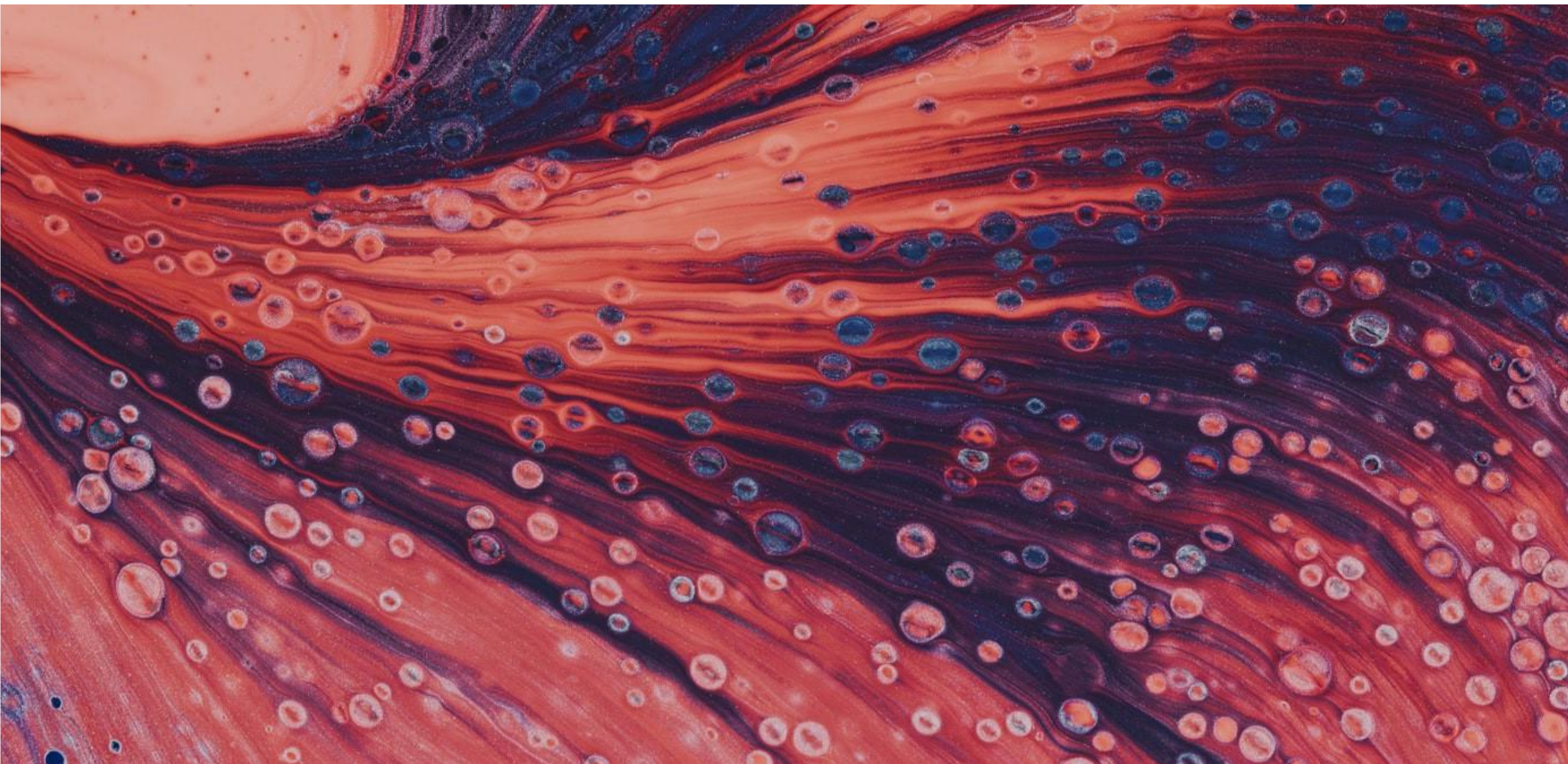
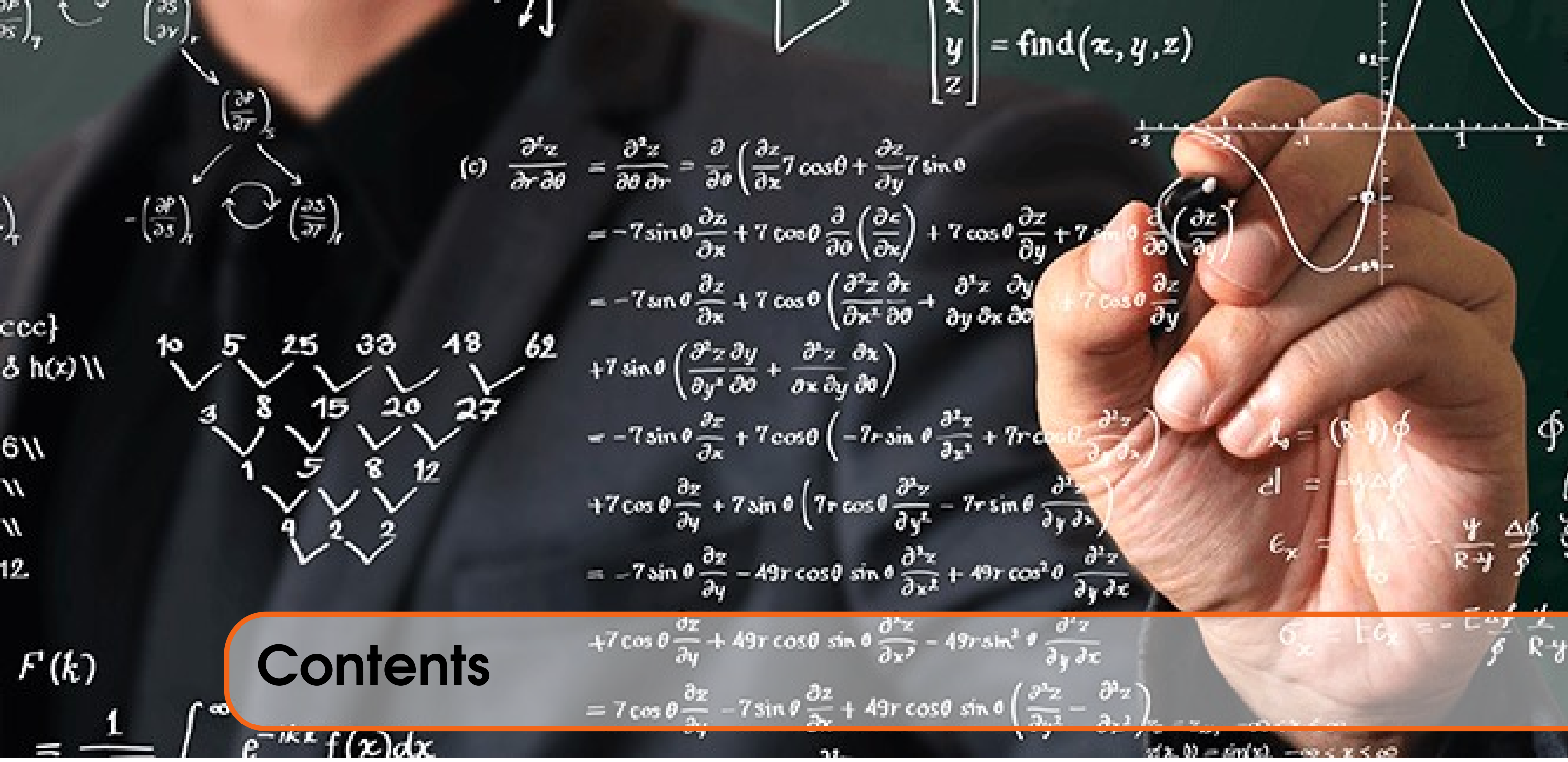
PMATH 365 Course Notes

University of Waterloo

The One And Only
Waterloo 76er
Bill Zhuo





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1. Multilinear Algebra

General Outline

1. Office Hour: MC5326 Mon/Wed 2:35pm-3:55pm
2. 6 bi-weekly assignments
3. February 14 Friday Midterm
4. *Differential Forms*—**Must Read**

Exterior Differential Calculus on Manifolds in $\mathbb{R}^n$

Main Result: Stokes's Theorem

Applications:

1. Brouwer fixed point theorem ($\mathbb{R}^n$)
2. The fundamental theorem of algebra ($\mathbb{R}^n$)
3. Hairy Ball Theorem (manifolds in $\mathbb{R}^n$)
4. Jordan-Brouwer Separation Theorem (manifolds in $\mathbb{R}^n$)
5. Gauss-Bonnet Theorem (manifolds in $\mathbb{R}^n$)
6. Poincare-Hopf Index Theorem (manifolds in $\mathbb{R}^n$)

Along the way, we'll prove the change of variable theorem from multivariable calculus.

1.1 Review on Linear Algebra

1.1.1 General Review

In this course, every vector space is real and finite-dimensional. Let V be a vector space in that regard. If $\dim V = n > 1$ and there exists a basis

$$\{e_1, e_2, \ldots, e_n\}$$

of V consisting of n elements.
Let $A : V \to W$ be a linear map of vectors spaces which means

$$A(t_1 v_1 + t_2 v_2) = t_1 A(v_1) + t_2 A(v_2), t_1, t_2 \in \mathbb{R}, v_1, v_2 \in V$$

and

$$\ker(A) = \{v \in V : Av = 0\} \subseteq V$$
$$\mathrm{Im}(A) = \{Av \in W, , v \in V\} \subseteq W$$

and

$$A \text{ is injective} \iff \ker(A) = \{0\}$$
$$A \text{ is surjective} \iff \mathrm{Im}(A) = W$$

Theorem 1 — Rank-Nullity Theorem.

$$\dim(V) = \dim(\ker(A)) + \dim(\mathrm{Im}(A))$$

Let $\{e_1, e_2, \dots, e_n\}$ be a basis of V and let $\{f_1, f_2, \dots, f_m\}$ be a basis of W let $A : V \to W$ be linear, then there exists $a_{ij} \in R, 1 \leq i \leq m = \dim(W), 1 \leq j \leq n = \dim V$, then

$$Ae_j = \sum_{i=1}^{m} a_{ij} f_i$$

Then $m \times n$ real matrix a_{ij} is the matrix representation of A with respect of these bases (**change of coordinate matrix**).

$$\begin{aligned}
V \ni v &= \sum_{j=1}^{n} t_j e_j \\
Av &= \sum_{j=1}^{n} t_j A_j \\
&= \sum_{j=1}^{n} t_j \sum_{i=1}^{m} a_{ij} f_j \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} t_j \right) f_j
\end{aligned}$$

where the sum in braket is the coordinate of Av

Definition 1.1.1 — Inner Product. An inner product on V which will always be positive-definite in this course) is a map

$$B : V \times V \to \mathbb{R}$$

such that

1. $B(v_1, v_2)$ is linear in V_1 and V_2
2. $B(v_1, v_2) = B(v_2, v_1)$ symmetric
3. $B(v, v) \geq 0$ with equality if and only if $v = 0$

1.1.2 Quotient Spaces

Let V be vector space and W be a subspace of V.

■ **Example 1.1 — Equivalence Relation.** A equivalence relations $\sim$ on V using W as follows we say $v_1 \sim v_2$ if $v_1 - v_2 \in W$

1. **Reflexity** $v \sim v$ since $v - v = 0 \in W$
2. **Symmetry** suppose $v_1 \sim v_2$ so $v_1 - v_2 \in W$, then

$$W \ni (-1) \cdot (v_1 - v_2) = v_2 - v_1 \to v_2 \sim v_1$$

3. **Transitivity** suppose $v_1 \sim v_2$ and $v_2 \sim v_3$

$$(v_1 - v_2 + (v_2 - v_3) = v_1 - v_3 \in W \to v_1 \sim v_3$$

■

Definition 1.1.2 — Quotient Space. The set of equivalence classes of this equivalence relation is denoted as ${}^{V}\big/_{W}$ called the quotient of V by W

■ **Example 1.2** Let $v \in V$,

$$\begin{aligned}
[v] &= \{\tilde{v} \in V : \tilde{v} \sim v\} \\
&= \{\tilde{v} \in V : \tilde{v} - v \in W\} \\
&= \{v\} + W = \{v + w : w \in W\} \\
[0] &= \{w + W\} = W
\end{aligned}$$

■

Theorem 2 — Equivalence Class Decomposition. $[v_1] = [v_2] \iff v_1 \sim v_2$. If $v_1 \not\sim v_2$, then $[v_1], [v_2]$ are disjoint subsets of V

Proposition 1.1.1 The set ${}^{V}\big/_{W}$ admits the structure of a vector space where we define:

$$[v_1] + [v_2] = [v_1 + v_2], v, v_1, v_2 \in V \tag{1.1.1}$$

$$t[v] = [tv], t \in \mathbb{R} \tag{1.1.2}$$

Proof. We need to show $(1), (2)$ are well-defined. Suppose $v_1 \sim \tilde{v}_1$ and $v_2 \sim \tilde{v}_2$ and

$$v_1 - \tilde{v}_1, v_2 - \tilde{v}_2 \in W \to (v_1 + v_2) - (\tilde{v}_1 + \tilde{v}_2) \in W$$

so, $[v_1 + v_2] = [\tilde{v}_1 + \tilde{v}_2]$
If $v \sim \tilde{v} \to v - \tilde{v} \in W$, so $t(v - \tilde{v}) \in W \to tv - t\tilde{v} \to [tv] = [t\tilde{v}]$ ■

The ${}^{V}\big/_{W}$ with this vector space structure is a called the quotient space of V by the subspace W. Note that

$${}^{V}\Big/_{\{0\}} \cong V, {}^{V}\big/_{V} \cong \{0\}$$

Definition 1.1.3 $\pi : V \to {}^{V}\big/_{W}$ by $\pi(v) = [v]$

Proposition 1.1.2 π is linear and surjective so $\ker(\pi) = W$

Proof. The linearity is done by previous proposition. Let $[v] \in {}^{V}\big/_{W}$ and $[v] = \pi(v)$ so clearly π is surjective. Now, suppose $\pi(v) = 0 \in {}^{V}\big/_{W}$ so $v \sim 0 \to v - 0 = v \in W$. Thus, $\ker(\pi) = W$. ■

π is called the quotient map from V to ${}^{V}\big/_{W}$.

Corollary 1.1.3

$$\dim\left({}^{V}\big/_{W}\right) = \dim(V) - \dim(W)$$

Proof. Apply rank-nullity theorem to π we have

$$\dim(V) = \dim(\ker(\pi)) + \dim(\operatorname{Im}(\pi)) = \dim(W) + \dim\left({}^{V}\big/_{W}\right)$$

■

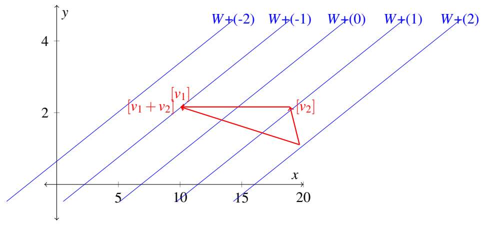


Figure 1.1.1: *Graphical Representation of the Quotient Space Operations*

■ **Example 1.3** Let $V \in \mathbb{R}^2$ and W= any 1-dimensional subspace, the $[v_1]$= line through v_1 parallel to W, $W = [0]$ ■

From last time...

$$\dim(V/W) = \dim(V) - \dim(W)$$

Lemma 1.2 Let $n = \dim(V), k = \dim(W)$. Let $\{v_1, \ldots, v_k\}$ be a basis of W. Extend to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$, then

$$\{\pi(v_{k+1}), \ldots, \pi(v_n)\}$$

are basis of V/W

Proof. Exactly the proof of rank-nullity ■

Definition 1.2.1 — Direct Complement & Direct Sum. Let U be a direct complement to W in V. This means U is a subspace of V, $U \cap W = \{0\}$ and $U + W$ spans V. And $v \in V$ can be written **uniquely** as $v = u + w$ for $u \in U, w \in W$. We write

$$V = U \bigoplus W$$

there exists **infinitely** many direct complements to W. A direct complement is not unique.

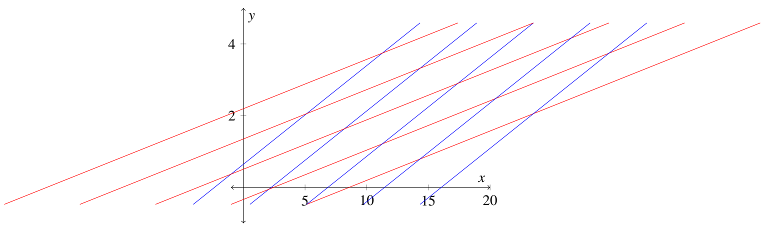


Figure 1.2.1: *Direct complement is not unique*

Proposition 1.2.1 Let U be a direct complement of W in V

$$\pi|_U : U \to V/W$$

this is a linear isomorphism. Leave as an exercise to show injectivity and surjectivity. Hence, any direct complement of W is ismorphic V/W.

Proposition 1.2.2 Let $A : V \to Y$ be a linear map and let $W \subseteq \ker(A)$, then $\exists!$ linear $\hat{A} : V/W \to Y$ such that $A = \hat{A} \circ \pi$.

1.3 Dual Space

Definition 1.3.1 — Dual Space. Let V be a vector space, the dual space of V is V^*, the set of linear maps from V to $\mathbb{R}$. Note V^* itself is a real vector space. For $a_1, a_2 \in V^*$ and $t_1, t_2 \in \mathbb{R}$,

$$t_1 a_1 + t_2 a_2 : V \to \mathbb{R}$$

is defined as

$$(t_1 a_1 + t_2 a_2)(v) = t_1 a_1(v) + t_2 a_2(v)$$

It is linear, trivial to prove.

Proposition 1.3.1 — $\dim(V) = \dim(V^*)$. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Define $\forall 1 \leq i \leq n$,

$$e_i^* \in V^*$$

as follows

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

For $v \in V$,

$$e_i^*(v) = t_1 e_i^*(e_1) + \cdots + t_n e_i^*(e_n) = t_i$$

which is the coordinate of v of e_i. We claim that

$$\{e_1^*, \ldots, e_n^*\}$$

is the basis of V^*, which will be sufficient.

Proof. Let $\alpha \in V^*$, let $\alpha_i = \alpha(e_i) \in \mathbb{R}$, let $\beta = \alpha_1 e_1^* + \cdots + \alpha_n e_n^* \in V^*$. We will show $\beta = \alpha$.

$$\beta(e_k) = \alpha_k = \alpha(e_k), \forall k$$

So, $\beta = \alpha$. Hence, $\{e_1^*, \ldots, e_n^*\}$ spans V^*.

$$\alpha = \alpha(e_1)e_1^* + \cdots + \alpha(e_n)e_n^*$$

Suppose $\alpha = c_1 e_1^* + \cdots + c_n e_n^* = 0 \in V^*$. $0 = \alpha(e_n) = c_k$, thus, linearly independent. ■

R

$$\{e_1^*, \ldots, e_n^*\}$$

is called the dual basis to $\{e_1, \ldots, e_n\}$. Hence $V \cong V^*$ but subject to a basis of V (canonically)

Definition 1.3.2 — Dual Map. Let $A : V \to W$ be linear. Define a map $A^* : W^* \to V^*$ as follows. Let $B \in W^*$, then $A^* B \in V^*$ is the map $V \to \mathbb{R}$.

$$(A^* B)(v) = BAv \in \mathbb{R}$$

Observe that $A^* B = B \circ A$, so the composition of linear map is linear. A^* is called the **dual map** of $A : V \to W$.

R Some texts call it the *Transpose Map*. Let $\{e_1,\dots,e_n\}$ be basis of V let $\{e_1^*,\dots,e_n^*\}$ be the dual basis of V^*. Let $\{f_1,\dots,f_m\}$ be a basis of W. And $\{f_1^*,\dots,f_m^*\}$ be the dual basis of W^*. For $A: V \to W$, there exists a $m \times n$ matrix $[a_{ij}]$ such that

$$Ae_j = \sum_{i=1}^{m} a_{ij} f_i$$

For $A^*: W^* \to V^*$, there exists $n \times m$ matrix $[c_{ji}]$, then

$$A^* f_i^* = \sum_{j=1}^{n} c_{ji} e_j^*$$

Now,

$$(A^* f_i^*)(e_j) = \left(\sum_{k=1}^{n} c_{ki} e_k^*\right)(e_j) = c_{ji}$$

by definition of A^*,

$$f_i^*(Ae_j) = f_i^*\left(\sum_{i=1}^{m} a_{kj} f_k\right) = a_{ij}$$

so $a_{ij} = c_{ji} \to c = a^T$.

Lemma 1.4 Let $A: V \to W$, $A^*: W^* \to V^*$, $B: W \to U$, $B^*: U^* \to W^*$. Then, $BA: V \to U$, $A^*B^*: U^* \to V^*$, $(BA)^*: U^* \to V^*$.

1. $(BA)^* = A^*B^*$
2. $I_{V^*} = (I_V)^*$
3. If $P: V \to V$ is invertible, so it $P^*: V^* \to V^*$, and $(P^*)^{-1} = (P^{-1})^*$

Definition 1.4.1 — Double Dual. Let V be a vector space and V^* is its dual space. Let $V^{**} = (V^*)^*$ be the dual space of V^* called the double dual of V. We know

$$\dim(V) = \dim(V^*) = \dim(V^{**})$$

Let $v \in V$, define $E_v \in V^{**}$ as a linear maps from V^* to $\mathbb{R}$. This is trivial again... for $\alpha \in V^*$

$$E_v(\alpha) = \alpha(v)$$

Hence, $E: V \to V^{**}$ that sends $v \mapsto E_v$. We claim that E is linear as well. Exercise

$$E_{t_1\alpha_1 + t_2\alpha_2} = t_1 E_{v_1} + t_2 E_{v_2}$$

Proposition 1.4.1 $E: V \to V^{**}$ is an isomorphism

Proof. We know $\dim(V) = \dim(V^{**})$, we just need to show $\ker(E) = \{0\}$. Suppose $v \in \ker(E)$, then

$$E_v = 0 \in V^{**} \to E_v(\alpha) = 0 \forall \alpha \in V^* \to \alpha(v) = 0 \forall \alpha \in V^*$$

Take $\alpha = e_1^*,\dots,e_n^*$, coordinates of v with respect to $\{e_1,\dots,e_n\}$ are all zero, so $v = 0$, so $\ker(E) = \{0\}$. v is canonically isomorphic to V^{**}. ■

Definition 1.4.2 — Annihilators. Let $W \subseteq V$ be a subspace. Define

$$W^0 = \{\alpha \in V^*, \alpha(w) = 0, \forall w \in W\}$$

all elements of V^* that annihilate all elements of W. The W^0 is the annihilator of W.

Proposition 1.4.2 W^0 is a subspace of V^*

Proof. Let $\alpha_1, \alpha_2 \in W^*$ and $w \in W$, then

$$(t_1\alpha_1 + t_2\alpha_2)(w) = t_1\alpha_1(w) + t_2\alpha_2(w) = 0$$

hence, $t_1\alpha_1 + t_2\alpha_2 \in W^0$. ■

Proposition 1.4.3 Let V be a vector space and U is a subspace of V, then

$$\dim(V) = \dim(U) + \dim(U^0)$$

Proof. Consider $I_U : U \to U \subseteq V$ with $I_U(u) = u, \forall u \in U$. Thus, $(I_U)^* \in V^* \to U^*$. Then, by Rank-Nullity Theorem,

$$\dim(V^*) = \dim(\ker((I_U)^*)) + \dim(\text{im}((I_U)^*))$$

by part (a), we know

$$\dim(V^*) = \dim((\text{im}(I_U))^0) + \dim(\text{im}((I_U)^*))$$

from MATH245, we know that the dual basis of V is the basis of the dual space, so $\dim(V) = \dim(V^*)$. We claim the following:

1. $(\text{im}(I_U)) = U$

   Proof. This is apparent since I_U is the inclusion or say identity map from U to U, which is bijective ■

2. $\text{im}((I_U)^*) = U^*$

   Proof. Let $\varphi \in \text{im}((I_U)^*)$, then there exists $\psi \in U*$ such that $(I_U)^*(\psi) = \varphi$. But note that

   $$\varphi = \psi \circ I_U = \psi \in U^*$$

   Thus, $\text{im}((I_U)^*) \subseteq U^*$.
   Now, for $\psi \in U^*$, then, for $u \in U$

   $$(I_U)^*(\psi)(u) = \psi \circ I_U u = \psi(u)$$

   Thus, $\psi \in \text{im}((I_U)^*)$. Therefore, the claim is true. ■

Finally, by the two claims above

$$\dim(V^*) = \dim(U^0) + \dim(U^*) \to \dim(V) = \dim(U^0) + \dim(U)$$

■

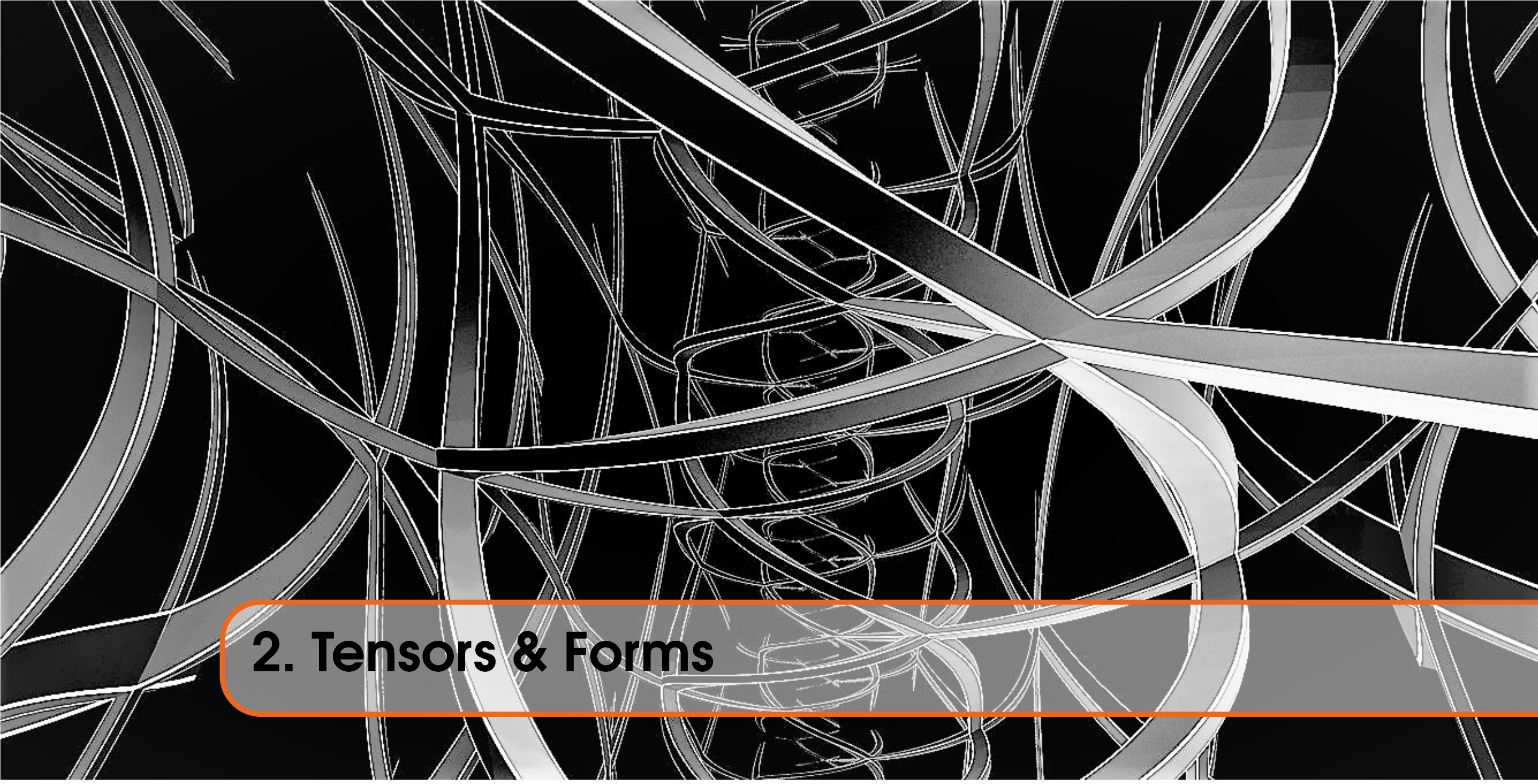
Proof. **Spiro's Proof**: Let $\{e_1, \dots, e_k\}$ be a basis of W and extend to basis $\{e_1, \dots, e_n\}$ of V. Let $\{e_1^*, \dots, e_n^*\}$ be the dual basis of V^*. Let $\alpha \in W^0 \subseteq V^*$,

$$\alpha = \alpha_1 e_1^* + \cdots + \alpha_n e_n^*, a_i \in \mathbb{R}$$

so, $\alpha_i = \alpha(e_i)$. But $\alpha(e_i) = 0, \forall i = 1, \dots, k$. Hence,

$$\alpha = \alpha_{k+1} e_{k+1}^* + \dots \alpha_n e_n^*$$

so $\{e_{k+1}^*, \dots, e_n^*\}$ spans W^* and also linearly independent. ■



2. Tensors & Forms

2.1 Tensors on V

Definition 2.1.1 — Tensors on V. Let V be a n-dimensional vector space. Let $T : V^k \to \mathbb{R}$. We say T is a k-tensor on V for $k \geq 1$ if and only if $v_i \mapsto T(v_1, \ldots, v_k)$ is linear in v_i if $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ are fixed.
A k-tensor is also called a **k-linear map** or a **multilinear map**

Proposition 2.1.1 Let T_1, T_2 be k-tensors, then $c_1T_1 + c_2T_2$ is a k-tensor where

$$(c_1T_1 + c_1T_2)(v_1, \ldots, v_k) = c_1T_1(v_1, \ldots, v_k) + c_2T_2(v_1, \ldots, v_k)$$

Let $\mathscr{L}^k(V)$ denote the set of all k-tensors in V. Then $\mathscr{L}^k(V)$ is a real vector space.

R

1. If $k = 1$, $\mathscr{L}^1(V) = v^*$
2. If $k = 2$, $\mathscr{L}^2(V) = V^* \otimes V^*$ is the bilinear forms on V.
3. If $k = 0$, we define $\mathscr{L}^0(V) = \mathbb{R}$

Definition 2.1.2 — Tensor Product. Let $T \in \mathscr{L}^k(V)$ and $S \in \mathscr{L}^l(V)$, define

$$T \otimes S : V^{k+l} \to \mathbb{R}$$

by

$$(T \otimes S)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_n) = T(v_1, \ldots, v_k)S(v_{k+1}, \ldots, v_{k+l}) \in \mathbb{R}$$

Thus, $T \otimes S \in \mathscr{L}^{(k+l)}(V)$.

R [Distributivity] Let $T_1, T_2, T \in \mathscr{L}^k(V)$ and $S_1, S_2, S \in \mathscr{L}^l(V)$.

$$(T_1 + T_2) \otimes S = T_1 \otimes S + T_2 \otimes S$$
$$T \otimes (S_1 + S_2) = T \otimes S_1 + T \otimes S_2$$

R If $\lambda \in \mathscr{L}^0(V) = \mathbb{R}$, then

$$\lambda \otimes T = T \otimes \lambda = \lambda T$$

if $k, l \geq 1$, in general

$$T \otimes S \neq S \otimes T$$

R [Associativity] Let $R \in \mathscr{L}^m(V)$. Then,

$$T \otimes (S \otimes R) = (T \otimes S) \otimes R = T \otimes S \otimes R$$

Definition 2.1.3 — Decomposable. Let $l_1, \ldots, l_k \in V^* = \mathscr{L}^1(V)$, then

$$l_1 \otimes l_2 \otimes \ldots l_k \in \mathscr{L}^k(V)$$

so,

$$(l_1 \otimes l_2 \otimes \cdots \otimes l_k)(v_1, \ldots, v_k) = l_1(v_1) l_2(v_2) \ldots l_k(v_k)$$

Such a k-tensor is called a **decomposable** k-tensor

R We'll see that not every k-tensor is decomposable if $k > 1$. But any k-tensor is a finite sum of decomposable k-tensor (not in a unique way)

Theorem 3 Let $\{e_1, \ldots, e_n\}$ be a basis of V and $\{e_1^*, \ldots, e_n^*\}$ be the basis of V^*, then

$$\mathscr{B} := \left\{ e_{i_1}^* \otimes \cdots \otimes e_{i_k}^* : 1 \leq i_1, \ldots, i_k \leq n \right\}$$

is a basis of $\mathscr{L}^k(V)$

Proof. Let $T \in \mathscr{L}^k(V)$. Define $T_{i_1,\ldots,i_k} := T(e_{i_1}, \ldots, e_{i_k}), \forall 1 \leq i_1, \ldots, i_k \leq n$. Let

$$S = \sum_{i_1,\ldots,i_k=1}^{n} T_{i_1,\ldots,i_k} e_{i_1}^* \otimes \cdots \otimes e_{i_k}^* \in \mathscr{L}^k(V)$$

1. Claim: $S = T$

   Proof.

$$\begin{aligned} S(e_{j_1}, \ldots, e_{j_k}) &= \sum_{i_1,\ldots,i_k=1}^{n} T_{i_1,\ldots,i_k} (e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*)(e_{j_1}, \ldots, e_{j_k}) \\ &= \sum_{i_1,\ldots,i_k=1}^{n} T_{i_1,\ldots,i_k} \big((e_{i_1}^*(e_{j_1})) \ldots (e_{i_k}^*(e_{j_k})) \big) \\ &= T_{i_1,\ldots,i_k} \end{aligned}$$

   The last line is true since the only output $T_{i_1,\ldots,i_k}$ is achieved when $(e_{j_1}, \ldots, e_{j_k}) = (e_{i_1}, \ldots, e_{i_k})$, which will happen once in the whole sum. Since S, T both k-linear and they agree on $\{e_1, \ldots, e_n\}$ which is a basis, so they agree everywhere $S = T$. So, $\mathscr{B}$ spans $\mathscr{L}^k(V)$

   ■

2. Claim: $\mathscr{B}$ linearly independent:

Proof. Suppose

$$\sum_{i_1,\dots,i_k=1}^{n} c_{i_1,\dots,i_k}(e^*_{i_1}\otimes\cdots\otimes e^*_{i_k}) = 0 \in \mathscr{L}^k(V)$$

evaluate both sides on $e_{j_1},\dots,e_{i_k}$, so $0 = c_{i_1,\dots,i_k}$. So, linearly independent. ∎

Done. ∎

R When $k=1$, this is exactly the proof in previous class.

Proposition 2.1.2

$$\dim\left(\mathscr{L}^k(V)\right) = n^k$$

R If follows that any k-tensor is a linear combination of decomposable k-tensors, but not in a unique way if $k>1$ because

$$(\lambda T)\otimes S = T\otimes(\lambda S) = \lambda(T\otimes S)$$

R Because of the previous remark, some books write

$$\mathscr{L}^k(V) = V^*\otimes\cdots\otimes V^* = \bigotimes^k V^*$$

Definition 2.1.4 — Pullback of k-tensors. Let $A: V\to W$ be linear. Given a k-tensor $T\in\mathscr{L}^k(V)$, we obtain an element $A^*T: V\times\cdots\times V\to\mathbb{R}$ by

$$(A^*T)(v_1,\dots,v_k) = T(Av_1,\dots,Av_k)$$

since A is linear and T multilinear, so

$$A^*T\in\mathscr{L}^k(V)$$

and A^*T is called the pullback of T by A. If $k=1$, thhis is just the dual map $A^*: W^*\to V^*$

Proposition 2.1.3 — Pullback Properties.
1. Let $A^*: \mathscr{L}^k(W)\to\mathscr{L}^k(V)$ is linear
2. $A^*(T_1\otimes T_2) = (A^*T_1)\otimes(A^*T_2)$
3. and more on assignments.

2.2 Review of Group Theory

Definition 2.2.1 — Permutation. Let $\Sigma_k = \{1,\dots,k\}$. A permutation of Σ_k (also known as a permutation n k letters) is a bijection $\sigma:\Sigma_k\to\Sigma_k$ (i.e it is a rearrangement). We usually writes

$$(\sigma(1),\dots,\sigma(k))$$

is a rearrangement of $(1,\dots,k)$, sometimes written as

$$\begin{pmatrix} 1 & 2 & \dots & k \\ \sigma(1) & \sigma(2) & \dots & \sigma(k)\end{pmatrix}$$

■ **Example 2.1** Let S_k be the set of all permutations of Σ_k, is $\sigma, \tau \in S_k$, so is $\sigma \circ \tau$ and σ^{-1} also the identity. So, S_k is a group. S_k has $k!$ elements.
For example,

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, 1! = 1$$

$$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}, 2! = 2$$

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \right\}, 3! = 6$$

■

Definition 2.2.2 — Elementary Transposition. Let $i < j$ and let $\tau_{ij} \in S_k$ be the permutation

$$\tau_{ij}(i) = j$$

$$\tau_{ij}(j) = i$$

$$\tau_{ij}(d) = d, d \neq i, j$$

if $j = i+1$, $\tau_{i,i+1}$ is called elementary transpoition.

Proposition 2.2.1 Let $k \geq 2$, and $\sigma \in S_k$, then σ is a product of transpositions.

Proof. True for $k = 2$ and induction on k. Let $k \geq 3$ and suppose true for $k-1$. Let $\sigma \in S_k$ and $\sigma(k) = i \in S_k$, then

$$(\tau_{ik} \circ \sigma)(k) = k$$

hence $\tau_{ik} \circ \sigma \in S_{k-1}$. By induction hypothesis, $\tau_{ik} \circ \sigma$ is a product of transpositions. Hence,

$$\sigma = \tau_{ki} \circ \tau_{ik} \circ \sigma$$

■

Proposition 2.2.2 Any transposition is a product of elementary transposition.

Proof. Let i be fixed, and $j > i$. It is clear if $j = i+1$, induction on j. Suppose $j \geq i+2$ and it time for $\tau_{i,j-1}$, then

$$\tau_{i,j} = \tau_{j-1,j} \circ \tau_{i,j-1} \circ \tau_{j-1,i}$$

and we are done. ■

Corollary 2.2.3 Any $\sigma \in S_k$ is a product of elementary transposition (but NOT in a unique way).

Definition 2.2.3 — Sign of a Permutation. Let $X_1, \ldots, X_k$ be the coordinate functions of $\mathbb{R}^k$. Let $\sigma \in S_k$, we define

$$\operatorname{sgn}(\sigma) = (-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j} \in \{-1, 1\}$$

all terms in the numerators are the factors in the denominators except possibly a sign of -1.

■ **Example 2.2** Let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \in S_3$. Then,

$$\operatorname{sgn}(\sigma) = \frac{X_{\sigma(1)} - X_{\sigma()}}{X_1 - X_2} \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} = (-1)$$

■

Definition 2.2.4 σ is called even if $\operatorname{sgn}(\sigma) = +1$; σ is called odd if $\operatorname{sgn}(\sigma) = -1$

Proposition 2.2.4 Let $\sigma, \tau \in S_k$, then

$$(-1)^{\sigma\tau} = (-1)^{\sigma}(-1)^{\tau}$$

or equivalently,

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$$

Proof.

$$\begin{aligned}
(-1)^{\sigma\tau} &= \prod_{i<j} \frac{X_{\sigma\tau(i)} - X_{\sigma\tau(j)}}{X_i - X_j} = \prod_{i<j} \frac{X_{\sigma\tau(i)} - X_{\sigma\tau(j)}}{X_{\tau(i)} - X_{\tau(j)}} \frac{X_{\tau(i)} - X_{\tau(j)}}{X_i - X_j} \\
&= \prod_{i<j} \frac{X_{\sigma\tau(i)} - X_{\sigma\tau(j)}}{X_{\tau(i)} - X_{\tau(j)}} \prod_{i<j} \frac{X_{\tau(i)} - X_{\tau(j)}}{X_i - X_j} \\
&= \prod_{i<j} \frac{X_{\sigma\tau(i)} - X_{\sigma\tau(j)}}{X_{\tau(i)} - X_{\tau(j)}} \operatorname{sgn}(\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)
\end{aligned}$$

Let $i < j$

1. If $\tau(i) < \tau(j)$, let $p = \tau(i), q = \tau(j)$, $p < q$
2. If $\tau(i) > \tau(j)$, let $p = \tau(j), q = \tau(i)$, $p < q$

either way,

$$\frac{X_{\sigma\tau(i)} - X_{\sigma\tau(j)}}{X_{\tau(i)} - X_{\tau(j)}} = \frac{X_{\tau(p)} - X_{\tau(q)}}{X_p - X_q}$$

■

R This says $\operatorname{sgn} : S_k \to \{-1, 1\}$ is a group homomorphism.

Exercise 2.1 If τ is a transposition, $\operatorname{sgn}(\tau) = -1$. ■

Corollary 2.2.5 $\sigma \in S_k$ is even if and only if it is the product of an even number of tranposition. $\sigma \in S_k$ is odd if and only it is the product of odd number of transpositions.

Corollary 2.2.6 Even though decomposition of σ into product of transpositions is not unique, the number of factors is always either even or odd.

2.3 Alternating Tensors

Definition 2.3.1 Let $n = \dim V$ and $T \in \mathscr{L}^k(V)$ for $k \geq 1$, for $\sigma \in S_k$, define $T^{\sigma} : V^k \to \mathbb{R}$ by

$$T^{\sigma}(v_1, \ldots, v_k) = T\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)$$

R note that $T^\sigma \in \mathscr{L}^k(V)$, we can fix all arguments except i-th in T^σ, that's the same as fixing all arguments except the $\sigma^{-1}(i)$-th in T. Thus, T is linear in the $\sigma^{-1}(i)$-th argument. Thus, T^σ is linear in i-th argument.

Proposition 2.3.1 1. Let $T = l_1 \otimes \cdots \otimes l_k$ be decomposable, then

$$T^\sigma = l_{\sigma(1)} \otimes \ldots l_{\sigma(k)}$$

2. $T \to T^\sigma$ is a linear map of $\mathscr{L}^k(V)$ to itself
3. Let $\sigma, \tau \in S_k$, $(T^\sigma)^\tau = T^{\sigma\tau}$

Proof. 1. $T = l_1 \otimes \cdots \otimes l_k$, then

$$\begin{aligned} T^\sigma(v_1, \ldots, v_k) &= T\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) \\ &= l_1 \otimes \cdots \otimes l_k\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) \\ &= l_{\sigma(1)}(v_1) \ldots l_{\sigma(k)}(v_k) \\ &= (l_{\sigma(1)} \otimes \ldots l_{\sigma(k)})(v_1, \ldots, v_k) \end{aligned}$$

since we can let $\sigma^{-1}(i) = p$ if and only if $i = \sigma(p)$, the i-th term is $l_i(v_{\sigma^{-1}(i)}) = l_{\sigma(p)}(v_p)$ and real numbers are commutative.

2. Exercise.
3. WLOG by 2, we can assume T is decomposable, for $T = l_1 \otimes \cdots \otimes l_k$. And

$$T^\sigma = l_{\sigma(1)} \otimes \ldots l_{\sigma(k)} = l'_1 \otimes \cdots \otimes l'_k, l'_j = l_{\sigma(j)}, \forall j$$

then,

$$(T^\sigma)^\tau = (l'_1 \otimes \cdots \otimes l'_k, l'_j)^\tau = l'_{\tau(1)} \otimes \cdots \otimes l'_{\tau(k)} = l_{\sigma\tau(1)} \otimes \cdots \otimes l_{\sigma\tau(k)} = T^{\sigma\tau}$$

■

Definition 2.3.2 — Alternating Tensors. Let $k \geq 1$ and $T \in \mathscr{L}^k(V)$, we say T is an alternating k-tensor if

$$T^\sigma = (-1)^\sigma T, \forall \sigma \in S_k$$

■ **Example 2.3** If $k = 1$ and $S_1 = \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ and T is alternating.

If $k = 2$ and $S_2 = \left\{\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\right\}$ Then, T is alternating if and only if $T(v_1, v_2) = -T(v_2, v_1), \forall v_1, v_2 \in V$. ■

R From 2 in the previous proposition, suppose T_1, T_2 are alternating, then

$$\begin{aligned} (c_1T_1 + c_2T_2)^\sigma &= c_1T_1^\sigma + c_2T_2^\sigma \\ &= c_1(-1)^\sigma T_1 + c_2(-1)^\sigma T_2 \\ &= (-1)^\sigma(c_1T_1 + c_2T_2) \end{aligned}$$

Hence, a alternating k-tensors on V are a subspace of $\mathscr{L}^k(V)$ denotes $\mathscr{A}^k(V)$.
Also, $\mathscr{A}^1(V) = \mathscr{L}^1(V) = V^*$ and $\mathscr{A}^2(V) \subsetneq \mathscr{L}^2(V)$

Definition 2.3.3 — Alternation of T. Define a map $\text{Alt} : \mathscr{L}^k(V) \to \mathscr{L}^k(V)$ by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

clear that Alt is linear since T^σ is linear. It is called the alternation of T. Note that if $k = 1$, we have $\text{Alt}(T) = T$

Proposition 2.3.2 Let $T \in \mathscr{L}^k(V), \sigma \in S_k$

1. $[\text{Alt}(T)]^\sigma = (-1)^\sigma \text{Alt}(T)$ and $\text{Alt}(T) \in \mathscr{A}^k(V)$
2. If $T \in \mathscr{A}^k(V)$, then $\text{Alt}(T) = k!T$
3. $\text{Alt}(T^\sigma) = [\text{Alt}(T)]^\sigma$
4. Linearity: this has been done.

1.

$$\begin{aligned}
[\text{Alt}(T)]^\sigma &= \left[\sum_{\tau \in S_k} (-1)^\tau T^\tau\right]^\sigma \\
&= \sum_{\tau \in S_k} (-1)^\tau (T^\tau)^\sigma \\
&= \sum_{\tau \in S_k} (-1)^\tau (-1)^\sigma (-1)^\sigma T^{\tau\sigma} \\
&= (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\
&= (-1)^\sigma \text{Alt}(T)
\end{aligned}$$

2. Suppose $T \in \mathscr{A}^k(V)$ and $T^\tau = (-1)^\tau T$,

$$\begin{aligned}
\text{Alt}(T) &= \sum_{\tau \in S_k} (-1)^\tau T^\tau \\
&= \sum_{\tau \in S_k} (-1)^\tau (-1)^\tau T^\tau \\
&= k!T
\end{aligned}$$

3.

$$\begin{aligned}
\text{Alt}(T^\sigma) &= \sum_{\tau \in S_k} (-1)^\tau (T^\sigma)^\tau \\
&= (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\sigma\tau} T^{\sigma\tau} \\
&= (-1)^\sigma \text{Alt}(T) \\
&= [\text{Alt}(T)]^\sigma
\end{aligned}$$

We want to construct a bassis of $\mathscr{A}^k(V)$ which will determine its dim

Definition 2.3.4 Let $I = (i_1, \ldots, i_k)$ be a multi-index and $1 \le i_1 \le \cdots \le i_k \le n = \dim V$

1. We say I is repeating if $i_r = i_s$ for some $r \neq s$
2. We say I is strictly increasing if

$$i_1 < i_2 < \cdots < i_k$$

for $\sigma \in S_k$, define

$$I^\sigma = \left(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\right)$$

Note that if I is non-repeating, then there exists uniquely $\sigma \in S_k$ such thatt I^σ is strictly increasing

Let I be the a k multi-index and $\{e_1, \dots, e_n\}$ be a basis of V and define $e_I^* = e_1^* \otimes \cdots \otimes e_k^*$. We know that

$$\{e_I^* : I \text{ is a multi-index}\}$$

is a basis of $\mathscr{L}^k(V)$. Let $\Phi_I = \mathrm{Alt}(e_I^*) \in \mathscr{A}^k(V)$. Then,

$$\begin{aligned}(e_I^*)^\sigma &= (e_{I^\sigma}^*) \\ &\to \mathrm{Alt}(e_{I^\sigma}^*) = [\mathrm{Alt}(e_I^*)]^\sigma \\ &= (-1)^\sigma \mathrm{Alt}(e_I^*)\end{aligned}$$

Proposition 2.3.3 1. $\Phi_{I^\sigma} = (-1)^\sigma \Phi_I$

2. If I is repeating, $\Phi_I = 0$

3. If I, J both strictly increasing, then

$$\Phi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

Proof. 1.

$$\begin{aligned}\Phi_{I^\sigma} &= \mathrm{Alt}(e_{I^\sigma}^*) \\ &= \mathrm{Alt}(e_I^*)^\sigma \\ &= [\mathrm{Alt}(e_I^*)]^\sigma \\ &= (-1)^\sigma \mathrm{Alt}(e_I^*) = (-1)^\sigma \Phi_I\end{aligned}$$

2. Suppose $i_r = i_s$ for $r \neq s$ and let $\tau = \tau_{i_r, i_s}$ and $I^\tau = I$

$$\begin{aligned}\Phi_I &= \Phi_{I^\tau} = (-1)^\tau \Phi_I \\ &= -\Phi_I \to \Phi_I = 0\end{aligned}$$

3.

$$\Phi_I(e_{j_1}, \dots, e_{j_k}) = \sum_{\sigma \in S_k} (-1)^\sigma e_{I^\sigma}^*(e_{j_1}, \dots, e_{j_k})$$

Note that

$$e_{I^\sigma}^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I^\sigma = J \\ 0 & I^\sigma \neq J \end{cases}$$

Thus, if I, J are strictly increasing, I^σ is strictly increasing if and only if $I^\sigma = I$. And, non-zero if and only if $I = J$.

■

Theorem 4

$$\mathscr{B} = \{\Phi_I : I \text{ is strictly increasing}\}$$

is a basis of $\mathscr{A}^k(V)$

Proof. let $T \in \mathscr{A}^k(V) \subseteq \mathscr{L}^k(V)$. Then,

$$T = \sum_{\text{multi-index } J} a_J e_J^*$$

since $T \in \mathscr{A}^k(V)$,

$$k!T = \text{Alt}(T) = \sum_J c_J \text{Alt}(e_J^*) = \sum_J c_J \Phi_J$$

Thus, $T = \frac{1}{k!}\sum_J c_J \Phi_J$. If J is repeating, $\Phi_I = 0$, if J is non-repeating, there exists a unique σ such that J^σ is strictly increaseing and

$$\Phi_{J^\sigma} = (-1)^\sigma \Phi_J$$

therefore, the previous result is a lienar combination of the elements of $\mathscr{B}$ and $\mathscr{B}$ spans $\mathscr{A}^k(V)$. Suppose

$$\sum_{I \text{ str inc}} c_I \Phi_I = 0 \in \mathscr{L}^k(V)$$

apply both sides to $e_{j_1}, \dots, e_{j_k}$ when J is strictly increasing, then $C_J = 0$. Thus, $\mathscr{B}$ is linearly independent. Hence, a basis. ■

Corollary 2.3.4 Let $n = \dim V$ and let $1 \le k \le n$

$$\begin{aligned}\dim(\mathscr{A}^k(V)) &= \binom{n}{k} = \frac{n!}{k!(n-k)!} \\ &= \binom{n}{n-k} = \dim(\mathscr{A}^{n-k}(V))\end{aligned}$$

Notice that if $k > n$, then any multi-index is repeating, so $\mathscr{A}^k(V) = \{0\}$ if $k > n$. If $k = 0$, we define $\mathscr{L}^0(V) = \mathbb{R}$.

2.4 The Space $\wedge^k(V)$

We want to show $\mathscr{A}^k(V)$ is canonically isomorphic to a quotient of $\mathscr{L}^k(V)$ by another subspace $\mathscr{S}^k(V)$.

Theorem 5

$$\mathscr{L}^k(V) = \mathscr{A}^k(V) \oplus \mathscr{S}^k(V)$$

2.4.1 Digressions on Projections

Definition 2.4.1 — Projector. Let $P : V \to V$ be a linear map such that $P^2 = P$. This is a projector.

Proposition 2.4.1 1. Let $Q = I - P$, this is also a projector.

2. $PQ = P(I-P) = P - P^2 = P - P = 0$ hence, $\text{im}Q \subseteq \ker P$ and $\text{im}P \ker Q$, thus, $\text{im}P = \ker Q$ and $\ker P = \text{im}Q$. By rank-nullity, let $v \in V$

$$\begin{aligned} v &= Pv + v - Pv \\ &= Pv + Qv \end{aligned}$$

suppose $w \in \text{im}P \cap \ker P$, $w = Pu$ for some u. Then,

$$0 = Pw = P^2 u = Pu = w$$

so $w = 0$, so $\text{im}P \cap \ker P = \{0\}$.

3. Upshot

$$V = (\text{im}P) \oplus (\ker P)$$

Recall that $\text{Alt}(T) = k!T$ if $T \in \mathscr{A}^k(V)$. Let $P = \frac{1}{k!}\text{Alt}() : \mathscr{L}^k(V) \to \mathscr{L}^k(V)$, then

$$\begin{aligned} P^2(T) &= \frac{1}{k!}\text{Alt}\left(\frac{1}{k!}\text{Alt}(T)\right) \\ &= \frac{1}{(k!)^2}k!\text{Alt}(T) = P(T) \end{aligned}$$

Thus,

$$\mathscr{L}^k(V) = \mathscr{A}^k(V) \oplus \ker(\text{Alt})$$

we are going to call this $\ker(\text{Alt}) = \mathscr{I}^k(V)$.
Hence, by earlier results or quotient spaces,

$$\mathscr{A}^k(V) \cong \mathscr{L}^k(V) \Big/ \mathscr{I}^k(V) = \wedge^k(V^*)$$

$$\mathscr{A}^k(V) \cong \mathscr{L}^k(V) \Big/ \mathscr{I}^k(V) = \wedge^k(V^*)$$

Definition 2.4.2 — Redundant. For $k = 0, 1$, set $\mathscr{S}^k(V) = \{0\}$, then $\mathscr{L}^k(V) = \mathscr{A}^k(V)$ and

$$\mathscr{A}^k(V) \cong \mathscr{L}^k(V) \Big/ \{0\} = \wedge^k(V^*) = \mathscr{L}^k(V) = \mathscr{A}^k(V)$$

Assume $k = 2$, a decomposable k-tensor $l_1 \otimes \cdots \otimes l_k$ is called redundant if $l_i = l_{i+1}$ FOR SOME $i = 1, \ldots, k-1$

Let $\mathscr{S}^k(V)$ be the linear space of all redundant k-ternsors.
We want to show $\mathscr{S}^k(V) = \ker(\text{Alt})$

Proposition 2.4.2

$$\mathscr{S}^k(V) \subseteq \ker(\text{Alt})$$

Proof. Since Alt is linear, enough to consider redundant k-tensors. Let $T = l_1 \otimes \cdots \otimes l_k$ with $l_i = l_{i+1}$ let $\tau = \tau_{i,i+1}$. Thus, $T^\tau = T$ and $(-1)^\tau = -1$. Hence,

$$\begin{aligned} \text{Alt}(T) &= \text{Alt}(T^\tau) \\ &= (\text{Alt}(T))^\tau \\ &= (-1)^\tau \text{Alt}(T) \\ &= -\text{Alt}(T) \to \text{Alt}(T) = 0 \end{aligned}$$

■

Proposition 2.4.3 Let $T \in \mathscr{S}^k(V)$ and $S \in \mathscr{L}^l(V)$, then $T \otimes S$ and $S \otimes T \in \mathscr{S}^{k+l}(V)$

Proof. Again WLOG, since $\otimes$ is bilinear, we can assume both T, S are both decomposable. Say

$$T = l_1 \otimes \cdots \otimes l_k, l_i = l_{i+1}$$

$$S = u_1 \otimes \cdots \otimes u_l$$

then,

$$T \otimes S = l_1 \otimes \cdots \otimes l_k \otimes u_1 \otimes \cdots \otimes u_l, l_i = l_{i+1}$$

so, $T \otimes S \in \mathscr{S}^{k+l}(V)$. ■

Lemma 2.5 Let $l,u \in V^* = \mathscr{L}^1(V)$ and $l \otimes u + u \otimes l \in \mathscr{S}^2(V)$

Proof.

$$l \otimes u + u \otimes l = (l+u) \otimes (l+u) - l \otimes l - u \otimes u \in \mathscr{S}^2(V)$$

since each part is redundant. ■

Proposition 2.5.1 Let $T \in \mathscr{L}^k(V)$ and $\sigma \in S_k$, then

$$T^\sigma = (-1)^\sigma T + K, K \in \mathscr{S}^k(V)$$

Proof. Since $T \mapsto T^\sigma$ is linear, enough to assume T is decomposable. Let $T = l_1 \otimes \cdots \otimes l_k$ let $\sigma = \tau_{i,i+1}$ be an elementary transposition. Then,

$$T - (-1)^\sigma T^\sigma = l_1 \otimes \cdots \otimes l_{i-1} \otimes (l_i \otimes l_{i+1} + l_{i+1} \otimes l_i) \otimes l_{i+2} \otimes \cdots \otimes l_k \in \mathscr{S}^k(V)$$

by previous lemma. ■

Now, assume $\sigma \in S_k$ is finite product of elementary tranposition $\sigma = \tau_1 \circ \cdots \circ \tau_m$. We will prove by induction on m we proved $m = 1$. Suppose $m \geq 2$ and the proposition is true for $m-1$.

$$\sigma = \tau_1 \circ \cdots \circ \tau_{m-1} \circ \tau_m = \beta\tau$$

$$\begin{aligned} T^\sigma = T^{\beta\tau} &= (-1)^\tau T^\beta + K, K \in \mathscr{S}^k(V) \\ &= (-1)^\tau \Big[(-1)^\beta T + K' \Big] + K \\ &= (-1)^\sigma T + K'', K'' \in \mathscr{S}^k(V) \end{aligned}$$

Corollary 2.5.2 Let $T \in \mathscr{L}^k(V)$, then

$$\mathrm{Alt}(T) = k!T + K, K \in \mathscr{S}^k(V)$$

Proof.

$$\begin{aligned} \mathrm{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\ &= \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + K_\sigma] = k!T + K \end{aligned}$$

■

Corollary 2.5.3

$$\mathscr{S}^k(V) = \ker \mathrm{Alt}$$

Proof. We showed $\mathscr{S}^k(V) \subseteq \ker(Alt)$, suppose $T \in \ker Alt$, by previous corollary,

$$0 = \mathrm{Alt}(T) = k!T + K \to T = -\frac{1}{k!} K \in \mathscr{S}^k(V)$$

■

Hence,

$$\wedge^k(V^*) = \mathscr{L}^k(V) \Big/ \mathscr{S}^k(V) \cong \mathscr{A}^k(V)$$

where $\wedge^k(V^*)$ is the space of k-forms on V. And

$$\pi : \mathscr{A}^k(V) \to \mathscr{L}^k(V) \Big/ \mathscr{S}^k(V)$$

Now, we want to define algebraic operations on $\wedge^k(V^*)$.

2.6 Wedge Product

Definition 2.6.1 — Wedge Product. Let $\omega \in \wedge^k(V^*), \eta \in \wedge^k(V^*)$, then

$$[T] = \omega = \pi(T), T \in \mathscr{L}^k(V) \qquad \eta = \pi(S), S \in \mathscr{L}^l(V)$$

define the wedge product of ω, η, denote

$$\omega \wedge \eta = \pi(T \otimes S) \in \wedge^{k+l}(V^*)$$

Need to show this is well-defined. Suppose $T \sim T'$ and $S \sim S'$. This means $T = T' + K_1$ and $S = S + K_2$, $K_1 \in \mathscr{S}^k(V), K_2 \in \mathscr{S}^l(V)$. Then,

$$T \otimes S = T' \otimes S' + T' \otimes K_2 + K_1 \otimes S' + K_1 \otimes K_2$$

note that $T' \otimes K_2, K_1 \otimes S', K_1 \otimes K_2 \in \mathscr{S}^{k+l}(V)$ then,

$$\pi(T \otimes S) = \pi(T' \otimes S')$$

More generally, if $\omega_i \in \wedge^{k_i}(V^*)$, let $\omega_i = \pi[T_i], T_i \in \mathscr{L}^{k_i}(V)$, define

$$\omega_1 \wedge \cdots \wedge \omega_m = \pi(T_1 \otimes \cdots \otimes T_m)$$

By construction, we have **associative**

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$$

on assignment 2, we will show wedge product is bilinear and

$$(t\omega) \wedge \eta = \omega \wedge (t\eta) = t(\omega \wedge \eta), \forall t \in \mathbb{R}$$

If $\omega \in \wedge^k(V^*)$, we say degree of ω is k, or $\omega \in \wedge^k(V^*), \eta \in \wedge^l(V^*)$, then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$

so wedge product is not commutative. An even form commutes with any form but two odd degree forms are anti-commutative.

■ **Example 2.4** Let $l_1, \ldots, l_k \in V^*$, so

$$l_1 \wedge \cdots \wedge l_k = \pi(l_1 \otimes \cdots \otimes l_k) \in \wedge^1(V^*)$$

with $\pi : \mathscr{L}^1(V) \to \mathscr{A}^1(V)$ is the identity. ■

Proposition 2.6.1 $l_{\sigma(1)} \wedge \cdots \wedge l_{\sigma(k)} = (-1)^\sigma l_1 \wedge \cdots \wedge l_k$

Proof. For all T, $T^\sigma = (-1)^\sigma T + K$ for some $K \in \mathscr{S}^k(V)$, apply π to this

$$\pi(T^\sigma) = (-1)^\sigma \pi(T)$$

let $T = l_1 \otimes \cdots \otimes l_k$, then

$$\pi(l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(k)}) = (-1)^\sigma \pi(l_1 \otimes \cdots \otimes l_k)$$

and we are done. ■

Theorem 6 Let $\{e_1,\dots,e_n\}$ be a basis of V, then

$$\beta_\wedge = \left\{e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* : I = (i_1,\dots,i_k) \text{ str inc}\right\}$$

is a basis of $\wedge^k(V^*)$.

Proof. We know $\{\Phi_I = \alpha e_I^*\}, I$ str inc is a basis of $\mathscr{A}^k(V)$. And $\pi : \mathscr{A}^k(V) \to \wedge^k(V^*)$ is an isomorphism, so

$$\{\pi(\Phi_I) : I = (i_1,\dots,i_k) \text{ str inc}\}$$

is a basis of $\wedge^k(V^*)$. Then, note that

$$\begin{aligned}\pi(\Phi_I) &= \pi(\mathrm{Alt}(e_I^*)) \\ &= \pi()\end{aligned}$$

■

R

$$\wedge^{\cdot}(V^*) = \bigoplus_{k=0}^{n} \wedge^k(V^*)$$

with $n = \dim V$ is called the exterior algebra of V. The wedge product makes $\wedge^{\cdot}(V^*)$ into an assciatieve algebra with identity. And,

$$\dim\left(\wedge^{\cdot}(V^*)\right) = \sum_{k=0}^{n} \dim\left(\wedge^k(V^*)\right) = \sum_{k=0}^{n}\binom{n}{k} = 2^n$$

2.7 Interior Product

Definition 2.7.1 Let $v \in V$ and $T \in \mathscr{L}^k(V)$ and $k \geq 1$, define $\iota_v T : V^{k-1} \to \mathbb{R}$. by

$$(\iota_v T)(v_1,\dots,v_{k-1}) = \sum_{j=1}^{k} (-1)^{j-1}(v_1,\dots,v_{j-1},v,v_j,\dots,v_{k-1})$$

for $k = 0$, define $\iota_v T = 0$. Notice if $k = 1$, $\iota_v T = T(v)$. It's clear that $\iota_v T \in \mathscr{L}^{k-1}(V)$.
When $k = 2$, $(\iota_v T)(w) = T(v,w) - T(w,v)$
When $k = 3$, $(\iota_w T)(w_1,w_2) = T(v,w_1,w_2) - T(w_1,v,w_2) + T(w_1,w_2,v)$

Proposition 2.7.1

$$\iota_{v_1+v_2} T = \iota_{v_1} T + \iota_{v_2} T$$
$$\iota_v(T_1 + T_2) = \iota_v T_1 + \iota_v T_2$$
$$\iota_v(cT) = c\iota_v T \in \mathbb{R}$$

Lemma 2.8 Let T be a decomposable tensor, then

$$\iota_v T = \sum_{j=1}^{k} (-1)^{j-1} l_j(v) l_1 \otimes \dots \hat{l}_j \otimes \dots l_k)$$

where the hat is for omit. And if

$$T_1 \in \mathscr{L}^k(V) \qquad T_2 \in \mathscr{L}^l(V)$$

then

$$\iota_v(T_1 \otimes T_2) = (\iota_v T_1) \otimes T_2 + (-1)^k T_1 \otimes (\iota_v T_2)$$

Lemma 2.9

$$\iota_v(\iota_v T) = 0, \forall T \in \mathscr{L}^k(V)$$

Proof. Induction on k, $k = 0, 1$ trivia. Then, let $k \geq 2$, assume true for $k-1$, since ι_v is linear, we can WLOG assume T is decomposable. Let $T = l_1 \otimes \cdots \otimes l_{k-1} \otimes l_k$ and the first k-1 term is $S \in \mathscr{L}^{k-1}(V)$. Then

$$\iota_v T = (\iota_v S) \otimes l_k + (-1)^{k-1} S \otimes \iota_v l_k = (\iota_v S) \otimes l_k + (-1)^{k-1}(l_k(v))S$$

then

$$\iota_v(\iota_v T) = (\iota_v(\iota_v S)) \otimes l_k + (-1)^{k-2}(\iota_v S) \otimes \iota_v l_k + (-1)^{k-1} l_k(v)(\iota_v S) = 0$$

■

Corollary 2.9.1

$$\iota_{v_1}\iota_{v_2} = -\iota_{v_2}\iota_{v_1}$$

Proof.

$$0 = (\iota_{v_1+v_2})^2 = (\iota_{v_1} + \iota_{v_2})^2 = \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1}$$

■

Lemma 2.10 Let $T \in \mathscr{S}^k(V)$, then $\iota_v T \in \mathscr{S}^{k-1}(V)$

Proof. By linearity, it is enough to show for a redundant k-tensor

$$T = l_1 \otimes \cdots \otimes l_k, l_i = l_{i+1}$$

Then, let $T_1 = l_1 \otimes \cdots \otimes l_{i-1}$ and $T_2 = l_{i+2} \otimes \cdots \otimes l_k$, then

$$T = T_1 \otimes l_i \otimes l_i \otimes T_2$$

$$\begin{aligned}
\iota_v T &= (\iota_v T_1) \otimes l_i \otimes l_i \otimes T_2 \\
&\quad + (-1)^{i-1} T_1 \otimes \iota_v(l_i \otimes l_i) \otimes T_2 \\
&\quad + (-1) T_1 \otimes l_i \otimes l_i \otimes (\iota_v T_2) \in \mathscr{S}^{k-1}(V)
\end{aligned}$$

since

$$\iota_v(l_i \otimes l_i) = (\iota_v l_i) \otimes l_i - l_i \otimes (\iota_v l_i) = l_i(v) l_i - l_i(v) l_i = 0$$

■

Definition 2.10.1 Let $v \in V$, $\omega \in \wedge^k(V^*)$, we define $\iota_v \omega \in \wedge^{k-1}(V^*)$ called the interior product of v with ω, as follow

$$\omega = \pi(T), T \in \mathscr{L}^k(V)$$

define

$$\iota_v \omega = \pi(\iota_v T) \in \wedge^{k-1}(V^*)$$

well-defined: if $T = T' + S, S \in \mathscr{S}^k(V)$ and $\iota_v T = \iota_v T' + \iota_v S, \iota_v S \in \mathscr{S}^{k-1}(V)$ and

$$\pi(\iota_v T) = \pi(\iota_v T') + 0$$

Automatically, we get

$$\iota_{v_1+v_2}\omega = \iota_{v_1}\omega + \iota_{v_2}\omega$$

$$\iota_{v_1}(\omega_1 \wedge \omega_2) = (\iota_v \omega_1) \wedge \omega_2 + (-1)^{\deg(\omega_1)} \omega_1 \wedge (\iota_v \omega_2)$$

2.11 Pullback of Forms

Let $A : V \to W$

Lemma 2.12 If $T \in \mathscr{I}^k(W)$, then $A^*T \in \mathscr{I}^k(V)$

Proof. Since A^* is linear, enough to show for T redundant

$$T = l_1 \otimes \cdots \otimes l_k, l_i = l_{i+1}$$

then

$$A^*T = (A^*l_1) \otimes \cdots \otimes (A^*l_k), (A^*l_i) = (A^*l_{i+1})$$

is redundant. ∎

Definition 2.12.1 Let $\omega \in \wedge^k(W^*)$ and $\omega \in \pi(T)$. Then,

$$A^*\omega := \pi(A^*T)$$

well-defined by lemma.

Proposition 2.12.1 1. $A^* : \wedge^k(W*) \to \wedge^k(V^*)$ is linear
2. $A^*(\omega_1 \wedge \omega_2) = (A^*\omega_1) \wedge (A^*\omega_2)$
3. $B : U \to V$ and $AB : U \to W$, then $(AB)^*\omega = B^*(A^*\omega) \in \wedge^k(V^*), \forall w \in \wedge^k(W^*)$

2.12.1 Application of Pullback

Definition 2.12.2 — Determinant. Let $n = \dim V$, $\wedge^n(V^*)$ is one-dimensional. Let $A : V \to V$ linear, then

$$A^* : \wedge^n(V^*) \to \wedge^n(V^*)$$

is a one-dimensional pullback map. There exists a scalar which we called the determinant of A, $\det A$ such that

$$A^*\omega = (\det A)\omega, \forall \omega \in \wedge^n(V^*)$$

Proposition 2.12.2

$$\det(AB) = \det(A)\det(B)$$

$$\det(I_v) = 1$$

Proof.

$$\begin{aligned}
\det(AB)\omega(AB)^*\omega \\
&= B^*A^*\omega = (B^*)(\det A)\omega \\
&= \det(A)\det(B)\omega, \forall \omega \in \wedge^n(V^*)
\end{aligned}$$

and

$$\det(I_v)\omega = I_v^*\omega = \omega = 1\omega$$

∎

Corollary 2.12.3 If A is invertible, then $\det A \neq 0$.

Proposition 2.12.4 Suppose $A : V \to V$ is not invertible, then $\det A = 0$

Proof. A is not invertible if and only if A is not surjective. Then, $W = \text{im}A \subsetneq V$, so $\dim W < \dim V = n$. Let $B : V \to W$ such that $B(v) = A(v)$ that maps A with $\text{im}A = W$ as codomain) Let $i_w : W \to V$ be the inclusion map. Then

$$
\begin{aligned}
(\det A)\omega &= A^*\omega = (i_w B)^*\omega \\
&= B^*(i_w^*)\omega \in \wedge^n(W^*) = \{0\}
\end{aligned}
$$

Thus, $B^*(0) = 0$, so $\det(A) = 0$. ■

Theorem 7 — This really is the determinant. More generally, we need this next time, let $A : V \to W$ be linear where $\dim V = \dim W = n$. Let $\{e_1, \dots, e_n\}$ be a basis of V and $\{f_1, \dots, f_n\}$ be a basis of W, and *dualbn* be the dual basis of V^* and $\{f_1^*, \dots, f_n^*\}$ be the dual basis of W^*. We already say that

$$
Ae_j = \sum_{i=1}^{n} a_{ij} f_i \to A^* f_j^* = \sum_{i=1}^{n} a_{ji} e_i^*
$$

Proof. $\omega = f_1^* \wedge \cdots \wedge f_n^*$ be a basis vector of $\wedge^n(W^*)$ then

$$
\begin{aligned}
A^*(f_1^* \wedge \cdots \wedge f_n^*) &= (A^* f_1^*) \wedge \cdots \wedge (A^* f_n^*) \\
&= \left(\sum_{k_1=1}^{n} a_{i,k_1} e_{k_1}^* \right) \wedge \cdots \wedge \left(\sum_{k_n=1}^{n} a_{i,k_n} e_{k_n}^* \right) \\
&= \sum_{k_1,\dots,k_n=1}^{n} a_{1,k_1} \dots a_{n,k_n} e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*
\end{aligned}
$$

If $(k_1, \dots, k_n)$ is repeating, then $e_{k_1}^* \wedge \cdots \wedge e_{k_n}^* = 0$. If $(k_1, \dots, k_n)$ is non-repeating, there exists a unique $\sigma \in S_n$ such that $\sigma(1) = k_1, \dots, \sigma(n) = k_n$, then

$$
\begin{aligned}
&= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)} e_{\sigma(1)}^* \wedge \cdots \wedge e_{\sigma(n)}^* \\
&= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)} e_{\sigma(1)}^* \wedge \cdots \wedge e_{\sigma(n)}^* \\
&= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)} (-1)^{\sigma} e_1^* \wedge \cdots \wedge e_n^*
\end{aligned}
$$

■

2.13 Orientations

Let L be a one dimensional vector space then $L \cong \mathbb{R}$. We have

$$
\mathbb{R} \backslash \{0\}
$$

is a disjoint union of two subsets, the positive reals, and the negative reals. Similarly, $L \backslash \{0\}$ consists of two disjoint subsets. Explicitly, given any $v \neq 0 \in L$. We can define

$$
L^1 = \{tv, t > 0\}
$$

$$
L^1 = \{tv, t < 0\}
$$

and $L = L^1 \cup \{0\} \cup L^2$

Definition 2.13.1 — Orientation on L. An orientation of L is a choice of one of the two disjoint subsets, L_1, L_2, of $L\backslash\{0\}$ to be called the positive part. Here, we choose

$$L^+ = L_1, L^- = L_2$$

Definition 2.13.2 — Positively/Negatively Orientation. Let L be one dimensional equipped with an orientation L^+. Any $v \neq 0 \in L$, is called positively oriented if $v \in L^+$ and called negatively oriented if $v \in L^-$.

R Equivalently, an orientation on L is a choice of basis vector v of L up to positive scaling.

Definition 2.13.3 Given an orientation on L, we get an orientation on L^*, by demanding that

$$(L^*)^+ = \left\{\alpha \in L^* : \alpha(v) > 0, \forall v \in L^+\right\}$$

R That is, if e_1 is a positively oriented vector in L, then e_1^* is declared to be positively oriented L^*.

Definition 2.13.4 — n-dimensional Orientation. Let $n = \dim V$, an orientation on V is defined to be an orientation on the one dimensional space $\wedge^n(V^*)$.

Lemma 2.14 — Consistency Check. When $n = 1$, $\wedge^1(V^*) = V^*$.

An orientation on V^* when $\dim V = 1$ is equivalent to an orientation of V by the remark above.

2.14.1 How to Assign An Orientation to V

Let $\{e_1, \dots, e_n\}$ be a basis of V with dual basis $\{e_1^*, \dots, e_n^*\}$ of V^*. Choose the orientation of $\wedge^n(V^*)$ such that

$$e_1^* \wedge \cdots \wedge e_n^*$$

is positively oriented and it is a basis vector of $\wedge^n(V^*)$.
Suppose $\{f_1, \dots, f_n\}$ is another basis of V with dual basis $\{f_1^*, \dots, f_n^*\}$ is the dual basis of V^*. Then,

$$e_j = \sum_{i=1}^{n} a_{ij} f_i$$

then, from previous material, we have

$$f_1^* \wedge \cdots \wedge f_n^* = \det(a_{ij}) e_1^* \wedge \cdots \wedge e_n^*$$

with $W = V$ and $A = I$. **So, $\{f_1, \dots, f_n\}$ determines the same orientation as $\{e_1, \dots, e_n\}$ if and only if $\det(a_{ij}) > 0$; and it determines the opposite orientation if and only if $\det(a_{ij}) < 0$.**

■ **Example 2.5** If $\{e_1, \dots, e_n\}$ is positively oriented, then $\{-e_1, e_2, \dots, e_n\}$ is negatively oriented. Similarly, $\{e_2, e_1, \dots, e_n\}$ is negatively oriented. ■

Theorem 8 Let $n = \dim V$ and $W \subseteq V$ be a subspace and $k = \dim W$ and $1 \leq k \leq n-1$. Given an orientation on V and given an orientation on V/W, we get a natural orientation on W. "Natural" will become clear in the proof.

Proof. Let $r = n-k = \dim \left({}^{V}\!/_{W} \right)$. Choose a basis $\{e_{r+1},\dots,e_n\}$ of W and extend it to a basis $\{e_1,\dots,e_n\}$ of V. Let $\pi : V \to {}^{V}\!/_{W}$, we know $\{\pi(e_1),\dots,\pi(e_r)\}$ is a basis of ${}^{V}\!/_{W}$. By replacing, e_1 by $-e_1$, we can assume $\{\pi(e_1),\dots,\pi(e_r)\}$ is positively oriented for ${}^{V}\!/_{W}$. And by replacing, e_n by $-e_n$ if necessary, we can assume $\{e_1,\dots,e_n\}$ is positively oriented for V.
Give W the orientation determined by $\{e_{r+1},\dots,e_n\}$.
We need to show this is well-defined. Suppose $\{f_1,\dots,f_n\}$ is another basis of V such that $f_{r+1},\dots,f_n$ is a basis of W. So, $\{\pi(f_1),\dots,\pi(f_r)\}$ is a basis of ${}^{V}\!/_{W}$ and suppose $\{f_1,\dots,f_n\}$ is positively oriented for V and $\{\pi(f_1),\dots,\pi(f_r)\}$ is positivel oriented for ${}^{V}\!/_{W}$. We need to show that $f_{r+1},\dots,f_n$ determines the same orientation of W as $\{e_{r+1},\dots,e_n\}$.
We know that

$$e_j = \sum_{i=1}^{n} a_{ij} f_i$$

$$A = \begin{pmatrix} B_{r\times r} & 0_{r\times(n-r)} \\ C_{(n-r)\times r} & D_{(n-r)\times(n-r)} \end{pmatrix}$$

where B is the $r\times r$ invertible matrix that takes $\{\pi(f_1),\dots,\pi(f_r)\}$ to $\{\pi(e_1),\dots,\pi(e_r)\}$.
where D is the $(n-r)\times(n-r)$ invertible matrix that takes $f_{r+1},\dots,f_n$ to $\{e_{r+1},\dots,e_n\}$.
Then, by block triangular matrix,

$$\det(A) = \det(B)\det(D)$$

$\det(A)$ is positive since $f_1,\dots,f_n$ and $e_1,\dots,e_n$ are of the same orientation. $\det(B)$ is positive since $\{\pi(f_1),\dots,\pi(f_r)\}$ and $\{\pi(e_1),\dots,\pi(e_r)\}$ are of the same orientation. Thus, $\det(D)$ is positive. ■

R **How to Remember?**
Recall ${}^{V}\!/_{W}$ is ismorphic to any direct complement of W in V,

$$V \cong \left({}^{V}\!/_{W} \right) \oplus W$$

Special Case

Let $k = \dim W = n-1$ called the codimension one subspace. A choice of $v \in V\backslash W$ is equivalent to a choice of basis $\{\pi(v)\}$ for ${}^{V}\!/_{W}$, which is also equivalent to an orientation for ${}^{V}\!/_{W}$. By the proved theorem,

$$\{v, e_1,\dots,e_{n-1}\}$$

is positively oriented for V if and only if $\{e_1,\dots,e_{n-1}\}$ is positively oriented for W.

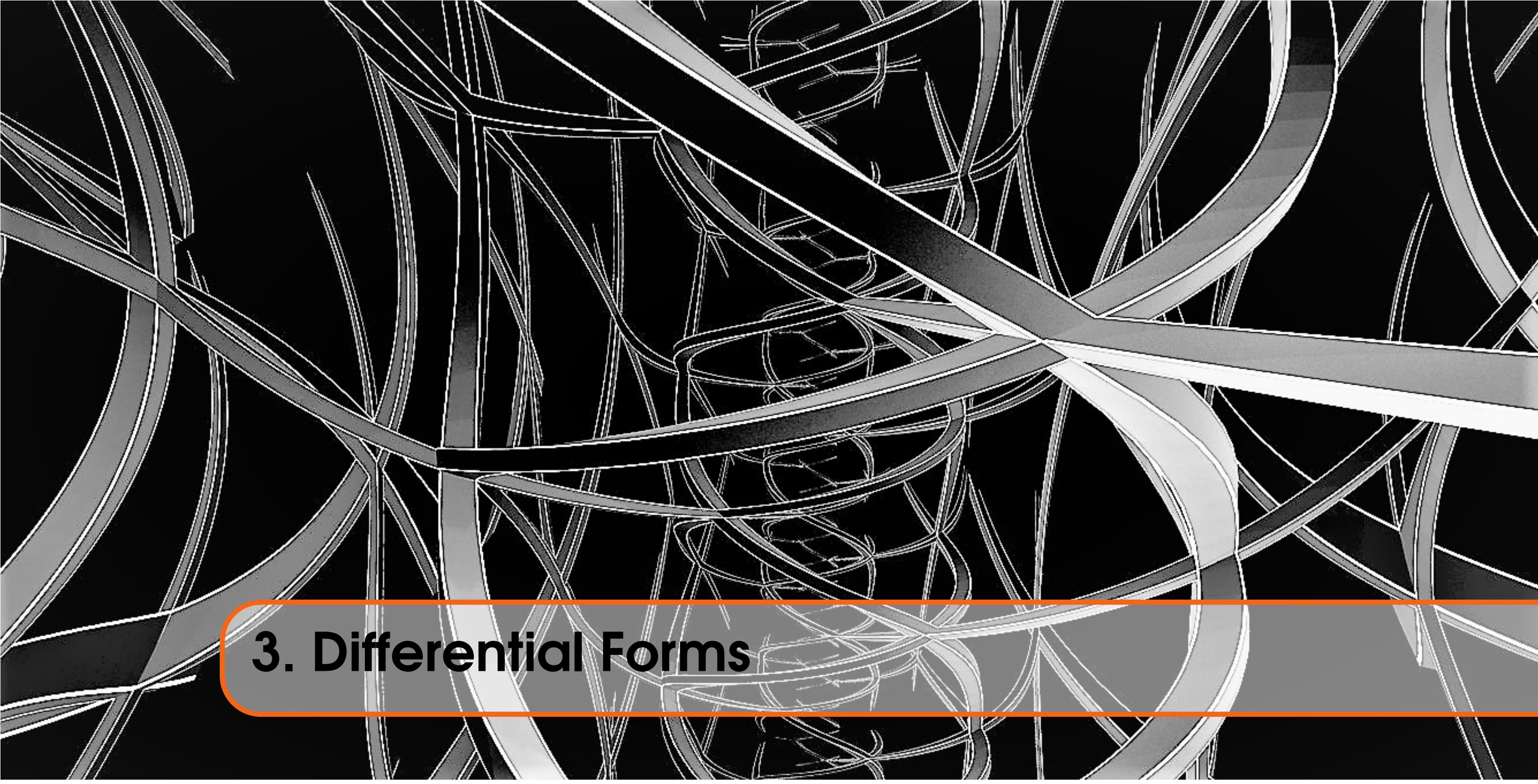
■ **Example 2.6** If $V = \mathbb{R}$, then the standard basis $\{e_1,\dots,e_n\}$ determines the standard orientation such that

$$e_1^* \wedge \cdots \wedge e_n^* \in \wedge^n (V^*)^+$$

■

Definition 2.14.1 — Orientation Preserving/Reversing. Let $A : V \to W$ be a linear isomorphism. Suppose V, W both oriented. Then A is called orientation preserving if for $A^* : \wedge^n(W^*) \to \wedge^n(V^*)$ $A^*\omega \in \wedge^n(V^*)^+$ whenever $\omega \in \wedge^n(W^*)^+$;
Similarly, A is orientation reversing if $A^*\omega \in \wedge^n(V^*)^-$ whenever $\omega \in \wedge^n(W^*)^+$;
Special Case: $V = W$, then A is orientation perserving if and only if $\det A > 0$ and orientation reversing if and only if $\det A < 0$.

Exercise 2.2 The composition of orientation perserving isomorphisms is orientation perserving. ■



3. Differential Forms

3.1 Tangent Spaces and Smoothness

Definition 3.1.1 — Tangent Space. Let $p \in \mathbb{R}^n$. The tangent space to $\mathbb{R}^n$ at p is denoted $T_p\mathbb{R}^n$ and is the set

$$T_p\mathbb{R}^n = \{(p,v), v \in \mathbb{R}^n\}$$

$T_p\mathbb{R}^n$ is given the structure of a n-dimensional real vector space by declaring

$$t_1(p,v_1) + t_2(p,v_2) := (p, t_1v_1 + t_2v_2)$$

There is a bijection

$$T_p : \mathbb{R}^n \to T_p\mathbb{R}^n$$
$$v \mapsto (p,v)$$

be our definition of the vector space structure on $T_p\mathbb{R}^n$, this map is a linear isomorphism and $T_p\mathbb{R}^n$ is canonically isomorphic to $\mathbb{R}^n$.

Definition 3.1.2 — Smoothness. Let $U \subseteq \mathbb{R}^n$ be open

$$C^\infty(U) = \{f : U \to \mathbb{R} : f \text{ is } C^\infty\}$$

this means

$$\frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}$$

exists and are continuous on U for all $k \geq 0, \forall 1 \leq i_1, \dots, i_k \leq n$.
C^∞ is also called **smooth**.

R

1. $C^\infty(U)$ is a real vector space.
2. $C^\infty(U)$ is **infinite dimensional**.

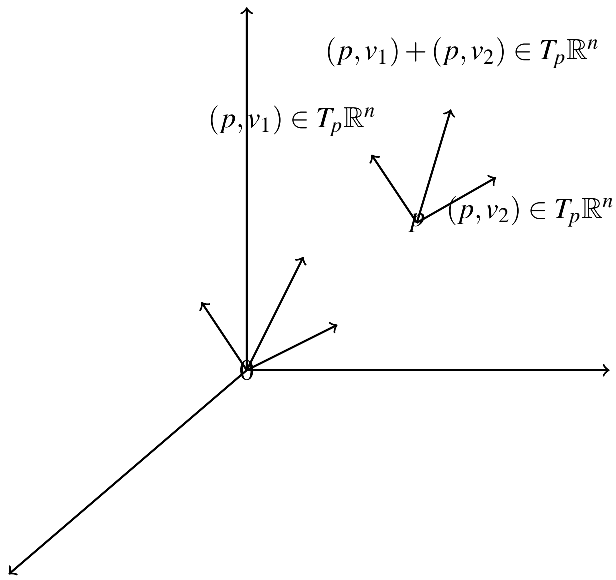


Figure 3.1.1: *Translation of the Origin from* 0 *to* p

3. Moreover, if $f,g \in C^\infty(U)$, then
$$(fg)(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)g(x_1,\ldots,x_n) \in C^\infty(U)$$
by product rule. Thus, $C^\infty(U)$ is a ring with scalar multiplication or an algebra.
4. Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then
$$F(p) = (F_1(p),\ldots,F_m(p)), \forall p \in U$$
we say F is smooth if all its component functions $F_j \in C^\infty(U), \forall 1 \le j \le m$.
5. Recall from MATH247 that the derivatives of F at p is denoted $(DF)_p$, and is the real $m \times n$ matrix whose i,j-th entry is
$$\frac{\partial F_i}{\partial x_j}(p), 1 \le i \le m, 1 \le j \le n$$
since F is C^1, then F is differentiable.

Definition 3.1.3 We define a **linear** map
$$(dF)_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$$
by $(dF)_p(p,v) := (F(p),(DF)_p v)$. Note that
$$T_p\mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n \longrightarrow \mathbb{R}^m \longrightarrow T_{F(p)}\mathbb{R}^m$$

Let $V \subseteq \mathbb{R}^m$ be open and $G : V \to \mathbb{R}^q$ be smooth such that $F(U) \subseteq V$. Hence, we can define $H : G \circ F : U \to \mathbb{R}^p$ by $H(p) = G(F(p))$. We have H smooth by Chain rule.
$$(DH)_p = (DG)_{F(p)}(DF)_p$$

Proposition 3.1.1
$$(dH)_p = (dG)_{F(p)}(dF)_p$$

Proof.

$$
\begin{aligned}
&(dH)_p(p,v) = (H(p),(DH)_p v)\\
&= (G(F(p)),(DG)_{F(p)}(DF)_p v)\\
&= (dG)_{F(p)}(F(a),(DF)_p v) = (dG)_{F(p)}(dF)_p(p,v)
\end{aligned}
$$

■

3.2 Smooth Vector Fields

Definition 3.2.1 Let $U \subseteq \mathbb{R}^n$ be open. A vector field on U is a map X that assigns to every $p \in U$, an element $X(p) =: X_p \in T_p\mathbb{R}^n, \forall p \in U$.

■ **Example 3.1**

1. Let $v \in \mathbb{R}^n$ be a fixed vector and define X be

$$X_p = (p,v), \forall v \in \mathbb{R}^n$$

this is called a constant vector field.

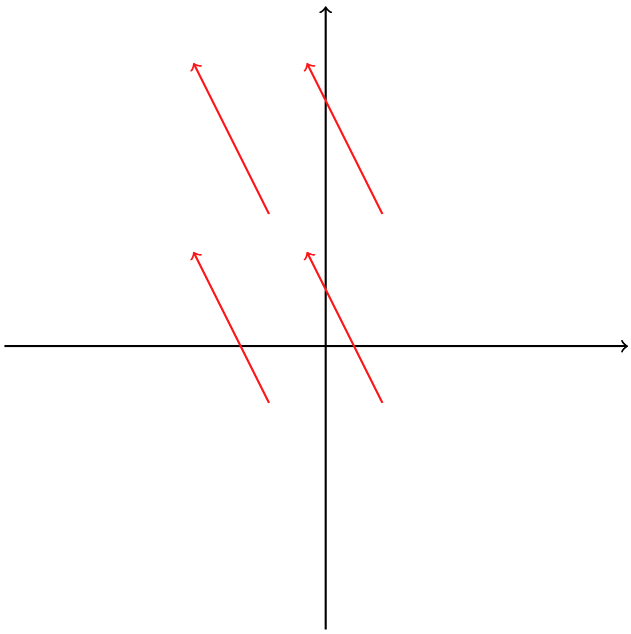


Figure 3.2.1: *Constant Vector Field*

2. Suppose X is a vector field on U and let $f : U \to \mathbb{R}$ be a function, then we can define a vector field on U denoted fX by

$$(fX)_p = f(p)X_p \in T_p\mathbb{R}^n, \forall p \in U$$

Definition 3.2.2 Let $\{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{R}^n$ the constant vector field corresponding to e_i is denoted $\frac{\partial}{\partial x_i}$. We mean that

$$\frac{\partial}{\partial x_i}$$

is a vector field on $\mathbb{R}^n$ defined by

$$\left.\frac{\partial}{\partial x_i}\right|_p = (p, e_i), \forall p \in \mathbb{R}^n$$

3. Suppose X, Y both vector fields on U, we define a vector field on U, denoted $X+Y$ by

$$(X+Y)_p = X_p + Y_p$$

since $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$, it follows that $\{(p, e_1), \dots, (p, e_n)\}$ is a basis of $T_p\mathbb{R}^n$. Let X be a vector field on U and let $p \in U$,

$$T_p\mathbb{R}^n \ni X_p = \sum_{i=1}^{n} a_i(p)(p, e_i)$$

where $a_i : U \to \mathbb{R}$ function for all $1 \leq i \leq n$. Then,

$$X_p = \sum_{i=1}^{n} a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p = \left(\sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}\right)_p$$

hence, we can write

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

Definition 3.2.3 We say X is a C^∞ smooth vector field if all the component functions $a_i \in C^\infty(U)$.

■

■ **Example 3.2** Let $n = 2$ and $X_1 = x$ and $X_2 = y$, then

$$X = x^2 y \frac{\partial}{\partial x} - e^y \cos(x) \frac{\partial}{\partial y}$$

be the a smooth vector field on $\mathbb{R}^2$ at $(2, \pi)$,

$$X_{(2,\pi)} = 4\pi \left.\frac{\partial}{\partial x}\right|_{(2,\pi)} - e^2 \left.\frac{\partial}{\partial y}\right|_{(2,\pi)}$$

■

3.3 Lie Differentiation

Definition 3.3.1 Lie Differentiation of a function by a tangent vector . Let $X_p \in T_p\mathbb{R}^n$. (In general, X_p will be the value at $p \in U$ of some vector field on U but it needs not be). Let $f : U \to \mathbb{R}$ be a smooth function defined on some open set containing p, we define

$$\mathscr{L}_{X_p} f$$

to be the real number

$$\mathscr{L}_{X_p} f = (df)_p(X_p) = a_1(p) \frac{\partial f}{\partial x_1}(p) + \cdots + a_n(p) \frac{\partial f}{\partial x_n}(p)$$

This is the **directional derivative** of f at p, in the direciton of $X_p = (p, (a_1, \dots, a_n))$. Thus,

$$\left.\frac{d}{dt}\right|_{t=0} f(p, (a_1, \dots, a_n))$$

$$= \lim_{t \to 0} \frac{f(p,(a_1,\dots,a_n)) - f(p)}{t}$$

(if $X_p = (p,(a_1,\dots,a_n)) = a_1(p)\frac{\partial}{\partial x_1}\Big|_p + \cdots + a_n(p)\frac{\partial}{\partial x_n}\Big|_p$)

R If

$$X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\Big|_p$$

then,

$$\mathscr{L}_{X_p} f = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}(p) \in \mathbb{R}$$

this is called the Lie derivative of f in X_p direction. More generally, let X be a vector field on U and let $f \in C^\infty(U)$, we define $\mathscr{L}_X f$ to be a function from U to $\mathbb{R}$ given by

$$p \mapsto \mathscr{L}_{X_p} f := \left(\mathscr{L}_{X_p} f\right)(p) \in \mathbb{R}$$

explicitly, if $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, a_i : U \to \mathbb{R}$, then

$$\mathscr{L}_{X_p} f = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}$$

is a smooth function from U to $\mathbb{R}$ as well.

We use $\mathfrak{X}(U)$ to denote the space of smooth vector fields on U. If $f \in C^\infty(U)$, then $fX \in \mathfrak{X}(U)$ and $(fX)_p = f(p)X_p$. We can also compute

$$\mathscr{L}_X f \in C^\infty(U)$$

Proposition 3.3.1 let $f,g \in C^\infty(U)$, let $X \in \mathfrak{X}(U)$, then

$$\mathscr{L}_X(fg) = (\mathscr{L}_X f)g + f(\mathscr{L}_X g)$$

Proof. Let $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, then

$$\begin{aligned}
\mathscr{L}_X(fg) &= \sum_{i=1}^n a_i \frac{\partial fg}{\partial x_i} \\
&= \sum_{i=1}^n a_i \left[\frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}\right] \\
&= (\mathscr{L}_X f)g + f(\mathscr{L}_X g)
\end{aligned}$$

■

3.4 Cotangent Vectors & 1-Forms

Definition 3.4.1 Let $p \in \mathbb{R}^n$, we define the **cotangent space** to $\mathbb{R}^n$ at p to be the dual space of $T_p\mathbb{R}^n$. And we denote it $T_p^*\mathbb{R}^n$ where

$$T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$$

space of linear maps from $T_p\mathbb{R}^n$ to $\mathbb{R}$.

Definition 3.4.2 Let $U \in \mathbb{R}^n$ be open. A 1-form α on U is a map

$$p \mapsto \alpha_p \in T_p^*\mathbb{R}^n, \forall p \in U$$

(a 1-form on U is also claled a contangent vector field on U. An element $\alpha_p \in T_p^*\mathbb{R}^n$ is called a cotangent vector at p).

■ **Example 3.3** Let $f \in C^\infty(U)$ define a 1-form df on U

$$(df)_p : T_p\mathbb{R}^n \to \mathbb{R}$$

thus, $(df)_p \in T_p^*\mathbb{R}^n$ and df is a 1-form on U. ■

Proposition 3.4.1 — Properties of 1-Forms. Let α, β be 1-forms on U and let h be real-valued function on U, define

1. Define
$$(\alpha+\beta)_p = \alpha_p + \beta_p \in T_p^*\mathbb{R}^n$$
2. Define $h\alpha$ is a 1-form on U
$$(h\alpha)_p = h(p)\alpha_p \in T_p^*\mathbb{R}^n$$

Special Case: let $x_i \in C^\infty(U)$ be the i-th coordinate function

$$x_i(p) = p_i, p = (p_1, \dots, p_n)$$

R Recall previous example, dx_i is a 1-form on U,

$$(dx_i)_p : T_p\mathbb{R}^n \to \mathbb{R}$$

then,

$$(dx_i)_p(X_p) = (Dx_i)_p(v), X_p = (p, v)$$

equivalently

$$\left(\frac{\partial x_i}{\partial x_1}(p), \dots, \frac{\partial x_i}{\partial x_n}(p)\right) = (0, \dots, 1, \dots, 0)$$

the i-th component. Then,

$$(dx_i)_p\left(\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\bigg|_p\right) = a_i$$

hence, $(dx_i)_p\left(\frac{\partial}{\partial x_i}\bigg|_p\right) = \delta_{ij}$. we see that

$$\{(dx_1)_p, \dots, (dx_n)_p\}$$

is the dual basis of $\left\{\frac{\partial}{\partial x_1}\bigg|_p, \dots, \frac{\partial}{\partial x_n}\bigg|_p\right\}$.

Therefore, if α is any 1-form on U, we can write

$$\alpha = \sum_{i=1}^n s_i dx_i$$

where $s_i : U \to \mathbb{R}$ functions. For $p \in U$,

$$\alpha_p = \sum_{i=1}^n s_i(p)(dx_i)_p \in T_p^*\mathbb{R}^n$$

we say α is a smooth 1-form on U if $S_i \in C^\infty(U)$.

Let α be a 1-form on U and let X be a vector field on U (both are not necessarily smooth). Then, $\forall p \in U$, $X_p \in T_p\mathbb{R}^n$ and $\alpha_p \in T_p^*\mathbb{R}^n$. Define $\alpha(X) : U \to \mathbb{R}$ function

$$(\alpha(X))(p) := \alpha_p(X_p) \in \mathbb{R}$$

if $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $\alpha = \sum_{j=1}^n s_j dx_j$. Then

$$(\alpha(X))(p) = \left(\sum_{j=1}^n s_j(p)(dx_j)_p\right)\left(\sum_{i=1}^n a_i(p)\frac{\partial}{\partial x_i}\bigg|_p\right) = \sum_{i=1}^n s_i(p)a_i(p)$$

thus,

$$\alpha(X) = \sum_{i=1}^n s_i a_i$$

hence, if X is smooth and α is smooth, then $\alpha(X) \in C^\infty(U)$. Let $\Omega^1(U)$ denote the **smooth 1-forms** on U, if $\alpha, \beta \in \Omega^1(U)$ and $\alpha + \beta \in \Omega^1(U)$, if $h \in C^\infty(U)$, then $h\alpha \in \Omega^1(U)$.
We also write

$$(\iota_X \alpha) = \iota_{X_p}\alpha_p = \alpha_p(X_p)$$

Special Case $\alpha = df$, $f \in C^\infty(U)$. And $\alpha_p = (df)_p : T_p\mathbb{R}^n \to \mathbb{R}$. Then,

$$(df)_p(X_p) = (Df)_p v$$

If $X_p = \frac{\partial}{\partial x_i}\Big|_p = (p, e_i)$. Then,

$$(df)_p\left(\frac{\partial}{\partial x_i}\bigg|_p\right) = \frac{\partial f}{\partial x_i}(p) \to df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

thus, $df \in \Omega^1(U)$.

Corollary 3.4.2 We also observe that if

$$X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$$

recall:

$$\mathscr{L}_X f = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i} = (df)(X) = \iota_X(df)$$

Definition 3.4.3 A **0-form** on U is a function $h : U \to \mathbb{R}$. This makes sense because 1-form at p is in $\wedge^1(T_p^*\mathbb{R}^n) = T_p^*\mathbb{R}^n$. Then,

$$\wedge^0(T_p^*\mathbb{R}^n) = \mathbb{R}$$

a smooth 0-form on U is just a smooth function on U

$$\Omega^0(U) = C^\infty(U)$$

Proposition 3.4.3 Let $f, g \in C^\infty(U) = \Omega^0(U)$, $fg \in C^\infty(U)$, then

$$\Omega^0(U) \ni d(fg) = fdg + gdf$$

Proof.

$$\begin{aligned} d(fg) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(fg)dx_i \\ &= \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i} g + f\frac{\partial g}{\partial x_i}\right] dx_i \end{aligned}$$

■

3.5 Pullbacks of 0-forms and 1-forms

3.5.1 Pullbacks of 0-forms

Let U be open in $\mathbb{R}^n$ and V be open in $\mathbb{R}^m$ and $F : U \to V$ be smooth. (F needs not be injective nor surjective) Let $h \in \Omega^0(V)) = C^\infty(V)$, we define the pullback of h by F, denoted by F^*h to be the function on U

$$F^*h = h \circ F : U \to \mathbb{R}$$

Moreover,

$$\forall h \in \Omega^0(V), F^*h \in \Omega^0(V)$$

and

$$(F^*h)(p) = h(F(p))$$

The following is clear: for $a, b \in \mathbb{R}$, $h, g \in \Omega^0(V)$, then

$$F^*(ah + bg) = aF^*h + bF^*g$$

also,

$$F^*(gh) = (F^*g)(F^*h)$$

3.5.2 Pullbacks of 1-forms

Let $\omega \in \Omega^1(V)$ we want to define $F^*\omega$ to be a 1-form on U. If $p \in U$, $(dF)_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$. Then, the dual map is

$$(dF)_p^* : T_{F(p)}^*\mathbb{R}^m \to T_p^*\mathbb{R}^n$$

define

$$(F^*\omega)_p := (dF)_p^* \omega_{F(p)} \in T_p^*\mathbb{R}^n$$

equivalently,

$$(F^*\omega)_p X_p = \omega_{F(p)}((dF)_p X_p)$$

Proposition 3.5.1 Clearly,

$$F^*(\omega_1 + \omega_2) = F^*\omega_1 + F^*\omega_2$$

and for $s \in C^\infty(V)$

$$\begin{aligned} F^*(s\omega)_p &= (dF)_p^*((S\omega)_{F(p)}) \\ &= (dF)_p^*\big(S(F(p))\omega_{F(p)}\big) \\ &= S(F(p))\big((dF)_p^*\omega_{F(p)}\big) \\ &= (S \circ F)(p)(F^*\omega)_p \\ &= [(F^*S)F^*\omega]_p \end{aligned}$$

Special Case: let $h \in \Omega^0(V) = C^\infty(V)$ and $dh \in \Omega^1(V)$. F^*dh is a one-form on U with $p \in U$,

$$(F^*dh)_p(X_p) = (dh)_{F(p)}((dF)_p X_p) = \big((dh)_{F(p)}(dF)_p\big)(X_p)$$

$$= [d(h \circ F)]_p(X_p), \forall X_p \in T_p\mathbb{R}^n$$

Thus,

$$(F^*dh)_p = [d(h \circ F)]_p, \forall p \in U$$

hence,

$$F^*dh = d(h \circ F) = d(F^*h)$$

Hence, if $h \in C^\infty(U)$,

$$F^*(dh) = d(F^*h) \in \Omega^1(U)$$

Corollary 3.5.2

$$F^* : \Omega^1(V) \to \Omega^1(U)$$

Proof. Let $\alpha = \sum_{i=1}^n s_i dx_i$ and $F^*\alpha = \sum_{i=1}^n (S \circ F) d(X_i \circ F) \in \Omega^1(U)$ ■

3.6 Differential k-forms

Definition 3.6.1 Let U be open in $\mathbb{R}^n$ with $k \geq 1$. A k-form on U is a mapp ω such that ω sends $p \in U$ to $\omega_p \in \wedge^k(T_p^*\mathbb{R}^n)$

Concretely, by identifying the canonical isomorphism

$$\wedge^k(T_p^*\mathbb{R}^n) \cong \mathscr{A}^k(T_p\mathbb{R}^n)$$

$\omega \in \mathscr{A}^k(T_p\mathbb{R}^n)$. Since $\{(dx_1)_p, \ldots, (dx_n)_p\}$ is a basis of $T_p^*\mathbb{R}^n$, we know that

$$\{(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p, 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

is a basis of $\wedge^k(T_p^*\mathbb{R}^n)$ where $I = (i_1, \ldots, i_k)$ is the strictly increasing multindex. Then, define

$$(dx_I)_p := (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p$$

hence, any k-form ω on U can be written as

$$\omega = \sum_I C_I dx_I$$

we say ω is a smooth f-form on U if $C_I \in C^\infty(U) \forall I$.

Proposition 3.6.1 If $\omega, \omega_1, \omega_2$ k-forms on U and f is a function on U where $\omega_1 + \omega_2$ is k-form on U. And $(f\omega)_p = f(p)\omega_p$. Let $\Omega^k(U)$ be the smooth k-forms on U. $\Omega^k(U)$ is a infinite dimensionnal real vector spae and a module on $C^\infty(U)$.

Proposition 3.6.2 Given α is a k-form on U and β is a l-form on U, we define $\alpha \wedge \beta$ to be a $k+l$ form on U given by

$$(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p$$

wedge product is bilinear.

Proof.

$$\alpha = \sum_I \alpha_I dx_I \qquad \beta = \sum_J \beta_J dx_J$$

then,

$$\alpha \wedge \beta = \sum_{I,J} \alpha_I \beta_J dx_I \wedge dx_J$$

each is either 0 or $\pm dx_M$ for some strictly increasing multindex M of length $k+l$. Hence, coefficients of $\alpha \wedge \beta$ with respect to the dx_M's are the sums of products of smooth functions. ■

■ **Example 3.4** Let $U = \mathbb{R}^3$ and $x_1 = x, x_2 = y, x_3 = z$.

$$\begin{aligned}
\alpha &= x^2 dy - \sin(y)dz \\
\beta &= e^y dx \wedge dz + z dy \wedge dz \\
\gamma &= dx
\end{aligned}$$

α, γ are smooth 1-forms and β is a smooth two-form

$$\begin{aligned}
\alpha \wedge \beta &= (x^2 dy - \sin(y)dz)(e^y dx \wedge dz + z dy \wedge dz) \\
&= (x^2 dy) \wedge (e^y dx \wedge dz) + (x^2 dy) \wedge (z dy \wedge dz) \\
&- (\sin(y)dz) \wedge (e^y dx \wedge dz) - (\sin(y)dz) \wedge (z dy \wedge dz) \\
&= -x^2 e^y dx \wedge dy \wedge dz
\end{aligned}$$

$$\begin{aligned}
\alpha \wedge \gamma &= x^2 dy \wedge dx - \sin(y)dz \wedge dx \\
&= -x^2 dx \wedge dy + \sin(y)dx \wedge dz
\end{aligned}$$

$$\beta \wedge \gamma = z dx \wedge dy \wedge dz$$

■

3.7 Exterior Derivative

Definition 3.7.1 — Exterior Derivative. Let U in $\mathbb{R}^n$ be open. We seek to derive a map

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

called the exterior derivative

1. When $k = 0$, $\Omega^0(U) = C^\infty(U)$ then

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

2. d is linear
3. **Product Rule:** for $\alpha \in \Omega^k(U), \beta \in \Omega^l(U)$

$$d(\alpha \wedge \beta) = (dx) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$$

4. $d^2 = 0$ which means

$$d(d\alpha) = 0, \forall \alpha \in \Omega^k(U)$$

We will first show such a map is unique if it exists. Observe

$$dx_i = d(x_i)$$

hence, by 4

$$d^2 x_i = d(dx_i) = 0$$

by 2

$$d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = 0$$

Let $\alpha \in \Omega^k(U)$

$$\alpha = \sum_I \alpha_I dx_I$$

where

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

it's irrelevant whether or not I is strictly increasing.

$$\begin{aligned}
d\alpha &= \sum_I d(\alpha_I dx_I) && \textbf{Linearity} \\
&= \sum_I (d\alpha_I) \wedge dx_I + \sum_I \alpha_I \wedge d(dx_I) && \textbf{Product Rule} \\
&= \sum_I (d\alpha_I) \wedge dx_I && \textbf{4} \\
&= \sum_{I,l} \frac{\partial \alpha_I}{\partial x_l} dx_l \wedge dx_I = d\alpha
\end{aligned}$$

Say

$$\sum_{I,l} \frac{\partial \alpha_I}{\partial x_l} dx_l \wedge dx_I$$

is our candidate

Proof. Property 1 is by definition when $k = 0$

1. **Property 2** let $\alpha = \sum_I \alpha_I dx_I$ and $\beta = \sum_I \beta_I dx_I$, I str inc, then

$$\begin{aligned}
d(t\alpha + s\beta) &= d\left(\sum_I (t\alpha_I + s\beta_I) dx_I\right) \\
&= \sum_I \frac{\partial}{\partial x_l}(t\alpha_I + s\beta_I) dx_l \wedge dx_I \\
&= t d\alpha + s d\beta
\end{aligned}$$

2. **Property 3** let $\alpha = \sum_I \alpha_I dx_I$ and $\beta = \sum_J \beta_J dx_J$,

$$\alpha \wedge \beta = \sum_{I,J} \alpha_I \beta_J dx_I \wedge dx_J$$

note that

$$dx_I \wedge dx_J$$

is either 0 or $\pm dx_M$ for some multi-index M of length $k+l$.

$$\begin{aligned}
d(\alpha\wedge\beta) &= \sum_{I,J,a}\frac{\partial}{\partial x^a}(\alpha_I\beta_J)dx_a\wedge dx_I\wedge dx_J\\
&= \sum_{I,J,a}\left(\frac{\partial\alpha_I}{\partial x^a}\beta_J+\alpha_I\frac{\partial\beta_J}{\partial x^a}\right)dx_a\wedge dx_I\wedge dx_J\\
&= \left(\sum_{I,a}\left(\frac{\partial\alpha_I}{\partial x^a}dx_a\wedge dx_I\right)\right)\wedge\left(\sum_J\beta_J dx_J\right)+(-1)^k\left(\sum_I\alpha_I dx_I\right)\wedge\left(\sum_{J,a}\frac{\partial\beta_J}{\partial x^a}dx_a\wedge dx_J\right)\\
&= (d\alpha)\wedge\beta+(-1)^k\alpha\wedge d\beta
\end{aligned}$$

3. **Property 4** for $\alpha=\sum_I\alpha_I dx_I$ and $d\alpha=\sum_I(d\alpha_I)\wedge dx_I$ then,

$$d^\alpha = d(d\alpha)=\sum_I\left(d^2\alpha_I\right)\wedge dx_I$$

this will be done if we can show $d^2f=0$. True by Clairaut Theorem.

■

■ **Example 3.5** 1.

$$\begin{aligned}
d\alpha &= d(x^2)\wedge dy - d(\sin(y))\wedge dz\\
&= 2xdx\wedge dy-\cos(y)dy\wedge dz
\end{aligned}$$

2.

$$\begin{aligned}
d\beta &= e^y dy\wedge dx\wedge dz+dz\wedge dy\wedge dz\\
&= e^y dy\wedge dx\wedge dz
\end{aligned}$$

■

Since $d^2=0$, this means the image of d is contained in the kernel of d.

Definition 3.7.2 — Closed and Exact. A smooth k-form α on U is called **closed** if $d\alpha=0$, $\alpha\in\ker(d)$;
It is called **exact** if $\alpha=d\sigma$ for some $\sigma\in\Omega^{k-1}(U)$, so $\alpha\in\operatorname{im}(d)$.

$d^2=0$ says any exact form is closed. But **are all closed forms exact?** Answer is no. By Assignment 2, there exists a closed 1-form α on $U=\mathbb{R}^2\backslash\{0\}$

$$\alpha=\frac{xdy-ydx}{x^2+y^2}$$

and α is not exact on U, but it is exact on a smaller open set.

Definition 3.7.3 — k-th De Rham Cohomology of U. Let $U\subseteq\mathbb{R}^n$ be open

$$Z^k(U)=\left\{\alpha\in\Omega^k(U):d\alpha=0\right\}$$

the set of closed k-forms and

$$B^k(U)=\left\{\alpha\in\Omega^k(U):\alpha=d\sigma,\sigma\in\Omega^{k-1}(U)\right\}$$

the set of exact k-forms with $B^0(U)=\{0\}$. Then, clearly

$$B^k(U)\subseteq Z^k(U)\subseteq\Omega^k(U)$$

subspaces. Let

$$H^k(U) = Z^k(U) \Big/ B^k(U)$$

is the k-th de Rham cohomology of U.

$H^k(U)$ is a real vector space depending on k and U. And $H^k(U) = \{0\}$ if and only if any closed k-form on U must be exact. such as

$$H^1\left(\mathbb{R}^2 \backslash \{0\}\right) \neq \{0\}$$

de Rham cohomology $H^k(U)$ measure global topology of U. Aside if $U = \mathbb{R}^n$, then

$$H^k(\mathbb{R}^n) = \{0\}, \forall k > 0$$

and $H^0(\mathbb{R}^n) \cong \mathbb{R}$.

Theorem 9 — The Poincare Lemma. Let $U \subseteq \mathbb{R}^n$ be open and $k \geq 1$. Suppose $\omega \in \Omega^k(U)$ and $d\omega = 0$. Then, for all $p \in U$, there exists an open neighbourhood W of $p \in U$ such that ω is exact on W. This means $p \in W \subseteq U$, there exists $\sigma \in \Omega^{k-1}(W)$ such that

$$\omega\big|_W = d\sigma$$

taking the inclusion map $\iota : W \to U$.

Proof. Let $p \in (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \subseteq U$. In this case, we suppose $k = n$ and let

$$\omega = f dx_1 \wedge \cdots \wedge dx_n, f \in C^\infty(U)$$

we invoke the following lemma to finish the proof. ■

Lemma 3.8 Let $V \subseteq \mathbb{R}^{n-1}$ and $A \subseteq \mathbb{R}$ be an open interval. If any closed k-form on V is exact, then any closed k-form or $V \times A \subseteq \mathbb{R}^n$ is exact. $k \geq 1$

Proof. Let $\omega \in \Omega^k(V \times A)$ such that $d\omega = 0$, we need to show $\omega\sigma$ for some $\sigma \in \Omega^{k-1}(V \times A)$. For α as a (k-1)-form on $V \times A$ and β as a k-form on $V \times A$, both have no dt.

$$\omega = dt \wedge \alpha + \beta$$

$$\begin{aligned}
0 = d\omega &= -dt \wedge d\alpha + d\beta \\
&= -dt \wedge \left[\frac{\partial \alpha_I}{\partial t} dt \wedge dx_I + \sum_{l=1}^{n-1} \frac{\partial \alpha_I}{\partial x_l} dx_l \wedge d\alpha_I\right] \\
&+ \left[\frac{\partial \beta_J}{\partial t} dt \wedge dx_J + \sum_{l=1}^{n-1} \frac{\partial \beta_J}{\partial x_l} dx_l \wedge d\beta_J\right]
\end{aligned}$$

$$0 = d\omega = -dt \wedge [d_v\alpha] + dt \wedge \frac{\partial \beta}{\partial t} + d_v\beta$$

$$d_v\alpha = \sum_I \sum_{l=1}^{n-1} \frac{\partial \alpha_I}{\partial x^l} dx_l \wedge dx_I$$

$$d_v\beta = \sum_J \sum_{l=1}^{n-1} \frac{\partial \beta_J}{\partial x^l} dx_l \wedge dx_J$$

then,

$$\frac{\partial \beta}{\partial t} = \sum_J \frac{\partial \beta_J}{\partial t} dx_J$$

where d_v is the exterior derivative on forms on V treating t as an independent parameter. $\frac{\partial}{\partial t}$ is the partial differentiation with respect to this parameter t. Then,

$$0 = dt \wedge \left[d_v \alpha - \frac{\partial \beta}{\partial t} \right] d_v \beta \longrightarrow d_v \beta = 0, d_v \alpha = \frac{\partial \beta}{\partial t}$$

Then,

$$\alpha = \sum_I \alpha_I(x_1, \ldots, x_{n-1}, t) dx_I$$

define $\mu \in \Omega^{k-1}(V \times A)$ by

$$\mu = \sum_I \left(\int_a^t \alpha_I(x_1, \ldots, x_{n-1}, s) ds \right) dx_I \in \Omega^{k-1}(V \times A)$$

By construction of μ,

$$\frac{\partial \mu}{\partial t} = \alpha$$

by FTC. Thus,

$$d\mu = dt \wedge \frac{\partial \mu}{\partial t} + d_v \mu = dt \wedge \alpha + d_v \mu$$

define $\tilde{\omega} = \omega - d\mu = dt \wedge \alpha + \beta - (dt \wedge \alpha + d_v \mu) = \beta - d_v \mu$ Then, note that

$$d\tilde{\omega} = d(\omega - d\mu) = d\omega - d^2 \mu = 0$$

since $dt \wedge \frac{\partial \tilde{\omega}}{\partial t} + d_v \tilde{\omega} = 0$. Coefficients of $\tilde{\omega}$ are independent of t and $\tilde{\omega}$ has no dt's. Thus, $\tilde{\omega}$ is actually a smooth k-form on V. By hypothesis of lemma, there exists $\tilde{\sigma} \in \Omega^{k-1}(V)$ such that $\tilde{\omega} = d\tilde{\sigma}$. and

$$\omega = \tilde{\omega} + d\mu = d\tilde{\sigma} + d\mu = d(\mu + \tilde{\sigma})$$

Thus, $\sigma = \mu + \tilde{\sigma} \in \Omega^{k-1}(V \times A)$ and $\omega = d\sigma$. ■

3.9 Interior Product on k-Forms

Definition 3.9.1 — Interior Product on k-Forms. Let $X \in \mathfrak{X}(U)$ and $U \subseteq \mathbb{R}^n$ be open. Let $\omega \in \Omega^k(U)$, we define $\iota_X(U)$ called the interior product with ω to be the $(k-1)$-form on U given by

$$(\iota_X \omega)_p = \iota_{X_p} \omega_p \in \wedge^{k-1}(T_p^* \mathbb{R}^n)$$

since $\iota_{X_p} \in T_p \mathbb{R}^n$ and $\omega_p \wedge^k (T_p^* \mathbb{R}^n)$

Proposition 3.9.1

$$\iota_X \omega \in \Omega^{k-1} U$$

Proof. Let $\omega = \sum_I \omega_I dx_I$ where I is a multi-index. Let $X = \sum_l a_l \frac{\partial}{\partial x_l}$. Then,

$$\begin{aligned} \iota_X \omega &= \iota_{X_p} \omega_p \\ &= \sum_{I,l} a_l \omega_I \left(\iota_{\frac{\partial}{\partial x_l}} dx_I \right) \in \Omega^{k-1} U \end{aligned}$$

since $a_l \omega I \in C^\infty(U)$ and $\iota_{\frac{\partial}{\partial x_l}} dx_I$ is either 0 or $\pm dx_M$ with the length of M is $k+l$ as a multi-index. ■

Proposition 3.9.2 — Properties of Interior Product on k-Forms.
1. $\iota_X(\omega_1+\omega_2)=\iota_X\omega_1+\iota_X\omega_2$
2. $\iota_{X_1+X_2}\omega=\iota_{X_1}\omega+\iota_{X_2}\omega$
3. $\iota_{fX}\omega=\iota_X(f\omega)=f(\iota_v\omega), f\in C^\infty(U)$
4. $$\iota_X(\omega\wedge\eta)=(\iota_X\omega)\wedge\eta+(-1)^{|\omega|}\omega\wedge(\iota_X\eta)$$
5. $(\iota_X)^2=0$ and $\iota_X\iota_Y=-\iota_Y\iota_X$
6. $\iota_X f=0$ if $f\in\Omega^0(U)=C^\infty(U)$

Proof. True because they hold at every $p\in U$ by properties of interior product from Chapter 1 ■

3.10 Lie Derivative on Forms

Definition 3.10.1 — Lie Derivative. The Lie derivative of ω with respect to X is denoted to be $\mathscr{L}_X\omega$ and defined by

$$\mathscr{L}_X\omega=\iota_X(d\omega)+d(\iota_X\omega)\in\Omega^k(U)$$

clearly,
1. $\mathscr{L}_X:\Omega^k(U)\to\Omega^k(U)$
2. $\mathscr{L}_X$ is real linear

R **Special Case:** when $k=0$, $f\in\Omega^0(U)=C^\infty(U)$. Then,

$$\begin{aligned}\mathscr{L}_X f&=d(\iota_X f)+\iota_X(df)\\&=0+\iota_X(df)\\&=\iota_X(df)=(df)X=\mathscr{L}_X f=Xf\end{aligned}$$

Proposition 3.10.1

$$\mathscr{L}_X(\omega\wedge\eta)=(\mathscr{L}_X\omega)\wedge\eta+\omega\wedge(\mathscr{L}_X\eta)$$

Proof. Assignment 3 ■

R When $k=0$, $\mathscr{L}_X f$ is just differentiate f. But if $k\geq 1$, then $\mathscr{L}_X\omega$ differentiate both ω and X. (This is not a "directional derivative of ω in X direction")

Proposition 3.10.2

$$d\mathscr{L}_X\omega=\mathscr{L}_X d\omega,\forall\omega\in\Omega^k(U)$$

and

$$\iota_X\mathscr{L}_X\omega=\mathscr{L}_X\iota_X\omega,\forall\omega\in\Omega^k(U)$$

and

$$\mathscr{L}_X\iota_Y\neq\iota_Y\mathscr{L}_X$$

if $X\neq Y$

Proof. Assignment 3 ■

■ **Example 3.6** Let $\omega=dx_1\wedge\cdots\wedge dx_n\in\Omega^n(\mathbb{R}^n)$ and $d\omega=0$ and $X=\sum_{i=1}^n a_i\frac{\partial}{\partial x_i}$, then

$$\mathscr{L}_X\omega=\iota_X d\omega+d(\iota_X\omega)$$

then,

$$\iota_X \omega = \sum_{i=1}^{n} a_i \iota_{\frac{\partial}{\partial x_i}}(dx_1 \wedge \cdots \wedge dx_n) = \sum_{i=1}^{n} a_i (-1)^{i-1} dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$$

Then,

$$\begin{aligned}
\mathscr{L}_X \omega &= d(\iota_X \omega) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial a_j}{\partial x_i} dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n \\
&= \left(\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n = (\mathbf{div}(X))\omega
\end{aligned}$$

■

3.11 Pullback of k-Forms

Definition 3.11.1 — Pullback of k-Forms. Let $U \subseteq \mathbb{R}^n$ be open and $V \subseteq \mathbb{R}^m$ open and $F : U \to V$ smooth. Let $p \in U$, then

$$(dF)_p : T_p\mathbb{R}^n \xrightarrow{\text{linear}} T_{F(p)}\mathbb{R}^m$$

we define the pullback by F to be the map

$$\omega \in \Omega^k(V) \mapsto F^*\omega \in \Omega^k(U)$$

define by $(F^*\omega)_p = (dF)_p^* \omega_{F(p)} \in \wedge^k(T_p^*\mathbb{R}^n)$. $F^*\omega$ is a k-form on V, still need to show $F^*\omega$ is smooth.

Proposition 3.11.1 — Properties of Pullback of k-Forms.

1. $$F^*(t_1\omega_1 + t_2\omega_2) = t_1 F^*\omega_1 + t_2 F^*\omega_2$$

   for $\omega_1, \omega_2 \in \Omega^k(V)$ and $t_1, t_2 \in \mathbb{R}$.
2. $$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$$
3. If $W \subseteq \mathbb{R}^q$ is open, and $G : V \to W$ is smooth, then

   $$(G \circ F)^*\sigma = F^*G^*\sigma, \sigma \in \Omega^k(V)$$

Proof. All these holds automatically because they hold point wise. ■

Theorem 10 — Explicit Formula of Pullback. Let's derive an explicit formula for pullback which will show pullback of a smooth forms is smooth.
For multi-index $I = \{i_1, \ldots, i_k\}$

$$\begin{aligned}
\Omega^k(V) \ni \omega &= \sum_I \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
F^*\omega &= \sum_I (F^*\omega_I)(F^*dx_{i_1}) \wedge \cdots \wedge (F^*dx_{i_k}) \\
&= \sum_I (\omega_I \circ F) d(x_{i_1} \circ F) \wedge \cdots \wedge d(x_{i_k} \circ F) \in \Omega^k(V)
\end{aligned}$$

Proposition 3.11.2

$$d(F^*\omega) = F^*(d\omega), \forall \omega \in \Omega^k(V)$$

it is worth noting that the d on the LHS is the differential on U and the d on the RHS is the differential on V.

Proof.

$$\begin{aligned} d(F^*\omega) &= d\left(\sum_I (\omega_I \circ F) d(x_{i_1} \circ F) \wedge \cdots \wedge d(x_{i_k} \circ F)\right) \\ &= \sum_I d(\omega_I \circ F) d(x_{i_1} \circ F) \wedge \cdots \wedge d(x_{i_k} \circ F) \in \Omega^k(V) \quad d(dx_{i_k}) = 0 \end{aligned}$$

while,

$$\begin{aligned} d\omega &= \sum_I (d\omega_I) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ F^*(d\omega) &= \sum_I d(\omega_I \circ F) d(x_{i_1} \circ F) \wedge \cdots \wedge d(x_{i_k} \circ F) \in \Omega^k(V) \end{aligned}$$

Thus,

$$d(F^*\omega) = F^*(d\omega)$$

■

■ **Example 3.7** This example will be important for the change of variable theorem in Chapter 3. Let U, V be open subsets of $\mathbb{R}^n$ and let $F : U \to V$ be smooth. Then,

$$F^*(dx_1 \wedge \cdots \wedge dx_n) = \det\left(\frac{\partial F_i}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n$$

this is shown in Assignment 3. ■

Appendix A: Bump Functions & Partition of Unity

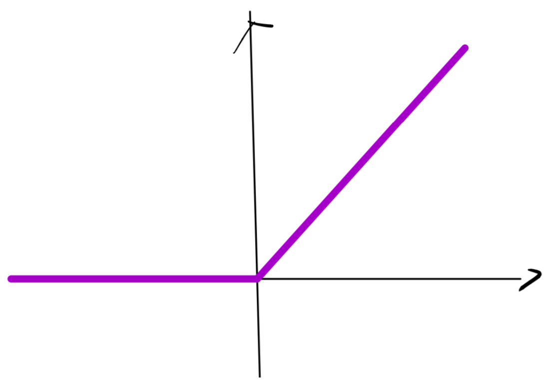
Motivation:
we have Poincare Lemma that can give pretty good local properties. We need machinery to break down global states to local states.

3.12 Basic Ingredients

Lemma 3.13 There exists a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that

1. $f(x) > 0, \forall x > 0$
2. $f(x) = 0, \forall x \leq 0$

(Note that such a function is definitely not unique)



Proof. Define

$$\rho(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

we need to show $\rho^{(k)}(x)$ exists and continuous for all $x \in \mathbb{R}$.

1. Clearly smooth at $x > 0$ and $x < 0$, we need to show $\rho^{(k)}(0) = 0$
2. We have for $x > 0$,
$$\rho^{(k)}(x) = \frac{e^{-\frac{1}{x}} P_k(x)}{x^{2k}}$$
where P_k is some polynomial. (This is an induction proof, exercise), then by L'Hopital
$$\lim_{x \to 0^+} \frac{e^{-\frac{1}{x}} P_k(x)}{x^{2k}} = 0$$

■

R This cannot be done in the category of analytic functions. Using ρ, we can build many useful smooth functions.

■ **Example 3.8** Let $a > 0$, define $\rho_a : \mathbb{R} \to \mathbb{R}$ by
$$\rho_a(x) = \frac{\rho(x)}{\rho(x) + \rho(a - x)}$$
note that $\rho_a(x)$'s denominator is never 0 and it satisfies two properties
1. $\rho_a(x) = 0, \forall x \leq 0$
2. $0 \leq \rho_a(x) \leq 1, \forall x \in \mathbb{R}$
3. $\rho_a(x) = 1, \forall x \geq a$

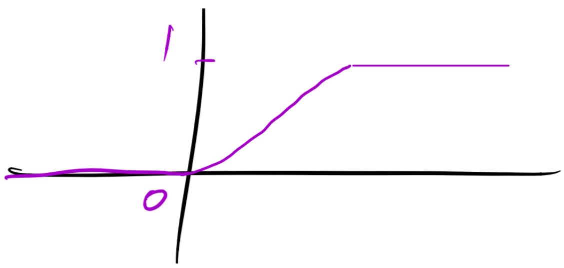


Figure 3.13.1: $\rho_a 9(x)$

■

■ **Example 3.9** For $I = (a, b) \subseteq \mathbb{R}$. Let $\rho_I(x) = \rho(x - a)\rho(b - x)$. We can see that $\rho_I \in C^\infty(\mathbb{R})$. If $x \in (a, b)$ we have $x - a > 0$ and $b - x > 0$, so $\rho_I(x) > 0$. Otherwise, $\rho_I(x) = 0$. ■

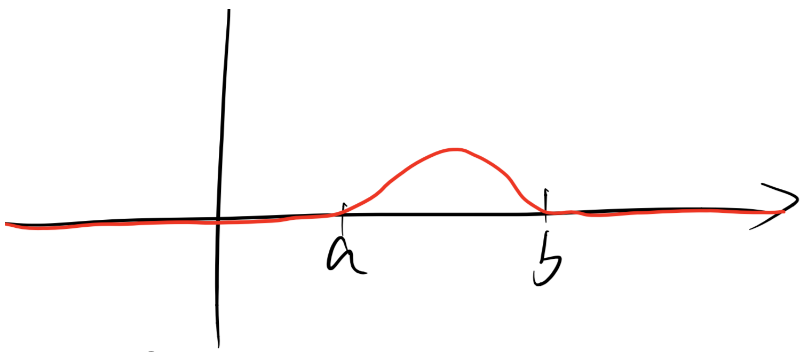


Figure 3.13.2: *Bump Function*

■ **Example 3.10 — General Bump Function.** Let $I_1, \dots, I_n$ be open intervals in $\mathbb{R}$ with $I_k = (a_k, b_k)$ for $1 \le k \le n$. Then, $Q = I_1 \times \cdots \times I_n$ is an open rectangle.

$$x \in Q \iff a_k < x_k < b_k, \forall 1 \le k \le n$$

we define $\rho_Q : \mathbb{R}^n \to \mathbb{R}$ by $\rho_Q(x) = \rho_{I_1}(x_1) \dots \rho_{I_n}(x_n)$. We can see $\rho_Q \in C^\infty(\mathbb{R}^n)$ and again

1. $\rho_Q(x) > 0 \iff x \in Q$
2. $\rho_Q(x) = 0 \iff x \notin Q$

■

3.14 Compactly Supported Functions

Definition 3.14.1 — Support. Let $U \subseteq \mathbb{R}^n$ be open and $f \in C^\infty(U)$. The support of f defined to be

$$\mathbf{supp}(f) = \overline{\{x \in U : f(x) \neq 0\}}$$

Definition 3.14.2 — Space of Compactly Supported Functions.

$$C_0^\infty(U) = \{f \in C^\infty(U) : \text{ such that } \mathbf{supp}(f) \text{ is compact and contained in } U\}$$

or called as the space of smooth functions on U that are compactly supported in U.

3.14.1 Some Results From Real Analysis

Theorem 11 — Compactness in $\mathbb{R}^n$. A subset $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Definition 3.14.3 — Compactness. If $K \subseteq T$ a topological space, then K is compact means for every open cover of K, K admits a finite subcover.

Theorem 12 — Heire-Borel Theorem. In any metric space T, $K \subseteq T$ is compact (in the topological notion) if and only if K is closed and bounded.

Lemma 3.15 Let $K \subseteq \mathbb{R}$ be compact in an open subset U. There exists $\phi \in C_0^\infty(U)$ such that

1. $\phi(x) \ge 0, \forall x \in U$
2. $\phi(x) > 0, \forall x \in K$

Proof. Let $p \in K$, there exists an open rectangle Q_p such that

$$p \in Q_p \subseteq \overline{Q_p} \subseteq U$$

then, $K \subseteq \bigcup_{p \in K} Q_p$ is an open cover of K. By compactness, there exists a finite subcover. Say $P_1, \dots, P_N \in K$, such that

$$K \subseteq \bigcup_{j=1}^{N} Q_{P_j}$$

let $\phi = \sum_{j=1}^{N} \rho_{Q_{P_j}} \in C^\infty(\mathbb{R}^n)$. Then, note that

$$\{x \in U : \phi(x) \neq 0\} \subseteq Q_{P_1} \cup \dots \cup Q_{P_N}$$

then,

$$\mathbf{supp}(\phi) \subseteq \overline{\{x \in U : \phi(x) \neq 0\}} \subseteq \overline{Q_{P_1} \cup \dots \cup Q_{P_N}} \subseteq \overline{Q_{P_1}} \cup \dots \cup \overline{Q_{P_N}} \subseteq U$$

thus, $\mathbf{supp}(\phi)$ is compact since closed subset of a compact set is compact. Thus, $\phi \in C_0^\infty(U)$. Also, by constrution, $\phi \ge 0$ on U. If $x \in K$, then $x \in Q_{P_j}$ for some j, so $\phi(x) > 0$. ■

Theorem 13 — Existence of Smooth Bump Function on a Compact Set. Let K be compact subset of an open set U in $\mathbb{R}^n$. There exists $\psi \in C_0^\infty(U)$ such that
1. $0 \le \psi(x) \le 1, \forall x \in U$
2. $\psi(x) = 1, \forall x \in K$

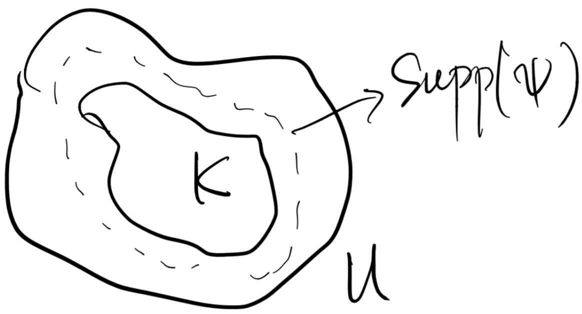


Figure 3.15.1: *Intuition for such a proof*

Proof. Let $\phi \in C_0^\infty(U)$ be as in the previous lemma. Define

$$a = \inf_{x \in K}\{\phi(x)\} > 0$$

since K is compact and ϕ is continuous. Then, by extreme value theorem and infinimum property, we should have this result. Let $\rho_a : \mathbb{R} \to \mathbb{R}$ be as what we constructed before. Then, clearly,
1. $\rho_a \in C^\infty(\mathbb{R})$
2. $0 \le \rho_a(t) \le 1, \forall t \in \mathbb{R}$
3. $\rho_a(t) = 0, t \le 0$
4. $\rho_a(t) = 1, t \ge a$

Let $\psi = \rho_a \circ \phi : U \to \mathbb{R}$. Thus, $\psi \in C^\infty(U)$ by composition. Then, by construction, we have
1. $0 \le \psi(x) \le 1, \forall x \in U$
2. For $x \in K$, $\psi(x) = \rho_a(\phi(x)) = 1$ since $\phi(x) \ge a$.
3. Suppose $\psi(x) = \rho_a(\phi(x)) \ne 0$, then $\phi(x) \ge a > 0$, then $\mathbf{supp}(\psi) \subseteq \mathbf{supp}(\phi) \subseteq U$

Thus, $\psi \in C_0^\infty(U)$. ■

3.16 Partition of Unity

3.16.1 Basic Set-up

Let $U \subseteq \mathbb{R}^n$ be open, let

$$\mathscr{U} = \{U_\alpha : \alpha \in A\}$$

be a collection of open sets covering U. This means, for all $\alpha \in A$, U_α is open and $U_\alpha \subseteq U$, then

$$U = \bigcup_{\alpha \in A} U_\alpha$$

A could be of any cardinality.

Theorem 14 — Point-Set Topology Theorem. Let $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ be as before, there exists a countable covering of U with open rectangle $\{Q_k : k \in A\}$ and $Q_k \subseteq U$. Say,

$$U = \bigcup_{k=1}^{\infty} Q_k$$

such that

A $\overline{Q_k} \subseteq U, \forall k \in A$

B For all $k \in \mathbb{N}$, there exists $\alpha(k) \in A$ such that $\overline{Q_k} \subseteq U_{\alpha(k)}$

C For all $p \in U$, there exists U_p open set, $p \in U_p \subseteq U$ and $N_p \in \mathbb{N}$ such that

$$Q_k \cup U_p = \emptyset, \forall k > N_p$$

Theorem 15 — Existence of a Partition of Unity Suborinate to $\mathscr{U}$. There exists a countable collection of functions ρ_k for some $k \in \mathbb{N}$ with $\rho_k : U \to \mathbb{R}$ such that

1. $\rho_k \in C_0^\infty(U)$ for all k
2. $\rho_k \geq 0$ on U for all k
3. For all $k \in \mathbb{N}$, there exists at least one $\alpha(k) \in A$ such that

$$\rho_k \in C_0^\infty\left(U_{\alpha(k)}\right)$$

4. For all $p \in U$, there exists U_p as an open subset with $p \in U_p \subseteq U$ and $N_p \in \mathbb{N}$ such that $\rho_k\big|_{U_p} = 0$ for all $k > N_p$.

   R This says that there exists an open neighbourhood of each $p \in U$ on which all but finitely many ρ_k will vanish.

5. $\sum_{k=1}^{\infty} \rho_k = 1$ as the constant function.

   R By 4, we should see that this is actually a finite sum. This construction is extremely non-unique.

3.16.2 Application of Partitions of Unity

Definition 3.16.1 Let $X \subseteq \mathbb{R}^n$ be any subsets (not necessarily open). Let $f : X \to \mathbb{R}^m$ be continuous. We say f is C^∞ on X if and only if for all $p \in X$ there exists an open neighbourhood W_p of p in $\mathbb{R}^n$ and a smooth map $g_p : W_p \to \mathbb{R}^m$ such that

$$g_p = f$$

on $W_p \cap X$.

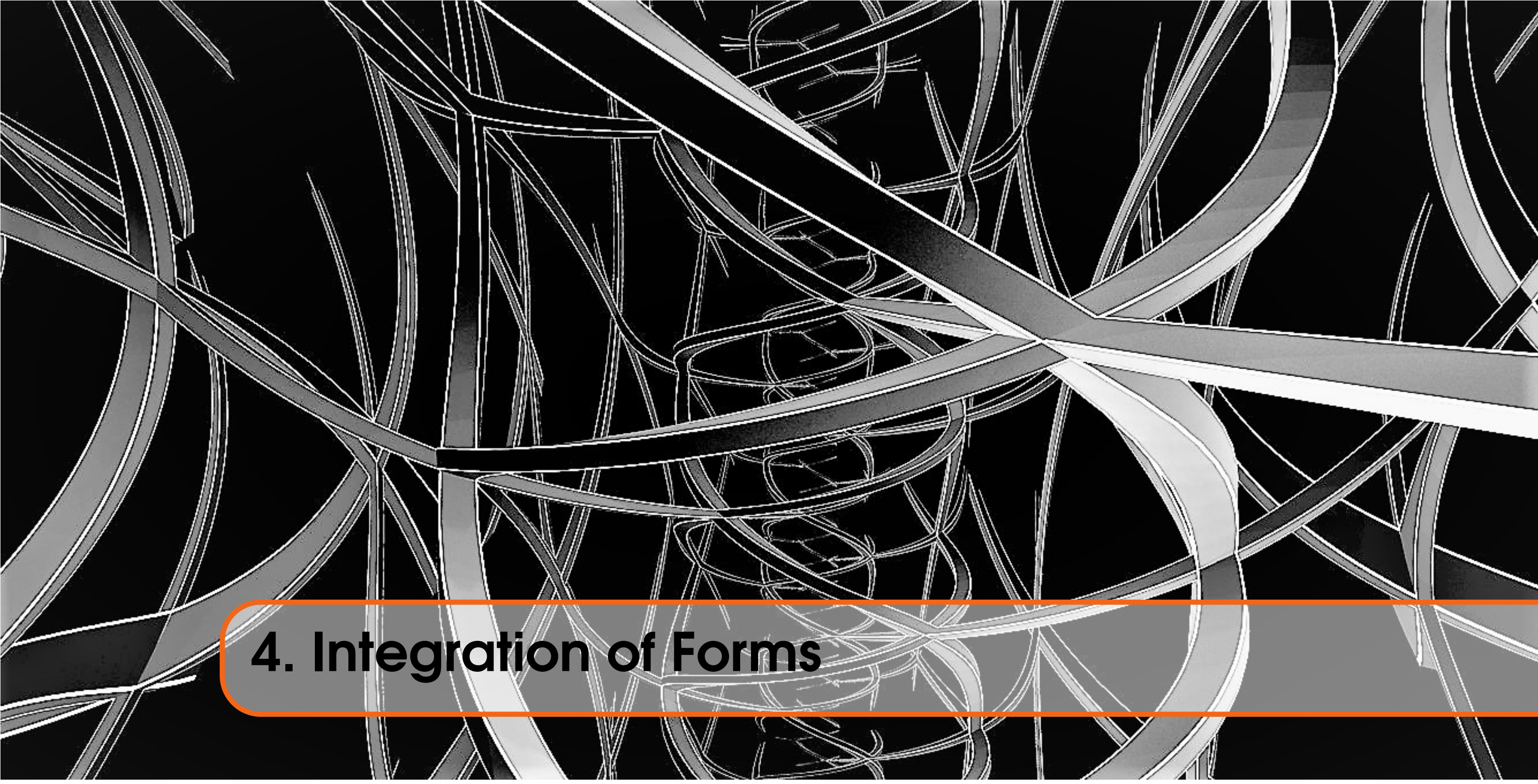
Theorem 16 — Extension Theorem. Let $f : X \to \mathbb{R}^m$ be smooth in the sense just defined, then there exists W open subset in $\mathbb{R}^n$ with $X \subseteq W$ and a smooth map $g : W \to \mathbb{R}^m$ such that $f = g$ on X

Proof. Let $W = \bigcup_{p \in X} W_p$ and let $\{\rho_k : k \in \mathbb{N}\}$ be the partition of unity subordinate to this open cover. In this case, $X = A$ is the index set. Then, for all k, there exists $\alpha(k) \in X$ such that $\rho_k \in C_0^\infty(W_{\alpha(k)})$ which means

$$\textbf{supp}(\rho_k) \subseteq W_{\alpha(k)}$$

Let $g_k(x) = \rho_k(x) g_{\alpha(k)}(x)$ for all $x \in W_{\alpha(k)}$. Note that $g_k(x) = 0$ for all $x \neq W_{\alpha(k)}$. Then, $g_k \in C^\infty(\mathbb{R}^m)$. Let $g = \sum_{k=1}^{\infty} g_k$ to be our desired extension. ■

R This means every map $f : X \to \mathbb{R}^m$ can be extended to a smooth map on an open neighbourhood W of X. Being locally, the restriction of a smooth map is equivalent to being locally the restriction of a smooth map.



4. Integration of Forms

4.1 Compact Differential k-forms

Definition 4.1.1 — Space of Compact Differential k-forms.

$$\Omega_c^k(U) = \left\{\omega \in \Omega^k(U) : \mathbf{supp}(\omega) \subseteq U \text{ is compact}\right\}$$

It is clear that $\Omega_c^k(U) \subseteq \Omega^k(U)$.

R Note that

$$C_0^\infty(U) = C_c^\infty(U) = \Omega_c^0(U)$$

if $f \in \Omega_c^0(U)$ and $\omega \in \Omega^k(U)$, then $f\omega \in \Omega_c^k(U)$. More generally, if $\omega \in \Omega_c^k(U), \eta \in \Omega^l(U)$, then

$$\omega \wedge \eta \in \Omega_c^{k+l}(U)$$

Definition 4.1.2 Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ be a compactedly supported top-form. Then,

$$\omega = f dx_1 \wedge \cdots \wedge dx_n$$

for some $f \in \Omega_c^0(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ and $\mathbf{supp}(\omega) = \mathbf{supp}(f)$. We define

$$\int \omega = \int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f dx_1 \wedge \cdots \wedge dx_n$$

This is well-defined because f is continuous on $\mathbb{R}^n$ and $\mathbf{supp}(f)$ is compact implies that there exists a closed rectangle $\bar{Q}$ such that $f \equiv 0$ on $\left(\bar{Q}\right)^c$, so

$$\int_{\mathbb{R}^n} f dx_1 \wedge \cdots \wedge dx_n = \int_{\bar{Q}} f dx_1 \wedge \cdots \wedge dx_n$$

Theorem 17 — Poincare Lemma For Open Rectangle. Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ and let $Q = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ be an open rectangle in $\mathbb{R}^n$ such that $\mathbf{supp}(\omega) \subseteq Q$. Then,

$$\int_{\mathbb{R}^n} \omega = 0 \iff \text{there exists } \mu \in \Omega_c^{n-1}(\mathbb{R}^n) \text{ with } \mathbf{supp}(U) \subseteq Q \text{ such that } d\mu = \omega$$

Proof. 1. ($\Longleftarrow$): suppose $\mu \in \Omega_c^{n-1}(\mathbb{R}^n)$ and $\mathbf{supp}(U) \subseteq Q$. Then,

$$\mu = \sum_{j=1}^{n} (-1)^{j-1} f_j dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n$$

where $f_j \in C_c^\infty(\mathbb{R}^n)$ and $\mathbf{supp}(f_j) \subseteq Q$. Then,

$$d\mu = \left(\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} \right) dx_1 \wedge \cdots \wedge dx_n$$

where $\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} \in C_0^\infty(\mathbb{R}^n)$ and its support is a subset of Q.
We claim that

$$\int_{\mathbb{R}^n} d\mu = 0$$

Note that

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} \right) dx_1 \wedge \cdots \wedge dx_n = \sum_{j=1}^{n} \left(\int_{\mathbb{R}^n} \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n \right)$$

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n &= \int_{\bar{Q}} \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n \\ &= \int_{\bar{Q}} \left[\int_{a_j}^{b_j} \frac{\partial f_j}{\partial x_j} dx_j \right] dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n \\ &\quad \text{by Fubini's Theorem} \end{aligned}$$

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} \right) dx_1 \wedge \cdots \wedge dx_n = 0$$

since $f_j(x_j)\big|_{x_j=a_j}^{b_j} = 0$ since $\mathbf{supp}(f_j) \subseteq Q$ if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_j = a_j$ or b_j. If $x \in \bar{Q} \backslash Q$, then $f_j(x) = 0$.

2. ($\Longrightarrow$): we need to define something and prove a lemma first.

Definition 4.1.3 — Property P. Let $V \subseteq \mathbb{R}^m$ be open, we say V has property P if and only if for every $\omega \in \Omega_c^m(V)$ such that $\int \omega = 0$, there exists $\mu \in \Omega_c^{m-1}(V)$ with $\omega = d\mu$.

R We need to show an open rectangle has property P.

Lemma 4.2 Let $V \subseteq \mathbb{R}^{n-1}$ be open and $A \subseteq \mathbb{R}$ be an open interval. If V has property P, then $V \times A$ has property P.

Proof. Write $x = (x_1, \ldots, x_{n-1}) \in V \subseteq \mathbb{R}^{n-1}$ and $x_n = t \in A \subseteq \mathbb{R}$. Let $\omega \in \Omega_c^n(V \times A)$ such that $\int \omega = 0$ where $dt \wedge \alpha$ and $\alpha = f(x,t) dx_1 \wedge \cdots \wedge dx_{n-1}$. By Induction Hypothesis, we have that $\mathbf{supp}(f) \subseteq V \times A$ and it is compact. Define θ to be an $(n-1)$-form on V by

$$\theta = \left(\int_A f(x,t) dt \right) dx_1 \wedge \cdots \wedge dx_{n-1}$$

we claim that $\theta \in \Omega_c^{n-1}(V)$, which means, we need to show it is

(a) **Smooth:** it is smooth since

$$\frac{\partial}{\partial x_j}\left(\int_A\right) = \int_A \frac{\partial f}{\partial x_j}(x,t)dt \quad A4$$

(b) **Compact and supp$(\theta) \subseteq V$:** Let $K = \mathbf{supp}(f) \subseteq V \times A$ and have the following continuous maps to be

$$\begin{cases} \pi_1(x,t) = x \in \mathbb{R}^{n-1} \\ \pi_2(x,t) = t \in \mathbb{R} \end{cases}$$

Then, $K_1 = \pi_1(K) \subseteq V$ and $K_2 = \pi_2(K) \subseteq A$ are both compact. Then, the cartesian product

$$K \subseteq K_1 \times K_2 \subseteq V \times A$$

Suppose $x \notin K_1 \implies (x,t) \notin K_1 \times K_2, \forall t \in A$. Therefore, $(x,t) \in K \implies f(x,t) = 0, \forall t \in A, x \notin K_1$. Hence, we have

$$\int_A f(x,t)dt = 0, x \notin K_1$$

so, $\mathbf{supp}(\theta) \subseteq K_1 \subseteq V$ is compact since a closed subset of a compact set is compact. Therefore $\theta \in \Omega_c^{n-1}(V)$

Thus, $\theta \in \Omega_c^{n-1}(V)$. Then, by Fubini, we have

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \theta &= \int_{\mathbb{R}^{n-1}} \left(\int_A f(x,t)dt\right) dx_1 \wedge \cdots \wedge dx_{n-1} \\ &= \int_{\mathbb{R}^n} f dx_1 \wedge \cdots \wedge dx_n = \int \omega = 0 \end{aligned}$$

by hypothesis. Thus, $\int \theta = 0$, but V has property P. Hence, there exists $\beta \in \Omega_c^{n-2}(V)$ such that

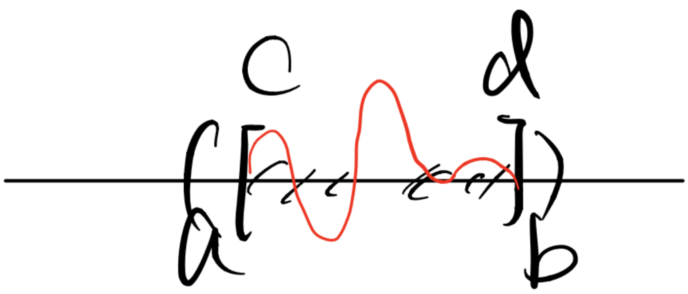
$$\theta = d\beta$$

■

Back to the proof using this lemma. Induction on n. Suppose $n = 1$ and $Q = (a,b)$ and $\omega = f dx$ for some $f \in C_c^\infty(Q)$ and $\mathbf{supp}(f)$ is compact on Q. Then, there exists $a < c < d < b$ such that

$$\mathbf{supp}(f) \subseteq [c,d] \subsetneq (a,b)$$

By Hypothesis, $\int \omega = 0 \implies \int_a^b f(s)dx = 0$. We define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \int_a^x f(s)ds$ and



$g \in C^\infty(\mathbb{R})$ by the Fundamental Theorem of Calculus.

1. If $a < x < c$, then $g(x) = 0$

2. If $b < xd$, then $g(x) = \int_a^x f(s)dx = \int_a^b f(s)ds = 0$ by induction hypothesis.

Thus, $\mathbf{supp}(g) \subseteq [c,d] \subseteq Q = (a,b)$ and

$$dg = \frac{\partial g}{\partial x}dx \underset{FTC}{=} f dx = \omega$$

By the proved lemma, since (a_1, b_1) has property P, this implies that

$$(a_1,b_1) \times (a_2,b_2) \times \cdots \times (a_n,b_n)$$

has property P. Thus, we are done. ■

R In summary, if Q is an open rectangle and $\omega \in \omega_c^n(Q)$, we have

$$\int \omega = 0 \iff \omega = d\sigma, \sim \in \Omega_c^{n-1}(Q)$$

Next step is to appreciate this theorem by replacing Q with an arbitrary connected subset of $\mathbb{R}^n$.

We will prove a general result. To state it, we need a definition.

Definition 4.2.1 Let $U \subseteq \mathbb{R}^n$ be open, define an equivalence relation on $Z_c^k(U)$ as follows:

$$\omega \sim \eta \iff \omega - \eta = d\sigma, \sigma \in \Omega_c^{k-1}(U)$$

This is basically saying that

$$d : \Omega_c^{k-1}(U) \to \Omega_c^k(U)$$

support is non-increasing.

This is an equivalence relationship since $\Omega_c^{k-1}(U)$ is a subspace of $\Omega^{k-1}(U)$.

Definition 4.2.2 — k-th Compactly Supported Cohomology. We define

$$H_c^k(U) = \frac{\ker\left(d : \Omega_c^{k-1}(U) \to \Omega_c^k(U)\right)}{\operatorname{im}\left(d : \Omega_c^{k-1}(U) \to \Omega_c^k(U)\right)}$$

to be the k-th compactly supported cohomology of U.

R So, our equivalence relationship says that

$$\omega \sim \eta \iff [\omega] = [\eta] \in H_c^k(U)$$

when $k = n$, any n-form is closed, so $\sim$ is a relationship on $\Omega_c^k(U)$.

Theorem 18 Let $U \subseteq \mathbb{R}^n$ be open and connected. Fix an open rectangle $Q \subseteq U$. Let $\omega_0 \in \Omega_c^n(Q)$ such that $\int \omega_0 = 1$, then if $\omega \in \Omega_c^n(U)$ with

$$\int \omega = c \in \mathbb{R}$$

then,

$$\omega \sim c\omega_0 \iff [\omega] = c[\omega_0]$$

Proof. Let $\mathscr{U} = \{Q_k : k \in \mathbb{N}\}$ be a countable collection of open rectangles such that

$$\bigcup_{k=1}^{\infty} Q_k = U$$

Let $\{\rho_k : k \in \mathbb{N}\}$ be a Partition of Unity subordinate to $\mathscr{U}$ and $\rho_k \in C_c^\infty(U)$ $0 \leq \rho_k \leq 1$, $\mathbf{supp}(\rho_k) \subseteq Q_k$, and $\sum_k \rho_k = 1$. Then,

$$\omega = \sum_{k=1}^{\infty} \rho_k \omega$$

Let K be the support of ω is compact and $K \subseteq U$. By the properties of Partition of Unity, for all $p \in K$, there exists U_p open such that $p \in U_p \subseteq K$ such that

$$\rho_k\big|_{U_p} \neq 0$$

for only finitely many k. Then $\{U_p : p \in K\}$ is an open cover of K implies it has a finite subcover since K is compact. Thus,

$$K \subseteq \bigcup_{l=1}^{N} U_{P_l}$$

for finitely many ρ_k are non-zero on each U_{P_l} covering $K = \mathbf{supp}(\omega)$. Thus, there exists M such that

$$\omega = \sum_{k=1}^{M} \rho_k \omega$$

each $\rho_k \omega$ is compactly supported in an open rectangle Q_k. Let

$$c = \int \omega = \int \sum_{k=1}^{M} \rho_k \omega = \sum_{k=1}^{M} \int \rho_k \omega = \sum_{k=1}^{M} c_k$$

where $c_k = \int \rho_k \omega$. Then,

$$c = \sum_{k=1}^{M} c_k$$

suppose we can show that

$$\rho_k \omega \sim c_k \omega_0$$

which means the theorem is true for forms compactly supported in some open rectangle in U. Then,

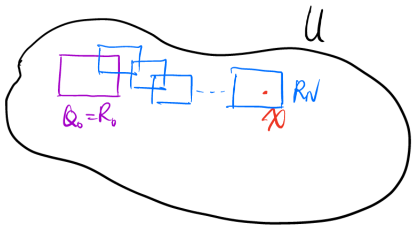
$$\omega = \sum_{k=1}^{M} \rho_k \omega \sim \left(\sum_{k=1}^{M} c_k \right) \omega_0 = c\omega_0$$

This reduces the proof to the case where $\mathbf{supp}(\omega) \subseteq Q_0$ where Q_0 is some open rectangle in U and Q_0 needs not to be Q. We need a Lemma, where we crucially use U is connected.

Lemma 4.3 Let U be open, non-empty and connected. Let

$$A = \{x \in U : \text{ there exists a sequence of } R_0, \ldots, R_N \text{ of open rectangle in } U\}$$

with $x \in R_N$ and $R_0 = Q_0$ and $R_i \cap R_{i+1} \neq$ for all $i = 0, \ldots, N-1$. Then, $A = U$.



Proof. It is clear that A is open, we claim $U \backslash A$ is also open. Let $x \in U \backslash A$, then $x \in U$, so there exists an open rectangle R such that $x \in R \subseteq U$. If $R \cap A \neq \emptyset$, then there exists $y \in R \cap A$ and

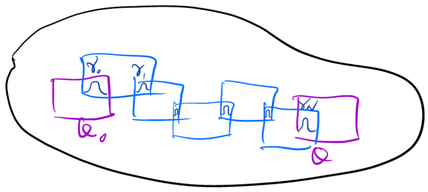
$$y \in A, y \in R, x \in R \Longrightarrow x \in A$$

contradiction! Thus, $R \cap A = \emptyset$, so $R \subseteq U \backslash A$. Thus, $U \backslash A$ is open. Since A is open and non-empty, $U \backslash A$ is open, and U is connected, these give us

$$U = A$$

■

Back to the proof, let Q_0 and Q be as before. Let $x \in Q_0$, by Lemma, there exists a chain of open rectangle $R_0 = Q, R_1 \ldots, R_N = Q_0$ such that $x \in R_N$ and $R_i \cap R_{i+1} \neq \emptyset$ for all i. For each



$i = 0, 1 \ldots, N$, let γ_i be a compactly supported n-form on U with

$$\mathbf{supp}(\gamma_i) \subseteq R_i \cap R_{i+1} \neq \emptyset$$

and $\int \gamma_i = 1$. Let $\theta_i = \gamma_i - \gamma_{i+1}$. Since $\mathbf{supp}(\gamma_i) \subseteq R_i \cap R_{i+1}$ and $\mathbf{supp}(\gamma_{i+1}) \subseteq R_{i+1} \cap R_{i+2}$, we have

$$\mathbf{supp}(\theta_i) \subseteq R_{i+1}$$

By construction, wee have $\int \theta_i = \int \gamma_i - \int \gamma_{i+1} = 0$. By the result of the previous theorem, which was the version of this theorem of open rectangles, we have

$$\gamma_i \sim \gamma_{i+1} \iff \theta_i = \gamma_i - \gamma_{i+1} = d\sigma_i, \sigma_i \in \Omega_c^{n-1}(R_{i+1})$$

Also, $\mathbf{supp}(\omega_0) \subseteq Q = R_0$ and $\mathbf{supp}(\gamma_0) \subseteq R_0$, then

$$\int (\omega_0 - \gamma_0) = 0 \Longrightarrow \omega_0 \sim \gamma_0$$

also, $\mathbf{supp}(\omega) \subseteq Q_0 = R_N$ and $\mathbf{supp}(\gamma_N) \subseteq R_N$, then

$$\int (\omega - c\gamma_N) = 0 \Longrightarrow \omega \sim c\gamma_N$$

In summary,

$$\begin{aligned} &c\omega_0 \sim c\gamma_0 \\ &c\gamma_i \sim c\gamma_{i+1}, \forall i \implies \omega \sim c\gamma_N \sim c\gamma_{N-1} \sim \cdots \sim c\gamma_1 \sim c\gamma_0 \sim c\omega_0 \\ &c\gamma_N \sim \omega \end{aligned}$$

■

R So, we have shown if $\omega_0 \in \Omega_c^n(U)$ and U is a connected, open, and non-empty, then for any $\omega \in \omega_c^n(U)$. There exists a unique $c \in \mathbb{R}$ such that (with $\int \omega_1 \neq 0$)

$$\omega = c\omega_0 + d\sigma, \sigma \in \Omega_c^{n-1}(U)$$

and

$$\int \omega = c \int \omega_0 + \int d\sigma = c \int \omega_0$$

Corollary 4.3.1 If $U \subseteq \mathbb{R}^n$ is open and connected, then

$$H_c^n(U) \cong \mathbb{R}$$

has basis $\{[\omega_0]\}$.

4.4 The Degree Theorem

This will have the following corollaries

1. Change of Variable Theorem
2. Brouwer Fixed Point Theorem
3. Fundamental Theorem of Algebra

Definition 4.4.1 — Proper. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open. Let $f: U \to V$ be continuous. We say f is proper if $f^{-1}(K)$ is compact whenever $K \subseteq V$ is compact.

R Note that if f is continuous $f(K)$ is compact whenever K is compact, but f needs not to be proper.

Theorem 19 Let $f: U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be proper. Let $B \subseteq V$ be compact, then

$$A = f^{-1}(B) \subseteq U$$

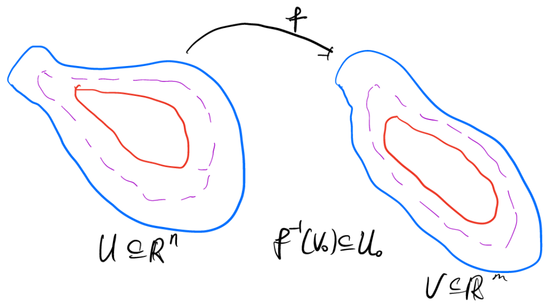
is compact. For every U_0 open with $A \subseteq U_0 \subseteq U$. There exists V_0 open with $B \subseteq V_0 \subseteq V$ and $f^{-1}(V_0) \subseteq U_0$.

Proof. Let $W = f^{-1}(C) \backslash U_0 = f^{-1}(C) \cap U_0^c$ compact. Since f is proper, U_0^c is closed. Thus, W is compact subset of U.
Since f is continuous, $f(W)$ is a compact subset of V. Suppose $y \in f(W)$, then $y = f(x)$ for some $x \in W = f^{-1}(C) \cap U_0$. Then, $x \notin U_0 \implies x \notin A$. Thus, $y = f(x) \notin B$ and $f(W) \cap B = \emptyset$.
Let $V_0 = \operatorname{int}(C) \backslash f(W) \subseteq V$ be open and $B \subseteq \operatorname{int}(C), B \cap f(W) = \emptyset \implies B \subseteq V_0 \subseteq V$.
It remains to show that $f^{-1}(V_0) \subseteq U_0$. Suppose that $x \in f^{-1}(V_0)$, then $f(x) \in V_0 = \operatorname{int}(C) \backslash f(W)$. Thus, $f(x) \in \operatorname{int}(C) \subseteq C \implies x \in f^{-1}(C)$. Suppose $x \notin U_0$, then $x \in W, f(x) \in f(W)$ yields a contradiction.

Hence, $x \in U_0$, so

$$f^{-1}(V_O) \subseteq U_0$$

■



Corollary 4.4.1 Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ bee proper. If $X \subseteq U$ with $U \backslash X$ open, then $V \backslash f(X)$ is open.

Proof. Let $y \in V \backslash f(X)$, we need to find V_0 open suc that $y \in V_0 \subseteq V \backslash f(X)$. Thus, $V_0 \subseteq V$ and $V_0 \cap f(X) = \emptyset$. Let $B = \{y\}$ which is compact. Let $f^{-1}(B) = A = f^{-1}(y)$ which is compact as well. Since f is proper, let $x \in A \implies f(x) = y \notin f(X)$. So, $x \notin X$ and $x \in U_0 = U \backslash X$. Thus, $A \subseteq U_0 \subseteq U$. By previous Theorem, there exists V_0 open such that $B \subseteq V_0 \subseteq V$ and $f^{-1}(V) \subseteq U_0$. This implies

$$f^{-1}(V_0) \cap X = \emptyset \implies V_0 \cap f(X) = \emptyset$$

■

Now, let $f : U \to V$ be a **smooth proper map**.

Lemma 4.5 If $\omega \in \Omega_c^k(V)$, then $f^*\omega \in \Omega_c^k(U)$.

> Proper smooth maps pullback compactly supported forms to compactly supported forms.

Proof.

$$\omega = \sum_I \omega_I dx_i$$

where I is a strictly increasing multi-index of k and $\omega_I \in C_c^\infty(V)$. Then,

$$f^*\omega = \sum_I (f^*\omega_I) f^* dx_I = \sum_I (\omega_I \circ f) f^* dx_I$$

Suppose $(f^*\omega)_x \neq 0 \implies$ at least one of the $\omega_I(f(x)) \neq 0 \iff \omega_{f(x)} \neq 0$. Thus,

$$\begin{aligned} \{x \in U : (f^*\omega)_x \neq 0\} &\subseteq \{x \in U : f(x) \in \{y \in V : \omega_y \neq 0\}\} \\ &\subseteq \{x \in U : f(x) \in \mathbf{supp}(\omega)\} \\ &= f^{-1}(\mathbf{supp}(\omega)) \end{aligned}$$

Note that $\mathbf{supp}(\omega)$ is compact and this leads to $f^{-1}(\mathbf{supp}(\omega))$ being compact. Then, taking closure, we have

$$\mathbf{supp}(f^*\omega) \subseteq f^{-1}(\mathbf{supp}(\omega))$$

thus, $f^*\omega \in \Omega_c^k(U)$. ■

Theorem 20 — The Degree Theorem. Let $U, V \subseteq \mathbb{R}^n$ be non-empty, open, connected in the same $\mathbb{R}^n$. Let $f : U \to V$ be smooth and proper, there exists a real number called the degree of f, which we denote $\deg(f) \in \mathbb{R}$ such that

$$\int_U f^*\omega = \deg(f) \int_V \omega, \forall \omega \in \Omega_c^n(V)$$

R

1. $\deg(f) \in \mathbb{Z}$
2. Explicitly, the theorem says let $\omega = h(y_1, \ldots, y_n) dy_1 \wedge \cdots \wedge dy_n$ and $h \in C_c^\infty(V)$,

$$f^*\omega = (h \circ f)(x_1, \ldots, x_n) f^*(dy_1 \wedge \cdots \wedge dy_n) = h(f(x_1, \ldots, x_n)) \det\left(\frac{\partial f_i}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n$$

Theorem 21 Let $\omega = h(y_1, \ldots, y_n) dy_1 \wedge \cdots \wedge dy_n$ and $h \in C_c^\infty(V)$,

$$f^*\omega = (h \circ f)(x_1, \ldots, x_n) f^*(dy_1 \wedge \cdots \wedge dy_n) = h(f(x_1, \ldots, x_n)) \det\left(\frac{\partial f_i}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n$$

Then,

$$\int_U h(f(x_1, \ldots, x_n)) \det\left(\frac{\partial f_i}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n = \deg(f) \int_V h(y_1, \ldots, y_n) dy_1 \wedge \cdots \wedge dy_n$$

Proof. Fix $\omega_0 \in \Omega_c^n(V)$ with $\int_v \omega_0 = 1$. Define $\deg(f) \in \mathbb{R}$ by

$$\int_U f^*\omega_0 = \deg(f) \int_V \omega_0 = \deg(f)$$

Let $\omega \in \Omega_c^n(V)$ and let $c = \int_V \omega$, then $\omega \sim c\omega_0$. So,

$$\int_V (\omega - c\omega_0) = 0 \Longrightarrow \omega - c\omega_0 = d\sigma, \sigma \in \Omega_c^{n-1}(V)$$

Then, we know that $\omega = c\omega + d\sigma \Longrightarrow f^*\omega = cf^*\omega_0 + f^*d\sigma = cf^*\omega + d(f^*\sigma), f^*\sigma \in \Omega_c^{n-1}(U)$ by the previous proposition.
Then,

$$\begin{aligned}
\int_U f^*\omega &= \int_U cf^*\omega + \int_U d(f^*\sigma) \\
&= \int_U cf^*\omega = c\deg(f) = \deg(f) \int_V \omega
\end{aligned}$$

■

Proposition 4.5.1 Let U, V, W be open, non-empty, connected subsets of $\mathbb{R}^n$. $f : U \to V$, $g : V \to W$ be smooth and proper, then $g \circ f : U \to W$ is also smooth and proper by composition and

$$\deg(g \circ f) = \deg(g)\deg(f)$$

Proof. Let $K \subseteq W$ be compact. Then,

$$(g \circ f)^{-1}(K) = f^{-1}(g^{-1}(K))$$

is compact since $g^{-1}(K)$ is compact by g is proper. Then, for $\omega \in \Omega_c^n(W)$,

$$\int_U (g\circ f)^*\omega = \deg(g\circ f)\int_W \omega$$
$$= \int_U f^*(g^*\omega) = \deg(f)\int_V g^*\omega = \deg(f)\deg(g)\int_W \omega$$

Thus,

$$\deg(g\circ f) = \deg(f)\deg(g)$$

■

4.5.1 Examples of Computing the Degree

Proposition 4.5.2 Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be $f(x) = x+a$ for some $a\in\mathbb{R}^n$ (translation by a), then $\deg(f) = +1$.

Proof. Note that f is clearly smooth and proper. We approach this by induction on n.

1. When $n=1$, let $\omega\in\Omega_c^1(\mathbb{R})$ then $\omega = h(x)dx$ for some $h\in C_c^\infty(\mathbb{R})$. Then,

$$\int f^*\omega = \int f^*(h(x)dx) = \int h(f(x))d(f^*x)$$
$$= \int_{-\infty}^{\infty} h(x+a)dx$$
$$= \int_{-\infty}^{\infty} h(y)dy \qquad \begin{cases} y = x+a \\ dy = dx \end{cases}$$
$$= \int \omega \Longrightarrow \deg(f) = +1$$

2. **General case:** let $a = (a_1,\dots,a_n)$ and $x = (x_1,\dots,x_n)$. Let $\phi(x) = h(x_1)h(x_2)\dots h(x_n)$ for same h in the case of $n=1$. Then, $\phi\in C_c^\infty(\mathbb{R}^n)$ since the cartesian product of compact set is compact. Let $\omega = \phi(x_1,\dots,x_n)dx_1\wedge\dots\wedge dx_n \in \Omega_c^n(\mathbb{R}^n)$. Then,

$$\int f^*\omega = \int_{\mathbb{R}^n}(f^*\omega)d(x_1+a)\wedge\dots\wedge d(x_n+a_n)$$
$$= \int_{\mathbb{R}^n} h(x_1+a_1)\dots h(x_n+a_n)dx_1\wedge\dots\wedge dx_n$$
$$\underset{\text{Fubini}}{=} \prod_{i=1}^{\infty}\left(\int_{-\infty}^{\infty} h(x_i+a_i)dx_i\right)$$
$$\underset{\text{case 1}}{=} \prod_{i=1}^{\infty}\left(\int_{-\infty}^{\infty} h(y_i)dy_i\right)$$
$$= \int_{\mathbb{R}^n} h(x)dx_1\wedge\dots\wedge dx_n = \int\omega$$

Thus, $\deg(f) = +1$.

■

Proposition 4.5.3 Let $\lambda_1,\dots,\lambda_n$ be non-zero real numbers. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be

$$f(x_1,\dots,x_n) = (\lambda_1 x_1,\dots,\lambda_n x_n)$$

or in matrix form

$$f(x) = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}$$

then, f is smooth and proper and

$$\deg(f) = (-1)^{\text{number of negative } \lambda_i} = \operatorname{sgn}(\lambda_1 \dots \lambda_n)$$

Proof. Induction on degree n.

1. $n = 1$: let $\omega = h(x)dx$ and $h \in C_c^\infty(\mathbb{R})$. Then

$$f^*\omega = h(\lambda x)d(\lambda x) = h(\lambda x)\lambda dx$$

By substitution $y = \lambda x, dy = \lambda dx$, we have

$$\int f^*\omega = \int_{-\infty}^{\infty} h(\lambda x)\lambda dx$$

(a) When $\lambda > 0$, we have

$$\int_{-\infty}^{\infty} h(y)dy = \int \omega$$

(b) When $\lambda < 0$, we have

$$\int_{\infty}^{-\infty} h(y)dy = -\int \omega$$

Thus,

$$\deg(f) = \begin{cases} +1 & \lambda > 0 \\ -1 & \lambda < 0 \end{cases}$$

2. **General Case:** Let $\phi(x) = h(x_1)\dots h(x_n) \in C_c^\infty(\mathbb{R}^n)$ and $\omega = \phi(x)dx_1 \wedge \dots \wedge dx_n \in \Omega_c^n(\mathbb{R}^n)$. Then,

$$\begin{aligned} \int f^*\omega &= \int_{\mathbb{R}^n} h(\lambda_1 x_1)\dots h(\lambda_n x_n) d(\lambda_1 x_1) \wedge \dots \wedge d(\lambda_n x_n) \\ &\underset{\text{Fubini}}{=} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} h(\lambda_i x_i)\lambda_i dx_i \right) \\ &\underset{\text{case 1}}{=} \operatorname{sgn}(\lambda_1 \dots \lambda_n) \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} h(y_i)dy_i \right) \\ &= \operatorname{sgn}(\lambda_1 \dots \lambda_n) \int \omega \end{aligned}$$

Note that

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

is invertible (diagonal), $f(x) = Ax$, then $\deg(f) = \operatorname{sgn}(\det(A))$.

■

Linear Algebra Preliminaries

Lemma 4.6 Let $A \in M_n(\mathbb{R})$ be invertible. There exists a unique P that is orthogonal ($P^T = P^{-1}$) and unique S that is self-adjoint ($S^T = S$) such that $A = PS$.

Proof. We consider A^TA which is symmetric. By Spectral Theorem, there exists Q orthogonal such that

$$Q^TA^TAQ = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = B$$

Claim: $\lambda_i > 0, \forall i$
Note that

$$\begin{aligned}
A^T Ax &= \lambda x \\
\langle A^T Ax, x\rangle &= \lambda x^2 \implies \lambda \geq 0 \\
\langle Ax, Ax\rangle &\geq 0
\end{aligned}$$

Since A is invertible, $\lambda > 0$.

Define $\sqrt{B} = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$ and $S = Q\sqrt{B}^T$. Then,

$$S^T = Q\left(\sqrt{B}\right)^T Q^T = Q\sqrt{B}Q^T = S$$

thus, S is self-adjoint and $\det(S) > 0$. Then,

$$\begin{aligned}
S^2 &= \left(Q\sqrt{B}Q^T\right)(Q\sqrt{B}Q^T) = QBQ^T = A^T A \\
(A^T A)S &= A^T AQ\sqrt{B}Q^T = QBQ^T Q\sqrt{B}Q^T = QB\sqrt{B}^T
\end{aligned}$$

Then, $S(A^T A) = (A^T A)S$. Define $P = AS^{-1}$, then $P^T = (S^{-1})^T A^T = (S^T)^{-1} A^T = S^{-1} A^T$.
Then,

$$P^T P = S^{-1} A^T AS^{-1} = S^{-1} S^2 S^{-1} = I$$

Thus, P is orthogonal.

$$A = PS$$

■

Theorem 22 Let $A \in M_n(\mathbb{R})$ be invertible and define $f : \mathbb{R}^n \to \mathbb{R}^n$. by $f_A(x) = Ax$. f_A is smooth and proper since f_A is invertible with $(f_A)^{-1} = f_{A^{-1}}$, then

$$\deg(f_A) = \begin{cases} +1 & \det(A) > 0 \text{ orientation preserving} \\ -1 & \det(A) < 0 \text{ orientation reversing} \end{cases}$$

Proof. Let $A \in M_n(\mathbb{R})$ be invertible. Then, there exists S self-adjoint and $\det(S) > 0$ and P is orthogonal such that $A = PS$. Then,

$$f_A = f_{PS} = f_P \circ f_S$$

then, by the multiplicative property, we have

$$\deg(f_A) = \deg(f_P)\deg(f_S)$$

then,

$$\det(A) = \det(P)\det(S)$$

note that $\det(S) > 0$. We only need to show the following

1. $\deg(f_P) = \det(P) = \pm 1$: Let $\psi \in C_c^\infty(\mathbb{R}^n)$. Define $h \in C_c^\infty(\mathbb{R}^n)$ by $h(x_1, \ldots, x_n) = \psi(x_1^2 + \cdots + x_n^2) = \psi\left(\|x\|^2\right)$. Let

$$\omega = h(x)dx_1 \wedge \cdots \wedge dx_n \in \Omega_c^n(\mathbb{R}^n)$$

then,

$$\begin{aligned}
f_P^*\omega &= (h\circ f_P)_{(x_1,\dots,x_n)}(f_P^*)(dx_1\wedge\cdots\wedge dx_n)\\
&= (h\circ f_P)_{(x_1,\dots,x_n)}\det(P)dx_1\wedge\cdots\wedge dx_n\\
&= \psi\left(\|Px\|^2\right)\det(P)dx_1\wedge\cdots\wedge dx_n\\
&\underset{P\text{ orthogonal}}{=} \psi\left(\|x\|^2\right)\det(P)dx_1\wedge\cdots\wedge dx_n\\
&= \det(P)\omega
\end{aligned}$$

Then,

$$\int f_P^*\omega = \det(P)\int\omega \Longrightarrow \deg(f_P) = \det(P) = \pm 1$$

2. $\deg(f_S) = +1$: Note that S is self-adjoint and $\det(S) > 0$ with all eigenvalue(s) being positive. By Spectral Theorem, there exists an orthogonal Q such that

$$Q^TSQ = L = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$$

Then,

$$\begin{aligned}
S = QLQ^T &\Longrightarrow f_S = f_Q\circ f_L\circ f_{Q^T}\\
\deg(f_S) &= \deg(f_Q)\deg(f_L)\deg(f_{Q^T})\\
&\underset{\text{previous proposition}}{=} \det(Q)(+1)\det\left(Q^T\right)\\
&= +1
\end{aligned}$$

■

Proposition 4.6.1 We need this for Brouwer Fixed Point Theorem. Let U be an open connected, non-empty subset of $\mathbb{R}^n$. Let $K\subseteq U$ be compact. Suppose $f: U\to U$ such that $f(x) = x, \forall x\in U\backslash K$, which means f is the identity function outside of the compact set K, then f is proper and $\deg(f) = +1$.

Proof. 1. **Proper:** Let $A\subseteq U$ and $f^{-1}(A) = \{x\in U : f(x)\in A\}$. Let $x\in f^{-1}(A)$, if $x\in U\backslash A$, then $f(x) = x\in A$. Hence, $f^{-1}(A)\subseteq K\cup A$. If A is compact, $f^{-1}(A)\subseteq K\cup A$ since $K\cup A$ is compact. Hence, $f^{-1}(A)$ is compact, so f is proper.

2. $\deg(f)$: Let $U\backslash K$ is open, by earlier corollary on proper map, $U\backslash f(K)$ is open and non-empty since $f(K)$ is compact, so there exists $\omega\in\Omega^n(U\backslash f(K))$ with $\int\omega = 1$. Then,

$$\omega = h(x_1,\dots,x_n)dx_1\wedge\cdots\wedge dx_n, h\in C_c^\infty(U\backslash f(K))$$

with $x\in U$,

$$(f^*\omega) = h(f(x_1,\dots,x_n))\det(DF(x))dx_1\wedge\cdots\wedge dx_n$$

If $x\in K\Longrightarrow f(x)\in f(K)$ and $h(f(x)) = 0$.
If $x\notin K\Longrightarrow f(y) = y, \forall y$ in a neighbourhood of K. Then, $\det(DF(y)) = 1, \forall y$ in a neighbourhood of $x\notin K$. Hence,

$$(f^*\omega)_x = \omega_x, \forall x\notin K$$

so

$$f^*\omega = \begin{cases}0 & K\\ \omega & U\backslash K\end{cases}$$

then,

$$\int_U f^*\omega = \int_{U\setminus K} f^*\omega = \int_{U\setminus K}\omega = \int_U \omega = 1$$

Thus, $\deg(f) = +1$.

■

4.7 Change of Variable Theorem

Definition 4.7.1 — Diffeomorphism. Let $f : U \to V$ be smooth where $U \subseteq \mathbb{R}^n$ open and $V \subseteq \mathbb{R}^m$ open, say f is a diffeomorphism if f is invertible and $f^{-1} : V \to U$ is also smooth.

R

1. Note that when $m = n$, $f \circ f^{-1} : V \to V$ and $f^{-1} \circ f : U \to U$ are identities.
2. **Chain Rule:**
$$D\big(f \circ f^{-1}\big) = (Df)(Df^{-1}) = I_{m\times n}$$
$$(Df)(Df^{-1}) = I_{n\times n}$$
Thus, Df is both injective and surjective when $m = n$ and
$$(Df)(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right), 1 \le i, j \le n$$
is an $n \times n$ invertible matrix of smooth functions on U. Then,
$$\det((Df)(x)) \neq 0, \forall x \in U$$
If U is connected, then either
$$\begin{cases} \det((Df)(x)) > 0 & \forall x \in U \text{ orientation preserving} \\ \det((Df)(x)) < 0 & \forall x \in U \text{ orientation reversing} \end{cases}$$
On **summary**, let U be open and connected, then any diffeomorphisms $f : U \to V$ must be either or preserving/reserving.
3. It is clear that any diffeomorphism is proper.

Theorem 23 — Main Ingredient for Change of Variable Formula. Let $f : U \to V$ be a diffeomorphism between non-empty, open, connected subsets of $\mathbb{R}^n$, then

$$\deg(f) = \begin{cases} 1 & f \text{ is orientation preserving} \\ -1 & f \text{ is orientation reversing} \end{cases}$$

Proof. **Simplification 1:**
First, we shall show that we can simplify this to a special case. Let $a \in U$ and define $b = -f(a) \in \mathbb{R}^n$. Define $T_a : \mathbb{R}^n \to \mathbb{R}^n$ by $T_a(x) = x + a$ as a translation. Then, by previous result, we have

$$\deg(T_a) = \deg(T_a) = 1$$

Let $\tilde{f}(x) = T_b(f(T_a(x)))$. Then, $\tilde{f}(0) = f(a) + b = 0$. And by degree multiplicity, we have

$$\deg\big(\tilde{f}\big) = (\deg(T_b))(\deg(f))(\deg(T_a)) = \deg(f)$$

Thus, WLOG, we may assume $0 \in U$ and $f(0) = 0 \in V$.

Simplification 2:
Let $A = (DF)(0)$ be an $n \times n$ matrix. Let $\tilde{f} = A^{-1} \circ f$ is a diffeomorphism of U onto $A^{-1}(V)$. Then, note that

$$D\tilde{f} = \left(DA^{-1}\right)(Df) = A^{-1} \circ (Df) \Longrightarrow \det\left(D\tilde{f}\right) = \det\left(A^{-1}\right)\det(Df) > 0$$

Thus,

$$\deg\left(D\tilde{f}\right) = \deg\left(A^{-1}\right)\deg(f)$$

So, WLOG, we need to show $\deg\left(\tilde{f}\right) = +1$. Note that

$$\begin{aligned}\left(D\tilde{f}\right)(x) &= A^{-1}(Df)(x) \\ \left(D\tilde{f}\right)(0) &= A^{-1}(Df)(0) \\ &= A^{-1}A = I\end{aligned}$$

In summary, we reduced the theorem to be the following special case.

Special Case:
Let $f : U \to V$ be a diffeomorphism between non-empty, open, connected subsets of $\mathbb{R}^n$, then

$$\deg(f) = \begin{cases} 1 & f \text{ is orientation preserving} \\ -1 & f \text{ is orientation reversing} \end{cases}$$

where $f(0) = 0$ and $(Df)(0) = I$.

Preliminaries to show this
For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $|x| = \max_{1 \le k \le n} |x)k|$ to be the l_∞ norm. It induces the same topology as $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ since

$$|x|^2 \le \|x\|^2 \le n|x|^2$$

Let $Q_r = \{x \in \mathbb{R}^n : |x| < r\}$ is an open rectangle centred at 0 with side length r.

Lemma 4.8 Let $g(x) = f(x) - I$ and $g(0) = 0$ and $(Dg)(0) = 0$, then there exists $\delta > 0$ such that

$$|g(x)| \le \frac{1}{2}|x|, \forall |x| < \delta$$

Proof. For $g(x) = (x_1, \ldots, x_n)$ we have

$$(Dg)(x) = \left(\frac{\partial g_i}{\partial x_j}(x)\right)$$

and in this case

$$(Dg)(0) = \left(\frac{\partial g_i}{\partial x_j}(0)\right) = 0, \forall i, j$$

By continuity, there exists $\delta > 0$ such that

$$|x| < \delta \Longrightarrow \left|\frac{\partial g_i}{\partial x_j}(x)\right| \le \frac{1}{2n}, \forall i, j$$

By Mean Value Theorem, there exists $c = (c_1, \ldots, c_n)$ such that

$$g_i(x) = g_i(0) + \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}(c_i)x_j$$

where $c_i = t_i x, 0 < t_i < 1$. Then,

$$|g_i(x)| \leq \sum_{j=1}^{n} \left| \frac{\partial g_i}{\partial x_j}(c_i) \right| |x_j| \leq \sum_{j=1}^{n} \frac{1}{2}|x_j| \leq \frac{1}{2}|x|$$

Thus,

$$|g(x)| = \max_{1 \leq i \leq n} |g_i(x)| \leq \frac{1}{2}|x|$$

■

Back to the proof:
Let $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \rho \leq 1$ and

$$\rho(x) = \begin{cases} 0 & |x| \geq \delta \\ 1 & |x| \leq \frac{\delta}{2} \end{cases}$$

this can be done due to all the work we have done in the bump function part. This means $\rho = 1$ on $\overline{Q_{\frac{\delta}{2}}}$ and $\mathbf{supp}(\rho) \subseteq Q_\delta$.
Define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$\tilde{f}(x) = x + \rho(x)g(x) = x + \rho(x)(f(x) - x)$$

By construction,

$$\tilde{f}(x) = \begin{cases} x & |x| \geq \delta \\ f(x) & |x| \leq \frac{\delta}{2} \end{cases}$$

this is a smoothly interpolated function that is f when it is close to the origin and being identity for far away.

Claim 1: $|\tilde{f}(x)| \geq \frac{1}{2}|x|, \forall x \in \mathbb{R}^n$
Proof. If $x \geq \delta$ then $\left|\tilde{f}(x)\right| = |x| > \frac{1}{2}|x|$ already.
If $x < \delta$, by the previous lemma, we have

$$\left|\tilde{f}(x)\right| \geq |x| - \rho(x)|g(x)| \geq |x| - |g(x)| \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x|$$

■

Now, let $\overline{Q_r} = \{x \in \mathbb{R}^n : |x| \leq r\}$ and $\overline{Q_r}^c = \{x \in \mathbb{R}^n : |x| > r\}$. Suppose $x \in \tilde{f}^{-1}(\overline{Q_r})$ such that $\left|\tilde{f}(x)\right| \leq r$. By the first claim. we have

$$\frac{1}{2}|x| \leq \left|\tilde{f}(x)\right| \leq r \Longrightarrow |x| \leq 2r$$

Hence, we have $x \in \overline{Q_{2r}}$ and the

Claim 2: $\tilde{f}^{-1}(\overline{Q_r}) \subseteq \overline{Q_{2r}}$
Aside, $\tilde{f}$ is proper, since whenever $K \subseteq \mathbb{R}^n$ is compact, we have

$$K \subseteq \overline{Q_r}$$

for some $r > 0$, then

$$\tilde{f}^{-1}(K) \subseteq \tilde{f}^{-1}(\overline{Q_r}) \subseteq \overline{Q_{2r}}$$

thus, $\tilde{f}^{-1}(K)$ is also compact.
Now, suppose $x \in Q_\delta$, we have

$$|x| < \delta \Longrightarrow \left|\tilde{f}(x)\right| \leq |x| + |g(x)| \leq \frac{3}{2}|x| < \frac{3}{2}\delta$$

So if $|f(x)| \geq \frac{3}{2}\delta$, we have $|x| \geq \delta$. Then, we have

Claim 3: $\tilde{f}^{-1}\left(\overline{Q_{\frac{3}{2}\delta}}^c\right) \subseteq Q_\delta^c$
Since f is a diffeomorphism and $f(0) = 0 \in V$, it maps an open neighbourhood of 0 to an open neighhourhood of $0 \in V_0$. WLOG, we can shrink U_0 arbitrarily to have

$$U_0 \subseteq \overline{Q_{\frac{\delta}{2}}}, \qquad V_0 \subseteq \overline{Q_{\frac{\delta}{4}}}$$

Let $\omega \in \Omega_c^n(V_0)$ with $\int \omega = 1$. Note that $\mathbf{supp}(\omega) \subseteq V_0$ is compact and

$$f : u_0 \to V_0 \qquad f^{-1}(V_0) = U_0$$

Thus,

$$\mathbf{supp}(f^*\omega) \subseteq U_0 \subseteq \overline{Q_{\frac{\delta}{2}}}$$

is compact.
Now, if $\left(\tilde{f}^*\omega\right)_x \neq 0$, then

$$\omega_{\tilde{f}(x)} \neq 0 \Longrightarrow \tilde{f}(x) \in \mathbf{supp}(\omega) \subseteq V_0 \subseteq \overline{Q_{\frac{\delta}{4}}}$$

by Claim 2, we have $x \in \overline{Q_{\frac{\delta}{2}}}$. Thus, $\mathbf{supp}\left(\tilde{f}^*\omega\right) \subseteq \overline{Q_{\frac{\delta}{2}}}$ is also compact. Note that on $\overline{Q_{\frac{\delta}{2}}}$, by construction of $\tilde{f}$, we have $\tilde{f}(x) = f(x)$. Then, $\tilde{f}^*\omega = f^*\omega$ on $\mathbb{R}^n$. We only need show that

$$\deg\left(\tilde{f}\right) = +1$$

Now let, $\omega \in \Omega_c^n\left(\overline{Q_{\frac{3}{2}\delta}}^c\right)$ with $\int \omega = 1$. By Claim 3, we have

$$\mathbf{supp}\left(\tilde{f}^*\omega\right) \subseteq (Q_\delta)^c$$

and $\left(\tilde{f}^*\omega\right)_x \neq 0$ leads to

$$\omega_{\tilde{f}(x)} \in \left(\overline{Q_{\frac{3}{2}\delta}}^c\right) \subseteq \left(Q_{\frac{3}{2}\delta}^c\right) \Longrightarrow x \in (Q_\delta)^c$$

Now, on Q_δ^c,

$$|x| \geq \delta \Longrightarrow \tilde{f}(x) = x \Longrightarrow \tilde{f}^*\omega = \omega$$

So,

$$\deg\left(\tilde{f}\right) = \int_{\mathbb{R}^n} \tilde{f}^*\omega = \int_{Q_\delta^c} \tilde{f}^*\omega = \int_{Q_\delta^c} \omega = \int_{\mathbb{R}^n} \omega = 1$$

The orientation-reversing case is similar and we are done. ■

Theorem 24 — Change ov Variable (Smooth Version). Let U, V be non-empty, open, and connected subsets of $\mathbb{R}^n$. Let $f : U \to V$ be a diffeomorphism (either orientation-preserving or orientation-reversing). Let $\phi \in C_c^\infty(V)$ be $\phi : V \to \mathbb{R}$ a smooth map with compact support in V, then

$$\int_U (\phi \circ f)(x)|\det(Df(x))|dx = \int_V \phi(y)dy$$

Proof. Let $\omega = \phi(y)dy_1 \wedge \cdots \wedge dy_n \in \Omega_c^n(V)$. By the Degree Theorem, we have

$$\int f^*\omega = (\deg f)\int \omega$$

LHS:

$$\int_U (\phi \circ f)(x)\det(Df(x))dx_1 \wedge \cdots \wedge dx_n$$

RHS:

$$(\deg(f))\int_V \phi(y)dy_1\wedge\cdots\wedge dy_n$$

then, if f is orientation preserving, then

$$\deg(f)=+1,\det(Df(x))>0$$

if f is orientation reversing, then

$$\deg(f)=-1,\det(Df(x))<0$$

in either case,

$$\int_U(\phi\circ f)(x)|\det(Df(x))|dx=\int_V\phi(y)dy$$

■

To get the continuous version, we need a lemma.

Lemma 4.9 — Uniform Approximation Lemma. Let $V\subseteq\mathbb{R}^n$ be open and let $\phi\in C_c^\infty(V)$ be a continuous and compactly supported function on V. For any $\varepsilon>0$, there exists $\psi\in C_c^\infty(V)$ with

$$\sup_{x\in V}|\psi(x)-\phi(x)|<\varepsilon$$

Proof. From PMATH351 Real Analysis. ■

Theorem 25 — Change of Variable Formula (Continuous Version). Let $\phi\in C_c^0(V)$ and $\phi:V\to\mathbb{R}$. Then,

$$\int_U(\phi\circ f)|\det(Df(x))|dx=\int_V\phi(y)dy$$

Proof. Let $\gamma:\mathbb{R}^n\to\mathbb{R}$ be a smooth cut-off function such that $\gamma(x)=1,\forall x\in V_1$ where V_1 is some open neighbourhood of

$$\mathbf{supp}(\phi)\subseteq V_1\subseteq V$$

and $\gamma\geq 0$. Let $c=\int_{\mathbb{R}^n}\gamma(y)dy>0$.

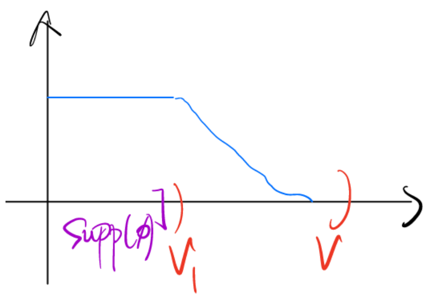


Figure 4.9.1: *Topological Picture*

By the Uniform Approximation Lemma applying to V_1 and $\mathbf{supp}(\phi)\subseteq V_1$, there exists $\psi\in C_c^\infty(V_1)$ such that

$$|\phi(x)-\psi(x)|\leq\frac{\varepsilon}{2c},\forall x\in V_1$$

Hence,

$$\begin{aligned}
\left|\int_V (\phi-\psi)(y)dy\right| &\le \int_V |(\phi-\psi)(y)|dy \\
&= \int_V \gamma|(\phi-\psi)(y)|dy && \text{since } \phi,\psi \text{ are compactly supported in } V_1 \\
&\le \frac{\varepsilon}{2c}\int_v \gamma(y)dy = \frac{\varepsilon}{2c}c = \frac{\varepsilon}{2}
\end{aligned}$$

Thus,

$$\left|\int_V (\phi-\psi)(y)dy\right| = \left|\int_V \phi(y)dy - \int_V \psi(y)dy\right| \le \frac{\varepsilon}{2} \tag{4.9.1}$$

Similarly,

$$\begin{aligned}
&\left|\int_U (\phi-\psi)\circ f\right| \det(Df(x))\,|dx| \\
&\le \int_U (\gamma\circ f)(x)|(\phi-\psi)\circ f(x)||\det(Df(x))|dx \\
&\le \frac{\varepsilon}{2c}\int_U (\gamma\circ f)(x)|\det(Df(x))\,|dx \\
&= \frac{\varepsilon}{2c}\int_V \gamma(y)dy && \text{smooth version CoV} \\
&= \frac{\varepsilon}{2}
\end{aligned}$$

Thus,

$$\left|\int_U (\phi\circ f)(x)\,|\det(Df(x))|\,dx - \int_U (\psi\circ f)(x)|\det(Df(x))|dx\right| \le \frac{\varepsilon}{2} \tag{4.9.2}$$

Combine both equations we have

$$\left|\int_V \phi(y)dy - \int_u (\phi\circ f)(x)()\right| \det(Df(x))|dx| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

■

4.10 Applications of the Degree Theorem

R Let $f: U \to V$ be a continuous proper map with $X \subseteq U$, if $U \backslash X$ is open, then $V \backslash f(X)$ is open.

Theorem 26 Let U, V be non-empty subsets of $\mathbb{R}^n$ (not necessarily connected) and $f: U \to V$ is a proper smooth map. If f is not surjective, then $\deg(f) = 0$.

Proof. Let $X = U \subseteq U$ and $U \backslash X = U \backslash U = \emptyset$ is open, so by the earlier result, we have

$$V \backslash f(U)$$

is open.
If f is not surjective, then $V \backslash f(U)$ is non-empty and open. Thus, there exists $\omega \in \Omega_c^n(V \backslash f(U))$ such that $\int \omega = 1$. Since $\omega = 0$ on $f(U)$, we get

$$f^*\omega = 0$$

everywhere on U. Hence,

$$\deg(f) = \int f^*\omega = 0$$

■

R Equivalently, let $f : U \to V$ be a smooth and proper map. If $\deg(f) \neq 0$, then f is surjective! This means $y = f(x)$ has at least one solution $x \in U$ for any $y \in V$. This is the basic idea for the **Fundamental Theorem of Algebra**.

4.10.1 Brouwer Fixed Point Theorem

Theorem 27 — Brouwer Fixed Point Theorem. Let $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be a closed unit ball. Let $f : B^n \to B^n$ be continuous, then f has at least one fixed point. That is there exists at least one $x \in B^n$ such that $f(x) = x$.

Proof.

Lemma 4.11 Let $U \subseteq \mathbb{R}^n$ be open. Let $C \subseteq U$ be compact. Let $\phi : U \to \mathbb{R}$ be continuous on U and smooth on $U \backslash C$, then, for any $\varepsilon > 0$, there exists $\psi : U \to \mathbb{R}$ that is smooth on U and

$$\sup_{x \in U} |\phi(x) - \psi(x)| < \varepsilon$$

Proof. Let $\rho \in C_c^\infty(U)$ such that $\rho = 1$ on a neighbourhood of C. By the Uniform Approximation Lemma, there exists $\psi_0 \in C_c^\infty(U)$ such that

$$|\rho\phi - \psi_0| < \varepsilon$$

let $\psi = (1-\rho)\phi + \psi_0$, then ψ is smooth since ϕ is smooth away from C and $1 - \rho = 0$ on C.

$$\begin{aligned} \phi - \psi &= (1-\rho)\phi + \rho\phi - [(1-\rho)\phi + \psi_0] \\ &= \rho\phi - \psi_0 \end{aligned}$$

Thus, $|\phi - \psi| < \varepsilon$. And

$$\mathbf{supp}(\phi - \psi) \subseteq \mathbf{supp}(\phi) \cup \mathbf{supp}(\psi) \subseteq U$$

Note that $\mathbf{supp}(\phi) \cup \mathbf{supp}(\psi)$ is compact, so $\mathbf{supp}(\phi - \psi)$ is compact. ■

Back to prove the theorem

By contradiction, suppose f has no fixed points, then for all $x \in B^n$, $f(x) \in B^n$ and $f(x) \neq x$. Let $l(x) = f(x) + t(x - f(x))$ be the unique straight line through $f(x)$,x and $l(0) = f(x0$ and $l(1) = x$. intersects with the boundary

$$\partial B^n = S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$$

at a point $\gamma(x)$. We can solve

$$\|f(x) + t(x - f(x))\|^2 = 1$$

for t. There exists a positive t_+ and negative t_-. We plug back in

$$\gamma(x) = l(t_+)$$

then, $\gamma(x)$ is smooth (Assignment 4) such that $\gamma : B^n \to S^{n-1}$ is continuous.

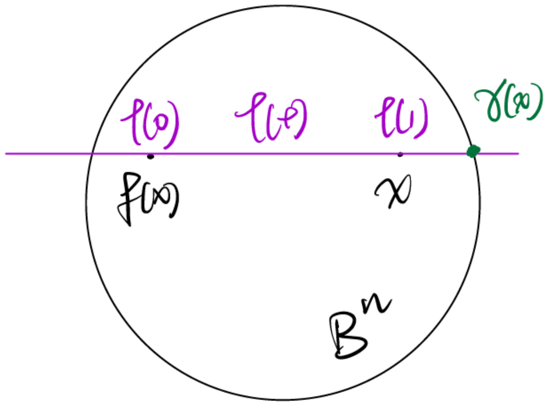


Figure 4.11.1: *Construction of $l(t)$*

If $x \in S^{n-1} = \partial B^n$, then $\gamma(x) = x$. Extend γ to be a map $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ by $\gamma(x) = x$ if $\|x\| > 1$. By construction, $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n$ and smooth outside of $C = B^n$ which is compact. Again, apply the Uniform Approximation Lemma to each component function

$$\gamma_k : \mathbb{R}^n \to \mathbb{R}$$

we get $\psi_k \in C^\infty(\mathbb{R}^n)$ such that

$$\gamma_k - \psi_k$$

has compact support and

$$\sup_{x \in \mathbb{R}^n} |\gamma_k(x) - \psi_k(x)| < \sqrt{\frac{\varepsilon}{n}}$$

Let $\psi = (\psi_1, \ldots, \psi_n)$ be the smooth map and $(\gamma - \psi)$ is compact and

$$\sup_{x \in \mathbb{R}^n} \|\phi(x) - \psi(x)\| < \varepsilon < 1$$

(note that $\varepsilon < 1$ can be assumed WLOG). Thus, $\gamma = \psi$ outside of a compact set but $\gamma(x) = x$ outside of B^n. Thus, $\psi(x) = x$ outside of a compact set. Then,

$$\deg(\psi) = +1$$

We also have

$$\|\psi(x)\| \geq \|\gamma(x)\| - \|\gamma(x) + \psi(x)\| \geq 1 - \varepsilon$$

where $\|\gamma(x)\| \geq, \forall x \in \mathbb{R}^n$ if $x \in B^n$ we have $\gamma(x) \in S^{n-1}$, so $\|\gamma(x)\| = 1$. If $x \notin B^n$, we have $\|\gamma(x)\| = \|x\| > 1$. Thus, the inequality above holds along with triangle inequality.
Hence, $\|\psi(x)\| > 0, \forall x \in \mathbb{R}^n$, so $\psi(x) \neq 0, \forall x \in \mathbb{R}^n$. Thus, ψ is not a surjective map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus, $\deg(\psi) = 0$ by a previous theorem. This yields a contradiction. ■

4.11.1 Techniques for Computing the Degree

Definition 4.11.1 — Critical Points and Regular Points. Let $U, V \subseteq \mathbb{R}^n$ be open and $f : U \to V$ be a smooth and proper. A point $x \in U$ is called a critical point of f if

$$(Df)(x) : \mathbb{R}^n \to \mathbb{R}^n$$

is not an isomorphism. This means $x \in U$ is a critical point if and only if $\det(Df(x)) = 0$.

Let
$$C_f = \{x \in U : x \text{ is a critical point of } f\}$$
and
$$U \backslash C_f = \{x \in U : \det(Df(x)) \neq 0\}$$
be the set of regular points of f and clearly $U \backslash C_f$ is open because $\det(Df(x))$ is continuous given f is smooth.

Definition 4.11.2 — Regular Values and Critical Values. From now on, assume f to be proper. By a previous result, we have
$$V \backslash f(C_f)$$
is open. This is the set of **regular values** of f.
And $f(C_f)$ is called the set of **critical values** of f. So, $y \in V$ is a critical value of f if there exists $x \in C_f$ such that $y = f(x)$.

R Note that there might be other $x \notin C_f$ can do this as well.
Also, by definition, if $y \in V$ and $f \notin \mathrm{im}(f)$, then y is a regular value since
$$f(C_f) \subseteq f(U)$$
So, $V \backslash f(U) \subseteq V \backslash f(C_f)$.

Theorem 28 — Sard's Theorem. Let $U, V \subseteq \mathbb{R}^n$ be open, non-empty. Let $f : U \to V$ be a proper/smooth map. The set $V \backslash f(C_f)$ of regular values of f is an open dense subset of V

Proof. Require Baire Category Theorem from PMATH351 Real Analysis. ■

Given a regular value of f, there exists a procedure for computing $\deg(f)$ which we now describe.

Theorem 29 Let $U, V \subseteq \mathbb{R}^n$ be open. Let $f : U \to V$ be a smooth and proper map. Let $q \in V$ be a regular value of f ($\forall p \in f^{-1}(q), (Df(p))$ is bijective), then,
1. $f^{-1}(q) = \{P_1, \ldots, P_N\}$ is a finite set. (N depends on q)
2. There exists an open connected neighbourhoods $U_1, \ldots, U_N$ of $P_1, \dot{;} P_N$ respectively such that $U_1 \subseteq U, \forall 1 \leq i \leq N$ and $U_i \cap U_j = \emptyset$ if $i \neq j$ (disjoint)
3. There exists an open and connected neighbourhood W of q in V such that
   (a) $U_i \cap U_j = \emptyset$ if $i \neq j$
   (b) $f^{-1}(W) = U_1 \cup \cdots \cup U_N$
   (c) f maps each U_i diffeomorphically onto W.

Proof. Let $p \in f^{-1}(q)$ since q is a regular value $(Df(q))$ is bijective. Hence, by the **inverse function theorem**, f maps an open neighbourhood U_p of p in U diffeomorphically onto an open neighbourhood Y_p of q in V. Let $\mathscr{U} = \left\{U_p : p \in f^{-1}(q)\right\}$ be an open cover of $f^{-1}(q)$. Note that $f^{-1}(q)$ is compact since $\{q\}$ is compact and f is proper. Hence, there exists a $P_1, \ldots, P_N$ in $f^{-1}(q)$ such that
$$f^{-1}(q) \subseteq \bigcup_{i=1}^{N} U_{P_i}$$
for each $i = 1, \ldots, N$. for each $i = 1, \ldots, N$,
$$f\big|_{U_{P_i}} : U_{P_i} \to f(U_{P_i}) = Y_{P_i}$$

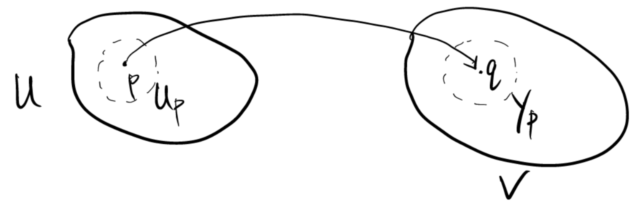


Figure 4.11.2: *U_p and Y_p*

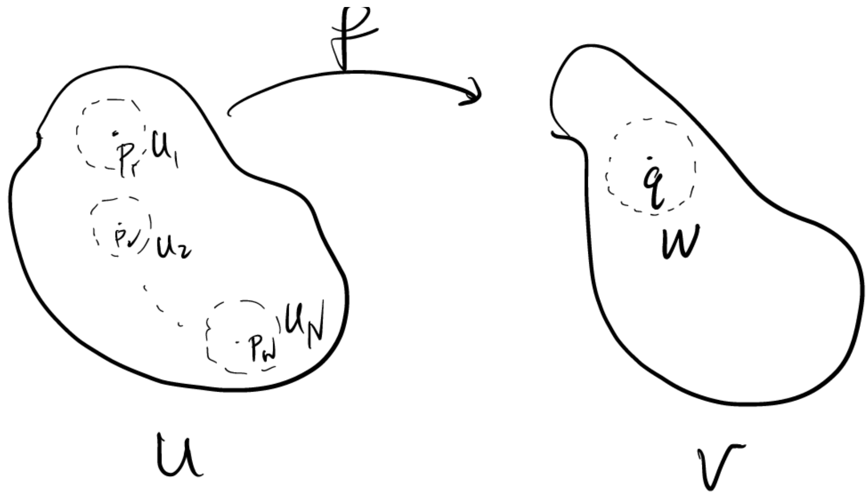


Figure 4.11.3: *Proof Picture*

is a diffeomorphism onto its image taking P_i to q and nothing else in U_{P_i} is sent to q. Hence,

$$f^{-1}(q) = \{P_1, \dots, P_N\}$$

By shrinking, U_{P_i}s, we can assume them to be disjoint if necessary.
Recall, a theorem on proper map from proper map. Let $B = \{q\}$ be compact on V and let $A = f^{-1}(B) = f^{-1}(q)$ be compact on U (since f is proper). Let $U_0 = U_{P_1} \cup \dots \cup U_{P_N}$ be an open set with $A \subseteq U_0 \subseteq U$. Then, there exists V_0 open such that $\{q\} = B \subseteq V_0 \subseteq V$ and $f^{-1}(V_0) \subseteq U_0 \subseteq U$. Let W be this V_0 from the Theorem. So $q \in W \subseteq V$ is open and $f^{-1}(W) \subseteq U_{P_1} \cup \dots \cup U_{P_N}$. Let $U_i = f^{-1}(W) \cap U_{P_i}, \forall 1 \le i \le N$. Note that each U_i is open and $p \in U_i \subseteq U$ and $U_i \cap U_j = \emptyset$ if $i \neq j$ (a) is done. Also,

$$f^{-1}(W) = \bigcup_{i=1}^{N} U_i$$

(b) is done.
f maps U_1 diffeomorphically onto W. (c) is done. ■

Corollary 4.11.1 Consider the same hypothesis as the theorem. For each $P_i \in f^{-1}(q)$ let

$$\sigma_{P_i} = +1$$

if $f\big|_{U_i} : U_i \to W$ is orientation-preserving.

$$\sigma_{P_i} = -1$$

if $f\big|_{U_i} : U_i \to W$ is orientation-reversing.
Then,

$$\deg(f) = \sum_{i=1}^{N} \sigma_{P_i}$$

in particular, $\deg(f) \in \mathbb{Z}$.

Proof. Let $\omega \in \Omega_c^n(W)$ with $\int \omega = 1$. Since

$$f^{-1}(W) = U_1 \cup \cdots \cup U_N$$

we get $\mathbf{supp}(f^*\omega) \subseteq U_1 \cup \cdots \cup U_N$. So,

$$\deg(f) = \int f^*\omega = \sum_{i=1}^{N} \int_{U_i} f^*\omega = \sum_{i=1}^{N} \sigma_{P_i} \in \mathbb{Z}$$

■

R Suppose $f : U \to V$ is not surjective, then any $q \in V \backslash f(U)$ is a regular value. So, the above theorem will also give $\deg(f) = 0$ since $f^{-1}(q) = \emptyset$.

4.12 Topological Invariance of The Degree

Definition 4.12.1 — Smooth Homotopy. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open. Let A be an open interval in $\mathbb{R}$ contained in $[0,1]$. Let $f_0, f_1 : U \to V$ be smooth maps. Then, a smooth map

$$F : U \times A \to V$$

is called a smooth homotopy between f_0, f_1 if $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in U$.

R Think of $f_t(x) = F(x,t)$ is a one parameter family of smooth maps from U to V that starts at f_0 and ends at f_1.

Now, suppose further that f_0, f_1 are both proper, then **proper**.

Definition 4.12.2 — Smooth & Proper Homotopy. A smooth homotopy F between f_0, f_1 is called a proper homotopy if the map

$$F^\# : U \times A \to V \times A$$

given by $F^\#(x,t) = (F(x,t),t)$ is proper.

R Suppose F is a proper homotopy between f_0, f_1, then $f_t : U \to V$ is proper for all $t \in A$. To see this, let $K \subseteq V$ be compact. Then, $K \times \{t\}$ is compact in $V \times A$. Then,

$$\left(F^\#\right)^{-1}(K \times \{t\}) = \left\{(x,s) \in U \times A : F^\#(x,s) = (F(x,s),s) \in K \times \{t\}\right\} = f_t^{-1}(K), \forall t \in A$$

which is compact.

Theorem 30 Let $f_0, f_1 : U \to V$ be smooth and proper maps. Suppose there exists a proper smooth homotopy between them, then

$$\deg(f_0) = \deg(f_1)$$

Proof. Let $\omega = h(y)dy_1 \wedge \cdots \wedge dy_n \in \Omega_c^n(V)$ and $\int \omega = 1$. Then,

$$\deg(f_t) = \int_U f_t^* \omega = \int_U h(F_1(x,t), \ldots, F_n(x,t)) \det(DF_x(x,t)) dx_1 \wedge \cdots \wedge dx_n$$

note that $h(F_1(x,t), \ldots, F_n(x,t)) \det(DF_x(x,t))$ is continuous in t. Let $t \in [0,1]$, if $h(F(x,t)) \neq 0$, then $F(x,t) \subseteq \mathbf{supp}(h)$. Then, $F^\#(x,t) \subseteq \mathbf{supp}(h) \times [0,1]$. Then,

$$(x,t) \in \left(F^\#\right)^{-1}(\mathbf{supp}(h) \times [0,1])$$

since $\mathbf{supp}(h) \times [0,1]$ is compact and $\left(F^\#\right)^{-1}(\mathbf{supp}(h) \times [0,1])$ is compact since $F^\#$ is a proper homotopy. Thus, the integrand is supported in a compact subset of $U \times [0,1]$. By differentiation under the an integral sign, we know that $\deg(f_t)$ is continuous in $t \in [0,1]$. But $\deg(f_t) \in \mathbb{Z}$. Thus, it must be a constant. ■

4.12.1 Examples of Computing the Degree Using the Algorithm

■ **Example 4.1** Let $f : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C} \cong \mathbb{R}^2$ given by $f(z) = z^n$ and $n \in \mathbb{Z}^+$. Note that f is smooth since each component function is a homogenous polynomial in x, y of degree n. Also $(Df)(z) : \mathbb{R}^2 \to \mathbb{R}^2$ is the unique linear map such that

$$\frac{f(z+h) - f(z) - h(Df)(z)}{\|h\|} \xrightarrow{h \to 0} 0, h \in \mathbb{R}^2 \cong \mathbb{C}$$

for $f(z) = z^n$ in particular,

$$\frac{(z+h)^n - z^n - nz^{n-1}h}{\|h\|} \xrightarrow{h \to 0} 0$$

so $\det(Df(z)) = \left|nz^{n-1}\right|^2 = n^2|z|^{2n-2} > 0$ whenever $z \neq 0$.
Note that f is a proper map. Since

$$z^n = Re^{i\theta} \iff z = R^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}$$
$$|f(z)| \leq R \iff |z| \leq R^{\frac{1}{n}}$$

Thus, $f^{-1}(B_R) \subseteq B_{R^{\frac{1}{n}}}$.
Hence, any $w \neq 0$ in $\mathbb{C}$ is a regular value of f because if $w \neq 0$, $f^{-1}(w) \neq 0$ and $(Df(z)) > 0, \forall z \in f^{-1}(w)$ implies that $w \neq 0$ is a regular value. Thus, $z = 0$ is the only critical point of f. If follows from the algorithm that

$$\deg(f) = n$$

since there are n roots of unity. ■

4.12.2 Fundamental Theorem of Algebra

Definition 4.12.3 Let $P : \mathbb{C} \times A \to \mathbb{C}$ where A is a bounded open interval containing $[0,1]$ by

$$P(z,t) = z^n + t \sum_{k=0}^{n-1} a_k z^k$$

being smooth in z,t. Note that $P(z,0)=z^n=f(z)$ and $P(z,1)=p(z)$ given below in the theorem.

Theorem 31 — Fundamental Theorem of Algebra. Let $p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0, a_k\in\mathbb{C}$ be a degree n polynomial with complex coefficients. Then, there exists $z_0\in\mathbb{C}$ such that $p(z_0)=0$.

Proof. We want to show $P:\mathbb{C}\times A\to\mathbb{C}$ is proper. It follows that $P_t(z)=P(z,t)$ is proper for all $t\in A$.
Let $M=\max_{0\le k\le n-1}|a_k|$ and let $A\subseteq[-L,L]$. Thus, $|t|\le L,\forall t\in A$.
If $|z|^k\le|k|^{n-1},\forall k=0,1,\ldots,n-1$, then

$$\left|a_0+a_1z+\cdots+a_{n-1}z^{n-1}\right|\le|a_0|+|a_1||z|+\cdots+|a_{n-1}||z|^{n-1}\le nM|z|^{n-1}$$

when $|z|\ge1$. Then,

$$|P(z,t)|=\left|z^n+t\sum_{k=1}^{n-1}a_kz^k\right|\ge|z|^n-|t|\left|\sum_{k=0}^{n-1}a_kz_k\right|\ge|z|^n-LnM|z|^{n-1}$$

Suppose $|z|\ge 2LMn$, then

$$|P(z,t)|\ge LMn|z|^{n-1}$$

if $|z|\ge\max(1,2LMn)$.
Now, for $R>0$ such that $|P(z,t)|\le R$. We have several cases to discuss

1. $|z|\le1$
2. $|z|\le 2LMn$
3. $LMn|z|^{n-1}\le R$

Then,

$$P^{-1}(B_R)\subseteq\{|z|\le1\}\cup\{|z|\le 2LMn\}\cup\left\{LMn|z|^{n-1}\le R\right\}$$

implies $P^{-1}(B_R)$ is compact and P is a proper homotopy. Then, by the topological invariance, we have

$$\deg(p(z))=\deg(P(z,1))=\deg(P(z,0))=\deg(f(z))=n\neq0$$

Thus, we know p is surjective and there exists $z_0\in\mathbb{C}$ such that $p(z_0)=0$. ■

5. Manifold

5.1 Introduction to Sub-manifold in $\mathbb{R}^n$

Definition 5.1.1 — Manifolds. A k-dimensional manifold in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that "locally" looks like open sets in $\mathbb{R}^k$. We want to make this definition precise.

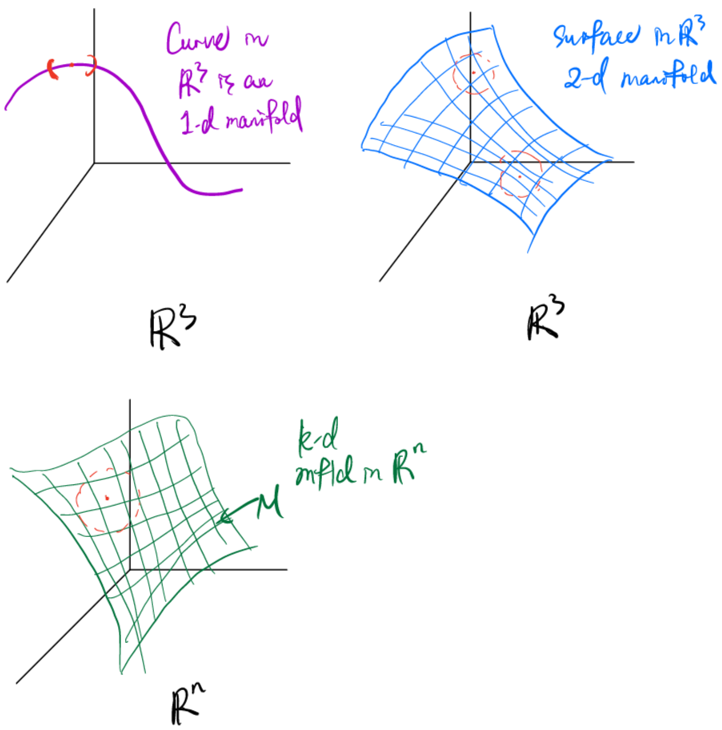


Figure 5.1.1: *Graphical Examples of Manifolds*

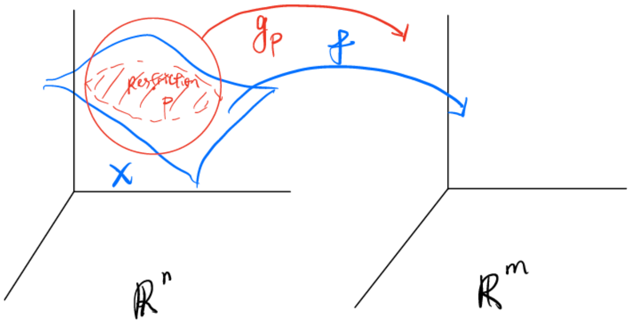


Figure 5.1.2: *What actually happened*

Definition 5.1.2 — Smooth (Manifold version). Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be subsets of $\mathbb{R}^n$. Suppose $f : X \to Y$ is continuous. We say that f is smooth if and only if for all $p \in X$ there exists U_p open in $\mathbb{R}^n$ such that $p \in U_p$ and there exists $g_p : U_p \to \mathbb{R}^m$ smooth such that

$$g_p\big|_{U_p \cap X} = f\big|_{U_p \cap x}$$

We say $f : X \to Y$ is smooth if it is locally the restriction to X of a smooth map.

In Appendix A, using partition of unity, we proved that

Theorem 32 If $f : X \to Y$ is smooth (in the above sense), then there exists $U \subseteq \mathbb{R}^n$ open, $X \subseteq U$ and $g : U \to \mathbb{R}^m$ being smooth such that

$$g\big|_U = f$$

again, U and g are not unique.

R This theorem says that the map $f : X \to Y$ is locally the restriction to X of a smooth map if and only if it is globally the restriction to X o a smooth map.

Definition 5.1.3 — Diffeomorphism. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. We say that a continuous map $f : X \to Y$ is a diffeomorphism if

1. f is smooth in the above sense
2. f is a bijection
3. $f^{-1} : Y \to X$ is smooth

R So a diffeomorphism between subsets of Euclidean spaces is a smooth bijection with smooth inverse.

Definition 5.1.4 — k-Manifold. Let $0 \le k \le n$ be an integer. A non-empty subset $M \subseteq \mathbb{R}^n$ is called a k-dimensional manifold if and only if for all $p \in M$, there exists an open neighbourhood V of p in $\mathbb{R}^n$ and an open subset U in $\mathbb{R}^k$ and a diffeomorphism:

$$\phi : U \to M \cap V$$

A k-manifold M is locally diffeomorphic to $\mathbb{R}^k$.

The map $\phi : U \to M \cap V$ is called a local **parametrization** of M and $\phi^{-1} : M \cap V \to U$ is called a **coordinate** chart. Informally, a k-manifold is locally like $\mathbb{R}^k$.

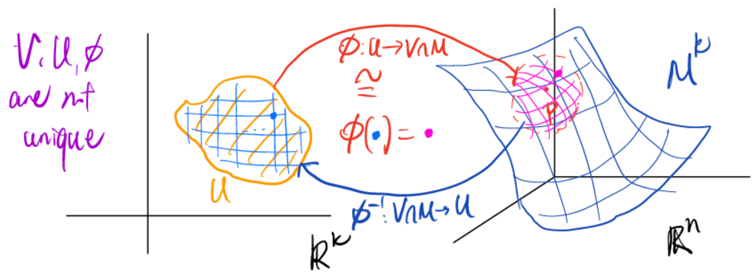


Figure 5.1.3: *Local Diffeomorphic*

5.1.1 Some Examples

■ **Example 5.1** Let $M \subseteq \mathbb{R}^n$ be an open subsets. Let $p \in M$ and $V = M$ be open neighbourhood of V. And $V \cap M = M$, take $U = V \subseteq \mathbb{R}^k = \mathbb{R}^n$ and $n = k$. Let $\varphi = Id : U = M \to M = V \cap M$ is a diffeomorphism (this is globally though...)
So, any open set in $\mathbb{R}^n$ is an n-dimensional manifold in $\mathbb{R}^n$. ■

■ **Example 5.2** Let $U \subseteq \mathbb{R}^n$ be open, let $f : U \subseteq \mathbb{R}^k \to \mathbb{R}^m$ be smooth

$$f(x_1, \ldots, x_k) = (f_1(x_1, \ldots, x_k), \ldots, f_m(x_1, \ldots, x_k))$$

each component is smooth.
Define

$$\Gamma_f := \left\{ (x, f(x)) \in \mathbb{R}^k \times \mathbb{R}^m \cong \mathbb{R}^{k+m} \right\}$$

to be the **graph** of f.

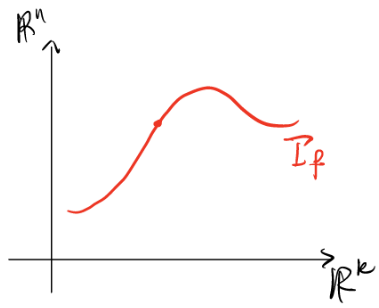


Figure 5.1.4: *Graph*

Claim: Γ_f is a k-manifold in $\mathbb{R}^{k+m}$
Proof. Define $\phi : U \to M \cap V$ by

$$\phi(x) = (x, f(x))$$

being smooth and let $\pi : \mathbb{R}^{k+m} \to \mathbb{R}^k$ given by $\pi(x, y) = x$ being smooth. Then, $\pi \circ \phi : U \to \mathbb{R}^k$ and,

$$(\pi \circ \phi)(x) = \pi(x, f(x)) = x$$

so, $\phi : U \to \Gamma_f$ is a diffeomorphism with inverse

$$\pi\big|_{\Gamma_f} : \Gamma_f \to U$$

■

■

> **R** Note that in both (a) and (b), we are able to cover the whole manifold with a single (local) parametrization. In general, this is not gonna happen.

■ **Example 5.3** Recall

$$S^{n-1} = \left\{x \in \mathbb{R}^n : \|x\|^2 = 1\right\}$$

the standard $(n-1)$-sphere in $\mathbb{R}^n$.

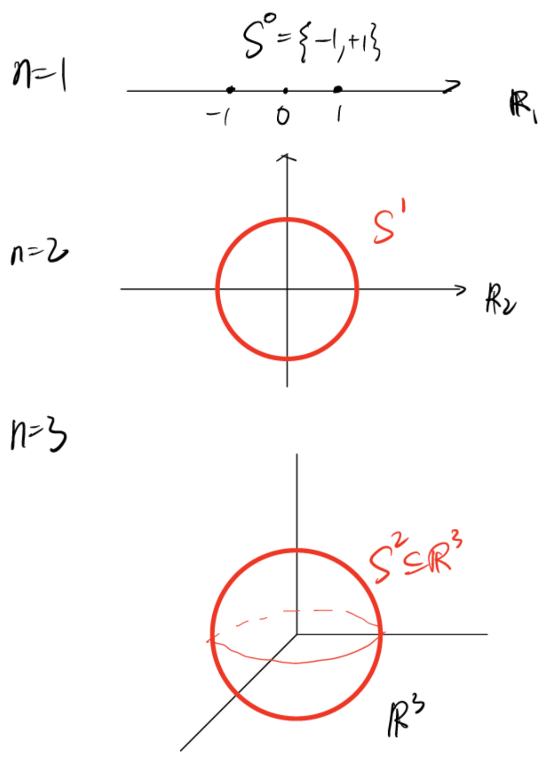


Figure 5.1.5: S^{n-1} *Spheres*

S^{n-1} is a $(n-1)$-manifold

Proof. We cannot cover S^{n-1} with a single parametrization since S^{n-1} is compact. (If we can do it with a single parametrization, then we cannot have U to be open since there is a diffeomorphism in between).

Let $P = (P_1, \ldots, P_n) \in S^{n-1}$ and $p \neq 0$, so there exists $P_i \neq 0$ for some i. Either $P_i > 0$ or $P_i < 0$. We assume $P_i > 0$ for now.

Let

$$V_i^{\pm} = \{P = (P_1, \ldots, P_n) \in \mathbb{R}^n : P_i > 0 \text{ or } P_i < 0\}$$

be open in $\mathbb{R}^n$.

Let $U = \{(y_1,\dots,y_{n-1}) \in \mathbb{R}^{n-1} : y_1^2+\dots+y_n^2 < 1\}$ be an open unit disk centred at the origin. Then,

$$U_i^+ \cap S^{n-1} = \left\{(x_1,\dots,x_n) \in \mathbb{R}^n : \|x\|^2 = 1, x_i > 0\right\}$$

on $V_i^+ \cap S^{n-1}$ where $x_i = \sqrt{1-(x_1^2+\dots+x_{i-1}^2+x_{i+1}^2+\dots+x_n^2)}$ on $(x_1,\dots,\hat{x}_i,\dots,x_n)$. Define $f_i^\pm : U \to V_i^\pm \cap S^{n-1}$ given by

$$f_i^\pm(y_1,\dots,y_{n-1}) = \left(y_1,\dots,y_{i-1}, \pm\sqrt{1-(y_1^2+\dots+y_{n-1}^2)}, y_i,\dots,y_{n-1}\right)$$

is a smooth diffeomorphism of U onto $V_i^\pm \cap S^{n-1}$ Notice $V_i^+ \cap S^{n-1}$ is the graph of a smooth

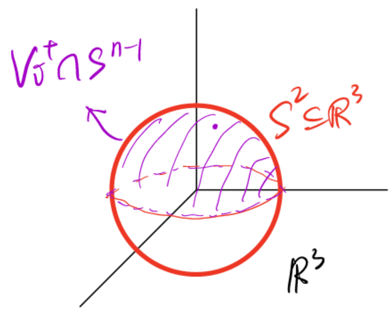


Figure 5.1.6: $V_i^+ \cap S^{n-1}$

function. Here we cover it with $2n$ local parametrization. ■

▪

> **R** In Assignment 5, we can cover S^{n-1} with only 2 local parametrizations.

To study more examples, we need a corollary of the **Implicit Function Theorem**.

Theorem 33 — Implicit Function Theorem. Let $N \subseteq \mathbb{R}^{k+l} \cong \mathbb{R}^k \times \mathbb{R}^l$ be open. Let $F : \mathbb{R}^{k+l} \to \mathbb{R}^l$ be smooth given by

$$(x,y) \mapsto F(x,y) \in \mathbb{R}^l$$

let $(x_0,y_0) \in W$ such that $F(x_0,y_0) = 0$ Then,

$$\underbrace{(Df)_{(x_0,y_0)}}_{l\times(k+l)\text{ matrix}} = \left(\underbrace{\frac{\partial f_i}{\partial x_a}}_{l\times k}, \underbrace{\frac{\partial f_i}{\partial y_j}}_{l\times l}\right)\Bigg|_{(x_0,y_0)} = (*, A)$$

where $A_{ij} = \frac{\partial f_i}{\partial y_j}(x_0,y_0)$ is invertible. Then, there exists an open neighbourhood W' of (x_0,y_0) in W. and an open neighbourhood U of x_0 in $\mathbb{R}^k$ and a smooth map $h : U \subseteq \mathbb{R}^k \to \mathbb{R}^l$ such that

$$\underbrace{\{(x,y) \in W', F(x,y) = 0\}}_{W\cap F^{-1}(0)} = \underbrace{\{(x,h(x)), x \in U\}}_{\text{Graph of a smooth function}}$$

Thus, $F^{-1}(0)$ is a k-manifold on $\mathbb{R}^{k+l}$.

Corollary 5.1.1 Let $F : \mathbb{R}^{k+l} \to \mathbb{R}^l$ be smooth. If $(DF)(x,y)$ is surjective for all $(x,y) \in F^{-1}(0)$, then $F^{-1}(0)$ is a k-manifold in $\mathbb{R}^{k+l}$.