ACTSC 372 Course Notes

University of Waterloo

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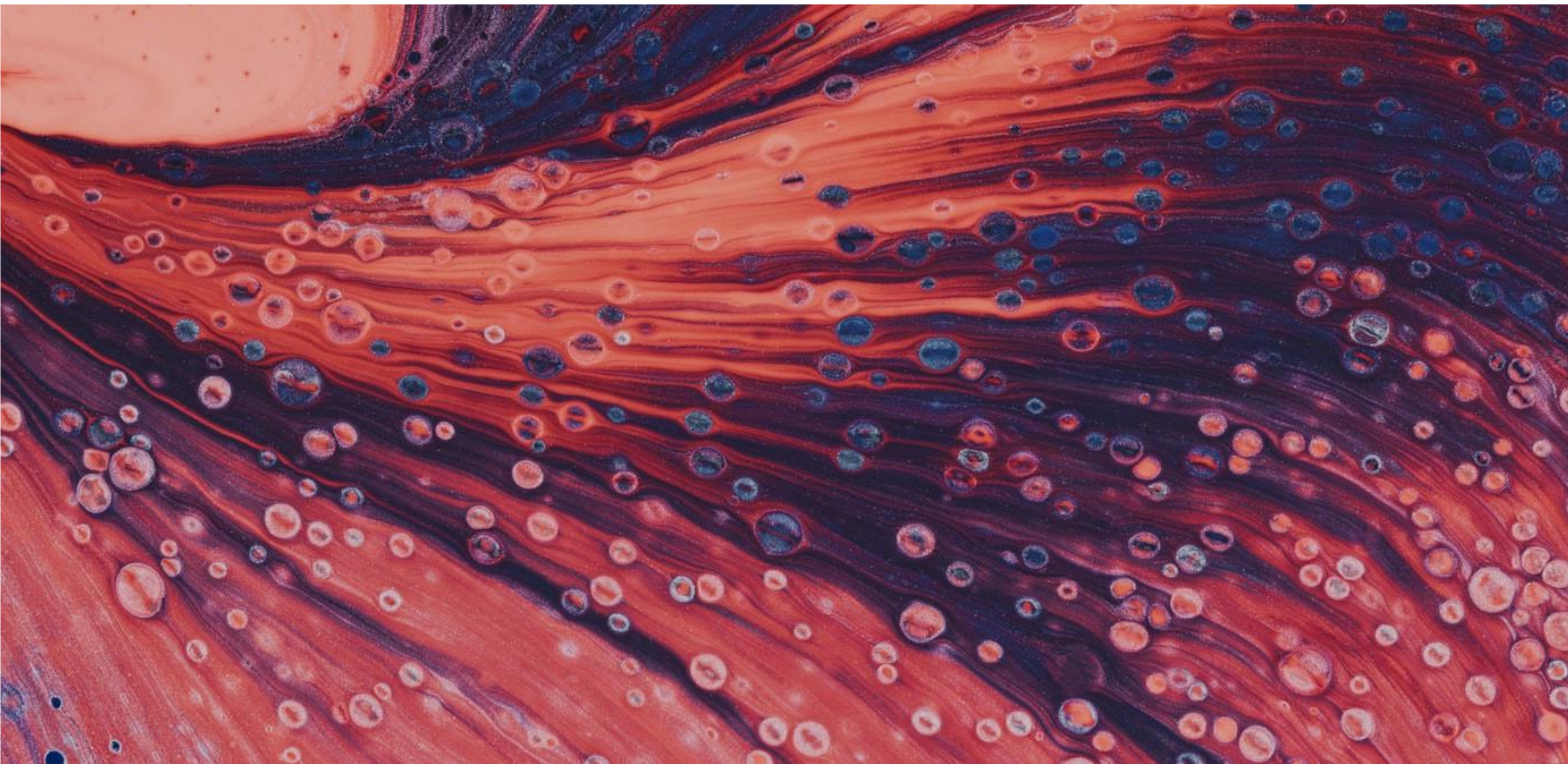

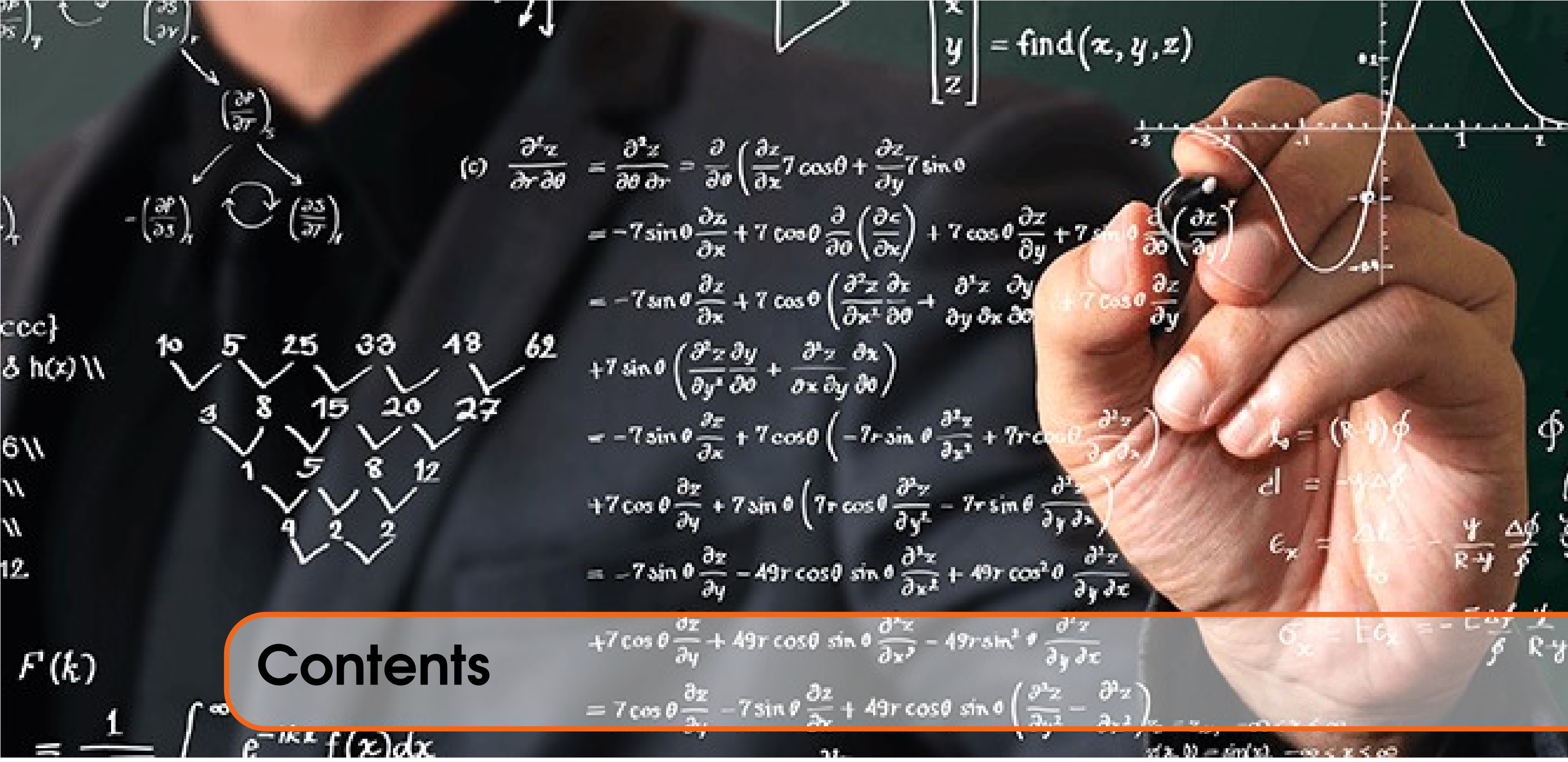

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1. Derivatives Trader: Options

1.1 Finanial vs. Real Investment

Definition 1.1.1 — Real Investment. In economics, we define real investment to be the function

$$Y_t = F(K_t, L_t)$$

where Y_t is GDP at time t and K_t is the amount of capital at time t while L_t is the amount of labour at time t.

Definition 1.1.2 — Financial Investment. Essentially exchange of financial assets, such as stocks and bonds.

1.2 A Two Period Model

Stage One: Without Bank

1. Consider two stages 1 (young) and 2 (old)
2. Say there is an endowment (born with) with (e_1, e_2)
3. Optimal consumption:

$$e_1 = c_1$$

$$e_2 = c_2$$

Stage Two: A Bank Has Entered the Chat

1. Suppose you can borrow at a cost of interest of r.
2. We can have other possibility of consumptions now. We can consume c_1 and used the saving/borrowing to have the following equation

$$\underbrace{(e_1 - c_1)}_{\text{Saving/Borrowing}}(1+r) + e_2 = c_2$$

then, our budget constraint is

$$c_1 + \frac{c_2}{1+r} = e_1 + \frac{e_2}{1+r}$$

we can visualize it as follow we see that comparing to the first stage, we have many more

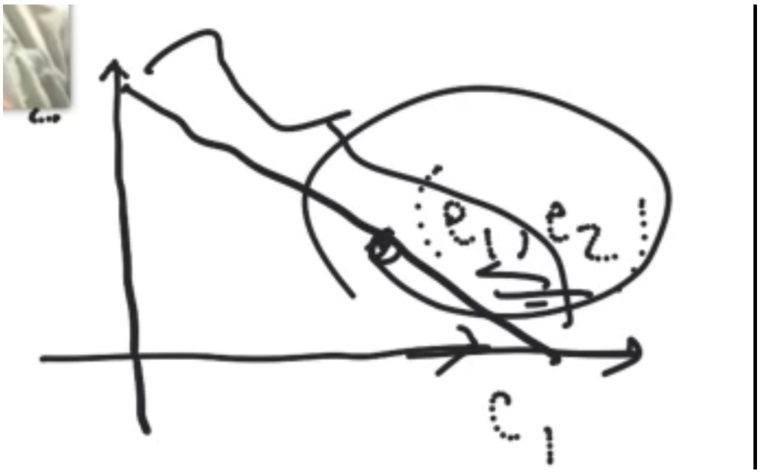


Figure 1.2.1: *Surya Banerjee's Drawing Not Mine*

choices. This implies **higher utility**.

How to Find the Optimal Consumption?

1. This depends on the utility function of the person
2. **Present Value Intuition:** the present value of your income is the present value of your consumption.

1.3 Options and Corporate Finance

1.3.1 Basic Definitions and Terminology

Option is a sub-asset class of the general derivative instruments. Let's talk about what is a derivative first.

Definition 1.3.1 — Derivative (Asset). A derivative is an asset whose value depends on the price of some other asset.

▪ **Example 1.1 — Surya Banerjee's House.** You are the house owner and Surya Banerjee is homeless. You two set up a contract **today** saying

> On **Dec 31, 2020**, Surya Banerjee has the **right** to **purchase** the **house** at the price of **$400,000**.

You understand that this is providing a convenience for Surya, thus, you charged him **$500** just for this contract to take place. ▪

Definition 1.3.2 — Option. An option gives the holder the right, but not the obligation, to buy or sell a given quantity of an asset on (or before) a given date at prices agreed upon.

Definition 1.3.3 — Exercising the Option. The act of buying or selling the underlying asset through the option contract.

Definition 1.3.4 — Strike/Exercise Price. Refers to the fixed price in the option contract at which the holder can buy or sell the underlying asset.

▪ **Example 1.2 — Expiration Date.** The maturity date of the option is referred to as the expiration date, or the expiry. ▪

Definition 1.3.5 — Premium. The price of the option contract.

■ **Example 1.3 — Surya Banerjee's House Revisit.** In our previous example, you can see there are a lot of bolded texts, since they resemble the features of an option.

1. Writer of the option: you
2. Holder of the option: Surya Banerjee
3. Right to buy the house, not obligation
4. Expiration date/Exercising date: Dec 31, 2020
5. Strike Price: $400,000
6. Evaluation date: today
7. Option Premium: $500

■

Definition 1.3.6 — Call Option. Call options give the holder the right, but not the obligation, to buy a given quantity of some asset on or before some time in the future, at prices agreed upon today.

R When you hope/think the price of the underlying asset will go up.

■ **Example 1.4 — Surya Banerjee's House Rerevisit.** Say Dec 31, 2020 comes by, should Surya Banerjee exercise the right to buy the house? **It depends.** If we let S_T be the price on expiration date. Recall that $X = \$400,000$ being the strike price. Suppose the $S_T > X$, Surya is happy to exercise it, but not the other way around. We can visualize the situation as below.

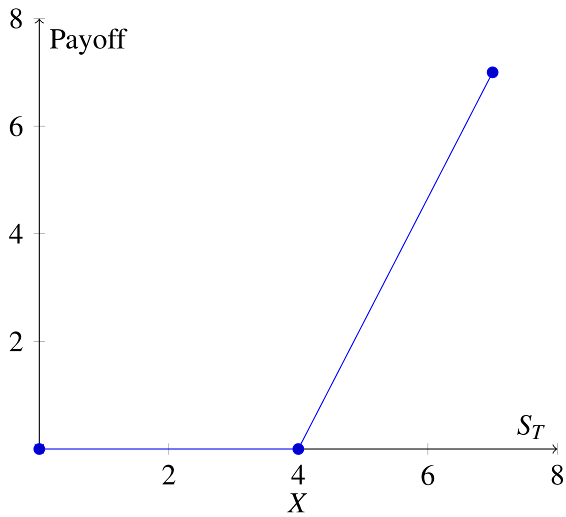


Figure 1.3.1: *Surya's Payoff Chart*

1. **In the money:** $S_T > X$
2. **At the money:** $S_T = X$
3. **Out of the money:** $S_T < X$

■

American vs. European Options

1. **American Options**: can be exercised before expiration date
2. **European Options:** can only be exercised on the expiration date

Definition 1.3.7 — Put Option. Call options give the holder the right, but not the obligation, to sell a given quantity of some asset on or before some time in the future, at prices agreed upon today.

R When you hope/think the price of the underlying asset will go down.

1.3.2 Option Payoffs on Expiry

■ **Example 1.5 — XYZ Company Option.** The company XYZ has the following stock option:

1. Current stock price is $S_0 = \$45$
2. Strike price is $X = \$50$
3. Expiration date is June 30, 2020

What is the payoffs on June 30?

This is simple,

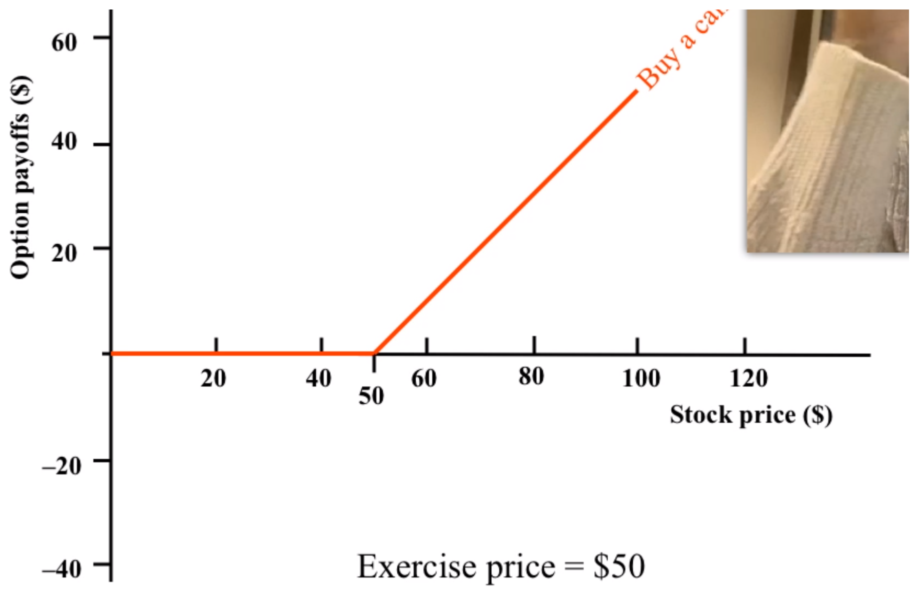


Figure 1.3.2: *Payoff of the Holder of the Option*

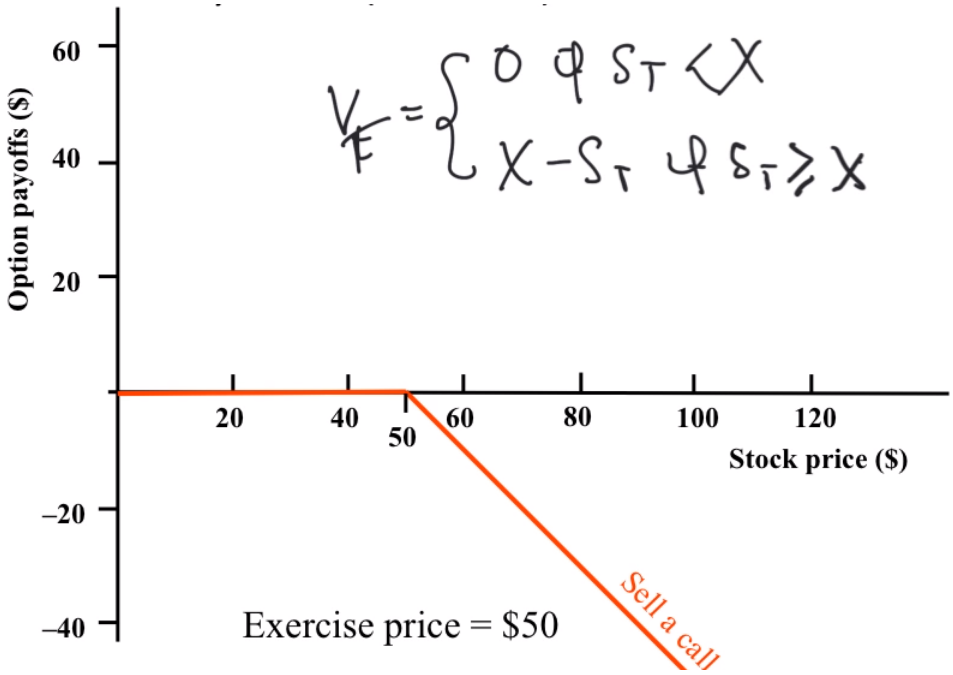


Figure 1.3.3: *Payoff of the Writer of the Option*

It is almost too good to be true, that's why it makes more sense to look at the actual profit chart.

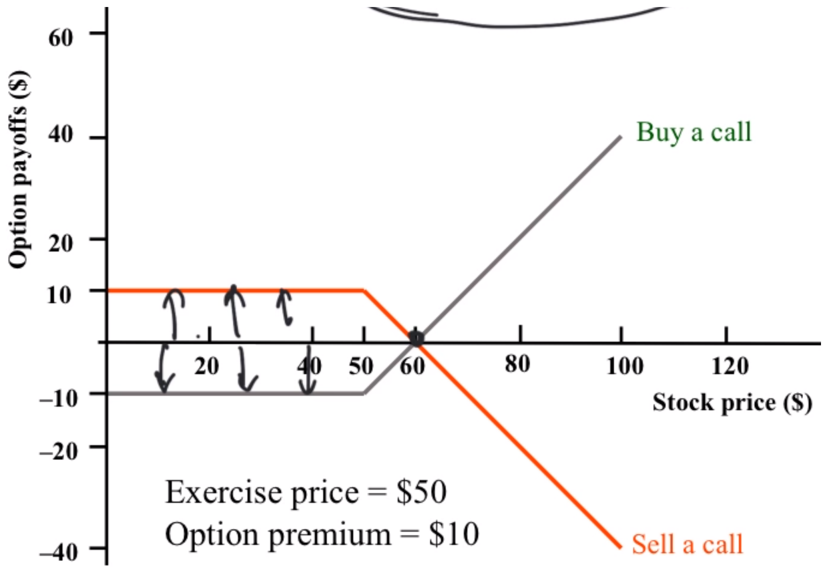


Figure 1.3.4: *Profit Chart*

■

R Remember that we need to consider the time value of money. The premium is paid up front but the actual payoff happens at later date. We will include this later.
The writer and holder of the option contract always have mirror payoff and profit charts.

Another analogous example of put option was given in the lecture. We will ignore it in the note here.

Theorem 1.3.1 — Payoff and Profit of Option Writer/Holder. Let c be the call option premium and p be the put option premium.

	Call	Put
Holder	$\max\{0, S_T - X\}$	$\max\{0, X - S_T\}$
Writer	$\min\{0, X - S_T\}$	$\min\{0, S_T - X\}$

Table 1.3.1: *Payoff Charts*

	Call	Put
Holder	$\max\{0, S_T - X\} - c$	$\max\{0, X - S_T\} - p$
Writer	$\min\{0, X - S_T\} + c$	$\min\{0, S_T - X\} + p$

Table 1.3.2: *Profit Charts*

R The above results can be easily derived using the following min/max function identities

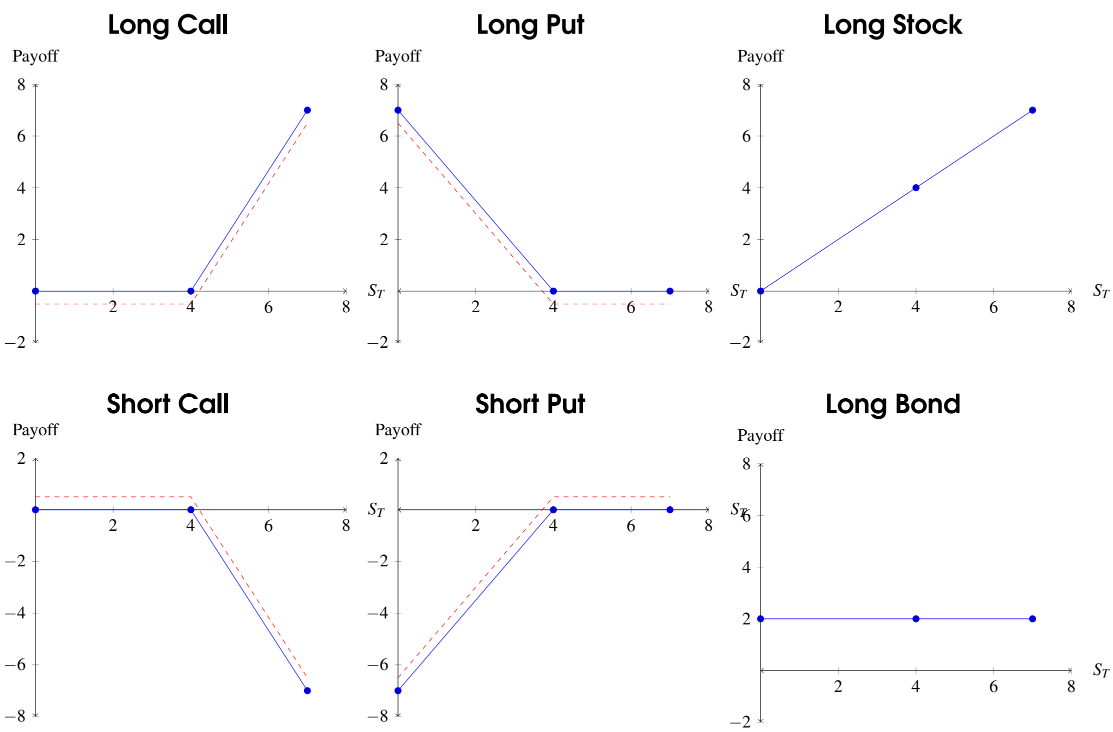
$$\min(a,b) = -\max(-a,-b)$$

and

$$\min(a+c, b+c) = \min(a,b) + c, \qquad \max(a+c, b+c) = \max(a,b) + c$$

1.4 Option Strategies

So far, we have encountered quite a bit of payoff/profit structures of different financial instruments. They have been summarized in the charts below:



R Options can be combined in various ways to create an unlimited number of payoff profiles.

1.4.1 Protective Put

Definition 1.4.1 — Protective Put. Long a stock and a put option.

Example 1.6 — Banerjee's Grandma. Banerjee's grandma saved up some money over the years and tried to invest it in the stock market, in particularly, a stock that she liked. The current price is \$50. She speculated the price will rise but she did not want to bear the market risk by losing it all. Banerjee recommended his grandma **protective put**, where

1. Puts can be used as insurance against stock price declines
2. Protective puts lock in a minimum portfolio value
3. The cost of the insurance is the put premium.

Banerjee's grandma seemed lost, "What the hell is he talking about? One more time!"

Mathematical Formulation

1. $X = 50$
2. Put premium = p
3. Stock price at time t is S_t

we will look at each component of the investment bundle under different conditions to answer the two questions:

1. What is the value of the strategy on expiration?
2. What are the values of the stock price where you will positive profits?

	$S_T < 50$	$S_T \geq 50$
Long Stock	S_T	S_T
Long Put	$X - S_T$	0
Total Payoff	$X = 50$	S_T

Table 1.4.1: *Payoff Chart*

The payoff/profit chart is illustrated below:

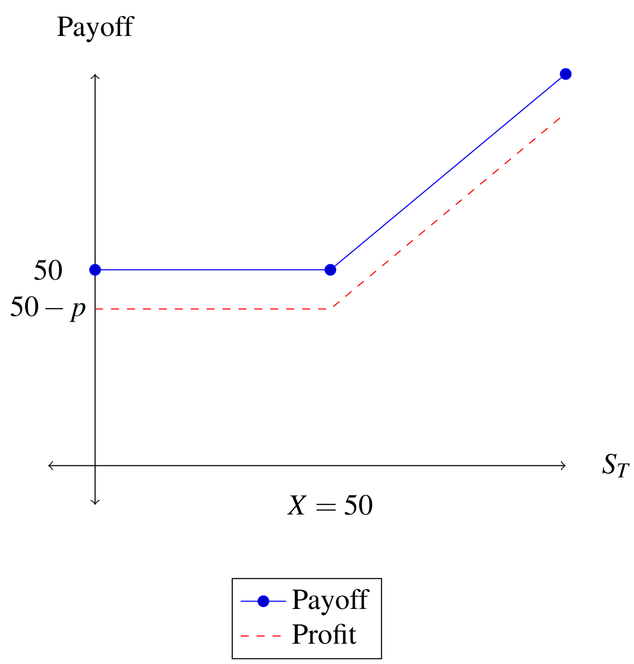


Figure 1.4.1: *Payoff/Profit Graph*

Geometric Interpretation:
we can explicitly look at the payoff chart of each instruments.

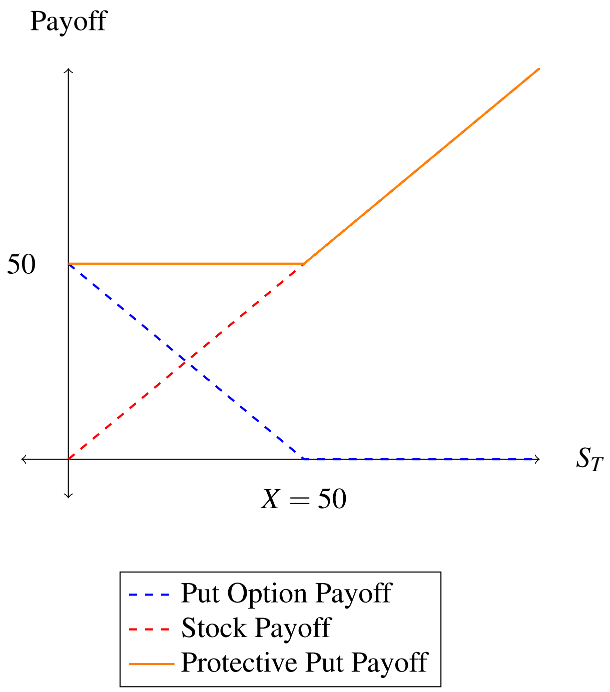
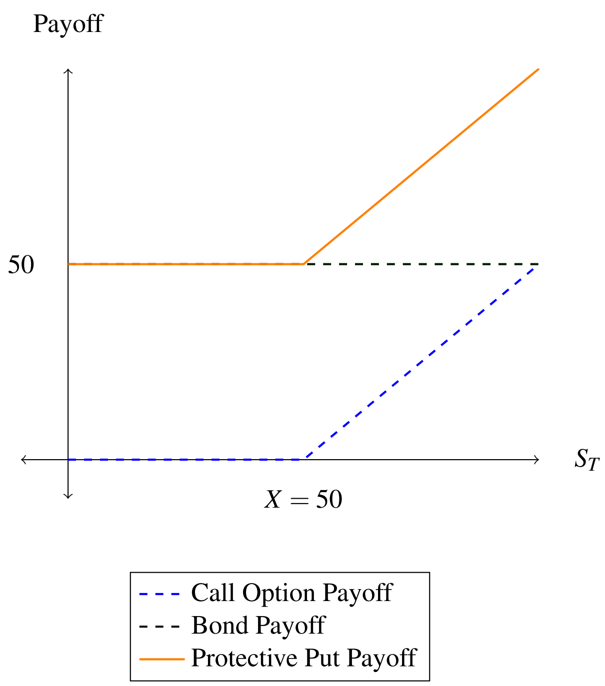


Figure 1.4.2: *Combination Graph Graph*

■

R One might find that the payoff graph of a protective put is really similar to a call option payoff chart being shifted up by p. This is no coincidence. One can also think of protective put as a bundle of long a call and long a bond. This is will motivate the **Put-Call Parity Theorem**.



1.4.2 Bull Spread

Definition 1.4.2 — Spread. A spread is a combination of two or more calls (or puts) on the same stock with
1. Differing exercise prices (bull/bear spread), or
2. Times to maturity (calendar spread)

Some options are long and some options are shorted/written.

Definition 1.4.3 — Bull Spread. A bull spread is a spread that profits when the stock price increases. It is constructed by options with different strike prices.

■ **Example 1.7** Suppose $S_0 = 40$, you have a rough idea about the valuation of the stock, and comes up with a thesis that the right price is between 45 and 50, but overall, feeling bullish on this stock. You set up a **bull spread** by
1. Long a call at $X_1 = 45$
2. Short a call at $X_2 50$

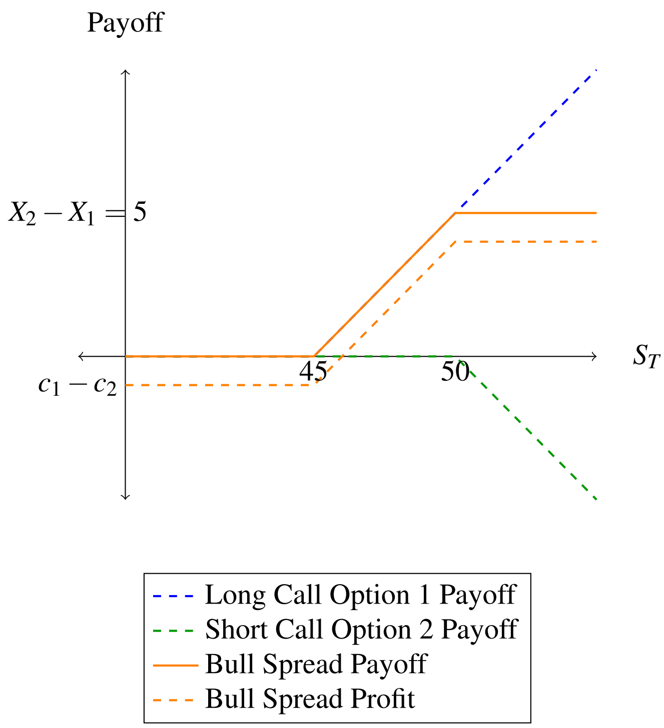
Let's consider its payoff:

	$S_T < 45$	$45 \geq S_T < 50$	$S_T > 50$
Long Call ($X_1 = 45$)	0	$S_T - 45$	$S_T - 45$
Long Call ($X_2 = 50$)	0	0	$50 - S_T$
Total Payoff	0	$S_T - 45 = S_T - X_1$	$5 = X_2 - X_1$

Table 1.4.2: *Payoff Chart*

It has the following payoff/profit graph combination: first note that the cost of the bull spread is

$$c_1 - c_2$$



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R Now, we can see graphically that as the stock price increases, our bull spread tends to make profits.

Why Do We Want to Use a Bull Spread?

Note that a call option has similar properties as a bull spread with unlimited upside. But bull spread is cheaper in terms of premium. With lower potential profit, it also comes with lower cost and more predictable return range.

Exercise 1.1
1. Try to draw the payoff/profit chart for the writer of the bull spread
2. Try to construct a bull spread using puts
3. Try to construct a bear spread.

1.4.3 Straddle

Definition 1.4.4 — Straddle. Buy call and put with the same strike price and maturity.

R

1. The straddle is a bet on **volatility**, it could be high or low, either way, we make money using straddle
   (a) To make a profit, the change in stock price must exceed the cost both options
   (b) You need a strong change in stock price in either direction, otherwise, still can incur a loss.
2. The writer of the straddle is betting the stock price will not change much.

Example 1.8 — XYZ Company Upheaval. The current stock price of XYZ company is at $S_0 = 50$, and there is an upheaval in the management since the current CEO posted on twitter the following quote.

"XYZ stock price is too high imo"

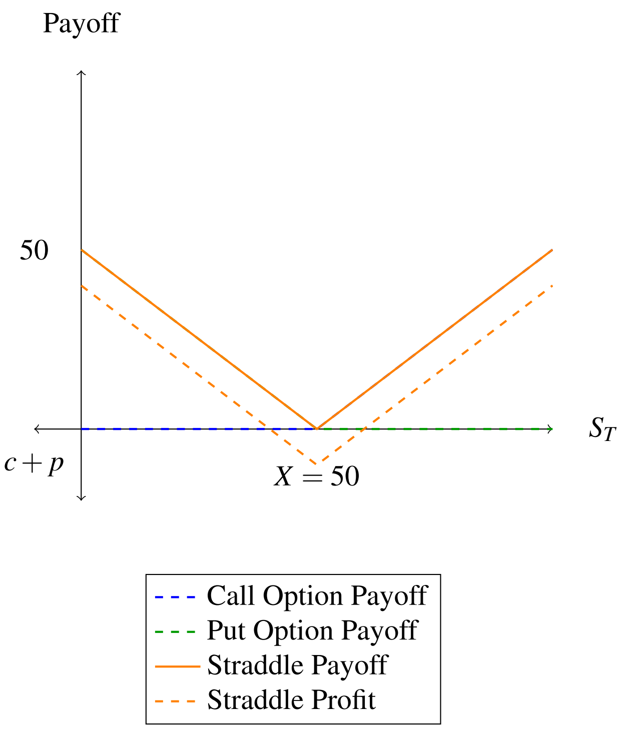
The board of directors decided to change the CEO to one of the two candidates, one of them is Barren Waffle, and the other one is Ndam Aeumann. Clearly, if Barren Waffle is appointed, the stock price will soar, while Ndam Aeumann's case will result in bankruptcy. How can we exploit this information? **Disclaimer: no insider-trading activities happening**

Mathematical Formulation:
1. $S_0 = 50$
2. You want to exploit the volatility, thus, you engage into a straddle
   (a) Long a call with premium c and long a put with premium p at strike price X
   (b) The cost of straddle is $c + p$
   (c) The detailed payoff/profit chart is left as an exercise to the board of directors. But in general

	$S_T < X$	$S_T \geq X$
Payoff of Call	0	$S_T - X$
Payoff of Put	$X - S_T$	0
Total Payoff	$X - S_T$	$S_T - X$

Table 1.4.3: *Payoff Chart*

Let's look at the payoff/profit combination geometrically.



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1.5 Put-Call Parity Theorem

Every mathematical model requires certain premises/assumptions, this one is no exception.

Assumptions:

1. All options involved are of the same strike price X
2. Same expiration date
3. No dividends are given on stocks
4. Bond matures on the expiration date as well

Non-Arbitrage Pricing Principle

This will be required to prove the put-call parity. In an ideal world, financial instruments with the same type of cash flows with corresponding payment dates should cost the same. Otherwise, arbitrage opportunity exists.

Theorem 1.5.1 — Put-Call Parity. The portfolio on the left (long call plus a long bond) must cost the same as the portfolio on the right (long stock plus a long put)

$$c+\frac{X}{(1+r_f)^T}=S_0+p$$

where c is the call premium, r_f is the interest rate, S_0 is today's price of the stock, and p is the put premium.

Proof. We shall prove the equivalent equation

$$c-p=S_0-\frac{X}{(1+r_f)^T}$$

consider two strategies:

1. **Strategy 1: Long a call, write a put**
   (a) Net cost is $c - p$
   (b) On maturity:

	$S_T < 50$	$S_T \geq 50$
Long Call	0	$S_T - X$
Short Put	$S_T - X$	0
Total Payoff	$S_T - X$	$S_T - X$

Table 1.5.1: *Payoff Chart*

2. **Strategy 2: Long a stock and borrow the present value of X**
   (a) Net cost is $S_0 - \frac{X}{(1+r_f)^T}$
   (b) On maturity: the payoff is $S_T - X$

Note that both strategies have the same payoff cash flow on Maturity. Thus, by the Non-Arbitrage Pricinple, the theorem holds. ■

R We can manipulate the Put-Call Parity equation to derive different option strategies:

$$\underbrace{c + \frac{X}{(1+r_f)^T}}_{\text{Remark from last section}} = \underbrace{S_0 + p}_{\text{Protective Put}}$$

alternatively, we can have

$$S_0 = c + \frac{X}{(1+r_f)^T} - p$$

this implies that we do not need to hold a stock directly since we can long a call and short a put while lending money (long a bond).

1.5.1 Arbitrage Exploitation

When the Put-Call Parity or the Non-Arbitrage Pricing Principle is violated, we can "buy low and sell high" to capture the **arbitrage**.

Definition 1.5.1 — Arbitrage. Riskless and immediate profit.

Let's see a concrete example (ACTSC371 example)

■ **Example 1.9**
1. $S_0 = 110$
2. $c = 17$
3. $p = 5$
4. $r_f = 5\%$ let's say annual interest rate for now
5. 1 year maturity
6. $X = 105$

We will check whether the Put-Call Parity holds first. Recall:

$$c + \frac{X}{(1+r_f)^T} = S_0 + p$$

$$LHS = c + \frac{X}{(1+r_f)^T} = 17 + \frac{105}{(1.05)^1} = 117$$

$$RHS = S_0 + p = 110 + 5 = 115$$

Since $LHS > RHS$, an arbitrage opportunity of \$2 exists. We can buy the cheaper one and sell the more expensive side.

1. Long the stock and long a put
2. Write a call and borrow $\frac{X}{(1+r_f)^T}$ amount

We can look at the pay off chart

Position	Immediate Cash Flows	Cash Flows in 1 Year	
		$S_T < 105$	$S_T \geq 105$
Buy Stock	-110	S_T	S_T
Borrow	$+100$	-105	-105
Sell Call	$+17$	0	$-(S_T - 105)$
Buy Put	-5	$105 - S_T$	0
Total Payoff	2	0	0

Table 1.5.2: *Arbitrage Finding*

■

1.5.2 Real Life Options

Corporate Liabilities

The company can be considered a call option.

Mathematical Formulation

Option can be written upon any asset, in this case, let V be the asset and B be the liability. Then,

1. **Shareholder Equity:** $E = \max\{0, V - B\}$
2. **Bondholder Debt:** $D = \min\{V, B\} = \min\{0, B - V\} + V = V - \max\{0, V - B\}$

Interpretation

1. Note that $V = E + D$ which makes sense
2. The V is like stock price and B is the strike price. The equity shareholders are the holder of the call option and feel bullish of the company
3. The bond holders are the writers of the call options

Exercise 1.2 Can you explain the company concept using put options?
Solution:

1. **Shareholder Equity:** $E = V - B + \max\{0, B - V\}$
2. **Bondholder Debt:** $D = \min\{V, B\} = B - \max\{0, B - V\}$

■

Tenure Track Professors

1. Are assistant professors/tenure track options for the university?
2. If you are the Dean, what kind of candidate would you hire?

R The short answer is you should hire the assistant professor with more volatile performance as there is no downside risk when holding this option.

1.6 Options Valuations

1.6.1 Restrictions on American Option Value

For an American call option (No Dividend Payment)

1. Call value cannot be negative $c \geq 0$

2. The option payoff is zero at worst
3. Call value cannot exceed the stock value $c \leq S_0$; otherwise, why not just buy the stock?
4. Value of the Call option is at least $S_0 - X$

   Proof. Suppose otherwise, that $c < S_0 - X$. Then, there is an arbitrage opportunity by exercising the option and sell the stock at S_0. This result in $S_0 - X - c > 0$ amount of arbitrage. ■

5. We can have something finer if the call option is **European**!

   Proposition 1.6.1 — Lower Bound for EUR Call. This is also know as the adjusted intrinsic value

   $$c \geq S_0 - \frac{X}{(1+r_f)^T}$$

   Proof. By Put-Call Parity,

   $$c - p = S_0 - \frac{X}{(1+r_f)^T}, p \geq 0$$

   we have the result immediately. ■

R

Approximation to c

Think that you have a really-in-the-money call option. Say $S_0 = 200$ and $X = 40$ and the expiration is in a month. This implies that $p \to 0$ having no value whatsoever. Then,

$$c \approx S_0 - PV(X)$$

What if Dividends are Allowed?

Immediately, we should have

$$c \geq S_0 - PV(X) - PV(D)$$

The way to justify this is that these dividend payments are not being received by you by the time of exercising. They decrease the value of the call option since you could have had them if you get the stock directly.

Determinants of Call Option Values

Assuming other factors are constant we have

If Increases...	**The Value of c**
Stock price, S	Increases
Exercise price, X	Decreases
Volatility, σ	Increases
Time to expiration, T	Increases
Interest rate, r_f	Increases
Dividend payouts, D	Decreases

Table 1.6.1: *Determinants of Call Option*

R These should be intuitive when you think of which direction is better for the payoff of the option, or the probability of the payoff to get better. In particular, let's talk about volatility

■ **Example 1.10 — An Example of Volatility.** You are given the following stock price outcomes chart for stock A and B. There are two stock call options c_A, c_B with the same at the money strike price \$50.

Prob. Price	0.2	0.2	0.2	0.2	0.2
A	40	45	50	55	60
B	30	40	50	60	70

Table 1.6.2: *Stock Price Outcome*

We can have the corresponding payoff for the options.

Prob. Price	0.2	0.2	0.2	0.2	0.2
c_A	0	0	0	5	10
c_B	0	0	0	10	20

Table 1.6.3: *Option Payoff Outcome*

Clearly, given that the downside risks are of the same for both c_A, c_B, the more volatile the underlying stock price is, the better the payoff of the option will be. ■

Determinants of Put Option Values

Assuming other factors are constant we have

If Increases...	**The Value of p**
Stock price, S	Decreases
Exercise price, X	Increases
Volatility, σ	Increases
Time to expiration, T	Increases
Interest rate, r_f	Decreases
Dividend payouts, D	Increases

Table 1.6.4: *Determinants of Call Option*

Exercise 1.3 Find the finest lower bound and upper bound for AMER p.
Solution:

1. **Upper bound:** note that $p \le X$. If $p > X$, what's the point of having the right of selling the stock at price X
2. **Lower bound:** since AMER put can be exercised at any time, we can have at least

$$\max\{0, K - S\}$$

■

Exercise 1.4 Try to show the following results for option price bounds. (Assume no dividends)

1. **European Options:**

$$S_0 \ge c \ge \max\{0, S_0 - PV(X)\}$$

$$PV(X) \ge p \ge \max\{0, PV(X) - S_0\}$$

2. **American Options:** Let S_t be the stock price and $0 \leq t \leq T$

$$S_t \geq c_{\geq} \max\{0, S_t - X\}$$

$$X \geq p \geq \max\{0, X - S_t\}$$

1.6.2 Intermission: Risk Appetite

One can think of investment cash flows as random variables. And it is common that people might be using **expected return** as metric for their investment. In economic sense, let's say the utility function $U(x) = \mathbb{E}(X)$ where X is the random investment return.

R This is clearly not general enough for a lot of cases. For example,

$$X = \begin{cases} 10 & \mathbf{P}(X = 10) = 0.5 \\ 2 & \mathbf{P}(X = 2) = 0.5 \end{cases} \Longrightarrow \mathbb{E}(X) = 6$$

while

$$Y = 6 \Longrightarrow \mathbb{E}(X) = 6$$

which one do you prefer more? **It depends on the investor's risk appetite.** This $U(X)$ will not give us a clear answer.

Another problem of this theory is that not all random variable has expectation just from a mathematical stand point.

■ Example 1.11 — St. Petersburg's Paradox. There is a casino in a back alley in St.Petersburg run by the Banerjee Gang. The casino has been prosperous mainly due to the following scheme.

Prizes	2	4	8	$\cdots$	2^k	$\cdots$
Probability	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\cdots$	$\frac{1}{2^k}$	$\cdots$

Table 1.6.5: *outcome and Probabilities*

The expected value of this process is

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k} = \infty$$

The Banerjee Gang figured out the alchemy in statistics. (Did they? Hopefully not.) ■

Risk Aversion and Risk Neutrality

Back to our utility model, we clearly did not cover every factor in the market and we can never achieve. However, one particular factor in the market, **the volatility**, is usually considered a rational choice for the measure of risk.

1. **Risk Adverse:** investors who think the amount of volatility of their investments should provide them risk premium. Just like stock return should be higher than the bond return
2. **Risk Neutral:** the amount of volatility will not affect the return (expected return) of any investment. This will result in the same amount of return for all asset classes
3. **Risk Loving:** investors who think the amount of volatility of their investments should provide them penalty on return.

R One thing to keep in mind is that we are probably in a risk adverse world but it does not mean risk is totally avoided or not preferred. What we are saying is that given the choice between the risky outcome and the average of the risky outcome, we will always choose the average.

1.7 Binomial Pricing Model

1.7.1 One-Period Binomial Model

Consider an one-period call option on Stock XYZ. The current stock price is $S_0 = \$50$, the strike price is $X = \$50$. The option expires at time 1, then the payoff is

$$c_1 = \max\{S_1 - X, 0\}$$

the question is: what is today's option price c_0?

Say there are two possible value of the stock at time 1 (shown in the graph below).

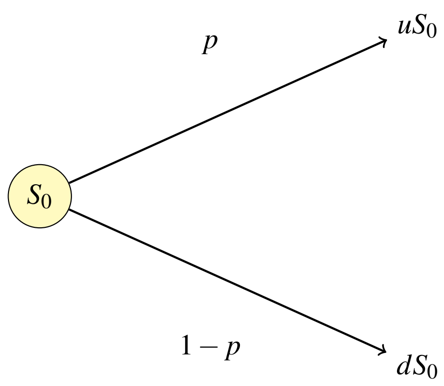


Figure 1.7.1: *Stock Price Outcome at Time* 1

where u is an upward factor and d is a downward factor, clearly $u > d$. The probability to go up is p and $1-p$ for going down. As for the corresponding option payoff, we have the following.

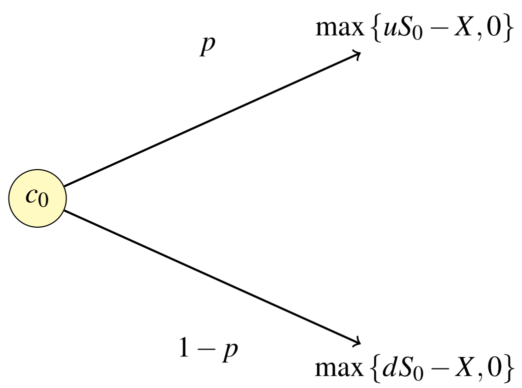


Figure 1.7.2: *Option Payoff at Time* 1

Alternative Strategy: Long Δ Stock and B Bonds Today

1. The net cost today is $V_0 = S_0\Delta + B$
2. The payoff V_1 based on the Binomial model assumption is

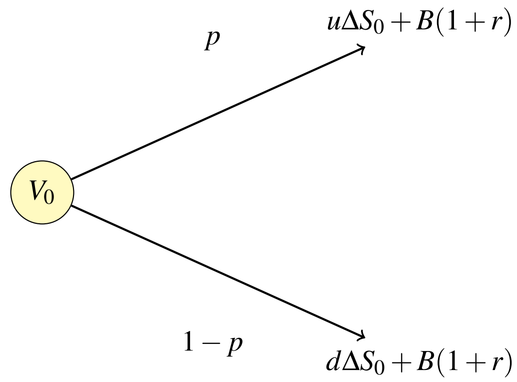


Figure 1.7.3: V_1

Since it is an alternative strategy, we should have

$$\begin{cases} \max\{uS_0 - X, 0\} = u\Delta S_0 + B(1+r) \\ \max\{dS_0 - X, 0\} = d\Delta S_0 + B(1+r) \end{cases}$$

and we can solve for Δ, B. Then, output $c_0 = \Delta S_0 + B$.

■ **Example 1.12** You are given the following information for one-stage Binomial model on a call option.

1. $S_0 = 50$
2. $u = 1.2, d = 0.8$
3. $p = 0.5$
4. $r = 10\%$

What is the price of the call option, c?

Solution:

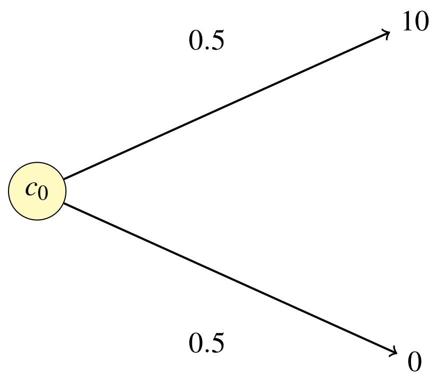


Figure 1.7.4: *Option Payoff at Time* 1

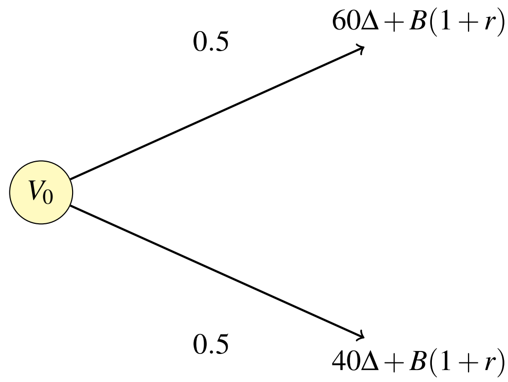


Figure 1.7.5: V_1

Then, we have

$$\begin{cases} 60\Delta + B(1+r) = 10 \\ 40\Delta + B(1+r) = 0 \end{cases}$$

we can solve for $\Delta = 0.5$ and $B = -18.18$. Then,

$$c = 0.5 \times 50 - 18.18 = 6.82$$

■

Exercise 1.5 Banerjee found out that the call option is actually priced at $c = \$5$ in the market right now. How can he capture this arbitrage opportunity?

Solution: He can purchase the call option at \$5 from the market, short 0.5 shares of the stock, and lend 18.18. ■

R

Delta Hedging Strategy

$$\Delta = \frac{c_u - c_d}{(u-d)S_0}$$

1. Δ is positive for call option
2. Δ is negative for put option

Theorem 1.7.1 — One-Stage Binomial Model-Rox, Ross, Rubinstein.

$$c_0 = V_0 = S_0\Delta^* + B^* = \frac{1}{R}\left[\frac{R-d}{u-d}c_u + \frac{u-R}{u-d}c_d\right]$$

where

1. $R = 1 + r$
2. $\Delta^* = \frac{c_u - c_d}{(u-d)S_0}$ is the relative volatility
3. $B^* = \frac{uc_d - dc_u}{(u-d)R}$

R Note that the pricing formula does not depend on the probability p. This means even though we might have different opinions or perspectives on the future of the stock, the call option should be valued the same.

Also, if we let $w_1 = \frac{R-d}{u-d}$ and $w_2 = \frac{u-R}{u-d}$, we see that $w_1 + w_2 = 1$. This is interesting. We can interpret the result as:

> the price of the call option is the present value of the “expected” value of the option payoffs.

1.7.2 Risk Neutral Probabilities

Imagine that we are in a risk neutral universe, we only cares about

1. Expected values of the investments
2. The consequence is that all assets give the **same expected return**.

Then, we reverse-engineer the problem to be the following.

■ **Example 1.13** You are given that $r = 10\%$ and the following one-stage Binomial model.

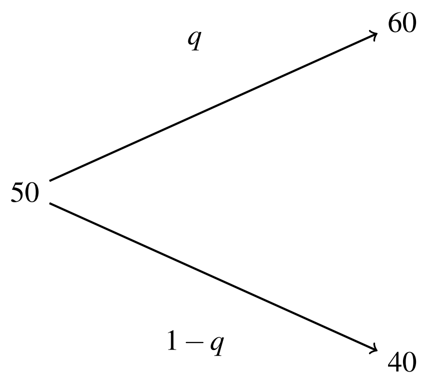


Figure 1.7.6: *Stock Price Outcome at Time* 1

The question is what value of q can result in the risk-free rate of return of 10%?
Solution: it would be

$$\frac{60q + 40(1-q) - 50}{50} = 10\% \Longrightarrow q = \frac{3}{4}$$

then, the price of the option is

$$c = \frac{1}{1+0.1}\left[\frac{3}{4} \times 10 + \frac{1}{4} \times 0\right] = 6.82$$

again. ■

R The more celebrated Black Scholes model is the continuous case of the Binomial pricing model.

2. Financial Markets & Net Present Value

2.1 The Financial Market Economy

Definition 2.1.1 — Finance. The process by which **special markets** deal with cash flows over time. These markets are called financial markets.

Financial markets exist so companies/individuals can

1. **Adjust consumption across time periods**, which means
   (a) Borrow and spend now, repay with interest later
   (b) Save and start earning interest now, spend later ("lend" to the bank)
2. For now, we shall assume that financial market only provides risk-free borrowing and lending
   (a) Price of borrowing and lending: r
   (b) At equilibrium, this r should be unique. Otherwise, there exists arbitrage opportunities to buy low and sell high.
3. Financial markets facilitate borrowing and lending between participants.

   Definition 2.1.2 — Financial Intermediaries. Usually banks, that match borrowers and lenders with the essential functions, including taking deposits and making loans.

   These financial intermediaries tend to perform financial intermediations:
   (a) **Size intermediation:** to provide funds of different amounts, such as credit card vs. mortgage.
   (b) **Term intermediation:** to provide funds of different maturities, again, credit card vs. mortgage.
   (c) **Risk intermediation:** based on depositors' risk appetite, they can choose either saving accounts of different yields.

2.2 Net Present Value

This is really a concept derived from the time value of money.

> Finance is the process by which special markets deal with **cash flows over time**

Definition 2.2.1 — Cash Flow. In general a cash flow should have the following features:
1. **Amount:** how much cash is transacted?
2. **Timing:** when does the transaction happen?
3. **Direction:** who is the payer and who is the receiver?
4. **Likelihood:** is the transaction surely to happen or somewhat uncertain?

For this class, we have the following definitions.

Definition 2.2.2 — Project/Proposal. A project/proposal is a combination of cash flows.

Definition 2.2.3 — Company/Firm. A company/firm is a combination of projects.

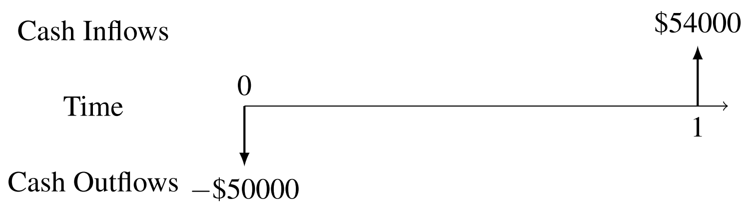
2.2.1 The Basic Financial Principle: Should I Invest?

The financial market provides the key test fro investment decisions.

> A project is worth taking if it is at least as desirable as the opportunity available in the financial market.

Individuals should never by made worse of by increasing the range of choices open to them, we are not taking in the decision-making cost here. For now, the risk-free borrowing/lending are the only market opportunities, so a project is "good" if it is better than just plainly borrowing/lending. How do we compare? We can use the Net Present Value (NPV)

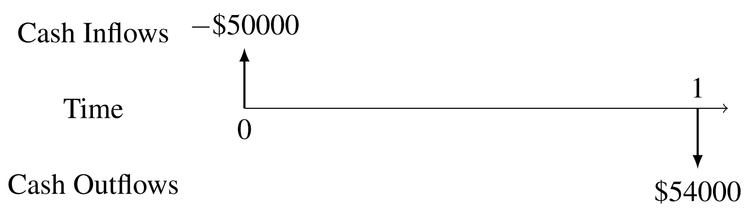
■ **Example 2.1 — Investment.** Consider a project investment that costs \$50K now and pays \$54K in one year, should we do it?



If you save \$50000 in a bank with interest rate r, how much can you withdraw in one year? Well, this is simple, $50000(1+r)$.
If the project investment has an 8% rate of return, is it good? It depends on the interest rate of in the financial market.
1. If $r < 8\%$, it is worthwhile.
2. If $r > 8\%$, don't do it.

■

■ **Example 2.2 — Financing.** Consider a project finance that pays \$50K now and costs \$54K in one year, should we do it?



If you borrow \$50000 in a bank with interest rate r, how much do you repay in one year? Well, this is simple, $50000(1+r)$.

If the project finance has an 8% rate of return, is it good? It depends on the interest rate of in the financial market.

1. If $r < 8\%$, finance from the market since the borrowing is cheaper
2. If $r > 8\%$, finance from project finance.

■

2.2.2 NPV: A Basic Fact in Finance

Theorem 2.2.1 — Present Value Pricing. The value of an asset is the sum of the present values of the cash flow generated by the asset.
The Net Present Value (NPV) of a project with cash flows $C_0, C_1, \cdots$.

$$\mathbf{NPV} = C_0 + \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{C_t}{(1+r)^t}$$

where

1. C_0 is the initial cash flow (often negative for investment projects)
2. C_t is a cash flow at time t, could be positive or negative
3. r is the discount rate (risk-free rate for now)

R For the case of investment,

1. if r is less than the project return, then **NPV** is positive.
2. if r is more than the project return, then **NPV** is negative.
3. The NPV of any financial opportunity in the financial market should be 0, otherwise, arbitrage opportunity arises and defies the efficient market hypothesis.

Proposition 2.2.2 — NPV Rule. A project should be accepted if $\mathbf{NPV} > 0$.

R

1. Given multiple projects, we should rank them in the order of NPVs
2. The NPV rule applies to corporates and individuals in the same way
3. Negative NPV projects should be rejected since you can just invest in the financial market.
4. All shareholders agree to the NPV rules and will ask their managers to following the rule as well. This is to ensure their return as shareholders. This provides a separation of ownership and management since the shareholders only need to worry about that single number, NPV, and the managers have all the freedom to carry out the projects.

■ **Example 2.3 — Project A.** Should we accept the following project?

1. Initial cost is 1 million
2. Cash flows: 150K at the **end** of year for the next 10 years.
3. Discount rate: 8%

recall from ACTSC231 that

$$a(n,r) = \frac{1-(1+r)^{-n}}{r}$$

thus,

$$\mathbf{NPV} = -1000000 + 150000 \times a(10, 8\%) = 6512.21 > 0$$

and we should accept project A by NPV rule.
What if we have **Project B** to be of same maturity as A but costs and incomes are **twice as much**? And another **Project C** of the same incomes as A but **twice as costly and the maturity is twice as long**? Let's take a look at the graph. ■

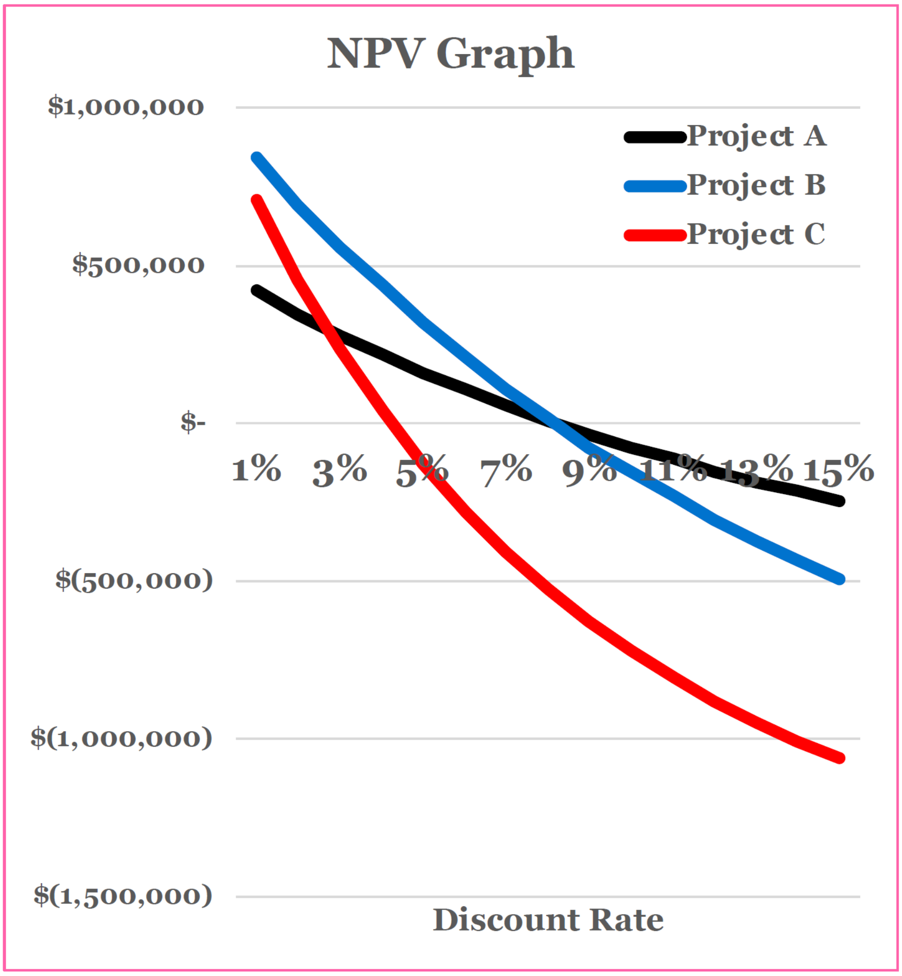


Figure 2.2.1: *NPV Graph*

R As we can see from the graph,

1. Investment NPVs decrease with r (finacing projects are the other way around)
2. Larger cash flow tends to give steeper NPV graph
3. Longer maturity tends to give steeper NPV graph

2.2.3 NPV Rule: Pros and Cons

Advantages

1. **Discount all cash flows properly** by taking the time value of money into consideration
2. Easy to evaluate NPV for combination of projects
3. Applied to any projects, as long as there are cash flows.

Lingering Problems

1. What are the cash flows? How much and when? How the hell do you know? (Capital budgeting)
2. At what rate should we discount the cash flow? That low risk-free rate seems to be too little.
3. Usually the initial costs are the easiest to estimate
4. Do we have alternatives?
   (a) Alternatives that still look at cash flows
   (b) Alternatives that consider the general "welfare".

2.2.4 NPV Rule from Company's Standpoint

We shall go through an example.

■ **Example 2.4 — Alpha Corporation.** The Alpha Corporation is considering investing in a risk-free project.

1. Initial cost 100, return of 107 in one year
2. The market interest rate is 6%

From

$$\mathbf{NPV} = -100 + \frac{107}{1.06} = 0.94 > 0$$

and the NPV rule says accept the project. Let's take a look at what the shareholders might be thinking. The choices for them are

1. Reject the project and get the 100 dollar now to invest at risk-free rate. This gives 106 after 1 year.
2. Accept the project and get 107 after 1 year.

Clearly, the shareholder will choose to accept the project as well. But how do we interpret NPV of 0.94 here? This is actually **the increase of the value of the firm by this project**! ■

R Summary on NPV

1. Accepting positive NPV projects benefits the shareholders
2. Value of the firm rises by the NPV of the project (value additivity)
3. NPV has both qualitative and quantitative information

$$\mathbf{NPV} > 0 \iff \text{Increase company value} \iff \text{Increase shareholder value}$$

4. The NPV rule discounts all cash flows properly
5. Assumes all cash flows are reinvested at the discount rate r

3. Other Investment Rules

What are some of the alternative investment rules besides the NPV rule?

3.1 Payback Period Rule

The **payback period** rule concerns about how long it takes a project to "payback" its initial investment.

■ **Example 3.1** Find the payback period of the following project

1. Initial cost is 1000000
2. Cash flows: 150K at the end of the year for eadch of the next 10 years
3. Discount rate: 8%

Since the total inflow after 6 years is 0.9 mil and is 1.05 mil after 7 years, so

$$6 < PP < 7$$

we can find precise value by linear interpolation but we shall not discuss it in this course. ■

■ **Example 3.2 — Projects Comparison.**

Year	Project A	Project B	Project C
0	-\$100	-\$100	-\$100
1	20	50	50
2	30	30	50
3	50	20	0
4	60	60	-600
PP (yrs)	3	3	2

1. Even though $\mathbf{NPV}_A < \mathbf{NPV}_B$, PP indicates indifferent. This is a problem of PP since it does care the amount of cash flow happens at what time.
2. Project C is clearly a bad project since NPV_C is probably negative but it has the shortest PP

■

Advantages

1. Easy to understand and communicate

2. More likely to approve a highly liquid project

Disadvantages

1. Ignores time value of money
2. Ignores all cash flows after the payback period
3. May result in accepting a project with a negative NPV
4. The decision rule is not clear when dealing with multiple projects

3.1.1 Discounted Payback Period Rule

This is an attempt to fix the PP's problems. Same idea, but we use discounted cash flows.

Advantages

1. ~~Easy to understand and communicate~~
2. More likely to approve a highly liquid project

Disadvantages

1. ~~Ignores time value of money~~
2. Ignores all cash flows after the payback period
3. ~~May result in accepting a project with a negative NPV~~
4. The decision rule is not clear when dealing with multiple projects

R Can fix some but increase some complexity as well.

3.2 Internal Rate of Return (IRR)

Definition 3.2.1 — IRR. IRR is the discount rate that sets the NPV of a project to 0.

$$\mathbf{NPV}_{\mathbf{IRR}} = C_0 + \sum_{t=1}^{\infty} \frac{C_t}{(1+\mathbf{IRR})^t} = 0$$

R To get IRR, usually, need to solve this using computational methods.

3.2.1 The IRR Rule

1. Given one project, accept it if **IRR** >required rate of return, which could be
   (a) Usually the discount rate that we would have used for NPV
   (b) Sometimes a target rate set by the management (hurdle rate)
2. Given multiple projects, rank them in the order of IRR, **the higher the better**.

Advantages

1. Easy to understand and communicate
2. Usually results in the correct decision

Disadvantages

1. Cash flow pattern matters:
   (a) Does not distinguish between investing and borrowing
   (b) IRR may not exist or there may be multiple IRRs
   (c) Problems with mutually exclusive investments. (What do you mean?)

3.2.2 Mutually Exclusive vs. Independent Projects

Definition 3.2.2 — Independent. Projects are called independent if selecting one does not affect our ability to accept others

Definition 3.2.3 — Mutually exclusive. Projects are call mutually exclusive if you can select only one.

R Independent projects need only to be "good enough" by themselves. But mutually exclusive project must be ranked.

Exercise 3.1
1. Can two projects be both mutually exclusive and independent? Not really, if two projects are mutually exclusive then they cannot be independent since one's selection means the other one's demise.
2. Can two projects be neither mutually exclusive nor independent? That's possible. Say you have project A that is selected if project B is selected while project B is selected based on its own quality.

3.2.3 Four Problems with the IRR Rule

1. IRR problems affecting independent and mutually exclusive projects:
   (a) Mutliple IRRs (or no IRR): when there are only cash inflows or outflows, the IRR does not really exist. Now, consider the case when you have

$$C_0 = -100, C_1 = 230, C_2 = -132$$

$$-100 + \frac{230}{1+\mathbf{IRR}} - \frac{132}{(1+\mathbf{IRR})^2} = 0$$

If we look at this from a function perspective, we will see that this function has two roots for the IRR.

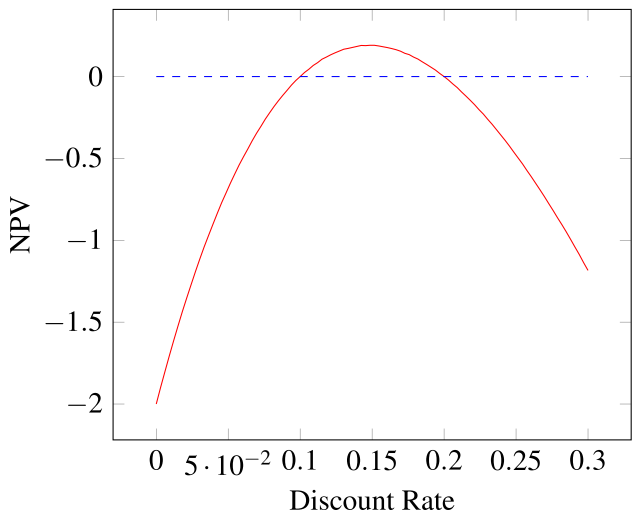


Figure 3.2.1: *NPV Graph*

We see there are two IRRs, 10% and 20%.
   i. Positive NPV for $10\% \le r \le 20\%$

ii. Negative NPV if r is outside this range

With more cashflows, there might be negative or even complex roots, which cannot be interpreted.

R If let $x = \frac{1}{1+r}$, then $\mathbf{NPV} = C_0 + \sum_{t=1}^{\infty} C_t x^t$, which is a polynomial of x.

> **Theorem 3.2.1 — Descartes Rule of Signs.** The number of positive roots of a polynomial is less than or equal to the number of sign changes.

(b) Investing vs. Financing:
   i. Investing, which is lending, then prefer higher r as the return of lending
   ii. Financing, which is borrowing, then prefer lower r as the cost of borrowing

■ **Example 3.3 — Pick the preferred project in each case.**

Investing:

Project A: $C_0 = -100, C_1 = 50, C_2 = 60$

Project B: $C_0 = -100, C_1 = 50, C_2 = 70$

Financing:

Project C: $C_0 = 100, C_1 = -50, C_2 = -60$

Project D: $C_0 = 100, C_1 = -50, C_2 = -70$

Then, by calculation, one might find that $\mathbf{IRR}_A = \mathbf{IRR}_C$ and $\mathbf{IRR}_B = \mathbf{IRR}_D$. However, if we compare projects purely by IRRs, then $\mathbf{IRR}_A < \mathbf{IRR}_B \Longrightarrow$ Prefer B over A and $\mathbf{IRR}_C < \mathbf{IRR}_D \Longrightarrow$ Prefer D over C. This is absurd since having more payout is usually considered bad. ■

R
   i. Pure investing projects: $C_0 < 0, C_t > 0, \forall t > 0$, we should accept the project if the IRR is higher than the required rate of return (hurdle rate)
   ii. Pure financing projects: $C_0 > 0, C_t < 0, \forall t < 0$, we should accept the project if the IRR is lower than the expected cost of financing
   iii. Mixed projects: cash flow at any time can have any sign
   iv. Only mixed projects can have multiple IRRs by Descartes Rule of Signs

2. IRR problems affecting only mutually exclusive projects:
   (a) The Scale Problem: the IRR rules ignores the scale of the investment and its actual return amount
   (b) The Timing Problem: IRR and NPV may give conflicting results due to timing of cash flows.

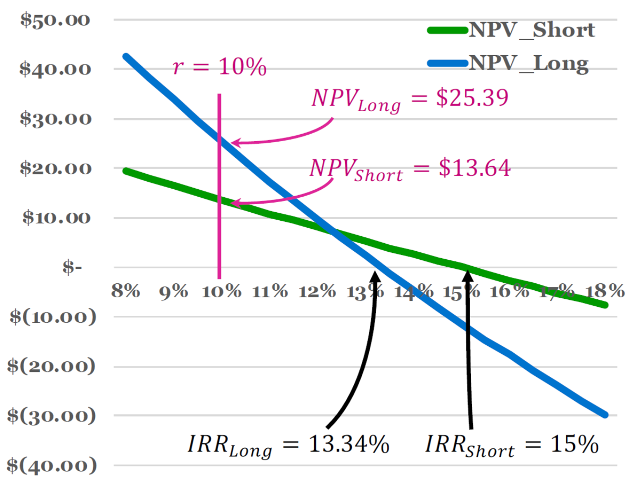


Figure 3.2.2: *NPV & IRR Comparison*

3.2.4 NPV vs. IRR

1. NPV assumes the cash flows can be reinvested at the discount rate
   (a) Usually reasonable
   (b) Kind of saying: you can invest in the market
2. IRR assumes the cash flows can be reinvested at the IRR
   (a) Usually unreasonable
   (b) Kind of saying: you can find another project as good as this one
3. NPV and IRR usually come to the same decision about a project, **except:**
   (a) Odd cash flow paterns (many sign changes)
   (b) Mutually exclusive projects with different scales or timing

3.3 Profitability Index

Definition 3.3.1 — Profitability Index (PI). The profitability index is defined to be

$$\textbf{PI} = \frac{\text{PV of Future Cash Flows}}{\text{Initial Cost}} = \frac{\sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}}{-C_0}$$

3.3.1 The PI Rule

1. Given one project, accept it if $\textbf{PI} > 1$
2. Given multiple projects, rank project in order of PI, the higher the better

Advantages

1. Somewhat easy to communicate
2. Correct decision for independent projects

Disadvantages

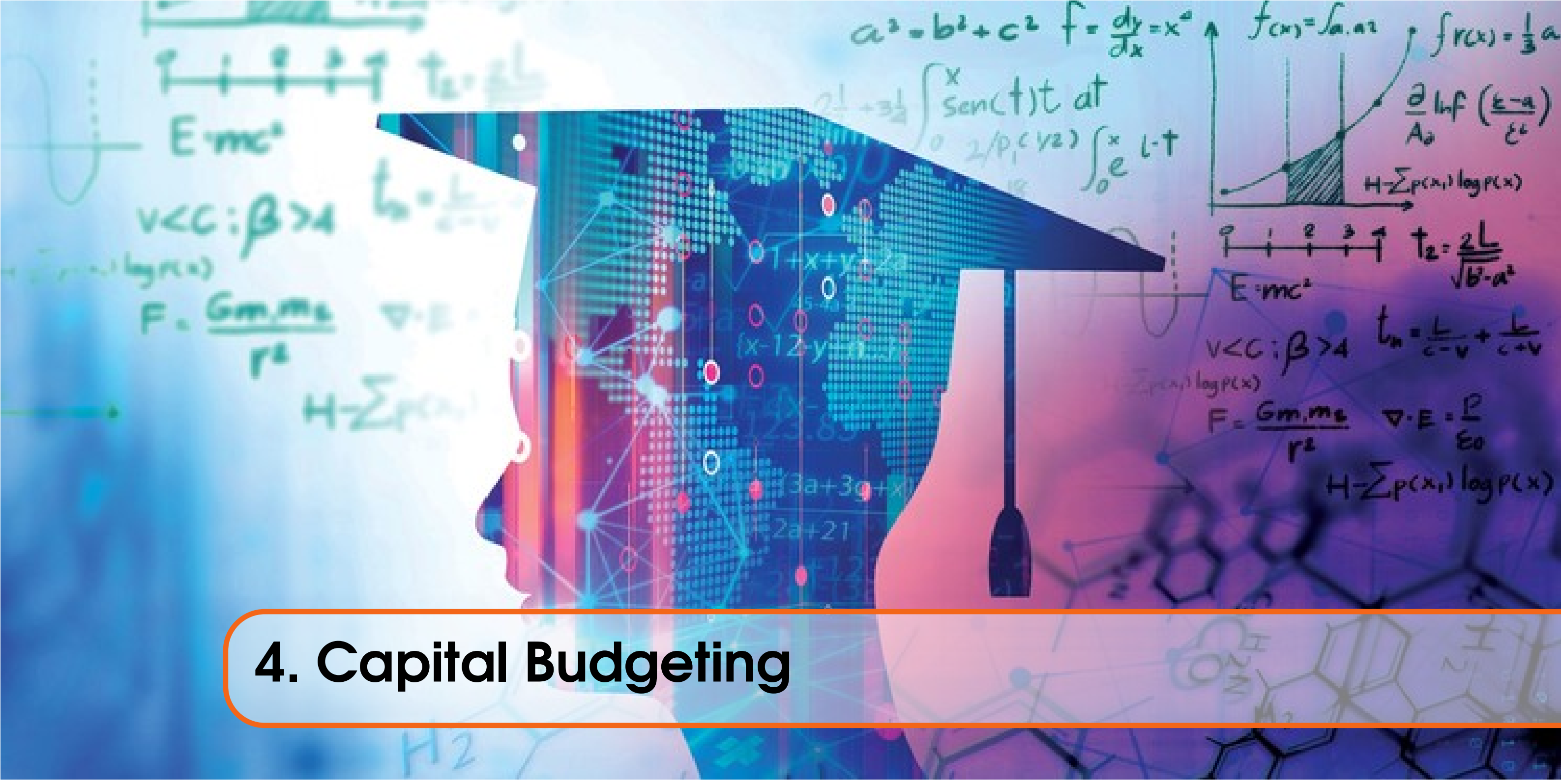
1. Scale problem again since it is taking a ratio and this might happen for mutually exclusive projects as shown in the IRR section

■ **Example 3.4** Suppose $r = 12\%$, calculate the PI and NPV for the following projects.

Projects	C_0	C_1	C_2	$PI@12\%$	$NPV@12\%$
A	−16	60	5	3.59	\$41.6
B	−20	70	10	3.52	\$50.5
C	−5	10	10	3.38	\$11.9

1. Rank the projects by PIs and by NPVs, are the ranks the same?
   (a) By PI: $A > B > C$
   (b) By NPV: $B > A > C$
   not the same.
2. If you can only pick one project, which one would you pick? Due to the scale problem, we probably should choose Project B over Project A by NPV rule.
3. **Capital Rationing:**
   (a) If we only have 20 dollar, then we choose B for sure to get 50.5 NPV
   (b) If we have 21 dollar, then we should choose A and C to get 53.5 NPV in total.

■



4. Capital Budgeting

4.1 Introduction

Definition 4.1.1 — Capital Budgeting. Capital budgeting is the decision-making process for accepting or rejecting new projects or investments.

Definition 4.1.2 — PV01. PV of annuity that pays 1 dollar at the end of next n years at discount rate r is

$$a(n,r) = \frac{1-(1+r)^{-n}}{r}$$

1. **1-year annuity:** $a(1,r) = \frac{1}{1+r}$
2. **Perpetuity:** $a(\infty,r) = \frac{1}{r}$
3. **Increasing perpetuity that pays 1 dollar at time 1, grows by g every year thereafter**

   $$IA(\infty,r,g) = \frac{1}{r-g}, g < r$$

   g could be negative if payments are shrinking.
4. For the sake of the examples later, we have

   $$a(5,10\%) = 3.791, a(8,10\%) = 5.335$$

4.2 Net Incremental Cash Flows

When assessing a project, we consider the **net incremental cash flows**.

> What is the difference between the cash flows of the entire firm with vs. without the project?

Definition 4.2.1 — Cash Flows. Not accounting income, we exclude depreciation, goodwill, uneared capital gains, etc.

Definition 4.2.2 — Incremental Cash Flows. Incremental cash flows only

1. cash flows that change as a **direct consequence** of accepting a project
2. Some costs are tax-deductible: say T_c is the corporate tax rate, then the net incremental cost is $X(1-T_c)$.

R We exclude/include the following from net incremental cash flows:

1. **Exclude Sunk Costs:** a cost that has occured in the past

   ■ **Example 4.1** A consulting fee paid before a construction project starts is considered a sunk cost. ■

2. **Include Opportunity Costs:** CFs that would have happened if the project is not accepted
3. **Include Side Effects:**
   (a) Synergy benefit: Michelin stars
   (b) Erosion costs: new iPhones
4. Taxes and inflation matter

4.2.1 Estimating New Incremental Cash Flows

This is pretty hard in general, need to look at BS, IS, etc. Then translate from earnings to cash flows.

Definition 4.2.3 — Operating Cash Flows (OCF).

$$\mathbf{OCF} = \mathbf{EBIT} - \mathbf{Taxes} + \mathbf{Depreciation}$$

$$\mathbf{EBIT} = \mathbf{Revenue} - \mathbf{Costs} - \mathbf{Depreciation}$$

$$\mathbf{Taxes} = \mathbf{EBIT} \times T_c$$

Then,

$$\mathbf{OCF} = (\mathbf{Rev} - \mathbf{Costs}) \times (1 - T_c) + \mathbf{Depre} \times T_c$$

we do know that profits generated by any project if accepted is subject to taxation. Depreciation is an accounting trick to get tax deduction, but there is no real cash inflows.

Three Approaches to Calculate OCF

1. **Bottom-up:** $\mathbf{OCF} = \mathbf{Net\ Income} + \mathbf{Depreciation}$
2. **Top-down:** $\mathbf{OCF} = \mathbf{Rev} - \mathbf{Costs} - \mathbf{Taxes}$
3. **Tax Shield:** $\mathbf{OCF} = (\mathbf{Rev} - \mathbf{Costs}) \times (1 - T_c) + \mathbf{Depre} \times T_c$

We are also reminded that OCF is on an annual basis, so the cashflows are annual.

Theorem 4.2.1

$$\mathbf{NPV} = C_0 + \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t} = C_0 + PV((\mathbf{Rev} - \mathbf{Costs}) \times (1 - T_c) + \mathbf{Depre} \times T_c)$$

Definition 4.2.4 — After Tax OCF (ATOCF).

$$ATOCF = (\mathbf{Rev} - \mathbf{Costs}) \times (1 - T_c)$$

Definition 4.2.5 — Present Value of Capital Allowance Tax Shield (PVCCATS).

$$PVCCATS = PV(\textbf{Depre} \times T_c)$$

R Thus, we can also write

$$\textbf{NPV} = -\textbf{Initial Costs} + PV(ATOCF) + PVCCATS$$

In this course, all capital costs occur the day the project is accepted. In real life, costs might spread over time or even have salvage value

In Canada, depreciation is referred to as **Capital Cost Allowance (CCA)**

1. CCA provides a tax shield
2. But also decreases EBIT, hence reducing taxes

Theorem 4.2.2

$$PVCCATS = \frac{C \cdot d \cdot T_c}{r+d} \times \frac{1+0.5r}{1+r} - \frac{S \cdot d \cdot T_c}{r+d} \times \frac{1}{(1+r)^n}$$

where

1. C =original price of the assets
2. d =CCA rate that applies to the asset class
3. T_c =corporate tax rate
4. r =discount rate
5. S =salvage value
6. n =the period when assets are sold

4.3 Equivalent Annual Costs (EAC)

There are cases where a direct NPV analysis is too simple

1. Comparing projects with unequal lives
2. Cheap but short-lived machine vs. expensive but high-quality machine

Definition 4.3.1 — Equivalent annual costs (EACs). Spread the NPV to each year in a financially equivalent way

$$\textbf{NPV} = EAC \times a(n,r) \Longrightarrow EAC = \frac{\textbf{NPV}}{a(n,r)}$$

R Usual examples are car loans, mortgage, and equipment financing.

4.4 Indirect NPV Analysis

R Why NPV rules can fail?

1. The NPVs cover different time periods, so cannot be directly compared
2. We can make it work by covering the same time period, say forever

In this case, we assume that operation is required forever. Replace every expired ones with new ones. Say $\textbf{NPV}_5$ is for the 5 year project and $r_5 = (1+r)^5 - 1$ is the 5-year discount rate. Then, we treat this as a perpetuity due! Thus,

$$\mathbf{NPV}_\infty = \mathbf{NPV}_5 \cdot a(\infty, r_5) \cdot (1 + r_5)$$

R In fact,

$$EAC \times a(\infty, r) = \mathbf{NPV}_\infty$$

Thus, the indirect NPV analysis and EAC analysis produce the same conclusion.

4.5 Inflation and Capital Budgeting

For long term projects, the purchasing power of 1 dollar changes by quite a lot. From ECON102, we know that

$$(1 + \textbf{Nominal Rate}) = (1 + \textbf{Real Rate}) \times (1 + \textbf{Inflation Rate})$$

and from ACTSC371, we know that when the rates are of small numbers

$$\textbf{Real Rate} \approx \textbf{Nominal Rate} - \textbf{Inflation Rate}$$

4.5.1 How to deal with inflation?

1. Consistency: discount real cash flows with real rates;discount nominal cash flows at nominal rates
2. Conversion: convert cash flows to match discount rates or convert discount rates to match cash flows, but not both
3. In practice, CCA tax shields are often calculated in nominal terms, recorded every year without adjustment to inflation.
4. It is wrong to compare EAC_{nominal} with EAC_{real}, but we can compare $\mathbf{NPV}_\infty$ (indirect NPV analysis) safely regardless discounted using nominal or real since that's a time-0 values.

5. Capital Budgeting: Risk Analysis

5.1 Capital Budgeting with Uncertainty

What if we have uncertain cash flows? Several practical methods:

1. Decision trees
2. Scenairo analysis
3. Monte Carlo Simulation
4. Sensitivity analysis
5. Breakeven analysis
6. Valuation of real options

Decision Tree Graphical representation of decision and information. We compare expected NPVs of different paths and make decision after.

Scenario Analysis

Suppose equal likely scenarios and compare NPV across all of them

MC Simulation

An extension of scenario analysis where some of the variables in the project are assumed to distributed normally with some mean and variance. We take NPV across all scenarios by doing random sampling and take average at the end.

Sensitivity Analysis

How "sensitive" is NPV to the uncertain variable in the project?
Break-Even Analysis What value of the uncertain variable can make the **NPV** $= 0$
Some AFM102 kids might say, wait a minute, I know something called **contribution margin!** (Weird flex, but okay.)

Definition 5.1.1 — Contribution Margin. Contribution margin is the increase in net income per unit increase in sales (usually after-tax)

$$CM = (\textbf{Unit Rev} - \textbf{UVC}) \times (1 - T_c)$$

Theorem 5.1.1 — CM Approach B/E.

$$V_{B/E} = \frac{\textbf{Fixed Costs} \times (1 - T_c) + [EAC_{initial} - EAC_{Salvage} - EAC_{PVCCATS}]}{(\textbf{Unit Rev} - \textbf{UVC}) \times (1 - T_c)}$$

R B/E can be used on any variable in the NPV formula depends on what the question wants. We have used this idea to derive IRR and payback period before.

Real Options

Rather than options in the first chapter, we focus on managerial options. The value of such options can be computed as the difference between the value of the project with such options, and value of the project ignoring such options.

$$\textbf{NPV}_{\text{with options}} = \textbf{NPV}_{\text{without options}} + \textbf{option value}$$

6. Financial Markets, Returns, and Risks

Type of Investments

1. Cash (or savings accounts, T-bills) item Bonds (corporate vs. government bonds)
2. Stocks (single name, mutual funds)
3. ETFs
4. Options
5. Commodities
6. Real estate

6.1 Risk and Return Trade-off

Definition 6.1.1 — Return. Absolute return can be thought of as dividend plus change in market value.

$$\Delta V_t = Div_{t+1} + (P_{t+1} - P_t)$$

in percentage-wise,

$$\%Return = \frac{\Delta V_t}{P_0} = \text{dividend yield} + \text{capital gains}$$

if we adjust the prices to dividends, we have

$$\%Return = R_t = \frac{P_{t+1} - P_t}{P_t}$$

Definition 6.1.2 — Mean Return. Let R_i be the %-return over the $i-$th period, for $i = 1, 2, \cdots, n$, then the mean return is

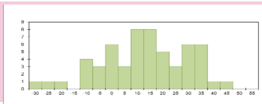
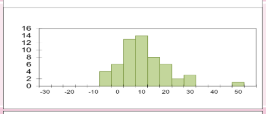
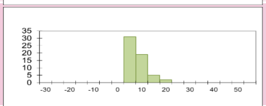
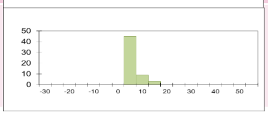
$$\mu = \frac{1}{n}\sum_{i=1}^{n} R_i$$

Definition 6.1.3 — Return Variance/S.D.

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(R_n - \mu)^2}{n-1}$$

where $\hat{\sigma}$ is the standard deviation.

■ Example 6.1 — Returns/Risk of Different Asset Classes.

Investment	Average Annual Return	Standard Deviation	Distribution
Canadian common shares	10.43%	16.64%	
Long Bonds	8.38%	9.81%	
Treasury Bills	5.97%	3.81%	
Inflation	3.85%	3.12%	

■

The stock price is empirically not normally distributed:

1. Heavier tail: extreme returns are likely
2. Asymmetric: stock prices are skewed

Definition 6.1.4 — Holding Period Return (HPR). Geometric average return.

R By the famous AM-GM inequality stating that

$$\frac{\sum_{i=1}^{n}(1+R_i)}{n} \geq \sqrt[n]{\prod_{i=1}^{n}(1+R_i)}$$

this implies that for the same investment returns, the HPR is less than or equal to the mean return.

R Drawbacks of sample variance as a risk measure:

1. Symmetric penalty for unexpected losses and profits
2. Risk measurement around the center of the distribution

If we let R be a return random varaible (assumed continuous)

Definition 6.1.5 — Downside Semi-Variance.

$$\sigma_{-}^2 = \mathbb{E}\left[\min(0, R - \mathbb{R})^2\right]$$

Definition 6.1.6 — Value-at-Risk (VaR).

$$VaR_\alpha = \inf\{x|\mathbb{P}(R \leq x) \leq 1-\alpha\}$$

Definition 6.1.7 — Conditional Value-at-Risk (Conditional Tail Expectation).

$$CVaR_\alpha = \mathbb{E}[R|R \leq VaR_\alpha]$$

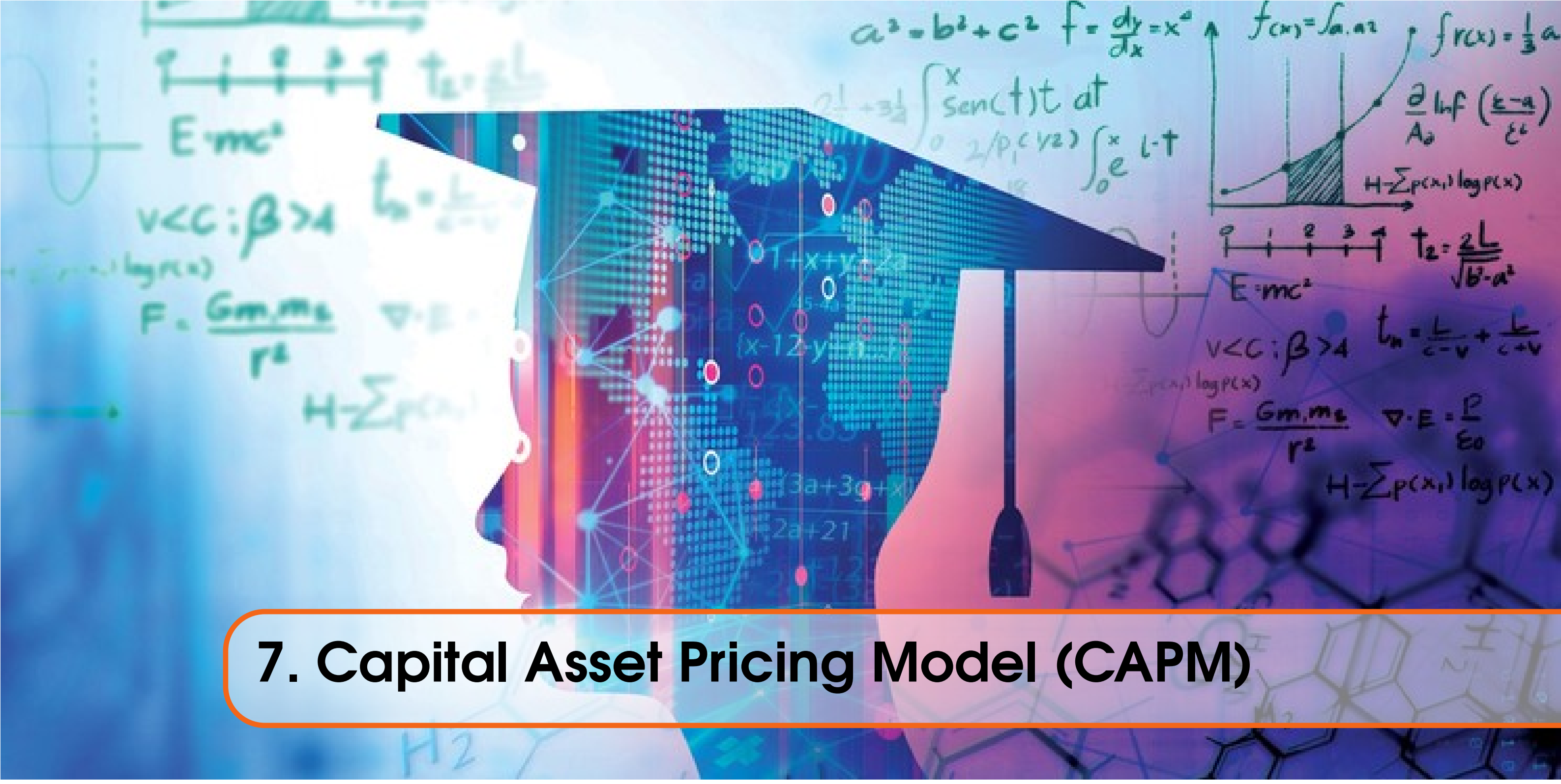
R If R_i is normal, then $VaR, CVaR$ can be calculated using μ, σ. For discrete r.v.s, look out for $\leq$ vs. $<$.

Risk Premium

Rational investors want high returns and low risk.

Definition 6.1.8 — Risk Premium. Risk Premium is the additional return an investment earns to compensate for the additional risk

R Often assume T-bills are risk-free, so excess returns over T-bills are the risk premium.



7. Capital Asset Pricing Model (CAPM)

7.1 Two Risky Assets

Suppose we invest x_A and x_B in stocks A, B

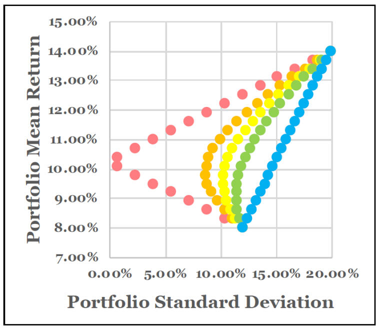
Assets	A	B	Portfolio (P)
\$ Invested	x_A	x_B	$x = x_A + x_B$
% Invested	$w_A = \frac{x_A}{x}$	$w_B = \frac{x_B}{x}$	$w_A + w_B = 1$

Table 7.1.1: *Caption*

1. Portfolio return: $R_p = w_A R_A + w_B R_B$
2. Variance:
$$\sigma_P^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \rho \sigma_A \sigma_B$$

■ Example 7.1 — Correlation Impact.

			σ_P (%) for given correlation ρ				
w_A	w_B	μ_P (%)	$\rho = -1$	$\rho = -0.3$	$\rho = 0$	$\rho = 0.3$	$\rho = 1$
0%	100%	14.00	20.00	20.00	20.00	20.00	20.00
20%	80%	12.80	13.60	15.45	16.18	16.88	18.40
40%	60%	11.60	7.20	11.51	12.92	14.20	16.80
60%	40%	10.40	0.80	9.02	10.76	12.26	15.20
80%	20%	9.20	5.60	9.23	10.40	11.45	13.60
100%	0%	8.00	12.00	12.00	12.00	12.00	12.00



Can have $\sigma_P < \min\{\sigma_A, \sigma_B\}$. We have

$$\sigma_P(\rho) = \sqrt{w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \rho \sigma_A \sigma_B}$$

1. If $\rho = 1$, then
$$\sigma_P(1) = w_A\sigma_A + w_B\sigma_B$$
if $0 \le w_A, w_B \le 1$ (what about shorting?)
2. If $\rho = -1$, then
$$\sigma_P(-1) = |w_A\sigma_A - w_B\sigma_B|$$
3. If $\rho = 0$, then
$$\sigma_P(0) = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2}$$

Thus, for $0 \le w_A, w_B \le 1$ and $- \le \rho \le 1$:
$$\sigma_P(-1) \le \rho_P(\rho) \le \sigma_P(1)$$

How to find the minimal risk portfolio w_A, w_B
$$w_A^+ = \frac{\sigma_B^2 - \rho\sigma_A\sigma_B}{\sigma_A^2 - 2\rho\sigma_A\sigma_B + \sigma_B^2}$$
and
$$w_B^* = 1 - w_A^*$$

■

7.1.1 Minimum Risk Portfolio

If apply w_A^*, w_B^*, we have the following result.

	$\rho = -1$	$\rho = -0.3$	$\rho = 0$	$\rho = 0.3$
w_A^*	62.50%	68.60%	73.53%	82.00%
w_B^*	37.50%	31.40%	26.47%	18.00%
μ_P^* (%)	10.25	9.88	9.59	9.08
σ_P^* (%)	0.00	8.73	10.29	11.45

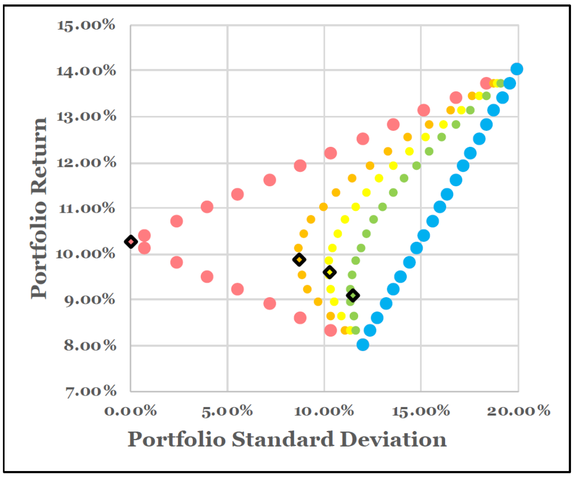


Figure 7.1.1: (σ_P^*, μ_P^*) *for different* ρ

We have the following observations:
1. σ_P^* increases with ρ while μ_P^* decreases with ρ

If $\mu_A \neq \mu_B$, we have
$$w_A = \frac{\mu_P - \mu_B}{\mu_A - \mu_B}, w_B = \frac{\mu_A - \mu_P}{\mu_A - \mu_B}$$

Definition 7.1.1 — Feasible Set/Opportunity Set. Set of portfolios that can be constructed from a given set of assets.

Definition 7.1.2 — Efficient Set/Efficient Frontier. Set of efficient portfolios, no other feasible portfolio can have both return and lowerer risk.