Lecture 6

Monday, June 3, 2019 11:45 AM

Random Vectors and Multivariate Normal Distribution

A random vector $\vec{y}$ is a vector of random variables:

$$\vec{y} = (y_1, y_2, \dots, y_n)^T$$

The probabilistic behaviour of $\vec{y}$ is summarized with the mean vector:

- $E[\vec{y}] = (E[y_1], E[y_2], \dots, E[y_n])^T$

and the variance-covariance matrix:

- $Var[\vec{y}] = \begin{bmatrix} Var[y_1] & Cov[y_1,y_2] & Cov[y_1,y_3] & \cdots & Cov[y_1,y_n] \\ Cov[y_2,y_1] & Var[y_2] & Cov[y_2,y_3] & \cdots & Cov[y_2,y_n] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Cov[y_n,y_1] & Cov[y_n,y_2] & Cov[y_n,y_3] & \cdots & Var[y_n] \end{bmatrix}$

$E[(\vec{y}-E[\vec{y}])(\vec{y}-E[\vec{y}])^T]$

*this is symmetric and positive semi-definite

Properties of random vectors

Let $\vec{a}$ be a 1xn row vector of constants and A be an nxn matrix of constants.

- $E[\vec{a}\vec{y}] = \vec{a}E[\vec{y}]$
- $E[A\vec{y}] = A E[\vec{y}]$
- $Var[\vec{a}\vec{y}] = \vec{a} Var[\vec{y}] \vec{a}^T$
- $Var[A\vec{y}] = A Var[\vec{y}] A^T$

$\hookrightarrow E[(A\vec{y} - E[A\vec{y}])(A\vec{y} - E[A\vec{y}])^T] = E[(A\vec{y} - AE[\vec{y}])(Ay - AE[\vec{y}])^T]$

$(AB)^T = B^T A^T$

$= E[A(\vec{y}-E[\vec{y}])(\vec{y}-E[\vec{y}])^T A^T]$

$= A\, E[(\vec{y}-E[\vec{y}])(\vec{y}-E[\vec{y}])^T]\, A^T$

$= A\, Var[\vec{y}]\, A^T$

Example: Suppose we have a random vector $\vec{y} = (\vec{y}_1, \vec{y}_2, \vec{y}_3)^T$ with mean vector and variance-covariance matrix:

$$E[\vec{y}] = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad Var[\vec{y}] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

- $E[y_3] = 2$
- $Var[y_2] = 5$
- $Cov[y_1, y_2] = 0$
- $Cov[y_1, y_3] = -1$

Let $\vec{a} = [2 \ 1 \ -1]$, $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ -1 & 4 & 1 \end{bmatrix}$

- $E[\vec{a}\vec{y}] = \vec{a}E[\vec{y}] = [2 \ 1 \ -1]\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = (2)(1) + (1)(3) + (-1)(2) = 3$

- $Var[\vec{a}\vec{y}] = \vec{a}\,Var[\vec{y}]\,\vec{a}^T = [2 \ 1 \ -1]\begin{bmatrix} 1 & 0 & -1 \\ 0 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$= [2 \ 1 \ -1]\begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}$

$= (2)(3) + (1)(4) + (-1)(-4)$

$= 14$

Exercise: $E[A\vec{y}]$ and $Var[A\vec{y}]$

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \sim MVN(\underset{\text{mean vector}}{\vec{\mu}}, \underset{\text{variance-covariance matrix}}{\Sigma})$

$$f(\vec{y}; \vec{\mu}, \Sigma) = \frac{1}{2\pi \underset{\text{determinant}}{|\Sigma|}^{1/2}} \exp\left\{-\frac{1}{2}(\vec{y}-\vec{\mu})^T \overset{\text{inverse}}{\Sigma^{-1}} (\vec{y}-\vec{\mu})\right\}$$