Optimal Execution with Non-linear Market Impact

ACTSC972 Final Project

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Agenda

1 Continuous Propagator Model for Nonlinear Market Impact

2 Homotopy Analysis Method

3 Numerical Nonlinear Programming

4 Empirical Tests

5 Possible Extensions

Continuous Propagator Model for Nonlinear Market Impact

Propagator Models

Discrete Propagator Model (Bouchaud 2004)[1]

$$S_j = S_0 + \sum_{k=0}^{j-1} \left(\mathcal{G}(j, k, v_k) + \eta_k \right), \quad \mathcal{G}(j, k, v_k) \cong f(v_k) \cdot G(j, k)$$

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Continuous Propagator Model (Gatheral 2009)[5]

$$S_t = S_0 + \int_0^t f(v_s) G(t-s) ds + \int_0^t \sigma dZ_s$$

Some Definitions

- $f(\cdot)$: (instantaneous) market impact function
- $G(\cdot)$: decay kernel
- Π: an execution strategy/scheme

$$\int_0^T v(t)dt = X$$

where X is the amount of shares to execute.

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Expected Cost Function $C(\Pi)$

$$C(\Pi) = \mathbb{E}\left[\int_0^T v(t)(S_t - S_0)dt\right] = \int_0^T v(t) \int_0^t f(v(s))G(t-s)dsdt$$

also known as **expected implementation shortfall**.

A Constrained Optimization Problem

We want to find such v that solves the following problem

$$\begin{array}{ll} \min & C(\Pi) \\ \text{s.t.} & \int_0^T v(t)dt = X \end{array}$$

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Sign requirement on $v(t)$?

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Sign requirement on $v(t)$**?**

Consistency requirement on f **and** G**?**

Principle of no-dynamic-arbitrage: price manipulation is not possible.

$$\nexists \Pi \;\; \text{s.t.} \;\; C(\Pi) < 0$$

Why Consider Nonlinear *f*, *G*?

- Empirical results on *f*: power-law [9] or log impact function [1]

Why Consider Nonlinear f, G?

- Empirical results on f: power-law [9] or log impact function [1]
- Observed autocorrelation of trade signs [1] motivates a decay model

$$G(t-s) = (t-s)^{-\gamma}, \quad \gamma > 0$$

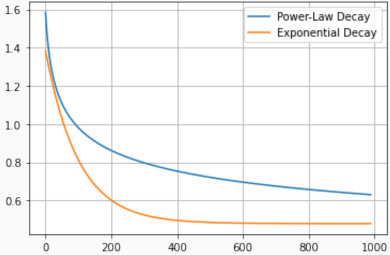

Figure: Power law vs. Exponential

Why Not Exponential Decay?

Obizhaeva and Wang 2005 [11]

Assume $f(x) \propto x$ and $G(t-s) = \exp\{-\rho(t-s)\}, \rho > 0$.

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Lemma (Gatheral[5])

If G is an exponential decay kernel, price manipulation is possible unless $f(x) \propto x$.

Clean Linear Case f

Assume f to be linear with the proposed G

Theorem (Gatheral, Schied[6])

$$v(t) = \frac{c}{[t(T-t)]^{\frac{1-\gamma}{2}}}$$

where

$$c = X / \left(\sqrt{\pi} \left(\frac{T}{2} \right)^{\gamma} \frac{\Gamma((1+\gamma)/2)}{\Gamma(1+\gamma/2)} \right)$$

This would be a benchmark solution called **GSS**.

Symmetric Execution

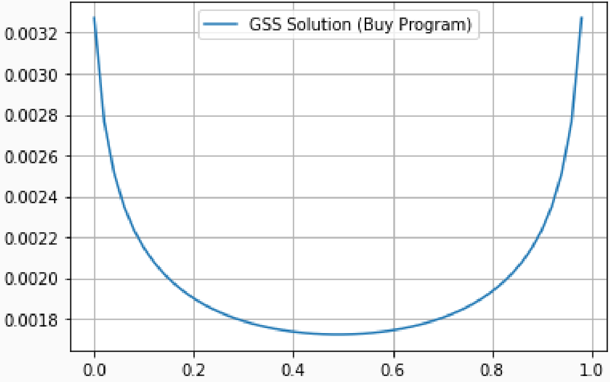

Figure: Linear impact function solution $v(t)$ with $T = 1$ and $X = 0.1, \gamma = 0.5$

Our Nonlinear Impact Function f

Nonlinear Impact Function f

We shall work with the following f

$$f(x) = \text{sign}(x)|x|^{\delta}, \delta > 0$$

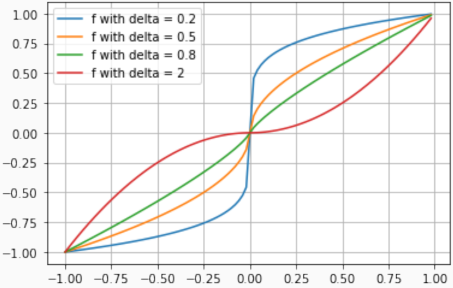

Figure: Impact function f with various $\delta > 0$ gives different nonlinearity properties

Our Problem to Solve

Theorem (Dang 2014[4])

The expected cost minimization problem is equivalent to solving the integral equation

$$\int_0^t f(v(s))G(t-s)ds + f'(v(t)) \int_t^T v(s)G(s-t)ds = \lambda$$

where λ is derived based on the constraint.

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Simplification

$$\int_0^T G(|t-s|)F(v(s),v(t))ds = \lambda$$

where $F(v(s),v(t)) = \begin{cases} f(v(s)) & s \leq t \\ v(s)f'(v(t)) & s > t \end{cases}$

What δ, γ to consider?

Theorem (Gatheral[5])

Dynamic-no-arbitrage imposes that

$$\gamma + \delta \geq 1$$

Homotopy Analysis Method

Homotopy Analysis Method (Liao)[8]

General Nonlinear Equation

$$\mathcal{N}[v(t)] = 0$$

Zero-Order Deformation Equation

$$(1-p)\mathcal{L}[\phi(t;p) - v^0(t)] = p\hbar H(t)\mathcal{N}[\phi(t;p)]$$

- $p \in [0,1]$ homotopy parameter *
- $\hbar \neq 0$ convergence control *
- $H(t)$ auxiliary function, $H(t) = 1$
- $\mathcal{L}$ linear operator $\mathcal{L} =$identity

We let $\mathcal{N}[v(t)] = -\lambda + \int_0^T G(|t-s|)F(v(s), v(t))ds$
Note that $\phi(t;0) = v^0(t)$ and $\phi(t;1) = v(t)$

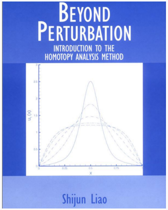

Figure: Cover page[7]

Homotopy Series and Homotopy Derivatives

Consider the Maclaurin series of ϕ with respect to p,

$$\phi(t;p) = v^0(t) + \sum_{m=1}^{\infty} v^m(t)p^m, \quad v^m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;p)}{\partial p^m}\bigg|_{p=0}$$

Since $\phi(t;1) = v(t)$, we have

$$v(t) = v^0(t) + \sum_{m=1}^{\infty} v^m(t)$$

We don't really know $\phi(t;p)$, how to compute $v^m(t)$?

*m*th-order Deformation Trick

We can take partials on the zero-order deformation equation iteratively to get the *m*th-order deformation equation

$$v^m(t) - v^{m-1}(t) = \hbar R^m, \quad R^m = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(t;p)]}{\partial p^{m-1}}\bigg|_{p=0}$$

This tells us

$$v^m(t) = v^{m-1}(t) + \hbar R^m$$

Approximate Solution

Theorem (Approximate Solution Convergence (Liao 1997)[8])

The nth-order approximate solution is

$$v^{(n)}(t) = v^{0}(t) + \sum_{m=1}^{n} v^{m}(t)$$

and

$$\lim_{n\to\infty} v^{(n)}(t) = v(t)$$

Discrete Homotopy Analysis Method

DHAM

Consider N subintervals of $[0, T]$ with $t_i = iT/N$. A discrete solution is $v_i = v(t_i)$ for $i \in \{1, \cdots, N\}$.

$$\int_0^T G(|t-s|)F(v(s), v(t))ds = \lambda \longrightarrow \sum_{j=1}^{N} G_{ij}F_{ij}(v) = \lambda$$

with

$$F_{ij}(v) = \begin{cases} f(v_j) & j \leq i \\ v_j f'(v_i) & j > i \end{cases}, \quad G_{ij} = \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} G(|t-s|)dsdt$$

DHAM Continues...

In our case, $G(t-s) = (t-s)^{-\gamma}$, we have

$$G_{ij} = \begin{cases} \frac{1}{(1-\gamma)(2-\gamma)}\left(\frac{T}{N}\right)^{2-\gamma}\{(i-j+1)^{2-\gamma} - 2(i-j)^{2-\gamma} + (i-j-1)^{2-\gamma}\} & i > j \\ \frac{1}{(1-\gamma)(2-\gamma)}\left(\frac{T}{N}\right)^{2-\gamma}\{(j-i+1)^{2-\gamma} - 2(j-i)^{2-\gamma} + (j-i-1)^{2-\gamma}\} & i < j \\ \frac{2}{(1-\gamma)(2-\gamma)}\left(\frac{T}{N}\right)^{2-\gamma} & i = j \end{cases}$$

Constraint becomes

$$\int_0^T v(t)dt = X \longrightarrow \sum_{i=1}^{N} v_i = \frac{NX}{T}$$

DHAM Continues...

The mth-order discrete homotopy derivatives at time t_i

$$v_i^m = v_i^{m-1} + \hbar \sum_{j=1}^{N} G_{ij} F_{ij}^{m-1}$$

where $F_{ij}^{m-1} = \frac{\partial^{m-1}}{\partial p^{m-1}} F\left(\sum_{k=0}^{\infty} v_j^k p^k, \sum_{l=0}^{\infty} v_i^l p^l \right) \Big|_{p=0}$.

nth-order approximated solution at time t_i

$$v_i^{(n)} = v_i^{(0)} + \sum_{m=1}^{n} v_i^m$$

DHAM Almost There...

Discrete expected cost

$$C(\Pi) \approx C(v^{(n)}) = \sum_{i=1}^{N}\sum_{j=1}^{N} v_i^{(n)} A_{ij} f\left(v_j^{(n)}\right) = v^{(n)} A f(v^{(n)})^\top$$

where

$$A_{ij} = \begin{cases} 0 & j > i \\ G_{ii}/2 & i = j \\ G_{ij} & j < i \end{cases}$$

Initial guess $v^{(0)}$ candidates

1. **VWAP:** $v_i^{(0)} = \frac{X}{TN}$
2. **GSS (linear solution)**

DHAM Results

Implementation Details

- Python 3.8.8[12]
- Symbolic computation of partials in SymPy[10]
- Dynamic programming for efficiency
- Obtained 6th order approximate solution

Quick result on DHAM implementation time.

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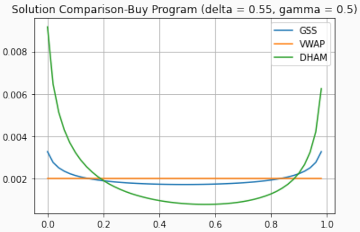
Lemma (Computational Complexity of DHAM)

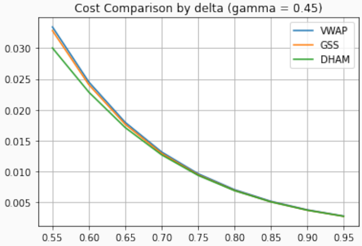
Let M be the pre-specified order of approximation. Using the polynomial dynamic programming approach, the DHAM solution is attainable in $\mathcal{O}(N^2 2^M)$ time.

Solution Comparison

We computed solutions on the (δ, γ) grid with $\delta \in \{0.55, 0.60, \cdots, 0.95\}$ and $\gamma \in \{0.45, 0.50, 0.55\}$ with $N = 50, X = 0.1, T = 1$.



(a) Solution shape comparison



(b) Solution expected cost comparison

Numerical Nonlinear Programming

Really a Nonlinear Program

$$\begin{array}{ll} \min & C(v) = vAf(v)^\top \\ \text{s.t.} & \sum_{i=1}^{N} v_i = \frac{XN}{T} \\ & v_i \in \mathbb{R} \end{array}$$

Implemented in Python[12].

Sequential Quadratic Programming

Really a Nonlinear Program

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 & v_i \in \mathbb{R}
\end{array}
$$

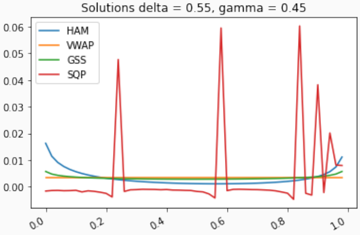
Implemented in Python[12].

Remarks

- When $0 < \gamma < 1$, not convex
- Aim to find local minima
- Constraints on v_i's sign will make it suffer convergence problem
- Iterative algorithm with an initial point

SQP Results

We computed SQP solutions on the same (δ, γ) grid with **GSS** solution as an initial guess.



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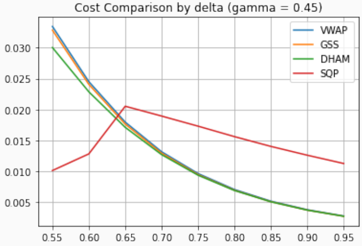
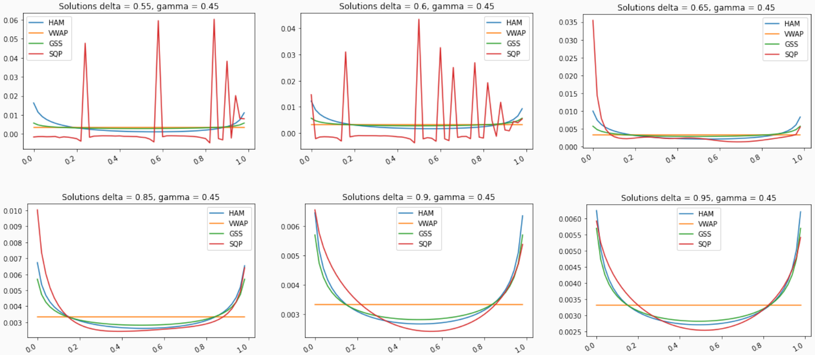


(b) Solution expected cost comparison

Convergence of Solutions



Transaction-triggered Price Manipulation

For a liquidation strategy v with $\int_0^T v(t)dt = X$, there exists t such that $v(t) < 0$.

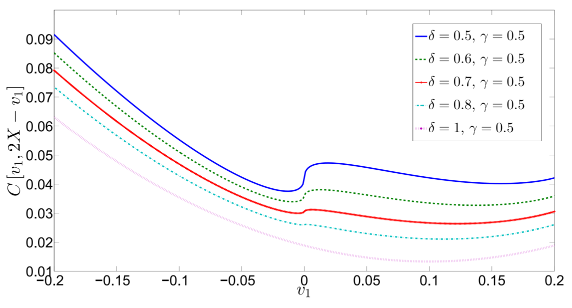


Figure: $N = 2$ example. SQP can find local minima by incuring $v_i < 0$ in a buy program near $\gamma + \delta = 1$[3]

Empirical Test

Market Data Calibration

Calibration of f

- Order data required (rare and expensive to find)
- Regression between ”impact” and ”volume”

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Propagator Model Calibration (Enzo 2012)[2]

Let $\mathbb{T}$ be an ordered partition of $[0, T]$ of size N. For $I_n \in \mathbb{T}, 1 \leq n \leq N$, the **aggregated volume** is

$$v_n = \sum_{i \in I_n} v_i^{tt}$$

the **normalized volume** is

$$v_n^* = \frac{\sum_{i \in I_n} v_i^{tt}}{\sum_{i \in I_n} |v_i^{tt}|}$$

Discrete Propagator Model

Recall that

$$S_j = S_0 + \sum_{k=0}^{j-1} \left(f(v_k^*) \cdot G(j-k) + \eta_k\right)$$

then,

$$r_j = S_{j+1} - S_j = \sum_{k=1}^{j-1} f(v_{j-k}^*)(G(k+1) - G(k)) + \eta_j$$

My Own Experiment

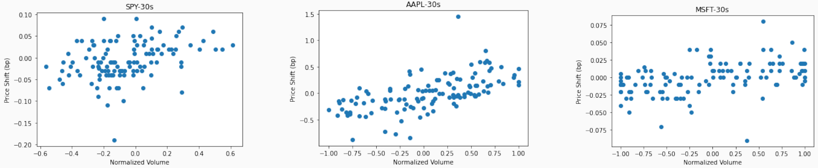

Figure: Scatter plots between r_j and v_j^* using 30-second interval among AAPL, MSFT, SPY on 2012-06-21 9:30AM-10:30AM EST. *Sample free data provided by NASDAQ Historical TotalView-ITCH*

Shape of f?

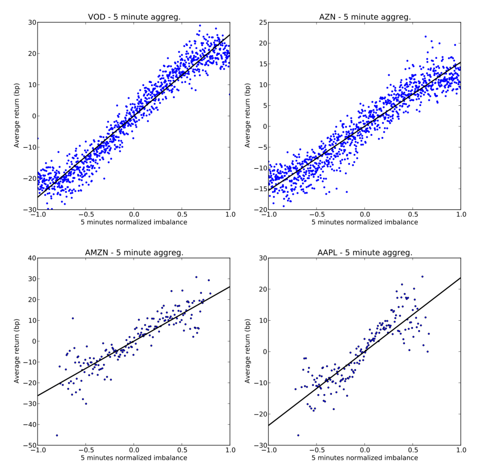

Figure: Similar tests on VOD, AZN, AMZN, AAPL with much larger data set (Enzo, Fabrizio 2012)[2]

TSLA Sunshine Event

CNBC

WATCH LIVE

TECH

Elon Musk sold about $6.9 billion in Tesla stock this week

PUBLISHED SAT, NOV 13 2021·9:11 AM EST UPDATED SAT, NOV 13 2021·10:00 AM EST

Todd Haselton
@ROBOTODD

WATCH LIVE

KEY POINTS

- Tesla CEO Elon Musk's trust sold about $1.2 billion in Tesla stock, according to financial filings posted Friday evening. He sold about $6.9 billion worth of stock in the company over the course of the week.
- Musk still holds more than 166 million shares in the company.
- Tesla stock declined 15.4% for the week, marking the company's worst one-week performance in 20 months.

TSLA Sunshine Event

Application of Execution Solutions

We get price series of TSLA from 11/08/2021 2:30PM to 11/12/2021 4:00PM. See how the price dynamic evolves based on our execution scheme versus what happened in real life. (Very hand-waving)

Simulated Result

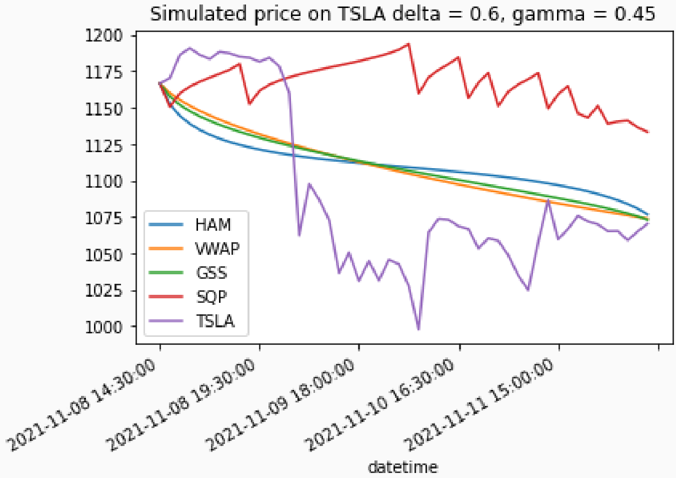

Figure: SQP solution seems suggesting the impossible

Possible Extensions

Non-exhaustive List

- Spread cost regularization (Lasso). See [3] Section 5.
- How to make HAM faster?
- How to have more rigorous test of execution scheme without goint directly to the market

Q&A Time!

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