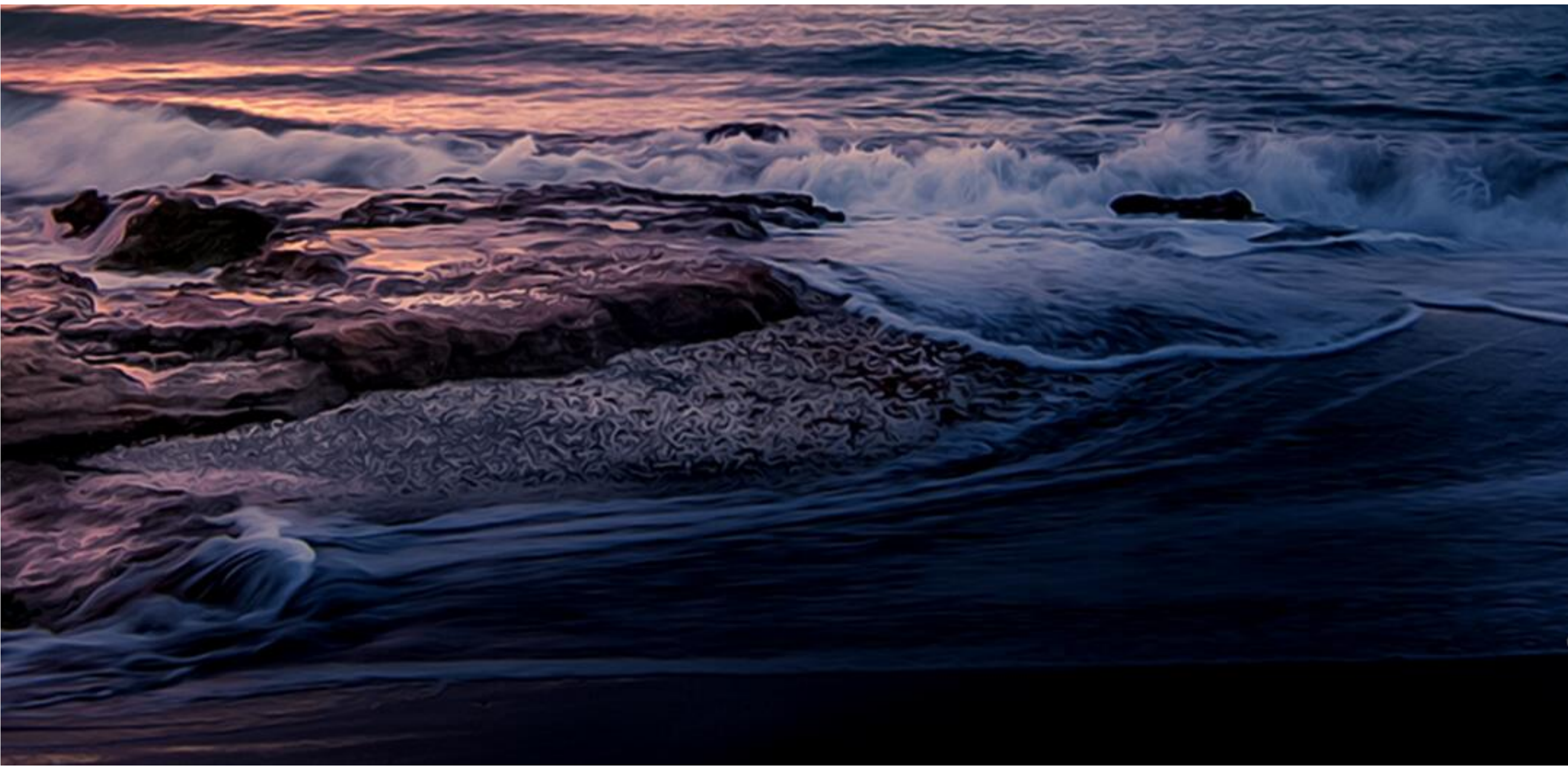
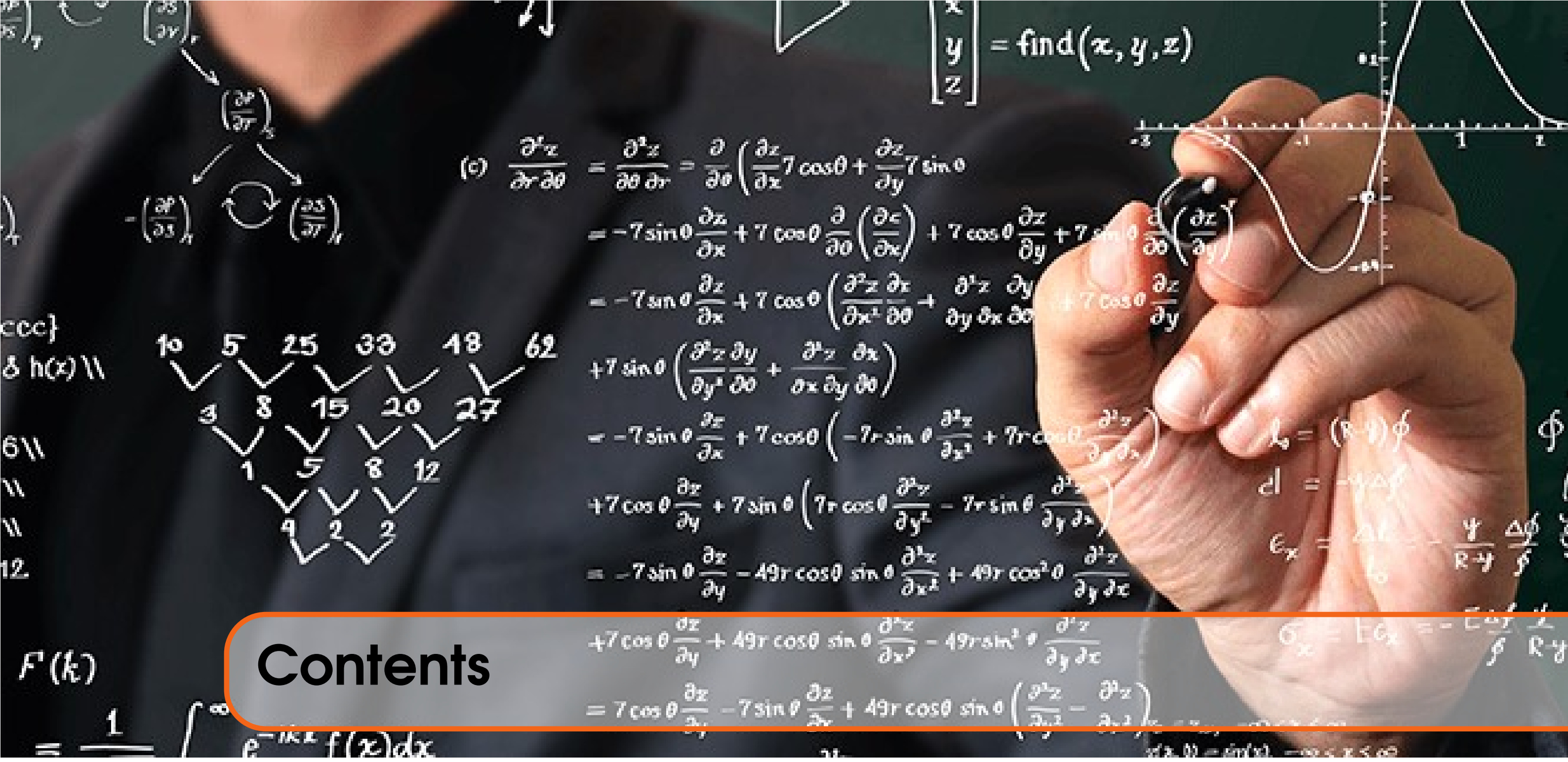
PMATH 451 Course Notes

University of Waterloo

The One And Only
Waterloo 76er
Bill Zhuo





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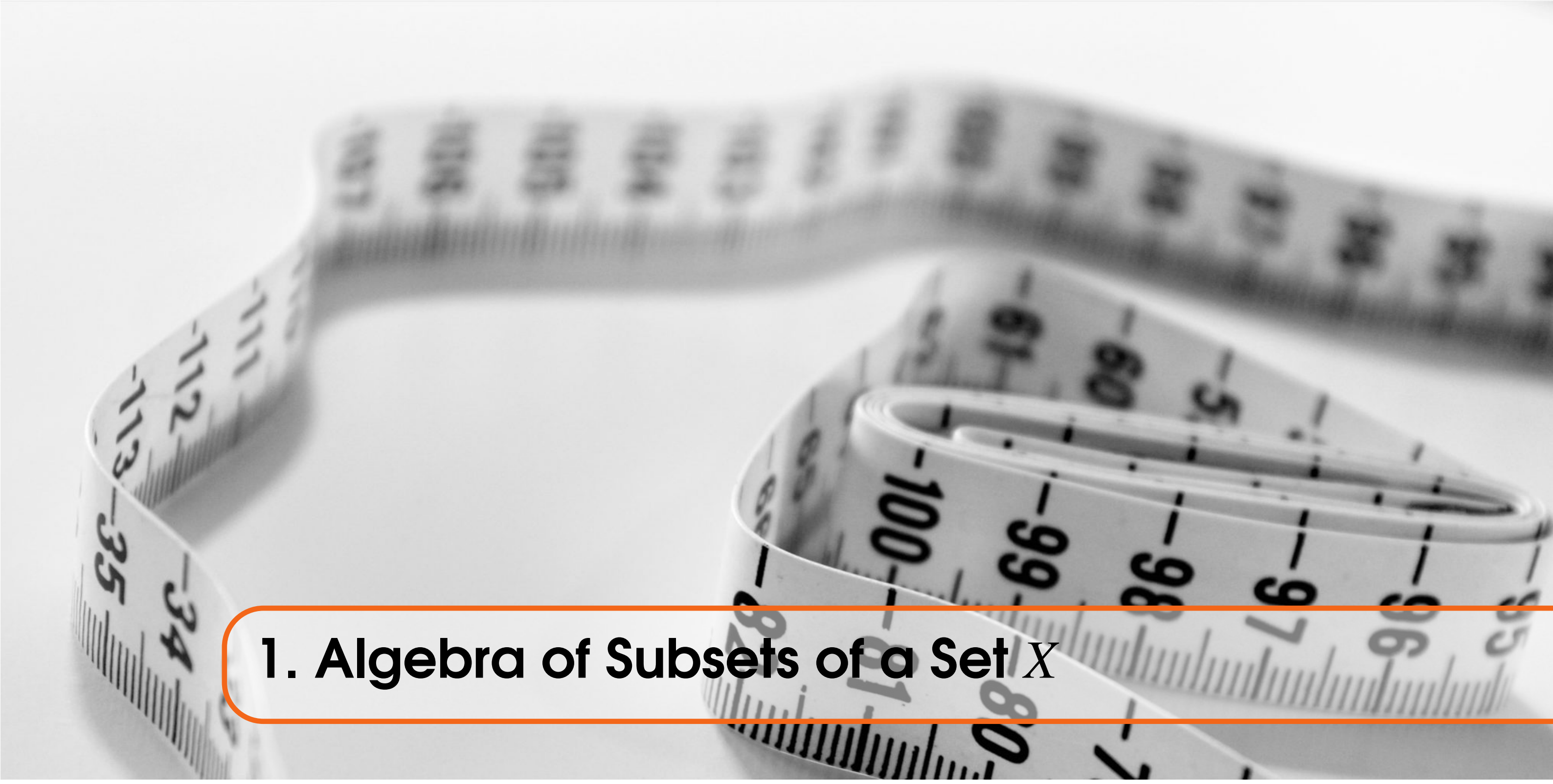

1. Algebra of Subsets of a Set X

1.1 Definitions, Properties, and Some Basic Examples

Definition 1.1.1 — Algebra of Sets. Let X be a non-empty set and let $\mathscr{A}$ be a collection of subsets of X. We say that $\mathscr{A}$ is an algebra of sets to mea that it satisfies the following conditions:

1. **(AS1)** $\emptyset \in \mathscr{A}$
2. **(AS2)** If $A \in \mathscr{A}$, then $X \backslash A \in \mathscr{A}$
3. **(AS3)** If $A, B \in \mathscr{A}$, then $A \cap B \in \mathscr{A}$.

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1. The $\mathscr{A}$ from Definition 1.1.1 is, in a certain sense, a set of sets. In fact, the first sentence in the definition could be told as:

   Let X be a non-empty set and let $\mathscr{A} \subseteq 2^X$, which is the power set of X.

   In this course, these calligraphic fonts, like $\mathscr{A}, \mathscr{B}, \mathscr{M}$ will be used to denote subsets of 2^X.
2. The notion of algebra of X sometimes also goes under the name of *field of subsets of* X.
3. Why we only care about the intersection in (AS3)? This is in fact sufficient for other things we want.

Proposition 1.1.1 — Properties of $\mathscr{A}$. Let X be a non-empty set and let $\mathscr{A}$ be an algebra of subsets of X.

1. The total set X must be in $\mathscr{A}$
2. If $A, B \in \mathscr{A}$, then $A \cap B \in \mathscr{A}$
3. If $A, B \in \mathscr{A}$, then $A \backslash B \in \mathscr{A}$
4. For all $n \geq 2$ and every $A_1, \cdots, A_n \in \mathscr{A}$, we have $\bigcap_{i=1}^n A_i \in \mathscr{A}$
5. For all $n \geq 2$ and every $A_1, \cdots, A_n \in \mathscr{A}$, we have $\bigcup_{i=1}^n A_i \in \mathscr{A}$.

Proof. 1. The combination of conditions (AS1) and (AS2) gives us that $X = X \backslash \emptyset \in \mathscr{A}$.

2. We have
$$\begin{aligned} A,B\in\mathscr{A} &\Longrightarrow X\backslash A, X\backslash B\in\mathscr{A} && \text{(AS2)}\\ &\Longrightarrow (X\backslash A)\cap(X\backslash B)\in\mathscr{A} && \text{(AS3)}\\ &\Longrightarrow X\backslash(A\cup B)\in\mathscr{A} && \text{De Morgan!}\\ &\Longrightarrow A\cup B\in\mathscr{A} && \text{(AS2)}\end{aligned}$$
3. We can write $A\backslash B = A\cap(X\backslash B)$ where $A\in\mathscr{A}$ and where $X\backslash B\in\mathscr{A}$ as well, due to (AS2). Hence, (AS3) assures us that $A\backslash B\in\mathscr{A}$.
4. Induction on n, where for the base case $n=2$ we use (AS3)
5. Induction on n, where for the base case $n=2$ we use the statement from part 2.

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R The moral coming from proposition 1.1.1 is hence this:

When dealing with an algebra of sets $\mathscr{A}$, we can do any kind of set operations we want, involving **finitely many** sets from $\mathscr{A}$, and the result of these operations will still belong to $\mathscr{A}$.

Alright, let's see some concrete examples.

■ **Example 1.1** 1. Let X be a non-empty set and let $\mathscr{A}=2^X$. This $\mathscr{A}$ obviously satisfies the conditions (AS1), (AS2), (AS3) and is thus an algebra of subsets of X.
2. Let X be an infinite set and let
$$\mathscr{A} := \{F\subseteq X : F \text{ is finite}\}\cup\{G\subseteq X : X\backslash G \text{ is finite}\}$$
we claim this is an algebra of subsets of X. Let's check. Let $\mathscr{F}=\{F\subseteq X : F \text{ is finite}\}$ and $\mathscr{G}=\{G\subseteq X : X\backslash G \text{ is finite}\}$, so $\mathscr{A}=\mathscr{F}\cup\mathscr{G}$.
   (a) (AS1): Since the empty set $\emptyset$ is a particular case of finite set, we have $\emptyset\in\mathscr{F}$ and therefore $\emptyset\in\mathscr{A}$.
   (b) (AS2): Let A be a set in $\mathscr{A}$. Then either $A\in\mathscr{F}$ or $A\in\mathscr{G}$. In both these cases we see that $X\backslash A\in\mathscr{A}$ since
$$\begin{cases} A\in\mathscr{F}\Longrightarrow X\backslash A\in\mathscr{G}\Longrightarrow X\backslash A\in\mathscr{A}\\ A\in\mathscr{G}\Longrightarrow X\backslash A\in\mathscr{F}\Longrightarrow X\backslash A\in\mathscr{A}\end{cases}$$
   Hence (AS2) holds.
   (c) (AS3) Let $A,B\in\mathscr{A}$. There are three cases:
      i. If A,B are finite, then clearly the intersection is finite. So, $A\cap B\in\mathscr{A}$
      ii. If $X\backslash A, X\backslash B$ are finite, then,
$$(X\backslash A\cup X\backslash B)\in\mathscr{A} \iff X\backslash(A\cap B)\in\mathscr{A} \iff A\cap B\in\mathscr{A}$$
      iii. If $A\in\mathscr{F}$ and $B\in\mathscr{G}$, then,
$$A\cap B=(X\backslash A)\cup(X\backslash B)\in\mathscr{A}$$
      since $X\backslash A\in\mathscr{G}$ and $X\backslash B\in\mathscr{F}$.

   Thus, $\mathscr{A}$ is an algebra of subsets of X.

3. In this example, consider $X = \mathbb{R}$. Let $\mathscr{J}$ be the collection of intervals of $\mathbb{R}$ defined as follow

$$\mathscr{J} := \{\emptyset\} \cup \{(a,b] : a < b \in \mathbb{R}\} \cup \{(-\infty, b] : b \in \mathbb{R}\} \cup \{(a,\infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}\}$$

and define

$$\mathscr{E} := \left\{ E \subseteq \mathbb{R} \middle| \begin{array}{l} \exists k \in \mathbb{N} \text{ and } J_1, \cdots, J_k \in \mathscr{J} \text{ such that} \\ J_i \cap J_j = \emptyset \text{ for } i \neq j \text{ and } J_1 \cup \cdots \cup J_k = E \end{array} \right\}$$

Then, $\mathscr{E}$ is an algebra of subsets of $\mathbb{R}$. It is called the **algebra of half-open intervals**.

4. Non-example: let (X,d) be a metric space, and let

$$\mathscr{T} := \{G \subseteq X | G \text{ is an open set}\}$$

. This is the notion of a topology from PMATH351. Well, a topology is not necessarily an algebra of subsets of X. Consider the Euclidean metric on $\mathbb{R}$. Then, then complement of an open set is actually closed (and not open), which is not in $\mathscr{T}$.

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1.2 The Trick of the Semi-Algebra

How do we generate algebra though? Not so clear? Let's consider the following.

Definition 1.2.1 — Semi-Algebra. Let X be a non-empty set and let $\mathscr{S}$ a be a collection of subsets of X. We will say that $\mathscr{S}$ is a semi-algebra of subsets of X when it satisfies:

1. **(Semi-AS1)** $\emptyset \in \mathscr{S}$
2. **(Semi-AS2)** for every $S \in \mathscr{S}$, there exists $p \in \mathbb{N}$ and $T_1, \cdots, T_p \in \mathscr{S}$ such that $T_i \cap T_j = \emptyset$ for $i \neq j$ and such that $X \backslash S = \bigcup_{i=1}^{p} T_i$
3. **(Semi-AS3)** whenever $S, T \in \mathscr{S}$, it follows that $S \cap T \in \mathscr{S}$.

R Comparing to the definition of algebra of subsets, we have the first and the third being identical. The only difference is that the second one more relaxed than (AS2). Clearly, (AS2) implies (Semi-AS2). This reminds me of all the half-open rectangles on $\mathbb{R}^2$ (a semi-algebra on $\mathbb{R}^2$) and all the finite unions of those half-open rectangles give us an algebra of subsets on $\mathbb{R}^2$. This is not a algebra though, originally. Since we can consider $(0,1] \times (0,1]$ but its complement is for sure not a half-open rectangle.

Let's generalize this proces.

Proposition 1.2.1 — Trick of the Semi-Algebra. Let X be a non-empty set and let $\mathscr{S}$ be a semi-algebra of subsets of X. Let

$$\mathscr{A}_0 := \left\{ A \subseteq X \middle| \begin{array}{l} \exists k \in \mathbb{N} \text{ and } S_1, \cdots, S_k \in \mathscr{J} \text{ such that} \\ S_i \cap S_j = \emptyset \text{ for } i \neq j \text{ and } S_1 \cup \cdots \cup S_k = A \end{array} \right\}$$

then $\mathscr{A}_0$ is an algebra of subsets of X.

Proof. 1. (AS1): consider $k = 1$ and we see that $\mathscr{A}_0 \supseteq \mathscr{S}$, so $\emptyset \in \mathscr{A}_0$.

2. (AS3): let $A, B \in \mathscr{A}_0$, then

$$A = \dot{\bigcup}_{i=1}^{k} S_i$$

$$B = \dot{\bigcup}_{j=1}^{l} T_j$$

then,

$$A \cap B = \left(\dot{\bigcup}_{i=1}^{k} S_i \right) \cap \left(\dot{\bigcup}_{j=1}^{l} T_j \right) = \dot{\bigcup}_{i=1}^{k} \dot{\bigcup}_{j=1}^{l} S_i \cap T_j$$

$S_i \cap T_j \in \mathscr{S}$ and they are disjoint for $(i, j) \neq (i', j')$, thus, we are done.

3. (AS2): Let $A \in \mathscr{A}_0$. This means there exists disjoint $T_i \in \mathscr{S}$ such that

$$A = \bigcup_{i=1}^{k} T_i$$

Then,

$$X \backslash A = X \backslash \bigcup_{i=1}^{k} T_i = \bigcap_{i=1}^{k} (X \backslash T_i)$$

then, for each i, we can write, by (Semi-AS2),

$$X \backslash T_i = \bigcup_{j=1}^{p} S_{(i,j)}, S_{(i,j)} \in \mathscr{S}$$

where $S_{(i,j)} \cap S_{(i,j')} = \emptyset$ for $j' \neq j$. Then, this means $X \backslash T_i \in \mathscr{A}_0$. By (AS3) that has been proved and induction on i, we have

$$X \backslash A = X \backslash \bigcup_{i=1}^{k} T_i = \bigcap_{i=1}^{k} (X \backslash T_i) \in \mathscr{A}_0$$

And now, we are done. ■

1.3 Algebra of Sets Generated by An Arbitrary $\mathscr{C} \subseteq 2^X$

In order to make sure there is no shortage of examples of algebras of sets, we now look a general method to produce such examples.

Lemma 1.4 Let X be a non-empty set, and let $(\mathscr{A}_i)_{i \in I}$ be a family of algebras of subsets of X. Denote $\mathscr{A} := \cap_{i \in I} \mathscr{A}_i$. Then, $\mathscr{A}$ is also an algebra of subsets of X.

Proof. Do it yourself. ■

Proposition 1.4.1 Let X be a non-empty set and let $\mathscr{C}$ be a collection of subsets of X. There exists a collection $\mathscr{A}_0$ of subsets of X, **uniquely** determined, such that

1. $\mathscr{A}_0$ is an algebra of subsets of X and $\mathscr{A}_0 \supseteq \mathscr{C}$.
2. Whenever $\mathscr{A}$ is an algebra of subsets of X such that $\mathscr{A} \supseteq \mathscr{C}$, it follows that $\mathscr{A} \supseteq \mathscr{A}_0$.

Proof.

1. **Existence:** Let $(\mathscr{A}_i)_{i \in I}$ be the family of all the algebras of subsets of X which contains $\mathscr{C}$. This is not empty since 2^X is in it. Then, let

$$\mathscr{A}_0 := \bigcap_{i \in I} \mathscr{A}_i$$

then, by Lemma 1.4, we know that $\mathscr{A}_0$ is also an algebra of subsets of X and it contains $\mathscr{C}$. This gives us the first condition in the proposition. Now, let $\mathscr{A}$ be some other algebra of subsets of X such that $\mathscr{A} \supseteq \mathscr{C}$. Then, $\mathscr{A} \in (\mathscr{A})_{i \in I}$ for some index and it must contains $\mathscr{A}_0$ by construction. Thus, the second condition is met.

2. **Uniqueness:** this is really using the second condition of the proposition and two-way inclusion to finish the uniqueness proof.

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Definition 1.4.1 — Algebra of Sets Generated by $\mathscr{C}$. Let X be a non-empty set and let $\mathscr{C}$ be a collection of subsets of X. The algebra of sets $\mathscr{A}_0$ shown in Proposition 1.4.1 is called the algebra of sets of generated by $\mathscr{C}$, and will be denoted as $\mathrm{Alg}(\mathscr{C})$.

Exercise 1.1 Let X be a non-empty set, let $\mathscr{S}$ be a semi-algebra of subsets of X, and let $\mathscr{A}_0$ be defined as in Proposition 1.2.1. Prove that $\mathscr{A}_0 = \mathrm{Alg}(\mathscr{S})$.

Proof. It is clear that $\mathscr{A}_0$ satisfies the first condition to be $\mathrm{Alg}(\mathscr{S})$. Let $\mathscr{A}$ be any algebra of subsets of X such that $\mathscr{A} \supseteq \mathscr{S}$. ■

Exercise 1.2 Let X be a non-empty set, let $\mathscr{S}$ be a semi-algebra of subsets of X, and let $\mathscr{A}_0$ be as defined as in Proposition 1.2.1. Prove that

$$\mathscr{A}_0 = \widetilde{\mathscr{A}_0} := \left\{ A \subseteq X \middle| \exists k \in \mathbb{N} \text{ and } S_1, \cdots, S_k \in \mathscr{S} \text{ such that } \bigcup_{i=1}^{k} S_i = A \right\}$$

Proof. Only need to show $\mathscr{A}_0 \supseteq \widetilde{\mathscr{A}_0}$ ■

Exercise 1.3 Let (X,d) be a metric space and let $\mathscr{T} := \{G \subseteq X | G \text{ is open}\}$. Can you give an explicit description of the algebra of subsets of X which is generated $\mathscr{T}$?

R One might say

$$\mathrm{Alg}(\mathscr{T}) \stackrel{?}{=} \{G \subseteq X | G \text{ is open}\} \cup \{F \subseteq X | F \text{ is closed}\}$$

this may not satisfy (AS3), since an intersection $F \cap G$ may be neither open nor closed.

2. Additive Set-Function on An Algebra of Sets

2.1 Definition, Properties, Some Basic Examples

As this is a course titled "abstract measure and integration", we now start on a systematic study of how to measure subsets of a given set X.

Definition 2.1.1 — Additive Set-Function. Let X be a non-empty set, and let $\mathscr{A}$ be an algebra of subsets of X. By additive set function on $\mathscr{A}$ we understand a function $\mu : \mathscr{A} \to [0,\infty]$ such that $\mu(\emptyset) = 0$ and such that

$$\text{(Add)} \qquad \begin{cases} \mu(A \cup B) = \mu(A) + \mu(B) \\ \forall A, B \in \mathscr{A}, A \cap B = \emptyset \end{cases}$$

R Note that μ may take the value ∞. When operating with the values of μ we will thus use the arithmetic of $[0,\infty]$, with rules such as

1. $a + \infty = \infty + a = \infty, \forall a \in [0,\infty]$
2. $a \times \infty = \infty \times a = \infty, \forall a \in (0,\infty]$
3. $0 \times \infty = \infty \times 0 = 0$
4. Avoid $\infty - \infty$
5. If $\mu \neq \infty$ identically everywhere, then $\mu(\emptyset) = 0$ can be derived from the additivity

Proposition 2.1.1 — Properties of Additive Set-Function. Let X be a non-empty set, let $\mathscr{A}$ be an algebra of subsets of X, and let $\mu : A \to [0,\infty]$ be a finitely additive set-function.

1. **General Finite Additivity:** one has that $\mu(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^n \mu(A_i)$, for every $n \geq 2$ and every $A_1, \cdots, A_n \in \mathscr{A}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$
2. **μ is an increasing set-function:** If $A, C \in \mathscr{A}$ and $A \subseteq C$, then $\mu(A) \leq \mu(C)$
3. **Finite Subadditivity:**

$$\text{(SubAdd)} \qquad \begin{cases} \mu(A_1 \cup \cdots \cup A_n) \leq \sum_{i=1}^n \mu(A_i) \\ \forall n \geq 2, \forall A_1, \cdots, A_n \in \mathscr{A} \end{cases}$$

Proof. Elementary. Omitted. ■

Example 2.1 — How to get μ?. Let X be a non-empty set, and consider the algebra of sets $\mathscr{A} = 2^X$. Suppose we are given a "weight-function" $w : X \to [0,\infty)$. We define $\mu : 2^X \to [0,\infty]$ as follows: for every $\emptyset \neq A \subseteq X$, we put

$$\mu(A) := \sup\left\{\sum_{x\in F} w(x) : F \subseteq A, \text{ finite set}\right\} \in [0,\infty]$$

For the case of $F = \emptyset$, we make the convention $\sum_{x\in\emptyset} w(x) = 0$.

Claim: The function $\mu : 2^X \to [0,\infty]$ defined above is an additive set-function

Proof. We first note that for $A = \emptyset$, this gives us

$$\mu(\emptyset) = \sum_{x\in\emptyset} w(x) = 0$$

by convention. Now, consider $A, B \subseteq X$ with $A \cap B = \emptyset$, we want to show $\mu(A\cup B) = \mu(A) + \mu(B)$.

1. If $\mu(A) + \mu(B) = \infty$, then clearly $\mu(A\cup B) \leq \infty$. Thus, we shall assume both $\mu(A), \mu(B)$ finite. Then for any finite $F \subseteq A \cup B$, we have

$$\sum_{x\in F} w(x) = \sum_{x\in F\cap A} w(x) + \sum_{x\in F\cap B} w(x) \leq \mu(A) + \mu(B)$$

since $A \cap B = \emptyset$ and $F \cap A \subseteq A, F \cap B \subseteq B$ are finite in A, B respectively. Since this is true for all $F \subseteq A \cup B$, by the supremum property, we have

$$\mu(A\cup B) \leq \mu(A) + \mu(B)$$

2. For the other direction. We assume $\mu(A\cup B) < \infty$ for similar reason. Since $A, B \subseteq A\cup B$, a finite subset of A, B is clearly a finite subset of $A \cup B$. Thus, $\mu(A), \mu(B) \leq \mu(A\cup B) < \infty$. Let $\varepsilon > 0$, then since $\mu(A) - \frac{\varepsilon}{2} < \mu(A)$. There exists a finite subset set $F_A \subseteq A$ such that $\mu(A) - \frac{\varepsilon}{2} \leq \sum_{x\in F_A} w(x) \leq \mu(A)$. Similarly, there exists a finite subset $F_B \subseteq B$ such that $\mu(B) - \frac{\varepsilon}{2} \leq \sum_{x\in F_B} w(x) \leq \mu(B)$. Then,

$$\mu(A) + \mu(B) - \varepsilon \leq \sum_{x\in F_A\cup F_B} w(x)$$

where $\sum_{x\in F_A} w(x) + \sum_{x\in F_B} w(x) = \sum_{x\in F_A\cup F_B} w(x)$ since $A \cap B = \emptyset$. We note that $F_A \cup F_B \subseteq A\cup B$ is a finite subset. Thus,

$$\mu(A) + \mu(B) - \varepsilon \leq \mu(A\cup B)$$

as desired for any $\varepsilon > 0$.

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1. In the example above, if let $w(x) = 1$ for all $x \in X$, then

$$\mu(A) = \begin{cases} k & |A| = k \\ \infty & |A| = \infty \end{cases}$$

In this case, μ is called the **counting measure** on X.

2. Consider the special case where we fix an element $x_0 \in X$ and we consider the weight function $w : X \to [0,\infty)$ defined by putting $w(x_0) = 1$ and $w(x) = 0$ for every $x \neq x_0$. Then,

$$\mu(A) = \begin{cases} 1 & x_0 \in A \\ 0 & x_0 \notin A \end{cases}$$

In this special case, μ is called the Dirac measure concentrated at x_0.

Proposition 2.1.2 — Measure the Length on $\mathbb{R}$. Let $\mathscr{E}$ be the algebra of half-open intervals of $\mathbb{R}$ that was considered in Example 1.1.3. There exists an additive set-function $\mu : \mathscr{E} \to [0,\infty]$, uniquely determined, such that

$$\mu((a,b]) = b-a, \forall a < b \in \mathbb{R}$$

2.2 An Addendum to the Trick of the Semi-Algebra

Definition 2.2.1 — Respect Decompositions. Let X be a non-empty set and let $\mathscr{S}$ be a semi-algebra of subsets of X. A function $\mu_0 : \mathscr{S} \to [0,\infty]$ will be said to respect decomposition when it has the following property

$$\text{(RespDec)} \qquad \begin{pmatrix} S, S_1, \cdots, S_p \in \mathscr{S} \\ \text{s.t. } S = S_1 \cup \cdots \cup S_p \\ \text{and s.t. } S_i \cap S_j = \emptyset \text{ for } i \neq j \end{pmatrix} \Longrightarrow \mu_0(S) = \sum_{i=1}^{p} \mu_0(S_i)$$

Lemma 2.3 Let X be a non-empty set, let $\mathscr{S}$ be a semi-algebra of subsets of X, and let $\mu_0 : \mathscr{S} \to [0,\infty]$ be a function which respects decompositions. Suppose we are given a set $A \subseteq X$ (not necessarily in $\mathscr{S}$) which was written as union in two ways,

$$A = \bigcup_{i=1}^{k} S_i \text{ and } A = \bigcup_{j=1}^{l} T_j$$

for some $S_1, \cdots, S_k, T_1, \cdots, T_l \in \mathscr{S}$, where $S_i \cap S_j = \emptyset$ for $i \neq j$ in $\{1, \cdots, k\}$ and $T_i \cap T_j = \emptyset$ for $i \neq j$ in $\{1, \cdots, l\}$. Then, it follows that

$$\sum_{i=1}^{k} \mu_0(S_i) = \sum_{j=1}^{l} \mu_0(T_j)$$

Proof. Fix for the moment an $i \in \{1, \cdots, k\}$ and observe that the set $S_i \in \mathscr{S}$ decomposes as

$$S_i = (S_i \cap T_1) \cup \cdots \cup (S_i \cap T_l)$$

Due to (Semi-AS3), we have $(S_i \cap T_j) \in \mathscr{S}, \forall 1 \leq j \leq l$. And it is clear that $(S_i \cap T_j) \cap (S_i \cap T_k) = \emptyset$ for $j \neq k$ in $\{1, \cdots l\}$. Then, by (RespDec), we have

$$\mu_0(S_i) = \sum_{j=1}^{l} \mu_0(S_i \cap T_j)$$

By similar argument, we have

$$\mu_0(T_j) = \sum_{i=1}^{k} \mu_0(S_i \cap T_j)$$

Finally,

$$\begin{aligned}\sum_{i=1}^{k}\mu_0(S_i) &= \sum_{i=1}^{k}\left(\sum_{j=1}^{l}\mu_0(S_i\cap T_j)\right)\\ &= \sum_{j=1}^{l}\left(\sum_{i=1}^{k}\mu_0(S_i\cap T_j)\right)\\ &= \sum_{j=1}^{l}\mu_0(T_j)\end{aligned}$$

as desired. ■

Proposition 2.3.1 Let X be a non-empty set, let $\mathscr{S}$ be a semi-algebra of subsets of X, and let $\mu_0 : \mathscr{S} \to [0,\infty]$ be a function which respects decompositions. Let $\mathscr{A}_0$ be the algebra of subsets of X defined in the way shown in Proposition 1.2.1 ($\mathscr{A}_I = \mathrm{Alg}(\mathscr{S})$). Then there exists an additive set-function $\mu : \mathscr{A}_0 \to [0,\infty]$, uniquely determined, such that $\mu(S) = \mu_0(S)$ for every $S \in \mathscr{S}$.

Proof. Recall that

$$\mathscr{A}_0 := \left\{ A \subseteq X \,\middle|\, \begin{array}{l} \exists k \in \mathbb{N} \text{ and } S_1,\cdots,S_k \in \mathscr{J} \text{ such that} \\ S_i \cap S_j = \emptyset \text{ for } i \neq j \text{ and } S_1 \cup \cdots \cup S_k = A \end{array} \right\}$$

We define $\mu : \mathscr{A}_0 \to [0,\infty]$ by

$$\mu(A) = \sum_{i=1}^{k}\mu_0(S_i)$$

where $A \in \mathscr{A}_0$ and there exists $S_1,\cdots,S_k \in \mathscr{S}$ such that $S_i \cap S_j = \emptyset$ for $i \neq j$ and $A = \cup_{i=1}^{k} S_i$ by $\mathscr{A}_0$'s definition. This is well-defined due to the lemma proved above, so our value of μ does not depend on the decompositions. For $\emptyset \in \mathscr{A}_0$, we define it to be $\mu(\emptyset) = 0$. It remains to check the following.

1. Let $A, B \in \mathscr{A}_0$ such that $A \cap B = \emptyset$. And say

$$A = \bigcup_{i=1}^{k} S_i, S_i \in \mathscr{S}$$

where $S_i \cap S_j = \emptyset, i \neq j$ and

$$B = \bigcup_{j=1}^{l} T_j, T_j \in \mathscr{S}$$

where $T_i \cap T_j = \emptyset, i \neq j$. Then,

$$A \cup B = \left(\bigcup_{i=1}^{k} S_i\right) \cup \left(\bigcup_{l=1}^{l} T_j\right)$$

where $S_i \cap T_j = \emptyset$ for any i, j since $A \cap B = \emptyset$. Then, using the definition of μ that we defined,

$$\begin{aligned}\mu(A\cup B) &= \sum_{i=1}^{k}\mu_0(S_i) + \sum_{j=1}^{l}\mu_0(T_j)\\ &= \mu(A)+\mu(B)\end{aligned}$$

2. For $S \in \mathscr{S}$, we have $S \in \mathscr{A}_0$ since it can be an one-element union and $\mu(S) = \mu_0(S)$.

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Exercise 2.1 Show how Proposition 2.1.2 follows from Proposition 2.3.1. ■

3. $\sigma-$Algebras of Sets, Positive Measures

3.1 Definitions, Properties, Some Basic Examples

Due to historical reasons, people like to use the prefix σ whenever doing an upgrade from a finite structure to its infinite countable version.

Definition 3.1.1 — $\sigma-$Algebra.
1. Let X be a non-empty set and let $\mathscr{M}$ be a collection of subsets of X. We say $\mathscr{M}$ is a $\sigma-$algebra of subsets of X to mean that it satisfies the following conditions:
   (a) **(Sigma-AS1)** $\emptyset \in \mathscr{M}$
   (b) **(Sigma-AS2)** $A \in \mathscr{M} \implies$
   (c) **(Sigma-AS3)** $(A_n)_{n=1}^{\infty} \subseteq \mathscr{M} \implies \cap_{n=1}^{\infty} A_n \in \mathscr{M}$
2. We use the name **measurable space** to refer to a pair $(X, \mathscr{M})$ where X is a non-empty set and $\mathscr{M}$ is a $\sigma-$algebra of subsets of X

Proposition 3.1.1 Let X be a non-empty set and let $\mathscr{M}$ be a $\sigma-$algebra of subsets of X.
1. $\mathscr{M}$ is, in particular, an algebra of sets. Therefore $\mathscr{M}$ enjoys the additional properties of algebras of sets
2. In parallel with the statement about countable intersections from (Sigma-AS3, $\mathscr{M}$ also has the corresponding property concerning unions: $(A_n)_{n=1}^{\infty} \subseteq \mathscr{M} \implies \cup_{n=1}^{\infty} A_n \in \mathscr{M}$

Proof.
1. Really nothing to prove here.
2. Let $(A_n)_{n=1}^{\infty} \subseteq \mathscr{M}$, then

$$\bigcup_{n=1}^{\infty} A_n = X \setminus \bigcap_{n=1}^{\infty} (X \setminus A_n) \in \mathscr{M}$$

■

Let us move now to upgrade the additive set-function to positive measure. We first talk about a suitable notion of convergence in this space.

R

1. Let $(s_n)_n$ be a sequence of elements of $[0,\infty]$ which is increasing in the sense that we have
$$0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq \cdots$$
such a sequence is sure to approach a limit $L \in [0,\infty]$. Say
$$L = \sup\{s_n | n \in \mathbb{N}\}$$
2. Let $(t_n)_n$ be any sequence in $[0,\infty]$. We can consider the finite sums
$$s_n = \sum_{i=1}^{n} t_i$$
which forms an increasing sequence in $[0,\infty]$, the limit is
$$L = \sum_{n=1}^{\infty} t_n$$

Definition 3.1.2 — Positive Measure.
1. Let X be a non-empty set, and let $\mathscr{M}$ be a $\sigma-$algebra of subsets of X. A positive measure on $\mathscr{M}$ is a function $\mu : \mathscr{M} \to [0,\infty]$ which has $\mu(\emptyset) = 0$ and satisfies:
$$\text{(Sigma-Add)} \begin{cases} \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n), \text{ whenever } A_1, \cdots, A_n, \cdots \in \mathscr{M} \\ \text{are such that } A_i \cap A_j = \emptyset, i \neq j \end{cases}$$
The property described is called $\sigma-$additivity
2. We use the name **measure space** to refer to a triple $(X, \mathscr{M}, \mu)$ where X is a non-empty set, $\mathscr{M}$ is a $\sigma-$algebra of subsets of X, and $\mu : \mathscr{M} \to [0,\infty]$ is a positive measure

Proposition 3.1.2 — Positive Measure Properties. Let X be a non-empty set, let $\mathscr{M}$ be a $\sigma-$algebra of subsets of X, and let μ be positive measure.
1. μ is a finitely additive set-function
2. μ possesses countable version of the sub-additivity:
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

1. We only need to show $\mu(A \cup B) = \mu(A) + \mu(B)$ for any $A, B \in \mathscr{M}$ with $A \cap B = \emptyset$. We can let $A_1 = A, A_2 = B, A_i = \emptyset$ for all $i \geq 3$. Then, by (Sigma-Add), we have
$$\mu(A \cup B) = \sum_{n=1}^{\infty} \mu(A_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n) = \mu(A) + \mu(B)$$
2. Let $(A_n)_n \subseteq \mathscr{M}$. Consider the $(B_n)_n$ defined by
$$B_1 = A_1, B_2 = A_2 \backslash A_1, \cdots, B_n = A_n \backslash \left(\bigcup_{i=1}^{n-1} A_i\right), \cdots$$
By (Sigma-AS2) (Sigma-AS3), we have $B_n \in \mathscr{M}$. Also note that for $m < n$, we have that $B_m \cap B_n = \emptyset$. And $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n$. Then,
$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \sum_{n=1}^{\infty} \mu(B_n) \qquad \text{(Sigma-Add)} \\ &\leq \sum_{n=1}^{\mu} (A_n) \end{aligned}$$

since $\mu(B_n) \leq \mu(A_n)$ for all $n \in \mathbb{N}$ with $B_n \subseteq A_n$.

■ **Example 3.1** The construction of a finitely additive set-function $\mu : 2^X \to [0,\infty]$ by starting from a weight-function $w : X \to [0,\infty)$ as described in example 2.1 is actually producing a positive measure. Thus, in particular, counting measure and Dirac measure are positive measures. We need to show that for $(A_n)_n \in \mathscr{M}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ and verify $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

1. Let $F \subseteq \cup_{n=1}^{\infty} A_n$ by a finite set. Then, we have

$$\sum_{x\in F} w(x) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

since F is finite then there exists $n_0 \in \mathbb{N}$ such that $F \subseteq \cup_{n=1}^{n_0} A_n$. Then, by finite additivity, we have

$$\sum_{x\in F} w(x) \leq \mu\left(\cup_{n=1}^{n_0} A_n\right) = \sum_{n=1}^{n_0} \mu(A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Thus,

$$\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

2. For $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} \mu(A_n) = \mu\left(\cup_{n=1}^{N} A_n\right) \leq \mu(\cup_{n=1}^{\infty} A_n)$$

since $\cup_{n=1}^{N} A_n \subseteq \cup_{n=1}^{\infty} A_n$ for all $N \in \mathbb{N}$. Thus,

$$\sum_{n=1}^{\infty} \mu(A_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n) \leq \mu(\cup_{n=1}^{\infty} A_n)$$

and we are done.

■

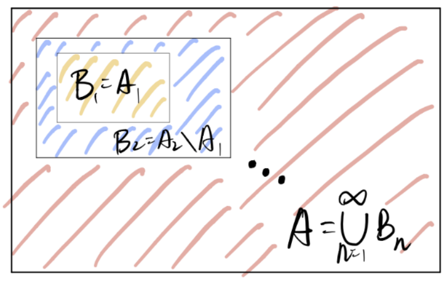
3.2 Continuity Along Increasing and Decreasing Chains

Proposition 3.2.1 — Continuity Along Increasing Chains. Let $(X,\mathscr{M},\mu)$ be a measure space and let $(A_n)_n$ be an increasing chain of sets from $\mathscr{M}$. Then, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

(limit in $[0,\infty]$)

Proof. Let $B_1 = A_1$ and for $n > 1$, let $B_n = A_n \backslash A_{n-1}$. Then, for each $B_n \in \mathscr{M}$ and these sets are disjoint, i.e. $B_i \cap B_j = \emptyset$ if $i \neq j$. We have a graph below to illustrate this.



Furthermore, $\cup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ and $\cup_{k=1}^{n} B_k = A_n$, then by (Sigma-Add), and let $n \to \infty$,

$$\mu(A_n) = \sum_{k=1}^{n} \mu(B_k) \Longrightarrow \sum_{k=1}^{\infty} \mu(B_k) \underset{(Sigma-Add)}{=} \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

■

■ **Example 3.2** Let $X = \mathbb{N}$ and let $\mathscr{M} = 2^{\mathbb{N}}$, and let $\mu : 2^{\mathbb{N}} \to [0,\infty]$ be the counting measure. For every $n \in \mathbb{N}$, consider the set $C_n := \{k \in \mathbb{N} : k \geq n\}$. Then $(C_n)_n$ is a decreasing chain of sets from $\mathscr{M}$, and we have $\mu(C_n) = \infty$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty} \mu(C_n) = \infty$. On the other hand, we have

$$\bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \{k \in \mathbb{N} : k \geq n\} = \{k \in \mathbb{N} : k \geq n, \forall n \in \mathbb{N}\} = \emptyset$$

therefore $\mu(\cap_{n=1}^{\infty} C_n) = 0$. This example tells us the decreasing chain with $\mu(\cap_{n=1}^{\infty} C_n)$ is not necessarily $\lim_{n\to\infty} \mu(C_n)$ when $\mu(C_1) = \infty$. ■

Proposition 3.2.2 — Continuity Along Decreasing Chains. Let $(X, \mathscr{M}, \mu)$ be a measure space and let $(C_n)_n$ be a decreasing chain of sets from $\mathscr{M}$, such that $\mu(C_1) < \infty$ (which forces $\mu(C_n) \leq \mu(C_1) < \infty, \forall n \in \mathbb{N}$). Then, we have

$$\mu\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mu(C_n)$$

(limit in $[0,\infty)$).

Proof. For every $n \in \mathbb{N}$, let $A_n = C_1 \backslash C_n \in \mathscr{M}$ by (Sigma-AS2) that A_n's form an increasing chain with

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (C_1 \backslash C_n) = C_1 \backslash \left(\bigcap_{n=1}^{\infty} C_n\right)$$

Then, by continuity along increasing chains, we have

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(C_1 \backslash (\bigcap_{n=1}^{\infty} C_n)\right)$$

We know that $\mu(A_n) = \mu(C_1) - \mu(C_n)$ given that $\mu(C_1) < \infty$. Then,

$$\begin{aligned} \mu(C_1) - \mu\left(\bigcap_{n=1}^{\infty} C_n\right) &= \mu\left(C_1 \backslash (\bigcap_{n=1}^{\infty} C_n)\right) \\ &= \lim_{n\to\infty} \mu(A_n) \qquad \Longrightarrow \mu\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mu(C_n) \\ &= \lim_{n\to\infty} \mu(C_1) - \mu(C_n) \\ &= \mu(C_1) - \lim_{n\to\infty} \mu(C_n) \end{aligned}$$

as desired. ■

The following exercise lead the well-unknown **Borel-Cantelli Lemma**.

Exercise 3.1 Let $(X, \mathscr{M}, \mu)$ be a measure space, and let $(A_n)_n$ be a family of sets from $\mathscr{M}$. Consider $T := \{x \in X : x \text{ belongs to infinitely many of the } A_n\text{'s}\}$

1. Prove that $T = \cap_{k=1}^{\infty} \left(\cup_{k=n}^{\infty} A_k \right)$ and that, as a consequence, $T \in \mathscr{M}$
2. Can you identify a decreasing chain of sets which appeared in the description of T from part 1?
3. Suppose that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that $\mu(T) = 0$.

Solution:

1. We claim that $T \in \mathscr{M}$. In fact, we claim that

$$T = \bigcap_{M=1}^{\infty} \bigcup_{n=M}^{\infty} A_n$$

Proof. (a) For $x \in \cap_{M=1}^{\infty} \cup_{n=M}^{\infty} A_n$, then $x \in \cup_{n=M}^{\infty} A_n$ for all $M \in \mathbb{N}$. Then, for $M_1 = 1$, there exists $n_1 \geq M_1$ such that $x \in A_{n_1}$. Then, we pick $M_2 = n_1 + 1$ and there must exists $n_2 \geq M_2$ such that $x \in A_{n_2}$. Inductively, this process can never be terminated and x is contained in infinitely many different A_n. Thus, $x \in T$.

(b) Conversely, if $x \in T$, then for all $M \in \mathbb{N}$, we can find $n \geq M$ such that $x \in A_n$ or say $x \in \cup_{n=M}^{\infty} A_n$ for all $M \in \mathbb{N}$. Thus, $x \in \cap_{M=1}^{\infty} \cup_{n=M}^{\infty} A_n$.

Thus,

$$T = \bigcap_{M=1}^{\infty} \bigcup_{n=M}^{\infty} A_n$$

and by countable union and intersection property of the $\sigma-$algebra, we have $T \in \mathscr{M}$ as $A_n \in \mathscr{M}$. ■

2. Let $E_N = \cap_{M=1}^{N} \cup_{n=M}^{\infty} A_n$. Note that

$$E_1 \supseteq E_2 \cdots \supseteq \bigcap_{M=1}^{\infty} \bigcup_{n=M}^{\infty} A_n = T$$

. Thus, $(E_N)_N$ forms a decreasing chain of sets.

3. We claim that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, we have $\mu(T) = 0$.

Proof. Take the constructed $(E_N)_N$. By $\sigma-$subadditivity,

$$\mu(E_1) = \mu\left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n) < \infty$$

. Then, by continuity along decreasing chains, we have

$$\lim_{N \to \infty} \mu(E_N) = \mu(T)$$

Now, we examine that

$$\bigcap_{M=1}^{N} \bigcup_{n=M}^{\infty} A_n \subseteq \bigcup_{n=N}^{\infty} A_n$$
$$\Longrightarrow \mu(E_N) = \mu\left(\bigcap_{M=1}^{N} \bigcup_{n=M}^{\infty} A_n\right) \leq \mu\left(\bigcup_{n=N}^{\infty} A_n\right)$$
$$\leq \sum_{n=N}^{\infty} \mu(A_n)$$

since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, for any $\varepsilon > 0$, there exists $H \in \mathbb{N}$ such that $N-1 \geq H$ implies

$$\left|\sum_{n=1}^{N-1} \mu(A_n) - \sum_{n=1}^{\infty} \mu(A_n)\right| = \left|\sum_{n=N}^{\infty} \mu(A_n)\right| < \varepsilon$$

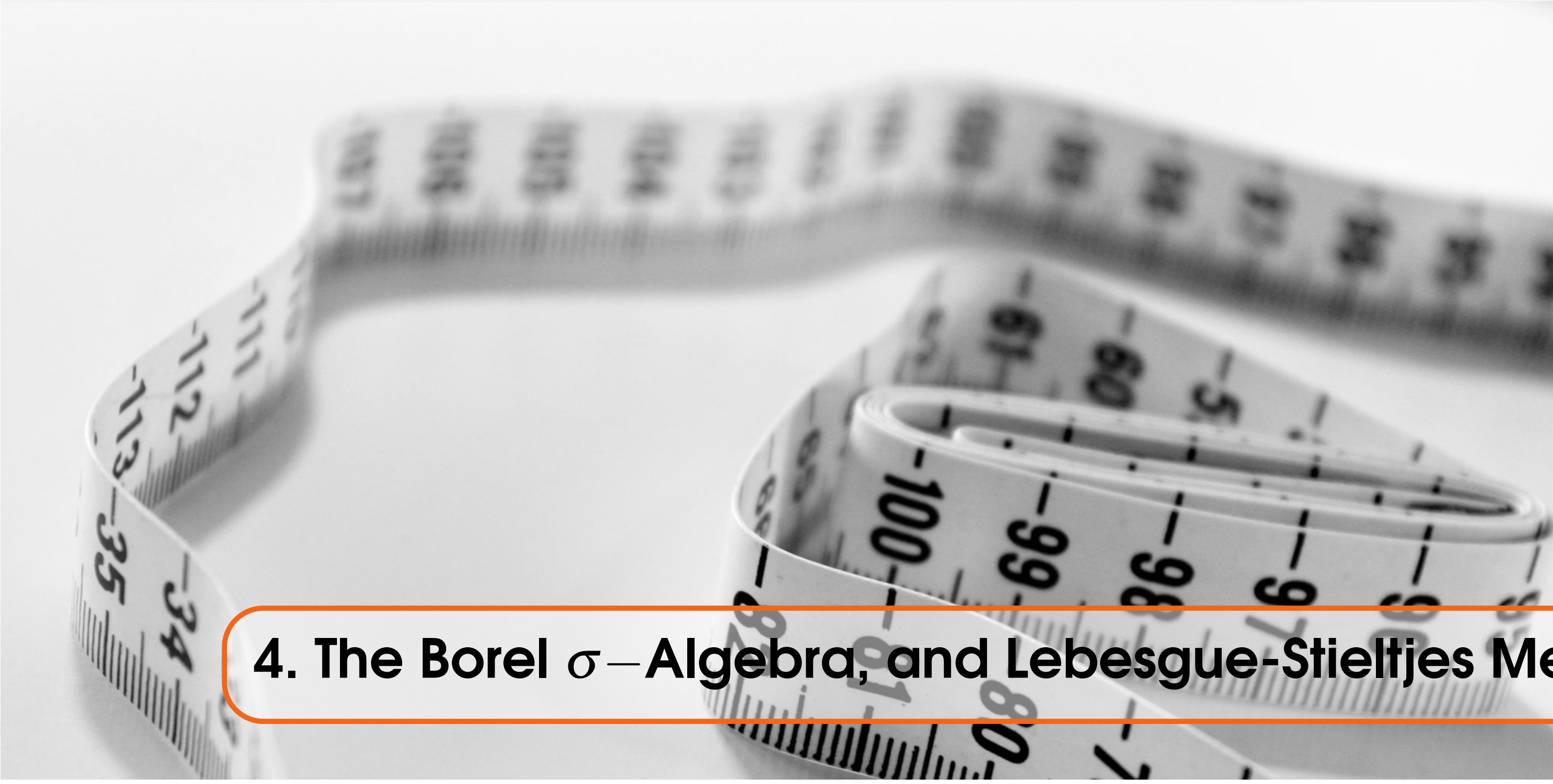
using this H and $N-1 \geq H$, we also have

$$\mu(E_N) < \varepsilon$$

Thus,

$$\mu(T) = \lim_{N\to\infty} \mu(E_N) = 0$$

as desired. ■



4. The Borel $\sigma-$Algebra, and Lebesgue-Stieltjes Me

4.1 $\sigma-$Algebra of sets generated by an arbitrary $\mathscr{C} \subseteq 2^X$. The notion of Borel $\sigma-$Algebra

Lemma 4.2 Let X be a non-empty set, and let $(\mathscr{M}_i)_{i\in I}$ a family of $\sigma-$algebra of subsets of X. Denote $\mathscr{M} := \cap_{i\in I}\mathscr{M}_i$. Then $\mathscr{M}$ is a $\sigma-$algebra as well.

Proof. Immediate. ■

Proposition 4.2.1 Let X be a non-empty set and let $\mathscr{C}$ be a collection of subsets of X. There exists a collection $\mathscr{M}_0$ fo subsets of X, uniquely determined, such that:

1. $\mathscr{M}_0$ is a $\sigma-$algebra of subsets of X such that $\mathscr{M} \supseteq \mathscr{C}$
2. Whenever $\mathscr{M}$ is a $\sigma-$algebra of subsets of X such that $\mathscr{M} \supseteq \mathscr{C}$, it follows that $\mathscr{M} \supseteq \mathscr{M}_0$

Proof. Analogous. ■

Definition 4.2.1 — $\sigma-$Algebra Genenerated by $\mathscr{C}$. Let X be a non-empty set and let $\mathscr{C}$ be a collection of subsets of X. The $\sigma-$algebra $\mathscr{M}_0$ found in the proposition above is called the $\sigma-$algebra of sets generated by $\mathscr{C}$, denote by $\sigma - \text{Alg}(\mathscr{C})$.

R

1. Let X be a non-empty set and let $\mathscr{C}$ be a collection of subsets of X. Then, $\text{Alg}(\mathscr{C}) \subseteq \sigma - \text{Alg}(\mathscr{C})$

   Proof. Let $\text{Alg}(\mathscr{C}) =: \mathscr{A}_0$ and $\sigma - \text{Alg}(\mathscr{C}) =: \mathscr{M}_0$. And note that $\mathscr{M}_0$ is an algebra of subsets of X containing $\mathscr{C}$. Thus, this is true. ■

2. Let X be a non-empty set and let $\mathscr{C}_1$ and $\mathscr{C}_2$ be collections of subsets of X such that $\mathscr{C}_1 \subseteq \mathscr{C}_2$. Then it follows that $\text{Alg}(\mathscr{C})_1 \subseteq \text{Alg}(\mathscr{C})_2$ and $\sigma - \text{Alg}(\mathscr{C})_1 \subseteq \sigma - \text{Alg}(\mathscr{C})_2$.

Exercise 4.1 Let X be an infinite uncountable set, and let $\mathscr{C} := \{\{x\}|x \in X\}$ (the collection of all subsets $A \subseteq X$ such that $|A| = 1$). Prove that

$$\sigma - \text{Alg}(\mathscr{C}) = \{A \subseteq X | A \text{ is countable}\} \cup \{B \subseteq X | X \backslash B \text{ is countable}\} =: \mathscr{M}$$

Proof. Note that $\mathscr{C} \subseteq \mathscr{M}$ since $\{x\}$ are finite (countable).

1. $\emptyset \in \mathscr{M}$
2. Suppose $A \in \mathscr{M}$. If A is countable, then $X \backslash A$ has a countable complement. Thus, $X \backslash A \in \mathscr{M}$. Suppose A has a countable complement. Then, $X \backslash A$ is countable. Thus, $X \backslash A \in \mathscr{M}$.
3. Let $(A_n)_n \subseteq \mathscr{M}$. Then,
$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcap_{i \in I} A_i\right) \cap \left(\bigcap_{j \in J} A_j\right)$$
where the first part is in the intersection of all countable sets is still countable. The second part is the intersection of all sets with countable complements. Then,
$$X \backslash \left(\bigcap_{j \in J} A_j\right) = \bigcup_{j \in J} (X \backslash A_j)$$
is countable since countable union of countable sets is countable. Thus, $\cap_{j \in J} A_j \in \mathscr{M}$. Then, $\cap_{n=1}^{\infty} A_n \in \mathscr{M}$.

Thus, $\mathscr{M}$ is a $\sigma-$algebra of subsets of X.
Let $\mathscr{A}$ be a $\sigma-$algebra of subsets of X such that $\mathscr{A} \supseteq \mathscr{C}$. Let $A \in \mathscr{M}$. Say A is countable. Then, we can have

$$A = \bigcup_{x \in I} \{x\} \in \mathscr{A}$$

where I is countable. If $X \backslash A$ is countable, then

$$X \backslash A = \bigcup_{x \in I} \{x\} \in \mathscr{A}$$

Then,

$$A = X \backslash \left(\bigcup_{x \in I} \{x\}\right) = \bigcap_{X \in I} X \backslash \{x\} \in \mathscr{A}$$

Thus, $\mathscr{A} \supseteq \mathscr{M}$. ■

Definition 4.2.2 — Borel $\sigma-$Algebra. Let (X,d) be a metric space, and let $\mathscr{T} := \{G \subseteq X | G \text{ is open}\}$. The $\sigma - \text{Alg}(\mathscr{T})$ is called the Borel $\sigma-$algebra of (X,d) and will be denoted as $\mathscr{B}_X$. The subsets of X which belong to $\mathscr{B}_X$ are called Borel subsets of X.

R Let (X,d) be a metric space and let $\mathscr{T} := \{G \subseteq X | G \text{ is open}\}$. We have seen that we can find an intrinsic description for the sets belonging to $\text{Alg}(\mathscr{T})$. But we don't have a similar result for $\mathscr{B}_X$. We note that by taking complement, we have

$$\mathscr{B}_X = \sigma - \text{Alg}(\{F \subseteq X | F \text{ is closed}\})$$

. Thus, we can change the generators to have the same $\mathscr{B}_X$.

Definition 4.2.3 We will reserve the notation $\mathscr{B}_{\mathbb{R}}$ for the Borel $\sigma-$algebra of the metric space $(\mathbb{R}, d_{usual})$, where d_{usual} is the absolute distance between real numbers.

R A convenient way to approach the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$ is to write it as

$$\mathscr{B}_{\mathbb{R}} = \sigma - \text{Alg}(\{(a,b] | a < b \in \mathbb{R}\})$$

This is a result from PMATH450.

4.3 Lebesgue-Stieltjes Measures on $\mathscr{B}_{\mathbb{R}}$

Definition 4.3.1 — Lebesgue-Stieltjes . A positive measure $\mu : \mathscr{B}_{\mathbb{R}} \to [0,\infty]$ with the property that

$$\mu(K) < \infty, \forall K \subseteq \mathbb{R} \text{ compact}$$

is called a Lebesgue-Stieltjes measure.

R

1. The condition $\mu(K) < \infty, \forall K \subseteq \mathbb{R}$ compact can be replaced by the apparently weaker condition that
$$\mu([-n,n]) < \infty, \forall n \in \mathbb{N}$$
Since for every compact K, we can find an $n \in \mathbb{N}$ such that $K \subseteq [-n,n]$.
2. A substantial family of examples of Lebesgue-Stieltjes measures is provided by **finite measures**, that is, measures $\mu : \mathscr{B}_{\mathbb{R}} \to [0,\infty]$ such that $\mu(\mathbb{R}) < \infty$ since $\mu(K) \leq \mu(\mathbb{R}) < \infty$ for all compact K. But Lebesgue measure is also a Lebesgue-Stieltjes measure even thought it is not a finite measure.
In order to keep track of a Lebesgue-Stieltjes measure, it turns out to be useful to consider a certain function associated to it.

Definition 4.3.2 — (Also a proposition) Lebesgue-Stieltjes function. Let $\mu : \mathscr{B}_{\mathbb{R}} \to [0,\infty]$ be a Lebesgue-Stieltjes measure. There exists a function $G : \mathbb{R} \to \mathbb{R}$, uniquely determined, such that $G(0) = 0$ and such that

$$\mu((a,b]) = G(b) - G(a), \forall a < b \in \mathbb{R}$$

This function G is the centred Lebesgue-Stieltjes function associated to the measure μ and denoted by G_μ.

Proof. Consider

$$G(t) = \begin{cases} \mu((0,t]) & t > 0 \\ 0 & t = 0 \\ -\mu((t,0]) & t < 0 \end{cases}$$

We proceed to check when $a < b$ that we have $\mu((a,b]) = G(b) - G(a)$. For example, when $a < 0 < b$, we have

$$\begin{aligned} G(b) - G(a) &= \mu((0,b]) - (-\mu((a,0])) \\ &= \mu((0,b]) + \mu((a,0]) \\ &= \mu((a,b]) && \text{Sigma-Add} \end{aligned}$$

Suppose G is not unique. Say $\tilde{G}:\mathbb{R}\to\mathbb{R}$ also satisfies $\tilde{G}(0)=0$ and

$$\mu((a,b]) = \tilde{G}(b)-\tilde{G}(a), \forall a<b\in\mathbb{R}$$

Then, for every $t>0$, let $a=0$ and $b=t$. We have

$$\tilde{G}(t)=\tilde{G}(t)-\tilde{G}(0)=\mu((0,t])=G(t)$$

while for $t<0$. It is similar. Thus, $\tilde{G}=G$. ■

Properties of Lebesgue-Stieltjes Function

1. Let $\mu:\mathscr{B}_{\mathbb{R}}\to[0,\infty]$ be a Lebesgue-Stieltjes measure. Then, the function G_μ is increasing, that is
$$(a,b\in\mathbb{R}, a\le b)\Longrightarrow G_\mu(a)\le G_\mu(b)$$
2. **Cadlag Property of G_μ:** continuous from the right, limit from the left. Let $\mu:\mathscr{B}_{\mathbb{R}}\to[0,\infty]$ be a Lebesgue-Stieltjes measure. Then for every $a\in\mathbb{R}$ we have
$$\lim_{t\searrow a} G_\mu(t) = G_\mu(a)$$
and
$$\lim_{t\nearrow a} G_\mu(t) \text{ exists (and is } \le G_\mu(a)\text{, buit can be } < G_\mu(a))$$

Proof. It suffices to verify the sequential continuity of G_μ at a from the right. So let $(t_n)_n$ be a decreasing sequence which converges to a. The sequence $\left(G_\mu(t_n)-G_\mu(a)\right)_n$ is decreasing since G_μ is an increasing function. Thus, $\lim_{n\to\infty} G_\mu(t_n)-G_\mu(a)$ exists. Note that

$$\begin{aligned}
\lim_{n\to\infty} G_\mu(t_n)-G_\mu(a) &= \lim_{n\to\infty}\mu((a,t_n]) \\
&= \mu\left(\bigcap_{n=1}^{\infty}(a,t_n]\right) && \text{continuity along a decreasing chain} \\
&= \mu(\emptyset)=0
\end{aligned}$$

The existence of the left limit follows as well. ■

■ **Example 4.1** 1. The centred Lebesgue-Stieltjes function associated to be Lebesgue measure μ_{Leb} is just $G(t)=t$.

2. Suppose you are given a positive measure $\mu:\mathscr{B}_{\mathbb{R}}\to[0,\infty]$ which is a probability measure with $\mu(\mathbb{R})=1$. Then, μ is a finite positive measure, hence a Lebesgue-Stieltjes measure, and therefore has a centred Lebesgue-Stieltjes function G_μ. We have seen CDF in probability theory $F:\mathbb{R}\to[0,1]$ defined by
$$F(t)=\mu((-\infty,t]), \forall t\in\mathbb{R}$$
The connection between F and G_μ is that
$$G_\mu(t)=F(t)-F(0), \forall t\in\mathbb{R}$$
They differ by a constant $F(0)$ so that $G(0)=0$ for being centred.

■

5. Follow-up on Lebesgue-Stieltjes Measures

Question 1: Does G_μ determine μ uniquely?
That is, if $\mu, \nu : \mathscr{B}_\mathbb{R} \to [0,\infty]$ are Lebesgue-Stieltjes measures such that $G_\mu(t) = G_\nu(t)$ for all $t \in \mathbb{R}$, can we conclude that $\mu = \nu$?

Question 2: Are we in control of what functions can appear as G_μ?
Say that someone is giving you a function $G : \mathbb{R} \to \mathbb{R}$ which is increasing, cadlag, and has $G(0) = 0$. Is it then certain that you can find a Lebesgue-Stieltjes measure $\mu : \mathscr{B}_\mathbb{R} \to [0,\infty]$ such that G_μ is equal to the given G?

5.1 $\pi-$systems, and why we have an affirmative answer to Q1?

R Suppose that μ and ν are two Lebesgue-Stieltjes measures such that $G_\mu = G_\nu$. It then follows that μ and ν agree on half-open intervals $(a,b]$ with $a < b$ in $\mathbb{R}$:

$$\mu((a,b]) = G_\mu(b) - G_\mu(a) = G_\nu(b) - G_\nu(a) = \nu((a,b]), \forall a < b \in \mathbb{R}$$

Definition 5.1.1 — $\pi-$System. Let X be a non-empty set. A collection $\mathscr{P}$ of subsets of X is said to be a $\pi-$system when it has the property that

$$(P, Q \in \mathscr{P}) \Longrightarrow P \cap Q \in \mathscr{P}$$

R Inductively, if $\mathscr{P}$ is a $\pi-$system of subsets of X, then $\mathscr{P}$ is stable under finite intersections:

$$(P_1, \cdots, P_n \in \mathscr{P}) \Longrightarrow \bigcap_{i=1}^{n} P_i \in \mathscr{P}$$

1. In particular, an algebra of sets is a $\pi-$system
2. **Look at Dynkin's $\pi - \lambda$ theorem for research topic**

Proposition 5.1.1 — A consequence of Dynkin's $\pi-\lambda$ theorem. Let $(X,\mathscr{M})$ be a measurable space, and suppose that $\mathscr{P}$ is a $\pi-$system of subsets of X such that $\sigma-\text{Alg}(\mathscr{P})=\mathscr{M}$. Let $\mu,\nu:\mathscr{M}\to[0,\infty]$ be positive measures such that

1. $\mu(P)=\nu(P),\forall P\in\mathscr{P}$
2. There exists an increasing chain $(P_n)_n$ in $\mathscr{P}$, with $\cup_{n=1}^{\infty}P_n=X$, and such that $\mu(P_n)<\infty$ for all $n\in\mathbb{N}$

Then, it follows that $\mu=\nu$.

Corollary 5.1.2 Let $\mu,\nu:\mathscr{B}_{\mathbb{R}}\to[0,\infty]$ be Lebesgue-Stieltjes measures such that $G_\mu=G_\nu$. Then, it follows that $\mu=\nu$.

Proof. It is immediately verified that

$$\mathscr{P}:=\{(a,b]|a<b\in\mathbb{R}\}\cup\{\emptyset\}$$

is a $\pi-$system of subsets of $\mathbb{R}$. The $\sigma-$algebra generated by this $\pi-$system is $\mathscr{B}_{\mathbb{R}}$. Since $G_\mu=G_\nu$, we have that μ and ν agrees on $\mathscr{P}$. We have the first condition of proposition 5.1.1. Consider the increasing chain obtained by putting

$$P_n=(-n,n],\forall n\in\mathbb{N}$$

will have $\cup_{n=1}^{\infty}P_n=\mathbb{R}$ and $\mu(P_n)<\infty$. Thus, by proposition 5.1.1, we have $\mu=\nu$ as required. ∎

5.2 Caratheodory's Extension Theorem

This is why have an affirmative answer to question 2.

R We denote

$$\mathscr{J}:=\{\emptyset\}\cup\{(a,b]:a<b\in\mathbb{R}\}\cup\{(-\infty,b]:b\in\mathbb{R}\}\cup\{(a,\infty):a\in\mathbb{R}\}\cup\{\mathbb{R}\}$$

and $\mathscr{E}:=\text{Alg}(\mathscr{J})$. We know that $\mathscr{J}$ is a semi-algebra of subsets of $\mathbb{R}$, and we know that this has nice consequences on how we describe the sets from $\mathscr{E}$ and on how we can define an additive set-function on $\mathscr{E}$. Also,

$$\sigma-\text{Alg}(\mathscr{J})=\sigma-\text{Alg}(\mathscr{E})=\mathscr{B}_{\mathbb{R}}$$

Lemma 5.3 Let $G:\mathbb{R}\to\mathbb{R}$ be a function which is increasing, cadlag, and has $G(0)=0$. Besides the values of G, we also consider its limits at $\pm\infty$, which exist due tot he hypothesis that G is increasing:

$$L_+:=\lim_{t\to\infty}G(t)\in\mathbb{R}\cup\{\infty\}\quad\text{and}\quad L_-:=\lim_{t\to-\infty}G(t)\in\mathbb{R}\cup\{-\infty\}$$

There exists an additive set-function $\mu_0:\mathscr{E}\to[0,\infty]$ such that

$$\begin{cases}\mu_0((a,b])=G(b)-G(a),\forall a<b\in\mathbb{R}\\ \mu_0((-\infty,b])=G(b)-L_-,\forall b\in\mathbb{R}\\ \mu_0((a,\infty))=L_+-G(a),\forall a\in\mathbb{R}\\ \mu_0(\emptyset)=0,\text{ and }\mu_0(\mathbb{R})=L_+-L_-\end{cases}$$

Proof. Let us momentarily denote by $\mu_{00} : \mathscr{J} \to [0,\infty]$ the function which is obtained by using the formulas stipulated above. It is clear that μ_{00} respects decompositions, which means if a $J \in \mathscr{J}$ can be written as $\cup_{i=1}^{p} J_i$ with $J_i \cap J_j = \emptyset$ for $i \neq j$. Then, we can align them from the left to right on the real line and get $\mu_{00}(J_i)$ and add them up to get $\mu_{00}(J)$ through a lot of cancellations. Then, by Proposition 2.3.1, we have the desired additive set function. ■

Definition 5.3.1 — Pre-measure. Let X be a non-empty set, let $\mathscr{A}$ be an algebra of subsets of X, and let $\mu_0 : \mathscr{A} \to [0,\infty]$ be an additive set-function. We say that μ_0 is a pre-measure when it satisfies the following condition:

$$\text{(Pre-Sigma-Add)} \begin{cases} \text{If } (A_n)_n \subseteq \mathscr{A}, A_i \cap A_j = \emptyset, i \neq j \\ \text{and } \cup_{n=1}^{\infty} A_n \in \mathscr{A}, \\ \text{then } \mu_0(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu_0(A_n) \end{cases}$$

R Note that we don't assume $\mathscr{A}$ to be a $\sigma-$algebra. So, what we are saying is that if we have the case when the countable union is still in $\mathscr{A}$ and they are pairwise disjoint, then the sigma-additivity will hold.

Theorem 1 — Caratheodory Extension Theorem. Let X be a non-empty set, let $\mathscr{A}$ be an algebra of subsets of X, and let $\mathscr{M} = \sigma - \text{Alg}(\mathscr{A})$. Suppose that $\mu_0 : \mathscr{A} \to [0,\infty]$ is an additive set function which is a pre-measure. Then there exists a positive measure $\mu : \mathscr{M} \to [0,\infty]$ which extends μ_0 (that is $\mu(A) = \mu_0(A)$ for all $A \in \mathscr{A}$).

Proposition 5.3.1 Consider the framework and notation from Lemma 5.3. The additive set-function $\mu_0 : \mathscr{E} \to [0,\infty]$ of that lemma is a pre-measure.

Corollary 5.3.2 Let $G : \mathbb{R} \to \mathbb{R}$ be an increasing cadlag function such that $G(0) = 0$. There eixsts a Lebesgue-Stieltjes measure $\mu : \mathscr{B}_{\mathbb{R}} \to [0,\infty]$, uniquely determined, such that $G = G_\mu$.

Proof. By lemma 5.3, we have the additive set-function μ_0 and it is a pre-measure. Then Caratheodory theorem extends μ_0 to a positive measure μ on $\mathscr{B}_{\mathbb{R}}$. Now, note that

$$\mu((a,b]) = \mu_0((a,b]) = G(b) - G(a), \forall a < b \in \mathbb{R}$$

then,

$$\mu([-n,n]) \leq \mu((-n-1,n]) = G(n) - G(-n-1) < \infty$$

Thus, μ is a Lebesgue-Stieltjes measure. Therefore, there is a uniquely determined centred Lebesgue-Stieltjes function G_μ. This $G_\mu = G$ since G checks all the boxes. By corollary 5.12, we have this μ being uniquely determined. ■

6. Extension Theorem of Caratheodory

R Caratheodory extension theorem is powerful since it is difficult to extend a finitely additive set-function $\mu_0 : \mathscr{E} \to [0,\infty]$ to a positive measure (hence a sigma-additive set-function) $\mu : \sigma - \mathrm{Alg}(\mathscr{E}) \to [0,\infty]$.

R

Uniqueness, the pre-measure condition

1. **Uniqueness of μ:** suppose we have the additional hypothesis that

   there exists an increasing chain $(A_n)_n \subseteq \mathscr{A}$ with $\mu_0(A_n) < \infty$ for all $n \in \mathbb{N}$ and such that $\cup_{n=1}^{\infty} A_n = X$.

   This condition on μ_0 is called $\sigma-$finiteness. It holds in our example of interest, say $\mathbb{R}$ and any $\mu_0(X) < \infty$. Suppose the $\sigma-$finiteness holds, then our desired extension $\mu : \mathscr{M} \to [0,\infty]$, if it exists, will be uniquely determined. Equivalently, if $\mu, \nu : \mathscr{M} \to [0,\infty]$ are positive measures such that $\mu(A) = \mu_0(A) = \nu(A)$ for all $A \in \mathscr{A}$, then it must be that $\mu(M) = \nu(M)$ for all $M \in \mathscr{M}$. This is because of the $\pi - \lambda$ theorem of Dynkin, and the fact that the algebra $\mathscr{A}$ is in particular a $\pi-$system.
2. **The pre-measure condition:** consider the following property which μ_0 may or may not have:

$$\text{(Pre-Sigma-Add)} \begin{cases} \text{If } (A_n)_n \subseteq \mathscr{A}, A_i \cap A_j = \emptyset, i \neq j \\ \text{and } \cup_{n=1}^{\infty} A_n \in \mathscr{A}, \\ \text{then } \mu_0(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu_0(A_n) \end{cases}$$

   If μ_0 is not a pre-measure, then it does not stand a chance to extend to a positive measure on $\mathscr{M}$. Or vice-versa: if there exists a positive measure $\mu : \mathscr{M} \to [0,\infty]$ which extends μ_0, then μ_0 is sure to be a pre-measure.

   Proof. Suppose μ exists. Then for $(A_n)_n \subseteq \mathscr{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ and with

$\cup_{n=1}^{\infty} A_n$ still in $\mathscr{A}$, we have

$$\begin{aligned}\mu_0(\cup_{n=1}^{\infty} A_n) &= \mu(\cup_{n=1}^{\infty} A_n) \\ &= \sum_{n=1}^{\infty} \mu(A_n) \\ &= \sum_{n=1}^{\infty} \mu_0(A_n)\end{aligned}$$

■

So the Caratheodory extension theorem shows that the pre-measure condition also is sufficient for the extension to exist.

One sentence punchline:

From $\mathscr{A}$ you overshoot all the way up to 2^X, and then you trim down, back to $\mathscr{M} = \sigma - \text{Alg}(\mathscr{A})$.

Definition 6.0.1 — Outer Measure. Let X be a non-empty set. A function $\mu^* : 2^X \to [0,\infty]$ is said to be an outer measure when it satisfies the following conditions:

1. **(OM1)** $\mu^*(\emptyset) = 0$
2. **(OM2)** μ^* is an increasing set-function: $E \subseteq F \subseteq X \implies \mu^*(E) \leq \mu^*(F)$
3. **(OM3)** μ^* is countably subadditive: we have $\mu(\cup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ for every family $(E_n)_n$ of subsets of X.

Overshoot from $\mu_0 \to \mu^*$.

Proposition 6.0.1 Let X be a non-empty set, let $\mathscr{A}$ be an algebra of subsets of X, and let $\mu_0 : \mathscr{A} \to [0,\infty]$ be a finitely additive set-function which has the pre-measure property. We define a function $\mu^* : 2^X \to [0,\infty]$ as follows: for every $E \subseteq X$ we put

$$\mu^*(E) := \inf\left\{\sum_{n=1}^{\infty} \mu_0(A_n) : (A_n)_n \subseteq \mathscr{A} \text{ s.t } \bigcup_{n=1}^{\infty} A_n \supseteq E\right\}$$

then:

1. μ^* is an outer measure
2. μ^* extends μ_0: $\mu^*(A) = \mu_0(A), \forall A \in \mathscr{A}$

Lemma 6.1 Consider

$$\text{(Pre-Sigma-SubAdd)} \begin{cases} \text{Whenever } A \text{ and } (A_n)_n \text{ are sets from } \mathscr{A} \text{ such that } A \subseteq \cup_{n=1}^{\infty} A_n \\ \text{it follows that } \mu_0(A) \leq \sum_{n=1}^{\infty} \mu_0(A_n) \end{cases}$$

then,

$$\text{(Pre-Sigma-Add)} \Longrightarrow \text{(Pre-Sigma-SubAdd)}$$

Proof. Suppose μ_0 satisfies (Pre-Sigma-Add) and let $A, (A_n)_n$ be sets from $\mathscr{A}$ such that $A \subseteq \cup_{n=1}^{\infty} A_n$. For $i \in \mathbb{N}$, consider

$$B_1 = A_1, B_2 = A_2 \backslash A_1, B_3 = A_3 \backslash (A_1 \cup A_2), \cdots$$

and $B_n = A_n \backslash \left(\cup_{i=1}^{n-1} A_i\right)$. Since $\mathscr{A}$ is an algebra of subsets of X. We have $B_n \in \mathscr{A}$ for all $n \in \mathbb{N}$. Moreover, by construction, $(B_n)_n$ is pairwise disjoint. We note further that

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \backslash (\cup_{i=1}^{n-1} A_i) = \bigcup_{n=1}^{\infty} A_n$$

. Then,

$$\mu_0(A) \overset{A\subseteq\cup_{n=1}^{\infty}A_n}{\subseteq} \mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu_0\left(\bigcup_{n=1}^{\infty} B_n\right) \overset{\text{(Pre-Sigma-Add)}}{=} \sum_{n=1}^{\infty}\mu_0(B_n) \overset{B_n\subseteq A_n}{\leq} \sum_{n=1}^{\infty}\mu_0(A_n)$$

Thus, we have the (Pre-Sigma-SubAdd). ■

Proposition 6.1.1 (Pre-Sigma-SubAdd)$\Longrightarrow$(Pre-Sigma-Add)

Proof. Suppose we have (Pre-Sigma-SubAdd) and assume $(A_n)_n \subseteq \mathscr{A}$ where $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$. Then, by (Pre-Sigma-SubAdd),

$$\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\mu_0(A_n)$$

For $N \in \mathbb{N}$, by finite additivity,

$$\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu_0\left(\bigcup_{n=1}^{N} A_n\right) + \mu_0\left(\bigcup_{n=N+1}^{\infty} A_n\right) \geq \mu_0\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N}\mu_0(A_n)$$

since this is true for all N, we take $N \to \infty$, we get

$$\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty}\mu_0(A_n)$$

Combine with previous inequality, we have the (Pre-Sigma-Add)

$$\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty}\mu_0(A_n)$$

■

Lemma 6.2 — The Trick of the Outer Measure. Let X be a non-empty set and let $\mu^* : 2^X \to [0,\infty]$ be an outer measure. Let us say that a subset $G \subseteq X$ is "good for μ" when it has the following property:

Whenever $E \subseteq G$ and $F \subseteq X \backslash G$, it follows that $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$

then:
1. The collection of sets $\mathscr{G} := \{G \subseteq X : G \text{ is good for } \mu^*\}$ is a $\sigma-$algebra of subsets of X
2. The restriction of μ^* to $\mathscr{G}$ is a positive measure.

Lemma 6.3 Let μ^* be the outer measure defined by starting from the finitely additive set-function $\mu_0 : \mathscr{A} \to [0,\infty]$. Then, every set from $\mathscr{A}$ is "good for μ^*".

Theorem 2 — Caratheodory Extension Theorem (Restated). Let X be a non-empty set, let $\mathscr{A}$ be an algebra of subsets of X, and let $\mathscr{M} := \sigma - \text{Alg}(\mathscr{A})$. Suppose that $\mu_0 : \mathscr{A} \to [0,\infty]$ is an additive set-function such that
1. μ_0 is sigma-finite
2. μ_0 is a pre-measure

Then there exists a positive measure $\mu : \mathscr{M} \to [0,\infty]$, uniquely determined, such that μ extends μ_0.

Exercise 6.1 Let (X,d) be a compact metric space and let $\mathscr{A}$ be an algebra of subsets of X where every set $A \in \mathscr{A}$ is clopen. Prove that any additive set-function $\mu_0 : \mathscr{A} \to [0,\infty]$ is a pre-measure.

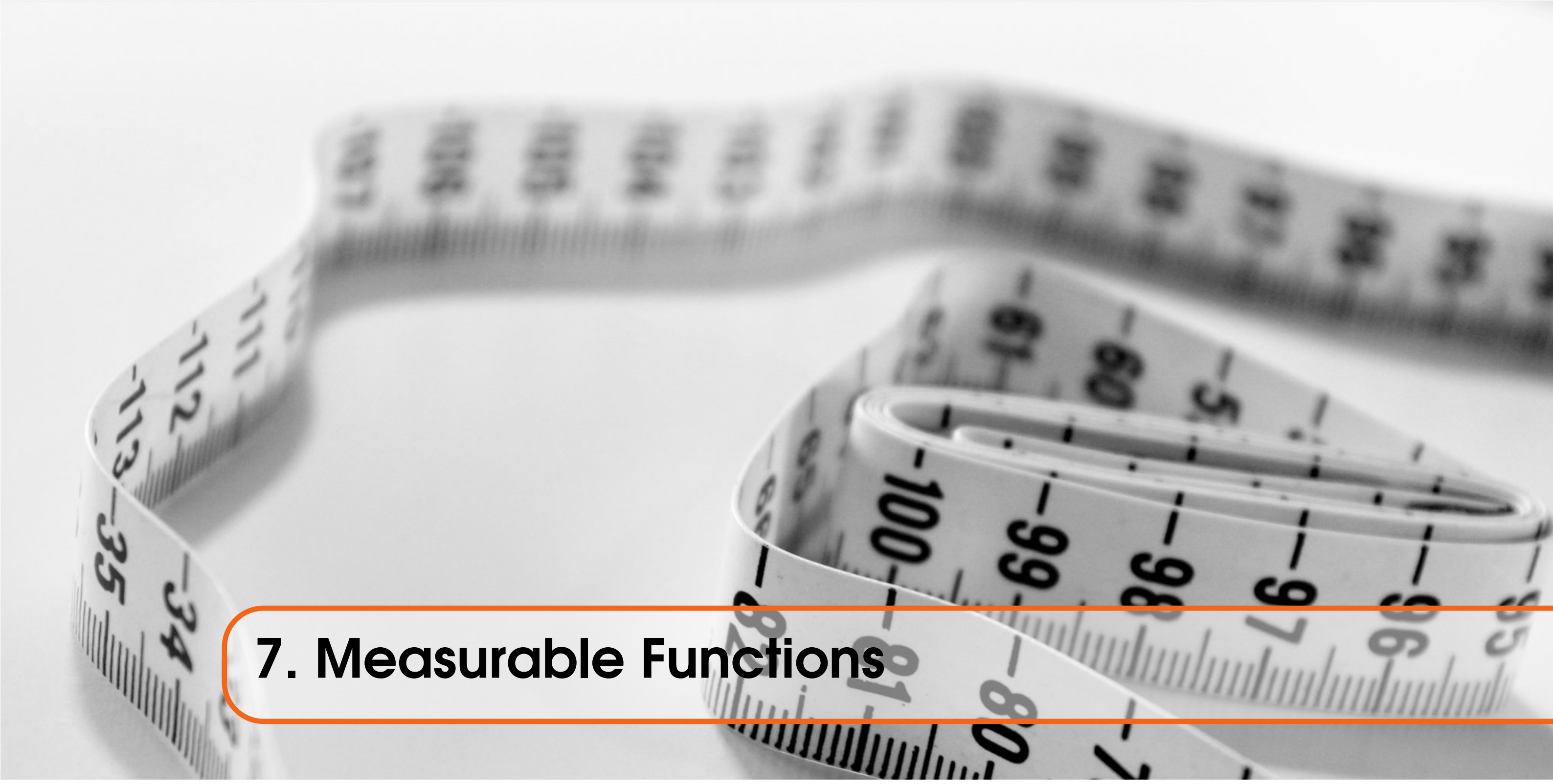
Proof. We shall show that μ_0 satisfies (Pre-Sigma-SubAdd) and invoke the lemma proved above to get (Pre-Sigma-Add). Let $A \in \mathscr{A}$ and $(A_n)_n \subseteq \mathscr{A}$ such that $A \subseteq \cup_{n=1}^{\infty} A_n$. Since A_n is clopen (open) for all $n \in \mathbb{N}$, $(A_n)_n$ is an open cover of A. Since (X,d) is a compact metric space, there exists a finite sub-cover $(A_{n_i})_{i=1}^{N} \subseteq (A_n)_n$ such that

$$A \subseteq \cup_{i=1}^{N} A_{n_i}$$

Then,

$$\begin{aligned}
\mu_0(A) &\leq \mu_0\left(\cup_{j=1}^{N} A_{n_i}\right) \\
&= \sum_{i=1}^{N} \mu_0(A_{n_i}) \qquad \text{finite additivity} \\
&\leq \sum_{n=1}^{\infty} \mu_0(A_n)
\end{aligned}$$

Thus, we have the (Pre-Sigma-SubAdd). Then, by the previous lemma proved, we have (Pre-Sigma-Add) and μ_0 is a pre-measure as required. ■



7. Measurable Functions

7.1 Measurable Functions

Definition 7.1.1 — $\mathscr{M}/\mathscr{N}$-Measurable. Let $(X,\mathscr{M}),(Y,\mathscr{N})$ be measurable spaces. A function $f : X \to Y$ is said to be $\mathscr{M}/\mathscr{N}-$ measurable when it has the property that $f^{-1}(N) \in \mathscr{M}$ for all $N \in \mathscr{N}$.

R Recall that $f : X \to Y$ and any family $(S_i)_{i\in I}$ of subsets of Y we have

$$f^{-1}\left(\bigcup_{i\in I} S_i\right) = \bigcup_{i\in I} f^{-1}(S_i)$$

and

$$f^{-1}\left(\bigcap_{i\in I} S_i\right) = \bigcap_{i\in I} f^{-1}(S_i)$$

moreover, when $S \subseteq Y$, we have

$$f^{-1}(Y \backslash S) = X \backslash f^{-1}(S)$$

Definition 7.1.2 — Space of Borel Functions. For $(X,\mathscr{M})$ and $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$, we define the space of Borel functions to be

$$\mathbf{Bor}(X,\mathbb{R}) := \{f : X \to \mathbb{R} : f \text{ is } \mathscr{M}/\mathscr{B}_{\mathbb{R}} - \text{measurable}\}$$

R It may be not clear that what $\sigma-$algebra that X has when using this notation.

Proposition 7.1.1 — Tool No.1. Let $(X,\mathscr{M}),(Y,\mathscr{N}),(Z,\mathbb{P})$ be measurable spaces. Let $f : X \to Y$ be an $\mathscr{M}/\mathscr{N}$-measurable function and let $g : Y \to Z$ be an $\mathscr{N}/\mathscr{P}$-measurable function. Consider the function $h := g \circ f$, then h is $\mathscr{M}/\mathscr{P}$-measurable.

Proof. For every $C \subseteq Z$ we have

$$h^{-1}(C) = (g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$$

so then,

$$C \in \mathscr{P} \overset{\mathscr{N}/\mathscr{P}-\text{measurable}}{\Longrightarrow} g^{-1}(C) \in \mathscr{N} \overset{\mathscr{M}/\mathscr{N}-\text{measurable}}{\Longrightarrow} f^{-1}(g^{-1}(C)) \in \mathscr{M} \Longrightarrow h^{-1}(C) \in \mathscr{M}$$

thus, h is $\mathscr{M}/\mathscr{P}-$ measurable as required. ∎

Proposition 7.1.2 — Tool No.2. Let $(X,\mathscr{M}),(Y,\mathscr{N})$ be measurable spaces, and let $\mathscr{C} \subseteq \mathscr{N}$ be a collection of subsets of Y such that $\sigma - \text{Alg}(\mathscr{C}) = \mathscr{N}$. Let $f : X \to Y$ be a function, and suppose that $f^{-1}(C) \in \mathscr{M}$ for all $C \in \mathscr{C}$. Then f is $\mathscr{M}/\mathscr{N}$-measurable.

Proof. Let $\mathscr{S} := \left\{S \subseteq Y : f^{-1}(S) \in \mathscr{M}\right\}$.

Claim 1: $\mathscr{S}$ is a $\sigma-$algebra of subsets of X
Proof. This is easily done by using basic properties of pre-images. We shall check (Sigma-AS3). Let $(S_n)_n \subseteq \mathscr{S}$. Then,

$$f^{-1}\left(\bigcap_{n=1}^{\infty} S_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(S_n)$$

Done with this claim. ∎

Claim 2: $\mathscr{S}$ contains $\mathscr{N}$
Proof. The hypothesis we have on the function f can be read as saying that $\mathscr{S} \supseteq \mathscr{C}$. So, $\mathscr{S}$ is a $\sigma-$algebra contains $\mathscr{C}$. While $\mathscr{N}$ is the smallest possible $\sigma-$algebra which contains $\mathscr{C}$. It follows that $\mathscr{S} \supseteq \mathscr{N}$. ∎

Now, for every $N \in \mathscr{N}$, we have $N \in \mathscr{S}$ and $f^{-1}(N) \in \mathscr{M}$. Thus, f is $\mathscr{M}/\mathscr{N}-$measurable. ∎

Corollary 7.1.3 Let $(X,d),(Y,d')$ be metric spaces, and let $f : X \to Y$ be a continuous function. Then f is $\mathscr{B}_X/\mathscr{B}_Y-$measurable.

Proof. Consider the topology of the space (Y,d'), $\mathscr{T}_Y = \{G \subseteq Y : G \text{ is open}\}$. Then, $\sigma-$ Alg$(\mathscr{T}_Y)$. Then, for any $G \in \mathscr{T}_Y$, we have $f^{-1}(G)$ is an open subset of X since f is continuous on X. Thus, $f^{-1}(G) \in \mathscr{B}_X$. Then, by Proposition 7.1.2, we have the result. ∎

Proposition 7.1.4 — Tool No.3. Let $(X,\mathscr{M})$ be a measurable space, and let $f : X \to \mathbb{R}^n$ be a function (for some $n \in \mathbb{N}$). For every $x \in X$, we write explicitly

$$f(x) = (f_1(x),\cdots,f_n(x)) \in \mathbb{R}^n$$

then we have

$$f \text{ is } \mathscr{M}/\mathscr{B}_{\mathbb{R}^n}-\text{measurable} \iff \text{each of } f_1,\cdots,f_n \text{ is in } \mathbf{Bor}(X,\mathbb{R})$$

Proof.
1. Fix $1 \leq i \leq n$, assume that f is $\mathscr{M}/\mathscr{B}_{\mathbb{R}^n}-$measurable. Let $P_i : \mathbb{R}^n \to \mathbb{R}$ be the $i-th$ projection map, this is a continuous map. Thus, P_i is $\mathscr{B}_{\mathbb{R}^n}/\mathscr{B}_{\mathbb{R}}-$measurable. Then, note that $f_i = P_i \circ f$. Then by Tool No.1, we have f_i is $\mathscr{M}/\mathscr{B}_{\mathbb{R}}-$measurable as required.
2. Now, assume that $f_1,\cdots,f_n$ are $\mathscr{M}/\mathscr{B}_{\mathbb{R}}-$measurable. Consider the collection of open cubes in $\mathbb{R}^n$ defined as

$$\mathscr{C} := \{(a_1-r,a_1+r)\times\cdots\times(a_n-r,a_n+r) : a_i \in \mathbb{R}, r \in (0,\infty)\}$$

Claim 1: $\sigma - \mathbf{Alg}(\mathscr{C}) = \mathscr{B}_{\mathbb{R}^n}$

Proof. Note that $\mathscr{C}$ is the collection of open balls with repsect to the d_∞ metric on $\mathbb{R}^n$. And the Borel $\sigma-$algebra is indeed generated by open balls. ■

R Shouldn't this always be the case in a separable metric space?

Claim 2: One has that $f^{-1}(C) \in \mathscr{M}$ for every $C \in \mathscr{C}$

Proof. Let $C \in \mathscr{C}$. Say

$$C = (a_1 - r, a_1 + r) \times \cdots \times (a_n - r, a_n + r)$$

where $a_i \in \mathbb{R}, r \in (0, \infty)$. Since f_i is $\mathscr{M}/\mathscr{B}_{\mathbb{R}}-$measurable, we have $f_i^{-1}((a_i - r, a_i + r)) \in \mathscr{M}, \forall 1 \leq i \leq n$. Then, we want to show that

$$f^{-1}(C) = \bigcap_{i=1}^{n} f_i^{-1}((a_i - r, a_i + r))$$

Let $x \in f^{-1}(C)$, we have $f(x) \in C$ and $f_i(x) \in (a_i - r, a_i + r), \forall 1 \leq i \leq n$. So, $x \in f_i^{-1}((a_i - r, a_i + r)), \forall 1 \leq i \leq n$ and $x \in \bigcap_{i=1}^{n} f_i^{-1}((a_i - r, a_i + r)) \Longrightarrow f^{-1}(C) \subseteq \bigcap_{i=1}^{n} f_i^{-1}((a_i - r, a_i + r))$. Now, for $x \in \bigcap_{i=1}^{n} f_i^{-1}((a_i - r, a_i + r))$, $x \in f_i^{-1}((a_i - r, a_i + r)), \forall 1 \leq i \leq n$. Thus, $f_i(x) \in (a_i - r, a_i + r), \forall 1 \leq i \leq n \Longrightarrow f(x) \in C \Longrightarrow x \in f^{-1}(C)$. Thus, our claim is true. Now, $f^{-1}(C)$ is finite intersection of elements in $\mathscr{M}$, which is still in $\mathscr{M}$. Thus, $f^{-1}(C) \in \mathscr{M}$. ■

Then, by Tool No.2, we are done.

■

R

Pointwise operation with $\mathbb{R}$-valued functions

We are now approaching the main point of the present lecture, which concerns the stability of $\mathbf{Bor}(X, \mathbb{R})$ under various algebraic operations that can be performed with functions $f : X \to \mathbb{R}$. We have the following pointwise operations. For $f, g : X \to \mathbb{R}$:

1. **Linear combinations:** for any two scalars $\alpha, \beta \in \mathbb{R}$, we can consider the function $\alpha f + \beta g : X \to \mathbb{R}$ defined by $(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x), \forall x \in X$.
2. **Product:** this is the function $f \cdot g : X \to \mathbb{R}$, $(f \cdot g)(x) = f(x) \cdot g(x), \forall x \in X$.
3. **Maximum and Minimum:**

$$f \wedge g : X \to \mathbb{R}, (f \wedge g)(x) = \min(f(x), g(x)), \forall x \in X$$
$$f \vee g : X \to \mathbb{R}, (f \vee g)(x) = \max(f(x), g(x)), \forall x \in X$$

It turns out that $\mathbf{Bor}(X, \mathbb{R})$ is stable under all these operations.

Proposition 7.1.5 Let $(X, \mathscr{M})$ be a measurable space, and let f, g be two functions in $\mathbf{Bor}(X, \mathbb{R})$. All the pointwise functions defined above have are contained in $\mathbf{Bor}(X, \mathbb{R})$.

Proof. Let's do it for $f \vee g$. Consider $F : X \to \mathbb{R}^2$ defined by $F(x) = (f(x), g(x)), \forall x \in X$. This function F is $\mathscr{M}/\mathscr{B}_{\mathbb{R}^2}-$measurable by Tool No.3. And consider $V : \mathbb{R}^2 \to \mathbb{R}$ defined by $V(s,t) = \max(s,t), \forall s,t \in \mathbb{R}$. The function V is continuous since $V(s,t) = \frac{s+t+|s-t|}{2}, s,t \in \mathbb{R}$. So, then V is $\mathscr{B}_{\mathbb{R}^2}/\mathscr{B}_{\mathbb{R}}-$measurable by corollay 7.1.3. Then, the composite function $V \circ F : X \to \mathbb{R}$ must be $\mathscr{M}/\mathscr{B}_{\mathbb{R}}-$measurable and $(V \circ F)(x) = \max(f(x), g(x)) = (f \vee g)(x)$. Thus, $f \vee g \in \mathbf{Bor}(X, \mathbb{R})$. ■

R $\mathbf{Bor}(X,\mathbb{R})$ is a unital algebra of functions since it contains the constant function 1. As a linear space of functions which is also stable under the operations $\vee$ and $\wedge$ is a structure called lattice of functions. Hence, $\mathbf{Bor}(X,\mathbb{R})$ is a lattice of functions.

R Let $(X,\mathscr{M})$ be a measurable space, and let $f : X \to \mathbb{R}$ be a Borel function. One can create new Borel function by starting just from this f. For instance,

$$|f| : X \to \mathbb{R}, |f|(x) = |f(x)|, \forall x \in X$$

or

$$\cos f : X \to \mathbb{R}, (\cos f)(x) = \cos(f(x))$$

8. Convergence Properties of Bor$(X,\mathbb{R})$

8.1 Bor$(X,\mathbb{R})$ is closed under pointwise convergence of sequences

Proposition 8.1.1 — Tool No.4. Let $(X,\mathscr{M})$ be a measurable space and let $f : X \to \mathbb{R}$ be a function. Suppose we could find a sequence $(f_n)_n$ of functions in $\mathbf{Bor}(X,\mathbb{R})$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. Then $f \in \mathbf{Bor}(X,\mathbb{R})$.

■ **Example 8.1** Consider $\mathbf{Bor}(\mathbb{R},\mathbb{R})$, the space of Borel functions from $\mathbb{R}$ to $\mathbb{R}$. Then, the following is true

> Whenever $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, it follows that the derivative f' belongs to $\mathbf{Bor}(\mathbb{R},\mathbb{R})$.

Proof. Consider f being differentiable and

$$f'(t) = \lim_{n\to\infty} f_n(t), \forall t \in \mathbb{R}$$

where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by putting

$$f_n(t) = \frac{f\left(t+\frac{1}{n}\right) - f(t)}{1/n}, \forall t \in \mathbb{R}$$

since f_n is a linear combination of f, $f_n \in \mathbf{Bor}(\mathbb{R},\mathbb{R})$ and $f' \in \mathbf{Bor}(\mathbb{R},\mathbb{R})$. ■

■

Proposition 8.1.2 — Tool No.4 sup, inf. Let $(X,\mathscr{M})$ be a measurable space.

1. Let $f : X \to \mathbb{R}$ be a function. Suppose we could find a sequence $(f_n)_n$ of functions in $\mathbf{Bor}(X,\mathbb{R})$ such that
$$\sup\{f_n(x) : n \in \mathbb{N}\} = f(x), \forall x \in X$$
then it follows that $f \in \mathbf{Bor}(X,\mathbb{R})$.

Proof. For fixed $t \in \mathbb{R}$, and for arbitrary $x \in X$, we have

$$f(x) \leq \sup\{f_n(x) : n \in \mathbb{N}\} \leq t \iff f_n(t) \leq t, \forall n \in \mathbb{N}$$

this implies that

$$\{x \in X : f(x) \leq t\} = \bigcap_{n=1}^{\infty} \{x \in X : f_n(x) \leq t\} = \bigcap_{n=1}^{\infty} f_n^{-1}((-\infty, t])$$

For every $n \in \mathbb{N}$, the set $f_n^{-1}((-\infty,t])$ is in $\mathscr{M}$ since f_n is $\mathscr{M}/\mathscr{B}_{\mathbb{R}}-$measurable. Thus, $\{x \in X : f(x) \leq t\} \in \mathscr{M}, \forall t \in \mathbb{R}$. Thus, $f \in \mathbf{Bor}(X,\mathbb{R})$. ■

2. Let $g : X \to \mathbb{R}$ be a function. Suppose we could find a sequence $(g_n)_n$ of functions in $\mathbf{Bor}(X,\mathbb{R})$ such that
$$\inf\{g_n(x) : n \in \mathbb{N}\} = g(x), \forall x \in X$$
then it follows that $g \in \mathbf{Bor}(X,\mathbb{R})$.

   Proof. Note that $-g(x) = \sup\{-g_n(x) : n \in \mathbb{N}\}, \forall x \in X$. ■

R

Review on $\limsup$, $\liminf$

$$\limsup t_n = \inf_{k\geq 1} \sup_{n\geq k} t_n \in [-\infty, \infty]$$

and

$$\liminf t_n = \sup_{k\geq 1} \inf_{n\geq k} t_n \in [-\infty, \infty]$$

Proposition 8.1.3 — Tool No.4 limsup, liminf. Let $(X,\mathscr{M})$ be a measurable space.

1. Let $f : X \to \mathbb{R}$ be a function. Suppose we could find a sequence $(f_n)_n$ of functions in $\mathbf{Bor}(X,\mathbb{R})$ such that
$$\limsup f_n(x) = f(x), \forall x \in X$$
then $f \in \mathbf{Bor}(X,\mathbb{R})$

   Proof. We note that for every $x \in X$, the sequence of real numbers $(f_n(x))_n$ is bounded above. Otherwise, we can find a subsequence diverges to ∞, contradicting $f(x) \in \mathbb{R}$. Thus, let $h_1 : X \to \mathbb{R}$ defined by
$$h_k(x) := \sup\{f_n(x) : n \geq k\}, \forall x \in X$$
we know that $h_k \in \mathbf{Bor}(X,\mathbb{R})$ for all $k \in \mathbb{N}$. And by remark on $\limsup$, we have $f \in \mathbf{Bor}(X,\mathbb{R})$. ■

2. Let $g : X \to \mathbb{R}$ be a function. Suppose we could find a sequence $(g_n)_n$ of functions in $\mathbf{Bor}(X,\mathbb{R})$ such that
$$\liminf g_n(x) = g(x), \forall x \in X$$
then $g \in \mathbf{Bor}(X,\mathbb{R})$

   Proof. Again, $f = -g$. ■

R For $f(x) = \lim_n f_n(x)$ for every $x \in X$. Then, for every $x \in X$, the number $f(x)$ is unique limit-point of the sequence $(f_n(x))_n$, which forces that we have

$$\limsup f_n(x) = \liminf f_n(x) = f(x)$$

we can invoke the proposition above to conclude that $f \in \mathbf{Bor}(X,\mathbb{R})$.

R Define

$$A_+ := \{x \in X : \limsup f_n(x) = \infty\}, A_- := \{x \in X : \limsup f_n(x) = -\infty\}$$

Claim: $A_+, A_- \in \mathscr{M}$

Proof. 1. For $t \in \mathbb{N}$, consider

$$A_t = \{x \in X : \limsup f_n(x) > t\}$$

and $X \backslash A_t = \{x \in X : \limsup f_n(x) \leq t\}$. Then, for any $x \in X \backslash A_t$, we can find a subsequence $(f_{n_k}(x))_k$ that converges to some real number. Since $f_n \in \mathbf{Bor}(X,\mathbb{R})$, by Proposition 8.5, $X \backslash A_t \in \mathscr{M}$. $\mathscr{M}$ is a $\sigma-$algebra, so $A_t \in \mathscr{M}$. Moreover, we have

$$A_+ = \bigcap_{t=1}^{\infty} A_t$$

$A_+ \subseteq \bigcap_{t=1}^{\infty} A_t$ is clear by construction. If $x \in \bigcap_{t=1}^{\infty} A_t$, then for $t = 1$, there exists a subsequence $(f_{n_k}(x))_{1,k}$ of $(f_n(x))_n$ that is eventually greater than 1. Say $f_{n_{k(1)}}(x) \geq 1$. Then, in general, for $t > 1$, there exists a subsequence $(f_{n_k}(x))_{t,k}$ of $(f_n(x))_n$ that is eventually greater than t and we can pick such element with index greater than $k(t-1)$. In this way, we have form the subsequence $(f_{n_{k(t)}}(x))_t$ of $(f_n(x))_n$ such that $f_{n_{k(t)}}(x) > t, \forall t \in \mathbb{N}$, which diverges to ∞. This means $x \in A_+$. Thus, $A_+ \in \mathscr{M}$ as $\mathscr{M}$ is a $\sigma-$algebra.

2. For $t \in \mathbb{N}$, consider

$$A_t = \{x \in X : \limsup f_n(x) \leq t\}$$

Then, for any $x \in X \backslash A_t$, we can find a subsequence $(f_{n_k}(x))_k$ that converges to some real number. Since $f_n \in \mathbf{Bor}(X,\mathbb{R})$, by Proposition 8.5, $A_t \in \mathscr{M}$. Moreover, we have

$$A_- = \bigcap_{t=1}^{\infty} A_t$$

$A_- \subseteq \bigcap_{t=1}^{\infty} A_t$ is clear by construction. then for $t = 1$, there exists a subsequence $(f_{n_k}(x))_{1,k}$ of $(f_n(x))_n$ that is eventually greater than 1. Say $f_{n_{k(1)}}(x) \geq 1$. Then, in general, for $t > 1$, there exists a subsequence $(f_{n_k}(x))_{t,k}$ of $(f_n(x))_n$ that is eventually greater than t and we can pick such element with index greater than $k(t-1)$. In this way, we have form the subsequence $(f_{n_{k(t)}}(x))_t$ of $(f_n(x))_n$ such that $f_{n_{k(t)}}(x) \geq t, \forall t \in \mathbb{N}$, which diverges to ∞. This means $x \in A_-$. Thus, $A_- \in \mathscr{M}$ as $\mathscr{M}$ is a $\sigma-$algebra. ■

Consider the following function $f : X \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & x \in A_+ \cup A_- \\ \limsup f_n(x) & x \in X \backslash (A_+ \cup A_-) \end{cases}$$

Claim: $f \in$ **Bor**$(X,\mathbb{R})$

Proof. For $t \in \mathbb{R}$, if $t < 0$, then consider

$$f^{-1}((-\infty, t]) = \{x \in X : \limsup f_n(x) \leq t\} \in \mathscr{M}$$

since $\limsup f_n(x) \in \mathbf{Bor}(X,\mathbb{R})$ when $x \in X \backslash (A_+ \cup A_-)$. If $t \geq 0$, then consider

$$f^{-1}((-\infty, t]) = (A_+ \cup A_-) \cup \{x \in X : \limsup f_n(x) \leq t\} \in \mathscr{M}$$

since we have shown $A_+, A_- \in \mathscr{M}$. Either way, this suffices to show that $f \in \mathbf{Bor}(X,\mathbb{R})$. ■

These two claims implies that

Bor$(X,\mathbb{R})$ being well-behaved limsup.

8.2 Approximation by Simple Functions

Definition 8.2.1 — Simple. Let $(X,\mathscr{M})$ be a measurable space. A function $f \in \mathbf{Bor}(X,\mathbb{R})$ is said to be simple when it only takes finitely many values, that is, the image $f(X)$ is a finite subset of $\mathbb{R}$. We denote

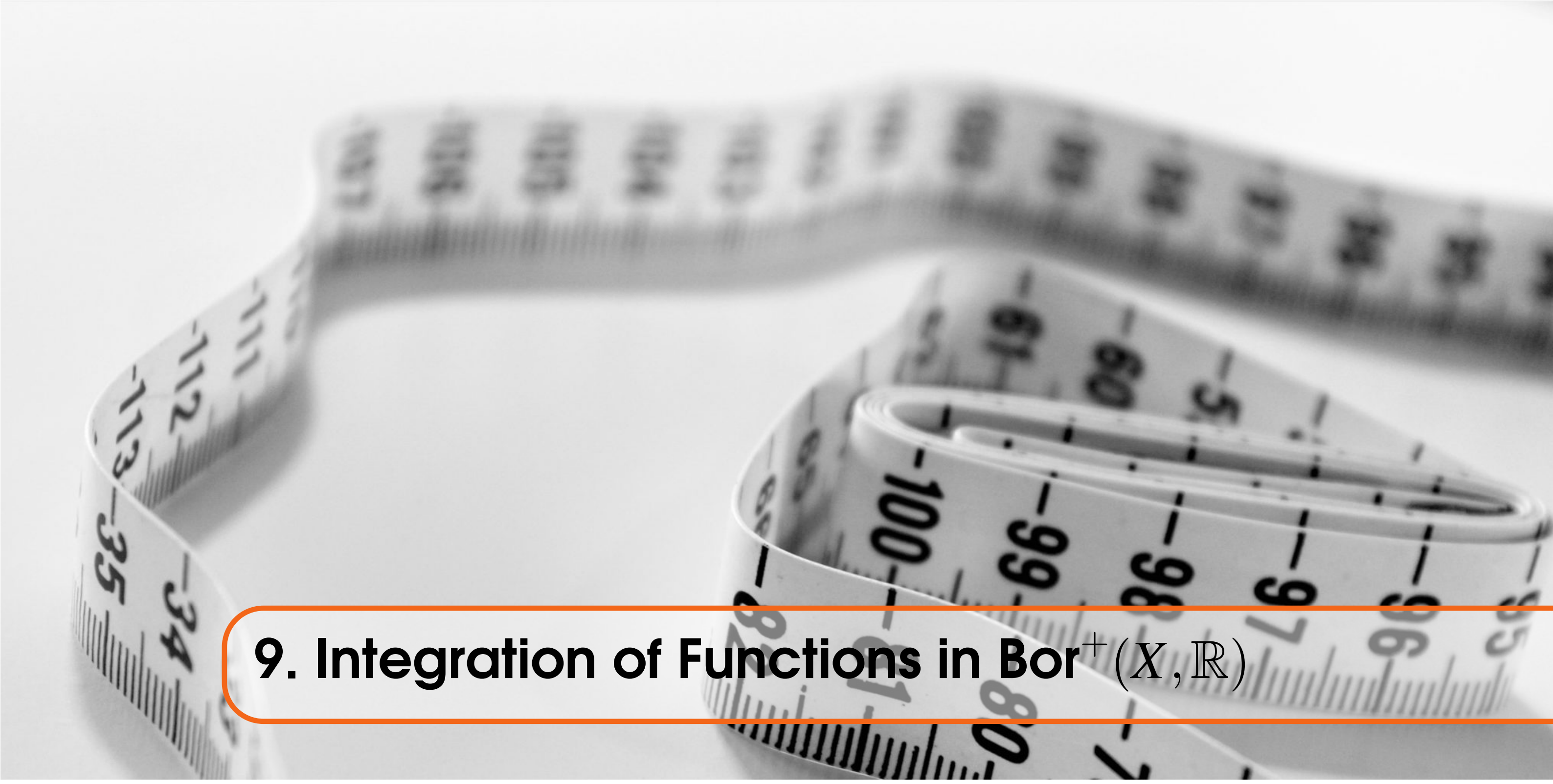
$$\mathbf{Bor}_s(X,\mathbb{R}) := \{f \in \mathbf{Bor}(X,\mathbb{R}) : f \text{ is simple}\}$$

R Let $(X,\mathscr{M})$ be a measurable space. One has a natural way of writing a general function $f \in \mathbf{Bor}(X,\mathbb{R})$ which approximate f can be thought of as a special way of binning the values of f. Consider the following function $f_n : X \to \mathbb{R}$ defined as follows:

1. If $x \in X$ and $f(x) \geq n$, then define $f_n(x) = n$
2. If $x \in X$ and $f(x) < n$, then we pick the unique $k \in \{1,2,\cdots,n2^n\}$ such that $(k-1)/2^n \leq f(x) < k/2^n$, and define $f_n(x) = (k-1)/2^n$

Proposition 8.2.1 Let $(X,\mathscr{M})$ be a measurable space and let f be a function $\mathbf{Bor}(X,\mathbb{R})$ such that $f(x) \geq 0$ for all $x \in X$. For every $n \in \mathbb{N}$, let f_n be the $n-$th binning of the values of f, then

1. $\forall n \in \mathbb{N}$, $f_n \in \mathbf{Bor}_s(X,\mathbb{R})$
2. $\forall n \in \mathbb{N}$, we have $f_n \leq f_{n+1}$
3. $\forall x \in X$, the increasing sequence $(f_n(x))_n$ has $\lim_n f_n(x) = f(x)$.



9. Integration of Functions in $\mathbf{Bor}^+(X,\mathbb{R})$

Definition 9.0.1 — The Framework. 1. We fix a measure space $(X,\mathscr{M},\mu)$.

$$\mathbf{Bor}^+(X,\mathbb{R}) := \{f \in \mathbf{Bor}(X,\mathbb{R}) : f(x) \geq 0, \forall x \in X\}$$

2. $\mathbf{Bor}^+(X,\mathbb{R})$ is not a linear subspace but nevertheless,

$$af + bg \in \mathbf{Bor}^+(X,\mathbb{R}), \forall f,g \in \mathbf{Bor}^+(X,\mathbb{R}), a,b \in [0,\infty)$$

define $\underline{0} : X \to \mathbb{R}$ such that $\underline{0}(x) = 0, \forall x \in X$. Clearly, $\underline{0} \in \mathbf{Bor}^+(X,\mathbb{R})$.

3. $\mathbf{Bor}^+(X,\mathbb{R})$ is also stable under other lattice operations besides the "linear" operation shown above.
4. Consider a partial order, for $f,g \in \mathbf{Bor}(X,\mathbb{R})$, we have

$$f \leq g \iff (f(x) \leq g(x), \forall x \in X)$$

in particular,

$$\mathbf{Bor}^+(X,\mathbb{R}) = \{f \in \mathbf{Bor}(X,\mathbb{R}) : f \geq \underline{0}\}$$

9.1 Integral for Simple Non-negative Borel Functions

Definition 9.1.1 — Simple. A function $f \in \mathbf{Bor}(X,\mathbb{R})$ is said to be simple when it only takes finitely many values. We denote

$$\mathbf{Bor}_s(X,\mathbb{R}) := \{f \in \mathbf{Bor}(X,\mathbb{R}) : f \text{ is simple}\}$$

we will put

$$\mathbf{Bor}_s^+(X,\mathbb{R}) := \mathbf{Bor}_s(X,\mathbb{R}) \cap \mathbf{Bor}^+(X,\mathbb{R})$$

R It follows that $\mathbf{Bor}_s^+(X,\mathbb{R})$ is a unital subalgebra and a sublattice of $\mathbf{Bor}(X,\mathbb{R})$. For now, we examine a map

$$L_s^+ : \mathbf{Bor}_s^+(X,\mathbb{R}) \to [0,\infty]$$

Definition 9.1.2 — L_s^+. Let $f \in \mathbf{Bor}_s^+(X,\mathbb{R})$. In connection to this f, we consider the following items:

We let $0 \le a_i < \cdots < a_p$ be the complete list of values assumed by f, and for every $1 \le i \le p$ we denote $A_i := \{x \in X : f(x) = \alpha_i\}$

we then put

$$L_s^+(f) := \sum_{i=1}^{p} \alpha_i \mu(A_i) \in [0,\infty]$$

R There are some questions that we want to resolve:

1. **First Question:** how do we know that $A_1,\cdots,A_p$ belong to $\mathscr{M}$? This is because f is $\mathscr{M}/\mathscr{B}_{\mathbb{R}}-$ measurable, and A_i is the pre-image by f of the closed set (hence Borel) $\{a_i\} \subseteq \mathbb{R}$.
2. **Second Question:** what if for some $1 \le i \le p$ we have $\mu(A_i) = \infty$? then $L_s^+(f) = \infty$
3. **Third Question:** what if $\alpha_i = 0$? Then, $0 \times \infty = 0$.

Lemma 9.2 Let f be a function in $\mathbf{Bor}_s^+(X,\mathbb{R})$. Suppose we found some sets $C_1,\cdots,C_r \in \mathscr{M}$ and some numbers $\gamma_1,\cdots,\gamma_r \in [0,\infty)$ such that

1. we have $C_1 \cup \cdots \cup C_r = X$ and $C_i \cap C_j = \emptyset$ for $i \neq j$
2. for every $1 \le j \le r$, we have $C_j \subseteq f^{-1}(\{\gamma_j\})$

Then it follows that

$$L_s^+(f) = \sum_{j=1}^{r} \gamma_j \mu(C_j)$$

Proof. We have to prove that

$$\sum_{j=1}^{r} \gamma_j \mu(C_j) = \sum_{i=1}^{p} \alpha_i \mu(A_i)$$

the rest is housekeeping. ■

Proposition 9.2.1 — Properties of L_s^+. The map $L_s^+ : \mathbf{Bor}_s^+(X,\mathbb{R}) \to [0,\infty]$, has the following properties:

1. **Additivity:** $L_s^+(f+g) = L_s^+(f) + L_s^+(g), \forall f,g \in \mathbf{Bor}_s^+(X,\mathbb{R})$

   Proof. Let $0 \le \alpha_1 < \cdots < \alpha_p$ be the list of values assumed by f, and let us denote $A_i = \{x \in X : f(x) = \alpha_i\}$. Likewise, let $0 \le \beta_1 < \cdots < \beta_q$ be the list of values assumed by g and denote $B_j = \left\{x \in X : g(x) = \beta_j\right\}$. We consider the sets

$$\left\{C_{i,j} : 1 \le i \le p, 1 \le j \le q\right\}, C_{i,j} = A_i \cap B_j$$

   then

$$\left\{\gamma_{i,j} : 1 \le i \le p, 1 \le j \le q\right\}, \gamma_{i,j} = \alpha_i + \beta_j$$

then,

$$\begin{aligned} L_s^+(f+g) &= \sum_{\substack{1\le i\le p\\ 1\le j\le q}} \gamma_{i,j}\mu(C_{i,j}) = \sum_{\substack{1\le i\le p\\ 1\le j\le q}} (\alpha_i+\beta_j)\mu(A_i\cap B_j) \\ &= \sum_{i=1}^{p}\alpha_i\left(\sum_{j=1}^{q}\mu(A_i\cap B_j)\right) + \sum_{j=1}^{q}\beta_j\left(\sum_{i=1}^{p}\mu(A_i\cap B_j)\right) \\ &= \sum_{i=1}^{p}\alpha_i\mu(A_i) + \sum_{j=1}^{q}\beta_j\mu(B_j) = L_s^+(f)+L_s^+(g) \end{aligned}$$

■

2. **Positive homogeneity:** $L_s^+(\alpha f) = \alpha L_s^+(f), \forall\alpha\in[0,\infty), f\in \mathbf{Bor}_s^+(X,\mathbb{R})$

   Proof. Follows from the lemma. ■

3. **Increasing:** if $f,g\in\mathbf{Bor}_s^+(X,\mathbb{R})$ are such that $f\le g$, then $L_s^+(f)\le L_s^+(g)$

   Proof. Let $h = g-f\in\mathbf{Bor}_s^+(X,\mathbb{R})$. Then, $h+f=g$, by 1, we have

$$L_s^+(g) = L_s^+(h)+L_s^+(f)\ge L_s^+(f)$$

■

■ **Example 9.1** Let $A\in\mathscr{M}$ and let χ_A be the indicator function of A, defined by

$$\chi_A(x) = \begin{cases} 1 & x\in A\\ 0 & x\in X\backslash A\end{cases}$$

then, $\chi_A\in\mathbf{Bor}_s^+(X,\mathbb{R})$ and $L_s^+(\chi_A) = \mu(A)\in[0,\infty]$. ■

R We can write

$$L_s^+(f) = \sum_{i=1}^{p}\alpha_i L_s^+(\chi_{A_i})$$

9.3 Moving to Bor$^+(X,\mathbb{R})$

R $\sup(S)=\infty$ means that either $\infty\in S$ or that is an unbounded subset of $[0,\infty)$.

Definition 9.3.1 — $L^+(f)$. For evever $f\in\mathbf{Bor}^+(X,\mathbb{R})$,

$$L^+(f) := \sup\left\{L_s^+(u) : u\in\mathbf{Bor}_s^+(X,\mathbb{R}), u\le f\right\}\in[0,\infty]$$

R Note that $\underline{0}$ is always in the set on the RHS. No need to worry about taking the sup of an empty set.

Proposition 9.3.1 L^+ extends the map L_s^+. That is

$$L^+(f) = L_s^+(f), \forall f\in\mathbf{Bor}_s^+(X,\mathbb{R})$$

Proof. If $f\in\mathbf{Bor}_s^+(X,\mathbb{R})$, then we can pick $u=f$ and $L^+(f)\ge L_s^+(f)$. If $f\in\mathbf{Bor}_s^+(X,\mathbb{R})$, $L_s^+(f0$ is an upper bound for the collection of numbers on the RHS. Thus, $L^+(f)\le L_s^+(f)$. ■

R

1. L^+ respects the partial order: if $f,g \in \mathbf{Bor}^+(X,\mathbb{R})$ are such that $f \leq g$, then it follows that $L^+(f) \leq L^+(g)$ since we are taking sup of a larger set of functions.
2. **Homogeneity:** one has
$$L^+(\alpha f) = \alpha L^+(f), \forall f \in \mathbf{Bor}^+(X,\mathbb{R}), \alpha \in [0,\infty)$$
3. **Additivity:** half of it for now
$$L^+(f+g) \geq L^+(f) + L^+(g), \forall f,g \in \mathbf{Bor}^+(X,\mathbb{R})$$

Proof. If $L^+(f+g) = \infty$, the required inequality is clear. Suppose it is finite. Then, $L^+(f), L^+(g)$ are finite as well. For $\varepsilon > 0$, we can find $u,v \in \mathbf{Bor}_s^+(X,\mathbb{R})$ such that $u \leq f, v \leq g$ and
$$L_s^+(u) > L^+(f) - \frac{\varepsilon}{2}, L_s^+(v) > L^+(g) - \frac{\varepsilon}{2}$$
then, $u+v \in \mathbf{Bor}_s^+(X,\mathbb{R})$ and $u+v \leq f+g$, thus,
$$L^+(f+g) \geq L_s^+(u+v) = L_s^+(f) + L_s^+(g) > L^+(f) + L^+(g) - \varepsilon$$
■

What about the other direction? We need the Monotone Convergence Theorem.

Definition 9.3.2 Let f and $(f_n)_n$ be functions in $\mathbf{Bor}^+(X,\mathbb{R})$. We will use the notation $f_n \nearrow f$ means that

1. $f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$ and
2. $\lim_n f_n(x) = f(x), \forall x \in X$

Theorem 3 — Monotone Convergence Theorem (MCT). Let f and $(f_n)_n$ be functions in $\mathbf{Bor}^+(X,\mathbb{R})$ such that $f_n \nearrow f$. Then, $\lim_n L^+(f_n) = L^+(f)$.

R It is not hard to see that
$$L^+(f_1) \leq L^+(f_2) \leq \cdots \leq L^+(f_n) \leq \cdots$$
and $L^+(f_i) \leq L^+(f), \forall i \geq 1$.

R What is for some $x \in X$, we have $\lim_n f_n(x) = \infty$? We can still do it but it depends on the measure of
$$M := \left\{x \in X : \lim_n f_n(x) = \infty\right\} \in \mathscr{M}$$

1. if $\mu(M) > 0$, then it is an addendum to the MCT
2. if $\mu(M) = 0$, sets of measure 0 can be ignored. We consider the adjusted functions
$$\tilde{f}_n := f_n \chi_{X \setminus M} \in \mathbf{Bor}^+(X,\mathbb{R}), n \in \mathbb{N}$$
which are increasing towards the function $\tilde{f} : X \to [0,\infty)$ defined by
$$\tilde{f}(x) := \begin{cases} \lim_n f_n(x) & x \in X \setminus M \\ 0 & X \in M \end{cases}$$
then $\tilde{f} \in \mathbf{Bor}^+(X,\mathbb{R})$ since it is a pointwise limit of a sequence in $\mathbf{Bor}^+(X,\mathbb{R})$. Moreover, $L^+(f) = L^+(\tilde{f})$ since we are modifying the functions on a set of measure 0. Thus,
$$\lim_n L^+(f_n) = L^+(f)$$

Proposition 9.3.2 — Behaviour of L^+ on linear combinations. The map $L^+ : \mathbf{Bor}^+(X,\mathbb{R}) \to [0,\infty]$ has the property that

$$L^+(af+bg) = aL^+(f) + bL^+(g), \forall f,g \in \mathbf{Bor}^+(X,\mathbb{R}), a,b \in [0,\infty)$$

Proof. We only need to show

$$L^+(f+g) \le L^+(f) + L^+(g)$$

We can find $(f_n)_n, (g_n)_n$ in $\mathbf{Bor}_s^+(X,\mathbb{R})$ such that $f_n \nearrow f$ and $g_n \nearrow g$. Then, $(f_n+g_n)_n$ is a sequence in $\mathbf{Bor}_s^+(X,\mathbb{R})$ such that $f_n + g_n \nearrow f+g$. Then, by MCT

$$\begin{aligned} L^+(f+g) &= \lim_n L^+(f_n+g_n) \\ &= \lim_n L_s^+(f_n+g_n) \\ &= \lim_n L_s^+(f_n) + L_s^+(g_n) \\ &= \lim_n L^+(f_n) + L^+(g_n) \\ &\le L^+(f) + L^+(g) \end{aligned}$$

■

9.4 Discussion around the proof of MCT

Proposition 9.4.1 Let f and $(f_n)_n$ be in $\mathbf{Bor}^+(X,\mathbb{R})$ such that $f_n \nearrow f$. Consider $\Lambda = \lim_n L^+(f_n) \in [0,\infty]$. We have $\Lambda \ge L^+(f)$.

Proof. It suffices to show that

$$\Lambda \ge L_s^+(u), \forall u \in \mathbf{Bor}_s^+(X,\mathbb{R}), u \le f$$

fix u, note that $(f_n \wedge u) \nearrow (f \wedge u) = u$. Then,

$$\lim_n L^+(f_n \wedge u) \ge L_s^+(u)$$

But for every $n \ge 1$, we have $f_n \wedge u \le f_n$, we have $L^+(f_n \wedge u) \le L^+(f_n)$. Thus,

$$\Lambda = \lim_n L^+(f_n) \ge \lim_n L^+(f_n \wedge u) \ge L_s^+(u)$$

■

R MCT follows from the proposition above. One thing to remember is that

MCT is an embellished version of the continuity of μ along increasing chains!

Lemma 9.5 Let A be a set in $\mathscr{M}$ and let $(f_n)_n$ be functions in $\mathbf{Bor}^+(X,\mathbb{R})$ such that $f_n \nearrow \chi_A$. Consider the limit $\Lambda = \lim_n L^+(f_n) \ge \mu(A)$.

Proof. Pick $\theta \in (0,1)$ and look at the sets $A_n = \{x \in X : f_n(x) \ge \theta\}$. These sets form an increasing chain with $\cup_{n=1}^{\infty} A_n = A$, so continuity of μ along increasing chains gives $\lim_n \mu(A_n) = \mu(A)$. By playing a bit with how $L^+(f_n)$ is defined as a sup, you see that $L^+(f_n) \ge \theta\mu(A_n)$ ■

Lemma 9.6 Let f be a function in $\mathbf{Bor}_s^+(X,\mathbb{R})$ and let $(f_n)_n$ be functions in $\mathbf{Bor}^+(X,\mathbb{R})$ and let $(f_n)_n$ be functions in $\mathbf{Bor}^+(X,\mathbb{R})$ such that $f_n \nearrow f$. Consider $\Lambda = \lim_n L^+(f_n) \ge L_s^+(f)$

10. The Space $\mathscr{L}^1(\mu)$ of integrable functions with re

10.1 Integrable Function

To extend $L^+ : \mathbf{Bor}^+(X,\mathbb{R}) \to [0,\infty]$ to more functions $f \in \mathbf{Bor}(X,\mathbb{R})$, we have

$$f_+ := f \vee \underline{0} \text{ and } f_- := (-f) \vee \underline{0}$$

it is clear that $f_+, f_- \in \mathbf{Bor}^+(X,\mathbb{R})$.

R We have

$$f_+ - f_- = |f|, f_+ - f_- = f$$

and

$$f_+ = \frac{f+|f|}{2}, f_- = \frac{|f|-f}{2}$$

Lemma 10.2 For $f \in \mathbf{Bor}(X,\mathbb{R})$, we have the equivalence:

$$L^+(|f|) < \infty \iff L^+(f_+) < \infty \text{ and } L^+(f_-) < \infty$$

Definition 10.2.1 — Integrable Function w.r.t μ. A function $f : X \to \mathbb{R}$ is said to be integrable with respect to μ when $f \in \mathbf{Bor}(X,\mathbb{R})$ and the equivalent conditions of the lemma are holding. We denote

$$\mathscr{L}^1(\mu) := \{f : X \to \mathbb{R} : f \text{ is integrable with respect to } \mu\}$$

we define a map $L : \mathscr{L}^1(\mu) \to \mathbb{R}$ by putting, for every $f \in \mathscr{L}^1(\mu)$:

$$L(f) := L^+(f_+) - L^-(f_-)$$

R $\mathbf{Bor}^+(X,\mathbb{R}) \cap \mathscr{L}^1(\mu) = \{f \in \mathbf{Bor}^+(X,\mathbb{R}) : L^+(f) < \infty\}$. For f in the latter set we see moreover that

$$L(f) = L^+(f)$$

Lemma 10.3 Let $f \in \mathscr{L}^1(\mu)$ and suppose that we could write $f = h_1 - h_2$ with $h_1, h_2 \in \mathbf{Bor}^+(X, \mathbb{R})$ such that $L^+(h_1), L^+(h_2) < \infty$. Then it follows that $L(f) = L^+(h_1) - L^+(h_2)$.

Proof. We have $f = f_+ - f_- = h_1 - h_2$, so $f_+ + h_2 = f_- + h_1 \in \mathbf{Bor}^+(X, \mathbb{R})$. Apply L^+ both sides

$$L^+(f_+) + L^+(h_2) = L^+(f_-) + L^+(h_1)$$

By the lemma, we can rearrange. ■

Theorem 4

1. $\mathscr{L}^1(\mu)$ is stable under linear combinations, hence is a linear subspace of $\mathbf{Bor}(X, \mathbb{R})$.

   Proof. Let $f, g \in \mathscr{L}^1(\mu)$ and $a, b \in \mathbb{R}$, and form the linear combination $h = af + bg \in \mathbf{Bor}(X, \mathbb{R})$. Note that

   $$|h| = |af + bg| \leq |a||f| + |b||g|$$

   then,

   $$L^+(|h|) \leq L^+(|a||f| + |b||g|) = |a|L^+(|f|) + |b|L^+(|g|) < \infty$$

   since $f, g \in \mathscr{L}^1(\mu)$. ■

2. The map $L : \mathscr{L}^1(\mu) \to \mathbb{R}$ is linear.

   Proof. (a) Let $f, g \in \mathscr{L}^1(\mu)$ and

   $$f + g = (f_+ - f_-) + (g_+ - g_-) = (f_+ + g_+) - (f_- + g_-) = h_1 - h_2$$

   Then, $h_1, h_2 \in \mathbf{Bor}^+(X, \mathbb{R})$ and $L^+(h_1), L^+(h_2) < \infty$. It follows that

   $$\begin{aligned} L(f+g) &= L^+(h_1) - L^+(h_2) \\ &= \big(L^+(f_+) + L^+(g_+)\big) - \big(L^+(f_-) + L^+(g_-)\big) \\ &= L(f) + L(g) \end{aligned}$$

   (b) $L(af) = aL(f)$. Not hard.

   ■

Proposition 10.3.1

1. If $f, g \in \mathscr{L}^1(\mu)$ are such that $f \leq g$, then it follows that $L(f) \leq L(g)$

   Proof. We have $L(g) - L(f) = L(g - f)$ and $g - f \in \mathscr{L}^1(\mu) \cap \mathbf{Bor}^+(X, \mathbb{R})$, then

   $$L(g - f) = L^+(g - f) \in [0, \infty)$$

   we conclude that

   $$L(g) - L(f) \geq 0 \Longrightarrow L(g) \geq L(f)$$

   ■

2. If $f \in \mathscr{L}^1(\mu)$, then $|f| \in \mathscr{L}^1(\mu)$ and

   $$|L(f)| \leq L(|f|)$$

   Proof. We have $-|f| \leq f \leq |f|$ and $|f| \in \mathscr{L}^1(\mu)$. Thus,

   $$-L(|f|) \leq L(f) \leq L(|f|)$$

   Thus,

   $$|L(f)| \leq L(|f|)$$

   ■

R We can use $\int$ sign now.

10.4 Equality a.e.-μ for functions in Bor$(X,\mathbb{R})$

Definition 10.4.1 — Equal a.e.-μ.

$$f = g \text{ a.e.-}\mu \iff \mu(\{x \in X : f(x) \neq g(x)\}) = 0$$

R Later on, we shall see that

$$f = g \text{ a.e.-}\mu \iff \int |f-g| d\mu = 0$$

R If $f, g \in \mathbf{Bor}(X,\mathbb{R})$ are such that $f = g$ a.e.-μ and if one of $f, g \in \mathscr{L}^1(\mu)$, then so is the other and $\int f d\mu = \int g d\mu$. Then, $f - g \in \mathscr{L}^1(\mu)$ and $\int (f-g) d\mu = 0$. Then,

$$g = f - (f-g) \in \mathscr{L}^1(\mu)$$

and

$$\int g d\mu = \int f d\mu - \int (f-g) d\mu$$

R This equality a.e.-μ forms a equivalence relation on $\mathbf{Bor}(X,\mathbb{R})$.

Exercise 10.1 Let $(X, \mathscr{M}, \mu)$ be a measure space and let h be be a function in $\mathbf{Bor}^+(X,\mathbb{R})$. We define a set-function $\nu : \mathscr{M} \to [0,\infty]$ by putting

$$\nu(A) := \int_A h d\mu, A \in \mathscr{M}$$

Prove that ν is a positive measure on $\mathscr{M}$.

Proof. 1. Let $A = \emptyset \in \mathscr{M}$. Then,

$$\nu(A) = \int_A h d\mu = \int h\chi_{\emptyset} d\mu = \int 0 d\mu = 0$$

2. Let $(A_n)_n \subseteq \mathscr{M}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$.

$$\begin{aligned}\nu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \int_{\bigcup_{i=1}^{\infty} A_i} h d\mu \\ &= \int h\chi_{\bigcup_{i=1}^{\infty} A_i} d\mu\end{aligned}$$

since $A_i \cap A_j = \emptyset$ for $i \neq j$ we have

$$\chi_{\bigcup_{i=1}^{\infty} A_i}(x) = \begin{cases} 1 & x \in \bigcup_{i=1}^{\infty} A_i \\ 0 & \text{otherwise} \end{cases} = \sum_{i=1}^{\infty} \chi_{A_i}(x)$$

Then,

$$h\chi_{\bigcup_{i=1}^{\infty} A_i} = h\sum_{i=1}^{\infty} \chi_{A_i} = \sum_{i=1}^{\infty} h\chi_{A_i} = \lim_n F_n$$

where $F_n = \sum_{i=1}^{n} h\chi_{A_i}$. Since $h \in \mathbf{Bor}^+(X,\mathbb{R})$ and $\chi_{A_i} \in \mathbf{Bor}^+(X,\mathbb{R}), \forall i$, we have $F_n \in \mathbf{Bor}^+(X,\mathbb{R})$. Moreover, $F_n \nearrow h\chi_{\bigcup_{i=1}^{\infty} A_i}$ since we are adding more indicators as $n \to \infty$. Then, by MCT, we have (in the sense of L^+)

$$\lim_n \int \sum_{i=1}^{n} h\chi_{A_i} d\mu = \lim_n \int F_n d\mu = \int \lim_n F_n d\mu = \nu\left(\bigcup_{i=1}^{\infty} A_i\right)$$

now, by linearity of L^+, we have

$$\begin{aligned}
\nu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_n \int \sum_{i=1}^{n} h\chi_{A_i} d\mu \\
&= \lim_n \sum_{i=1}^{n} \int h\chi_{A_i} d\mu \\
&= \lim_n \sum_{i=1}^{n} \nu(A_i) \\
&= \sum_{i=1}^{\infty} \nu(A_i)
\end{aligned}$$

Thus, ν is a positive measure. ■

10.5 Integral on a subset

Definition 10.5.1 Let $(X,\mathscr{M},\mu)$ be a measure space.

1. For any $f \in \mathbf{Bor}^+(X,\mathbb{R})$ and $A \in \mathscr{M}$ we denote

$$\int_A f d\mu := \int f\chi_A d\mu \in [0,\infty]$$

2. For any $f \in \mathscr{L}^1(\mu)$ and $A \in \mathscr{M}$ we observe that $f\chi_A \in \mathscr{L}^1(\mu)$ and

$$\int_A f d\mu := \int f\chi_A d\mu \in \mathbb{R}$$

Exercise 10.2 Let the measure space $(X,\mathscr{M},\mu)$, the function $h \in \mathbf{Bor}^+(X,\mathbb{R})$, and the new positive measure $\nu : \mathscr{M} \to [0,\infty]$ as in Exercise 11.13.

(a) Prove that for every $f \in \mathbf{Bor}^+(X,\mathbb{R})$ one has

$$\int f d\nu = \int fh d\mu \in [0,\infty]$$

(equality of integrals considered in L^+ sense)

Proof. We start with the simple case, let $f \in \mathbf{Bor}_s^+(X,\mathbb{R})$. Say

$$\int f d\nu = \sum_{i=1}^{p} \alpha_i \nu(A_i) = \sum_{i=1}^{p} \alpha_i \int h\chi_{A_i} d\mu$$

where α_i are the finite values of f and $A_i = \{x \in X : f(x) = \alpha_i\} \in \mathscr{M}$. Then, by linearity of L^+, we have

$$\int f d\nu = \int \left(\sum_{i=1}^{p} \alpha_i \chi_{A_i} \right) h d\mu = \int f h d\mu$$

Now, for $f \in \mathbf{Bor}^+(X,\mathbb{R})$, there eixsts an sequence $(f_n)_n$ in $\mathbf{Bor}_s^+(X,\mathbb{R})$ such that $f_n \nearrow f$. Meanwhile, $f_n h \nearrow f h$ pointwise and $fh, f_n h \in \mathbf{Bor}^+(X,\mathbb{R})$. By MCT, we have

$$\lim_n \int f_n d\nu = \int \lim_n f_n d\nu = \int f d\nu$$

also,

$$\lim_n \int f_n h d\mu = \int \lim_n f_n h d\mu = \int f h d\mu$$

by the simple case, we have

$$\int f_n d\nu = \int f_n h d\mu, \forall n \in \mathbb{N}$$

since the limit is unique in $[0,\infty] \subseteq \mathbb{R}$, we have

$$\int f d\nu = \lim_n \int f_n d\nu = \lim_n \int f_n h d\mu = \int f h d\mu$$

as required. ∎

(b) Let f be a function in $\mathbf{Bor}(X,\mathbb{R})$. Prove that

$$f \in \mathscr{L}^1(\nu) \iff fh \in \mathscr{L}^1(\mu)$$

moreover, prove that if the equivalent statements are true then

$$\int f d\nu = \int f h d\mu \in \mathbb{R}$$

(equality of integrals considered in L sense)

Proof. (a) Suppose $f \in \mathscr{L}^1(\nu)$, then $\int |f| d\nu < \infty$. We know that $|f| \in \mathbf{Bor}^+(X,\mathbb{R})$. Since $h \in \mathbf{Bor}^+(X,\mathbb{R})$, we have $|h| = h$ and

$$\int |fh| d\mu = \int |f| h d\mu = \int |f| d\nu < \infty$$

Thus, $fh \in \mathscr{L}^1(\mu)$.

(b) Suppose $fh \in \mathscr{L}^1(\mu)$, then $\int |fh| d\mu < \infty$. Similar to the previous case,

$$\int |fh| d\mu = \int |f| h d\mu = \int |f| d\nu < \infty$$

Thus, $f \in \mathscr{L}^1(\nu)$.

Now, provided that $f \in \mathscr{L}^1(\nu)$, in the L sense, we have

$$\int f d\nu = \int f_+ d\nu - \int f_- d\nu = \int f_+ h d\mu - \int f_- h d\mu$$

we know that $h = h_+ = |h|$ and $h_- = \underline{0}$. Moreover,

$$(fh)_+ = \frac{|fh| + fh}{2} = \frac{|f|h + fh}{2} = \left(\frac{|f| + f}{2}\right) h = f_+ h$$

and

$$(fh)_- = \frac{|fh| - fh}{2} = \frac{|f|h - fh}{2} = \left(\frac{|f| - f}{2}\right) h = f_- h$$

Thus,

$$\int f d\nu = \int (fh)_+ d\mu - \int (fh)_- d\mu = \int fh d\mu$$

as required. ■

11. Lebesgue's Dominated Convergence Theorem

We fix a measure space $(X, \mathscr{M}, \mu)$ and consider the space of functions $\mathscr{L}^1(\mu)$.

11.1 The statements(s) of LDCT

Theorem 5 — LDCT. Let f and $(f_n)_n$ be functions from $\mathbf{Bor}(X, \mathbb{R})$ such that $\lim_n f_n(x) = f(x), \forall x \in X$. Suppose that there exists a function $h \in \mathscr{L}^1(\mu) \cap \mathbf{Bor}^+(X, \mathbb{R})$ which dominates all the f_n, in the sense that we have $|f_n| \leq h, \forall n \in \mathbb{N}$. Then $f, f_1, \cdots, f_n, \cdots$ are all in $\mathscr{L}^1(\mu)$, and

$$\lim_n \int f_n d\mu = \int f d\mu$$

For a function $f \in \mathscr{L}^1(\mu)$, we denote

$$\|f\|_1 := \int |f| d\mu \in [0, \infty)$$

The integral on the RHS can be viewed either in the L^+ sense or in the L sense, since $|f| \in \mathscr{L}^1(\mu) \cap \mathbf{Bor}^+(X, \mathbb{R})$. Then,

$$\left| \int f d\mu \right| \leq \|f\|_1, \forall f \in \mathscr{L}^1(\mu)$$

$\|\cdot\|_1$ is a semi-norm on the vector space $\mathscr{L}^1(\mu)$, which means

$$\begin{cases} \|f+g\|_1 \leq \|f\|_1 + \|g\|_1, \forall f, g \in \mathscr{L}^1(\mu) \\ \|\alpha f\|_1 = |\alpha| \|f\|_1, \forall f \in \mathscr{L}^1(\mu), \alpha \in \mathbb{R} \end{cases}$$

We need to consider an equivalence relation to get $\|\cdot\|_1$ to a norm. We shall do that by

$$\|f\| = 0 \iff f = \underline{0} \text{ a.e.-}\mu$$

Theorem 6 — LDCT with $\|\cdot\|_1$. Let f and $(f_n)_n$ from $\mathbf{Bor}(X,\mathbb{R})$ such that $\lim_n f_n(x) = f(x), \forall x \in X$. Suppose moreover that there exists $h \in \mathscr{L}^1(\mu) \cap \mathbf{Bor}^+(X,\mathbb{R})$ such that $|f_n| \leq h, \forall n \in \mathbb{N}$. Then f and $f_1, \cdots, f_n, \cdots$ are all in $\mathscr{L}^1(\mu)$ and

$$\lim_n \|f_n - f\|_1 = 0$$

Proof. Note that

$$\int |f_n| d\mu \leq \int h d\mu < \infty$$

same for f. Let $g_n = |f_n - f| \in \mathbf{Bor}^+(X,\mathbb{R})$. Note that

1. For $x \in X$, $\lim_n g_n(x) = 0$
2. Note that $h = \sup\{g_n\}$.

∎

(R) This is a bit stronger than the first version.

Lemma 11.2 — Reverse MCT. Let u and $(u_n)_n$ be functions in $\mathbf{Bor}^+(X,\mathbb{R})$ such that $u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots$ and such that $\lim_n u_n(x) = u(x), \forall x \in X$. We assume, in addition, that $\int u_1 d\mu < \infty$. Then

$$\lim_n \int u_n d\mu = \int u d\mu$$

Proof. Note that $(\int u_n d\mu)_n$ is decreasing. Thus, the limit exists. For $n \in \mathbb{N}$, $f_n = u_1 - u_2 \in \mathbf{Bor}^+(X,\mathbb{R})$, let $f = u_1 - u$. We note that $f_n \nearrow f$. By MCT, we have

$$\lim_n \int f_n d\mu = \int f d\mu$$

Moreover,

$$\int f_n d\mu + \int u_n d\mu = \int u_1 d\mu$$

The hypothesis $\int u_1 d\mu < \infty$ implies

$$\int u_n d\mu = \int u_1 d\mu - \int f_n d\mu \in [0,\infty), \forall n \in \mathbb{N}$$

Let $n \to \infty$,

$$\lim_n \int u_n d\mu = \int u_1 d\mu - \lim_n \int f_n d\mu = \int u_1 d\mu - \int f d\mu = \int u d\mu$$

∎

Proposition 11.2.1 — LDCT$_0$. Let $(g_n)_n$ be a sequence of functions from $\mathbf{Bor}^+(X,\mathbb{R})$ such that $\lim_n g_n(x) = 0, \forall x \in X$. Suppose there exists $h \in \mathscr{L}^1(\mu) \cap \mathbf{Bor}^+(X,\mathbb{R})$ such that $g_n \leq h, \forall n \in \mathbb{N}$. Then, $\lim_n \int g_n d\mu = 0$.

Proof. If $(g_n)_n$ is decreasing, everything would be easy by invoking the reverse MCT.

Claim 1: for every $k \in \mathbb{N}$, define $u_k : X \to \mathbb{R}$ defined by $u_k(x) = \sup\{g_n(x) : n \geq k\}, \forall x \in X$. Moreover, $u_k \in$ Bor$^+(X, \mathbb{R})$ and $u_k \leq h$

Proof. $\{g_n(x) : n \geq k\}$ is bounded from above by $h(x)$. Our definition makes sense and $u_k \in$ **Bor**$(X, \mathbb{R})$. Since $u_k(x) \geq 0, \forall x \in X$, we have $u_k \in$ **Bor**$^+(X, \mathbb{R})$ and $u_k \leq h$. ■

$(u_k)_k$ is decreasing. For $x \in X$,

$$\begin{aligned}\lim_k u_k(x) &= \lim_k (\sup\{g_n(x) : n \geq k\}) \\ &= \limsup_k g_n(x) \\ &= \lim_n g_n(x) = 0\end{aligned}$$

Now, by the reverse MCT, we have $\lim_k \int u_k d\mu = 0$. Now, $u_k(x) \geq g_k(x), \forall x \in X$. Also, $0 \leq \int g_k d\mu \leq \int u_k d\mu, \forall k \in \mathbb{N}$. By Squeeze Theorem, we have the result. ■

Proposition 11.2.2 — Fatou's Lemma. Let f and $(f_n)_n$ be functions from **Bor**$^+(X, \mathbb{R})$ such that $f(x) = \liminf_n f_n(x), \forall x \in X$. Then,

$$\int f d\mu \leq \liminf_n \left(\int f_n d\mu \right)$$

Proof. Let $g_n = \inf_{k \geq n} f_k$. Note that $f_k \geq 0$ for all $k \geq n \in \mathbb{N}$. Thus, 0 is a lower bound for $(f_k)_{k \geq n}$. Then, $g_n \geq 0$. By taking the inf, we have $g_n \in$ **Bor**$(X, \mathbb{R})$. Therefore, $g_n \in$ **Bor**$^+(X, \mathbb{R})$. By construction,

$$\int g_n d\mu \leq \inf_{k \geq n} \int f_k d\mu$$

Moreover, $(g_n)_n$ is increasing and $\lim_n \int g_n$ exists. Likewise, the sequence $(\inf_{k \geq n} \int f_k)_n$ is increasing in n and hence has a limit. Thus,

$$\lim_n \int g_n d\mu \leq \lim_n \left(\inf_{k \geq n} \int f_k \right) = \liminf_n \left(\int f_n d\mu \right)$$

Let $f(x) = \liminf_n f_n(x)$. Note that by definition, $g_n \nearrow F(x)$. Then, by MCT, we have

$$\lim_n \int g_n d\mu = \int f d\mu = \int \liminf_n f_n d\mu$$

Thus,

$$\int f d\mu \leq \liminf_n \left(\int f_n d\mu \right)$$

■

12. The Spaces $\mathscr{L}^2(\mu)$

We shall let $p = [1, \infty)$.

12.1 What is $\mathscr{L}^p(\mu)$

Definition 12.1.1 — $\mathscr{L}^p(\mu)$. Let $(X, \mathscr{M}, \mu)$ be a measure space and let $p \in [1, \infty)$

1. we denote

$$\mathscr{L}^p(\mu) := \left\{ f \in \mathbf{Bor}(X, \mathbb{R}) : \int |f|^p d\mu < \infty \right\}$$

2. For $f \in \mathscr{L}^p(\mu)$, we denote

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{1/p} \in [0, \infty)$$

R $\mathscr{L}^p(\mu)$ is a linear subspace of $\mathbf{Bor}(X, \mathbb{R})$ and $\|\cdot\|_p$ is a semi-norm.

R

$$a \cdot b \leq \frac{1}{p} a^p + \frac{1}{q} b^q, \forall a, b \in [0, \infty)$$

Proposition 12.1.1 — Holder's Inequality. Let $(X, \mathscr{M}, \mu)$ be a measure space and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

1. For every $f \in \mathscr{L}^p(\mu)$ and $g \in \mathscr{L}^q(\mu)$ one has that $f \cdot g \in \mathbb{L}^1(\mu)$, and

$$\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q$$

2. For any $f \in \mathscr{L}^p(\mu)$,

$$\|f\|_p = \sup \left\{ \|f \cdot h\|_1 : h \in \mathscr{L}^q(\mu), \|h\|_q \leq 1 \right\}$$

Proposition 12.1.2 Let $(X,\mathscr{M},\mu)$ be a measure space and let $p \in [1,\infty)$. Then $\mathscr{L}^p(\mu)$ is a linear subspace of $\mathbf{Bor}(X,\mathbb{R})$.

Proof. 1. **Stability under addition:** note that

$$|a+b|^p \le (2\max\{|a|,|b|\})^p \le 2^p(|a|^p+|b|^p), \forall a,b \in \mathbb{R}$$

Let $f,g \in \mathbf{Bor}^+(X,\mathbb{R})$, for $x \in X$, we have

$$|f+g|^p \le 2^p|f|^p + 2^p|g|^p$$

in L^+ sense. Then,

$$\int |f+g|^p d\mu \le 2^p \int |f|^p d\mu + 2^p \int |g|^p d\mu$$

Then, $f+g \in \mathscr{L}^p(\mu)$.

2. **Stability under scalar multiplication:** note that

$$\alpha \in \mathbb{R}, f \in \mathscr{L}^p(\mu) \Longrightarrow \alpha f \in \mathscr{L}^p(\mu), \|\alpha f\|_p = |\alpha|\|f\|_p$$

■

Theorem 7 — Minokowski's Inequality. Let $(X,\mathscr{M},\mu)$ be a measure space, let $p \in [1,\infty)$, and consider the linear subspace $\mathscr{L}^p(\mu)$ of $\mathbf{Bor}(X,\mathbb{R})$. The map $\|\cdot\|_p : \mathscr{L}^p(\mu) \to [0,\infty)$ is a semi-norm. That is

$$\|f+g\|_p \le \|f\|_p + \|g\|_p, \forall f,g \in \mathscr{L}^p(\mu)$$

and homogeneity property that

$$\|\alpha f\|_p = |\alpha|\|f\|_p, \forall \alpha \in \mathbb{R}, f \in \mathscr{L}^p(\mu)$$

Proof. We shall show the case for $p > 1$. Let $q = \frac{p}{p-1}$. Let $f,g \in \mathscr{L}^p(\mu)$. Then,

$$\|f+g\|_p = \sup\left\{\|(f+g)\cdot h\|_1 : h \in \mathscr{L}^q(\mu), \|h\|_q \le 1\right\}$$

For $h \in \mathscr{L}^q(\mu)$ with $\|h\|_q \le 1$, we have

$$\begin{aligned}\|(f+g)h\|_1 &\le \|fh\|_1 + \|gh\|_1 \\ &\le \|f\|_p\|h\|_q + \|g\|_p\|h\|_q \\ &\le \|f\|_p + \|g\|_p\end{aligned}$$

■

Exercise 12.1 Let $(X,\mathscr{M},\mu)$ be a measure space where $0 < \mu(X) < \infty$. Let $p_1, p_2 \in [1,\infty)$, such that $p_1 \le p_2$. Prove that $\mathscr{L}^p(\mu) \supseteq \mathscr{L}^{p_2}(\mu)$, and that for every $f \in \mathscr{L}^{p_2}(\mu)$,

$$\|f\|_{p_1} \le \|f\|_{p_2}(\mu(X))^{\frac{1}{p_1}-\frac{1}{p_2}}$$

Proof. 1. Let $f \in \mathscr{L}^{p_2}(\mu)$. Then,

$$\int |f|^{p_2} d\mu < \infty$$

Let $r = \frac{p_2}{p_1}$ and $v = \frac{p_2}{p_2-p_1}$ which are at least 1 and $\frac{1}{r}+\frac{1}{v} = 1$. Moreover, $1^{p_2-p_1} = 1 \in$

$\mathscr{L}^v(\mu)$ since

$$\int (|1|^{p_2-p_1})^v d\mu = \int 1 d\mu = \mu(X) < \infty$$

Then,

$$\begin{aligned}\|f\|_{p_1}^{p_1} &= \int |f|^{p_1} d\mu = \int |f|^{p_1} |1|^{p_2-p_1} d\mu \\ &\underset{\text{holder}}{\leq} \left[\int |f|^{p_2} d\mu\right]^{1/r} \left[\int |1|^{p_2} d\mu\right]^{1/v} \\ &= \|f\|_{p_2}^{p_1} \mu(X)^{\frac{p_2-p_1}{p_2}}\end{aligned}$$

Take $1/p_1$ exponents both sides,

$$\|f\|_{p_1} \leq \|f\|_{p_2} \mu(X)^{\frac{1}{p_1}-\frac{1}{p_2}}$$

Thus, if $f \in \mathscr{L}^{p_2}(\mu)$, we have $\|f\|_{p_2} < \infty$, then

$$\|f\|_{p_1} \leq \|f\|_{p_2} \mu(X)^{\frac{1}{p_1}-\frac{1}{p_2}} < \infty \Longrightarrow f \in \mathscr{L}^{p_1}(\mu)$$

For the case of a probability space, we have $\mu(X) = 1$ and

$$\|f\|_{p_1} \leq \|f\|_{p_2}$$

■

12.2 Equality almost everywhere, and discussion of $L^p(\mu)$ vs. $\mathscr{L}^p(\mu)$

We have not yet get

$$f \in \mathscr{L}^p(\mu), \|f\|_p = 0 \Longrightarrow f(x) = 0, \forall x \in X$$

since $\|\cdot\|_p$ is just a semi-norm.

Definition 12.2.1 — Quotient by Null.

$$\mathscr{N} := \left\{ f \in \mathscr{L}^p(\mu) : \|f\|_p = 0 \right\}$$

then,

$$L^p(\mu) = \frac{\mathscr{L}^p(\mu)}{\mathscr{N}}$$

R

1. $\mathscr{N}$ is stable under linear combinations
2. There is a surjective map:

$$\mathscr{L}^p(\mu) \ni f \mapsto \hat{f} \in \mathscr{L}^p(\mu)/\mathscr{N} = L^p(\mu)$$

and

$$\hat{f} = \hat{g} \iff f - g \in \mathscr{N}$$

and

$$f \in \mathscr{N} \iff \|f\|_p = 0 \iff \int |f|^p d\mu = 0$$

thus, we can say

$$\mathscr{N} = \{ f \in \mathbf{Bor}(X, \mathbb{R}) : f = 0 \text{ a.e.-}\mu \}$$

3. If we define
$$\left\|\hat{f}\right\|_p := \|f\|_p, \forall f \in \mathscr{L}^p(\mu)$$
it becomes a norm on $L^p(\mu)$. And $L^p(\mu)$ is a Banach space.

13. $\mathscr{L}^p(\mu)$ Completeness

13.1 Completeness in a seminormed vector space

We let $\mathscr{V} = \mathscr{L}^p(\mu)$ and $(\mathscr{V}, \|\cdot\|_p = \|\cdot\|)$ be a seminormed vector space.

Definition 13.1.1 — Convergence in $(\mathscr{V}, \|\cdot\|)$. Let $(\mathscr{V}, \|\cdot\|)$ be a seminormed vector space, let $(v_n)_n$ be a sequence of vectors in $\mathscr{V}$ and let v be a vector in $\mathscr{V}$. We say that $(v_n)_n$ converges to v mean that one has

$$\lim_n \|v_n - v\| = 0$$

R Sequence might not have a unique limit. Say $(v_n)_n$ converges to v, v'. Then,

$$\|v - v'\| \leq \|v - v_n\| + \|v_n - v'\|, \forall n \in \mathbb{N} \implies \|v - v'\| = 0$$

but this does not necessarily imply $v = v'$.

Definition 13.1.2 — Cauchy Sequence. Let $(v_n)_n$ be a sequence in $\mathscr{V}$.

1. **(Cauchy)** We say that $(v_n)_n$ is Cauchy if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $m, n \geq n_0$ implies $\|v_m - v_n\| \leq \varepsilon$
2. We say that $(v_n)_n$ is 1/2-geometric sequence to mean

$$\|v_n - v_{n+1}\| \leq \frac{1}{2^n}, \forall n \in \mathbb{N}$$

R From PMATH351, we know that we have the following results:

1. If $(v_n)_n$ is convergent, then it is Cauchy
2. If $(v_n)_n$ is 1/2-geometric, then it is Cauchy
3. If $(v_n)_n$ is Cauchy, then we can find a subsequence $(v_{n(k)})_k$ that is 1/2-geometric.

R Another result from PMATH351,

> Let $(\mathscr{V}, \|\cdot\|)$ be a seminormed space. Whenever $(v_n)_n$ is 1/2-geometric sequence in $\mathscr{V}$, there exists $v \in V$ (not necessarily unique) such that $v_n \to v$. Then, $(\sqsubseteq, \|\cdot\|)$ is complete.

This is supported by

> a Cauchy sequence which has a cluster point is sure to converge to the cluster point.

13.2 Completeness of $(\mathscr{L}^p(\mu), \|\cdot\|_p)$

Lemma 13.3 — Convergence Mechanism - How to find a candidate for limit. Let X be a non-empty set and let $f_1, f_2, \cdots, f_n, \cdots$ be a sequence of functions from X to $\mathbb{R}$. Consider $h_n : X \to [0, \infty)$ defined by

$$h_1 = |f_1|, h_2 = |f_1| + |f_2 - f_1|, \cdots, h_n = |f_1| + \sum_{m=2}^{n} |f_m - f_{m-1}|, \cdots$$

we observe that $0 \le h_1 \le h_2 \le \cdots \le h_n \le \cdots$ and we denote

$$Z = \left\{ x \in X : \lim_n h_n(x) = \infty \right\}$$

Then, for every $x \in X \backslash Z$, the sequence $(f_n(x))_n$ is convergent in $\mathbb{R}$. Consider $f : X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \in Z \\ \lim_n f_n(x) & x \in X \backslash Z \end{cases}$$

Proof. Note that, for $x \in X$, we have

$$|f_n(x) - f_m(x)| \le h_n(x) - h_m(x), \forall m \le n \in \mathbb{N}$$

when $x \in X \backslash Z$, $(f(x)_n)_n$ is a Cauchy sequence in $\mathbb{R}$ and therefore convergent in $\mathbb{R}$. ■

Proposition 13.3.1 Let $(X, \mathscr{M}, \mu)$ be a measure space, let $p \in [1, \infty)$ and let $(f_n)_n$ be a sequence of functions in $\mathscr{L}^p(\mu)$ such that $\|f_n - f_{n+1}\| \le \frac{1}{2^n}, \forall n \in \mathbb{N}$. We have the function f described in the lemma above. Then, $f \in \mathscr{L}^p(\mu)$ and $\lim_n \|f_n - f\|_p = 0$.

Proof. Consider $h : X \to [0, \infty)$ defined by

$$h(x) = \begin{cases} 0 & x \in Z \\ \lim_n h_n(x) & x \in X \backslash Z \end{cases}$$

Claim 1: Denote $\|f_1\|_p =: c \in [0, \infty)$. We have $h_n \in \mathscr{L}^p(\mu)$ and $\|h_n\|_p \le 1 + c, \forall n \in \mathbb{N}$
Proof. When $n = 1$, the result is automatic as $f_1 \in \mathscr{L}^p(\mu)$. For every $2 \le m \le n$, we have $f_m - f_{m-1} \in \mathscr{L}^p(\mu)$. Then, $u_m = |f_m - f_{m-1}| \in \mathscr{L}^p(\mu)$ and $\|u_m\|_p < \frac{1}{2^{m-1}}$. We can write $h_n = |f_1| + \sum_{i=2}^{n} u_i \in \mathscr{L}^p(\mu)$ and

$$\|h_n\|_p \le \|f_1\|_p + \sum_{i=2}^{n} \|u_i\|_p \le c + \sum_{i=1}^{n} \frac{1}{2^{i-1}} \le 1 + c$$

■

Claim 2: $Z \in \mathscr{M}$ **and** $\mu(Z) = 0$

Proof. Consider the increasing sequence $0 \le h_1^p \le \cdots \le h_n^p \le \cdots$ to which we apply the addendum to MCT. It is useful to note that

$$\int h_n^p d\mu = \|h_n\|_p^p \le (1+c)^p, \forall n \in \mathbb{N}$$

then,

$$\lim_n \int h_n^p d\mu \le (1+c)^p < \infty$$

Also, $\left\{x \in X : \lim_n h_n^p(x) = \infty\right\} = \{x \in X : \lim_n h_n(x) = \infty\}$ implies $Z \in \mathscr{M}, \mu(Z) = 0$. Then, by MCT, we have

$$\lim_n \int h_n^p d\mu = \int h^p d\mu \le (1+c)^p \Longrightarrow \|h\|_p \le 1+c$$

■

Claim 3: $f \in$ **Bor**$(X,\mathbb{R})$

Proof. We can write f as the pointwise limit of $(f_n \cdot (1-\chi_Z))_n$. Each one of them is in $\mathbf{Bor}(X,\mathbb{R})$. Then, the pointwise limit $f \in \mathbf{Bor}(X,\mathbb{R})$. ■

Claim 4: For $n_0 \in \mathbb{N}$ **consider** $g = f_{n_0} - f \in$ **Bor**$(X,\mathbb{R})$**, we have taht** $g \in \mathscr{L}^p(\mu)$ **and** $\|g\|_p \le \frac{1}{2^{n_0-1}}$

Proof.
1. Since $f_{n_0}, f \in \mathbf{Bor}(X,\mathbb{R})$, we have g as a linear combination of these two, thus, $g \in \mathbf{Bor}(X,\mathbb{R})$.
2. We note that by construction

$$|f_{n_0} - f_n| \le h_n - h_{n_0} \le h - h_{n_0}, \forall n \ge n_0 \in \mathbb{N}$$

Then, by monotonicity with $p \in [1,\infty)$ and $u_i = |f_i - f_{i-1}| \in \mathscr{L}^p(\mu)$,

$$\begin{aligned}
\|f_{n_0} - f_n\|_p &\le \|h_n - h_{n_0}\|_p \\
&= \left\| |f_1| + \sum_{i=2}^{n} u_i - \left(|f_1| + \sum_{i=2}^{n_0} u_i \right) \right\|_p \\
&= \left\| \sum_{i=n_0+1}^{n} u_i \right\|_p \\
&\underset{\text{Minkowski}}{\le} \sum_{i=n_0+1}^{n} \|u_i\|_p \\
&< \frac{1}{2^{n_0}} + \frac{1}{2^{n_0+1}} + \cdots + \frac{1}{2^{n-1}} \qquad\qquad \|u_i\|_p < \frac{1}{2^{i-1}} \\
&\le \sum_{i=n_0}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n_0-1}} \qquad\qquad (*)
\end{aligned}$$

Moreover, $h, h_{n_0} \in \mathscr{L}^p(\mu)$. Then, $|h - h_{n_0}|^p \in \mathscr{L}^1(\mu) \cap \mathbf{Bor}^+(X,\mathbb{R})$ since

$$\int |h - h_{n_0}|^p d\mu < \infty$$

and

$$|f_n - f_{n_0}|^p \le |h - h_{n_0}|^p, \forall n \in \mathbb{N}$$

Given $\lim_n f_n(x) = f(x) \iff \lim_n |f_{n_0}(x) - f_n(x)| = |f_{n_0}(x) - f(x)|$ pointwise, by LDCT, we get

$$
\begin{aligned}
&= (\lim_n \|f_n - f_{n_0}\|_p)^p \\
&= \lim_n \|f_n - f_{n_0}\|_p^p \qquad \text{sequential continuity} \\
&= \lim_n \int |f_n - f_{n_0}|^p d\mu \\
&= \int \lim_n |f_n - f_{n_0}|^p d\mu \\
&= \int |f_n - f_{n_0}|^p d\mu \\
&= \|f - f_{n_0}\|_p^p = \|g\|_p^p
\end{aligned}
$$

By comparison theorem and (*), we get

$$\|g\|_p \leq \frac{1}{2^{n_0-1}}$$

This also implies $g \in \mathscr{L}^p(\mu)$.

■

Not hard to see that $f \in \mathscr{L}^p(\mu)$ and our claim 4 can give us $\|f_n - f\| \to 0$ very easily. ■

Corollary 13.3.2 The seminormed vector space $(\mathscr{L}^p(\mu), \|\cdot\|_p)$ is complete.

Corollary 13.3.3 The normed linear space $(L^p(\mu), \|\cdot\|_p)$ is Banach space.

13.4 Modes of Convergence

Definition 13.4.1 — a.e. convergence. Let $(X, \mathscr{M}, \mu)$ be a measure space, and let f and $(f_n)_n$ be functions in $\mathbf{Bor}(X, \mathbb{R})$. We say that $(f_n)_n$ converges to f almost everywhere with respect to μ or a.e.-μ for short to mean that there exists a set $Z \in \mathscr{M}$ with $\mu(Z) = 0$ and such that

$$\lim_n f_n(x) = f(x), \forall x \in X \backslash Z$$

Corollary 13.4.1 Let $(X, \mathscr{M}, \mu)$ be a measure space, let p be in $[1, \infty)$ and let $(f_n)_n$ be a sequence of functions in $\mathscr{L}^p(\mu)$ such that $\|f_n - f_{n+1}\| < \frac{1}{2^n}, \forall n \in \mathbb{N}$. Then there exists a functions $f \in \mathscr{L}^p(\mu)$ such that $\lim_n f_n(x) = f$ a.e.$-\mu$.

Definition 13.4.2 — Convergence in Probability. Let $(X, \mathscr{M}, \mu)$ be a probability space, and let f and $(f_n)_n$ be functions in $\mathbf{Bor}(X, \mathbb{R})$. We say that $(f_n)_n$ converges to f in probability with respect to μ to mean that for any fixed $\sigma > 0$ one has

$$\lim_n \mu(\{x \in X : |f_n(x) - f(x)| \geq \sigma\}) = 0$$

Proposition 13.4.2 Let $(X, \mathscr{M}, \mu)$ be a probability space.

1. Let $p \in [1, \infty)$ and let f and $(f_n)_n$ be functions in $\mathscr{L}^p(\mu)$ such that $\lim_n \|f_n - f\|_p = 0$. Then, $(f_n)_n$ converges to f in probability

Proof. Fix $\sigma > 0$. For $n \in \mathbb{N}$, we let

$$A_n := \{x \in X : |f_n(x) - f(x)| \geq \sigma\}$$

we have $|f_n - f| \geq \sigma \chi_{A_n}$ in $\mathbf{Bor}^+(X, \mathbb{R})$. Then,

$$\|f_n - f\|_p^p \geq \sigma^p \mu(A_n)$$

Hence,

$$\mu(A_n) \leq \left(\frac{1}{\sigma}\|f_n - f\|_p\right)^p, \forall n \in \mathbb{N}$$

By squeeze theorem, we get $\lim_n \mu(A_n) = 0$ as required. ■

2. Let f and $(f_n)_n$ be functions in $\mathbf{Bor}(X, \mathbb{R})$ such that $(f_n)_n$ converges to f a.e.-μ. Then $(f_n)_n$ converges to f in probability.

Proof. Suppose $f_n \to f$ a.e. then there exists $X \supseteq E \in \mathscr{M}$ such that $\mu(E) = 0$ and for all $x \notin E$, we have $f_n \to f$ pointwise. Let $\delta > 0$, we let $B = X \backslash E$ and define

$$B_N = \bigcup_{n \geq N} \{x \in B : |f_n(x) - f(x)| > \delta\} = \{x \in B : \exists n \geq N : |f_n(x) - f(x)| > \delta\}$$

Then, $B_N \supseteq B_{N+1} \supseteq \cdots B' = \cap_{N=1}^{\infty} B_N \in \mathscr{M}$ since $f_n, f, |f_n - f| \in \mathbf{Bor}(X, \mathbb{R})$. Then, by continuity decreasing chain and $\mu(B_N) \leq \mu(X) = 1 < \infty$ since $B_N \subseteq X$,

$$\lim_N \mu(B_N) = \mu(B')$$

we claim that $B' \subseteq E$. Let $x \in B'$, then, $x \in B_N, \forall 1 \leq N$. In particular, this implies $f_n \not\to f$ at x since we cannot find a N such that $n \geq N$ implies $|f_n(x) - f(x)| < \delta$. Thus,

$$B' \subseteq E \implies \mu(B') = 0$$

Note that

$$\{x \in B : |f_N(x) - f(x)| > \delta\} \subseteq B_N$$

then,

$$\mu\{x \in B : |f_N(x) - f(x)| > \delta\} \leq \mu(B_N)$$

Then, by limit property (squeeze theorem),

$$\lim_N \mu\{x \in B : |f_N(x) - f(x)| > \delta\} = \lim_N \mu(B_N) = 0$$

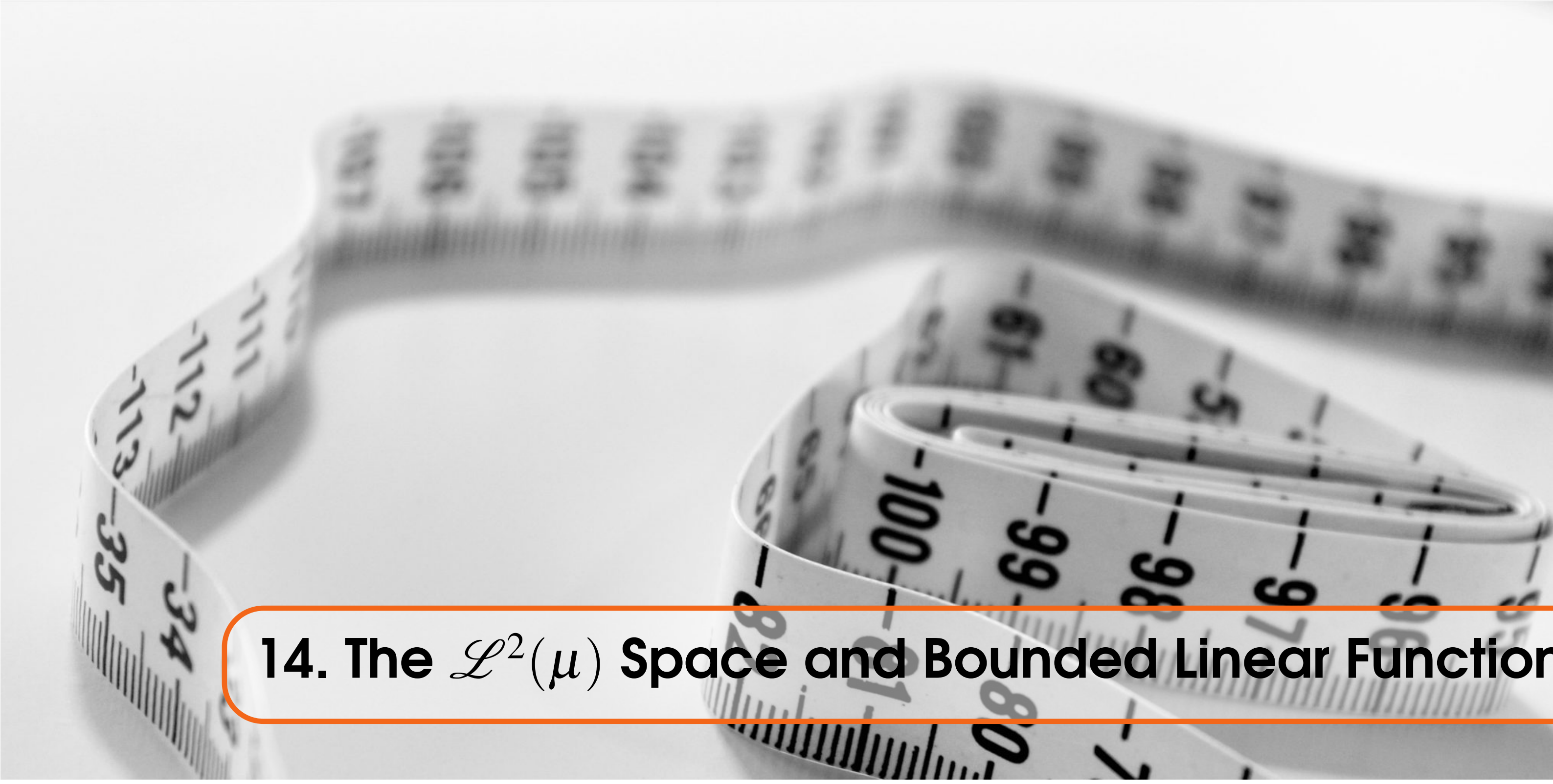
As for $\{x \in E : |f_N(x) - f(x)| > \delta\}$, it is clear that

$$\mu\{x \in E : |f_N(x) - f(x)| > \delta\} \leq \mu(E) = 0 \implies \mu\{x \in E : |f_N(x) - f(x)| > \delta\} = 0$$

Finally, by $\sigma-$additivity,

$$\begin{aligned}
&\lim_N \mu\{x \in X : |f_N(x) - f(x)| > \delta\} \\
=&\lim_N \mu\{x \in B : |f_N(x) - f(x)| > \delta\} + \mu\{x \in E : |f_N(x) - f(x)| > \delta\} \\
=&\lim_N \mu\{x \in B : |f_N(x) - f(x)| > \delta\} + \lim_N \mu\{x \in E : |f_N(x) - f(x)| > \delta\} = 0 + 0 = 0
\end{aligned}$$

■



14. The $\mathscr{L}^2(\mu)$ Space and Bounded Linear Function

14.1 The Inner Product Structure on $\mathscr{L}^2(\mu)$

Proposition 14.1.1

1. $f,g \in \mathscr{L}^2(\mu) \Longrightarrow f \cdot g \in \mathscr{L}^1(\mu)$. And we define

$$\langle f,g\rangle = \int fg d\mu$$

as the **inner product** of f and g. This is bilinear, symmetric, and non-negative definite.

2. The seminorm map $\|\cdot\|_2 : \mathscr{L}^2(\mu) \to [0,\infty)$ can be retrieved from the inner product by

$$\|f\|_2 = \sqrt{\langle f,f\rangle}, f \in \mathscr{L}^2(\mu)$$

3. Cauchy-Schwarz inequality:

$$|\langle f,g\rangle| \leq \|f\|_2\|g\|_2, \forall f,g \in \mathscr{L}^2(\mu)$$

R

1. Note that the inner product on $\mathscr{L}^2(\mu)$ is only claimed to be non-negative definite since positive definite will imply

$$f \in \mathscr{L}^2(\mu), \langle f,f\rangle = 0 \Longrightarrow f(x) = 0, \forall x \in X$$

which is not necessarily true.

2. Cauchy-Schwarz is a special case of Holder's inequality

Definition 14.1.1 — Bounded linear functional. Let $\varphi : \mathscr{L}^2(\mu) \to \mathbb{R}$ be a linear function. We say that φ is bounded to mean that there exists a constant $c \geq 0$ such that

$$|\varphi(f)| \leq c\|f\|_2, \forall f \in \mathscr{L}^2(\mu)$$

Proposition 14.1.2 Let a function $g \in \mathscr{L}^2(\mu)$ be given. We define $\varphi : \mathscr{L}^2(\mu) \to \mathbb{R}$ by the formula

$$\varphi(f) = \langle f, g \rangle = \int f g d\mu, f \in \mathscr{L}^2(\mu)$$

then φ is a bounded linear functional.

Proof.
1. **Linearity:** follows from the bilinearity of the inner product
2. **Boundedness:** we have
$$|\varphi(f)| = |\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$$
let $c = \|g\|_2$.

■

Theorem 8 Let $\varphi : \mathscr{L}^2(\mu) \to \mathbb{R}$ be a bounded linear functional. Then there exists $g \in \mathscr{L}^2(\mu)$ such that

$$\varphi(f) = \langle f, g \rangle = \int f g d\mu, \forall f \in \mathscr{L}^2(\mu)$$

R This is one of several theorems of Riesz which describe spaces of linear functionals. In particular, Riesz has such a theorem for every space $\mathscr{L}^p(\mu)$ with $p \in (1, \infty)$. It say

Theorem 9 — Riesz for $\mathscr{L}^p(\mu)$. If $\varphi : \mathscr{L}^p(\mu) \to \mathbb{R}$ is a linear functional which is bounded with respect to $\|\cdot\|_p$

The case of $p = 2$ is much easier to prove and considered as some kind of a lever which is used to get at the case of general $p \in (1, \infty)$. The order of proof is:

$$\textbf{Riesz for } \mathscr{L}^2(\mu) \implies \textbf{Radon-Nikodym} \implies \textbf{Riesz for } \mathscr{L}^p(\mu)$$

14.2 Proof of Riesz for $\mathscr{L}^2(\mu)$

Consider a the function $q : \mathbb{R} \to \mathbb{R}$ defined by putting $q(t) = \|v - tw\|^2$. Note that

$$\|v - tw\|^2 = \langle v - tw, v - tw \rangle = \|v\|^2 + \|w\|^2 t^2 - 2\langle v, w \rangle t$$

is a quadratic function with non-positive discriminant.
We will consider linear functionals $\varphi : \mathscr{V} \to \mathbb{R}$ ($(\mathscr{V}, \|\cdot\|)$ is a semi-normed space) Such a a linear functional φ is said to be bounded when there exists $c \geq 0$ such that

$$|\varphi(v)| \leq c\|v\|, \forall v \in V$$

We wish to pick $v_0 \in V$ such that $c = \|v_0\|$.

Lemma 14.3 — Parallelogram Law. For every $v, w \in \mathscr{V}$, we have

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

Lemma 14.4 Let $\varphi : \mathscr{V} \to \mathbb{R}$ be a linear functional which is not identically equal to 0. Consider the set $\mathscr{W} := \{w \in \mathscr{V} : \varphi(w) = 1\}$. Then:
1. The set $\mathscr{W}$ is not empty
2. Denoting $\alpha_0 := \inf\{\|w\| : w \in \mathscr{W}\}$, we have that $\alpha_0 > 0$.
3. There exists a vector $w_0 \in \mathscr{W}$ such that $\|w_0\| = \alpha_0$.

Proof. 1. Since φ is not identically 0, we pick $x \in \mathscr{V}$ such that $\varphi(x) \neq 0$. Let $w = \lambda x$ where $\lambda = \frac{1}{\varphi(x)}$. Then, $\varphi(w) = \varphi(\lambda x) = \lambda \varphi(x) = 1$. Hence $w \in \mathscr{W}$.
2. Let $c \geq 0$ such that $\varphi(v) \leq c\|v\|, \forall v \in V$. We have $c > 0$ otherwise identically 0. Let $\alpha_0 \geq \frac{1}{C} > 0$. Note that $\|w\| \geq \frac{1}{c}, \forall w \in \mathscr{W}$ and we are done.
3. We can pick a sequence $(w_n)_n$ in $\mathscr{W}$ such that

$$\alpha_0 \leq \|w_n\| < \left(1 + \frac{1}{n}\right)\alpha_0$$

using parallelogram law, we can show $(w_n)_n$ is Cauchy. Since $(\mathscr{V}, \|\cdot\|)$ is complete. There exists $w_0 \in \mathscr{V}$ that is the limit of $(w_n)_n$. By squeeze theorem, $\|w_0\| = \alpha_0$ and $\mathscr{W}$ is closed. Thus, $w_0 \in \mathscr{W}$.

■

Lemma 14.5 Let $\varphi : \mathscr{V} \to \mathbb{R}$ be a linear functional which is not identically equal to 0. Let $\mathscr{W} = \{w \in \mathscr{V} : \varphi(w) = 1\}$ and the vector $w_0 \in \mathscr{W}$ be as in the previous lemma and

$$\mathscr{K} := \{x \in \mathscr{V} : \varphi(x) = 0\}$$

is the kernel of φ. Then, $\langle w_0, x\rangle = 0, \forall x \in \mathscr{K}$.

Proof. For the sake of contradiction, say we could find an $x \in \mathscr{K}$ such that $\langle w_0, x\rangle \neq 0$. By replacing x with $-x$ if necessary, we can then assume that

$$\langle w_0, x\rangle < 0$$

Moreover, this means $x \neq 0$ in $\mathscr{K}$. Otherwise

$$2\langle w_0, 0\rangle = \langle w_0, 2 \cdot 0\rangle = \langle w_0, 0\rangle \implies \langle w_0, 0\rangle = 0$$

Consider the vectors of the form $w_0 + tx$ where $t > 0$ and consider $q : \mathbb{R} \to \mathbb{R}$ defined by $q(t) = \|w_0 + tx\|^2 - \|w_0\|^2$. Note that

$$\begin{aligned}
q(t) &= \|w_0 + tx\|^2 - \|w_0\|^2 \\
&= \langle w_0 + tx, w_0 + tx\rangle - \|w_0\|^2 \\
&= \|w_0\|^2 + 2\langle w_0, x\rangle t + \|x\|^2 t^2 - \|w_0\|^2 \\
&= 2\langle w_0, x\rangle t + \|x\|^2 t^2 \\
&= t(2\langle w_0, x\rangle + \|x\|^2 t)
\end{aligned}$$

note that $t = -\frac{2\langle w_0, x\rangle}{\|x\|^2} > 0$. Since $\|x\|^2 > 0$, we can pick $t_0 \in \left(0, -\frac{2\langle w_0, x\rangle}{\|x\|^2}\right)$ so that

$$q(t_0) < 0 \iff \|w_0 + t_0 x\|^2 < \|w_0\|^2 \iff \|w_0 + t_0 x\| < \|w_0\|$$

However,

$$\varphi(w_0 + t_0 x) = \varphi(w_0) + t_0 \varphi(x) = \varphi(w_0) = 1 \implies w_0 + t_0 x \in \mathscr{W}$$

However, $\|w_0\| = \alpha_0 = \inf\{\|w\| : w \in \mathscr{W}\} \geq \|w_0 + t_0 x\|$ yields a contradiction. Thus,

$$\langle w_0, x\rangle = 0, \forall x \in \mathscr{K}$$

■

Theorem 10 Let $\varphi : \mathscr{V} \to \mathbb{R}$ be a bounded linear functional. Then there exists $v_0 \in \mathscr{V}$ such that

$$\varphi(v) = \langle v, v_0 \rangle, \forall v \in \mathscr{V}$$

Proof. If φ is identically 0, we can choose $v_0 = 0 \in \mathscr{V}$. Suppose not identically equal to 0, we consider

$$\mathscr{W} = \{w \in \mathscr{V} : \varphi(w) = 1\}, \qquad \mathscr{K} = \{x \in \mathscr{V} : \varphi(x) = 0\}$$

Let $w_0 \in \mathscr{W}$ be as in the previous lemmas. We claim

$$v_0 = \frac{1}{\|w_0\|^2} w_0$$

Let $v \in \mathscr{V}$, let $\varphi(v) = \lambda$. Consider $x = v - \lambda w_0$. Then,

$$\varphi(x) = \varphi(v) - \lambda \varphi(w_0) = \lambda - \lambda = 0 \Longrightarrow x \in \mathscr{K}$$

Then, $\langle w_0, x \rangle = 0$. So, we have

$$0 = \langle w_0, v - \lambda w_0 \rangle = \langle w_0, v \rangle - \lambda \langle w_0, w_0 \rangle$$

then

$$\lambda = \frac{1}{\|w_0\|^2} \langle w_0, v \rangle = \left\langle \frac{1}{\|w_0\|^2} w_0, v \right\rangle = \langle v_0, v \rangle$$

This gives us $\varphi(v) = \lambda = \langle v, v_0 \rangle$ as required. ∎

15. Inequalities Between Positive Measures

15.1 Linear Combinations and Inequalities for Positive Measures on $(X,\mathscr{M})$

Definition 15.1.1 — Linear Combinations.

1. Given two positive measures $\mu,\nu:\mathscr{M}\to[0,\infty]$, we define $\mu+\nu:\mathscr{M}\to[0,\infty]$ by

$$(\mu+\nu)(A)=\mu(A)+\nu(A),\forall A\in\mathscr{M}$$

2. Given a positive measure $\mu:\mathscr{M}\to[0,\infty]$ and an $\alpha\in[0,\infty)$, we define $\alpha\mu:\mathscr{M}\to[0,\infty]$ by putting

$$(\alpha\mu)(A)=\alpha\mu(A),\forall A\in\mathscr{M}$$

   It is immediately verified that $\alpha\mu$ is a positive measure.
3. The map $\zeta:\mathscr{M}\to[0,\infty]$ defined by $\zeta(A)=0$ for all $A\in\mathscr{M}$ is a positive measure which is the neutral element for the addition operation and also has the property that $0\mu=\zeta$ for no matter what positive measure μ on $\mathscr{M}$

R It requires some rigour to have the following result:

$$\int fd(\mu+\nu)=\int fd\mu+\int fd\nu \text{ and } \int fd(\alpha\mu)=\alpha\int fd\mu$$

for every $f\in\mathbf{Bor}^+(X,\mathbb{R})$. We can do this for $\mathbf{Bor}(X,\mathbb{R})$ as well but requires the notion of "signed measure".

Definition 15.1.2 — Inequalities. Let $\mu,\nu:\mathscr{M}\to[0,\infty]$ be positive measures. We will write $\nu\le\mu$ to mean that one has

$$\nu(A)\le\mu(A),\forall A\in\mathscr{M}$$

then (with some work)

$$\nu \le \nu \Longrightarrow \int f d\nu \le \int f d\mu, \forall f \in \mathbf{Bor}^+(X,\mathbb{R})$$

Proposition 15.1.1 Let $\mu,\nu : \mathscr{M} \to [0,\infty]$ be positive measures such that $\nu \le \mu$, and let p be an exponent in $[1,\infty)$. We have an inclusion of $\mathscr{L}^p-$spaces:

$$\mathscr{L}^p(\mu) \subseteq \mathscr{L}^p(\nu)$$

Moreover, for every $f \in \mathscr{L}^p(\mu)$ we have the inequality

$$\|f\|_{p,\nu} \le \|f\|_{p,\mu}$$

Proof. Pick $|f|^p \in \mathbf{Bor}^+(X,\mathbb{R})$. ■

15.2 Integration of Densities from $\mathbf{Bor}^+(X,\mathbb{R}) \cap \mathscr{L}^1(\mu)$

Let $h \in \mathbf{Bor}^+(X,\mathbb{R})$ which we think of as a density to be used in connection to the measure μ. Let $\nu : \mathscr{M} \to [0,\infty]$ be

$$\nu(A) = \int_A h d\mu = \int \chi_A h d\mu \in [0,\infty], A \in \mathscr{M}$$

we have see that this ν is a positive measure and for every $f \in \mathbf{Bor}^+(X,\mathbb{R})$, we have

$$\int_X f(x) d\nu(x) = \int_X f(x)h(x)d\mu(x)$$

It seems like

$$d\nu(x) = h(x)d\mu(x)$$

but what does this mean?

R

1. ν does not change if the density h is replaced by $\hat{h} \in \mathbf{Bor}^+(X,\mathbb{R})$ such that $h = \hat{h}$ a.e.-μ. Since we have
$$\chi_A h = \chi_A \hat{h} \text{ a.e.-}\mu$$
2. If $h \in \mathbf{Bor}^+(X,\mathbb{R})$ has
$$0 \le h(x) \le 1, \forall x \in X$$
then, $\nu \le \mu$ since $\chi_A h \le \chi_A$

We shall assume that $h \in \mathbf{Bor}^+(X,\mathbb{R}) \cap \mathscr{L}^1(\mu)$ and $\nu(X) = \int h d\mu < \infty$.

Lemma 15.3
1. Let f be a function in $\mathscr{L}^1(\mu)$ for which we know that
$$\int_A f d\mu \ge 0, \forall A \in \mathscr{M}$$
Then it follows that $f \ge 0$ a.e.$-\mu$ (that is, there exists $Z \in \mathscr{M}$ with $\mu(Z) = 0$ and such that $f(x) \ge 0$ for all $x \in X \backslash Z)$
2. Let $g \in \mathscr{L}^1(\mu)$ such that
$$\int_A g d\mu \le \mu(A), \forall A \in \mathscr{M}$$
then $g \le 1$ a.e.μ (that is, there exists $Z \in \mathscr{M}$ with $\mu(Z) = 0$ and such that $g(x) \le 1$ for all $x \in X \backslash Z)$

Proof. For $n\in\mathbb{N}$, let

$$Z_n=\left\{x\in X : g(x)\geq 1+\frac{1}{n}\right\}$$

$Z_n\in\mathscr{M}$ since $g\in\mathscr{L}^1(\mu)\subseteq \mathbf{Bor}(X,\mathbb{R})$ where

$$Z_n=g^{-1}\left(\left[1+\frac{1}{n},\infty\right)\right)\in\mathscr{M}$$

Moreover, we note that since $g\in\mathscr{L}^1(\mu)$

$$\mu(Z_n)=\int_{Z_n}1d\mu\overset{|g(x)|>1,x\in Z_n}{<}\int_{Z_n}|g|d\mu\leq\int_X|g|d\mu<\infty$$

Now, for each $n\in\mathbb{N}$, we have

$$\int_{Z_n}1d\mu\leq\int_{Z_n}gd\mu\leq\mu(Z_n)\Longrightarrow\int_{Z_n}gd\mu=\mu(Z_n)$$

then,

$$0=\int_{Z_n}(g-1)d\mu\geq\int_{Z_n}\frac{1}{n}d\mu=\frac{1}{n}\mu(Z_n)\Longrightarrow\mu(Z_n)=0,\forall n\in\mathbb{N}$$

Now, we note that

$$Z=\bigcup_{i=1}^{\infty}Z_n=\{x\in X : g(x)>1\}\in\mathscr{M}$$

and $Z_1\subseteq Z_2\subseteq\cdots\subseteq Z$ forms an increasing chain. Since μ is a positive measure, by continuity along an increasing chain, we have

$$\mu(Z)=\lim_n\mu(Z_n)=\lim_n 0=0$$

We have found the desired $Z\in\mathscr{M}$ such that $g(x)\leq 1$ for $x\in X\backslash Z$. We conclude that $g\leq 1$ a.e.$-\mu$. ■

Proposition 15.3.1 Let f be a function in $\mathscr{L}^1(\mu)$ such that $\int_A fd\mu=0$ for every $A\in\mathscr{M}$. Then, it follows that $f=0$ a.e.-μ.

Proof. Try f and $-f$ with the previous lemma. ■

Corollary 15.3.2 Let $\nu:\mathscr{M}\to[0,\infty)$ be a finite positive measure, and suppose we are told that ν can be obtained out of μ via the procedure of integrating a density which was reviewed. Then, the density which gives ν is determined up to an a.e.-μ equality. If $h,\hat{h}\in\mathbf{Bor}^{+}(X,\mathbb{R})$ such that

$$\int_A hd\mu=\nu(A)=\int_A\hat{h}d\mu,\forall A\in\mathscr{M}$$

then, it follows that $h=\hat{h}$ a.e.$-\mu$.

Proof. Consider $h-\hat{h}=0$ a.e.-μ and invoke previous proposition. ■

R Since h is unique up to a.e.$-\mu$, it is formal to write

$$d\nu(x)=h(x)d\mu(x)\Longrightarrow\frac{d\nu(x)}{d\mu(x)}=h(x)$$

h is often referred to as the **Radon-Nikodym derivative of ν with respect to μ**

15.4 An Instance of the Radon-Nikodym Theorem

This is like the converse of ν's construction.

Theorem 11 — A version of the Radon-Nikodym Theorem. Consider the measurable space $(X, \mathscr{M})$ fixed throughout this lecture. Let $\mu, \nu : \mathscr{M} \to [0, \infty)$ be finite positive measure such that $\nu \leq \mu$. Then there exists an $h \in \mathbf{Bor}^{+}(X, \mathbb{R})$, with $0 \leq h(x) \leq 1$ for all $x \in X$, and such taht $d\nu(x) = h(x) d\mu(x)$.

Proof. We note that $\mathscr{L}^2(\mu) \subseteq \mathscr{L}^2(\nu) \subseteq \mathscr{L}^1(\nu)$. So every $f \in \mathscr{L}^2(\mu)$ is integrable with respect to ν, then we can define $\varphi : \mathscr{L}^2(\mu) \to \mathbb{R}$ by the formula

$$\varphi(f) = \int f d\nu, \forall f \in \mathscr{L}^2(\mu)$$

this is a linear functional and for every $f \in \mathscr{L}^2(\mu)$ we have

$$\begin{aligned}
|\varphi(f)| &= \left| \int f d\nu \right| \\
&\leq \int |f| d\nu \\
&= \|f\|_{1,\nu} \\
&\leq \|f\|_{2,\nu} (\nu(X))^{1/2} \\
&\leq \|f\|_{2,\mu} (\nu(X))^{1/2}
\end{aligned}$$

Thus, φ is a bounded linear functional. By Riesz representation theorem, there exists $h \in \mathscr{L}^2(\mu)$ such that

$$\varphi(f) = \int f h d\mu, \forall f \in \mathscr{L}^2(\mu)$$

Note that $h \in \mathscr{L}^1(\mu)$ as well. Then,

$$\varphi(\chi_A) = \int \chi_A d\nu = \nu(A) = \int \chi_A h d\mu, \forall A \in \mathscr{M}$$

Then,

$$0 \leq \int_A h d\mu \leq \mu(A), \forall A \in \mathscr{M}$$

then, by our previous lemma, there exists $Z \in \mathscr{M}$ such that $\mu(Z) = 0$ and $0 \leq h(x) \leq 1$ for all $x \in X \backslash Z$. The function

$$\hat{h} := h \chi_{X \backslash Z} \in \mathscr{L}^1(\mu)$$

and $0 \leq \hat{h}(x) \leq 1$ for all $x \in X$. ■

16. Radon-Nikodym Theorem

16.1 The notion of absolute continuity

Definition 16.1.1 — Absolute Continuity. Let $(X,\mathscr{M})$ be a measurable space, and let $\mu,\nu : \mathscr{M} \to [0,\infty]$ be positive measures. We say ν is absolutely continuous with respect to μ, denoted as $\nu << \mu$, to mean that

$$\textbf{(Abs-Cont)} \qquad A \in \mathscr{M}, \mu(A) = 0 \Longrightarrow \nu(A) = 0$$

■ **Example 16.1** Let $(X,\mathscr{M})$ be a measurable space and let $\mu,\nu : \mathscr{M} \to [0,\infty]$ be positive measures such that $d\nu(x) = h(x)d\mu(x)$, where h is a density in $\mathbf{Bor}^+(X,\mathbb{R})$. If $A \in \mathscr{M}$ has $\mu(A) = 0$ then it follows that

$$\nu(A) = \int_A h d\mu = \int_X \chi_A(x)h(x)d\mu(x) = \int_X 0 d\mu(x) = 0$$

since $\chi_A h = 0$ a.e.$-\mu$. Thus, $\nu << \mu$ ■

Proposition 16.1.1 Let $(X,\mathscr{M})$ be a measurable space, and let $\mu,\nu : \mathscr{M} \to [0,\infty]$ be positive measures, where ν is finite (that is, on has $\nu(X) < \infty$). Then the following statements are equivalent:

1. $\nu << \mu$
2. An $\varepsilon - \delta$ condition:

$$\text{For every } \varepsilon > 0, \text{ there exists a } \delta > 0 \text{ such that } A \in \mathscr{M}, \mu(A) < \delta \Longrightarrow \nu(A) < \varepsilon$$

Proof. 1. (1) $\Longrightarrow$ (2): For $\varepsilon > 0$ and the sake of contradiction, there is no such $\delta > 0$. Then, consider $\delta_n = \frac{1}{2^n}$ for each $n \in \mathbb{N}$. Then, there exists $B_n \in \mathscr{M}$ such that $\mu(B_n) < \frac{1}{2^n}$ and $\nu(B_n) \geq \varepsilon$. Consider

$$T = \bigcap_{n=1}^{\infty} C_n \in \mathscr{M}, C_n = \bigcup_{k=n}^{\infty} B_k$$

Claim 1: $\mu(T) = 0$

Proof. For $n \in \mathbb{N}$, the countable sub-additivity of μ gives us that

$$\mu(C_n) \leq \sum_{k=n}^{\infty} \mu(B_n) \leq \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}$$

We note that $T \subseteq C_n, \forall n \in \mathbb{N}$. Thus, $\mu(T) \le \mu(C_n) \le \frac{1}{2^{n-1}} \implies \mu(T) = 0$. ■

Claim 2: $\nu(T) \ge \varepsilon$

Proof. $(C_n)_n$ forms a decreasing chain in $\mathscr{M}$. Since it is assumed that ν is a fintie measure (!) we have in particular $\nu(C_1) < \infty$, we can invoke the continuity along a decreasing chain to conclude that $\nu(T) = \lim_n \nu(C_n) \ge \lim_n \nu(B_n) \ge \lim_n \varepsilon = \varepsilon$. ■

Combine claim 1 and 2, we have a contradiction to $\nu << \mu$.

2. $(2) \implies (1)$: for $n \in \mathbb{N}$, setting $\varepsilon = \frac{1}{n}$ gives us a $\delta_n > 0$ such that every $A \in \mathscr{M}$ with $\mu(A) < \delta_n$ is sure to have $\nu(A) < \frac{1}{n}$. For $A \in \mathscr{M}$, we have

$$\mu(A) = 0 \implies \mu(A) < \delta_n, \forall n \in \mathbb{N} \implies \nu(A) < \frac{1}{n}, \forall n \in \mathbb{N} \implies \nu(A) = 0$$

Thus, $\nu << \mu$.

■

R The direction of $(2) \implies (1)$ does not require $\nu(X) < \infty$. But $(1) \implies (2)$ indeed requires it. Otherwise, consider the measurable space $(X, \mathscr{M})$ where $X = \mathbb{N}$ and $\mathscr{M} = 2^{\mathbb{N}}$. Let $\mu, \nu : \mathscr{M} \to [0, \infty]$ be the positive measures which correspond to the weight-functions $w, w' : \mathbb{N} \to [0, \infty)$ defined by

$$w(n) = \frac{1}{2^n}, \forall n \in \mathbb{N}, \qquad w'(n) = 1, \forall n \in \mathbb{N}$$

For every $A \subseteq \mathbb{N}$ we have

$$\mu(A) = \sum_{n \in A} \frac{1}{2^n} \qquad \nu(A) = |A| \in [0, \infty]$$

Note that $\nu << \mu$ since

$$A \subseteq \mathbb{N}, \mu(A) = 0 \implies A = \emptyset \implies \nu(A) = 0$$

but for $\varepsilon = \frac{1}{2}$ and any $\delta > 0$, $\mu(A) < \delta$ means A is not empty, thus $\nu(A) \ge 1$ and cannot have $\nu(A) < \frac{1}{2}$. This fails since $\nu(\mathbb{N}) = \infty$.

Theorem 12 — Radon-Nikodym for finite measure $\nu << \mu$. Let $(X, \mathscr{M})$ be a measurable space and let $\mu, \nu : \mathscr{M} \to [0, \infty)$ be finite positive measures such that $\nu << \mu$. Then there exists a density $h \in \mathbf{Bor}^+(X, \mathbb{R}) \cap \mathscr{L}^1(\mu)$ such that $d\nu(x) = h(x) d\mu(x)$.

Proof. We have finite positive measures $\mu, \nu : \mathscr{M} \to [0, \infty)$ such that $\nu << \mu$. Let g be a connecting function between μ, ν and let $N = \{x \in X : g(x) = 1\}$. We know that $\mu(N) = 0$. Since $\nu << \mu$ $\nu(N) = 0$ as well. Consider the function $\tilde{g} := g(1 - \chi_N) \in \mathbf{Bor}^+(X, \mathbb{R})$. So $\tilde{:}X \to \mathbb{R}$ acts by

$$\tilde{g}(x) = \begin{cases} g(x) & x \in X \backslash N \\ 0 & x \in N \end{cases}$$

so $0 \le \tilde{g}(x) \le 1, \forall x \in X$. This is still a connecting function between μ, ν that is we have

$$\int f \tilde{g} d\mu = \int f(1 - \tilde{g}) d\nu, \forall f \text{ bounded in } \mathbf{Bor}^+(X, \mathbb{R})$$

Then,

$$\begin{aligned} \int f\tilde{g}d\mu &= \int fgd\mu && f\tilde{g} = fg \text{ a.e.-}\mu \\ &= \int f(1-g)d\nu \\ &= \int f(1-\tilde{g})d\nu && f(1-g) = f(1-\tilde{g}) \text{ a.e.-}\mu \end{aligned}$$

Now, let $h : X \to \mathbb{R}$ be defined by $h(x) = \frac{\tilde{g}(x)}{1-\tilde{g}(x)}$. Note that $h \in \mathbf{Bor}^+(X,\mathbb{R})$. Now, we check: for $A \in \mathscr{M}$ and every $n \in \mathbb{N}$, let us put

$$f_n := \chi_A(1+\tilde{g}+\tilde{g}^2+\cdots+\tilde{g}^n)$$

which forms a sequence of bounded functions in $\mathbf{Bor}^+(X,\mathbb{R})$. Then,

$$\int f_n\tilde{g}d\mu = \int f_n(1-\tilde{g})d\nu$$

Then,

$$f_n(x) \nearrow \frac{\chi_A(x)}{1-\tilde{g}(x)}, \forall x \in X$$

Similarly, $(f_n\tilde{g})_n$ and $(f_n(1-\tilde{g}))_n$ are also increasing sequences in $\mathbf{Bor}^+(X,\mathbb{R})$ with pointwise limits equal to $\chi_A h$ and χ_A respectively. By MCT, we have

$$\int \chi_A h d\mu = \int \chi_A d\nu = \nu(A)$$

■

Definition 16.1.2 — Concentrated and Mutually singular. Let $(X,\mathscr{M})$ be a measurable space.

1. Let $\mu : \mathscr{M} \to [0,\infty]$ be a positive measure, and let P be the set in $\mathscr{M}$. We say that μ is concentraded on P to mean that $\mu(X \backslash P) = 0$
2. Let $\mu, \nu : \mathscr{M} \to [0,\infty]$ be positive measures. We say that μ, ν are mutually singular, denoted as $\mu \perp \nu$, to mean that there exists some sets $P, Q \in \mathscr{M}$ with $P \cap Q = \emptyset$ such that μ is concentrated on P and ν is concentrated on Q.

Theorem 13 — Lebesgue Decomposition. Let $(X,\mathscr{M},\mu)$ be a measure space, where $\mu(X) < \infty$. Then any other finite positive measure $\nu : \mathscr{M} \to [0,\infty)$ can be written as $\nu = \nu_1 + \nu_2$ where $\nu_1, \nu_2 : \mathscr{M} \to [0,\infty)$ are finite positive measures such that $\nu_1 << \mu, \nu_2 \perp \mu$.

Proof. Let g be a connecting function between μ, ν and let $N = \{x \in X : g(x) = 1\}$ with $\mu(N) = 0$. Let $\nu_1, \nu_2 : \mathscr{M} \to [0,\infty)$ be defined by putting

$$\nu_1(A) = \nu(A \cap (X \backslash N)) \qquad \nu_2(A) = \nu(A \cap N), \forall A \in \mathscr{M}$$

Note that ν_1, ν_2 are finite positive measures on $\mathscr{M}$ such that $\nu_1 + \nu_2 = \nu$.

1. $\nu_1 << \mu$: by our previous exercise, we have $\mu(A) = 0 \implies \nu_1(A) = 0$
2. $\nu_2 \perp \mu$: we know that $\mu(N) = 0$ and this can be construed as saying that μ is concentrated on $X \backslash N$. Meanwhile,

$$\nu_2(X \backslash N) = \nu((X \backslash N) \cap N) = \nu(\emptyset) = 0$$

Thus, ν_2 concentrates on N and μ, ν_2 are mutually singular where $P = X \backslash N$ and $Q = N$. ■

R It is not hard to show that the measures ν_1, ν_2 from the Lebesgue Decomposition are uniquely determined.

16.2 The Trick of the Connecting Function

Proposition 16.2.1 — Trick of the Connecting Function. Let $(X, \mathscr{M})$ be a measurable space and let $\mu, \nu : \mathscr{M} \to [0, \infty)$ be finite positive measures. There exists a function $g \in \mathbf{Bor}(X, \mathbb{R})$ with $0 \leq g(x) \leq 1$ for all $x \in X$ and such that

$$\int fgd\mu = \int f(1-g)d\nu, \forall f \text{ bounded function in } \mathbf{Bor}^{+}(X, \mathbb{R})$$

In what follows, a function $g \in \mathbf{Bor}(X, \mathbb{R})$ with these properties will be called by the (ad-hoc) name of connecting function between μ, ν.

Proof. Consider the finite positive measure $\rho = \mu + \nu$. It is clear that $\nu \leq \rho$, so using the previous version of Radon-Nikdym, we have a density $g \in \mathbf{Bor}^{+}(X, \mathbb{R})$ such that $0 \leq g(x) \leq 1$ for all $x \in X$ and such that $d\nu(x) = g(x)d\rho(x)$. For a bounded function $f \in \mathbf{Bor}^{+}(X, \mathbb{R})$, for which we will verify the equality stated

$$\begin{aligned} \int fd\nu &= \int fgd\rho \\ &= \int fgd\mu + \int fgd\nu \end{aligned}$$

we can write $f = fg + f(1-g)$ and

$$\int fgd\nu + \int f(1-g)d\nu = \int fgd\mu + \int fgd\nu$$

Since f is bounded and μ, ν are finite measures. We have

$$\int f(1-g)d\nu = \int fgd\mu$$

■

Lemma 16.3 Let μ, ν and g be as in the previous propositiion and consider the set $N := \{x \in X : g(x) = 1\}$. Then, $\mu(N) = 0$.

Proof. Let $f = \chi_N$ which is bounded and in $\mathbf{Bor}(X, \mathbb{R})$. By definition, we have $\chi_N g = \chi_N$ while $\chi_N(1-g) = 0$. Then,

$$\int \chi_N d\mu = \int 0 d\nu = 0 \Longrightarrow \mu(N) = 0$$

■

R We cannot quite get $\nu(N) = 0$ from the proposition.

Exercise 16.1 Consider the framework and notation in the lemma. Prove the following statement:

if $A \in \mathscr{M}$ has $\mu(A) = 0$, then it follows that $\nu(A \cap (X \backslash N)) = 0$.

Proof. Recall that g is the connecting function between two finite positive measures $\mu, \nu : \mathscr{M} \to [0, \infty)$ and $N = \{x \in X : g(x) = 1\}$ with $\mu(N) = 0$. Suppose $A \in \mathscr{M}$ has $\mu(A) = 0$. Consider $A \cap (X \backslash N) \in \mathscr{M}$ and $\chi_{A \cap (X \backslash N)} \in \mathbf{Bor}^+(X, \mathbb{R})$ and it is inherently bounded. Then, by the Trick of the connecting function,

$$\int \chi_{A \cap (X \backslash N)} g d\mu = \int \chi_{A \cap (X \backslash N)} (1 - g) d\nu$$

In particular, since $0 \leq g(x) \leq 1$, we note that

$$0 \leq \int \chi_{A \cap (X \backslash N)} g d\mu \leq \int \chi_{A \cap (X \backslash N)} d\mu = \mu(A \cap (X \backslash N)) \leq \mu(A) = 0$$

Thus,

$$\int \chi_{A \cap (X \backslash N)} (1 - g) d\nu = \int_{A \cap (X \backslash N)} (1 - g) d\nu = 0$$

For the sake of contradiction, say $\nu(A \cap (X \backslash N)) > 0$. Then, we note that when $x \in A \cap (X \backslash N)$ we have $0 \leq g(x) < 1$ and $0 < 1 - g(x) \leq 1$. Thus, $1 - g$ is positive on $A \cap (X \backslash N)$. But this yields a contradiction to the lemma proved below. Thus, $\nu(A \cap (X \backslash N)) = 0$.

Lemma 16.4 Let $\nu : \mathscr{M} \to [0, \infty)$ be a finite positive measure. For $E \in \mathscr{M}$ we have $\nu(E) > 0$. If $f : X \to \mathbb{R}$ is a function positive on E and $f \in \mathbf{Bor}(X, \mathbb{R})$. Then, $\int_E f d\nu > 0$.

Proof. Since $f \in \mathbf{Bor}(X, \mathbb{R})$, the following sets are in $\mathscr{M}$,

$$E_n = \left\{ x \in E : f(x) > \frac{1}{n} \right\}, \forall n \in \mathbb{N}$$

Then, $E = \cup_{n=1}^{\infty} E_n$. Then, since $\nu(E) > 0$, there exists $N \in \mathbb{N}$ such that $\nu(E_N) > 0$, then we have

$$\int_E f d\nu \geq \int_{E_N} f d\nu > \int_{E_n} \frac{1}{N} d\nu = \frac{\nu(E_N)}{N} > 0$$

as claimed. ■

■

17. Direct Product of Two Finite Positive Measures

17.1 Direct Product of Two Measurable Spaces

We fix two measurable spaces $(X,\mathscr{M})$ and $(Y,\mathscr{N})$.

Definition 17.1.1 — Measurable Rectangle and Direct Product. $X \times Y := \{(x,y) : x \in X, y \in Y\}$. For $A \in \mathscr{M}$ and $B \in \mathscr{N}$, the set

$$A \times B := \{(x,y) : x \in A, y \in B\} \subseteq X \times Y$$

is referred to as a measurable rectangle. We will denote

$$\mathscr{R} := \{A \times B : A \in \mathscr{M} \text{ and } B \in \mathscr{N}\} \subseteq 2^{X \times Y}$$

The collection $\mathscr{R}$ of measurable rectangles generates a $\sigma-$algebra of subsets of $X \times Y$, which is called the direct product of $\mathscr{M}$ and $\mathscr{N}$, and is denoted as $\mathscr{M} \times \mathscr{N}$:

$$\mathscr{M} \times \mathscr{N} := \sigma - \mathrm{Alg}(\mathscr{R})$$

In this way, we have a measurable space $(X \times Y, \mathscr{M} \times \mathscr{N})$, which is called the direct product of $(X,\mathscr{M})$ and $(Y,\mathscr{N})$.

Exercise 17.1 Consider two separable metric spaces (X,d') and (Y,d'') and by putting

$$\mathscr{M} = \mathscr{B}_X, \qquad \mathscr{N} = \mathscr{B}_Y$$

Then, $X \times Y$ can be turned into a metric space by putting for instance

$$d((x_1,y_1),(x_2,y_2)) := \max\left(d'(x_1,x_2), d''(y_1,y_2)\right), \forall x_1,x_2 \in X, y_1,y_2 \in Y$$

Since the norms on a metric space are equivalent, different distance functions on $X \times Y$ yield the same topology and hence the same Borel $\sigma-$algebra. Prove that the direct product of $(X,\mathscr{B}_X)$ and $(Y,\mathscr{B}_Y)$ is $(X\times Y,\mathscr{B}_{X\times Y})$, where $\mathscr{B}_{X\times Y}$ is the Borel $\sigma-$algebra of the metric space $(X\times Y,d)$.

Proof. ■

Proposition 17.1.1 The collection of measurable rectangles $\mathscr{R}$ introduced is a semi-algebra.

Proof.
1. **(Semi-AS1)** $A=\emptyset\in\mathscr{M}$ leads to $\emptyset\times B\in\mathscr{R}$
2. **(Semi-AS3)** for $A_1,A_2\in\mathscr{M}$ and $B_1,B_2\in\mathscr{N}$ we have

$$(A_1\times B_1)\cap(A_2\times B_2)=(A_1\cap A_2)\times(B_1\cap B_2)$$

with $A_1\cap A_2\in\mathscr{M}$ and $B_1\cap B_2\in\mathscr{N}$.

3. **(Semi-AS2)** For $R=A\times B\in\mathscr{R}$, we have

$$(X\times Y)\backslash R=T_1\cup T_2$$

where $T_1=(X\backslash A)\times Y$ and $T_2=A\times(Y\backslash B)$ in $\mathscr{R}$. We note that $T_1\cap T_2=\emptyset$.

■

R Let $\mathscr{U}=\mathrm{Alg}(\mathscr{R})$. Explicitly,

$$\mathscr{U}=\{U\subseteq X\times Y:\exists k\in\mathbb{N}\text{ and }R_1,\cdots,R_k\in\mathscr{R}\text{ s.t. }R_1\cup\cdots\cup R_k=U\}$$

alternatively,

$$\mathscr{U}=\left\{U\subseteq X\times Y:\exists l\in\mathbb{N}\text{ and }S_1,\cdots,S_l\in\mathscr{R}\text{ with }S_i\cap S_j=\emptyset,i\neq j\text{ s.t. }S_1\cup\cdots\cup S_l=U\right\}$$

Then,

$$\sigma-\mathrm{Alg}(\mathscr{U})=\sigma-\mathrm{Alg}(\mathscr{R})=\mathscr{M}\times\mathscr{N}$$

17.2 Direct Product of Two Finite Positive Measure

Consider measure spaces $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$.

Theorem 14 There exists a positive measure $\rho:\mathscr{M}\times\mathscr{N}\to[0,\infty)$, uniquely determined, such that

$$\rho(A\times B)=\mu(A)\cdot\nu(B),\forall A\in\mathscr{M},B\in\mathscr{N}$$

Definition 17.2.1 — Direct Product. The positive measure ρ on $\mathscr{M}\times\mathscr{N}$ is called the direct product of the measures μ,ν and it is denoted by $\mu\times\nu$. The resulting measure space

$$(X\times Y,\mathscr{M}\times\mathscr{N},\mu\times\nu)$$

is called the direct product spaces of the measure spaces $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$.

Lemma 17.3 Suppose we have sets A and $(A_i)_i$ from $\mathscr{M}$ and sets B and $(B_i)_i$ from $\mathscr{N}$ such that

$$(A_i\times B_i)\cap(A_j\times B_j)=\emptyset,i\neq j\in\mathbb{N}$$

and

$$\bigcup_{i=1}^{\infty}A_i\times B_i=A\times B$$

consider the functions g and $(g_n)_n$ in $\mathbf{Bor}^+(Y,\mathbb{R})$ which are defined as follow:

$$g = \mu(A)\cdot\chi_B \text{ and } g_n = \sum_{i=1}^{n}\mu(A_i)\cdot\chi_{B_i}, n\in\mathbb{N}$$

Then, $g_n \nearrow g$.

Proof. For $y\in Y$, the sequence of non-negative numbers $(g_n(y))_n$ is clearly increasing, and we need

$$\lim_n g_n(y) = \begin{cases}\mu(A) & y\in B\\ 0 & y\in Y\backslash B\end{cases}$$

The case when $y\in Y\backslash B$ is immediate, because it has $g_n(y)=0,\forall n\in\mathbb{N}$. When $y\in B$, we want to show

$$\lim_n g_n(y) = \mu(A)$$

We note that we can write

$$\begin{aligned} g_n(y) &= \sum_{i=1}^{n}\left(\int_X \chi_{A_i}(x)d\mu(x)\right)\chi_{B_i}(y)\\ &= \int_X\left(\sum_{i=1}^{n}\chi_{A_i}(x)\chi_{B_i}(y)\right)d\mu(x)\end{aligned}$$

Let $h_n(x)=\sum_{i=1}^{n}\chi_{A_i}(x)\chi_{B_i}(y)$. Note that for every $x\in X$, the sequence of non-negative numbers $(h_n(x))_n$ is clearly increasing. We claim that $h_n\nearrow h$ where

$$h(x) = \begin{cases}\chi_A(x) & x\in A\\ 0 & x\in X\backslash A\end{cases}$$

The case when $x\in X\backslash A$ is clear since $h_n(x)=0,\forall n\in\mathbb{N}$. On the other hand, for $x\in A$, since $\cup_{i=1}^{\infty}A_i\times B_i = A\times B$ and $(x,y)\in A\times B$, we can find $(x,y)\in A_N\times B_N$. This order pair is only in $A_N\times B_N$ since $(A_i\times B_i)\cap(A_j\times B_j)=\emptyset, i\neq j$. Then,

$$\lim_n h_n(x) = \sum_{i=1}^{N}\chi_{A_i}(x)\chi_{B_i}(y) = \chi_{A_N}(x)\chi_{B_N}(y) = 1$$

since this is true for any $x\in A$. Then,

$$\lim_n h_n(x) = \chi_A(x)$$

Then, we note that $h_n\in\mathbf{Bor}^+(X,\mathbb{R})$. By MCT, we have

$$\mu(A) = \int_X \chi_A(x)d\mu(x) = \int_X \lim_n h_n(x)d\mu(x) = \lim_n\int_X h_n(x)d\mu(x)$$

This translates to

$$\lim_n\int_X h_n(x)d\mu(x) = \lim_n g_n(y) = \mu(A)$$

as required. ■

Proposition 17.3.1 1. It makes sense to define a set-function $\rho_{oo}:\mathscr{R}\to[0,\infty)$ by putting

$$\rho_{oo}(A\times B) = \mu(A)\nu(B), \forall A\in\mathscr{M}, B\in\mathscr{N}$$

Proof. Not so much to prove. Just check well-definedness. ■

2. Let R and $(R_i)_i$ be sets from $\mathscr{R}$ such that $\cup_{i=1}^{n} R_i = R$ and such $R_i \cap R_j = \emptyset$ for $i \neq j$. Then it follows that

$$\rho_{oo}(R) = \sum_{i=1}^{\infty} \rho_{oo}(R_i)$$

Proof. Let $R = A \times B$ with $A \in \mathscr{M}$ and $B \in \mathscr{N}$. For $i \in \mathbb{N}$, let us write

$$R_i = A_i \times B_i, A_i \in \mathscr{M}, B_i \in \mathscr{N}$$

Then, by the lemma proven, we have $g_n \nearrow g$, by MCT, we have

$$\lim_n \int_Y g_n(y) d\nu(y) = \int_Y g(y) d\nu(y) = \mu(A)\nu(B) = \rho_{oo}(R)$$

while

$$\begin{aligned} \lim_n \int_Y g_n(y) d\nu(y) &= \lim_n \sum_{i=1}^{n} \int_Y \mu(A_i) \chi_{B_i}(y) d\nu(y) \\ &= \lim_n \sum_{i=1}^{n} \mu(A_i)\nu(B_i) \\ &= \sum_{i=1}^{\infty} \rho_{oo}(R_i) \end{aligned}$$

■

3. The set-function ρ_{oo} has the property (RespDec).

Proof. Direct consequence of 2. ■

Proposition 17.3.2 There exists a finitely additive set-function $\rho_o : \mathscr{U} \to [0, \infty)$ such that

$$\rho_o(A \times B) = \mu(A)\nu(B), \forall A \in \mathscr{M}, B \in \mathscr{N}$$

Proof. We can extend ρ_{oo} to a finitely additive set function ρ_o ■

Lemma 17.4 Suppose that $R \in \mathscr{R}$ is written as $R = \cup_{n=1}^{\infty} V_n$, with V_n's from $\mathscr{U}$ such that $V_m \cap V_n = \emptyset, m \neq n$. Then it follows $\rho_{oo}(R) = \sum_{n=1}^{\infty} \rho_o(V_n)$.

Proof. Immediate by extension. ■

Proposition 17.4.1 ρ_o is a pre-measure.

Proof. Let U and $(U_n)_n$ in $\mathscr{U}$ such that $\cup_{n=1}^{\infty} U_n = U$ with $U_m \cap U_n = \emptyset, m \neq n$. Let $U = \cup_{i=1}^{k} R_i$ disjoint union in $\mathscr{R}$. Pick an i and observe that

$$R_i = R_i \cap U = R_i \cap \left(\bigcup_{n=1}^{\infty} U_n \right) = \bigcup_{n=1}^{\infty} (R_i \cap U_n) = \bigcup_{n=1}^{\infty} V_n^{(i)}$$

where for every $n \in \mathbb{N}$ we put $V_n^{(i)} = R_i \cap U_n \in \mathscr{U}$. By the lemma proved above, we have

$$\rho_{oo}(R_i) = \sum_{n=1}^{\infty} \rho_o(R_i \cap U_n)$$

Finally,

$$U = \sum_{i=1}^{k} \sum_{n=1}^{\infty} \rho_o(R_i \cap U_n) = \sum_{n=1}^{\infty} \sum_{i=1}^{k} \rho_o(R_i \cap U_n) = \sum_{n=1}^{\infty} \rho_o(U_n)$$

■

18. Calculations related to $\mu \times \nu$

18.1 Measurability for Slices of Sets and Functions

Definition 18.1.1

$$\mathbf{Bor}(X \times Y, \mathbb{R}) := \{f : X \times Y \to \mathbb{R} : f \text{ is } (\mathscr{M} \times \mathscr{N})/\mathscr{B}_{\mathbb{R}} - \text{measurable}\}$$

Definition 18.1.2 — Slicing for Sets. Let E be a subset of $X \times Y$. Then:

1. **The vertical slice cut through E at level x:** $\forall x \in X$, $E_{[x]} := \{y \in Y : (x,y) \in E\}$. So $E_{[x]}$ is a subset of Y
2. **The horizontal slice cut through E at level y:** $\forall y \in Y$, $E^{[y]} := \{x \in X : (x,y) \in E\}$. So $E^{[y]}$ is a subset of X.

Definition 18.1.3 — Slicing for Functions. Let $f : X \times Y \to \mathbb{R}$ be a function. Then:

1. **Vertical slice of f at level x:** $\forall x \in X$, $f_{[x]} : Y \to \mathbb{R}$ defined by $f_{[x]}(y) = f(x,y), \forall y \in Y$
2. **Horizontal slice of f at level y:** $\forall y \in Y$, $f^{[y]} : X \to \mathbb{R}$ defined by $f^{[y]}(x) = f(x,y), \forall x \in X$

Lemma 18.2
1. Fix an $x \in X$ and consider the map $V_x : Y \to X \times Y$ defined by $V_x(y) = (x,y), \forall y \in Y$. Then, V_x is $\mathscr{N}/(\mathscr{M} \times \mathscr{N})-$measurable
2. Fix an $y \in Y$ and consider the map $H_y : X \to X \times Y$ defined by $H_y(x) = (x,y), \forall x \in X$. Then, H_y is $\mathscr{M}/(\mathscr{M} \times \mathscr{N})-$measurable

Proof. The proof of 1 and 2 are similar. We focus on 1. Let $x \in X$ and consider $V_x : Y \to X \times Y$. We know that $\mathscr{M} \times \mathscr{N} = \sigma - \mathrm{Alg}(\mathscr{R})$. Let $S = A \times B \in \mathscr{R}$, then

$$V_x^{-1}(S) = \{y \in Y : (x,y) \in S\} = \begin{cases} B & x \in A \\ \emptyset & x \in X \backslash A \end{cases}$$

Since both $B, \emptyset \in \mathscr{N}$, we have the result. ■

Proposition 18.2.1
1. For a set $E \in \mathscr{M} \times \mathscr{N}$ we have that:

$$E_{[x]} \in \mathscr{N}, \forall x \in X, E^{[y]} \in \mathscr{M}, \forall y \in Y$$

Proof. Let $E\in\mathscr{M}\times\mathscr{N}$. We know that

$$E_{[x]}=V_x^{-1}(E),\forall x\in X,E^{[y]}=H_y^{-1}(E),\forall y\in Y$$

By lemma, we have $V_x^{-1}(E)\in\mathscr{N},\forall x\in X$ and $H_y^{-1}(E)\in\mathscr{M},y\in Y$. ■

2. For a function $f\in\mathbf{Bor}(X\times Y,\mathbb{R})$ we have that:

$$f_{[x]}\in\mathbf{Bor}(Y,\mathbb{R}),\forall x\in X,f^{[y]}\in\mathbf{Bor}(X,\mathbb{R}),\forall y\in Y$$

Proof. Let $f\in\mathbf{Bor}(X\times Y,\mathbb{R})$. We note that

$$f_{[x]}=f\circ V_x,\forall x\in X,f^{[y]}=f\circ H_y,\forall y\in Y$$

By tool no.1, we have the desired result. ■

18.3 How to Calculate $\mu\times\nu(E)$ via Slicing

Proposition 18.3.1 Let $(X,\mathscr{M},\mu),(Y,\mathscr{N},\nu)$ be measure spaces, where μ,ν are finite measures. We consider the direct product space $(X\times Y,\mathscr{M}\times\mathscr{N},\mu\times\nu)$, as defined before. The values of the measure $\mu\times\nu$ can be obtained as integrals with respect to μ or to ν:

1. Let $E\in\mathscr{M}\times\mathscr{N}$. For $x\in X$, let $E_{[x]}\in\mathscr{N}$, we put

$$u(x):=\nu(E_{[x]})\in[0,\infty)$$

then: the function $u:X\to[0,\infty)$ defined in this way belongs to the space $\mathbf{Bor}^+(X,\mathbb{R})$, and has $\int_X ud\mu=\mu\times\nu(E)$

2. Let $E\in\mathscr{M}\times\mathscr{N}$. For $y\in Y$, consider $E^{[y]}\in\mathscr{M}$, and we put

$$v(y):=\mu(E^{[y]})\in[0,\infty)$$

Then: the function $v:Y\to[0,\infty)$ defined in this way belongs to the space $\mathbf{Bor}^+(Y,\mathbb{R})$, and has $\int_Y vd\nu=\mu\times\nu(E)$

Proof. Let $Z=X\times Y$. The collection of measurable rectangles $\mathscr{R}$ is a $\pi-$system of subsets of Z, where it is also clear that $Z\in\mathscr{R}$. We invoke the $\pi-\lambda$ system result to get $\mathscr{G}\supseteq\sigma-\mathrm{Alg}(\mathscr{R})$. ■

R We summarize the proposition as: for $E\in\mathscr{M}\times\mathscr{N}$, we have

$$\int_X\nu(E_{[x]})d\mu(x)=\mu\times\nu(E)=\int_Y\mu(E^{[y]})d\nu(y)$$

Exercise 18.1 Let $(X,\mathscr{M},\mu)$ be a probability space, and let $f\in\mathbf{Bor}^+(X,\mathbb{R})$. Suppose f is bounded, and let $c>0$ such that $f(x)<c,\forall x\in X$. Consider the function $\varphi:[0,c]\to[0,1]$ defined by

$$\varphi(t):=\mu(\{x\in X:f(x)\geq t\}),0\leq t\leq c$$

1. Prove that φ is a decreasing function with $\varphi(0)=1$ and $\varphi(c)=0$
2. Prove that φ is integrable with respect to the Lebesgue measure on $[0,c]$, and has

$$\int_0^c\varphi(t)dt=\int_X f(x)d\mu(x)$$

Proof. 1. Let $0\leq t_1\leq t_2\leq c$. Then,

$$\{x\in X:f(x)\geq t_2\}\subseteq\{x\in X:f(x)\geq t_1\}$$

where $f \in \mathbf{Bor}^+(X, \mathbb{R})$. Thus,

$$\varphi(t_1) = \mu(\{x \in X : f(x) \geq t_1\}) \geq \mu(\{x \in X : f(x) \geq t_2\}) = \varphi(t_2)$$

moreover, $\{x \in X : f(x) \geq 0\} = f^{-1}(0) = X$ since $f \in \mathbf{Bor}^+(X, \mathbb{R})$. This means $\varphi(0) = \mu(X) = 1$ since μ is a probability measure. Now, we note that

$$\{x \in X : f(x) \geq c\} = X \backslash \{x \in X : f(x) < c\}$$

Since μ is a probability measure and $f(x) < c, \forall x \in X$, we have

$$\varphi(c) = \mu(\{x \in X : f(x) \geq c\}) = \mu(X) - \mu(\{x \in X : f(x) < c\}) = \mu(X) - \mu(X) = 0$$

2. consider the direct product $\mu \times \lambda$ of the measures μ given and λ, the Lebesgue measure on $(Y = [0,c], \mathscr{N} = \mathscr{B}_{[0,c]})$. Consider $E = \{(x,y) \in X \times Y : f(x) \geq y\}$. Then, for each $x \in X$, we have $E_{[x]} = \{y \in [0,c] : f(x) \geq y\}$ and for $y \in [0,c]$, we have $E^{[y]} = \{x \in X : f(x) \geq y\}$. We note that $g(x,y) = f(x) - y$ is $\mathscr{M} \times \mathscr{N} / \mathscr{B}_{[-c,c]}-$measurable by Proposition 7.10. Then, $E = g^{-1}([0,c]) \in \mathscr{M} \times \mathscr{N}$. Then, by Proposition 19.3, we have each $x \in X$ $E_{[x]} \in \mathscr{N}$ and for each $y \in Y$ $E^{[y]} \in \mathscr{M}$. Finally, by Proposition 19.5, $\varphi \in \mathbf{Bor}^+(Y, \mathbb{R})$, and

$$\int_Y \varphi(t)dt = \int_0^c \varphi(t)dt = \int_X \lambda(E_{[x]})d\mu(x) = \int_X (f(x) - 0)d\mu(x) = \int_X f(x)d\mu(x)$$

■

Lemma 18.4 We say $E \in \mathscr{M} \times \mathscr{N}$ is a "good" if the statement in Proposition 18.3.1 holds true for E: the function $u : X \to [0, \infty)$ defined by $u(x) = \nu(E_{[x]})$ is in $\mathbf{Bor}^+(X, \mathbb{R})$ and has $\int_X u d\mu = \mu \times \nu(E)$. Let $\mathscr{G} := \{E \in \mathscr{M} \times \mathscr{N} : G \text{ is good}\}$, then:

1. $E \in \mathscr{G} \implies (X \times Y) \backslash E \in \mathscr{G}$
2. $E_1, E_2 \in \mathscr{G}, E_1 \cap E_2 = \emptyset \implies E_1 \cup E_2 \in \mathscr{G}$
3. If $(E_n)_n$ is an increasing chain of sets from $\mathscr{G}$, then $\cup_{n=1}^{\infty} E_n \in \mathscr{G}$

Lemma 18.5 Every set $R = A \times B \in \mathscr{R}$ is good.

Proof. For $E = A \times B$ with $A \in \mathscr{M}, B \in \mathscr{N}$, the function $u : X \to [0, \infty)$ considered previously has the form

$$u(x) = \nu(E_{[x]}) = \begin{cases} \nu(B) & x \in A \\ 0 & x \in X \backslash A \end{cases}$$

That is, we have $u = \nu(B)\chi_A \in \mathbf{Bor}^+(X, \mathbb{R})$ with $\int_X u d\mu = \nu(B)\mu(A) = \mu \times \nu(E)$ ■

Lemma 18.6 — A Form of Dynkin's $\pi - \lambda$ Theorem. Let Z be a non-empty set and let $\mathscr{R}$ be a collection of subsets of Z such that $Z \in \mathscr{R}$ and such that $\mathscr{R}$ is a $\pi-$system.
On the other hand, suppose that $\mathscr{G}$ is a collection of subsets of Z such that $\mathscr{G} \supseteq \mathscr{R}$ (those good ones). Under these hypothesis, we have $\mathscr{G} \supseteq \sigma - \text{Alg}(\mathscr{R})$.

Proof. Check Chpater 3 of the "Probability and Measure" textbook by Billingsley. ■

18.7 Theorem of Tonelli

Theorem 15 — Tonelli. Let $(X,\mathscr{M},\mu),(Y,\mathscr{N},\nu)$ be measure spaces, where μ,ν are finite measures, and consider the product space $(X\times Y,\mathscr{M}\times\mathscr{N},\mu\times\nu)$. Let f be a bounded function in $\mathbf{Bor}^{+}(X,\mathbb{R})$. Then the integral of f with respect to the measure $\mu\times\nu$ can be calculated by the formula

$$\int_{X\times Y} f(x,y)d\mu\times\nu(x,y) = \int_X \left(\int_Y f(x,y)d\nu(y)\right)d\mu(x)$$

or by the formula

$$\int_{X\times Y} f(x,y)d\mu\times\nu(x,y) = \int_Y \left(\int_X f(x,y)d\mu(x)\right)d\nu(y)$$

19. Sigma-Finite Product Measures

19.1 Direct Product of Sigma-Finite Positive Measures

Definition 19.1.1 — $\sigma-$finite. We say that a measure space $(X, \mathscr{M}, \mu)$ is $\sigma-$finite to mean that

> there exists an increasing chain $(U_n)_n$ of sets from $\mathscr{M}$ such that $\cup_{n=1}^{\infty} U_n = X$ and such that $\mu(U_n) < \infty$ for every $n \in \mathbb{N}$.

such chain of U_n will be called as **exhausting chain of sets of finite measures**

■ **Example 19.1**
1. A finite measure space $(X, \mathscr{M}, \mu)$ is in particular sigma-finite, we can let $U_n = X$ for all $n \in \mathbb{N}$.
2. The measure space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \mu_{Leb})$ is sigma-finite. More generally, any of the Lebesgue-Stieltjes measures $\mu : \mathscr{B}_{\mathbb{R}} \to [0, \infty]$ is sure to be sigma-finite, where our exhausting chain $(U_n)_n$ of sets of finite measure can be taken to be $U_n = [-n, n], \forall n \in \mathbb{N}$.

■

Definition 19.1.2 — Restriction to a Subspace.
1. Let $(X, \mathscr{M})$ be a measurable space and let Z be a non-empty set in $\mathscr{M}$. We denote

$$\mathscr{M}_{\downarrow Z} : \{A \in \mathscr{M} : A \subseteq Z\}$$

note that $\mathscr{M}_{\downarrow Z}$ is a sigma-algera. We call $(Z, \mathscr{M}_{\downarrow Z})$ the restriction of $(X, \mathscr{M})$ to the subspace Z

2. Let $(X, \mathscr{M}, \mu)$ be a measure space, and let Z be a non-empty set in $\mathscr{M}$. We let the sigma-algebra $\mathscr{M}_{\downarrow Z}$ as above, and we denote by $\mu_{\downarrow Z} : \mathscr{M}_{\downarrow Z} \to [0, \infty]$ the restriction of μ to $\mathscr{M}_{\downarrow Z}$. It is immediate that $\mu_{\downarrow Z}$ is a positive measure. The measure space $(Z, \mathscr{M}_{\downarrow Z}, \mu_{\downarrow Z})$ is called the restriction of $(X, \mathscr{M}, \mu)$ to the subspace Z.

Proposition 19.1.1 Let (Z, P) be a measurable space. Let $(Z_n)_n$ be an increasing chain of sets from $\mathscr{P}$ such that $\cup_{n=1}^{\infty} Z_n = Z$, and for every $n \geq 1$ let us denote

$$\mathscr{P}_n := \mathscr{P}_{\downarrow Z_n} = \{E \in \mathscr{P} : E \subseteq Z_n\}$$

suppose we are given a family of finite positive measures $(\rho_n : \mathscr{P}_n \to [0,\infty))_n$ which are coherent, in the sense that we have

$$\rho_{n+1}|\mathscr{P}_n = \rho_n, \forall n \geq 1$$

then there exists a positive measure $\rho : \mathscr{P} \to [0,\infty]$, uniquely determined, such that

$$\rho|\mathscr{P}_n = \rho_n, \forall n \geq 1$$

Exercise 19.1 Let $(X,\mathscr{M})$ and $(Y,\mathscr{N})$ be measurable spaces, and consider their product $(X \times Y, \mathscr{M} \times \mathscr{N})$, as discussed before. Consider two non-empty sets $U \in \mathscr{M}$ and $V \in \mathscr{N}$, and look at the restricted spaces $(U, \mathscr{M}_{\downarrow U})$ and $(V, \mathscr{N}_{\downarrow V})$, for which we can consider the direct product $(U \times V, \mathscr{M}_{\downarrow U} \times \mathscr{N}_{\downarrow V})$. Prove that

$$(\mathscr{M} \times \mathscr{N})_{\downarrow U \times V} = \mathscr{M}_{\downarrow U} \times \mathscr{N}_{\downarrow V}$$

■

Theorem 16 Let $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ be sigma-finite spaces. Consider the direct product $(X \times Y, \mathscr{M} \times \mathscr{N})$ of the measurable spaces $(X,\mathscr{M})$ and $(Y,\mathscr{N})$. There exists a positive measure $\mu \times \nu : \mathscr{M} \times \mathscr{N} \to [0,\infty]$, uniquely determined, such that

$$\mu \times \nu(A \times B) = \mu(A)\nu(A) \in [0,\infty], \forall A \in \mathscr{M}, B \in \mathscr{N}$$

$\mu \times \nu$ is sigma-finite.

19.2 Sigma-Finite Tonelli Theorem

Theorem 17 — Tonelli. Let f be a function in $\mathbf{Bor}^+(X \times Y, \mathbb{R})$. For every $x \in X$ the slice $f_{[x]}$ of f is in $\mathbf{Bor}^+(Y,\mathbb{R})$, so we can consider the integral $\int_Y f_{[x]}(y)d\nu(y) \in [0,\infty]$. Let us denote

$$T := \left\{ x \in X : \int_Y f_{[x]}(y)d\nu(y) = \infty \right\}$$

then the following hold:

1. $T \in \mathscr{M}$
2. Suppose that $\mu(T) > 0$, then $\int_{X \times Y} f(x,y)d\mu \times \nu(x,y) = \infty$
3. Suppose that $\mu(T) = 0$, Consider the function $F : X \to [0,\infty)$ defined by

$$F(x) = \begin{cases} \int_Y f_{[x]}(y)d\nu(y) & x \in X \setminus T \\ 0 & x \in T \end{cases}$$

   Then, $F \in \mathbf{Bor}^+(X,\mathbb{R})$ and we have

$$\int_X F(x)d\mu(x) = \int_{X \times Y} f(x,y)d\mu \times \nu(x,y)$$

19.3 Fubini Theorem

Definition 19.3.1 Let $f \in \mathbf{Bor}(X \times Y, \mathbb{R})$.

1. Suppose we found a set $W \in \mathscr{M}$ such that $\mu(W) = 0$ and such that the slice $f_{[x]} \in \mathscr{L}^1(\nu)$ for every $x \in X \setminus W$.

2. Suppose moreover that the function $F : X \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} \int_Y f_{[x]}(y) d\nu(y) & x \in X \backslash W \\ 0 & x \in W \end{cases}$$

then $F \in \mathscr{L}^1(\mu)$. Then, the integral $\int_X F(x) d\mu(x) \in \mathbb{R}$ is said to be an iterated integral of f, where we first integrate y and then integrate x. We can write the integral of $F(x)$ as

$$\int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x)$$

Exercise 19.2 Let $f \in \mathbf{Bor}(X \times Y, \mathbb{R})$. Come up with a quantity that

$$\int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y)$$

■

R

1. Suppose someone else find another $\hat{W}$ and define

$$\hat{F}(x) = \begin{cases} \int_Y f_{[x]}(y) d\nu(y) & x \in X \backslash \hat{W} \\ 0 & x \in \hat{W} \end{cases}$$

then we will have $F = \hat{F}$ a.e.$-\mu$. This does not change the iterated integral.
2. So far, we don't need $f \in \mathscr{L}^1(\mu \times \nu)$.
3. There are examples where some order of integration will not work or two ways of integrations will result in different answers.

Theorem 18 — Fubini. Let $f \in \mathscr{L}^1(\mu \times \nu)$. The the iterated integrals for f exist, in both orders, and are equal to $\int_{X \times Y} f(x,y) d\mu \times \nu(x,y)$

R The converse of Fubini is not generally true. We could have iterated integrals of f exist but $f \notin \mathscr{L}^1(\mu \times \nu)$. To have

$$\int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y) = \int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x)$$

we can use Tonelli to verify that

$$\int_{X \times Y} |f| d\mu \times \nu(x,y) < \infty$$

20. Upgrade Radon-Nikodym

20.1 Upgrade to Sigma-Finite in Radon-Nikodym

Theorem 19 — Radon-Nikodym. Let $(X, \mathscr{M})$ be a measure