+d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)^2\Bigg)\Big/ \\
&\qquad \left(2\,\text{chi1}^2\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,(-2+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)+ \\
&\quad \Bigg(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\Bigg(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+ \\
&\qquad \Bigg(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right) \\
&\qquad\quad \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Big/\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+\right.\right. \\
&\qquad\quad \left.\left.d^2\right)\right)\Bigg)\Bigg)\Big/\left(2\,\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,(-2+d)\right)\Bigg)\,v[1,1]-
\end{aligned}
$$

$$\left(\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-48\,\text{chi1}^4+\left(96\,\text{chi1}^3+72\,\text{chi1}^4\right)d+\left(-36\,\text{chi1}^2-\right.\right.\right.\right.$$

$$\left.144\,\text{chi1}^3\right)d^2+\left(-12\,\text{chi1}+66\,\text{chi1}^2\right)d^3+\left(4+6\,\text{chi1}-6\,\text{chi1}^2\right)d^4+$$

$$\left.\left.(-2+6\,\text{chi1})\,d^5+d^6\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Big/$$

$$\left(8\,\text{chi1}^2\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+$$

$$\left(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)\right.$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+\right.\right.$$

$$\left.\left.\left.\left.d^2\right)\right)\right)\right)\Big/\left(8\,\text{chi1}\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\right)+$$

$$\left((3-d)\left(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.$$

$$\left(32-32\,d+6\,d^2+d^3\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}$$

$$\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+$$

$$\left((-4+d)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\right.\right.\right.$$

$$\left.d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\text{chi2}\left(2\,\text{chi2}^2-\right.$$

$$\left.3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(4\left(6\,\text{chi1}^2-\right.\right.$$

$$6\,\text{chi1}\,d + d^2\Big) - d^3\,\left(\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\right)^2\Bigg)\Bigg)\Bigg)\Bigg/$$

$$((-2\,\text{chi2} + d)\,(-\text{chi2} + d))\Bigg)\Bigg)\Bigg/\,\Big(32\,\text{chi1}\,\text{chi2}$$

$$(-2 + d)^2\,d^2\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)\Big)\Bigg)\,v[2, 1] -$$

$$\Bigg(\Bigg(\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)\,\Big(128\,\text{chi1}^6 + (-384\,\text{chi1}^5 - 1344\,\text{chi1}^6)\,d +$$

$$(224\,\text{chi1}^4 + 4032\,\text{chi1}^5 + 1216\,\text{chi1}^6)\,d^2 +$$
$$(192\,\text{chi1}^3 - 3792\,\text{chi1}^4 - 3648\,\text{chi1}^5)\,d^3 + (-232\,\text{chi1}^2 + 864\,\text{chi1}^3 +$$
$$3520\,\text{chi1}^4)\,d^4 + (72\,\text{chi1} + 372\,\text{chi1}^2 - 960\,\text{chi1}^3)\,d^5 +$$
$$(-6 - 132\,\text{chi1} - 212\,\text{chi1}^2)\,d^6 + (5 + 84\,\text{chi1} + 24\,\text{chi1}^2)\,d^7 +$$

$$(-2 - 24\,\text{chi1})\,d^8\Big)\,\left(\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\right)^2\Bigg)\Bigg/$$

$$\Big(128\,\text{chi1}^2\,(-2 + d)\,d^4\,(-2\,\text{chi1} + d)\,(-\text{chi1} + d)\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)\Big) +$$

$$\Bigg(3\,\left(-\frac{1}{2} + d\right)\,\left(\text{chi1}^2\,(1 - d) + \frac{d^2}{12} + \text{chi1}\,(-d + d^2)\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\,\Bigg(24\,\text{chi2}^2 - 24\,\text{chi2}\,d - (-2 + d)\,d^2 +$$

$$\Bigg(\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)\,(-24\,\text{chi1}^2 + 24\,\text{chi1}\,d + (-2 + d)\,d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\Bigg)\Bigg/\,\Big(\text{chi1}\,\Big(2\,\text{chi1}^2 - 3\,\text{chi1}\,d +$$

$$d^2\Big)\Big)\Bigg)\Bigg)\Bigg/\,\Big(16\,\text{chi1}\,(-2 + d)\,d^2\,(-2\,\text{chi1} + d)\,(-\text{chi1} + d)\Big) +$$

$$\Bigg(3\,(-3 + d)\,\left(\text{chi1}^2 + (-\text{chi1} - \text{chi1}^2)\,d + \left(\frac{1}{12} + \text{chi1}\right)\,d^2\right)$$

$$\Bigg(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)$$

$$(32-32\,d+6\,d^2+d^3)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}^{\,2}$$

$$\Bigg(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\Bigg)+$$

$$\Bigg((-4+d)\,(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2)\,\Bigg(\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}$$

$$d+d^2)\,(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2)+\text{chi2}\,(2\,\text{chi2}^2-$$

$$3\,\text{chi2}\,d+d^2)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\,\Bigg(4\,(6\,\text{chi1}^2-$$

$$6\,\text{chi1}\,d+d^2)-d^3\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}^{\,2}\Bigg)\Bigg)\Bigg)\Bigg/$$

$$((-2\,\text{chi2}+d)\,(-\text{chi2}+d))\Bigg)\Bigg)\Bigg/\,(64\,\text{chi1}\,\text{chi2}\,(-2+d)^2$$

$$d^4\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2))\Bigg)\,v[3,1]\Bigg)\Bigg)\Bigg/$$

$$\Bigg(2\,\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}^{\,2}\Bigg)$$

In[ ]:=
```
MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2][[3]][[1 ;; 3, 1]], u[3, 1], 1]]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}} \\ 0 \\ 0 \end{pmatrix}$$

In[ ]:=
```
Solve[ExtDiff[d, chi1, chi2][[3]][[1, 1]] == 0, {u[3, 1]}]
```

In[ ]:=

$$u[3,1] := \Bigg((4\,d^2)\,/\,(\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d))\,-$$

$$\left(4\, d^2 \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right)}}\ v[1,1]\right)\Big/ (\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)) -$$

$$\left(\left(\left(-d^3+\text{chi1}^2\,(-6+6\, d)+\text{chi1}\left(6\, d-6\, d^2\right)\right) \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right)}}\right)\Big/\right.$$

$$(\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)) - \Bigg((3-d)\,(-4+d)$$

$$\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\, d+(-4+d)\, d^2\right)+$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right) \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right)}}$$

$$\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\, d+d^2\right)-d^3 \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left.\left(4\,\text{chi1}\,\text{chi2}\,(-2+d)\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)\right)\right)$$

$$v[2,1] - \left(\left(3\left(-\frac{1}{2}+d\right)\left(\text{chi1}^2\,(1-d)+\frac{d^2}{12}+\text{chi1}\left(-d+d^2\right)\right)\right.\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right)}}\right)\Big/ (2\,\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)) -$$

$$\Bigg(3\,(-4+d)\,(-3+d)\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right) d+\left(\frac{1}{12}+\text{chi1}\right) d^2\right)$$

$$\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\, d+(-4+d)\, d^2\right)+$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right) \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right)}}$$

$$\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\, d+d^2\right)-d^3 \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\, d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\, d+d^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left(8\,\text{chi1}\,\text{chi2}\,(-2+d)\,d^2\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)$$

$$v[3,1]\Bigg)\Bigg/\left(\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)$$

In[∘]:= MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2][[2]][[1 ;; 3, 1]], u[2, 1], 1]]]

Out[∘]//MatrixForm=

$$\begin{pmatrix} \sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}} \\ 0 \\ -\frac{\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2\right)\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}}{8\,(-2+d)\,d^2} \end{pmatrix}$$

In[∘]:= Solve[ExtDiff[d, chi1, chi2][[2]][[1, 1]] == 0, {u[2, 1]}]

In[∘]:= u[2, 1] :=

$$\Bigg(\Bigg(3\,(-1+d)\,d\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Big/\,(\text{chi2}\,(\text{chi2}-d)\,(-2+d))\,+$$

$$\left(12\,(-2+d)\left(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)\Big/$$

$$(\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d))\,-$$

$$\left(3\,(-1+d)\,d\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;v[1,1]\right)\Big/$$

$$(\text{chi1}\,(\text{chi1}-d)\,(-2+d))\,-\,\Bigg(\left(-4\,\text{chi1}^2+\left(4\,\text{chi1}+14\,\text{chi1}^2\right)d\,+\right.$$

$$\left.\left(-14\,\text{chi1}-16\,\text{chi1}^2\right)d^2+\left(-3+16\,\text{chi1}+2\,\text{chi1}^2\right)d^3+(3-2\,\text{chi1})\,d^4\right)$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;v[2,1]\Bigg)\Big/\,(4\,\text{chi1}\,(-2+d)\,d\,(-\text{chi1}+d))\,-$$

$$\Bigg(\Bigg((-1+d)\,\Bigg(-6\,\text{chi1}^4+\left(12\,\text{chi1}^3+6\,\text{chi1}^4\right)d-6\,\text{chi1}^2\,(1+\text{chi1})^2\,d^2\,+$$

$$\left(\frac{9\,\text{chi1}^2}{2}+12\,\text{chi1}^3\right)d^3+\left(\frac{3}{8}+\frac{3\,\text{chi1}}{2}+\frac{\text{chi1}^2}{2}\right)d^4\,+$$

$$\left(-\frac{3}{4}-\frac{13\,\text{chi1}}{2}-\text{chi1}^2\right)d^5+\text{chi1}\,d^6\Bigg)$$

$$\left.\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right/\left(8\,\text{chi1}\,(-2+d)\,d^3\,(-\text{chi1}+d)\right)-$$

$$\left((-4+d)\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(-8+10\,d-d^2-4\,d^3+d^4\right)\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\right/$$

$$\left.\left(32\,\text{chi2}\,(-2+d)\,d^4\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)$$

$$v[3,1]+\left((-4+d)\left(-12\,\text{chi1}^2+\left(12\,\text{chi1}+6\,\text{chi1}^2\right)d+(-1-6\,\text{chi1})\,d^2\right)\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)$$

$$\left(\left(4\,d^2\right)/(\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d))-\right.$$

$$\left.\left(4\,d^2\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;v[1,1]\right)\right/$$

$$(\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d))-\left(\left(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)\right.\right.$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Bigg/\ (\text{chi1}\,(-2\,\text{chi1}+d)$$

$$(-\text{chi1}+d))-\Bigg((3-d)\,(-4+d)\,\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)$$

$$\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ \Bigg(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-$$

$$d^3\ \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\ \Bigg)\Bigg)\Bigg)\Bigg/\ \Big(4\,\text{chi1}\,\text{chi2}$$

$$(-2+d)\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\Big)\Bigg)$$

$$v[2,1]-\Bigg(\Bigg(3\left(-\frac{1}{2}+d\right)\left(\text{chi1}^2\,(1-d)+\frac{d^2}{12}+\text{chi1}\left(-d+d^2\right)\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Bigg/$$

$$(2\,\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d))-\Bigg(3\,(-4+d)\,(-3+d)\,\Big(\text{chi1}^2+$$

$$\left(-\text{chi1}-\text{chi1}^2\right)d+\left(\frac{1}{12}+\text{chi1}\right)d^2\Big)\,\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)$$

$$\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ \Bigg(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-$$

$$d^3\ \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\ \Bigg)\Bigg)\Bigg)\Bigg/\ \Big(8\,\text{chi1}\,\text{chi2}\,(-2+d)$$

$$d^2\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\Big)\Bigg)\,v[3,1]\Bigg)\Bigg)\Bigg/$$

$$\Big(32\,\mathrm{chi1}\,\mathrm{chi2}\,d^2\,(-2\,\mathrm{chi2}+d)\,(-\mathrm{chi2}+d)\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)}{\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)}}\Big)+\Big((-4+d)\,(-2+d)$$

$$\Big(\mathrm{chi1}\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)\,(-24\,\mathrm{chi2}^2+24\,\mathrm{chi2}\,d+(-4+d)\,d^2)+$$

$$\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)\,\sqrt[3]{\frac{\mathrm{chi1}\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)}{\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)}}$$

$$\Big(4\,(6\,\mathrm{chi1}^2-6\,\mathrm{chi1}\,d+d^2)-d^3\,\sqrt[3]{\frac{\mathrm{chi1}\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)}{\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)}}^{\,2}\Big)\Big)\Big)$$

$$\Big(-\big((d^2\,(2+d))\big/(2\,\mathrm{chi2}\,(\mathrm{chi2}-d)\,(2\,\mathrm{chi2}-d)\,(-2+d))\big)-$$

$$\Big(-\Big(\Big(\mathrm{chi2}\,d^2\,(2+d)\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)}{\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)}}^{\,2}\Big)\Big/\,\big(2\,\mathrm{chi1}^2\,(\mathrm{chi1}-d)$$

$$(2\,\mathrm{chi1}-d)\,(-2+d)\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)\big)\Big)+$$

$$\Big(\sqrt[3]{\frac{\mathrm{chi1}\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)}{\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)}}\,\Big(24\,\mathrm{chi2}^2-24\,\mathrm{chi2}\,d-(-2+d)\,d^2+$$

$$\Big(\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)\,(-24\,\mathrm{chi1}^2+24\,\mathrm{chi1}\,d+(-2+d)\,d^2)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2)}{\mathrm{chi2}\,(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2)}}\Big)\Big/\,\big(\mathrm{chi1}\,(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+$$

$$d^2)\big)\Big)\Big)\Big/\,(2\,\mathrm{chi1}\,(\mathrm{chi1}-d)\,(2\,\mathrm{chi1}-d)\,(-2+d))\Big)\,v[1,1]-$$

$$\Bigg(\Bigg(\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,(-48\,\text{chi1}^4+(96\,\text{chi1}^3+72\,\text{chi1}^4)\,d+$$

$$(-36\,\text{chi1}^2-144\,\text{chi1}^3)\,d^2+(-12\,\text{chi1}+66\,\text{chi1}^2)\,d^3+$$

$$(4+6\,\text{chi1}-6\,\text{chi1}^2)\,d^4+(-2+6\,\text{chi1})\,d^5+$$

$$d^6)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}^{\,2}\Bigg)\Big/$$

$$(8\,\text{chi1}^2\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2))+$$

$$\Bigg((-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\,(6\,d-6\,d^2))$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\,\Bigg(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+$$

$$\Bigg(\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)\,(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\Bigg)\Big/\,(\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+$$

$$d^2))\Bigg)\Bigg)\Big/\,(8\,\text{chi1}\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d))+$$

$$\Bigg((3-d)\,\Bigg(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)$$

$$(32-32\,d+6\,d^2+d^3)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}^{\,2}$$

$$\Bigg(2\,\text{chi1}-d+(-2\,\text{chi2}+d)\,\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)}{\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)}}\Bigg)+$$

$$\Bigg((-4+d)\,(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-2+d)\,d^2)$$

$$\Bigg(\text{chi1}\,(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2)\,(-24\,\text{chi2}^2+24\,\text{chi2}\,d+$$

$$(-4+d)\,d^2)+\text{chi2}\,(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;\Bigg(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+\right.$$

$$\left.d^2\right)-d^3\;\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\;\Bigg)\Bigg)\Bigg)\Big/$$

$$((-2\,\text{chi2}+d)\;(-\text{chi2}+d))\Bigg)\Bigg)\Big/\;\Big(32\,\text{chi1}\,\text{chi2}$$

$$(-2+d)^2\,d^2\;\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\Big)\Bigg)\;v[2,1]\;-$$

$$\Bigg(\Bigg(\Bigg(\text{chi2}\;\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\;\Big(128\,\text{chi1}^6+\left(-384\,\text{chi1}^5-1344\,\text{chi1}^6\right)\,d+$$

$$\left(224\,\text{chi1}^4+4032\,\text{chi1}^5+1216\,\text{chi1}^6\right)\,d^2+\left(192\,\text{chi1}^3-3792\,\text{chi1}^4-\right.$$
$$\left.3648\,\text{chi1}^5\right)\,d^3+\left(-232\,\text{chi1}^2+864\,\text{chi1}^3+3520\,\text{chi1}^4\right)\,d^4+$$
$$\left(72\,\text{chi1}+372\,\text{chi1}^2-960\,\text{chi1}^3\right)\,d^5+\left(-6-132\,\text{chi1}-212\,\text{chi1}^2\right)$$
$$d^6+\left(5+84\,\text{chi1}+24\,\text{chi1}^2\right)\,d^7+(-2-24\,\text{chi1})\,d^8\Big)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\;\Bigg)\Big/\;\Big(128\,\text{chi1}^2\;(-2+d)$$

$$d^4\;(-2\,\text{chi1}+d)\;(-\text{chi1}+d)\;\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\Big)\;+$$

$$\Bigg(3\;\left(-\frac{1}{2}+d\right)\;\left(\text{chi1}^2\;(1-d)+\frac{d^2}{12}+\text{chi1}\;\left(-d+d^2\right)\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;\Bigg(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+$$

$$\Bigg(\text{chi2}\;\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\;\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;\Bigg)\Big/\;\Big(\text{chi1}\;\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+\right.$$

$$\left.d^2\right)\Big)\Bigg)\Bigg)\Big/\;\Big(16\,\text{chi1}\;(-2+d)\;d^2\;(-2\,\text{chi1}+d)\;(-\text{chi1}+d)\Big)\;+$$

$$\Bigg(3\,(-3+d)\,\Big(\mathrm{chi1}^2+\big(-\mathrm{chi1}-\mathrm{chi1}^2\big)\,d+\Big(\frac{1}{12}+\mathrm{chi1}\Big)\,d^2\Big)$$

$$\Bigg(\frac{1}{-16+8\,d+d^2}\,192\,\mathrm{chi2}^2\,\big(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\big)$$

$$\big(32-32\,d+6\,d^2+d^3\big)\,\left(\sqrt[3]{\frac{\mathrm{chi1}\,\big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\big)}{\mathrm{chi2}\,\big(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\big)}}\right)^2$$

$$\left(2\,\mathrm{chi1}-d+(-2\,\mathrm{chi2}+d)\,\sqrt[3]{\frac{\mathrm{chi1}\,\big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\big)}{\mathrm{chi2}\,\big(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\big)}}\right)+$$

$$\Bigg((-4+d)\,\big(-24\,\mathrm{chi2}^2+24\,\mathrm{chi2}\,d+(-2+d)\,d^2\big)$$

$$\Bigg(\mathrm{chi1}\,\big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\big)\,\big(-24\,\mathrm{chi2}^2+24\,\mathrm{chi2}\,d+$$

$$(-4+d)\,d^2\big)+\mathrm{chi2}\,\big(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\big)$$

$$\sqrt[3]{\frac{\mathrm{chi1}\,\big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\big)}{\mathrm{chi2}\,\big(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\big)}}\,\Bigg(4\,\big(6\,\mathrm{chi1}^2-6\,\mathrm{chi1}\,d+$$

$$d^2\big)-d^3\,\left(\sqrt[3]{\frac{\mathrm{chi1}\,\big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\big)}{\mathrm{chi2}\,\big(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\big)}}\right)^2\Bigg)\Bigg)\Bigg)\Big/$$

$$((-2\,\mathrm{chi2}+d)\,(-\mathrm{chi2}+d))\Bigg)\Bigg)\Big/\Big(64\,\mathrm{chi1}\,\mathrm{chi2}$$

$$(-2+d)^2\,d^4\,\big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\big)\Big)\Bigg)\,v[3,1]-$$

$$\Bigg(\Bigg(\Bigg(\mathrm{chi2}\,\big(12\,\mathrm{chi1}^2-12\,\mathrm{chi1}\,d-d^2\big)\,\big(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\big)$$

$$\left(\sqrt[3]{\frac{\mathrm{chi1}\,\big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\big)}{\mathrm{chi2}\,\big(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\big)}}\right)^2\Bigg)\Big/$$

$$\Big(8\,\mathrm{chi1}\,(-2+d)\,d\,\big(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\big)\Big)+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{chi1\left(2\,chi1^2-3\,chi1\,d+d^2\right)}{chi2\left(2\,chi2^2-3\,chi2\,d+d^2\right)}}\left(24\,chi2^2-24\,chi2\,d-(-2+d)\,d^2+\left(chi2\left(2\,chi2^2-3\,chi2\,d+d^2\right)\left(-24\,chi1^2+24\,chi1\,d+(-2+d)\,d^2\right)\sqrt[3]{\frac{chi1\left(2\,chi1^2-3\,chi1\,d+d^2\right)}{chi2\left(2\,chi2^2-3\,chi2\,d+d^2\right)}}\right)\Big/\left(chi1\left(2\,chi1^2-3\,chi1\,d+d^2\right)\right)\right)\Bigg)\Bigg(\left(4\,d^2\right)/\left(chi2\,(chi2-d)\,(2\,chi2-d)\right)-\left(4\,d^2\,\sqrt[3]{\frac{chi1\left(2\,chi1^2-3\,chi1\,d+d^2\right)}{chi2\left(2\,chi2^2-3\,chi2\,d+d^2\right)}}\;v[1,1]\right)\Big/\left(chi1\,(chi1-d)\,(2\,chi1-d)\right)-\left(\left(\left(-d^3+chi1^2\,(-6+6\,d)+chi1\left(6\,d-6\,d^2\right)\right)\sqrt[3]{\frac{chi1\left(2\,chi1^2-3\,chi1\,d+d^2\right)}{chi2\left(2\,chi2^2-3\,chi2\,d+d^2\right)}}\right)\Big/\left(chi1\,(-2\,chi1+d)\,(-chi1+d)\right)-\left((3-d)\,(-4+d)\left(chi1\left(2\,chi1^2-3\,chi1\,d+d^2\right)\left(-24\,chi2^2+24\,chi2\,d+(-4+d)\,d^2\right)+chi2\left(2\,chi2^2-3\,chi2\,d+d^2\right)\sqrt[3]{\frac{chi1\left(2\,chi1^2-3\,chi1\,d+d^2\right)}{chi2\left(2\,chi2^2-3\,chi2\,d+d^2\right)}}\left(4\left(6\,chi1^2-6\,chi1\,d+d^2\right)-d^3\,\sqrt[3]{\frac{chi1\left(2\,chi1^2-3\,chi1\,d+d^2\right)}{chi2\left(2\,chi2^2-3\,chi2\,d+d^2\right)}}^{\,2}\right)\right)\right)\right)\Big/\left(4\,chi1\,chi2\,(-2+d)\,(-2\,chi2+d)\,(-chi2+d)\left(2\,chi1^2-3\,chi1\,d+d^2\right)\right)$$

$$
\begin{aligned}
&\mathrm{v}[2,1]-\Bigg(\Bigg(\bigg(3\left(-\frac{1}{2}+d\right)\left(\mathrm{chi1}^2(1-d)+\frac{d^2}{12}+\mathrm{chi1}\left(-d+d^2\right)\right)\\
&\quad\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\bigg)\Big/\ (2\,\mathrm{chi1}\,(-2\,\mathrm{chi1}+d)\\
&\quad(-\mathrm{chi1}+d))-\bigg(3\,(-4+d)\,(-3+d)\,\left(\mathrm{chi1}^2+\left(-\mathrm{chi1}-\mathrm{chi1}^2\right)d+\right.\\
&\quad\left.\left(\frac{1}{12}+\mathrm{chi1}\right)d^2\right)\bigg(\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)\left(-24\,\mathrm{chi2}^2+\right.\\
&\quad\left.24\,\mathrm{chi2}\,d+(-4+d)\,d^2\right)+\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)\\
&\quad\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\bigg(4\left(6\,\mathrm{chi1}^2-6\,\mathrm{chi1}\,d+\right.\\
&\quad\left.d^2\right)-d^3\ \sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}^{\,2}\bigg)\bigg)\bigg)\Big/\\
&\quad\left(8\,\mathrm{chi1}\,\mathrm{chi2}\,(-2+d)\,d^2\,(-2\,\mathrm{chi2}+d)\,(-\mathrm{chi2}+d)\right.\\
&\quad\left.\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)\right)\bigg)\ \mathrm{v}[3,1]\Bigg)\Bigg/\\
&\quad\left(\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\right)\Bigg)\Bigg)\Big/\ \Big(8\,\mathrm{chi2}^2\\
&\quad(-2\,\mathrm{chi2}+d)\,(-\mathrm{chi2}+d)\,\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)\\
&\quad\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}^{\,2}\Big)\Big)\Big/\\
&\left(\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^2-3\,\mathrm{chi1}\,d+d^2\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^2-3\,\mathrm{chi2}\,d+d^2\right)}}\right)
\end{aligned}
$$

In[∘]:=
```
MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2][[3]][[1 ;; 3, 1]], v[2, 1], 1]]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 \\ \frac{3-d}{-2+d} \\ 0 \end{pmatrix}$$

In[∘]:=
```
Solve[ExtDiff[d, chi1, chi2][[3]][[2, 1]] == 0, v[2, 1]]
```

In[∘]:= 
```
v[2, 1] := -1/(8 d^2) (-12 chi1^2 + 12 chi1 d + 12 chi1^2 d - d^2 - 12 chi1 d^2) v[3, 1]
```

In[∘]:= 
```
MatrixForm[Simplify[Coefficient[ExtDiff[d, chi1, chi2]〚2〛〚1 ;; 3, 1〛, v[1, 1], 1]]]
```

Out[∘]//MatrixForm=

$$\left(\frac{8\,\text{chi1}\,\text{chi2}\,(2\,\text{chi1}-d)\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\,d\,\left(-16+8\,d+d^2\right)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-3\,d^2\,\left(-2+d+d^2\right)+2\,\text{chi}\ldots\right.}{}\right.$$

In[∘]:= 
```
Solve[ExtDiff[d, chi1, chi2]〚2〛〚2, 1〛 == 0, v[1, 1]]
```

In[∘]:= 

$$v[1,1] := \left(\left(\text{chi1}\,d^2\,(2+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Big/\left(4\,\text{chi2}^2\,(\text{chi2}-d)\right.\right.$$

$$\left.(2\,\text{chi2}-d)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)-$$

$$\left(-12\,\text{chi1}^2+\left(12\,\text{chi1}+6\,\text{chi1}^2\right)d+(-1-6\,\text{chi1})\,d^2\right)\Big/$$

$$\left(2\,\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)+$$

$$\left(2\,(-2+d)\,d^2\left(\left(\text{chi2}\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Big/$$

$$\left(8\,\text{chi1}\,(-2+d)\,d\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)\Bigg)\Bigg)\Bigg/\left(\text{chi2}\,(\text{chi2}-d)\,(2\,\text{chi2}-d)\right)-$$

$$\frac{1}{8\,(-2+d)\,d^4}\,\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(-8+10\,d-d^2-4\,d^3+d^4\right)\text{v}[3,1]-$$

$$\Bigg(\left(-12\,\text{chi1}^2+\left(12\,\text{chi1}+6\,\text{chi1}^2\right)d+(-1-6\,\text{chi1})\,d^2\right)$$

$$\left(-12\,\text{chi1}^2+12\,\text{chi1}\,d+12\,\text{chi1}^2\,d-d^2-12\,\text{chi1}\,d^2\right)$$

$$\Bigg(\Bigg(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)\Bigg/$$

$$(\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d))-\Bigg((3-d)\,(-4+d)$$

$$\Bigg(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\Bigg(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg)\Bigg)\Bigg/$$

$$\left(4\,\text{chi1}\,\text{chi2}\,(-2+d)\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\Bigg)$$

$$\text{v}[3,1]\Bigg)\Bigg/\Bigg(64\,d^4\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\Bigg)+$$

$$\frac{1}{16\,d^2}\,(-2+d)\left(-12\,\text{chi1}^2+12\,\text{chi1}\,d+12\,\text{chi1}^2\,d-d^2-12\,\text{chi1}\,d^2\right)$$

$$\Bigg(\Bigg(\text{chi2}\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg/$$

$$\left(8\,\text{chi1}\,(-2+d)\,d\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\;\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\;\left(\text{chi1}\right.$$

$$\left.\left.\left.\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)\right)$$

$$\left(\left(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\,\left(6\,d-6\,d^2\right)\right)\;\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\right.$$

$$(\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d))-\left((3-d)\,(-4+d)\right.$$

$$\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\,\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\;\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\,\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\;\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\right)\Big/$$

$$\left.\left(4\,\text{chi1}\,\text{chi2}\,(-2+d)\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)$$

$$v[3,\,1]+\left(\left(-12\,\text{chi1}^2+\left(12\,\text{chi1}+6\,\text{chi1}^2\right)\,d+(-1-6\,\text{chi1})\,d^2\right)\right.$$

$$\left(\left(3\,\left(-\frac{1}{2}+d\right)\,\left(\text{chi1}^2\,(1-d)+\frac{d^2}{12}+\text{chi1}\,\left(-d+d^2\right)\right)\right.\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\ (2\,\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d))\,-$$

$$\left(3\,(-4+d)\,(-3+d)\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)d+\left(\frac{1}{12}+\text{chi1}\right)d^2\right)\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\Big/$$

$$\left.\left(8\,\text{chi1}\,\text{chi2}\,(-2+d)\,d^2\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)$$

$$\left.\text{v[3, 1]}\right)\Big/\left(8\,d^2\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)-$$

$$\frac{1}{2}\,(-2+d)\left(\left(\text{chi2}\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Big/$$

$$\left(8\,\text{chi1}\,(-2+d)\,d\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\left.\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\ \left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)\right)$$

$$\left(\left(3\left(-\frac{1}{2}+d\right)\left(\text{chi1}^2\,(1-d)+\frac{d^2}{12}+\text{chi1}\left(-d+d^2\right)\right)\right.\right.$$

$$\left.\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right/\left(2\,\text{chi1}\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\right)-$$

$$\left(3\,(-4+d)\,(-3+d)\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)d+\left(\frac{1}{12}+\text{chi1}\right)d^2\right)\right.$$

$$\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(-24\,\text{chi2}^2+24\,\text{chi2}\,d+(-4+d)\,d^2\right)+\right.$$

$$\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}$$

$$\left.\left.\left.\left(4\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)-d^3\,\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right)\right)\right/$$

$$\left.\left(8\,\text{chi1}\,\text{chi2}\,(-2+d)\,d^2\,(-2\,\text{chi2}+d)\,(-\text{chi2}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)$$

$$v[3,1]-\left(\text{chi1}\,(-2+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right.$$

$$\left(-12\,\text{chi1}^2+12\,\text{chi1}\,d+12\,\text{chi1}^2\,d-d^2-12\,\text{chi1}\,d^2\right)$$

$$\left(\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.$$

$$\left(-48\,\text{chi1}^4+\left(96\,\text{chi1}^3+72\,\text{chi1}^4\right)d+\left(-36\,\text{chi1}^2-144\,\text{chi1}^3\right)d^2+\right.$$
$$\left(-12\,\text{chi1}+66\,\text{chi1}^2\right)d^3+\left(4+6\,\text{chi1}-6\,\text{chi1}^2\right)d^4+$$

$$\left.\left.(-2+6\,\text{chi1})\,d^5+d^6\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\right/$$

$$\left(8\,\text{chi1}^2\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+$$

$$\left(\left(-d^3+\text{chi1}^2\,(-6+6\,d)+\text{chi1}\left(6\,d-6\,d^2\right)\right)\right.$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\;\Bigg(24\,\text{chi2}^2 - 24\,\text{chi2}\,d - (-2 + d)\,d^2 +$$

$$\Bigg(\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)\,(-24\,\text{chi1}^2 + 24\,\text{chi1}\,d + (-2 + d)\,d^2)$$

$$\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\;\Bigg)\Big/\;\Big(\text{chi1}\,\Big(2\,\text{chi1}^2 - 3\,\text{chi1}\,d +$$

$$d^2\Big)\Big)\Bigg)\Bigg)\Big/\;\Big(8\,\text{chi1}\,(-2 + d)\,d^2\,(-2\,\text{chi1} + d)\,(-\text{chi1} + d)\Big) +$$

$$\Bigg((3 - d)\,\Bigg(\frac{1}{-16 + 8\,d + d^2}\,192\,\text{chi2}^2\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)$$

$$(32 - 32\,d + 6\,d^2 + d^3)\;\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}^{\,2}$$

$$\Bigg(2\,\text{chi1} - d + (-2\,\text{chi2} + d)\;\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\;\Bigg) +$$

$$\Bigg((-4 + d)\,(-24\,\text{chi2}^2 + 24\,\text{chi2}\,d + (-2 + d)\,d^2)\,\Bigg(\text{chi1}\,\Big(2\,\text{chi1}^2 - 3\,\text{chi1}$$

$$d + d^2\Big)\,(-24\,\text{chi2}^2 + 24\,\text{chi2}\,d + (-4 + d)\,d^2) + \text{chi2}\,\Big(2\,\text{chi2}^2 -$$

$$3\,\text{chi2}\,d + d^2\Big)\;\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}\;\Bigg(4\,\Big(6\,\text{chi1}^2 -$$

$$6\,\text{chi1}\,d + d^2\Big) - d^3\;\sqrt[3]{\frac{\text{chi1}\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)}{\text{chi2}\,(2\,\text{chi2}^2 - 3\,\text{chi2}\,d + d^2)}}^{\,2}\;\Bigg)\Bigg)\Bigg)\Big/$$

$$((-2\,\text{chi2} + d)\,(-\text{chi2} + d))\Bigg)\Bigg)\Big/\;\Big(32\,\text{chi1}\,\text{chi2}\,(-2 + d)^2$$

$$d^2\,(2\,\text{chi1}^2 - 3\,\text{chi1}\,d + d^2)\Big)\Bigg)\;v[3,\,1]\Bigg)\Big/$$

$$\left(16\,\text{chi2}\,d^2\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)+$$

$$\left(\text{chi1}\,(-2+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right.$$

$$\left(\left(\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(128\,\text{chi1}^6+\left(-384\,\text{chi1}^5-1344\,\text{chi1}^6\right)\,d+\right.\right.\right.$$

$$\left(224\,\text{chi1}^4+4032\,\text{chi1}^5+1216\,\text{chi1}^6\right)\,d^2+$$
$$\left(192\,\text{chi1}^3-3792\,\text{chi1}^4-3648\,\text{chi1}^5\right)\,d^3+\left(-232\,\text{chi1}^2+864\,\text{chi1}^3+\right.$$
$$\left.3520\,\text{chi1}^4\right)\,d^4+\left(72\,\text{chi1}+372\,\text{chi1}^2-960\,\text{chi1}^3\right)\,d^5+$$
$$\left(-6-132\,\text{chi1}-212\,\text{chi1}^2\right)\,d^6+\left(5+84\,\text{chi1}+24\,\text{chi1}^2\right)\,d^7+$$

$$\left.(-2-24\,\text{chi1})\,d^8\right)\,\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\Bigg)\Bigg/$$

$$\left(128\,\text{chi1}^2\,(-2+d)\,d^4\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+$$

$$\left(3\,\left(-\frac{1}{2}+d\right)\,\left(\text{chi1}^2\,(1-d)+\frac{d^2}{12}+\text{chi1}\,\left(-d+d^2\right)\right)\right.$$

$$\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\,\left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\,\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Bigg/\left(\text{chi1}\,\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+\right.\right.$$

$$\left.\left.\left.\left.d^2\right)\right)\right)\right)\Bigg/\left(16\,\text{chi1}\,(-2+d)\,d^2\,(-2\,\text{chi1}+d)\,(-\text{chi1}+d)\right)+$$

$$\left(3\,(-3+d)\,\left(\text{chi1}^2+\left(-\text{chi1}-\text{chi1}^2\right)\,d+\left(\frac{1}{12}+\text{chi1}\right)\,d^2\right)\right.$$

$$\left(\frac{1}{-16+8\,d+d^2}\,192\,\text{chi2}^2\,\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.$$

$$\left(32-32\,d+6\,d^{2}+d^{3}\right)\left(\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^{2}-3\,\mathrm{chi2}\,d+d^{2}\right)}}\right)^{2}$$

$$\left(2\,\mathrm{chi1}-d+(-2\,\mathrm{chi2}+d)\,\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^{2}-3\,\mathrm{chi2}\,d+d^{2}\right)}}\right)+$$

$$\Bigg((-4+d)\left(-24\,\mathrm{chi2}^{2}+24\,\mathrm{chi2}\,d+(-2+d)\,d^{2}\right)\Bigg(\mathrm{chi1}\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\right.$$

$$\left.d+d^{2}\right)\left(-24\,\mathrm{chi2}^{2}+24\,\mathrm{chi2}\,d+(-4+d)\,d^{2}\right)+\mathrm{chi2}\,\Big(2\,\mathrm{chi2}^{2}-$$

$$3\,\mathrm{chi2}\,d+d^{2}\Big)\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^{2}-3\,\mathrm{chi2}\,d+d^{2}\right)}}\;\Bigg(4\,\Big(6\,\mathrm{chi1}^{2}-$$

$$6\,\mathrm{chi1}\,d+d^{2}\Big)-d^{3}\left(\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^{2}-3\,\mathrm{chi2}\,d+d^{2}\right)}}\right)^{2}\Bigg)\Bigg)\Bigg)\Bigg/$$

$$((-2\,\mathrm{chi2}+d)\,(-\mathrm{chi2}+d))\Big)\Big)\Big/\;\Big(64\,\mathrm{chi1}\,\mathrm{chi2}\,(-2+d)^{2}$$

$$d^{4}\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)\Big)\Big)\;\mathrm{v}[3,\,1]\Big)\Big/$$

$$\left(2\,\mathrm{chi2}\left(2\,\mathrm{chi2}^{2}-3\,\mathrm{chi2}\,d+d^{2}\right)\left(\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^{2}-3\,\mathrm{chi2}\,d+d^{2}\right)}}\right)^{2}\right)\Big)\Big/$$

$$\Big(-\Big(\left(-12\,\mathrm{chi1}^{2}+\left(12\,\mathrm{chi1}+6\,\mathrm{chi1}^{2}\right)d+(-1-6\,\mathrm{chi1})\,d^{2}\right)\Big/$$

$$(2\,\mathrm{chi1}\,(\mathrm{chi1}-d)\,(2\,\mathrm{chi1}-d))\Big)-$$

$$\Big(\mathrm{chi1}\,(-2+d)\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)\Big(-\Big(\Big(\mathrm{chi2}\,d^{2}\,(2+d)$$

$$\left(2\,\mathrm{chi2}^{2}-3\,\mathrm{chi2}\,d+d^{2}\right)\left(\sqrt[3]{\frac{\mathrm{chi1}\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)}{\mathrm{chi2}\left(2\,\mathrm{chi2}^{2}-3\,\mathrm{chi2}\,d+d^{2}\right)}}\right)^{2}\Big)\Big/$$

$$\left(2\,\mathrm{chi1}^{2}\,(\mathrm{chi1}-d)\,(2\,\mathrm{chi1}-d)\,(-2+d)\left(2\,\mathrm{chi1}^{2}-3\,\mathrm{chi1}\,d+d^{2}\right)\right)\Big)+$$

$$\left(\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ \left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.\right.$$

$$\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\ \left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+\right.\right.$$

$$\left.\left.\left.\left.d^2\right)\right)\right)\right)\Big/\ \left(2\,\text{chi1}\,(\text{chi1}-d)\,(2\,\text{chi1}-d)\,(-2+d)\right)\Big)\Big)\Big/$$

$$\left(2\,\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\ \sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)+$$

$$\left(2\,(-2+d)\,d^2\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right.$$

$$\left(\left(\text{chi2}\left(12\,\text{chi1}^2-12\,\text{chi1}\,d-d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\right.\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}^{\,2}\right)\Big/$$

$$\left(8\,\text{chi1}\,(-2+d)\,d\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)+\frac{1}{8\,(-2+d)\,d^2}$$

$$\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\ \left(24\,\text{chi2}^2-24\,\text{chi2}\,d-(-2+d)\,d^2+\right.$$

$$\left(\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\left(-24\,\text{chi1}^2+24\,\text{chi1}\,d+(-2+d)\,d^2\right)\right.$$

$$\left.\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\Big/\ \left(\text{chi1}\right.$$

$$\left.\left.\left.\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\right)\right)\right)\Big)\Big/$$

$$\left(\text{chi2}\ (\text{chi1}-d)\ (2\ \text{chi1}-d)\ \left(2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2\right)\right.$$
$$\left.\left.\sqrt[3]{\frac{\text{chi1}\ \left(2\ \text{chi1}^2-3\ \text{chi1}\ d+d^2\right)}{\text{chi2}\ \left(2\ \text{chi2}^2-3\ \text{chi2}\ d+d^2\right)}}^{\,2}\right)\right)$$

```
(* We are left with only v[3,1] variable. We
 use collections of two linear equations of v[3,1]
 (from ExtDiff[d,chi1,chi2]⟦1⟧⟦1,1⟧ and ExtDiff[d,chi1,chi2]⟦2⟧⟦1,1⟧)
 to obtain constraints for the existence of the solution. *)
```

In[∘]:=
```
AA[d_, chi1_, chi2_] := Coefficient[ExtDiff[d, chi1, chi2]⟦1⟧⟦1, 1⟧, v[3, 1], 1]
BB[d_, chi1_, chi2_] := Coefficient[ExtDiff[d, chi1, chi2]⟦1⟧⟦1, 1⟧, v[3, 1], 0]
CC[d_, chi1_, chi2_] := Coefficient[ExtDiff[d, chi1, chi2]⟦1⟧⟦2, 1⟧, v[3, 1], 1]
DD[d_, chi1_, chi2_] := Coefficient[ExtDiff[d, chi1, chi2]⟦1⟧⟦2, 1⟧, v[3, 1], 0]
```

In[∘]:=
```
Constraint[d_, chi1_, chi2_] := Simplify[Together[
   AA[d, chi1, chi2] * DD[d, chi1, chi2] - BB[d, chi1, chi2] * CC[d, chi1, chi2]]]
```

In[∘]:=
```
(* Show that there are no (d,chi1,chi2) with coprime 0<chi1,
chi2<d unless chi2=chi1 or chi2=d-chi1. *)
```

In[∘]:= **AA[d, chi1, chi2]**

Out[∘]=

```
((64 chi1^6 - 192 chi1^5 d - 128 chi1^6 d + 208 chi1^4 d^2 + 384 chi1^5 d^2 + 128 chi1^6 d^2 - 96 chi1^3 d^3 -
    392 chi1^4 d^3 - 384 chi1^5 d^3 - 64 chi1^6 d^3 - 8 chi1^2 d^4 + 144 chi1^3 d^4 + 344 chi1^4 d^4 + 192 chi1^5 d^4 +
    24 chi1 d^5 + 94 chi1^2 d^5 - 48 chi1^3 d^5 - 160 chi1^4 d^5 - 4 d^6 - 102 chi1 d^6 - 196 chi1^2 d^6 +
    12 d^7 + 156 chi1 d^7 + 140 chi1^2 d^7 - 9 d^8 - 108 chi1 d^8 - 24 chi1^2 d^8 + 2 d^9 + 24 chi1 d^9)
   ((chi1 (2 chi1^2-3 chi1 d+d^2))/(chi2 (2 chi2^2-3 chi2 d+d^2)))^(1/3)) / (128 chi1 (chi1 - d) (2 chi1 - d) (-2 + d) d^4) -
 ((-4 + d) (⋯ 28 ⋯ + 48 chi1 d^7) (chi1 (2 chi1^2 - 3 chi1 d + d^2) (-24 chi2^2 + 24 chi2 d + (-4 + d) d^2) +
    chi2 (2 chi2^2 - 3 chi2 d + d^2) (⋯ 1 ⋯)^(1/3) (4 (6 chi1^2 - 6 chi1 d + d^2) - d^3 (⋯ 1 ⋯)^(1/3)^2)))/
  (1024 chi1 chi2 (-2 + d) d^5 (-2 chi2 + d) (-chi2 + d) (2 chi1^2 - 3 chi1 d + d^2)) -
 (⋯ 1 ⋯)/(8 ⋯ 1 ⋯) + ⋯ 1 ⋯ +
 ((chi1-d) ⋯ 4 ⋯ (-(⋯ 1 ⋯)/(⋯ 1 ⋯) + ⋯ 4 ⋯))/(32 chi2 ⋯ 3 ⋯ (⋯ 1 ⋯) ((⋯ 1 ⋯)/(⋯ 1 ⋯))^(1/3)) +
 (chi1 ⋯ 2 ⋯
   (-(chi2 (2 chi2^2-3 chi2 d+d^2) (128 chi1^6+(-384 chi1^5-1344 chi1^6) d+ ⋯ 5 ⋯ +(5+84 chi1+24 chi1^2) d^7+(-2-24 chi1) d^8) ((chi1 ⋯ 1 ⋯)/(chi2 ⋯ 1 ⋯))^(1/3)^2)/
     (128 chi1^2 (-2+d) d^4 (-2 chi1+d) (-chi1+d) (2 chi1^2-3 chi1 d+d^2)) -
    (3 ⋯ 3 ⋯ (⋯ 1 ⋯))/(16 chi1 ⋯ 3 ⋯ (-chi1+d)) - ⋯ 1 ⋯ + ⋯ 1 ⋯ + (⋯ 1 ⋯)/(⋯ 1 ⋯) -
    (((chi2 ⋯ 2 ⋯ ⋯ 1 ⋯^2)/(8 chi1 (⋯ 1 ⋯) d (⋯ 1 ⋯)) + ⋯ 1 ⋯) (⋯ 1 ⋯))/((chi1 ⋯ 1 ⋯)/(chi2 ⋯ 1 ⋯))^(1/3)))/
  (2 chi2 (2 chi2^2 - 3 chi2 d + d^2) ((chi1 (2 chi1^2-3 chi1 d+d^2))/(chi2 (2 chi2^2-3 chi2 d+d^2)))^(1/3))
```

Full expression not available (original memory size: 0.7 MB)

In[•]:=
```
Constraint[d, chi1, chi2]
```

Out[•]=

$$-\left(\left(\text{chi1}\,(\text{chi1}-d)\,\left(12-9\,d+2\,d^2\right)\left(6\,(-1+d)^2\,d^2\left(32-26\,d+5\,d^2\right)+4\,\text{chi1}^4\left(-120+150\,d+3\,d^2-47\,d^3+11\,d^4\right)-8\,\text{chi1}^3\,d\left(-120+150\,d+3\,d^2-47\,d^3+11\,d^4\right)+\text{chi1}\,d\left(-768+2256\,d-2184\,d^2+822\,d^3-153\,d^4+50\,d^5-11\,d^6\right)+\text{chi1}^2\left(768-2256\,d+1704\,d^2-222\,d^3+165\,d^4-238\,d^5+55\,d^6\right)\right)\left(\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)\left(6\,\text{chi2}^2-6\,\text{chi2}\,d+d^2\right)-\text{chi2}\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)\Big/\left(128\,\text{chi2}^2\,(\text{chi2}-d)^2\,(-2+d)^3\,d^4\,(-2\,\text{chi2}+d)^2\left(6\,\text{chi1}^2-6\,\text{chi1}\,d+d^2\right)\sqrt[3]{\frac{\text{chi1}\left(2\,\text{chi1}^2-3\,\text{chi1}\,d+d^2\right)}{\text{chi2}\left(2\,\text{chi2}^2-3\,\text{chi2}\,d+d^2\right)}}\right)\right)$$

## Type II - Analysing constraints to existence of solutions

```
(* We can ignore chi1 (chi1-d) (12-9d+2 d^2) and the
 denominator. The rest is a product of two parts P1 and P2. *)
```

In[•]:=
```
P1[d_, chi1_] :=
 6 (-1 + d)^2 d^2 (32 - 26 d + 5 d^2) + 4 chi1^4 (-120 + 150 d + 3 d^2 - 47 d^3 + 11 d^4) -
  8 chi1^3 d (-120 + 150 d + 3 d^2 - 47 d^3 + 11 d^4) +
  chi1 d (-768 + 2256 d - 2184 d^2 + 822 d^3 - 153 d^4 + 50 d^5 - 11 d^6) +
  chi1^2 (768 - 2256 d + 1704 d^2 - 222 d^3 + 165 d^4 - 238 d^5 + 55 d^6)
P2[d_, chi1_, chi2_] :=
 (chi1 (2 chi1^2 - 3 chi1 d + d^2) (6 chi2^2 - 6 chi2 d + d^2) - chi2 (6 chi1^2 - 6 chi1 d + d^2)
   (2 chi2^2 - 3 chi2 d + d^2) CubeRoot[(chi1 (2 chi1^2 - 3 chi1 d + d^2))/(chi2 (2 chi2^2 - 3 chi2 d + d^2))])
```

```
(* We first show that P2 is nonzero unless chi1=chi2 or chi1+chi2=
  d. Note that P2 vanishing is equivalent to the vanishing of the following
   by taking cubes. By the argument in the paper involving coprimality,
this is nonzero unless chi1=chi2 or chi1+chi2=d. *)
```

In[ ]:=
```
Factor[chi1 ^2 (2 chi1^2 - 3 chi1 d + d^2) ^2 (6 chi2^2 - 6 chi2 d + d^2) ^3 -
  chi2^2 (6 chi1^2 - 6 chi1 d + d^2) ^3 (2 chi2^2 - 3 chi2 d + d^2) ^2 ]
```

Out[ ]=

$$(chi1 - chi2)\,(chi1 + chi2 - d)\,d^2\,\big(216\,chi1^4\,chi2^4 - 432\,chi1^4\,chi2^3\,d - 432\,chi1^3\,chi2^4\,d + 288\,chi1^4\,chi2^2\,d^2 + 864\,chi1^3\,chi2^3\,d^2 + 288\,chi1^2\,chi2^4\,d^2 - 72\,chi1^4\,chi2\,d^3 - 576\,chi1^3\,chi2^2\,d^3 - 576\,chi1^2\,chi2^3\,d^3 - 72\,chi1\,chi2^4\,d^3 + 4\,chi1^4\,d^4 + 144\,chi1^3\,chi2\,d^4 + 382\,chi1^2\,chi2^2\,d^4 + 144\,chi1\,chi2^3\,d^4 + 4\,chi2^4\,d^4 - 8\,chi1^3\,d^5 - 94\,chi1^2\,chi2\,d^5 - 94\,chi1\,chi2^2\,d^5 - 8\,chi2^3\,d^5 + 5\,chi1^2\,d^6 + 22\,chi1\,chi2\,d^6 + 5\,chi2^2\,d^6 - chi1\,d^7 - chi2\,d^7\big)$$

```
(* Therefore, the Constraint vanishing is equivalent to P1[d,chi1]=
 0. By the symmetric role of chi1 and chi2, we also have P1[d,chi2]=0. *)
```

In[ ]:=
```
P1[d, chi1]
P1[d, chi2]
```

Out[ ]=

$$6\,(-1 + d)^2\,d^2\,(32 - 26\,d + 5\,d^2) + 4\,chi1^4\,(-120 + 150\,d + 3\,d^2 - 47\,d^3 + 11\,d^4) - 8\,chi1^3\,d\,(-120 + 150\,d + 3\,d^2 - 47\,d^3 + 11\,d^4) + chi1\,d\,(-768 + 2256\,d - 2184\,d^2 + 822\,d^3 - 153\,d^4 + 50\,d^5 - 11\,d^6) + chi1^2\,(768 - 2256\,d + 1704\,d^2 - 222\,d^3 + 165\,d^4 - 238\,d^5 + 55\,d^6)$$

Out[ ]=

$$6\,(-1 + d)^2\,d^2\,(32 - 26\,d + 5\,d^2) + 4\,chi2^4\,(-120 + 150\,d + 3\,d^2 - 47\,d^3 + 11\,d^4) - 8\,chi2^3\,d\,(-120 + 150\,d + 3\,d^2 - 47\,d^3 + 11\,d^4) + chi2\,d\,(-768 + 2256\,d - 2184\,d^2 + 822\,d^3 - 153\,d^4 + 50\,d^5 - 11\,d^6) + chi2^2\,(768 - 2256\,d + 1704\,d^2 - 222\,d^3 + 165\,d^4 - 238\,d^5 + 55\,d^6)$$

```
(* P1[d,x] is a degree four polynomial in x with a symmetry. Therefore
 it has four distinct roots chi1,chi2,d-chi1,d-chi2. *)
```

In[ ]:=
```
Simplify[P1[d, x] - P1[d, d - x]]
```

Out[ ]=

0

```
(* We check that P1[d,0]>0 and P1[d,1]<0. This implies that
  P1[d,x] must have one more additional root between 0 and 1,
contradicting that it is a degree four polynomial.  *)
```

In[ ]:=
```
Factor[P1[d, 0]]
Factor[P1[d, 1]]
```

Out[ ]=

$$6\ (-2+d)\ (-1+d)^2\ d^2\ (-16+5\ d)$$

Out[ ]=

$$-\left((-3+d)\ (-2+d)^2\ (-1+d)\ \left(-24+66\ d-47\ d^2+11\ d^3\right)\right)$$

In[ ]:=
```
Solve[P1[d, 0] == 0, d]
Solve[P1[d, 1] == 0, d]
```

Out[ ]=

$$\left\{\{d\to 0\},\ \{d\to 0\},\ \{d\to 1\},\ \{d\to 1\},\ \{d\to 2\},\ \left\{d\to \frac{16}{5}\right\}\right\}$$

Out[ ]=

$$\left\{\{d\to 1\},\ \{d\to 2\},\ \{d\to 2\},\ \{d\to 3\},\ \{d\to 0.554\ldots\},\ \{d\to 1.86\ldots-0.695\ldots\, i\},\ \{d\to 1.86\ldots+0.695\ldots\, i\}\right\}$$

In[ ]:=
```
(* This proves that there are no ring isomorphisms. *)
```