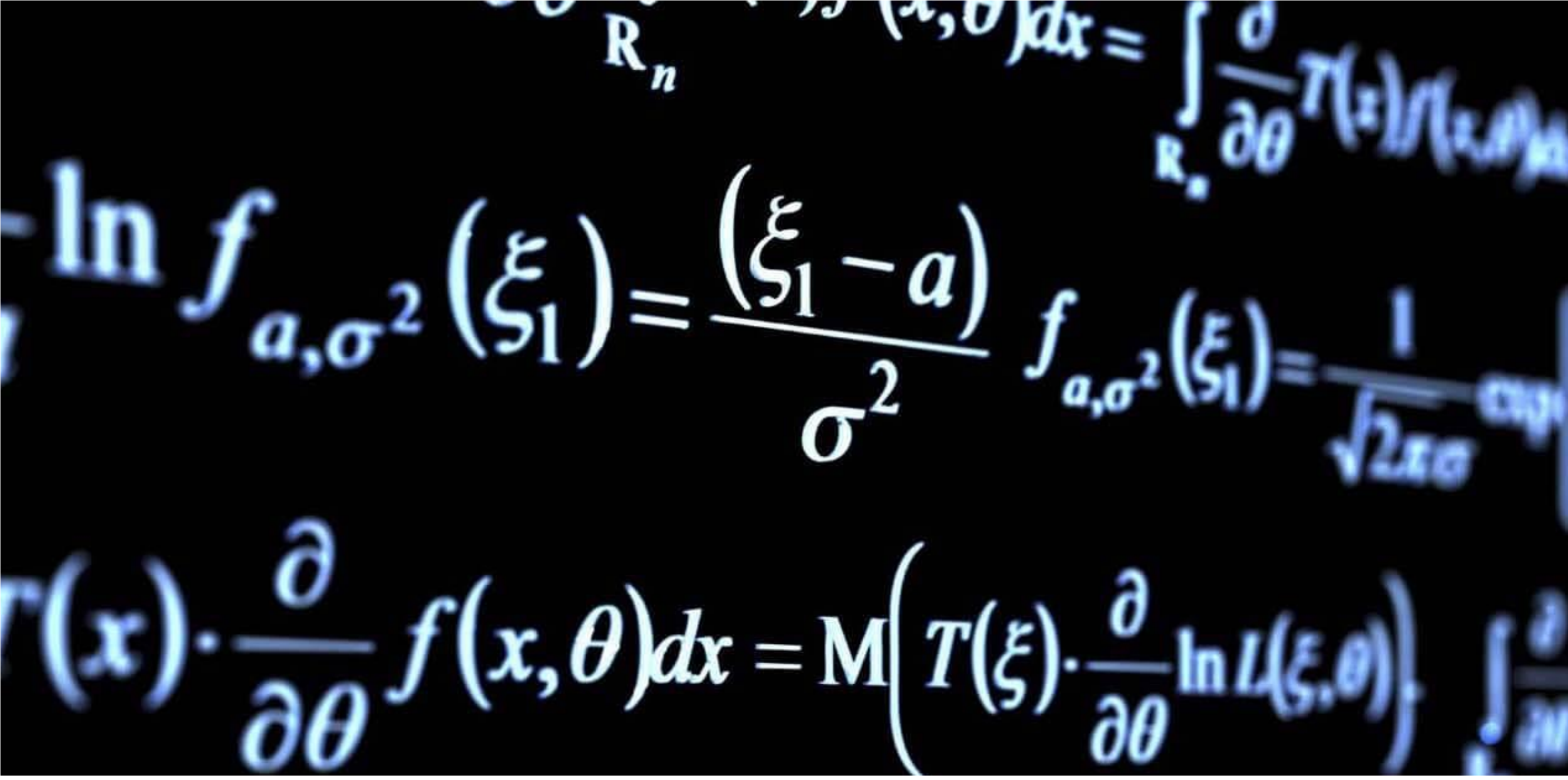

IEOR E4707 Course Notes

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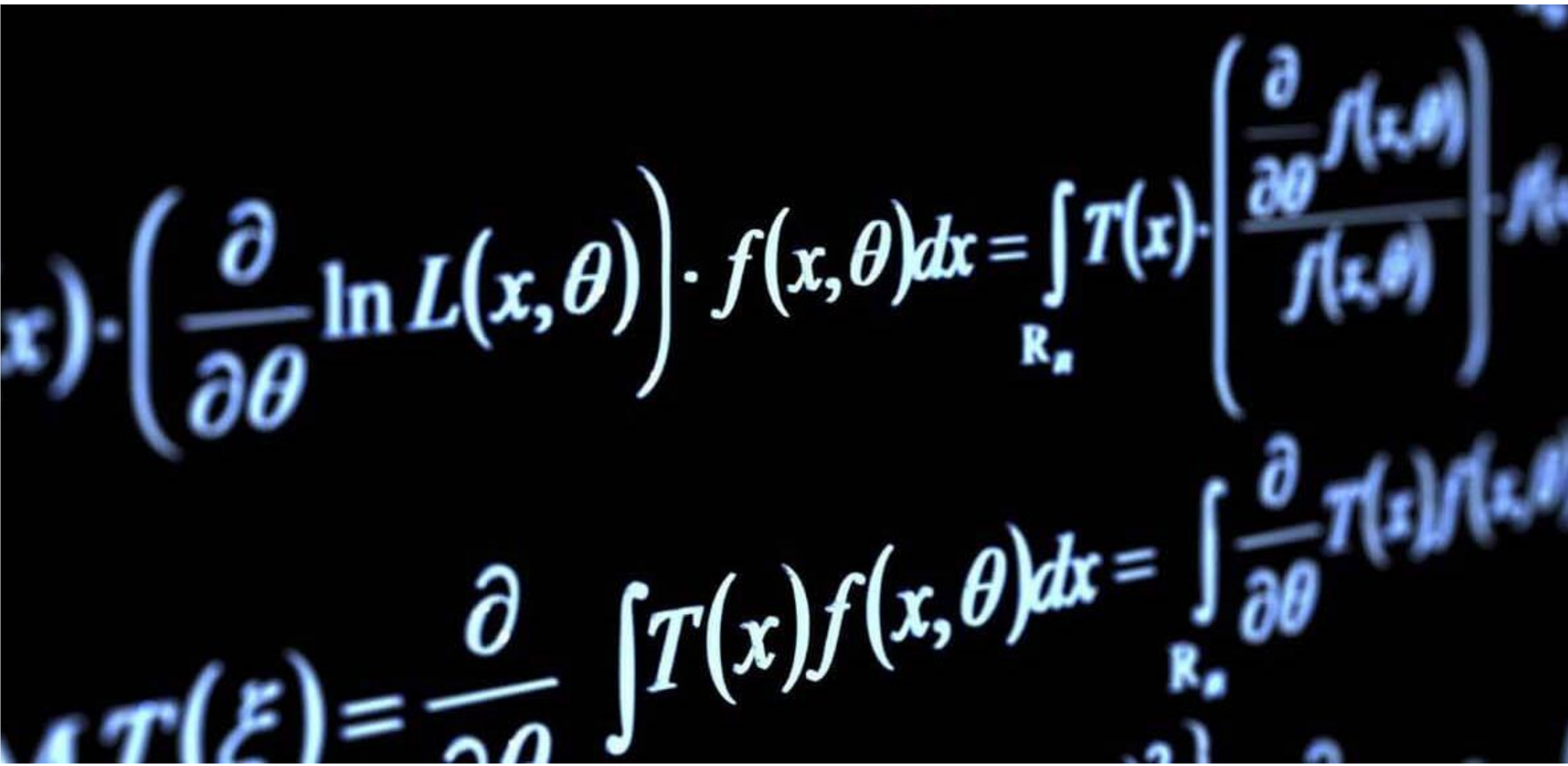
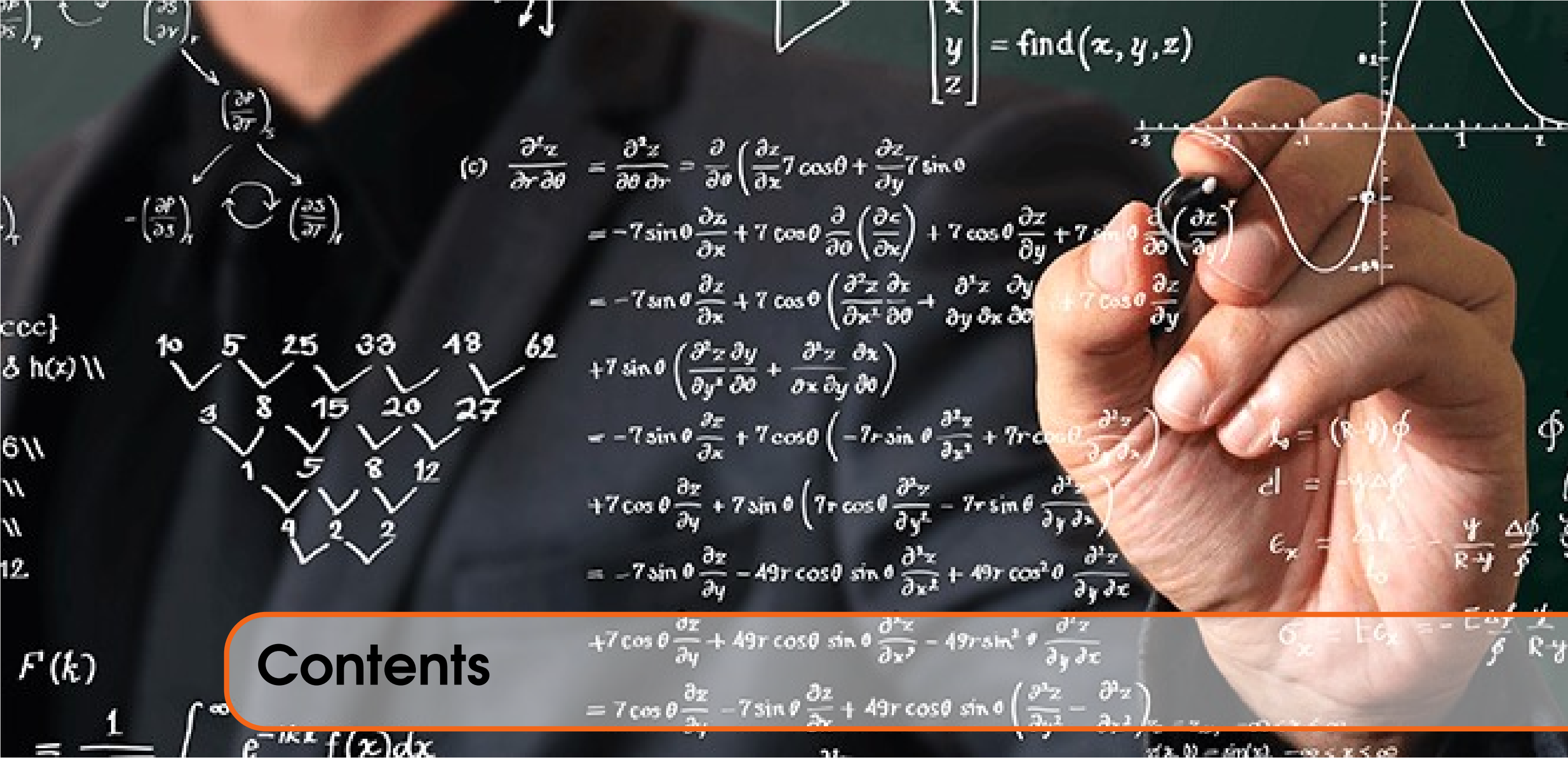


This note is for non-commercial use. Since this is not really used by a broad audience (probably just me), I will only include the necessary key points and figures. Lots of examples might be omitted. Use this at one's own discretion. This essentially records the lecture board writing by Prof. David D. Yao in IEOR E4707.

January 2023



Contents

1 Stock-Price Dynamics 5

1.1 Black-Scholes-Merton PDE 5

1.1.1 No-Arbitrage Principle 6

1.1.2 Solution of BSM PDE 6

1.2 Extension: Continuous Dividend Rate q 7

1.2.1 Applications 8

1.3 Delta Hedging 9

1.3.1 Hedging = Pricing 10

1.3.2 Self-Financing 11

2 Girsanov's Theorem 13

2.1 Change of Measure 13

2.1.1 Quadratic Variation 14

2.2 Change of Measure: BM 15

2.3 Risk-Neutral Measure 16

2.4 Comments on Ito's Integral Being MG 17

2.4.1 MG Representation Theorem 18

3 Multidimensional Case 19

3.1 Going Multidimensional 19

3.2 Numeraire & Applications 21

3.2.1 Numeraire 21

3.2.2 Applications 22

3.2.3 Domestic Risk-Neutral Measure $\mathbf{Q}$ 22

3.2.4 Quanto Option 23

4 Feyman-Kac Theorem 25
4.1 From SDE to PDE 25
4.1.1 Interest Rate Model 27
4.1.2 Hull-White Model 27
4.1.3 CIR Model 28
4.2 Applications 28
4.2.1 Asian Option 28
4.2.2 Linear Diffusion Models 29
4.2.3 Forward Rate Model 29

5 Portfolio Selection 33
5.1 Merton's Problem 33
5.2 Markowitz M-V Problem 35
5.2.1 Efficient Frontier 36
5.3 Commentaries 36

6 American Option 39
6.1 Basic Properties 39
6.2 Refresher on 4701 39
6.2.1 Asymptotic Analysis 40
6.3 Perpetual American Put 41
6.3.1 Analytical Properties 41
6.3.2 Probabilistic Properties 42
6.4 Martingale Review 43
6.4.1 Jenson's Inequality 43
6.5 American Call Paying Discrete Dividends 44
6.6 American Puts with Finite Maturity 44

1. Stock-Price Dynamics

1.1 Black-Scholes-Merton PDE

Recall that the stock follows a geometric brownian motion given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

Consider the replicating portfolio Π_t given by

$$\Pi_t = \begin{cases} \Delta = \frac{\partial f}{\partial S_t} \text{ shares of stock} \\ -1 \text{ share of the option} f = f(t, S_t) \end{cases} = \Delta S_t - f$$

Later on we see that $\Delta \in [0,1]$ and it is one of the greeks of an option. Moreover, notation-wise, we have

$$d\Pi_t = \Pi_{t+dt} - \Pi_t$$

From Ito's formula, we know that

$$\begin{aligned} d\Pi_t &= \Delta dS_t - df \\ &\overset{\text{Ito}}{=} \Delta dS_t - \left(\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} dt \right) \\ &= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} \right) dt \end{aligned}$$

R Every partial is deterministic, thus, every term ending with dt is deterministic. The only stochastic component of this PDE is the one ending with dS_t.

Note that, at the end, we do not have a stochastic PDE but a deterministic one. This is what we mean by "replicating", it means eliminating the randomness so that we can always replicate such a portfolio performance without any surprise. This is also why we have such number of shares to pair up with that short one option in our Π_t.

1.1.1 No-Arbitrage Principle

Based on our formulation, since our replicating portfolio has no randomness in its performance, its performance should be the same as saving the same amount of money in the bank to attain risk-free yield. Mathematically,

$$\begin{aligned} d\Pi_t &= \Pi_t r dt = (\Delta S_t - f) r dt \\ -\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} &= r\Delta S_t - rf \\ &\implies \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + r\Delta S_t = rf \end{aligned}$$

which is the famous **Black-Scholes-Merton** PDE. Why do we care about this PDE? This gives us the dynamics of the stock price with respect to t and S_t.

1.1.2 Solution of BSM PDE

To solve such a PDE, we need to have some well-defined boundary conditions. In this case, we have

$$f(T, S_T) = \begin{cases} (S_T - K)^+ & f \text{ is call} \\ (K - S_T)^+ & f \text{ is put} \end{cases}$$

which is the final payoff at maturity. From IEOR E4706, by the means of martingale pricing, we have the BSM PDE solution as, for a call,

$$c(t, S_t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)$$

where

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_1 = d_2 + \sigma\sqrt{T-t}$$

To verify this is indeed a solution of the BSM PDE, we first check the boundary conditions. Let $t \to T$, then $d_1 = d_2$ apparently. Secondly, the secondly term of $d_1 \to 0$. Now, we have some cases regarding S_T, K,

- $S_T > K$: then $d_1 = d_2 = +\infty$, then $c(T, S_T) = S_T - K$.
- $S_T < K$: then $d_1 = d_2 = -\infty$, then $c(T, S_T) = 0$.
- $S_T = K$: then regardless of the value of d_1, d_2, we have $c(T, S_T) = 0$.

We have verified the boundary conditions. Now, we want verify the PDE. The PDE can be rewritten in option greeks as

$$\Theta + rS_t\Delta + \frac{1}{2}\sigma^2 S_t^2 \Gamma = rf$$

where

$$\Delta = \Phi(d_1), \quad \Gamma = \frac{\phi(d_1)}{\sigma S_t \sqrt{T-t}}, \Theta = -\frac{\sigma S_t \phi(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2)$$

Note that

$$\Theta = -\frac{1}{2}\sigma^2 S_t^2 \Gamma - rKe^{-r(T-t)}\Phi(d_2)$$

we substitute this Θ expression into the PDE above to get

$$rS_t\Phi(d_1) - rKe^{-r(T-t)}\Phi(d_2) = rf \implies S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) = c(t, S_t)$$

This certainly assumed that $r \neq 0$, but the derivation is similar if $r = 0$. Now, we are done with the verification.

1.2 Extension: Continuous Dividend Rate q

What if the underlying stock pays a continuous dividend? First, how valid it is for us to assume a continuous dividend yield rate? In real life, companies pay out dividends at the board's discretion to its shareholders. The payout time is not necessarily pre-determined nor necessarily periodically. Here, it is better to think of the dividend as some kind of income that the underlying stock is generating for the stock holder, this is a great intuition for extension to other products as well. To justify the assumption of having a continuous dividend yield rate $q > 0$, a lot of the financial underlyings are actually indices, for which its constituents do pay out dividends quite frequently. In aggregation, it can be considered that the index itself has a continuous divident yield for the ease of modeling.
Now, we revisit the no-arbitrage equation that we had in the previous section. The only thing we need to modify are the terms related to S_t. In particular, our portfolio has an additional income coming from holding the stock. The following equation describe such a dynamic

$$d\Pi_t + \Delta S_t q dt = \Pi_t r dt$$

and again, we have the following new PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + (r-q)\Delta S_t = rf$$

We can still solve this analytically while using the result we had already as much as possible. Consider an auxiliary option $\tilde{f}$ such that $f = \tilde{f}e^{-q(T-t)}$.

Claim: $\tilde{f}$ **solves**

$$\frac{\partial \tilde{f}}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \tilde{f}}{\partial S_t^2} + (r-q)\Delta S_t = (r-q)\tilde{f}$$

In particular, if $\tilde{f} = \tilde{c}$, then

$$\tilde{c}(t, S_t) = S_t\Phi(d_1) - Ke^{-(r-q)(T-t)}\Phi(d_2)$$

where

$$d_2 = \frac{\ln(S_t/K) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_1 = d_2 + \sigma\sqrt{T-t}$$

Proof. To verify the claim, we first compute the 3 partials in the PDE.

$$\frac{\partial f}{\partial S_t} = \frac{\partial \tilde{f}}{\partial S_t} e^{-q(T-t)}$$
$$\frac{\partial^2 f}{\partial S_t^2} = \frac{\partial^2 \tilde{f}}{\partial S_t^2} e^{-q(T-t)}$$
$$\frac{\partial f}{\partial t} = \frac{\partial \tilde{f}}{\partial t} e^{-q(T-t)} + q\tilde{f}e^{-q(T-t)}$$

By substitution, we have

$$\frac{\partial \tilde{f}}{\partial t}e^{-q(T-t)} + q\tilde{f}e^{-q(T-t)} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \tilde{f}}{\partial S_t^2} e^{-q(T-t)} + (r-q)S_t \frac{\partial \tilde{f}}{\partial S_t} e^{-q(T-t)} = r\tilde{f}e^{-q(T-t)}$$
$$\frac{\partial \tilde{f}}{\partial t} + q\tilde{f} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \tilde{f}}{\partial S_t^2} + (r-q)S_t \frac{\partial \tilde{f}}{\partial S_t} = r\tilde{f}$$
$$\frac{\partial \tilde{f}}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \tilde{f}}{\partial S_t^2} + (r-q)S_t \frac{\partial \tilde{f}}{\partial S_t} = (r-q)\tilde{f}$$

and we are done. ■

> **R** How the hell we were so smart fo come up with claim in the first place? This is not so surprising if one can identify that the PDE above is a homogeneous PDE since $f = 0$ is an obvious solution. And such first degree coefficient changes can be retrieved by exponential shift.

Now, we can substitute f into the call $\tilde{f}$ to get

$$c(t, S_t) = S_t e^{-q(T-t))}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where

$$d_2 = \frac{\ln(S_t/K) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_1 = d_2 + \sigma\sqrt{T-t}$$

> **R** Note that we can write
>
> $$d_2 = \frac{\ln\left(S_t e^{-q(T-t)}/K\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
>
> this seems like we just replaced S_t by $S_t e^{-q(T-t)}$ in the original PDE solution. This gives further financial insights:
>
> 1. The underlying price drops by the dividend amount whenever it gets paid out. This is re-written version reflects such a discounting.
> 2. The underlying dividend does have impact on the option value.

1.2.1 Applications

Stock Index

Options over stock indices are some of the most liquid and highly frequently traded financial instruments. As we explained before, these are ideal underlyings to be modeled with continuous dividend rate. Usually, we can consider q being the average dividend rate of the index.

Foreign Currency

Consider the underlying being the Euro and the domestic curreny is USD. The option of Euro does not grant ownership to the option holder and the Euro itself enjoys some foreign risk-free rate. This is equivalent to the Euro, the underlying, generating some form of income with a continuous rate $q = r_f$. We can, therefore, apply our continuous dividend model fairly intuitively.

Options on Future

Consider the underlying being some future contract on some other underlyings. Recall a forward contract f's value with contract price K is given by

$$f(0, S_0) = e^{-rT}\mathbf{E}^{\mathbb{Q}}(S_T - K) = e^{-rT}(S_0 e^{rT} - K) = S_0 - Ke^{-rT}$$

On the other hand, the future price is the K value that makes $f(0, S_0) = 0$. So, at time 0, the future holder pays nothing but at time T, pay K. In this case, the fair value $K = S_0 e^{rT}$ and the future price at time t is $F_t = S_t e^{r(T-t)}$. In this case, the formula tells us the future generates income at rate of r, which means we set $q = r$.

> **R** Note that since $\frac{dS_t}{S_t} = rdt + \sigma dB_t$ for stock with no-dividend payment. One can derive that the dynamic for a future price is $\frac{dF_t}{F_t} = \sigma dB_t$. This is not only a GBM but also a martingale as there is no drift term.

> **R** **What if the underlying pays dividend?** The future value at time t is $F_t = S_t e^{(r-q)(T-t)}$ as the underlying's dividend cannot be materialized by the future holder. Then,
>
> $$F_t = S_0 e^{(r-q-\sigma^2/2)t+\sigma dB_t} e^{(r-q)(T-t)} = F_0 e^{-\sigma^2 t/2+\sigma B_t}$$
>
> Thus, $\frac{dF_t}{F_t} = \sigma dB_t$ and the future price process is still a martingale!

1.3 Delta Hedging

Let's consider a motivating example.

Example 1.1 — What should the bank do?. A bank sold 100k shares of a call option to a client. $S_0 = 49, K = 50, r = 0.05, \sigma = 0.2, T = \frac{20}{52} \approx 0.3846$. By BSM formula, the call option value is $c = 2.40$ per share. The stock does not pay dividends. Thus, in total, the option value is 240k and the bank marks it up to 300k with 60k service charge. What should the bank do between time 0 and the maturity, after 20 weeks?

1. Do nothing: this is called a **naked positions**, if $S_T = 55$, then the bank's loss is $5 \times 100k =$ 0.5m. This is bad.
2. Covered positions: buy 100k share of the stock at $t = 0$ at 49 dollar per share. If $S_T = 44$, again the loss will be 0.5m. This is also bad.
3. Delta hedging: take a position according to $\Delta = \Phi(d_1) = 0.522$, i.e., long $0.522 \times 100k =$ 52.2k shares at time 0. Then, adjust the position over $(0, T)$ since the Δ changes as the underlying changes.

```
%## Delta Hedging

import numpy as np
from scipy.stats.distributions import norm
from typing import Tuple
# np.random.seed(12300)

def BSd1(S: float, K: float, sigma: float, t: float, r: float, q: float) ->
    float:
    return (np.log(S / K) + (r - q + (sigma ** 2 / 2)) * t) / (sigma * np.
    sqrt(t))

def GBMPaths(S, T, mu, sigma, steps):
    """
    Inputs
    #S = Current stock Price
    #K = Strike Price
    #T = Time to maturity 1 year = 1, 1 months = 1/12
    #mu = yield
    # sigma = volatility

    Output
    # [steps,N] Matrix of asset paths
    """
    dt = T / steps
    ST = np.log(S) +  np.cumsum(((mu - (sigma**2/2))*dt + sigma*np.sqrt(dt)
     * np.random.normal(size = steps)),axis=0)

    return np.exp(ST)

def DeltaHedgingShort(
    S: float, K: float, sigma: float, t: float, r: float, q: float, mu:
    float, steps: int, contract_mult: int
) -> Tuple[float, float, float]:
```

```
    st_path = np.concatenate((np.array([S]), GBMPaths(S, t, mu, sigma,
   steps - 1)))
    dt = t / steps
    t_end = t
    prev_delta = 0
    prev_cost = 0
    prev_share = 0
    for st in st_path:
        delta = np.exp(-q * t_end) * norm.cdf(BSd1(st, K, sigma, t_end, r,
   q))
        t_end -= dt
        share = (delta - prev_delta) * contract_mult
        prev_delta = delta
        prev_share += share
        position = share * st
        prev_cost = prev_cost * np.exp(r * dt) + position
    return st_path[-1], prev_share, prev_cost

if __name__ == "__main__":
    S, K, r, sigma, t, q, mu, steps = 49, 50, 0.05, 0.2, 20/52, 0, 0.13,
   1000
    s_T, share_position, final_position_cost = (DeltaHedgingShort(S, K,
   sigma, t, r, q, mu, steps, 100000))
    if s_T <= K:
        final_cost = final_position_cost
    else:
        final_cost = final_position_cost - K * 100000
    print(f"Final stock price = {s_T:.2f}")
    print(f"Final share position = {int(share_position)}")
    print(f"Bank final hedging cost = {final_cost:.2f}")

'''
Final stock price = 60.04
Final share position = 100000
Bank final hedging cost = 244423.94
'''
```

■

1.3.1 Hedging = Pricing

Recall the hedging portfolio V_t

$$V_t = \begin{cases} \Delta_t \text{ shares of stock (may pay dividends)} \\ \text{rest in cash} \end{cases}$$

here we do not enforce $\Delta_t = \frac{\partial f}{\partial S_t}$. Then,

$$dV_t = \Delta_t dS_t + (V_t - \Delta_t S_t) r dt$$

To show that hedging is equivalent to pricing of an option, what we want is to start with $V_0 = f(0, S_0)$ and $V_t = f(t, S_t), \forall t \in [0, T]$. To this end, we can enforce $d(e^{-rt}V_t) = d(e^{-rt} f(t, S_t))$. By Ito's formula, we have the LHS

$$\begin{aligned} d(e^{-rt}V_t) &= e^{-rt}(-rdtV_t + dV_t) \\ &= e^{-rt}(\Delta_t dS_t - r\Delta_t S_t dt) \end{aligned}$$

The RHS is

$$d(e^{-rt}f(t,S_t)) = e^{-rt}\left(-rf\,dt + \underbrace{\frac{\partial f}{\partial t}dt + S_t\frac{\partial f}{\partial S_t}dS_t + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial S_t^2}dt}_{=df}\right)$$

Equating these two expression gives and compare coefficients,

$$\begin{cases} \Delta_t = \frac{\partial f}{\partial S_t} \\ -rf\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial S_t^2} = -r\Delta_t S_t \end{cases}$$

The first equation tells us the greek letter Δ_t is needed for the pricing to be correct and it solves the second PDE. Thus, this shows we do have some rigour when choosing Δ_t to be the number of shares to hedge.
More directly, with $f(0,S_0)$ initial capital, long Δ_0 in the underlying and hold the rest in cash, and $f(t,S_t)$ at t, to hold Δ_t in share an rest in cash. Note that

$$\begin{aligned} df(t,S_t) &= \Delta_t dS_t + (f-\Delta_t S_t)rdt \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S_t}dS_t + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S_t^2} \end{aligned}$$

Again, by comparing terms, we have

$$\begin{cases} \Delta_t = \frac{\partial f}{\partial S_t} \\ -rf\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial S_t^2} = -r\Delta_t S_t \end{cases}$$

1.3.2 Self-Financing

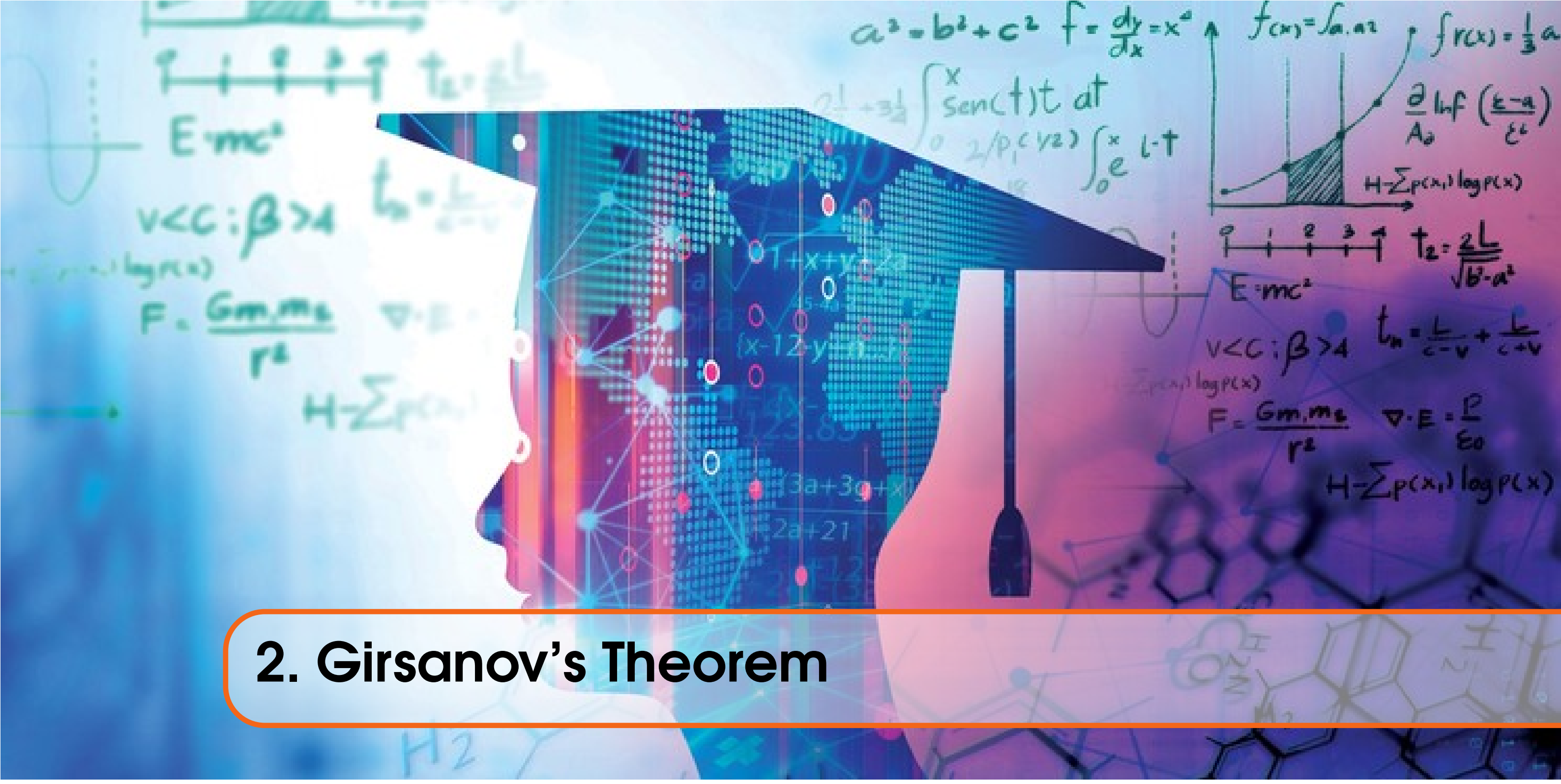
Rewrite $V_t = \Delta_t S_t + \gamma_t m_t$ where $m_t : e^{rt}$. At time t, Δ_t is the number of shares and γ_t is the amount in cash. Then,

$$\begin{aligned} dV_t &= d(\Delta_t S_t) + d(\gamma_t m_t) \\ &\overset{\text{Ito's Prod.}}{=} \Delta_t dS_t + S_t d\Delta_t + d\Delta_t dS_t + \gamma_t dm_t + m_t d\gamma_t + d\gamma_t dm_t \end{aligned}$$

On the other hand, $dV_t = \Delta_t dS_t + (V_t - \Delta_t S_t)rdt$ where $\gamma_t m_t = V_t - \Delta_t S_t$. We conclude that

$$(S_t + d\Delta_t)d\Delta_t = -(m_t + d\gamma_t)dm_t$$

LHS is how much we need to buy/short cash based on the change of the stock price and the RHS describes the cash position and interest accrued. This is the **self-financing** condition.



2. Girsanov's Theorem

2.1 Change of Measure

Let X be a continuous random variable with a pdf $f(x) > 0, \forall x$. Let $g(x)$ be another pdf. We can write

$$\mathbf{E}_g(X) = \int xg(x)dx = \int x\frac{g(x)}{f(x)}f(x)dx$$

This seems trivial. Define $\zeta := \frac{f}{g}$. Then, we can write $\mathbf{E}_g(X) = \mathbf{E}_f(X\zeta)$. This is what change of measure all about, to evaluate X's expectation under g we can introduce a new random variable ζ to do so under f.

Why do we do this?

When we do simulation, we care about the performance of a portfolio/trade under rare events. If we use a common measure, such event's occurrence is deemed to be rare. But maybe under another measure, such an event could occur more often.
In real life, changing any measurement units is considered as a change of measure, a scalar function ζ.

R

- $\zeta > 0$
- $\mathbf{E}_f(\zeta) = 1$

■ **Example 2.1** Let $Z \sim N(0,1)$ with $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ as the original measure. We want to change to a new measure denoted Q such that $Z \sim N(-\lambda, 1), \lambda > 0$ is given. In this case,

$$\zeta(z) = \frac{\phi(z+\lambda)}{\phi(z)} = e^{-\frac{1}{2}(z+\lambda)^2+\frac{1}{2}z^2} = e^{-\lambda z - \frac{1}{2}\lambda^2}$$

Consider $X = \lambda + Z \sim N(\lambda, 1)$ under $P = \phi$. Then $X \sim N(0,1)$ under Q. ■

Notation wise, for an event $A \in \mathscr{F}, \omega \in \Omega$,

$$Q(A) = \int_A \zeta(\omega)\mathbf{P}(d\omega), \quad \zeta := \frac{\mathbf{Q}(d\omega)}{\mathbf{P}(d\omega)}$$

where ζ is the Radon-Nikodym derivative. Moreover, to perform such a change of measure, these two measures need to agree on the null sets, namely,

$$\mathbf{P}(A) = 0 \equiv \mathbf{Q}(A) = 0, \forall A \in \mathscr{F}$$

2.1.1 Quadratic Variation

Recall that $d\langle B\rangle_t = (dB_t)^2 = dt$. Thus, Brownian motion has a positive (non-zero) QV.

Theorem 1 Let $t \in [0,T]$, let $0 < t_1 < t_2 < \cdots < t_n := T$. Define $\Delta_k := B(t_k) - B(t_{k-1})$. $TV(B) := \lim_{n\to\infty} \sum_k |\Delta_k|$ and $QV(B) = \langle B \rangle = \lim_{n\to\infty} \sum_k \Delta_k^2$.

- $\langle B \rangle = t$
- $TV(B) = +\infty$

(R) $TV(B) = +\infty$ implies that BM has infinite number of up and down over $[0,T]$. Moreover, we have $\langle B \rangle = t \implies TV(B) = +\infty$. Note that

$$\sum_k \Delta_k^2 \leq |\Delta_t| \sum_k |\Delta| \qquad |\Delta_l| = \sup_k |\Delta_k|$$

$$\sum_k |\Delta_k| \geq \frac{\sum_k \Delta_k^2}{|\Delta_l|}$$

as $n \to \infty$, then $\sum_k |\Delta_k| \geq \frac{T}{0} = +\infty$. On the other hand, if QV is positive, then TV blows up to ∞. The contrapositive shows, that if TV is finite, then QV is 0.

For a BM, it is a martingale with QV of t.

Theorem 2 — Levy's Theorem. If M_t is a continuous martingale and $\langle M \rangle_t = t$, then M_t is a BM.

Recall Ito's formula:

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds$$

in particular, $X_t := \int_0^t C_s dB_s$ is an Ito's integral where $C_s \in \mathscr{F}_s$ adapted to the filtration. For example, $C_s = B_s, B_s^2, s, \cdots$.

Proposition 2.1.1 Ito's Integral Properties

- X_t is a martingale if C_s is quadratic integrable, i.e., $\mathbf{E}\left(\int_0^t C_s^2 ds\right) < \infty$ for every given t.
- $dX_t = C_t dB_t \implies d\langle X \rangle_t = C_t^2 dt$.
- **Isometry**: $\mathbf{E}(X_t^2) = \int_0^t \mathbf{E}\left(C_s^2\right) ds$

Proof.

- Martingale property: C_t is adapted, i.e., $C_t \in \mathscr{F}_t, \forall t$

$$\begin{aligned}\mathbf{E}(dX_t|\mathscr{F}_t) &= \mathbf{E}(C_t dB_t|\mathscr{F}_t) \\ &\overset{C_t \in \mathscr{F}_t}{=} C_t \mathbf{E}(dB_t|\mathscr{F}_t) = 0\end{aligned}$$

since $\mathbf{E}(X_t^2) = \mathbf{E}\left(\int_0^t C_s^2 ds\right) < \infty$, we have X_t as a martingale.

■

■ **Example 2.2** • If $C_t = s$ for some constant, then $\int_0^t s^2 ds < \infty$, square-integrable.
- If $C_t = B_t$, then by Fubini,

$$\mathbf{E}\left(\int_0^t B_s^2 ds\right) = \int_0^t s ds < \infty$$

still square-integrable.

■

Recall from a previous example, that $X = \lambda + Z$, $\lambda > 0$. Under P, $X \sim N(\lambda, 1)$. We used $\zeta = e^{-\lambda Z - \frac{z^2}{2}}$ to change to Q and $X \sim N(0,1)$.

2.2 Change of Measure: BM

Consider $W_t = \int_0^t \lambda_s ds + B_t$ allowing λ_t to be an adapted process. Special case is $\lambda_t = \mu$, which results in $W_t = \mu t + B_t$, a BM with drift. How can we change this W_t into a drift-less process by changing the measure?
Here, the key is defining the ζ_t process

$$\zeta_t := \exp\left(-\int_0^t \lambda_s dB_s - \frac{1}{2}\int_0^t \lambda_s^2 ds\right)$$

in particular, when $\lambda_t = \mu$, then $\zeta_t = \exp\left(-\mu B_t - \frac{\mu^2}{2}t\right)$, which is a GBM. since

$$d\zeta_t = \mu dB_t$$

thus, also a martingale.

Theorem 3 Girsanov's Theorem Under $\mathbf{Q}$ measure, W_t is a driftless BM, hence a martingale. The condition is square-integrable: $\mathbf{E}\left(\int_0^t (\lambda_s \zeta_s)^2 ds\right) < \infty, \forall t$.
We have $\zeta_t > 0$, we first want to show that ζ_t is indeed a martingale with expectation 1 throughout. We started from a random variable ζ. Let $\zeta_t := \mathbf{E}(\zeta | \mathscr{F}_t)$. Then, ζ_t is a MG (Doob's martingale). Note that

$$\mathbf{E}(\zeta_t | \mathscr{F}_s) = \mathbf{E}(\mathbf{E}(\zeta | \mathscr{F}_t) | \mathscr{F}_s) \underset{\mathscr{F}_s \subset \mathscr{F}_t}{=} \mathbf{E}(\zeta | \mathscr{F}_s) = \zeta_s$$

Therefore, we let $\zeta = \zeta_T, t \in [0,T]$. We can write $\zeta_t = e^{X_t}$. By Ito's formula, we have

$$\begin{aligned}
d\zeta_t &= \zeta_t \left[dX_t + \frac{1}{2} d\langle X \rangle_t\right] \\
&= \zeta_t \left(-\lambda_t dB_t - \frac{1}{2}\lambda_t^2 dt + \frac{1}{2}\lambda_t^2 dt\right) \\
&= -\lambda_t \zeta_t dB_t
\end{aligned}$$

Thus, ζ_t is a martingale under the square-integrable condition. Also note that

$$\zeta_t = \mathbf{E}(\zeta_T | \mathscr{F}_t), \forall t \in [0,T]$$

ζ_t is called Radon-Nikodym process.

$\zeta_t W_t$ is a P-MG

By Ito's product rule,

$$\begin{aligned}d(\zeta_t W_t) &= \zeta_t dW_t + W_t d\zeta_t + dW_t d\zeta_t \\ &= \zeta_t(\lambda_t dt + dB_t) + W_t(-\lambda_t \zeta_t dB_t) - (\lambda_t \zeta_t)dt \\ &= \zeta_t(1 - \lambda_t W_t)dB_t\end{aligned}$$

By regularity condition, $\zeta_t W_t$ is a $\mathbf{P}-$MG.

W_t is $\mathbf{Q}-$MG

For any $t > s$,

$$\begin{aligned}\mathbf{E_Q}(W_t|\mathscr{F}_s) &\overset{\zeta_t:MG}{=} \frac{1}{\zeta_s}\mathbf{E}(W_t\zeta_t|\mathscr{F}_s) \\ &= \frac{W_s\zeta_s}{\zeta_s} = W_s\end{aligned}$$

Thus, W_t is a $\mathbf{Q}-$MG. Consider $s = 0$,

$$\begin{aligned}\mathbf{E_Q}(W_t|\mathscr{F}_0) &= \mathbf{E}(W_t\zeta_t|\mathscr{F}_0) \\ \mathbf{E_Q}(W_t) &= \mathbf{E}(W_t\zeta_t)\end{aligned}$$

Define $W := \frac{1}{\zeta_s}\mathbf{E}(W_t\zeta_t|\mathscr{F}_s)$. To verify the equation above, we check, for $A \in \mathscr{F}_s$

$$\begin{aligned}\mathbf{E_Q}(W1_A) &= \mathbf{E}(W1_A\zeta_s) \\ &= \mathbf{E}(\mathbf{E}(W_t\zeta_t|\mathscr{F}_s)1_A) \\ &= \mathbf{E}(W_t\zeta_t 1_A) = \mathbf{E_Q}(W_t 1_A)\end{aligned}$$

■

2.3 Risk-Neutral Measure

Recall the general stock dynamics model

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dB_t$$

where μ_t, σ_t are adapted processes. The exponential form is

$$S_t = S_0 \exp\left(\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dB_s\right)$$

In addition, the discount process is

$$D_t := \exp\left(-\int_0^t r_s ds\right)$$

where r_t is an adapted process. The differential form is $dD_t = -D_t r_t dt$.

Consider D_tS_t, analogous to $Y_t = e^{-rt}S_t$ previously. Then,

$$
\begin{aligned}
d(D_tS_t) &= S_t dD_t + D_t dS_t + dD_t dS_t \\
&= D_tS_t((\mu_t - r_t)dt + \sigma_t dB_t) \\
&= \sigma_t D_t S_t \underbrace{\left(\underbrace{\frac{\mu_t - r_t}{\sigma_t}}_{=:\lambda_t} dt + dB_t \right)}_{=:dW_t}
\end{aligned}
$$

D_tS_t is a $\mathscr{Q}-$MG by Girsanov's theorem. We call this $\mathscr{Q}$ measure, the risk-neutral measure and $\lambda_t := \frac{\mu_t - r_t}{\sigma_t}$ is the market price of risk. When μ_t, r_t, σ_t are all constants. λ is usually called the sharpe ratio.
Note that

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu_t dt + \sigma_t dB_t \\
&= \mu_t dt + \sigma_t (dW_t - \lambda_t dt) \\
&= r_t dt + \sigma_t dW_t
\end{aligned}
$$

Under $\mathbf{Q}$, W_t is the standard BM and this is equivalent to changing μ_t to r_t as we have seen in BSM formula derivation.
Let V_t be the hedging portfolio, Δ_t in S_t, and rest in cash. Then,

$$
\begin{aligned}
dV_t &= \Delta_t dS_t + (V_t - \Delta_t S_t) r_t dt \\
&= r_t V_t dt + \Delta_t dS_t - r_t \Delta_t S_t dt \\
&= r_t V_t dt + \Delta_t S_t [-r_t dt + r_t dt + \sigma_t dW_t] \\
&= r_t V_t dt + \Delta_t S_t \sigma_t dW_t
\end{aligned}
$$

Then,

$$
\begin{aligned}
d(D_tV_t) &= -r_t D_t V_t dt + D_t dV_t \\
&= D_t \Delta_t S_t \sigma_t dW_t
\end{aligned}
$$

This means D_tV_t is a Q-MG. This means, as we change the measure, not only the underlying process becomes a Q-MG, its derivatives are also Q-MGs.

Proof. Note that changing the measure is considered as changing the perspective. It DOES NOT change the physical reality, therefore, the underlying processes do not change. Only when we apply probability/expectation and other probabilistic calculations, the change of measure has its effect.

2.4 Comments on Ito's Integral Being MG

Recall that, for $\mathscr{F}_t$-adapted C_t, the Ito's integral

$$X_t := \int_0^t C_s dB_s$$

is a MG since $dX_t = C_t dB_t$ and $\mathbf{E}(dX_t | \mathscr{F}_t) = 0$. Moreover, we need $\mathbf{E}(|X_t|) < \infty$, which is implied by $\mathbf{E}(X_t^2) < \infty$. Even though the second-moment is an overkill, but it is nice and convenient to compute the second moment via Ito's isometry.

2.4.1 MG Representation Theorem

Turns out the converse of Ito's integrals' MG property is kind of true as well.

Theorem 4 — MG Representation Theorem. For any MG with continuous path M_t, it can be represented by an Ito's integral

$$M_t = M_0 + \int_0^t \Gamma_s dB_s, \quad \Gamma_t \in \mathscr{F}_t$$

i.e., such $\mathscr{F}_t$-adapted process Γ_t exists.

Applying this to $D_t V_t$, which we proved to be Q-MG, hence,

$$D_t V_t = D_0 V_0 + \int_0^t \tilde{\Gamma}_s dW_s$$

note that we are using W_t BM here to reflect $D_t V_t$ being a Q-MG. By MG representation theorem, such $\tilde{\Gamma}_t$ exists. From here,

$$d(D_t V_t) = \tilde{\Gamma}_t dW_t = \Delta_t D_t S_t \sigma_t dW_t$$

thus, we obtain a useful hedging strategy

$$\Delta_t = \frac{\tilde{\Gamma}_t}{D_t S_t \sigma_t}$$

This means $f_t (= V_t)$ can be hedged as above.

3. Multidimensional Case

3.1 Going Multidimensional

Consider

$$\frac{dS_{it}}{S_{it}} = \mu_{it}dt + \sigma_i(t)\cdot dB(t), \quad i = 1,\cdots,m$$

where $B(t) = (B_{it},\cdots,B_{lt})$. m is the number of assets and l is the number of random factors. Note that $\sigma_i(t)$ is also a vector, thus, we use the dot product sign in the definition above. It can be considered as the ith row vector of the matrix

$$\sigma(t) := [\sigma_{ijt}]_{i=1,\cdots,l}^{m}$$

Then, from Girsanov's theorem, we have

$$\zeta_t = \exp\left(-\int_0^t \lambda(s)\cdot dB(s) - \frac{1}{2}\int_0^t \|\lambda(s)\|^2 ds\right)$$

where $\lambda(t) := (\lambda_{1t},\cdots,\lambda_{lt})$ is another vector process. Note that ζ_t is still a scalar process.

■ **Example 3.1 — ζ_t is a Q-MG.** Let $\zeta_t = \exp(X_t)$. Then, by Ito's formula,

$$\begin{aligned} d\zeta_t &= \zeta_t\left(dX_t + \frac{1}{2}d\langle X\rangle_t\right) \\ &= \zeta_t\left[\lambda(t)\cdot dB(t) - \frac{1}{2}\|\lambda(t)\|^2 dt + \frac{1}{2}\|\lambda(t)\|^2 dt\right] \\ &= -\zeta_t\lambda(t)\cdot dB(t) \end{aligned}$$

Note that ζ_t is driftless and each λ_{it} is square-integrable. We must have ζ_t being a Q-MG. ■

Then, from the single-variable change of measure result, we would have

$$\frac{dS_{it}}{S_{it}} = r_t dt + \sigma_i(t)\cdot dW(t), \quad i = 1,\cdots,m$$

where $dW(t) = \lambda(t)dt + dB(t)$. And

$$d(D_tS_{it}) = (D_tS_{it})\sigma_i(t) \cdot dW(t)$$

$$dV_t = r_tV_tdt + \sum_{i=1}^{m} \Delta_{it}S_{it}\sigma_i(t) \cdot dW(t)$$

$$d(D_tV_t) = \sum_{i=1}^{m} \Delta_{it}(D_tS_{it})\sigma_i(t) \cdot dW(t)$$

These are analogous results compared to the single-asset-single-randomness case.
Here, $\lambda(t)$ is the solution to

$$\mu_{it} - r_t = \sigma_i(t) \cdot \lambda(t), \quad i = 1, \cdots, m$$

all of the results above coming from change of measure is ensured by the system above having a valid $\lambda(t)$ solution.

Theorem 5 — Solution $\Longrightarrow$ No Arbitrage. We define an (statistical) arbitrage exists if one can construct a portfolio V_t such that with $V_0 = 0$, there exists some $T > 0$ such that

$$\mathbf{P}(V_T \geq 0) = 1, \quad \mathbf{P}(V_T > 0) > 0$$

If there is a solution $\lambda(t)$, then

$$\mathbf{E}_{\mathbf{Q}}(D_TQ_T) \overset{Q-MG}{=} V_0 = 0$$

suppose, for the sake of contradiction, there exists an arbitrage such that $\mathbf{Q}(V_T \geq 0) = 1$. We can write this since $\mathbf{Q}$ and $\mathbf{P}$ are equivalent. Since $D_T > 0$, we have $V_T = 0$, meaning $\mathbf{Q}(V_T > 0) = 0 \Longrightarrow \mathbf{P}(V_T > 0) = 0$. This yields a contradiction.

■ **Example 3.2 — Converse is not necessarily true!.** If we do not have a solution $\lambda(t)$. Consider $m = 2, l = 1$ and all coefficients are constants. We would have

$$\frac{\mu_1 - r}{\sigma_1} = \lambda_1, \quad \frac{\mu_2 - r}{\sigma_2} = \lambda_2$$

Without loss of generality, suppose $\frac{\mu_1 - r}{\sigma_1} > \frac{\mu_2 - r}{\sigma_2}$, then there is no solution λ. From sharpe ratios perspective, you would long asset 1 and short asset 2 to construct such an arbitrage. We hold $\Delta_{1t} = \frac{1}{\sigma_1 S_{1t}}$ and $\Delta_{2t} = -\frac{1}{\sigma_2 S_{2t}}$. Then,

$$dV_t = \delta dt + rV_tdt, \quad d(e^{-rt}V_t) = \delta e^{-rt}dt$$

where $\delta = \frac{\mu_1 - r}{\sigma_1} - \frac{\mu_2 - r}{\sigma_2}$. The second equation implies an arbitrage as the discounted value of the portfolio is always positive. ■

Theorem 6 — Market Completeness. The solution $\lambda(t)$ is unique if and only if the market is complete, that every asset can be hedged.

Proof. $\Longleftarrow$ Suppose there are two measures Q_1, Q_2 that can be yielded from two different solutions. We want to argue that $\mathbf{Q}_1(A) = \mathbf{Q}_2(A), \forall A \in \mathscr{F}_T$ with $t \in [0, T]$. Consider an asset f_t can be represented by the hedging portfolio V_t. Let $D_Tf_T = \mathbf{1}(A)$. Since D_tV_t is a Q_1-MG, we have

$$V_0 = \mathbf{E}_{\mathbf{Q}_1}(D_TV_T) = \mathbf{E}_{\mathbf{Q}_1}(D_Tf_T) = \mathbf{Q}_1(A)$$

similarly, D_tV_t is also a Q_2-MG

$$V_0 = \mathbf{E}_{\mathbf{Q}_2}(D_TV_T) = \mathbf{E}_{\mathbf{Q}_2}(D_Tf_T) = \mathbf{Q}_2(A)$$

$\implies$ Now, suppose we have a unique solution and measure $\mathbf{Q}$. Recall that

$$D_tV_t = V_0 + \int_0^t \sum_{i=1}^m \Delta_{ik} D_s S_{is} \sigma_i(s) \cdot dW(s)$$

From MG-representation theorem, we can represent that Q-MG as

$$D_tV_t = V_0 + \int_0^t \tilde{\Gamma}(s) \cdot dW(s)$$

thus, $\sum_{i=1}^m \Delta_{it} D_t S_{it} \sigma_{ijt} = \tilde{\Gamma}_{jt}, \quad j = 1, \cdots, l$. We can write

$$\sum_{i=1}^m \Delta_{it} S_{it} \sigma_{ijt} = \frac{\tilde{\Gamma}_{jt}}{D_t}, \quad j = 1, \cdots, l$$

The unique solution $\lambda(t)$ implies that we can solve the system above for Δ_{it}. Thus, we have constructed a hedging portfolio.

■

3.2 Numeraire & Applications

3.2.1 Numeraire

Let S_t as before as a scalar process. The differential form is

$$d(D_tS_t) = D_tS_t\sigma(t) \cdot dW(t)$$

where $S_t := S_{it}$ and $\sigma(t) := \sigma_i(t)$. Consideer a new scalar process N_t, the numeraire, where

$$d(D_tN_t) = D_tN_t\nu(t) \cdot dW(t)$$

where $\nu(t)$ is like $\sigma(t)$ is the volatility process of N_t. We are interested in $S_t^N := S_t/N_t$. Typically, S_t is some stock and N_t is some foreign currency.
Recall that

$$D_tS_t = S_0 \exp\left[\int_0^t \sigma(s) \cdot dW(s) - \frac{1}{2}\int_0^t \|\sigma(s)\|^2 ds\right]$$

and

$$D_tN_t = N_0 \exp\left[\int_0^t \nu(s) \cdot dW(s) - \frac{1}{2}\int_0^t \|\nu(s)\|^2 ds\right]$$

Then,

$$S_t^N := \frac{S_t}{N_t} = \frac{S_0}{N_0} \exp\left[\int_0^t (\sigma(s) - \nu(s)) \cdot dW(s) - \frac{1}{2}\int_0^t (\|\sigma(s)\|^2 - \|\nu(s)\|^2) ds\right]$$

and

$$\begin{aligned}\frac{dS_t^N}{S_t^N} &= (\sigma(t) - \nu(t)) \cdot dW(t) - \frac{1}{2}\left(\|\sigma(t)\|^2 - \|\nu(t)\|^2\right)dt + \frac{1}{2}\left(\|\sigma(t) - \nu(t)\|^2\right)dt \\ &= (\sigma(t) - \nu(t)) \cdot dW(t) + \nu(t) \cdot (\sigma(t) - \nu(t))dt\end{aligned}$$

Note that S_t^N is not a Q-MG. We need to change measure again. Consider a new measure **N** such that S_t^N is an N-MG. Let $\lambda(t) := -v(t)$. The RN-derivative is $\zeta_t := \frac{D_t N_t}{N_0}$. We define $w^N(t)$ such that $dW^N(t) = -v(t)dt + dW(t)$ and

$$\frac{dS_t^N}{S_t^N} = (\sigma(t) - v(t))dW^N(t)$$

3.2.2 Applications

Before we proceed, we consider the case when $m = l = 2$. Recall that

$$\frac{dS_{it}}{S_{it}} = \mu_{i1}dt + \sigma_{i1t}dB_{1t} + \sigma_{i2t}dB_{2t}, \quad i = 1,2$$

so

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

alternatively, we can write it as

$$\frac{dS_{1t}}{S_{1t}} = \mu_{1t}dt + v_t dB_{1t}$$

and

$$\frac{dS_{2t}}{S_{2t}} = \mu_{2t}dt + \rho_t\eta_t dB_{1t} + \sqrt{1-\rho_t^2}\eta_t dB_{2t}$$

where

$$v_t^2 = \sigma_{11t}^2 + \sigma_{12t}^2, \ \eta_t^2 = \sigma_{21t}^2 + \sigma_{22t}^2, \ \rho_t = \frac{\sigma_{11t}\sigma_{21t} + \sigma_{12t}\sigma_{22t}}{v_t\eta_t}$$

Now, consider S_t, a stock in domestic currency; X_t, exchange rate, units of domestic currency per foreign currency. Then,

$$\frac{dS_t}{S_t} = \mu_{S_t}dt + \eta_{S_t}dB_{1t}$$

$$\frac{dX_t}{X_t} = \mu_{X_t}dt + \eta_{X_t}\rho_t dB_{1t} + \eta_{X_t}\sqrt{1-\rho_t^2}dB_{2t}$$

where $\eta_{S_t}, \eta_{X_t}, \rho_t$ are similarly defined as above. Moreover, we define $N_t := D_{f_t}^{-1}X_t$. $D_{f_t} = \exp\left(-\int_0^t r_{f_s}ds\right)$, the foreign risk-free discount rate. We use $D_t := \exp\left(-\int_0^t r_s ds\right)$ as the domestic risk-free discount rate. We focus on $S_t^N := \frac{S_t}{N_t}$. Note that N_t and X_t relates as follow,

$$\begin{aligned} dN_t &= r_{ft}D_{f_t}^{-1}X_t dt + D_{f_t}^{-1}dX_t \\ \frac{dN_t}{N_t} &= r_{f_t}dt + \frac{dX_t}{X_t} \end{aligned}$$

so the rate of return of the Numeraire here is just the exchange rate plus foreign risk free rate.

3.2.3 Domestic Risk-Neutral Measure Q

We have $\lambda(t) := (\lambda_{1t}, \lambda_{2t})$ being the solution of

$$\mu_{S_t} - r_f = \lambda_{1t}\eta_{S_t}, \quad \mu_{X_t} + r_{f_t} - r_t = \lambda_{1t}\eta_{X_t}\rho_t + \lambda_{2t}\eta_{X_t}\sqrt{1-\rho_t^2}$$

This yields a unique solution and define

$$dW(t) = \lambda(t)dt + dB(t)$$

and $\frac{d(D_tS_t)}{D_tS_t} = \eta_{S_t}dW_{1t}$, $\frac{d(D_tN_t)}{D_tN_t} = \eta_{X_t}\left(\rho_t dW_{1t} + \sqrt{1-\rho_t^2}dW_{2t}\right)$ are Q-MGs.

Foreign Risk-Neutral Measure N

We change the measure via

$$dW_{1t}^{\mathbf{N}} = -\rho_t \eta_{X_t} dW_{1t}, \;\; dW_{2t}^{\mathbf{N}} = -\sqrt{1-\rho_t^2}\eta_{X_t} dW_{2t}$$

and

$$\frac{dS_t^{\mathbf{N}}}{S_t^{\mathbf{N}}} = (\eta_{S_t} - \eta_{X_t}\rho_t)dW_{1t}^{\mathbf{N}} - \sqrt{1-\rho_t^2}\eta_{X_t} dW_{2t}^{\mathbf{N}}$$

so $S_t^{\mathbf{N}}$ is a N-MG. The stock return under the foreign risk-neutral measure is

$$\begin{aligned}
\frac{dS_t}{S_t} &= r_t dt + \eta_{S_t} dW_{1t} \\
&= r_t dt + \eta_{S_t}\left(dW_{1t}^{\mathbf{N}} + \rho_t \eta_{X_t} dt\right) \\
&= (r_t + \rho_t \eta_{S_t}\eta_{X_t})dt + \eta_{S_t} dW_{1t}^{\mathbf{N}}
\end{aligned}$$

Similarly,

$$\frac{dN_t}{N_t} = (r_t + \eta_{X_t}^2)dt + \cdots$$

3.2.4 Quanto Option

Definition 3.2.1 — Quanto Option. A quanto call gives the payoff of a Eur call at maturity $\left(\frac{S_T}{X_T} - K\right)^+$ in domestic currency.

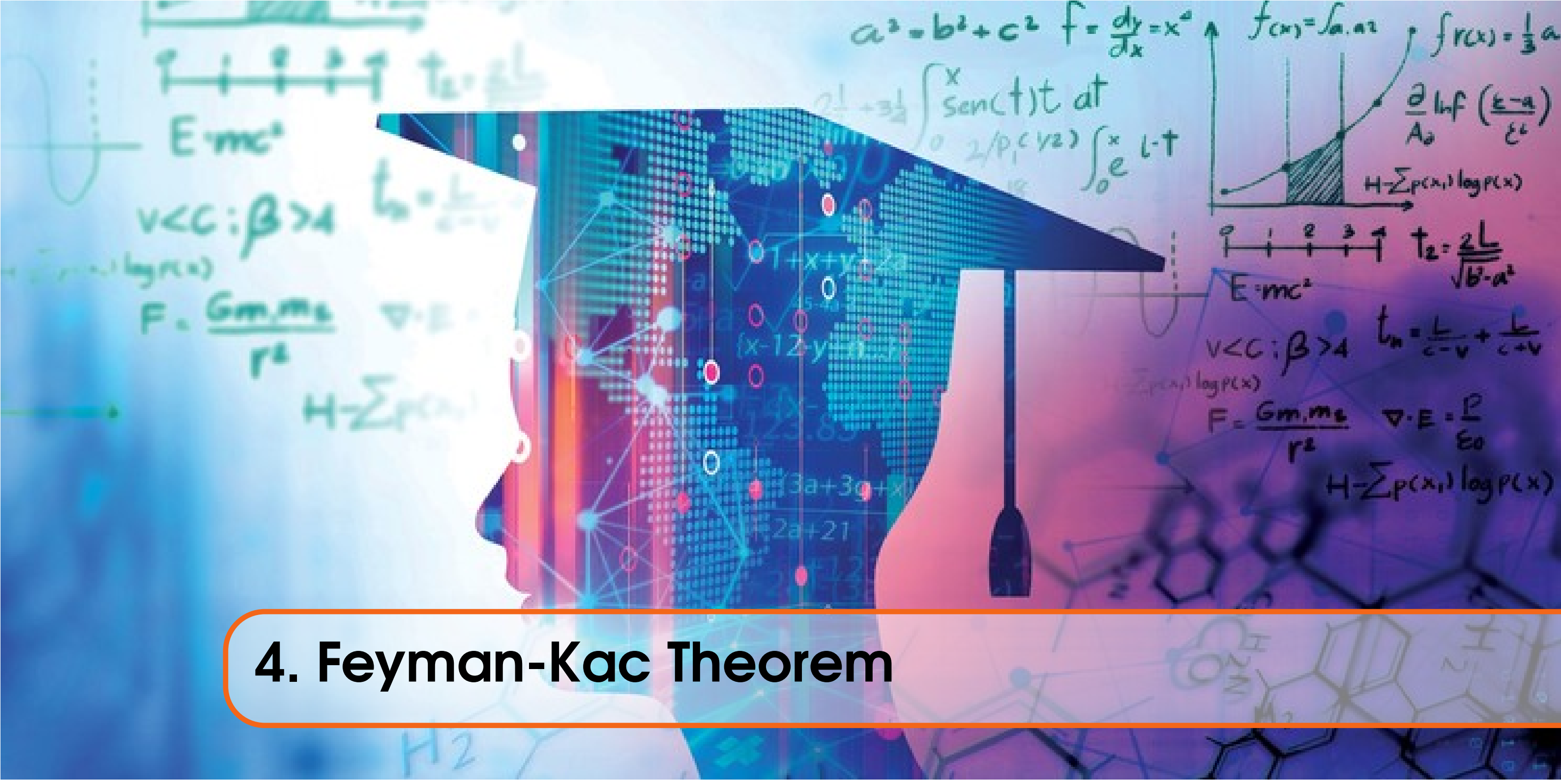
We want to price this call under $\mathbf{Q}$. Assuming all coefficients being constants and we are dealing with GBMs,

$$S_t = S_0 \exp\left[(r - \frac{1}{2}\eta_S^2)t + \eta_S W_{1t}\right]$$

$$X_t = X_0 \exp\left[(r - r_f - \frac{1}{2}\eta_X^2)t + \eta_X \rho dW_{1t} + \eta_X \sqrt{1-\rho^2} dW_{2t}\right]$$

and

$$S_t^X := \frac{S_t}{X_t} = \frac{S_0}{X_0}\exp\left[\left(r_f - \frac{1}{2}\eta_S^2 + \frac{1}{2}\eta_X^2\right)t + (\eta_S - \eta_X\rho)W_{1t} - \eta_X\sqrt{1-\rho^2}W_{2t}\right]$$



4. Feyman-Kac Theorem

4.1 From SDE to PDE

Recall the BSM model with GBM SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

we applied change of measure to $\mathbf{Q}$ via $dW_t = \lambda dt + dB_t$ where $\lambda = \frac{\mu - r}{\sigma}$ to have

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

For any pay off function h, the pricing formula of such an instrument is given by

$$f(t,S_t) = \mathbf{E_Q}\left[e^{-r(T-t)}h(S_T)|S_t\right]$$

in fact, it also solves the following PDE

$$f_t(t,x) = rxf_x(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) = rf(t,x)$$

with terminal condition $f(T,x) = h(x)$ for all x.

R So which approach of pricing is better? Sometimes, SDE gives explicit solution such as the BSM formula, which result in nice calculation. However, majority of the SDEs do not have explicit solutions and it requires monte carlo simulation of the underlying paths to generate a pricing result, which can be quite inefficient. For those cases, PDE can be handy as we can resort to numerical methods, such as grid approach.

Definition 4.1.1 — SDE. We say a general SDE (diffusion equation) is of the form

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dB_t$$

where $\mu(t,x)$ and $\sigma(t,x)$ are deterministic functions.

In particular, when $\mu(t,x) = \mu x, \sigma(t,x) = \sigma x$, we have the GBM case. Define an auxiliary function

$$g(t,x) := \mathbf{E}[h(X_T)|X_t = x]$$

Theorem 7 — Feyman-Kac Theorem. For a defined g, it satisfies

$$g_t(t,x) + \mu(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0$$

with terminal condition $g(T,x) = h(x), \forall x$.

Proof. We first want to show

$$g(t,X_t) := \mathbf{E}[h(X_T)|x_t] = \mathbf{E}[h(X_T)|\mathscr{F}_t]$$

which is the Markov property, this is true since μ, σ are deterministic. We claim $g(t,X_t)$ is also a MG, i.e., for any $t > s$,

$$\begin{aligned}\mathbf{E}[g(t,X_t)|\mathscr{F}_s] &= \mathbf{E}[\mathbf{E}[h(X_T)|\mathscr{F}_t]|\mathscr{F}_s] \\ &= \mathbf{E}[h(X_T)|\mathscr{F}_s] = g(s,X_s)\end{aligned}$$

By Ito,

$$\begin{aligned}dg(t,X_t) &= g_t dt + g_x dX_t + \frac{1}{2}g_{xx}\langle X\rangle_t \\ &= g_t dt + g_x(\mu(t,x)dt + \sigma(t,x)dB_t) + \frac{1}{2}g_{xx}\sigma^2(t,x)dt \\ &= \underbrace{\left(g_t + \mu(t,x)g_x + \frac{1}{2}\sigma^2(t,x)g_{xx}\right)}_{=0} dt + \sigma(t,x)g_x dB_t\end{aligned}$$

since $g(t,X_t)$ is a MG. Thus, we have our result. ■

For the discounted version, such as BSM solution, we have

$$f(t,x) = \mathbf{E}\left[e^{-r(T-t)}h(X_T)|X_t = x\right]$$

but $f(t,X_t)$ is not a MG in the original measure. But

$$e^{-rt}f(t,X_t) = e^{-rT}\mathbf{E}[h(X_T)|X_t]$$

is a MG. Then,

$$d\left(e^{-rt}f(t,X_t)\right) = e^{-rt}\left(-rfdt + f_t dt + f_x dX - t + \frac{1}{2}f_{xx}d\langle X\rangle_t\right)$$

Again, by equating the coefficients, we have

$$f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} = rf$$

as desired.

4.1.1 Interest Rate Model

Consider the interest rate model

$$dR_t = \mu(t,R_t)dt + \sigma(t,R_t)dW_t$$

where R_t is the interest rate process (instantaneous). Define $D_t := \exp\left(-\int_0^t R_s ds\right)$.

Definition 4.1.2 — Zero-Coupon Bond Price. With face value of 1,

$$f(t,R_t) = B(t,T) := \mathbf{E_Q}\left(\exp\left(-\int_t^T R_u du | \mathscr{F}_t\right)\right) = \mathbf{E_Q}\left(\exp\left(-\int_t^T R_u du | R_t\right)\right)$$

From previous deduction, it is clear that $D_t f(t,R_t)$ is Q-MG. The last equality is due to the Markov property of R_t.

Another look at the Markov property

Recall that

$$\begin{aligned} dR_t &:= R_{t+dt} - R_t \\ R_{t+dt} &= R_t + dR_t = R_t + \mu(t,R_t)dt + \sigma(t,R_t)dW_t \end{aligned}$$

we can use this recursive relationship to simulate the rate process. Consider $\mu(t,R_t) = \mu_t R_t, \sigma(t,R_t) = \sigma_t R_t$ but μ_t, σ_t are adapted process with respect to $\mathscr{F}_t$. Then,

$$\frac{dR_t}{R_t} = \mu_t dt + \sigma_t dW_t$$

we cannot simulate μ_t, σ_t with just given R_t. So Markov property is not satisfied when μ, σ are not deterministic functions.

4.1.2 Hull-White Model

$$dR_t = (a_t - b_t R_t)dt + \sigma_t dW_t$$

where a, b, σ are deterministic positive function. These being positive can avoid the model being trivial and it reflects the mean reversion property of the interest rate. The expression

$$b_t\left(\frac{a_t}{b_t} - R_t\right)$$

means the interest rate mean reverts to $\frac{a_t}{b_t}$ at the rate of b_t. This can be explicitly solved. In general linear diffusion models can be solved. To solve this, we consider, with $b(t_1,t_2) = \int_{t_1}^{t_2} b_u d_u$,

$$d[R_s \exp(b(0,s))] = e^{b(0,s)}[b_s R_s ds + dR_s] = e^{b(0,s)}[a_s ds + \sigma_s dW_s]$$

Taking integrals both sides from t to u and we are done.

PDE Approach

In this case, the PDE is

$$f_t(t,r) + (a_t - b_t r)f_t(t,r) + \frac{1}{2}\sigma_t^2 f_{rr}(t,r) = rf(t,r)$$

this PDE has an explicit solution as well.

$$f(t,r) = e^{-rC(t,T)-A(t,T)}$$

to derive A and C. We take derivatives,

$$f_t = f(-rC' - A'), \quad f_r = f(-C), \quad f_{rr} = fC^2$$

by substitution, we have

$$\begin{aligned}
&-rC' - A' - (a_t - b_t r)C + \frac{1}{2}\sigma^2 C^2 = r \\
&(b_t C - C' - 1)r - A' - a_t C + \frac{1}{2}\sigma^2 C^2 = 0
\end{aligned}$$

For any r, this needs to work, which means

$$\begin{cases} b_t C - C' - 1 = 0 \\ A' + a_t C - \frac{1}{2}\sigma^2 C^2 = 0 \end{cases} \implies \begin{cases} C' = b_t C - 1 \\ A' = -a_t C + \frac{1}{2}\sigma^2 C^2 \end{cases}$$

we can take integrals for C and find A subsequently.

4.1.3 CIR Model

$$dR_t = (a - bR_t)dt + \sigma\sqrt{R_t}dW_t$$

where $a, b, \sigma > 0$ are constants. This will always guarantees a non-negative interest rate model while Hull-White could result in negative interest rate. But this is no longer a linear SDE and it does not have an explicit solution. The PDE is

$$f_t(t,r) + (a - br)f_r(t,r) + \frac{1}{2}\sigma^2 r f_{rr}(t,r) = rf$$

it still has the solution form $f(t,r) = e^{-rC(t,T) - A(t,T)}$. By substitution, we have

$$\left(bC - C' - 1 + \frac{1}{2}\sigma^2 C^2\right)r - A' - aC = 0$$

we can solve for C and A analogously.

R Both HW and CIR model has **affine yield model**

$$f(t,r) = e^{-rC(t,T) - A(t,T)}$$

where r is the short-rate.

4.2 Applications

4.2.1 Asian Option

Consider an Asian call option,

$$V(T) = \left(\frac{1}{T}\int_0^T S_t dt - K\right)^+$$

this is a **path-dependent** option. Let $Y_t = \int_0^t S_u du$. This is not Markovian! Note that

$$Y_{t+dt} = Y_t + dY_t = Y_t + S_t dt$$

if only given Y_t, we cannot get the next step without knowing S_t. On the hand, we can use a coupling trick, (S_t, Y_t), the bivariate process, is Markovian. Then, we focus on and assume S_t follows GBM (not necessary),

$$f(t, S_t, Y_t) = \mathbf{E_Q}\left[e^{-r(T-t)}\left(\frac{Y_T}{T} - K\right)^+ \middle| \mathscr{F}_t\right]$$

the terminal condition is $f(T,x,y) = \left(\frac{y}{T} - K\right)^+, \forall x, y$. The corresponding PDE is

$$f_t(t,x,y) + rxf_x(t,x,y) + xf_y(t,x,y) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x,y) = rf(t,x,y)$$

note that there is no f_{yy} term since Y_t has 0 quadratic variation. This PDE might have numerical problem as there is no bound on y. Alternative approach could be using a Numeraire.

4.2.2 Linear Diffusion Models

Consider a special case of $\mu(t,x), \sigma(t,x)$ being linear.

$$dX_t = (a_t + b_t X_t)dt + (\gamma_t + \eta_t X_t)dB_t$$

allowing all coefficients a, b, γ, η to be adapted process with respect to $\mathscr{F}_t$. This has an explicit solution. Consider $X_t = Y_t Z_t$, then

$$d(Y_t Z_t) = (a_t + b_t Y_t Z_t)dt + (\gamma_t + \eta_t Y_t Z_t)dB_t = Y_t dZ_t + Z_t dY_t + dY_t dZ_t$$

we want the Z_t to capture the slope part of each term. Let

$$\frac{dZ_t}{Z_t} = b_t dt + \eta_t dB_t$$

then,

$$Y_t dZ_t = b_t Y_t Z_t dt + \eta_t Y_t Z_t dB_t$$
$$Z_t dY_t = a_t dt + \gamma_t dB_t$$

we still need $dY_t dZ_t$. Let's check.

$$dY_t dZ_t = \gamma_t \eta_t dt$$

So, we actually need $Z_t dY_t = (a_t - \gamma_t \eta_t)dt + \gamma_t dB_t$. Now, we are done.

4.2.3 Forward Rate Model

Recall zero-coupon bond

$$B(t,T) = \mathbf{E_Q}\left(\exp\left(-\int_t^T R_u du\right) \middle| \mathscr{F}_t\right)$$

R_t models the short-rate/instantaneous rate r_t.

Definition 4.2.1 — Forward Rate. Consider

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

note that, equivalently,

$$-\frac{\partial \ln B(t,T)}{\partial T} = f(t,T)$$

Next, we want to connect the forward rate and short rate. Consider

$$\begin{aligned}\frac{\partial \ln B(t,T)}{\partial T} &= \frac{1}{B(t,T)}\mathbf{E_Q}\left[-r_T \exp\left(-\int_t^T r_u du\right)\middle|\mathscr{F}_t\right]\\ &\overset{T\to t}{=} \frac{1}{B(t,t)}\mathbf{E_Q}(-r_t|\mathscr{F}_t)\\ &= -r_t\end{aligned}$$

so, $f(t,t) = r_t$.

HJM Model

Start with the SDE for $f(t,T)$

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)dB_t$$

this $f(t,T)$ is also different from $f(t,R_t)$ or $f(t,r)$ in bond pricing. Next, consider $B(t,T) = e^{X_t}$ where $X_t := -\int_t^T f(t,u)du$. Then,

$$\begin{aligned}dB(t,T) &= B(t,T)\left[dX_t + \frac{1}{2}d\langle X\rangle_t\right]\\ &= B(t,T)\left[f(t,t)dt - \int_t^T df(t,u)du + \frac{1}{2}d\langle X\rangle_t\right]\\ &= B(t,T)\left(r_t dt - \underbrace{\left[\int_t^T \mu(t,u)du\right]}_{=:\bar{\mu}(t,T)}dt - \underbrace{\left[\int_t^T \sigma(t,u)du\right]}_{=:\bar{\sigma}(t,T)}dB_t + \frac{1}{2}d\langle X\rangle_t\right)\\ &= B(t,T)\left(r_t dt - \bar{\mu}(t,T)dt - \bar{\sigma}(t,T)dB_t + \frac{1}{2}\bar{\sigma}^2(t,T)dt\right)\end{aligned}$$

Let $D_t = \exp\left(-\int_0^t r_s ds\right)$. Then,

$$\begin{aligned}d[D_t B(t,T)] &= D_t[-r_t B(t,T)dt + dB(t,T)]\\ &= D_t B(t,T)\left[-\mu(t,T)dt - \bar{\sigma}(t,T)dB_t + \frac{1}{2}\bar{\sigma}^2(t,T)dt\right]\end{aligned}$$

Let $dW_t = \lambda_t dt + dB_t$ such that $D_t B(t,T)$ is a Q-MG. Hence, we must have

$$-\bar{\mu}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T) = -\bar{\sigma}(t,T)\lambda_t, \forall t \in [0,T]$$

taking partials with respect to T on both sides,

$$-\mu(t,T) + \bar{\sigma}(t,T)\sigma(t,T) = -\sigma(t,T)\lambda(t)$$

Thus,

$$\lambda_t = \frac{\mu(t,T)}{\sigma(t,T)} - \bar{\sigma}(t,T), \forall t \in [0,T]$$

This is known as the **HJM condition** for uniqueness of the solution existence. Therefore, there is no arbitrage and we have a complete market. Every derivative of this short-rate can be hedged.

R $B(t,T)$ has a rate of return of $r_t - \bar{\mu}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T)$. While under Q-measure with HJM condition being satisfied, it has a rate of return of r_t. Under **Q**,

$$df(t,T) = [\mu(t,T) - \lambda_t\sigma(t,T)]dt + \sigma(t,T)dW_t = \sigma(t,T)\bar{\sigma}(t,T)dt + \sigma(t,T)dW_t$$

Do HW/CIR satisfy HJM conditions?

Short answer: yes. For affine-yield models, we can write

$$B(t,T) = e^{-R_t C(t,T) - A(t,T)}$$

with C, A explicit solutions.

$$\begin{aligned} f(t,T) &= -\frac{\partial \ln B(t,T)}{\partial T} = R_t \frac{\partial C}{\partial T} + \frac{\partial A}{\partial T} \\ df(t,T) &= dR_t \frac{\partial C}{\partial T} + \left(R_t \frac{\partial C'(t,T)}{\partial T} + \frac{\partial A'(t,T)}{\partial T} \right) dt \end{aligned}$$

recall $C'(t,T) = \frac{\partial C}{\partial t}$ and $A'(t,T) = \frac{\partial A}{\partial t}$.
Consider HW, $dR_t = (a_t - b_t R_t)dt + \sigma_t dW_t$. Then,

$$df(t,T) = \left(\frac{\partial C}{\partial T} + R_t \frac{\partial C'(t,T)}{\partial T} + \frac{\partial A'(t,T)}{\partial T} \right) dt + \frac{\partial C}{\partial T} \sigma_t dW_t$$

For HJM condtion to be satisfied, we need

$$\frac{\partial C}{\partial T} + R_t \frac{\partial C'(t,T)}{\partial T} + \frac{\partial A'(t,T)}{\partial T} = \sigma(t,T)\bar{\sigma}(t,T)$$

and

$$\frac{\partial C}{\partial T} \sigma_t = \sigma(t,T)$$

Recall in HW, with a, b, c being constant, we have

$$C(t,T) = \frac{1}{b}\left(1 - e^{-b(T-t)}\right), \quad A'(t,T) = -aC(t,T) + \frac{1}{2}\sigma^2 C^2(t,T)$$

then,

$$\frac{\partial C}{\partial T} = e^{-b(T-t)}, \quad \sigma(t,T) = \sigma e^{-b(T-t)}$$

we just compute $\bar{\sigma}$ additionally to check.

Exercise 4.1 — Check HJM for CIR model. Much more complicated.

5. Portfolio Selection

5.1 Merton's Problem

Given m assets

$$\frac{dS_{it}}{S_{it}} = \mu_{it}dt + \sigma_i(t)dB(t), \;\; i = 1, \cdots, m(=l)$$

V_t is the portfolio value, and $\pi(t) := (\pi_{it})_{i=1}^{m}$ which is the total dollar value of S_{it}. Then, recall

$$dV_t = (V_t - |\pi(t)|)r_t dt + \pi(t)[\mu(t)dt + \sigma(t)dB(t)]$$

where $\sigma(t) \in \mathbb{R}^{m\times m}$. We propose the following optimization problem,

$$\max_{\pi\in\mathscr{F}_t} \mathbf{E}[\mathscr{U}(V_T)], \;\; \text{s.t.} dV_t = \cdots$$

where $\mathscr{U}(\cdot)$ is a given utility function. Typically, $\mathscr{U}$ is a concave function.
First, consider an auxiliary problem,

$$\max_{\xi\in\mathscr{F}_T} \mathbf{E}[\mathscr{U}(\xi)], \;\; \text{s.t. } \mathbf{E}(\rho\xi) = v_0 = V_0$$

The original optimization wants to know the best way to drive to a destination. The auxiliary problem, however, want to figure out the best destination first. Then, we will map out a path to such a destination. ρ is called the **pricing kernel** defined as follows

$$\rho := \rho_T = \exp\left(-\int_0^T \lambda(s)dB(s) - \int_0^T \left(r_s + \frac{1}{2}\|\lambda(s)\|^2\right)ds\right) = D_T\zeta_T$$

Thus, $\mathbf{E}(\rho\xi) = \mathbf{E_Q}(D_T\xi)$. Also, note that $\xi := V_T$ at the terminal state. To solve this, we take the Lagrangian multiplier.

$$L(\xi) := \mathbf{E}\mathscr{U}(\xi) - \theta(\mathbf{E}(\rho\xi) - v_0), \;\; \theta > 0$$

treating $\mathbf{E}(\rho\xi) \le v_0$. θ is usually known as the penalty cost or **shadow price**. We want to solve the unconstrained problem

$$\max_{\xi,\theta} L(\xi;\theta)$$

Take derivatives with respect to ξ, we have

$$\dot{\mathscr{U}}(\xi) - \rho\theta = 0 \Longrightarrow \xi^* = \dot{\mathscr{U}}^{-1}(\rho\theta)$$

Take derivative with respect to θ, we have

$$\mathbf{E}(\rho\xi) = v_0 \Longrightarrow \mathbf{E}\big[\rho\dot{\mathscr{U}}^{-1}(\rho\theta)\big] = v_0$$

we can solve for θ^* as the single unknown in the last equation. Now, we have solved auxiliary problem, we know the best destination. Now, we need to find a way to drive there.
Next, we need to find $\{\pi(t), t \in [0,T]\}$ to replicate $V_T^* = \xi^*$. Recall the market price of risk process $\lambda(t)$ given by

$$\mu_{it} - r_t = \sigma_i(t)\lambda(t) \equiv \tilde{\mu}(t) := \mu(t) - r(t) = \sigma(t)\lambda(t)$$

We assume $\sigma(t)$ is invertible. Then, $\lambda(t) = [\sigma(t)]^{-1}\tilde{\mu}(t)$.

$$\begin{aligned}
dV_t &= r_t V_t dt - \pi(t)r(t)dt + \pi(t)[\mu(t)dt + \sigma(t)dB(t)] \\
&= r_t V_t dt + \pi(t)\sigma(t)\lambda(t)dt + \pi(t)\sigma(t)dB(t) \\
&= r_t V_t dt + \pi(t)\sigma(t)dW(t)
\end{aligned}$$

Now, by the fundamental theorem, it guarantees a replicating strategy $\pi^*(t)$ to give us $V_T^* = \xi^*$ as desired.

■ **Example 5.1 — Log Utility.** Let $\mathscr{U}(x) = \ln(x)$. $\dot{U}(x) = \frac{1}{x} = y$, $x = \frac{1}{y}$. Thus, $\dot{U}^{-1}(y) = \frac{1}{y}$. Then, $\xi^* = \frac{1}{\rho\theta}$. Then,

$$\mathbf{E}(\rho\xi^*) = \mathbf{E}\left[\rho\left(\frac{1}{\rho\theta}\right)\right] = \frac{1}{\theta} = v_0 \Longrightarrow \theta^* = \frac{1}{v_0}$$

Thus, $\xi^* = \frac{v_0}{\rho} = \frac{v_0}{\rho_T}$. This suggests $V_t = \frac{v_0}{\rho_t}$. Recall that

$$\begin{aligned}
\rho_t &= D_t\zeta_t = \exp\left[-\int_0^t \lambda(s)dB(s) - \int_0^t (r_s + \frac{1}{2}\|\lambda(s)\|^2)ds\right] \\
\frac{d\rho_t}{\rho_t} &= -r_t dt - \lambda(t)dB(t)
\end{aligned}$$

$$\begin{aligned}
dV_t &= -\frac{v_0 d\rho_t}{\rho_t^2} + \frac{v_0 d\langle\rho\rangle_t}{\rho_t^3} \\
&= \frac{v_0}{\rho_t}\left(-\frac{d\rho_t}{\rho_t} + \left(\frac{d\rho_t}{\rho_t}\right)^2\right) \\
&= V_t\left[\left(r_t + \|\lambda(s)\|^2\right)dt + \lambda(t)dB(t)\right]
\end{aligned}$$

Matching the volatility coefficient, we have

$$\pi(t)\sigma(t) = \frac{v_0}{\rho_t}\lambda(t) \Longrightarrow \pi^*(t) = \frac{v_0}{\rho_t}\lambda(t)[\sigma(t)]^{-1}$$

we still need to match the drift coefficient to show this replicating strategy suffices. ■

5.2 Markowitz M-V Problem

Consider

$$\min_{\pi \in \mathscr{F}_t} Var(V_T) \quad \text{,s.t. } \mathbf{E}(V_T) = \bar{v},\ dV_t = \cdots, V_0 = v_0$$

The auxiliary problem is

$$\min_{\xi \in \mathscr{F}_t} Var(\xi) \quad \text{,s.t. } \mathbf{E}(\xi) = \bar{v},\ \mathbf{E}(\rho\xi) = v_0$$

Again, the Lagrangian is

$$L(\xi;\theta,\eta) = \mathbf{E}(\xi^2) - 2\theta[\mathbf{E}(\xi) - \bar{v}] - 2\eta[v_0 - \mathbf{E}(\rho\xi)]$$

where $\theta, \eta > 0$ and we are treating $\mathbf{E}(\xi) \geq \delta v$ and $\mathbf{E}(\rho\xi) \leq v_0$. We want to solve the unconstrained problem

$$\min_{\xi,\theta,\eta} L(\xi;\theta,\eta)$$

Taking partials, we have

$$2\xi - 2\theta - 2\eta\rho = 0 \Longrightarrow \xi^* = \theta - \eta\rho$$

By substitution, we have

$$\theta - \eta\mathbf{E}(\rho) = \bar{v},\ \ \theta\mathbf{E}(\rho) - \eta\mathbf{E}(\rho^2) = v_0$$

we can solve for θ^*, η^*.

■ **Example 5.2 — Replicating Strategy Proposal.** Note that $V_T = \xi^* = \theta - \eta\rho_T$. Does $V_t = \theta - \eta\rho_t$ work? Unfortunately, no. It will fail the replicating strategy condition with one of the drift or volatility coefficient being unmatchable. Intuitively, θ plays a role in the path of V_t but taking derivative makes its effect void. What works is the following, consider

$$V_t = \frac{1}{\rho}\mathbf{E}(\rho_T\xi^*|\mathscr{F}_t) = \frac{\theta}{\rho_t}\mathbf{E}(\rho_T|\mathscr{F}_t) - \frac{\eta}{\rho_t}\mathbf{E}(\rho_T^2|\mathscr{F}_t)$$

clearly, as $t \to T$, we have $V_t \to V_T$. Assuming the short-rate process being deterministic,

$$\begin{aligned}
\frac{1}{\rho_t}\mathbf{E}(\rho_T|\mathscr{F}_t) &= \frac{1}{\rho_t}\mathbf{E}(D_T\zeta_T|\mathscr{F}_t) \\
&= \frac{D_T}{\rho_t}\mathbf{E}(\zeta_T|\mathscr{F}_t) \\
&= \frac{D_T S_t}{\rho_t} = \frac{D_T}{D_t} && MG
\end{aligned}$$

Let

$$Z_t = \exp\left(-\int_0^t 2\lambda(s)dB(s) - \frac{1}{2}\int_0^t \|2\lambda(s)\|^2 ds\right)$$

Note that Z_t is an exponential MG. Let

$$C_t = \exp\left(\int_0^t \|\lambda(s)\|^2 ds\right)$$

We can now write

$$\zeta_t^2 = C_t Z_t$$

$$
\begin{aligned}
\frac{1}{\rho_t}\mathbf{E}(\rho_T^2|\mathscr{F}_t) &= \frac{1}{\rho_t}(D_T^2\zeta_T^2|\mathscr{F}_t) \\
&= \frac{D_T^2 C_T}{\rho_t}\mathbf{E}(Z_T|\mathscr{F}_t) = \frac{D_T^2 C_T Z_t}{\rho_t} \\
&= \frac{D_T^2 C_T}{D_t^2 C_t}\rho_t
\end{aligned}
$$

Then,

$$\frac{D_T^2 C_T}{D_t^2 C_t} = \exp\left(-\int_t^T (2r_s - \|\lambda(s)\|^2)ds\right) =: A_{t,T}$$

Now,

$$dV_t = \left[r_t V_t + \eta\rho_t A_{t,T}\|\lambda(t)\|^2\right]dt + \eta\rho_t A_{t,T}\lambda(t)dB(t)$$

Matching the volatility coefficient, we get

$$\pi^*(t) = \eta\rho_t A_{t,T}\lambda(t)[\sigma(t)]^{-1}$$

Matching the drift term is left as an exercise. And we have solved the Markowitz M-V problem! ■

5.2.1 Efficient Frontier

Note that, under the Markowitz model,

$$
\begin{aligned}
\mathbf{E}(\rho) &= D_T\mathbf{E}(\zeta_T) = D_T \\
\mathbf{E}(\rho^2) &= D_T^2\mathbf{E}(\zeta_T^2) = D_T^2 C_T \\
Var(\rho) &= D_T^2(C_T - 1)
\end{aligned}
$$

We have $\eta^* = \frac{(\bar{v}\mathbf{E}(\rho) - v_0)}{Var(\rho)}$. Then, $\xi^* = \theta - \eta\rho$

$$
\begin{aligned}
Var(V_T) = Var(\xi^*) = \eta^2 Var(\rho) &= \frac{(\bar{v}\mathbf{E}(\rho) - v_0)^2}{Var(\rho)} \\
&= \frac{(\bar{v}D_T - v_0)^2}{D_T^2(C_T - 1)} = \frac{\left(\bar{v} - \frac{v_0}{D_T}\right)^2}{C_T - 1}
\end{aligned}
$$

The relationship between the target return $\bar{v}$ and the variance of the portfolio value is the efficient frontier. Also note that $C_T > 1$. We also want to set the target $\bar{v} \geq \frac{v_0}{D_T}$. Otherwise, we can just put all cash in the bank, as a trivial solution.

5.3 Commentaries

For the Merton's problem,,

$$\pi^*(t) = \frac{v_0}{\rho_t}\lambda(t)[\sigma(t)]^{-1}$$

and Markowitz yields

$$\pi^*(t) = \eta\rho_t A_{t,T}\lambda(t)[\sigma(t)]^{-1}$$

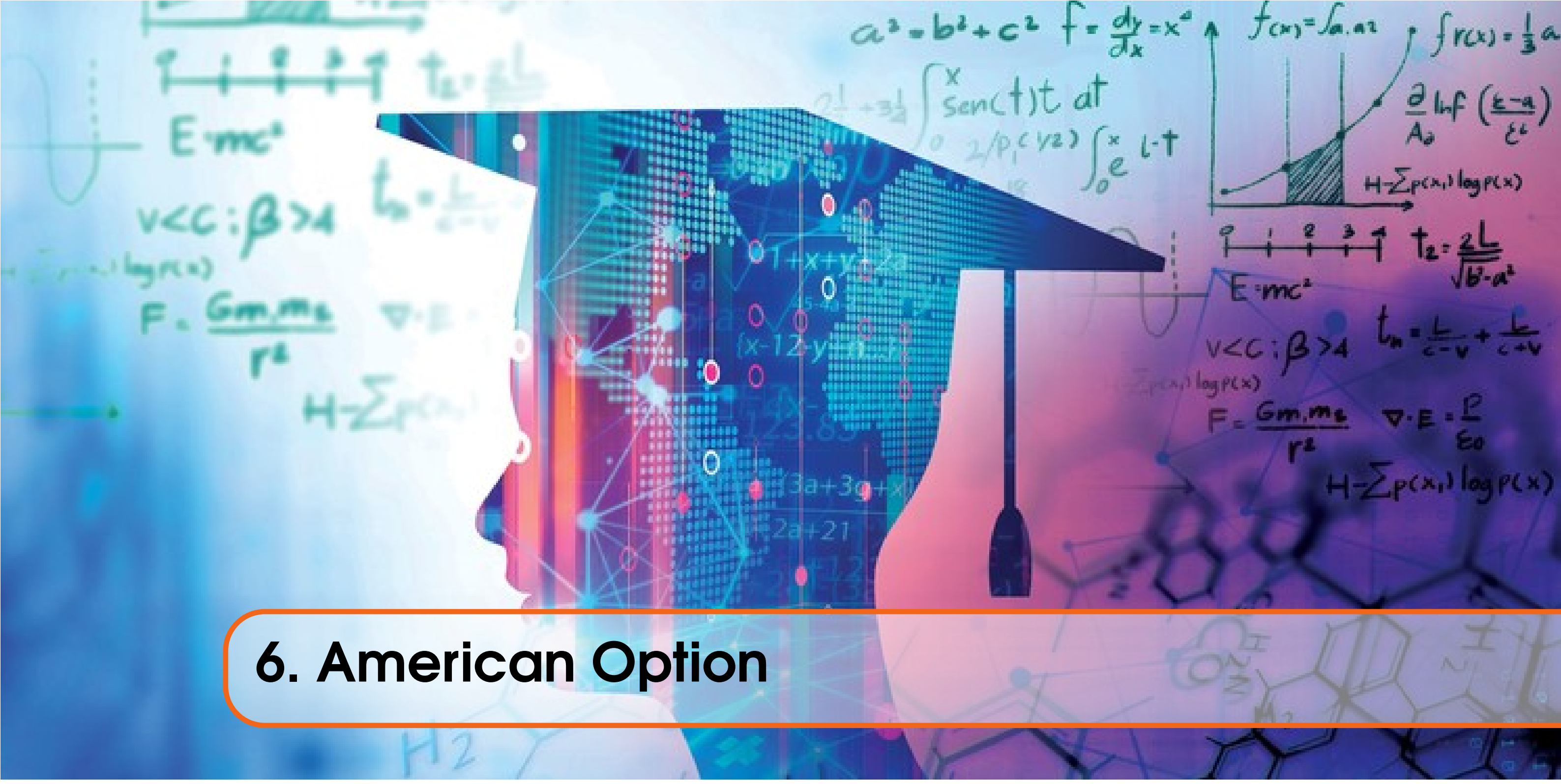
For both models,

$$\lambda(t) = [\sigma(t)]^{-1}\tilde{\mu}(t)$$

Then,

$$\begin{aligned}
\lambda(t)[\sigma(t)]^{-1} &= \tilde{\mu}(t)\left([\sigma(t)]^{\top}\right)^{-1}(\sigma(t))^{-1} \\
&= \tilde{\mu}(t)\left(\underbrace{\sigma(t)\sigma^{\top}(t)}_{=:\Gamma(t)}\right)^{-1}
\end{aligned}$$

where $\Gamma_{i,j,t} = \sigma_i(t)\sigma_j(t)$. Note that Γ is commonly known as the covariance matrix of the assets.



6. American Option

6.1 Basic Properties

Let C_t, P_t be American call/put price at t. Let c_t, p_t for European options.

Proposition 6.1.1 — Basic Properties.

1. $C_t \geq c_t, P_t \geq p_t, \forall t \in [0,T]$
2. Assuming underling S_t pays no income, then

$$C_t \geq c_t = p_t + S_t - Ke^{-r(T-t)} \geq S_t - K$$

The last term is the payoff if exercising at time t. This implies the selling price is better than the exercising payoff. Therefore, never exercise before maturity. $C_t = c_t$.

> R This does not apply to **American puts**! Note that
>
> $$P_t \geq p_t = c_t + Ke^{-r(T-t)} - S_t \ngeq K - S_t$$
>
> This does not apply if S_t pays income! Note that
>
> $$C_t \geq c_t = p_t + S_t e^{-q(T-t)} - Ke^{-r(T-t)} \ngeq S_t - K$$

6.2 Refresher on 4701

Let T be a stopping time, which is a random variable. In the Brownian motion setting,

$$T := \inf\{t : X_t := \mu t + \sigma B_t = -a, b\}, \quad a, b > 0$$

and

$$X_T = \begin{cases} b = & p_b \\ -a & 1 - p_b \end{cases}$$

we want p_b and $\mathbf{E}(T)$.

Theorem 8 — Optimal Stopping Theorem. If M_t is a MG, then $\mathbf{E}(M_T) = M_0$.

Apply OST to exp-MG $\exp\left(\theta B_t - \frac{1}{2}\theta^2 t\right)$. Then,

$$\mathbf{E}\left(\exp\left(\theta B_T - \frac{1}{2}\theta^2 T\right)\right) = \exp\left(\theta B_0 - \frac{1}{2}\theta^2 0\right) = 1$$
$$\mathbf{E}\left(\exp\left(\theta \frac{X_T - \mu T}{\sigma} - \frac{1}{2}\theta^2 T\right)\right) = 1$$

We pick θ such that

$$-\frac{\theta\mu}{\sigma} - \frac{1}{2}\theta^2 = 0 \iff \theta = -\frac{2\mu}{\sigma}$$

Then, $\mathbf{E}(e^{\frac{\theta}{\sigma}X_T}) - 1$ and

$$e^{\frac{\theta}{\sigma}(-a)}(1-p_b)e^{\frac{\theta}{\sigma}b}p_b = 1$$

write $\rho := e^{\frac{\theta}{\sigma}} = e^{-\frac{2\mu}{\sigma^2}}$. Thus, we have

$$p_b = \frac{1-\rho^{-a}}{\rho^b - \rho^{-a}} = \frac{1-\rho^a}{1-\rho^{a+b}}$$

To derive $\mathbf{E}(T)$, apply OST to B_t with $\mathbf{E}(B_T) = 0$. So,

$$\mathbf{E}\left(\frac{X_T - \mu T}{\sigma}\right) = 0 \Longrightarrow \mathbf{E}(T) = \frac{\mathbf{E}(X_T)}{\mu} = \frac{-a(1-p_b) + bp_b}{\mu}$$

6.2.1 Asymptotic Analysis

As $a \to \infty$, let p_b be the probability of ever reaching b.

1. Position drift $\mu > 0 \equiv \rho < 1$: $p_b = 1$
2. Zero drift $\mu = 0 \equiv \rho = 1$: $p_b = \frac{a}{a+b} \to 1$ as $a \to \infty$.
3. Negative drift $\mu < 0 \equiv \rho > 1$: $p_b = \frac{\rho^a}{\rho^{a+b}} = \rho^{-b} \in (0,1)$

■ **Example 6.1 — One-sided Random Walk Moment.** Let

$$T := \inf\{t : X_t = \mu t + \sigma B_t = b\}, \mu > 0, b > 0$$

want $\mathbf{E}(e^{-\gamma T})$ with $\gamma > 0$ being given. **This is the key idea to pricing the American option.**

Proof. Write $-\gamma T = \theta B_T - \frac{1}{2}\theta^2 T - \beta b$ where $\beta > 0, \theta > 0$ are parameters. Then,

$$\begin{aligned}\mathbf{E}(e^{-\gamma T}) &= \mathbf{E}\left(e^{\theta B_T - \frac{1}{2}\theta^2 T - \beta b}\right)\\ &= e^{-\beta b}\mathbf{E}\left(e^{\theta B_T - \frac{1}{2}\theta^2 T}\right)\\ &= e^{-\beta b}\end{aligned}$$

Then,

$$-\gamma T = \theta B_T - \frac{1}{2}\theta^2 T - \beta\mu T - \beta\sigma B_T$$

by mathcing coefficients, we get

$$\gamma = \frac{1}{2}\theta^2 + \beta\mu, \quad \theta = \beta\sigma$$

we get $\frac{\sigma^2}{2}\beta^2 + \beta\mu - \gamma = 0$ and

$$\beta = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\gamma}}{\sigma^2}$$

note that $\beta > 0$ is needed. ■

▪

6.3 Perpetual American Put

6.3.1 Analytical Properties

Assuming the underlying pays no income, we will start with American put with **no maturity**. This becomes a pure stopping-time decision problem. The stopping rule is captured by

$$\tau_l := \inf\{t \geq 0 : S_t = l\}, \quad l \in [0, K]$$

Put price at $t = 0$ denoted $v_l(x)$ and x is a placeholder for S_0. Then,

$$v_l(x) = \begin{cases} (K-l)\mathbf{E}(e^{-r\tau_l}) & x \geq l \\ K - x & x \leq l \end{cases}$$

the second case means exercising immediately. Note that this function is continuous at $x = l$. Let

$$X_t = -\left(r - \frac{\sigma^2}{2}\right)t - \sigma W_t = \ln(S_0/S_t)$$

and $b = \ln(x/l) > 0$. Using the previous one-sided random walk moment formula, we have

$$\mathbf{E}\left(e^{-r\tau_l}\right) = \left(\frac{x}{l}\right)^{-\frac{2l}{\sigma^2}}$$

and

$$v_l(x) = \begin{cases} (K-l)\left(\frac{x}{l}\right)^{-\frac{2l}{=}\sigma^2} = (K-l)l^\alpha x^{-\alpha} & x \geq l \\ K - x & x \leq l \end{cases}$$

where $\alpha = \frac{2r}{\sigma^2}$. Next we want to find l^* that maximize $v_l(x)$ for all given x. Suppose $x \geq l$, we want to maximize $(K-l)l^\alpha$ with respect to l. Taking derivative yields

$$K\alpha l^{\alpha-1} - (\alpha+1)l^\alpha = 0 \Longrightarrow l^* = \frac{\alpha K}{\alpha+1} < K$$

Moreover, $v_l(x)$ is strictly concave and it is the unique maxima.
Are we done here? Well, not so fast. $v_l(x)$ is a piece-wise function and x, l relationship could potentially change the result. We need to investigate further.
[INSERT FIGURE] We have 3 observations:

- Let $y_l(x) = (K-l)l^\alpha x^{-\alpha}$. Then,

  $$y_l'(x) = -\frac{\alpha}{x}y_l(x)$$

  implies that when $x = l_1$ being sufficiently small (close to 0), then $y_{l_1}'(l_1) = -\frac{\alpha}{l_1}(K - l_1) < -1$.
- On the other hand, $y_l(x)$ will cross $K - x$ again at $l_2 > l_1$.
- $y_{l_1}(x) = y_{l_2}(x), \forall x \in [l_2, K]$

Moreover, for $x \geq l$, $v_l'(x) = y_l'(x) = -\frac{\alpha}{x}v_l(x)$ and $v''(x) = \frac{\alpha(\alpha+1)}{x^2}v_l(x)$. For $x \leq l$, $v_l'(x) = -1$ and $v_l''(x) = 0$. Note that

$$v_l'(l-) = -1, v_l'(l+) = -\frac{\alpha}{l}(K-l)$$

and

$$l = l^* = \frac{\alpha K}{\alpha+1} \iff v_l'(l-) = v_l'(l+)$$

This is known as the **smooth pasting** point. Furthermore,

$$\begin{cases} rv_l(x) - rxv_l'(x) - \frac{1}{2}\sigma^2 x^2 v_l''(x) = 0 & x \geq l \\ rv_l(x) - rxv_l'(x) - \frac{1}{2}\sigma^2 x^2 v_l''(x) = rK & x \leq l \end{cases}$$

Linear Complementarity

(1) $v_l(x) \geq (K-l)^+, \forall x$

(2) $rv_l(x) - rxv_l'(x) - \frac{1}{2}\sigma^2 x^2 v_l''(x) \geq 0, \forall x$

(3) For each $x \geq 0$, (1) or (2) (or both) must hold as equality.

We cannot have both being strictly inequalities. One can verify that $v_{l^*}(x)$ is the only continuous function with continuous derivative that satisfies the linear complementarity condition.
Hence, let $\mathscr{S} := \{x : v_{l^*(}(x) = (K-x)^+\}$ (stop and exercise) and $\mathscr{C} := \{x : v_{l^*}(x) > (K-x)^+\}$ (continue).

6.3.2 Probabilistic Properties

Consider $e^{-rt}v_{l^*}(S_t)$, the discounted put price.

$$d\left(e^{-rt}v_{l^*}(S_t)\right) = e^{-rt}\left[-rv_{l^*}(S_t)dt + v_{l^*}'(S_t)dS_t + \frac{1}{2}v_{l^*}''(S_t)d\langle S\rangle_t\right]$$

$$= e^{-rt}\left[\underbrace{\left(-rv_{l^*}(S_t) + rS_t v_{l^*}'(S_t) + \frac{1}{2}\sigma^2 S_t^2 v_{l^*}''(S_t)\right)}_{=\begin{cases} 0 & S_t \geq l^* \ \text{MG} \ S_t := x \in \mathscr{S} \\ -rK & S_t \leq l^* \ \text{Super-MG} \ S_t := x \in \mathscr{C} \end{cases}} dt + v_{l^*}'(S_t)S_t\sigma dW_t\right]$$

This reinforces that l^* is the optimal stopping-rule since we should continue holding the option if $S_t \geq l^*$ as a MG; otherwise, we should exercise as it becomes a super-MG.

Theorem 9 Super-MG property implies τ_{l^*} is optimal among all $\tau \in \tilde{T}$. Recall a stopping time $\{\tau \leq t\} \in \mathscr{F}_t, \forall t$.

Proof.

$$v_{l^*}(S_0) \overset{\text{OST super-MG}}{\geq} \mathbf{E_Q}(e^{-r\tau}v_{l^*}(S_\tau)) \geq \mathbf{E_Q}(e^{-r\tau}(K-S_\tau)^+)$$
$$v_{l^*}(S_0) \geq \max_{\tau \in \tilde{T}}\left\{\mathbf{E_Q}(e^{-r\tau}(K-S_\tau)^+)\right\}$$

since $\tau_{l^*} \in \tilde{T}$, we have

$$v_{l^*}(S_0) = \max_{\tau \in \tilde{T}}\left\{\mathbf{E_Q}(e^{-r\tau}(K-S_\tau)^+)\right\}$$

Our stopping-time rule is indeed optimal among all possible stopping-time. ∎

6.4 Martingale Review

Definition 6.4.1 — MG/Sub-MG/Super-MG. $\{X_t\}$ is a sub-(super-) MG if

$$\mathbf{E}(X_t|\mathscr{F}_s) \geq (\leq) X_s, \forall t > s, \quad \mathbf{E}|X_t| < \infty$$

MG is a special case, both sub and super MG. Moreover,

$$\mathbf{E}(X_t) \geq (\leq)\mathbf{E}(X_s), \forall t > s$$

Theorem 10 — OST for Sub/Super MG. $\mathbf{E}(X_T) \geq (\leq)\mathbf{E}(X_0)$ for any stopping-time T with filtration $\mathscr{F}_t$ for any t.

6.4.1 Jenson's Inequality

Definition 6.4.2 — Convex function. $h(x)$ is convex if

$$h(\alpha x_1 + (1-\alpha)x_2) \leq \alpha h(x_1) + (1-\alpha)h(x_2)$$

for all $\alpha \in [0,1]$ and x_1, x_2 in the domain of h.

If X, r.v., has finte expectation, then

$$\mathbf{E}(h(X)) \geq h(\mathbf{E}(X))$$

Corollary 6.4.1 Suppose X_t is a MG and h is convex, then $h(X_t)$ is a sub-MG as a result.

Proof. For $t > s$,

$$\mathbf{E}h(X_t|\mathscr{F}_s) \geq h(\mathbf{E}(X_t|\mathscr{F}_s)) = h(X_s)$$

∎

■ **Example 6.2 — BM.** We know $\{B_t\}$ is a MG. Then, $\{B_t^2\}$ is a sub-MG. ■

Recall $(e^{-rt}S_t)$ is a MG under $\mathbf{Q}$. We know $h(x) = (x-K)^+$ is a convex function. The payoff of a call is indeed a sub-MG. True, but not useful. We want to check if $e^{-rt}h(S_t)$ is a MG. The convexity is not enough. Additionally, we need $h(0) = 0$. For example, the call payoff function indeed satisfies this.

Proposition 6.4.2 Let $e^{-rt}S_t$ be a MG and h convex and $h(0) = (0)$, then $e^{-rt}h(S_t)$ is a sub-MG.

Proof. Exercise. ∎

Then,

$$\mathbf{E}_{\mathbf{Q}}\left(e^{-rt}h(S_u)|\mathscr{F}_t\right) \geq e^{-rt}h(S_t), \forall u > t$$

Let $u = T$, then

$$\mathbf{E}_{\mathbf{Q}}\left(e^{-r(T-t)}h(S_T)|\mathscr{F}_t\right) \geq h(S_t)$$

If $h(x) = (x-K)^*$, this means the price of the American call is better than the payoff at time t. No early exercise! Again, the key is that S_t does not pay dividend. If it generates income, $e^{-rt}S_t$ is no longer a MG under $\mathbf{Q}$. In fact, a super-MG under $\mathbf{Q}$.

How About American Puts?

Cannot apply to American puts either since $h(x) = (K-x)^+$ yields $h(0) \neq 0$.

6.5 American Call Paying Discrete Dividends

Suppose the underlying pays dividends at $0 < t_1 < t_2 < \cdots < t_n < T$. The underlying price will drop at these dividend dates. At t_j, the stock price

$$S_{t_j} = S_{t_{j-}}(1-\alpha_j),\ \alpha_j \in (0,1)$$

The underlying process is not continuous, but right continuous. Consider the last period $t \in [t_n, T]$, there is no more dividend payout. Define

$$c_n(t, S_t) := \mathbf{E}_{\mathbf{Q}}\left(e^{-r(T-t)}h(S_T)|\mathscr{F}_t\right) \geq h(S_t)$$

which is the call option price in the n-th period. Thus, during the last period, there should not be any early exercise and $c_n(t,S_t)$ should follow the BSM formula and $c_n(t,x)$ follows BSM PDE with terminal condition $c_n(T,x) = h(x)$.

Now, consider $t \in [t_{n-1}, t_n)$.

$$c_{n-1}(t, S_t) := \mathbf{E}_{\mathbf{Q}}\left(e^{-r(t_n-t)}h_n(S_{t_{n-}})|\mathscr{F}_t\right) \geq h_n(S_t) \geq (S_t-K)^+$$

This is the price of the call at t in the $(n-1)$-th interval with

$$h_n(x) = \max\left\{\underbrace{(x-K)^+}_{\text{exercise}}, \underbrace{c_n(t_n,(1-\alpha_n)x)}_{\text{not exer. continue}}\right\},\quad x := S_t$$

note that $h_n(x)$ is a convex function and $h(0) = 0$. Again, we have the last inequality, the sub-MG property. Thus, we still do not exercise early until potentially the very last point at the dividend issuance time or continue the next period. $c_{n-1}(t,x)$ still satisfies BSM PDE (verify!)

$$\frac{\partial c_{n-1}}{\partial t} + rx\frac{\partial c_{n-1}}{\partial x} + \frac{1}{2}x^2\frac{\partial^2 c_{n-1}}{\partial x^2} = rc_{n-1}$$

with terminal condition

$$c_{n-1}(t_n, x) = h_n(x),\quad x := S_{t_{n-}}$$

We do not have close-form solution and we need to solve this PDE numerically.

6.6 American Puts with Finite Maturity

We do not have time stationary (decision-making indifferent of t). This complicates things as we need to look at $v(t,S_t)$, the put price at time t. Consider

$$v(t,x) = \max_{\tau\in\tilde{T}_{t,T}} \mathbf{E}_{\mathbf{Q}}\left(e^{-r(\tau-t)}(K-s_\tau)^+|S_t = x\right)$$

l_* will need to be changed to l_t^*. The good news is the linear complementary condition still holds.

(1) $v(t,x) \geq (K-l)^*, \forall x \geq 0, \forall t \in [0,T]$

(2) $rv(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2x^2v_{xx}(t,x) \geq 0, \forall x \geq 0, \forall t \in [0,T]$

(3) One or both of (1), (2) must be equality for each $x \geq 0$ and for each $t \in [0,T]$.

Similarly,

$$\mathscr{S} = \left\{(t,x) : v(t,x) = (K-x)^+\right\}, \ \mathscr{C} = \left\{(t,x) : v(t,x) > (K-x)^+\right\}$$

l_t^* is the stock price at optimal τ^*, which is S_{τ^*}.
Unfortunately, we cannot say more about the explicit form but resort to numerically solving the PDE using the linear complementary conditions.

Í—END OF NOTES—