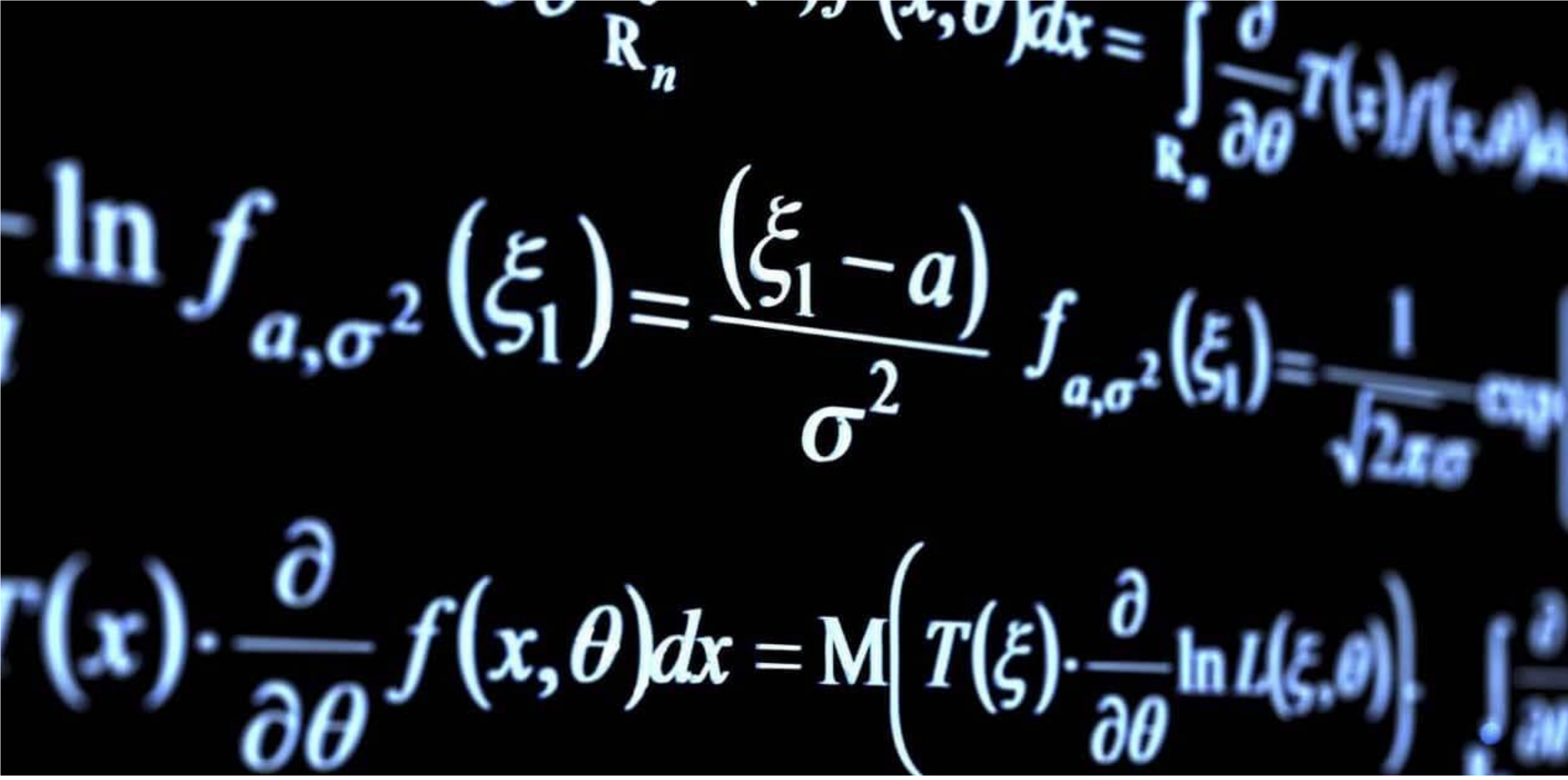

ACTSC 446 Course Notes

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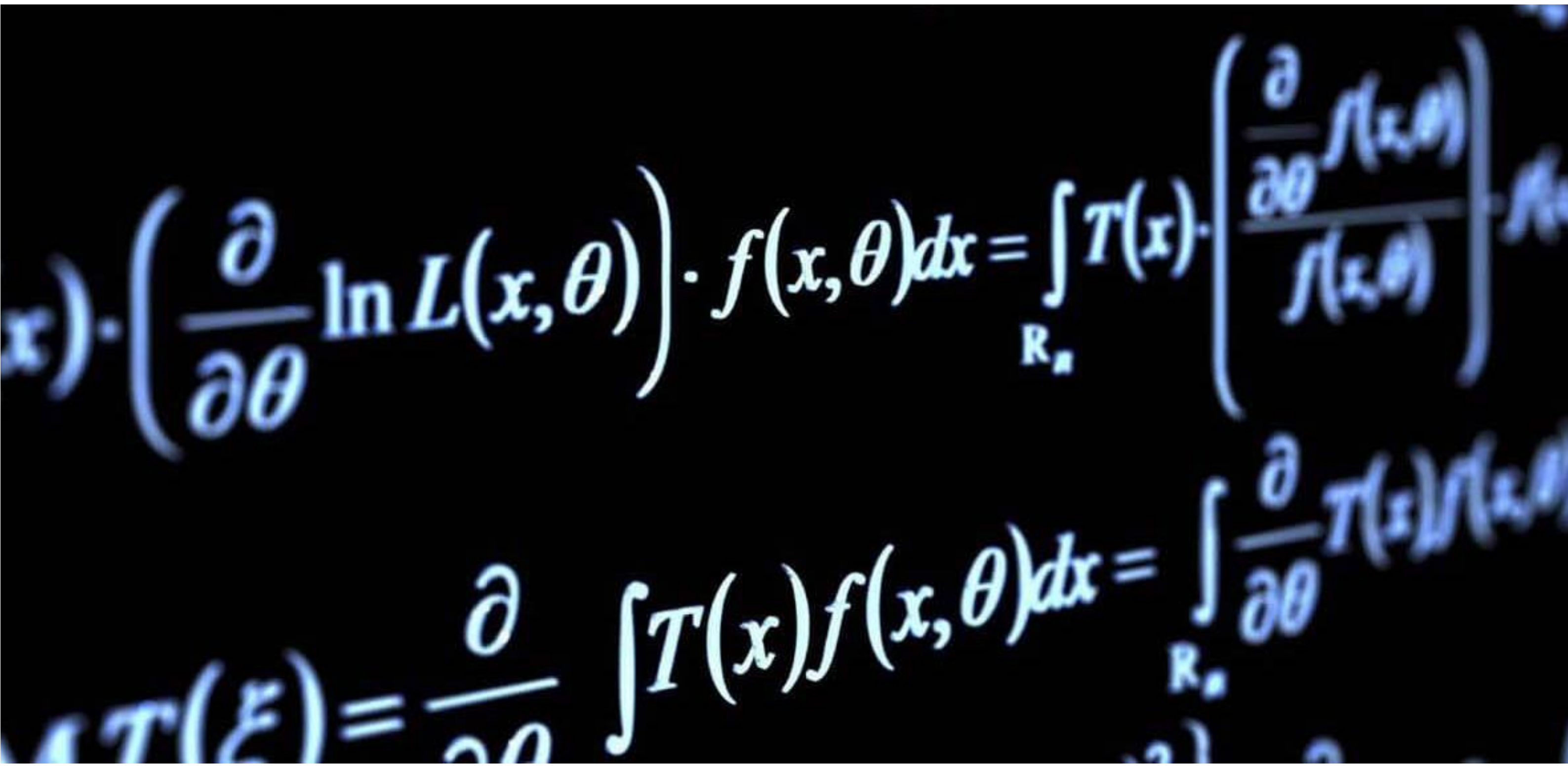

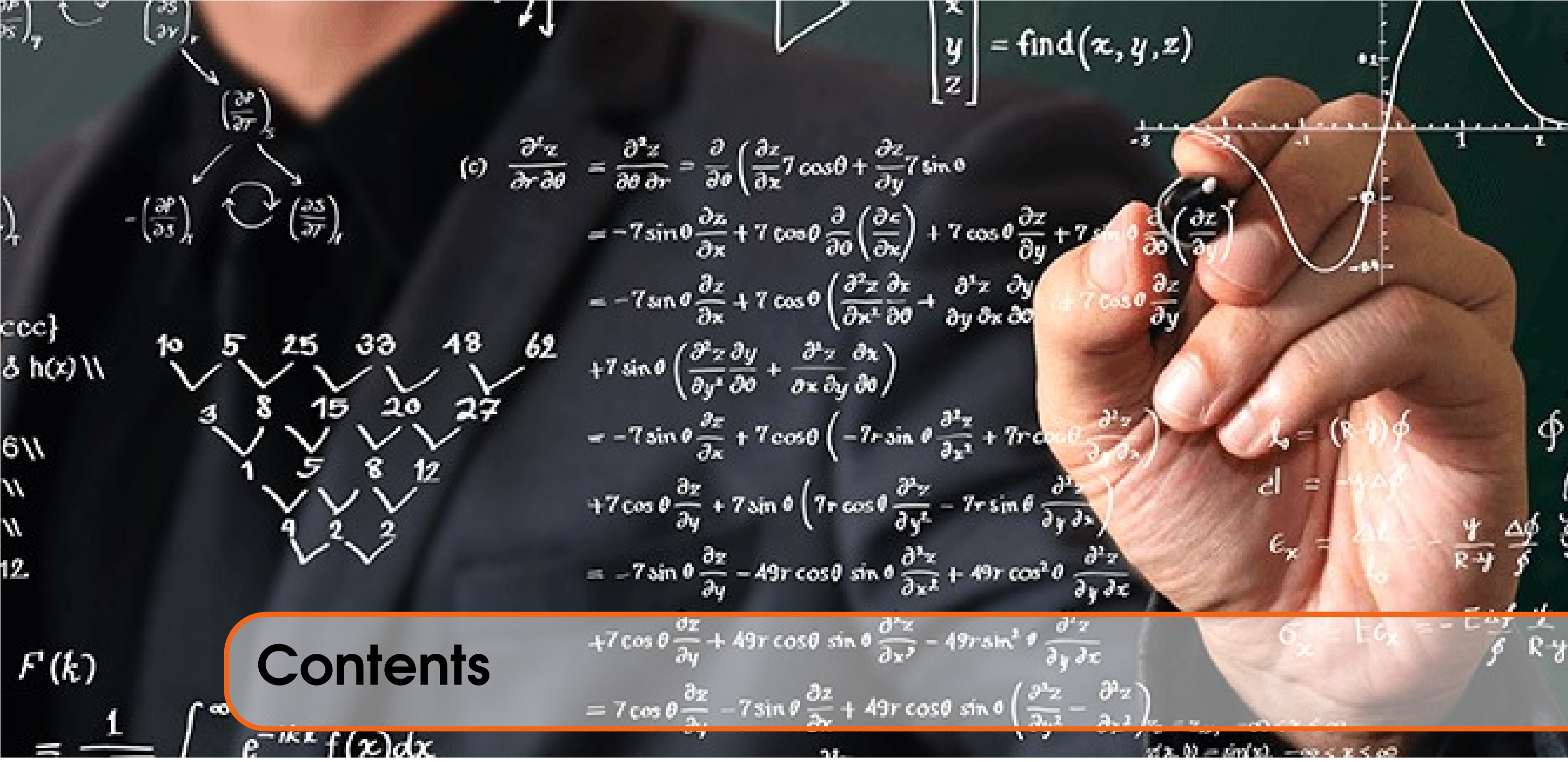

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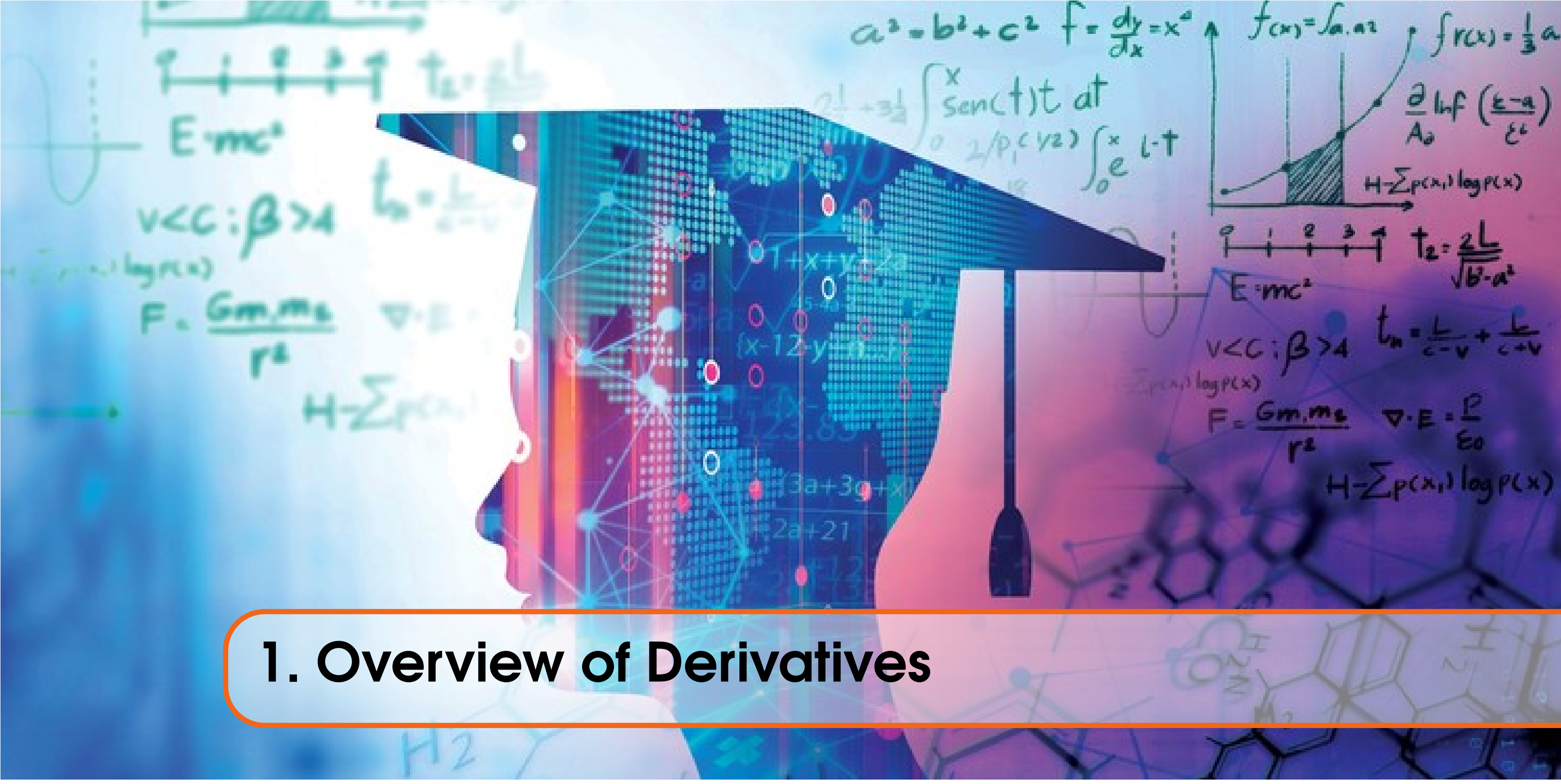

1. Overview of Derivatives

1.1 Defintions of Derivatives and Underlying Assets

Definition 1.1.1 — Derivative. A financial instrument whose price is derived by the price of one or more underlying assets

Definition 1.1.2 — Underlying Asset. An asset, basket of assets, or even another derivative.

R There must be an independent way to determine its value

■ **Example 1.1 — Examples of Underlying Assets.** ■

1.2 Uses of Derivatives

Common reasons to use derivatives

1. Provide insurance - hedging

Category	Description
Stock index	S&P 500 index, Dow-Jones Industrials
Interest rate	10-year Canada treasure bonds, LIBOR
Foreign exchange	U.S. dollar, Chinese Yuan, Euro
Commodity	Oil, natural gas, gold, corn, cattle
Other	Credit, real estate

Figure 1.1.1: *Underlying Assets*

2. Speculation - high leverage and risky
3. Reduced transaction costs
4. Regulatory arbitrage

OTC vs Exchange Traded

There are two groups of derivatives contracts.

1. Over-the-counter (OTC) derivatives:
   (a) Larger market
   (b) Privately traded
   (c) Less tranparent price
   (d) Low liquidity
   (e) Higher credit risk
   (f) Lower fees and taxes
   (g) Greater freedom
2. Exchange-traded derivatives (ETD)
   (a) Smaller market
   (b) Publicly traded
   (c) ...

Perspectives on Derivatives

1. **End-user:** they use derivatives to manage risk, speculate, reduce costs, or avoid regulation
2. **Market-maker:** they will buy derivatives from customers who wish to sell, and sell derivatives to customers who wish to buy. They make the **bid-ask spread**
3. **Economic observers:** the observe derivative markets and try to make sense of everything

R Just like the relationship between customers, car dealers, and ministry of transportation.

In this course, we care about

1. Non-abitrage pricing of OTC derivatives using discrete-time and continuous-time pricing.
2. We use derivatives for hedging (insurance)

R Why we care more about OTC instead of ETD? Since ETD facilitate pricing through its competitive market.

1.3 Types of Derivatives

There are four common types of derivative contracts: forwards, futures, **options**, and swaps.

Definition 1.3.1 — Forward. A non-standardized (OTC) contract between two parties to buy or to sell an asset at a specified future time at a fixed price agreed upon today.

1. Underlying assets: the asset on which the forward contract is based
2. Expiration date: the time at which the asset is delivered
3. Forward price: the price the buyer will pay at the expiration date

We usually let S_t be the spot price of the underlying asset at time $t \geq 0$, T be the expiration date, and K be the forward price.

R

1. Payoff to long forward $= S_T - K$
2. Payoff to short forward $= K - S_T$

Thus, its payoff structure looks like,

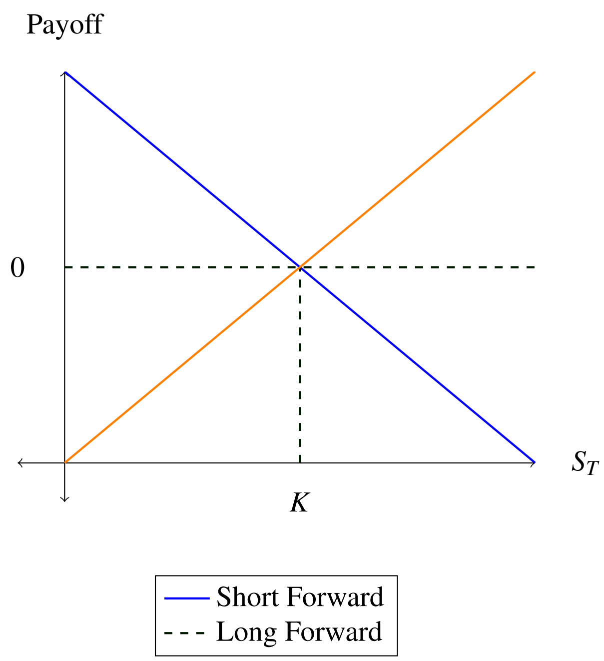

Figure 1.3.1: *Forward Payoff*

Definition 1.3.2 — Futures. A standardized (ETD) contract between two parties to buy or sell a specified asset of standardized quantity and quality at a specified future time at a given price.

Comparison Between Forwards and Futures

Forwards	Futures
OTC	ETD
Price does not change	Price changes daily
Delivery usually happens	Usually closed before maturity so delivery usually never happens
Mainly employed by hedgers	Mainly employed by speculators

Table 1.3.1: *Forwards vs. Futures*

Definition 1.3.3 — Option. A contract which gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a specified strike price on or before expiration.

R Exercise Styles:
1. **European:** the option can only be exercised at maturity
2. **American:** the option can be exercised any time at or before maturity
3. **Bermudan:** the option only be exercised on specified dates at or before maturity

R Payoff Types:
1. **Call option:** the buyer has the right to buy the underlying
2. **Put option:** the buyer has the right to sell the underlying

3. **Exotic:** an option including complex financial structures

Comparing Forwards/Future with Options

	Forwards/Futures	**Options**
Payoff type	Only one	Numerous
Exercise	Obligation	Right
Price	Usually zero	Usually positive

Table 1.3.2: *Forwards/Futures vs. Options*

Please read ACTSC372 first chapter `https://www.student.cs.uwaterloo.ca/~w3zhuo/Public_Notes/ACTSC372/ACTSC372_Lecture_Notes__Book_Version_.pdf` for detailed notes on the basics of options.

Insurance Strategies

1. **Floor:** long a stock and long a put
$$S_T + (K - S_T)_+$$
2. **Covered call writing:** long a stock and short a call
$$S_T - (S_T - K)_+$$
3. **Cap:** short a stock and long a call
$$-S_T + (S_T - K)_+ = -K + (K - S_T)_+$$
4. **Covered put writing:** short a stock and short a put
$$-S_T - (K - S_T)_+ = -K - (S_T - K)_+$$

R
$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

Speculation Strategies Using Options

Definition 1.3.4 — Spread. A position consisting of only calls or only puts.

1. **Bull spread:** long a call and short another call with higher strike to bet on up movement
2. **Bear spread:** long a call and short another call with lower strike to bet on down movement
3. **Box spread:** long a synthetic forward (long a call and short a put with the same strike and short another synthetic forward with different forward price to arbitrage if some options are underpriced.
4. **Ratio spread:** long m optioons and shrot n options at a different strike (to bet on stable but slightly bullish or bearish movement)

Definition 1.3.5 — Collar. Long a put and short a call with higher strike. The collar width is the difference between the strikes. Limiting positive and negative returns when in a long position of assets.

Definition 1.3.6 — Volatility Options.
1. **Straddle:** long an atm call and an atm put with the same strike, speculation on high vol
2. **Strangle:** long an otm call and otm put, speculation on high vol with lower cost
3. **Butterfly:** short a straddle and long a strangle, speculation on low vol

1.4 Relation of Options

Theorem 1 — Put-Call Parity.

$$C(K,T) - P(K,T) = PV(S_T) - e^{-rT}K$$

where
1. No dividend: $PV(S_T) = S_0$
2. Discrete dividend: $PV(S_T) = S_0 - PV(Div)$
3. Continuous dividend (with rate δ): $PV(S_T) = e^{-\delta T}S_0$

Theorem 2 — Generalized Parity. Consider two assets with prices: $\{S_t\}_{0\leq t\leq T}, \{Q_t\}_{0\leq t\leq T}$. Let
1. $C(S_0, Q_0, T)$ be the premium of the right to get one share of asset 1 by giving up one share of asset 2 at T (it will be exercised only if $S_T > Q_T$)
2. $P(S_0, Q_0, T)$ be the premium of the right to get one share of asset 2 by giving up one share of asset 1 at T (it will be exercised only if $Q_T > S_T$)

then

$$C(S_0, Q_0, T) - P(S_0, Q_0, T) = PV(S_T) - PV(Q_T)$$

Corollary 1.4.1 — Currency Parity. If $1f = x_0 d$ and all unit denoted in d,

$$C_d(x_0, K, T) - P_d(x_0, K, T) = P_d(x_0, K, T) = x_0 e^{-r_f T} - Ke^{-r_d T}$$

Proposition 1.4.2 — Different Strikes (Monotonicity). Suppose that $K_1 < K_2$, we have

$$C(K_1) \geq C(K_2) \qquad P(K_1) \leq P(K_2)$$

Proof. Since $K_1 \leq K_2$, we have

$$(S_T - K_1)_+ \geq (S_T - K_2)_+$$

then, $C(K_1) \geq C(K_2)$. The other one is similar. ∎

Proposition 1.4.3 — Different Strikes (Lipschitz). Suppose that $K_1 < K_2$ and r is the continuously compounded interest rate, we have

$$C(K_1) - C(K_2) \leq e^{-rT}(K_2 - K_1)$$
$$P(K_2) - P(K_1) \leq e^{-rT}(K_2 - K_1)$$

Proposition 1.4.4 — Different Strikes (Convexity). $C(K), P(K)$ are convex with respect to K.

Corollary 1.4.5 For $K_1 < K_2 < K_3$, we have

$$\frac{C(K_1) - C(K_2)}{K_1 - K_2} \leq \frac{C(K_2) - C(K_3)}{K_2 - K_3}$$

and

$$\frac{P(K_1)-P(K_2)}{K_1-K_2} \leq \frac{P(K_2)-P(K_3)}{K_2-K_3}$$

Proposition 1.4.6

$$S_0 \geq C_A(S_0,K,T) \geq C_E(S_0,K,T) \geq (PV(S_T)-PV(K))_+$$
$$K \geq P_A(S_0,K,T) \geq P_E(S_0,K,T) \geq (PV(K)-PV(S_T))_+$$

Proposition 1.4.7 If the stock pays no dividend, the American call option should **never be early exercised**

Proof. At any time t prior to maturity T, by the put-call parity

$$C_A(S_t,K,T-t) \geq C_E(S_t,K,T-t) = P_E(S_t,K,T-t) + S_t - e^{-r(T-t)}K > S_t - K$$

Thus, holding the american option is better. ∎

Corollary 1.4.8 Early exercise of American calls is not rational at time $t < T$, if

$$K + P_E(S_t,K,T-t) - PV_{t,T}(K) > PV_{t,T}(Div)$$

R This does not mean you should exercise it when the inequality is not true. But if dividend is high enough, you have incentive to exercise the American call early to get the stock.

Corollary 1.4.9 Early exercise of American puts is not rational at time $t < T$, if

$$PV_{t,T}(Div) > K - C_E(S_t,K,T-t) - PV_{t,T}(K)$$

Proposition 1.4.10 — Different Maturities. Suppose $0 < t < T_1 < T_2$,

1. **American options:**
$$C_A(S_t,K,T_1-t) \leq C_A(S_t,K,T_2-t)$$
$$P_A(S_t,K,T_1-t) \leq P_A(S_t,K,T_2-t)$$
2. **European calls (no dividend)**
$$C_E(S_t,K,T_1-t) \leq C_E(S_t,K,T_2-t)$$
3. **European calls with dividend and European puts:** indeterminate

1.5 Binomial Option Pricing Model

1.5.1 Single-Period Model

Definition 1.5.1 — Binomial Model. Stock price in the next moment only have two states (up and down) and the ratios of change are fixed

$$0 < d < u$$

where d is the down ratio and u is the up ratio

1. h: time to expire
2. r: cont com rf rate
3. δ : cont com div rate
4. $S(0)$: initial stock price

5. $S(h,u) = uS(0)$: stock price at time h in state u
6. $S(h,d) = dS(0)$: stock price at time h in state d
7. $X(h,u)$: option payoff at time h in the state u
8. $X(h,d)$: option payoff at time h in the state d

Replicating Portfolio

Let B be the amount of wealth invested in bond at time 0 and Δ be the number of stock shares purchased at time 0. Then (B,Δ) is called a replicating portfolio if

$$\begin{cases} e^{rh}B + \Delta e^{\delta h}S(h,u) = X(h,u) \\ e^{rh}B + \Delta e^{\delta h}S(h,d) = X(h,d) \end{cases} \implies \begin{cases} B = e^{-rh}\frac{uX(u,d)-dX(h,u)}{u-d} \\ \Delta = e^{-rh}\frac{X(u,d)-X(h,u)}{uS(0)-dS(0)} \end{cases}$$

By the law of one price, we have

$$X(0) = B + \Delta S(0) = e^{-rh}\left(\frac{e^{(r-\delta)h}-d}{u-d}X(h,u) + \frac{u-e^{(r-\delta)h}}{u-d}X(h,d)\right)$$

Definition 1.5.2 — Market Completeness. A binomial stock market model is called complete if any European type of option has a replicating portfolio (B,Δ).

R A binomial stock market model is complete if and only if $d < u$.

Definition 1.5.3 — Arbitrage Portfolio. A portfolio (B,Δ) is called an arbitrage portfolio if it satisfies all of the following three conditions:

1. $B + \Delta S(0) = 0$
2. $e^{rh}B + \Delta e^{\delta h}S(h,w) \geq 0$ for all $w \in \{u,d\}$
3. $e^{rh}B + \Delta e^{\delta h}S(h,w) > 0$ for some $w \in \{u,d\}$

A binomial stock market model is called arbitrage-free if there is no arbitrage portfolio.

Theorem 3 A binomial market model is arbitrage-free if and only if

$$d < e^{(r-\delta)h} < u$$

Definition 1.5.4 — Risk-Neutral Measure. The set (q_u, q_d) is called a risk-neutral measure if it satisfies the following three conditions:

1. $q_u e^{\delta h}S(h,u) + q_d e^{\delta h}S(h,d) = e^{rh}S(0)$
2. $q_u + q_d = 1$
3. $q_u, q_d > 0$

or we have

$$\begin{cases} q_u = \frac{e^{(r-\delta)h}-d}{u-d} \\ q_d = \frac{u-e^{(r-\delta)h}}{u-d} \end{cases}$$

R Risk-neutral measure exists if and only if

$$d < e^{(r-\delta)h} < u$$

or equivalently, the binomial market model is arbitrage-free

Theorem 4 — Risk-Neutral Pricing Formula of European Call/Put Option.

$$X(0) = e^{-rh}\mathbb{E}^{\mathbb{Q}}[X(h)]$$

where $\mathbb{Q}$ is the risk-neutral measure on the sample space $\{u,d\}$ such that $\mathbb{Q}\{u\} = q_u$ and $\mathbb{Q}\{d\} = q_d$.

R

1. Under risk-neutral measure, the expected return rate of any asset (stock, option, or bond) equals to the risk-free rate
2. Risk-neutral measure is an artificial measure for the convenience of option pricing.

1.5.2 Multi-period Model

We have trading times at $\{0,h,2h,\cdots,Nh\}$ with $Nh = T$ is the maturity and h is the time step. Then, the stock price at time nh in state $u^m d^{n-m}$ is given by

$$S(nh, u^m d^{n-m}) = u^m d^{n-m} S(0), 0 \le m \le n \le N$$

Theorem 5 — Risk-Neutral Pricing Formula for European Call/Put.

$$X(0) = e^{-rT}\sum_{m=0}^{N}\binom{N}{m} q_u^m q_d^{N-m} X(T, u^m d^{N-m})$$

where

$$\mathbb{Q}\{u^m d^{N-m}\} = \binom{N}{m} q_u^m q_d^{N-m}$$

1.5.3 Determination of u and d

Forward Tree

$$\begin{cases} u = e^{(r-\delta)h+\sigma\sqrt{h}} \\ d = e^{(r-\delta)h-\sigma\sqrt{h}} \end{cases}$$

where $\sigma > 0$ is called the annualized volatility of the stock

Cox-Ross-Rubinstain tree

$$\begin{cases} u = e^{\sigma\sqrt{h}} \\ d = e^{-\sigma\sqrt{h}} \end{cases}$$

Jarrow-Rudd Tree

$$\begin{cases} u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}} \\ d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}} \end{cases}$$

R Different results for finite N, but all three approaches to the same price as $N \to \infty$ $(h \to 0^+)$

1.5.4 Binomial Pricing for American Call/Put

Since American options allow early exercises, we have

1. American call:

$$X(nh,w) = \max \begin{cases} S(nh,w) - K & \text{early exercise} \\ e^{-rh}[q_u X((n+1)h, wu) + q_d X((n+1)h, wd)] & \text{continue} \end{cases}$$

2. American put:

$$X(nh,w) = \max \begin{cases} K - S(nh,w) & \text{early exercise} \\ e^{-rh}[q_u X((n+1)h, wu) + q_d X((n+1)h, wd)] & \text{continue} \end{cases}$$

1.5.5 Exotic Options

Payoffs of exotic options may depend on the path of stock price (path-dependent) not just one spot price at expiration.

Asian Options

Let

$$A(T) = \frac{1}{N}\sum_{n=1}^{N} S(nh), \qquad G(T) = \left(\prod_{n=1}^{N} S(nh)\right)^{1/N}$$

could be fixed strike and floating strike with $A(T), G(T)$

Lookback Options

1. Payoff of fixed strike call

$$\left(\max_{0\le t\le T} S(t) - K\right)_+$$

2. Payoff of fixed strike put

$$\left(K - \min_{0\le t\le T} S(t)\right)_+$$

3. Payoff of floating strike call

$$S(T) - \min_{0\le t\le T} S(t)$$

4. Payoff of floating strike put

$$\max_{0\le t\le T} S(t) - S(T)$$

All-or-nothing options

1. **Payoff of up-and-in call**

$$(S(T)-K)_+ \mathbf{1}_{\{\max_{0\le t\le T} S(t) > L\}}$$

2. **Payoff of up-and-out call**

$$(S(T)-K)_+ \mathbf{1}_{\{\max_{0\le t\le T} S(t) < L\}}$$

3. **Payoff of down-and-in call**

$$(S(T)-K)_+ \mathbf{1}_{\{\min_{0\le t\le T} S(t) < L\}}$$

4. **Payoff of down-and-out call**

$$(S(T)-K)_+ \mathbf{1}_{\{\min_{0\le t\le T} S(t) > L\}}$$

R

1. Up-and-in call plus up-and-out call is just a regular call
2. Down-and-in call plus down-and-out call is just a regular call

Gap Options

Gap options are options with discontinuous payoffs. Can be decomposed as call option + all-or-nothing option.

Compound options

$0 < T_1 < T_2$

1. Payoff of CoC at time T_1:
$$[C(S_{T_1}, K, T_2 - T_1) - x]_x$$
2. Payoff of PoC at time T_1:
$$[x - C(S_{T_1}, K, T_2 - T_1)]_+$$

Theorem 6 — Put-call Parity for Compound Option.

$$\begin{aligned} CoC(S_0, K, x, T_1, T_2) - PoC(S_0, K, x, T_1, T_2) &= PV_{0,T_1}[C(S_T, K, T_2 - T_1)] - PV_{0,T_1}[x] \\ &= C(S_0, K, T_2) - xe^{-rT_1} \end{aligned}$$

1.5.6 Binomial Pricing for Exotic Options

Since exotic option payoffs are path-dependent, we need to use uncombined trees.