The optional pricing model driven by Mixed sfbm

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Abstract

In order to depict the fractal features of real market, we verify the correlation increments and nonstationary increments of sub-fractional brownian motion (sfbm). The similar property have been verified for mixed sfbm which is linear combination of sfbm and independent brownian motion. We survey the European call option pricing model under the mixed sfbm. Empirical analysis on option contracts data from SSE verify accuracy of pricing results of the model comparing with BS model and modified BS model.

Keywords: Option Pricing, non-stationary increment, Mixed Sub-fractional brownian motion

1 Introduction

In recent years, the global trade frictions and the global epidemic have stimulated the uncertainty of the macroeconomic environment. The risk aversion of the market has obviously increased. Therefore, in order to avoid the investment risks brought by the sharp fluctuations of the capital market, both individual investors and financial institutions are in urgent need of a wealth of financial products to manage and avoid risks. Among many financial derivatives, options contract has become a preferred investment choice and risk management tool. Meanwhile, more meticulous research on the market indicates the changes financial asset prices show different degrees of long-term memory (Tan Z X & Zhang Q,2016;Nguyen D B B,2019). This property contrary to the assumptions of traditional optional pricing models. Consequently, how to accurately depict the dynamic changes of financial asset prices to improve the accuracy of option pricing is a worthy question to explore.

The sub-fractional Brownian motion(sfbm) is a kind of gaussian process. It's introduced for depiction of occupation time fluctuations of branching particle systems. As variants of Brownian motion, fractional Brownian motion(fbm) and sfbm both own the properties of long-range dependence and self-similarity. Comparing with fbm, nonstationary increment is a distinctive property of sfbm. Further, the increments of sfbm are weaklier correlated (Tomasz Bojdecki,2004; Constantin Tudor,2007).

Applications in not only mathematical finance but also hydrology, telecommunication and instrumental science motivate the involvement and study on the sfbm. Recently, there has been great interest involving fbm and sfbm as the input noises in the time-series represent similar properties. As the features of assets prices has been verified, sfbm emerges as a potential candidate.

The online of the paper is following. Firstly, some relevant researches are addressed, including their achievements and limitation. Later, we survey some properties of sfbm and the proof of existence. The next, we build up the optional pricing model driven by sfbm. Finally, a case study and result analysis are displayed.

2 Literature Review

Since the emergence of options, option pricing has become a momentous subject of finance. Core issue of option pricing is to establish a model that can describe the variation of assets prices well. B-S model (Black F& Scholes M, 1973) has been regarded as the fundament of modern optioning pricing theory and wildly affected practical pricing. It firstly introduced geometric Brownian motion (gbm) in characterization of variety of financial assets and built up a non-arbitrage market. Nevertheless, some of the hypothesizes deviate from the actual market. For example, underlying assets prices don't confirm gbm and the volatility varies instead of being constant. The empirical researches on the actual market show some features: the distribution of financial asset price return has the characteristics of "peak and fat tail"(Tan Z X & Zhang Q,2016); the change of financial asset price presents different degrees of long memory. Consequently, theoretical prices obtain from B-S model have some errors comparing with actual prices.

To reduce the error and make results more accurate, meanwhile, retain meaningful insights from B-S model, many scholars have proposed a series of improvement on B-S model, A mainstream research direction is based on Fractal Market Hypothesis (FMH). Pioneering work on fractional Brownian motion (fbm) of Hurst (1951) and Mandelbrot (1982) laid a foundation for subsequent analysis and applications. Using stochastic process with fractal characteristics is an effective method to describe the variation of assets prices to meet their realistic characteristics. Consequently, fbm was tried to substitute for gbm. However, fbm does not possess Markov property and isn't a semi-martingale. Therefore, directly using of It$\hat{o}$ integral in the same process of gbm generates arbitrage opportunities in the market (Rostek S,2009). For this stochastic process, a modified It$\hat{o}$ integral with Wick product was developed. Necula (2002) has figured out the option pricing through the martingale method under risk neutral measure with Wick-It$\hat{o}$ integral. The result shows Wick-It$\hat{o}$ integral eliminates the arbitrage opportunities. Unfortunately, the result obtained from fbm have no obvious improvement on accuracy comparing with result from B-S model(Liu S C,2011).

Recently, more researches focus on the feature of real market have been done and show the nonstationary increment of changes of assets prices. (Constantin Tudor & Tomasz Bojdecki, 2007; Raffaello Morales, 2013;Rogério L.,2003) The new feature motivates involving sfbm to the optional pricing model. Similar to the case of fbm, the option pricing driven by sfbm will generate arbitrage opportunities and can be solve through Wick-It$\hat{o}$ integral. Another way to eliminate arbitrage opportunities is to consider the mixed sfbm (the linear combination of bm and independent sfbm). Charles El-Nouty's (2015) research shows mixed sfbm not only possess the properties like long-range correlations, but also is semi-martingale when Hurst index $\in (\frac{3}{4}, 1)$. Some scholars have involved sfbm in the price fluctuation modeling(Constantin Tudor, 2008; Junfeng Liu,2010). All the relevant models mostly based on BS-model and improved the results.

To sum up, though many scholars have made explorations in various aspects, there are still some obvious deficiencies and directions worthy of further research.

1.Most of the modified models only focus on a unique feature of actual market, lacking reactions of the overall perspective of market. Base on new features of market discovered recently, assumptions of some models are unreasonable.

2.Compared with BS model, some existing option pricing models are more complex but less accurate. Some modified model lack validation under a certain amount of data.

3.Most of the research merely consider the diffusion process under gbm or fbm(sfbm) which is onefold and inconsistent with reality.

In the view of recent research, the main work of this paper is reflected in two aspects. Firstly, we involve the mixed sfbm to characterize the "peak and fat tail" and long memory characteristics of financial asset prices, and the explicit solution of option prices is obtained through Wick-It$\hat{o}$ integral and Fourier transform. Secondly, we select option data from SSE to implement the applicability and effectiveness of the model in comparing with traditional BS model and modified BS model (with sfbm).

3 On the sfbm and mixed sfbm

As a centered Gaussian process generate from brownian motion. The fractional brownian motion $B^H = \{B_t^H, t \geq 0\}$ with $H \in (0,1)$ holds:
(i)$\mathbb{E}B_t^H = 0$.
(ii)$B_0^H = 0$.
(iii)$\mathbb{E}(B_t^H B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$.

Similar, the sub-fractional brownian motion $\xi^H = \{\xi_t^H, t \geq 0\}$,is a centered Gaussian process which holds:
(i)$\mathbb{E}\xi_t^H = 0$.
(ii)$\xi_0^H = 0$.
(iii)$\mathbb{E}(\xi_t^H \xi_s^H) = s^{2H} + t^{2H} - \frac{1}{2}(|t-s|^{2H} + (s+t)^{2H})$.

The varification of the rationality of the definition is indispensable work. That is, to varify the existence of such a stochastic process holding these properties. The existence of BM can be proved through Kolomogorov theorem. Since a Gaussian process is completely determined by it's mathematical expectation and covariance function, we can directly construct fbm and with bm. We have

$$B_t^T = \frac{1}{C_1(H)} \int_R \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}} \right] dB_s.$$

$$C_1(H) = \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H} \right)^{\frac{1}{2}}.$$

Here B_s is the bm. It's easy to verified the construction satisfy the properties in the definition of fbm. The proof of existence of sfbm is through similar mechanisms and is omitted here.

Under the definition, we can get some properties of sfbm directly. Let $t_1 > t_2 \geq s_1 > s_2 \geq 0$, considering property (iii) of sfbm , we have:

$$\begin{aligned}\mathbb{E}[(\xi_{s_1}^H - \xi_{s_2}^H)(\xi_{t_1}^H - \xi_{t_2}^H)] =& \frac{1}{2}[(t_1+s_2)^{2H} + (t_1-s_1)^{2H} + (t_2+s_1)^{2H} + (t_2-s_2)^2] \\ &- \frac{1}{2}[(t_1+s_1)^{2H} + (t_1-s_1)^{2H} + (t_2+s_2)^{2H} + (t_2-s_2)^2].\end{aligned}$$

Thus, $\mathbb{E}[(\xi_{s_1}^H - \xi_{s_2}^H)(\xi_{t_1}^H - \xi_{t_2}^H)] > 0 (< 0)$ for $H > \frac{1}{2} (< \frac{1}{2})$. The equality implies the correlation of increments over the non overlapping time periods. For $H > \frac{1}{2} (< \frac{1}{2})$, the increments are positively(negatively) correlated.

Let $t > s \geq 0$, another direct inference is:

$$\mathbb{E}[(\xi_t^H - \xi_s^H)^2] = -2^{2H-1}(s^{2H} + t^{2H}) + (s+t)^{2H} - (t-s)^{2H}.$$

Thus, for $t' > s' \geq 0$ such that $t' - t = s' - s$, we have:

$$\mathbb{E}[(\xi_t^H - \xi_s^H)^2] \neq \mathbb{E}[(\xi_{t'}^H - \xi_{s'}^H)^2].$$

The inequality implies the non-stationary increments of sfbm. The increments over different isometric time periods is different.

As researches had indicated the real market shows the nonstationary increments and long-range correlations, directly use sfbm as the substitute of bm seems natural. However, this substitution result in overemphasising on these features and looking down opun other features of the market. Instead, use the linear combination of bm and sfbm is more rational which ensures more features are considered. Thus, we should have more meticulous disscusion on this linear combination.

Let $\alpha \geq 0, \beta \geq 0$, $M^{\alpha,\beta,H} = \{M_t^{\alpha,\beta,H} = \alpha B_t + \beta \xi_t^H, t \geq 0\}$ is a mixed sfbm. A mixed sfbm can degenerate to both bm and sfbm. From the properties of bm and sfbm, we have:

$$\mathbb{E}(M_t^{\alpha,\beta,H} M_s^{\alpha,\beta,H}) = \beta^2[s^{2H} + t^{2H} - \frac{1}{2}(|t-s|^{2H} + (s+t)^{2H})] + \alpha^2 \min(s,t)$$

Since bm holds independence of increments, let $t_1 > t_2 \geq s_1 > s_2 \geq 0$, we have:

$$\begin{aligned}\mathbb{E}[(M_{s_1}^{\alpha,\beta,H} - M_{s_2}^{\alpha,\beta,H})(M_{t_1}^{\alpha,\beta,H} - M_{t_2}^{\alpha,\beta,H})] =& \frac{\beta^2}{2}[(t_1+s_2)^{2H} + (t_1-s_1)^{2H} + (t_2+s_1)^{2H} + (t_2-s_2)^2 \\ &- (t_1+s_1)^{2H} - (t_1-s_1)^{2H} - (t_2+s_2)^{2H} - (t_2-s_2)^2].\end{aligned}$$

Thus, $\mathbb{E}[(M_{s_1}^{\alpha,\beta,H} - M_{s_2}^{\alpha,\beta,H})(M_{t_1}^{\alpha,\beta,H} - M_{t_2}^{\alpha,\beta,H})]$ is positive for $H \in (\frac{1}{2}, 1)$. The correlations of increments for mixed sfbm consistent with sfbm's.

Let $t > s \geq 0$, we have:

$$\mathbb{E}[(M_t^{\alpha,\beta,H} - M_s^{\alpha,\beta,H})^2] = \beta^2[-2^{2H-1}(s^{2H} + t^{2H}) + (s+t)^{2H} - (t-s)^{2H}] + \alpha^2(t-s)$$

Thus, mixed sfbm also holds nonstationary increments since:

$$\mathbb{E}[(M_t^{\alpha,\beta,H} - M_s^{\alpha,\beta,H})^2] \neq \mathbb{E}[(M_{t'}^{\alpha,\beta,H} - M_{s'}^{\alpha,\beta,H})^2]$$

Here, $t' > s' \geq 0$ and $t' - t = s' - s$.

4 European Call Option Pricing Model Driven by Mixed sfbm

The incompleteness of information disclosure makes most investors unable to obtain all the information. Consequently, the investors have no choice but to use part of past information to make investment decisions This fact results in the changes in the price of financial assets have a certain degree of long-term memory. Thus, mixed sfbm seems to be able to better depict the actual situation of real market. Taking BS model as the platform, a natural idea is to replace bm with mixed sfbm.

Before derivation, we make the following assumptions:
(1)There is no transaction cost for buying and selling stocks or options.
(2)Risk-free interest rate is known and remains unchanged.
(3)There is no short selling restriction.

Let $T \geq t \geq 0$, T is expiration date, we depict prices of risky assets with mixed sfbm. Specifically, the price of risky set S_t satisfy:

$$dS_t = \mu S_t dt + \sigma S_t dM_t^{\alpha,\beta,H}. \tag{1}$$

Here μ is expected rate of return on risk assets, σ is the volatility of the risky assets' prices.$\alpha \in (0,1)$,$\beta = \sqrt{1-\alpha^2}$. Under the risk-neutral measure Q, expected rate of return on risk assets is equal to risk-free rate r. Apply Wick-It$\hat{o}$ integral on (1), we have:

$$S_t = S_0 \exp[rt - \frac{1}{2}(\sigma^2\alpha^2 t + \beta^2\sigma^2 t^{2H}(2 - 2^{2H-1})) + \sigma M_t^{\alpha,\beta,H}]. \tag{2}$$

In addition, can obain the present value by discounting the expected value at r. Therefore, when the option contract expires, the value is $\max(S_T - K, t)$. Here K is the delivery price. According to the martingale pricing theory, the value of the option contract at t is equal to the present value of the expectation of $\max(S_T - K, t)$ discounted at r. Specifically, for the price of call option C_t, we have

$$C_t = \exp(-rT + rt)E_t^Q[\max(S_T - K, t)] \tag{3}$$

After Fourier transform and inverse Fourier transform (Yu Mei,2021), we can obtain the option pricing formula driven by mixed sfbm:

$$C_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{4}$$

with

$$d_1 = \frac{In\frac{S_t}{K} + r(T-t) + \frac{1}{2}\sigma^2\alpha^2(T-t) + \frac{1}{2}\sigma^2\beta^2(2-2^{2H-1})(T^{2H} - t^{2H})}{\sigma\sqrt{\alpha^2(T-t) + \beta^2(2-2^{2H-1})(T^{2H} - t^{2H})}}.$$

$$d_2 = d_1 - \sigma\sqrt{\alpha^2(T-t) + \beta^2(2-2^{2H-1})(T^{2H} - t^{2H})}.$$

5 Empirical Analysis

5.1 Selection of Comparison Model

To verify the accuracy of results, we select traditional BS model and modified BS model with sfbm as the comparison model. Since the mixed sfbm can degenerate to bm (sfbm) with $\alpha = 1, \beta = 0$ ($\alpha = 0, \beta = 1$), the option pricing model can driven by mixed sfbm can also degenerate to both comparison models. Therefore, the formulas of comparison models can be directly obtained from (4) and are omitted here.

5.2 Data Selection and Preprocessing

We select the SSE 50ETF option data from the Shanghai Stock Exchange as the experimental data. The data contains strike prices, clossing prices, days from due date, etc of the contracts with the largest trading volume from 2021.10.1 to 2022.8.31. The underlying asset of the option contract is SSE 50ETF index, and the contract unit is RMB. We select data from Wind Information Financial Terminal.

We preprocess the data with following process. Firstly, we exclude the data of European put options and data on non-trading days. Then we remove underpriced options to improve results. Finally, we select 174 option contracts which is temporally continuous for more than 44 trading days. Strike prices and closing prices of a specific option contract is shown in figure 1 below.

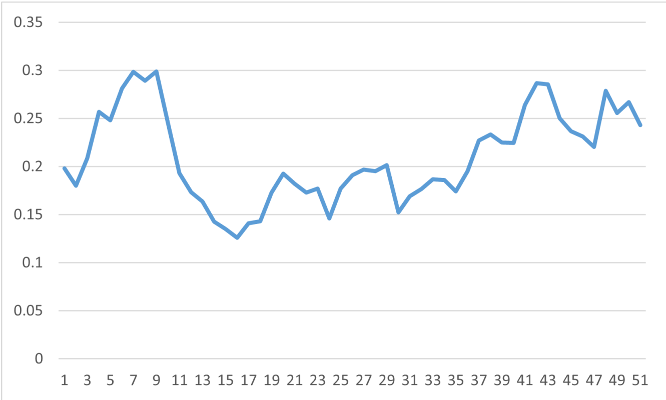

Fig. 1: closing prices data of contract #10003373 after preprocessing

These constracts are divided in two parts. We select 100 constracts randomly as the sample data to determine parameters of models. The remainds will be use to test accuracy of the results from models with determined parameters.

5.3 Parameters Estimation

In order to compare the performance of different models in option pricing, we need to estimate the parameters. We use simulated annealing algorithm (SAA) adjusting the parameters to minimize the mean square error of pricing results on the sample data we select. The result is shown in tabel 1 below.

Tab. 1: parameters and MSE on sample data

model	σ	H	α	MSE
mixed sfbm	0.1852	0.998	0.9972	1728
bs	0.1907			1792
sfbm	0.3068	0.5006		2352

5.4 Pricing Results and Analysis

With the parameters determined over the sample data, we applied the model on the remainders. The result over a specific contract is shown in tabel 2 below.

Tab. 2: Pricing results of constract #10003377

days to due date	44	43	42	41	40	37	36	35	34
mixed sfbm	120.3	108.6	84.6	164.8	139.5	124.6	118.5	90.2	57
bs	134.2	121.7	95.8	181.1	154.3	138.3	131.7	101.4	65.4
sfbm	146.6	131.2	102.2	187.7	157.8	134.5	125.7	94.3	58.5
real price	153	145	109	178	150	140	122	96	68

MSE of results from model driven by mixed sfbm is 1943, 2051 and 2649 for BS model and modified BS model. In addition, among 74 option contracts, the improvement is shown in tabel 3 belong precisely:

Tab. 3: Number of option contracts with different optimal extent

optimal extent	$\leq 5\%$	$5\% \sim 10\%$	$\geq 10\%$
compare with bs	1	0	24
compare with sfbm	7	2	36

In general, the mixed sfbm model improved results of BS model by 5.26%, 26% for sfbm model. It's worth mentioning SAA has certain random factors and may fall into local optimal solution. Thus, the parameters we determined way not be lobal optimal solution. However, parameters for our model and results have denoted the data we select have weak fractal characteristics.

Compared with the comparison model, the results of our model are more accurate. The model is more general and can degenerate to several traditional model including BS model. However, the model still has limitations. The model introduces more parameters and need prior expectation for parameters determination. Also, the effectiveness mainly depends on the whether the features of the market meet the assumptions. The jump diffusion process which contains the discontinuous jump is not considered in the model

6 Summary

In order to capture the long memory and fractal features of financial asset prices, we firstly verify the mixed sfbm is the stochastic process with similar properties:correlation increments and nonstationary increments. Taking BS model as the platform, we build up the European call option pricing model under mixed sfbm. With the pricing formula, we select SSE 50ETF option data for empirical analysis. Validity and accuracy of the model have been verified comparing with BS model and modified BS model with sfbm.

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