Principles of Communications

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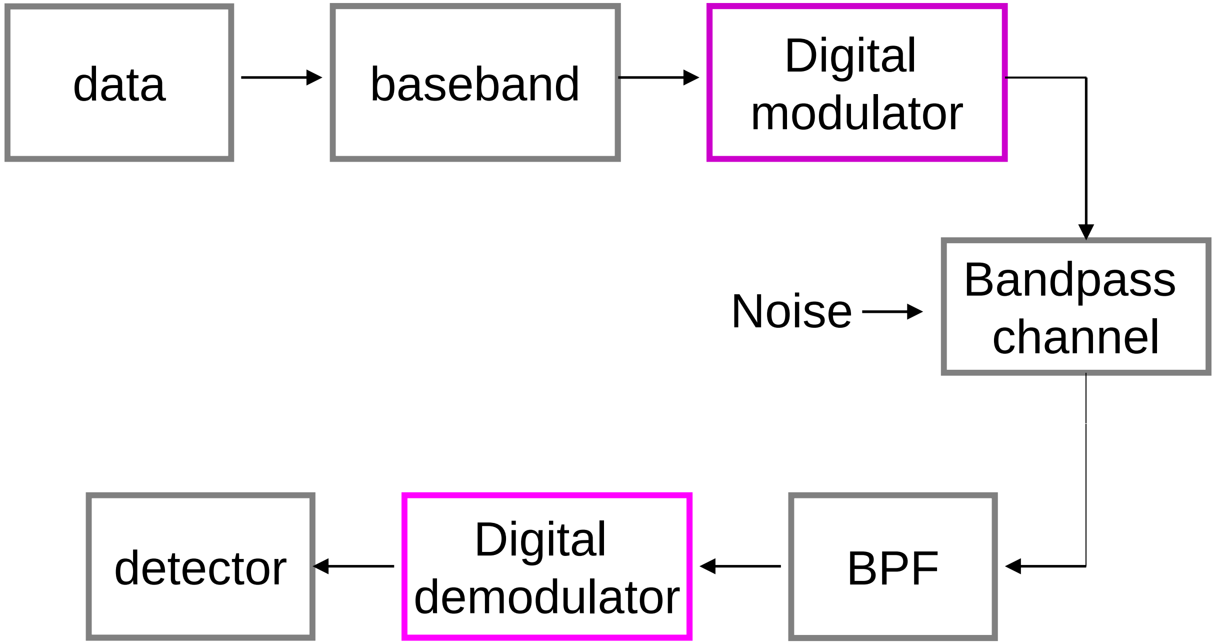
Chapter 7: Digital Modulation Techniques

Textbook: Ch 8.4 – 8 .7

Topics to be Covered



- Binary digital modulation
- M-ary digital modulation
- Tradeoff study

Digital Modulation

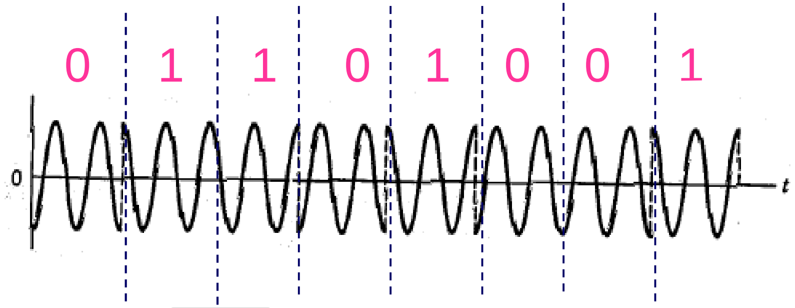
- The message signal is transmitted by a sinusoidal carrier wave
- In digital communications, the modulation process corresponds to switching or keying the amplitude, frequency, or phase of the carrier in accordance with the incoming digital data
- Three basic digital modulation techniques
  - Amplitude-shift keying (ASK) - special case of AM
  - Frequency-shift keying (FSK) - special case of FM
  - Phase-shift keying (PSK) - special case of PM
- Will use signal space approach in receiver design and performance analysis

7.1 Binary Modulation Types

- In binary signaling, the modulator produces one of two distinct signals in response to 1 bit of source data at a time.
- Binary modulation types
  - Binary PSK (BPSK)
  - Binary FSK
  - Binary ASK

Binary Phase-Shift Keying (BPSK)

- Modulation



"1" $\longrightarrow s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$

"0" $\longrightarrow s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$

  - $0 \leq t < T_b$, T_b bit duration
  - f_c : carrier frequency, chosen to be n_c/T_b for some fixed integer n_c or $f_c >> 1/T_b$
  - E_b: transmitted signal energy per bit, i.e.

$$\int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} s_2^2(t)dt = E_b$$

  - The pair of signals differ only in a relative phase shift of 180 degrees

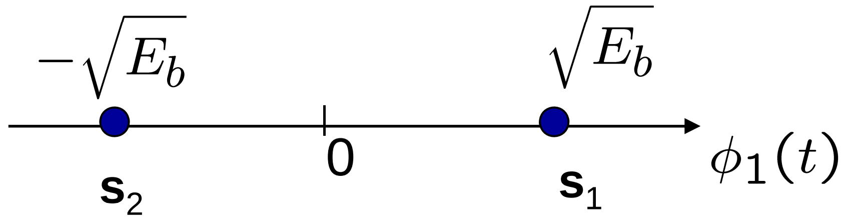
Signal Space Representation for BPSK

- Clearly, there is one basis function of unit energy

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \qquad 0 \le t < T_b$$

- Then $s_1(t) = \sqrt{E_b}\phi_1(t) \qquad s_2(t) = -\sqrt{E_b}\phi_1(t)$

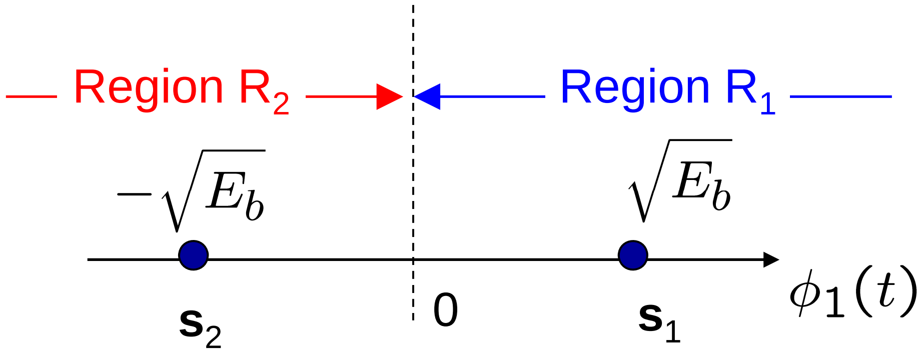
- A binary PSK system is therefore characterized by having a signal space that is one-dimensional (i.e. N=1), and with two message points (i.e. M = 2)



$$d_{12} = 2\sqrt{E_b}$$

Decision Rule of BPSK

- Assume that the two signals are equally likely, i.e. $P(s_1) = P(s_2) = 0.5$. Then the optimum decision boundary is the midpoint of the line joining these two message points



- Decision rule:
  - Guess signal $s_1(t)$ (or binary 1) was transmitted if the received signal point $\mathbf{r}$ falls in region R_1 $(r > 0)$
  - Guess signal $s_2(t)$ (or binary 0) was transmitted otherwise $(r \leq 0)$

Proof of the Decision Rule

- Observation scalar (output of the demodulator) r is

$$r = \begin{cases} s_{11} + n = \sqrt{E_b} + n & \text{If } s_1 \text{ is transmitted} \\ s_{21} + n = -\sqrt{E_b} + n & \text{If } s_2 \text{ is transmitted} \end{cases}$$

where n represents the AWGN component, which has mean zero and variance $\sigma_n^2 = N_0/2$

- Thus, the likelihood function of r is

$$f(r|s_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r-\sqrt{E_b})^2}{N_0}\right\}$$

$$f(r|s_2) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r+\sqrt{E_b})^2}{N_0}\right\}$$

- Recall ML decision criterion:

$$f(r|s_1) \underset{\text{Choose } s_2}{\overset{\text{Choose } s_1}{\gtrless}} f(r|s_2)$$

- Thus

$$\frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r-\sqrt{E_b})^2}{N_0}\right\} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r+\sqrt{E_b})^2}{N_0}\right\}$$

- And

$$(r-\sqrt{E_b})^2 \underset{s_2}{\overset{s_1}{\lessgtr}} (r+\sqrt{E_b})^2$$

- Finally

$$r \underset{s_2}{\overset{s_1}{\gtrless}} 0$$

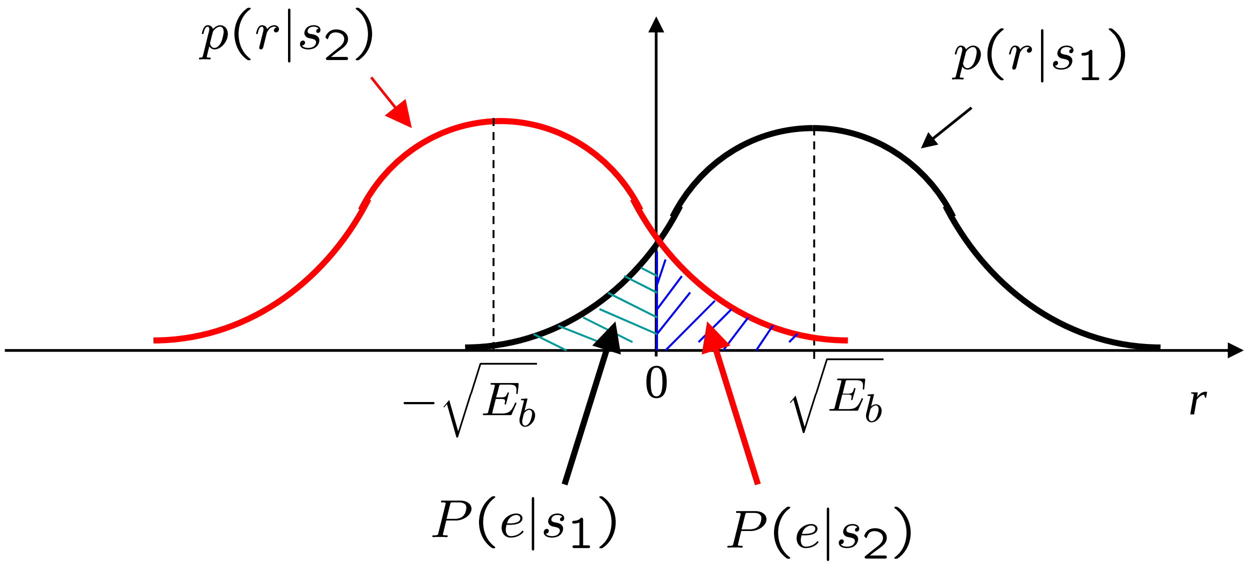
Probability of Error for BPSK

- The conditional probability of the receiver deciding in favor of symbol $s_2(t)$ given that $s_1(t)$ is transmitted is

$$
\begin{aligned}
P(e|s_1) &= P(r < 0|s_1) \\
&= \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(r-\sqrt{E_b})^2}{N_0}\right\} dr \\
&= Q\left(\sqrt{\frac{2E_b}{N_0}}\right)
\end{aligned}
$$

- Due to symmetry

$$
P(e|s_2) = P(r > 0|s_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)
$$

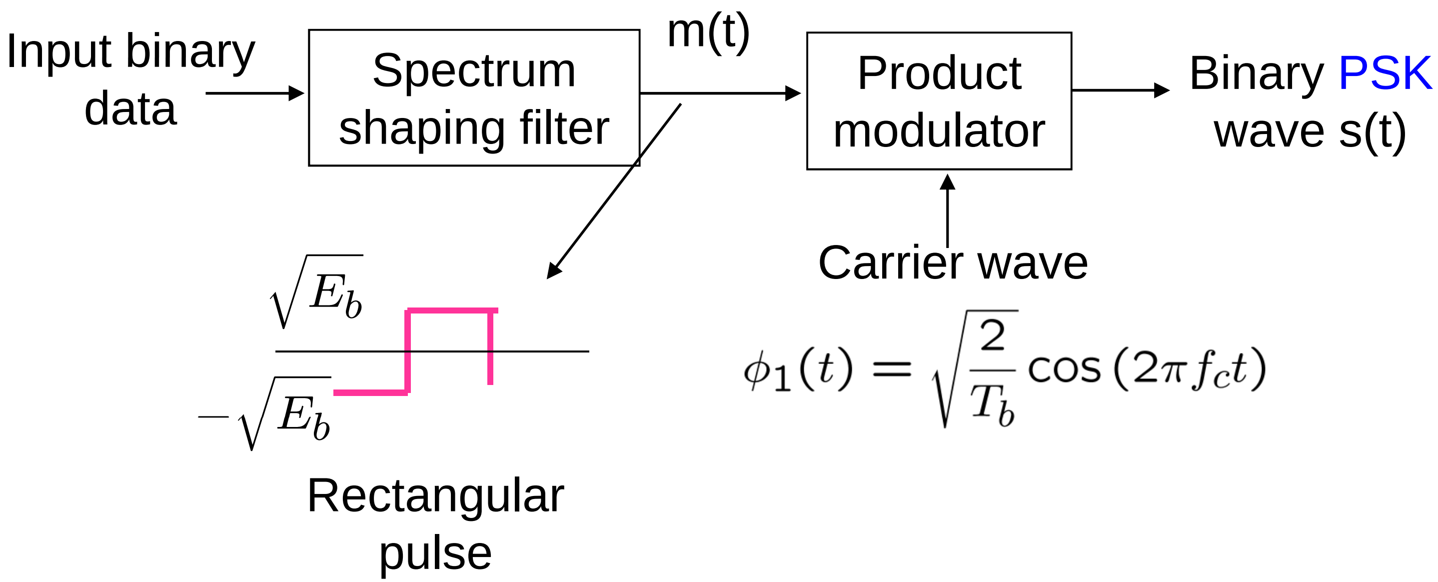


- Since the signals $s_1(t)$ and $s_2(t)$ are equally likely to be transmitted, the average probability of error is

$$P_e = 0.5P(e|\mathrm{s}_1) + 0.5P(e|\mathrm{s}_2) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

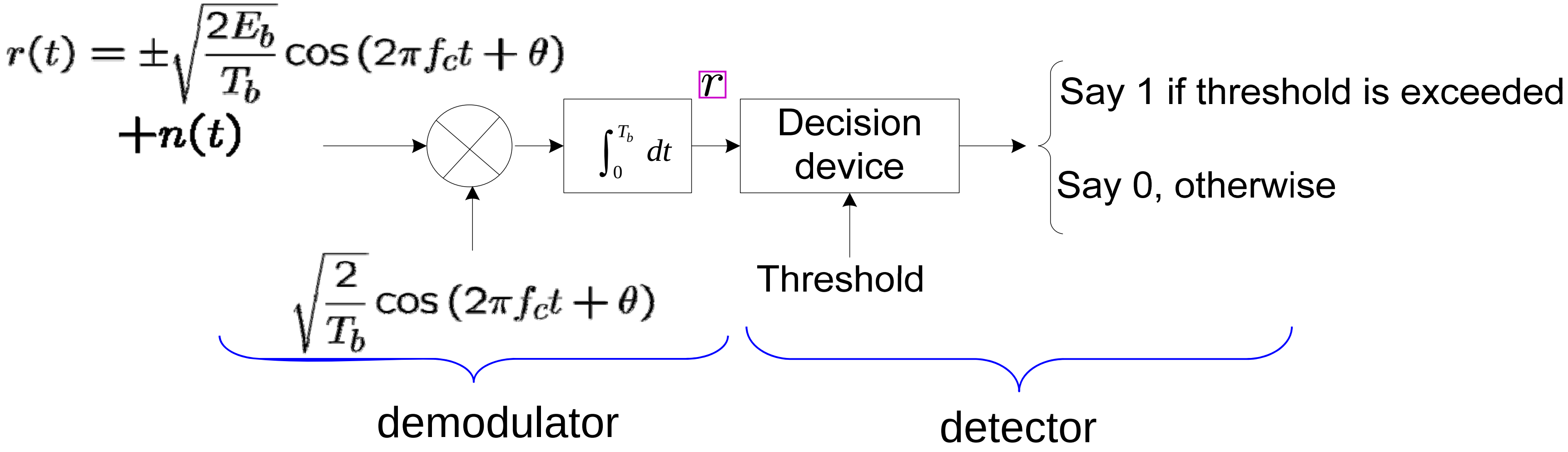
- Note: probability of error depends on ratio E_b/N_0. This ratio is normally called bit energy to noise density ratio (or SNR/bit)

BPSK Transmitter



$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$$

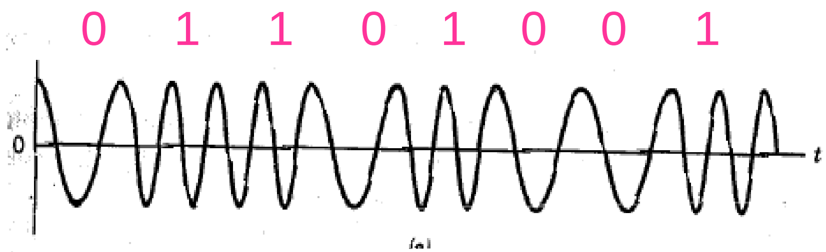
BPSK Receiver



- θ is the carrier-phase offset, due to propagation delay or oscillators at transmitter and receiver are not synchronous
- The detection is coherent in the sense of
  - Phase synchronization: ensure local oscillator output at the receiver is synchronized to the carrier in modulator
  - Timing synchronization: to ensure proper bit timing of the decision-making operation

Binary FSK

- Modulation



"1" $\longrightarrow s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t)$

$0 \leq t < T_b$

"0" $\longrightarrow s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t)$

  - E_b : transmitted signal energy per bit

$$\int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} s_2^2(t)dt = E_b$$

  - f_i : transmitted frequency with separation $\Delta f = f_1 - f_0$
  - Δf is selected so that $s_1(t)$ and $s_2(t)$ are orthogonal i.e.

$$\int_0^{T_b} s_1(t)s_2(t)dt = 0$$

(Example?)

Signal Space for BFSK

- Unlike BPSK, here two orthogonormal basis functions are required to represent $s_1(t)$ and $s_2(t)$.

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_1 t) \qquad 0 \le t < T_b$$

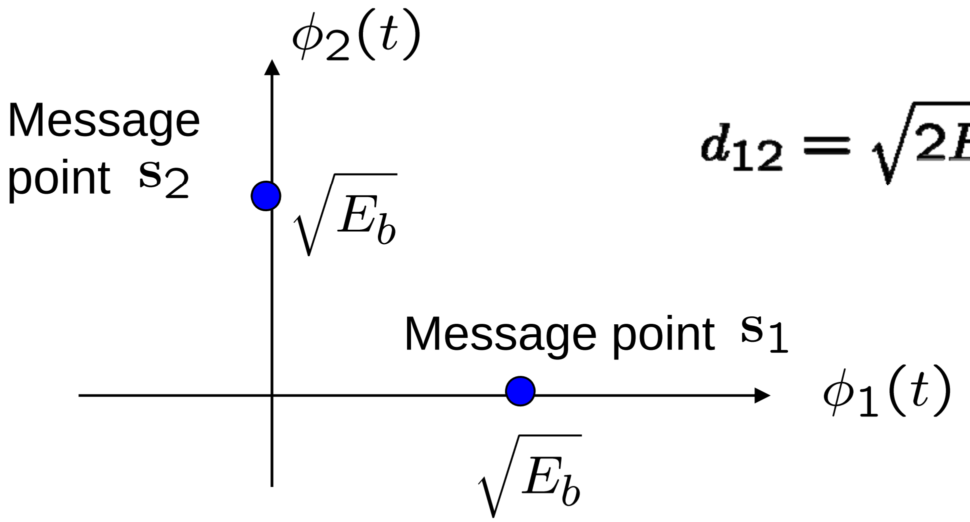
$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_2 t)$$

- Signal space representation

$$s_1(t) = \sqrt{E_b}\phi_1(t) \qquad \Rightarrow \qquad \mathbf{s}_1 = [\sqrt{E_b} \quad 0]$$

$$s_2(t) = \sqrt{E_b}\phi_2(t) \qquad \qquad \mathbf{s}_2 = [0 \quad \sqrt{E_b}]$$

- Signal space diagram for binary FSK



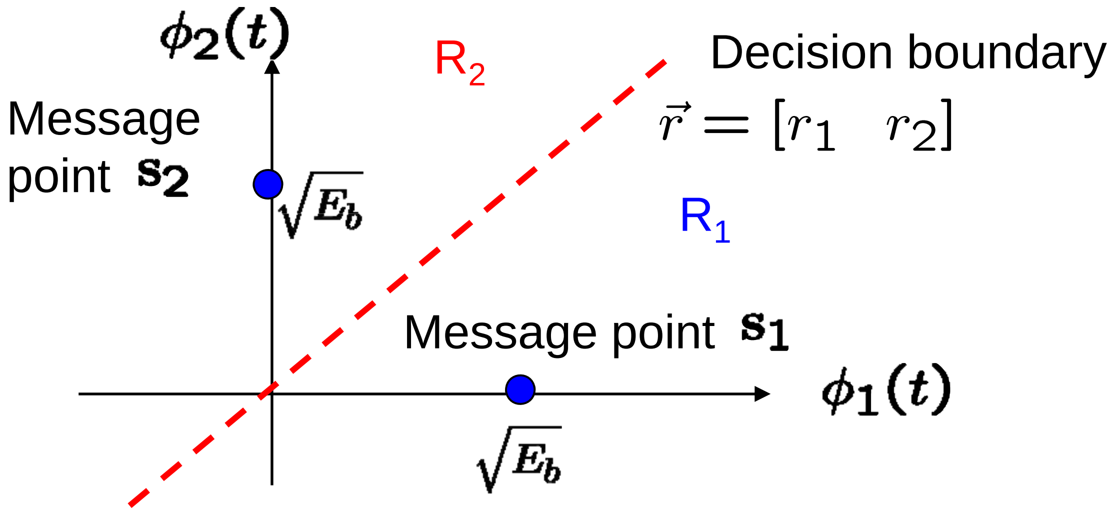
$$d_{12} = \sqrt{2E_b}$$

- Observation vector $\vec{r} = [r_1 \quad r_2]$

$$r_1 = \int_0^{T_b} r(t)\phi_1(t)dt$$

$$r_2 = \int_0^{T_b} r(t)\phi_2(t)dt$$

Decision Regions of Binary FSK



- The receiver decides in favor of $\mathbf{s}_1$ if the received signal point represented by the observation vector $\mathbf{r}$ falls inside region R$_1$. This occurs when $r_1 > r_2$
- When $r_1 < r_2$, $\mathbf{r}$ falls inside region R$_2$ and the receiver decides in favor of $\mathbf{s}_2$

Probability of Error for Binary FSK

- Given that $\mathbf{s}_1$ is transmitted,

$$r_1 = \sqrt{E_b} + n_1 \quad \text{and} \quad r_2 = n_2$$

- Since the condition $r_1 < r_2$ corresponds to the receiver making a decision in favor of symbol $\mathbf{s}_2$, the conditional probability of error given $\mathbf{s}_1$ is transmitted is given by

$$P(e|s_1) = P(r_1 < r_2|s_1) = P(\sqrt{E_b} + n_1 < n_2)$$

- Define a new random variable $n = n_1 - n_2$
- Since n_1 and n_2 are i.i.d with $n_1, n_2 \in \mathcal{N}(0, N_0/2)$
- Thus, n is also Gaussian with $n \in \mathcal{N}(0, N_0)$

$$\Rightarrow P(e|s_1) = P(n < -\sqrt{E_b}) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- By symmetry

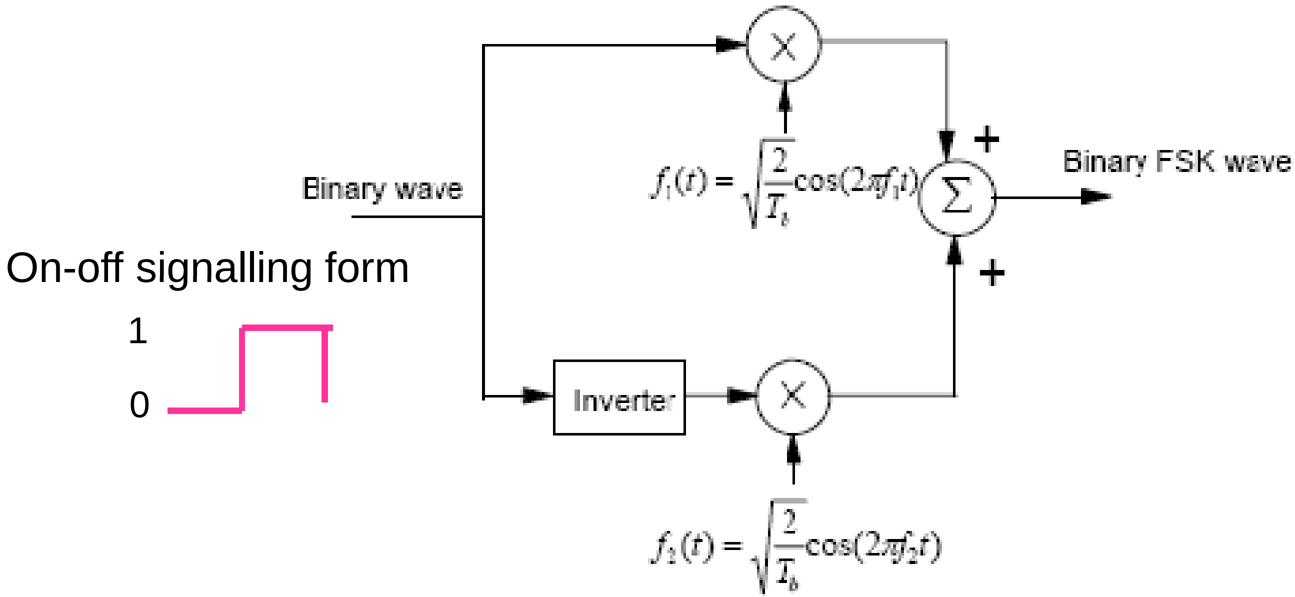
$$P(e|s_2) = P(r_1 > r_2|s_2) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- Since the two signals are equally likely to be transmitted, the average probability of error for coherent binary FSK is

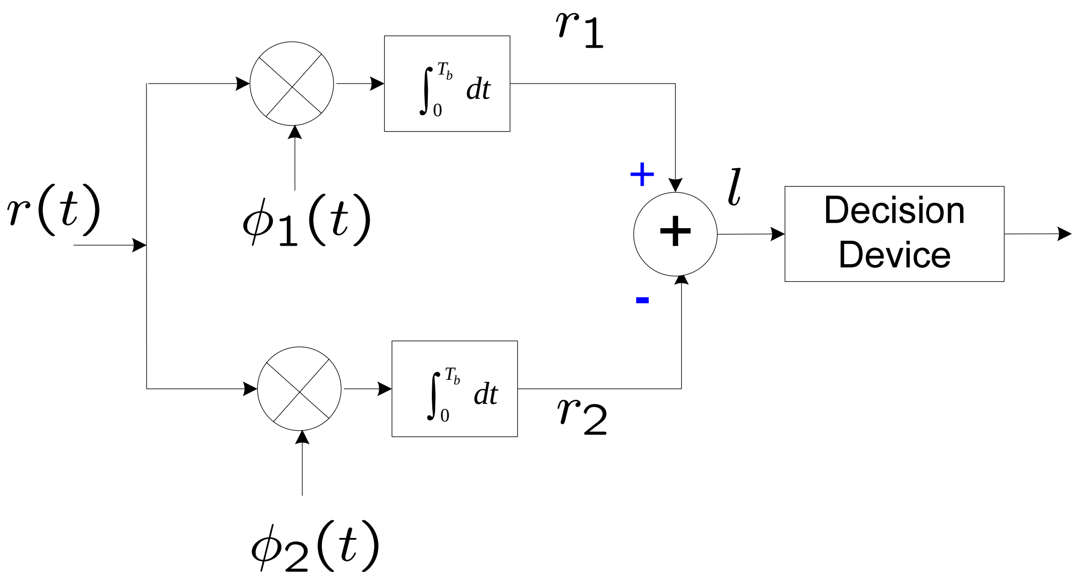
$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \longrightarrow \text{3 dB worse than BPSK}$$

i.e. to achieve the same P_e, BFSK needs 3dB more transmission power than BPSK

Binary FSK Transmitter



Coherent Binary FSK Receiver



Choose 1 if l>0
Choose 0 otherwise

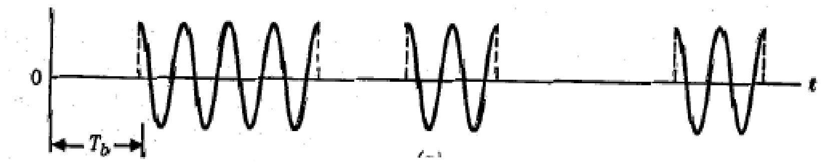
Binary ASK

- Modulation

"1" $\longrightarrow$ $s_1(t) = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_c t)$

"0" $\longrightarrow$ $s_2(t) = 0 \qquad 0 \le t < T_b$

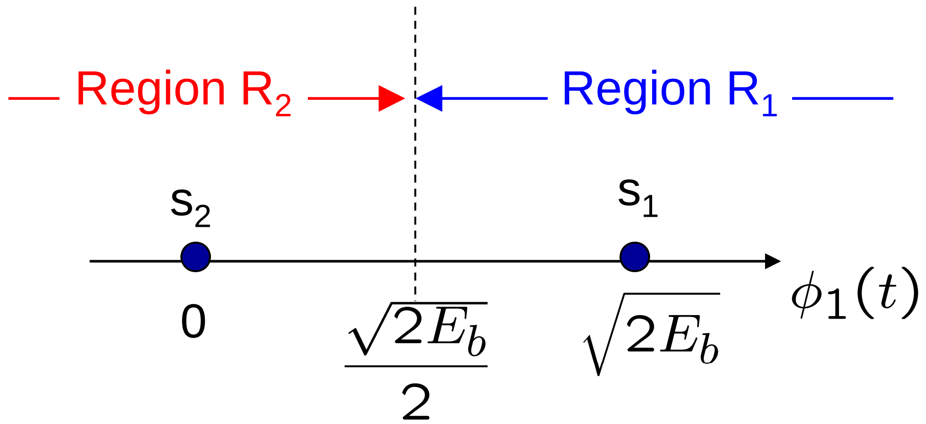
0 1 1 0 1 0 0 1



(On-off signalling)

- Average energy per bit

$$E_b = \frac{E + 0}{2} \qquad \text{i.e. } E = 2E_b$$



$$d_{12} = \sqrt{2E_b}$$

Probability of Error for Binary ASK

- Average probability of error is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Identical to that of coherent binary FSK

- <u>Exercise:</u> Prove P_e

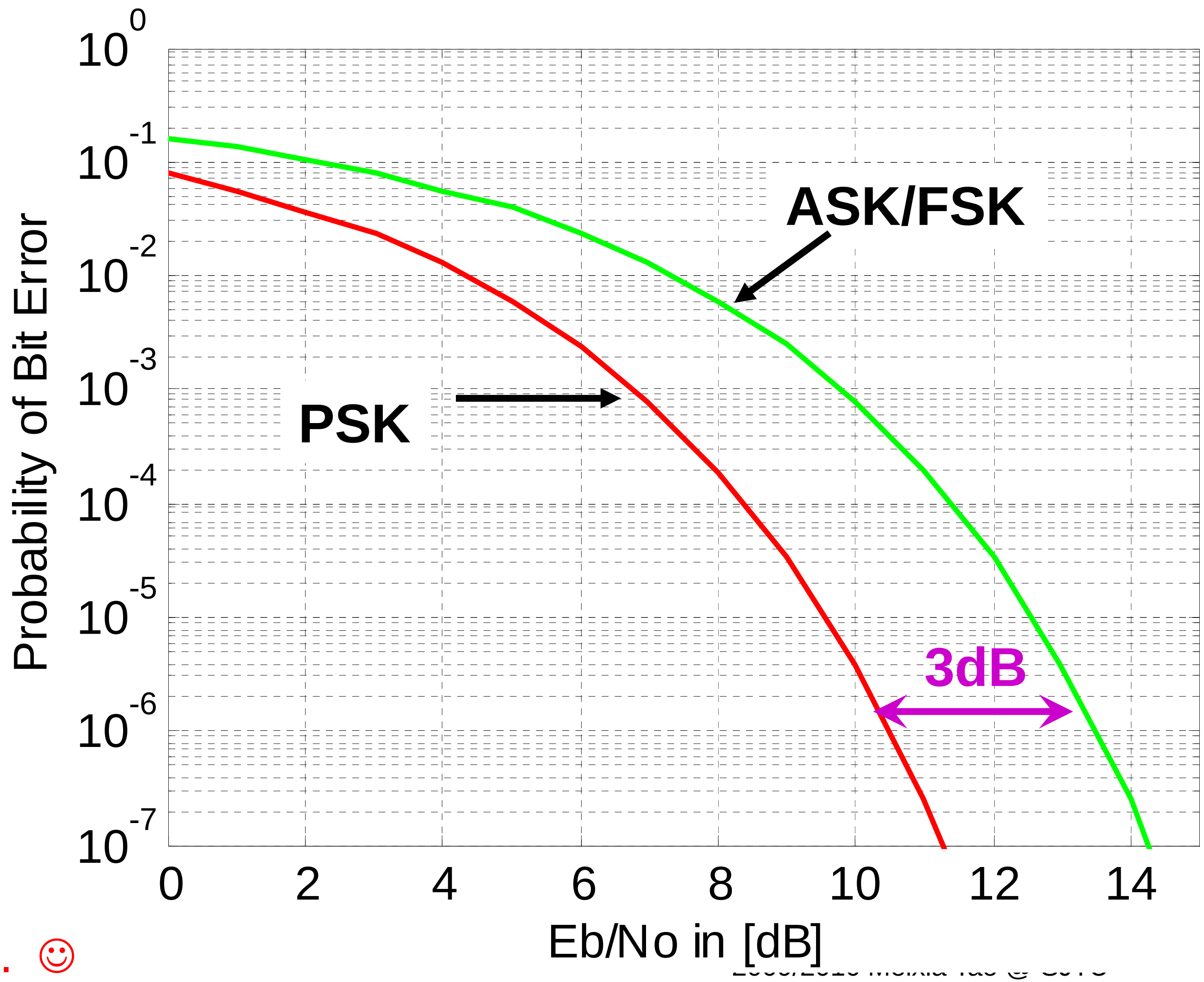
Probability of Error and the Distance Between Signals

BPSK	BFSK	BASK
$d_{1,2} = 2\sqrt{E_b}$	$d_{1,2} = \sqrt{2E_b}$	$d_{1,2} = \sqrt{2E_b}$
$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

- These expressions illustrate the dependence of the error probability on the distance between two signal points. In general,

$$\boxed{P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)}$$

Probability of Error Curve for BPSK and FSK/ASK



e.g. ☺

Example #1

Binary data are transmitted over a microwave link at the rate of 10^6 bits/sec and the PSD of the noise at the receiver input is 10^{-10} watts/Hz.

a) Find the average carrier power required to maintain an average probability of error $P_e \leq 10^{-4}$ for coherent binary FSK.

b) Repeat the calculation in a) for noncoherent binary FSK

- We have discussed
  - Coherent modulation schemes, .e.g. BPSK, BFSK, BASK
  - They needs coherent detection, assuming that the receiver is able to detect and track the carrier wave's phase

- In many practical situations, strict phase synchronization is not possible. In these situations, non-coherent reception is required.
- We now consider:
  - Non-coherent detection on binary FSK
  - Differential phase-shift keying (DPSK)

7.2: Non-coherent scheme: BFSK

- Consider a binary FSK system, the two signals are

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t + \theta_1) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t)\cos(\theta_1) - \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_1 t)\sin(\theta_1)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t + \theta_2) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t)\cos(\theta_2) - \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_2 t)\sin(\theta_2)$$

$$0 \le t < T_b$$

- Where θ_1 and θ_2 are unknown random phases with uniform distribution

$$p_{\theta_1}(\theta) = p_{\theta_2}(\theta) = \begin{cases} 1/2\pi & \theta \in [0, 2\pi) \\ 0 & \text{else} \end{cases}$$

Signal Space Representation

- No matter what the two phases are, the signals can be expressed as a linear combination of the four basis functions

$$\phi_{1c}(t) = \sqrt{2/T_b}\,\cos(2\pi f_1 t) \qquad \phi_{1s}(t) = -\sqrt{2/T_b}\,\sin(2\pi f_1 t)$$

$$\phi_{2c}(t) = \sqrt{2/T_b}\,\cos(2\pi f_2 t) \qquad \phi_{2s}(t) = \sqrt{2/T_b}\,\sin(2\pi f_2 t)$$

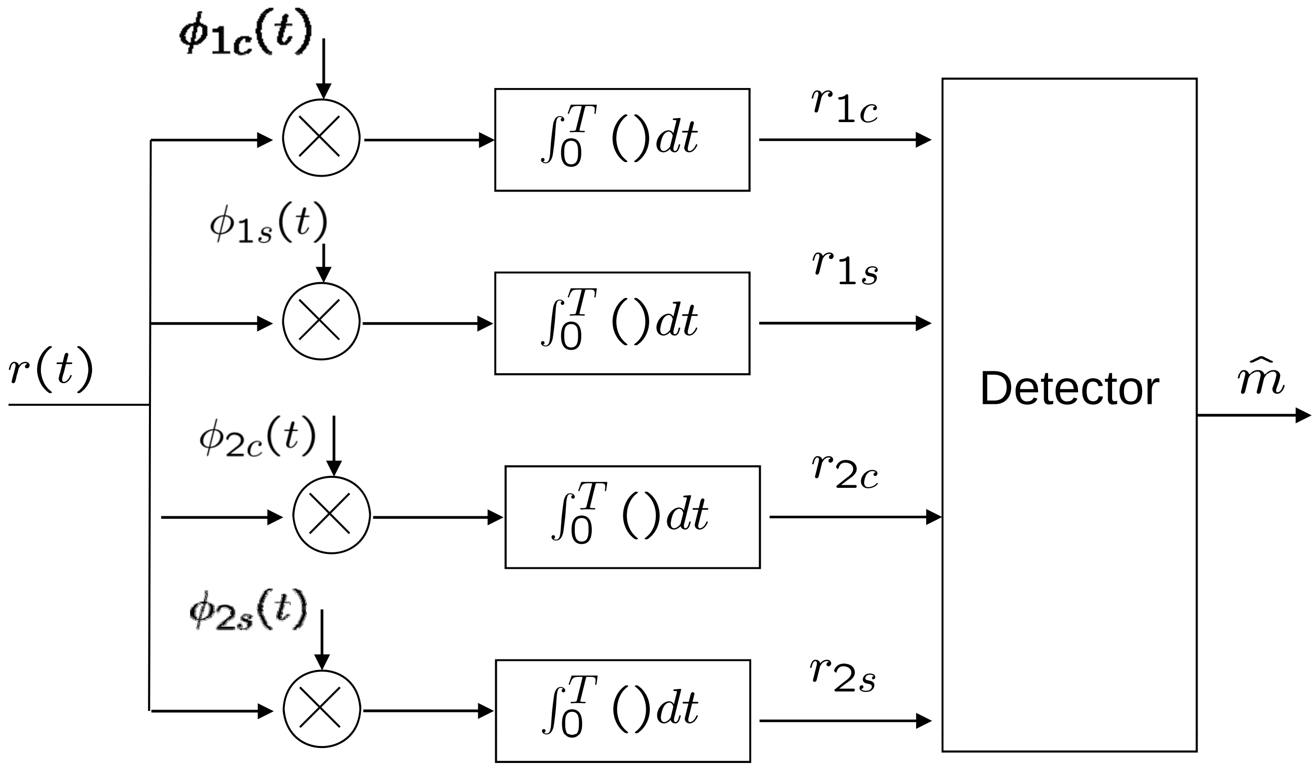
- Signal space representation

$$s_1(t) = \sqrt{E_b}\cos(\theta_1)\phi_{1c}(t) + \sqrt{E_b}\sin(\theta_1)\phi_{1s}(t)$$

$$s_2(t) = \sqrt{E_b}\cos(\theta_2)\phi_{2c}(t) + \sqrt{E_b}\sin(\theta_2)\phi_{2s}(t)$$

$$\vec{s}_1 = [\ \sqrt{E_b}\cos\theta_1 \quad \sqrt{E_b}\sin\theta_1 \quad 0 \quad 0\ ] \qquad \vec{s}_2 = [\ 0 \quad 0 \quad \sqrt{E_b}\cos\theta_2 \quad \sqrt{E_b}\sin\theta_2\ ]$$

- Correlating the received signal r(t) with the four basis functions produces the vector representation of the received signal $\vec{r} = [r_{1c} \quad r_{1s} \quad r_{2c} \quad r_{2s}]$



Decision Rule for Non-coherent FSK

- ML criterion, assume $P(s_1) = P(s_2)$:

$$f(\vec{r}|\vec{s}_1) \underset{\text{Choose } s_2}{\overset{\text{Choose } s_1}{\gtrless}} f(\vec{r}|\vec{s}_2)$$

- Conditional pdf

$$f(\vec{r}|\vec{s}_1,\theta_1) = \frac{1}{\pi N_0}\exp\left[-\frac{(r_{1c}-\sqrt{E_b}\cos\theta_1)^2+(r_{1s}-\sqrt{E_b}\sin\theta_1)^2}{N_0}\right] \times \frac{1}{\pi N_0}\exp\left[-\frac{r_{2c}^2+r_{2s}^2}{N_0}\right]$$

- Similarly,

$$f(\vec{r}|\vec{s}_2,\theta_2) = \frac{1}{\pi N_0}\exp\left[-\frac{r_{1c}^2+r_{1s}^2}{N_0}\right] \times \frac{1}{\pi N_0}\exp\left[-\frac{(r_{2c}-\sqrt{E_b}\cos\theta_2)^2+(r_{2s}-\sqrt{E_b}\sin\theta_2)^2}{N_0}\right]$$

- For ML decision, we need to evaluate

$$f(\vec{r}|\vec{s}_1) \geq f(\vec{r}|\vec{s}_2)$$

- i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} f(\vec{r}|\vec{s}_1,\theta_1)d\theta_1 \geq \frac{1}{2\pi}\int_0^{2\pi} f(\vec{r}|\vec{s}_2,\theta_2)d\theta_2$$

- Removing the constant terms

$$\left(\frac{1}{\pi N_0}\right)^2 \exp\left[-\frac{{r_{1c}}^2+{r_{1s}}^2+{r_{2c}}^2+{r_{2s}}^2+E}{N_0}\right]$$

- We have the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{1c}\cos(\phi_1)+2\sqrt{E}r_{1s}\sin(\phi_1)}{N_0}\right]d\phi_1$$
$$\geq \frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{2c}\cos(\phi_1)+2\sqrt{E}r_{2s}\sin(\phi_1)}{N_0}\right]d\phi_2$$

- By definition

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{1c}\cos(\phi_1)+2\sqrt{E}r_{1s}\sin(\phi_1)}{N_0}\right]d\phi_1 = I_0\left(\frac{2\sqrt{E(r_{1c}^{\ 2}+r_{1s}^{\ 2})}}{N_0}\right)$$

where $I_0(\cdot)$ is a modified Bessel function of the zeroth order

Decision Rule (cont'd)

- Thus, the decision rule becomes: choose s_1 if

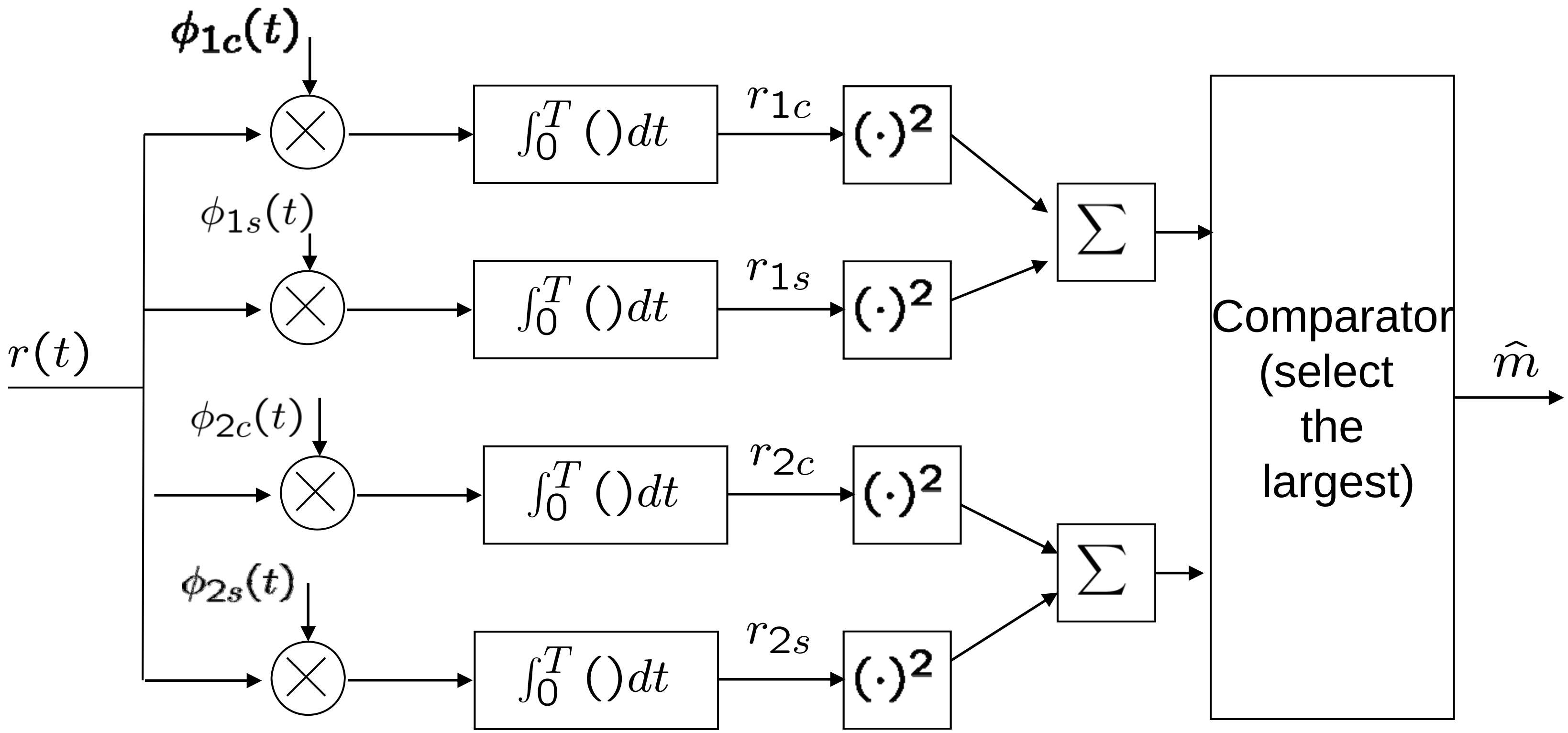
$$I_0\left(\frac{2\sqrt{E(r_{1c}^2+r_{1s}^2)}}{N_0}\right) \geq I_0\left(\frac{2\sqrt{E(r_{2c}^2+r_{2s}^2)}}{N_0}\right)$$

- But note that this Bessel function is monotonically increasing. Therefore we choose s_1 if

$$\sqrt{r_{1c}^2+r_{1s}^2} \geq \sqrt{r_{2c}^2+r_{2s}^2}$$

- Interpretation: compare the energy in the two frequencies and pick the larger => envelop detector
- Carrier phase is irrelevant in decision making

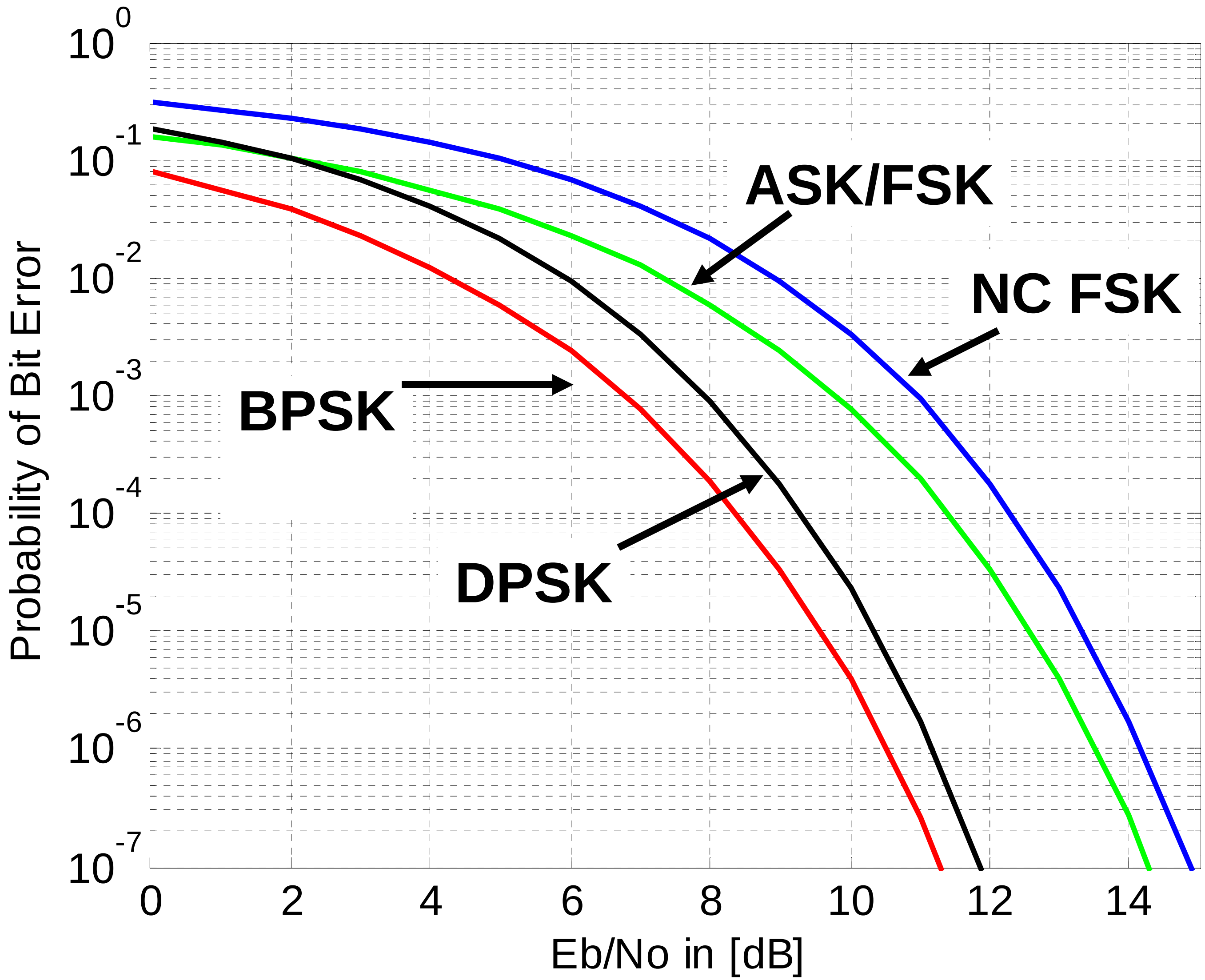
Structure of Non-Coherent Receiver for Binary FSK



- It can be shown that $$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$$

(For detailed proof, see Section 10.4.2 in the textbook)

Performance Comparison Between coherent FSK and Non-Coherent FSK



Differential PSK (DPSK)

- DPSK can be viewed as the non-coherent version of PSK.
- Phase synchronization is eliminated using differential encoding
  - Encoding the information in phase difference between successive signal transmission
- In effect:
  - to send “0”, we phase advance the current signal waveform by 180^0 ;
  - to send “1”, we leave the phase unchanged

DPSK (cont'd)

- Provided that the unknown phase θ contained in the received wave varies slowly (constant over two bit intervals), the phase difference between waveforms received in two successive bit interval will be independent of θ

Generation of DPSK signal

- We can generate DPSK signals by combining two basic operations
  - Differential encoding of the information binary bits
  - Phase shift keying
- The differential encoding process starts with an arbitrary first bit, serving as reference
- Let $\{m_i\}$ be input information binary bit sequence, $\{d_i\}$ be the differentially encoded bit sequence
  - If the incoming bit m_i is "1", leave the symbol d_i unchanged with respect to the previous bit d_{i-1}
  - If the incoming bit m_i is "0", change the symbol d_i with respect to the previous bit d_{i-1}

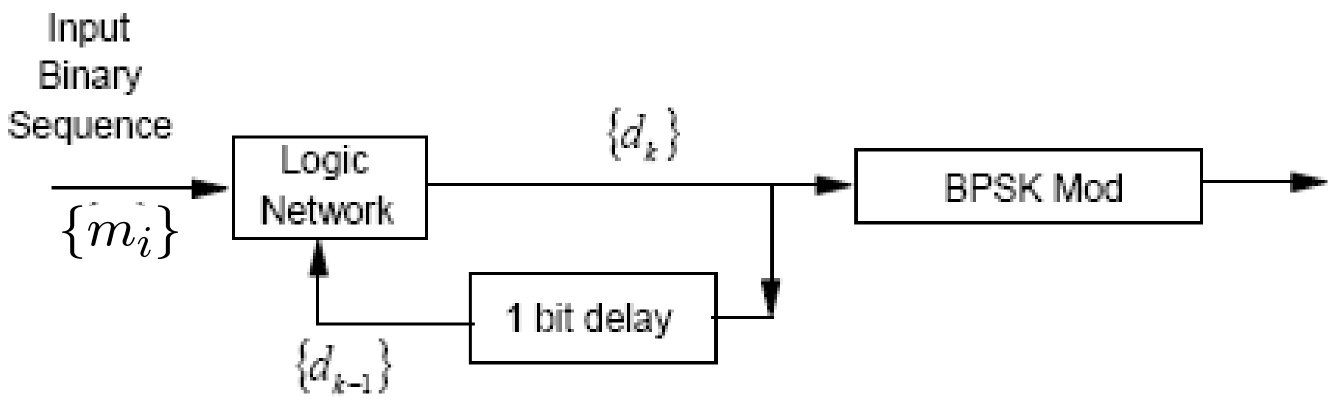
Illustration

- The reference bit is chosen arbitrary, here taken as 1

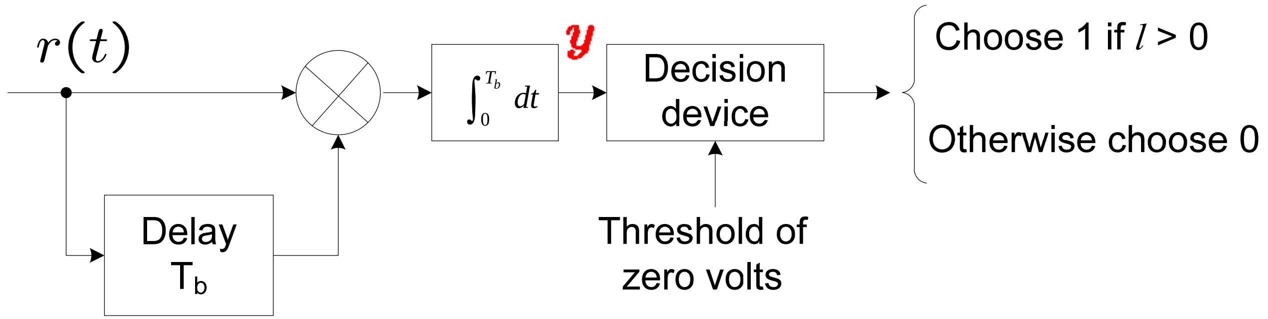
Binary data		1	0	0	1	0	0	1	1	m_i
Differentially encoded binary data	1 (Initial bit)	1	0	1	1	0	1	1	1	d_i
Transmitted Phase	0	0	π	0	0	π	0	0	0	

$$\boxed{d_i = \overline{d_{i-1} \oplus m_i}}$$

- DPSK transmitter diagram



Differential Detection of DPSK Signals



- Multiply the received DPSK signal with its delayed version
- Output of integrator (assume noise free)

$$y = \int_0^{T_b} r(t)r(t-T_b)dt = \int_0^{T_b} \cos(w_c t + \psi_k + \theta)\cos(w_c t + \psi_{k-1} + \theta)dt$$

$$\propto \cos(\psi_k - \psi_{k-1})$$

  - The unknown phase θ becomes irrelevant
  - If $\psi_k - \psi_{k-1}$= 0 (bit 1), the integrator output y is positive
  - if $\psi_k - \psi_{k-1}$=π (bit 0), the integrator output y is negative

Error Probability of DPSK

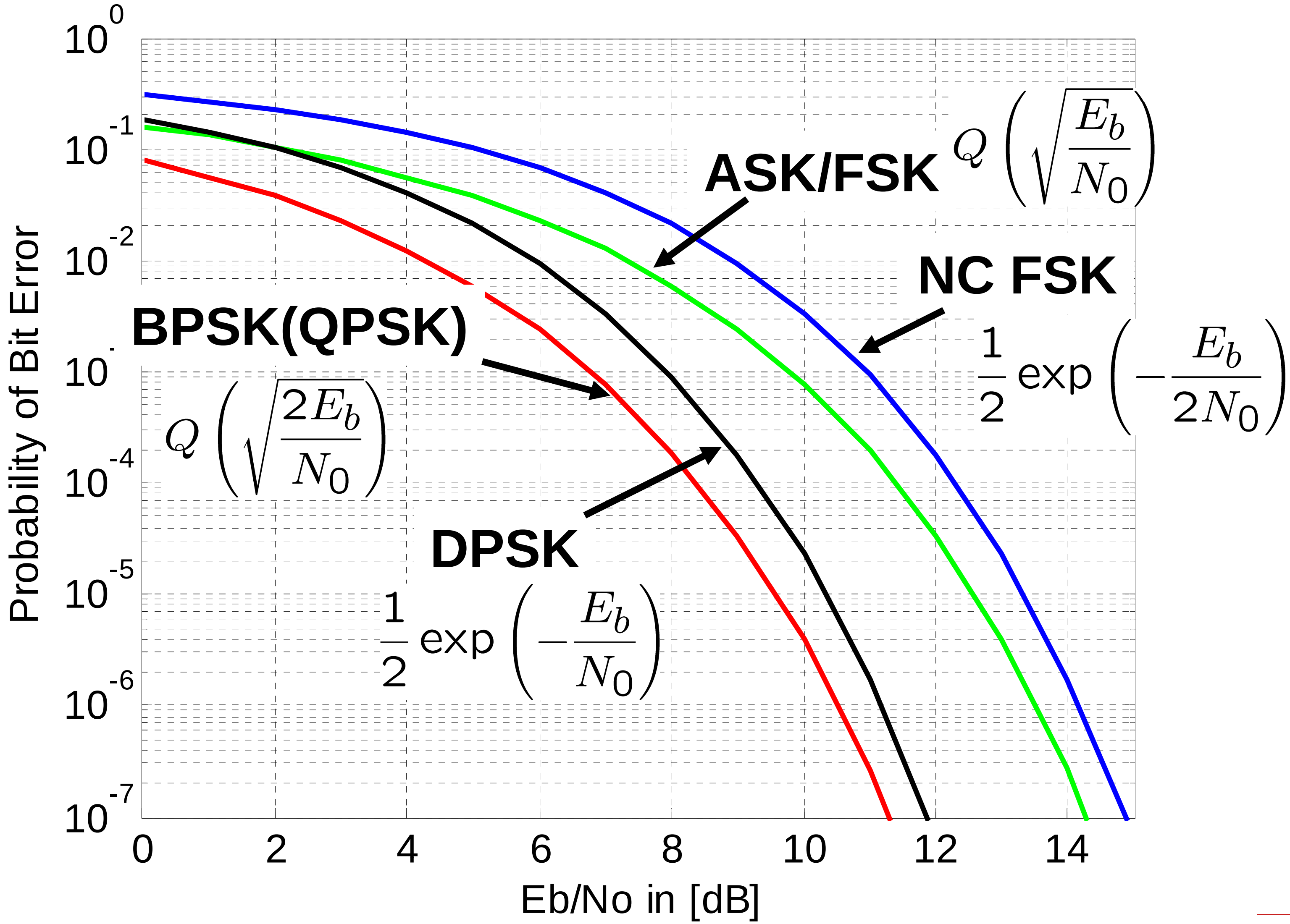
- The differential detector is suboptimal in the sense of error performance
- It can be shown that

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$$

Summary of P_e for Different Binary Modulations

Coherent PSK	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$
Coherent ASK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
Coherent FSK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
Non-Coherent FSK	$\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$
DPSK	$\frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$

P_e Plots for Different Binary Modulations



- We have discussed binary case
  - Coherent modulation techniques:

    BPSK, BFSK, BASK
  - Noncoherent modulation techniques:

    Non-coherent FSK, DPSK

- We now consider:

  M-ary modulation techniques
  - MPSK
  - MQAM
  - MFSK

7.3 M-ary Modulation Techniques

- In binary data transmission, send only one of two possible signals during each bit interval T_b
- In M-ary data transmission, send one of M possible signals during each signaling interval T
- In almost all applications, $M = 2^n$ and $T = nT_b$, where n is an integer
- Each of the M signals is called a symbol
- These signals are generated by changing the amplitude, phase or frequency of a carrier in M discrete steps.
- Thus, we have M-ary ASK, M-ary PSK, and M-ary FSK digital modulation schemes

- Binary is a special case of M-ary
- Another way of generating M-ary signals is to combine different methods of modulation into hybrid forms
- For example, we may combine discrete changes in both the amplitude and phase of a carrier to produce M-ary amplitude phase keying. A special form of this hybrid modulation is M-ary QAM (MQAM)

M-ary Phase-Shift Keying (MPSK)

- The phase of the carrier takes on M possible values:

$$\theta_m = 2\pi(m-1)/M, \;\; m = 1, \ldots, M$$

- Signal set:

$$s_m(t) = \sqrt{\frac{2E_s}{T}} \cos\left[2\pi f_c t + \frac{2\pi(m-1)}{M}\right] \qquad \begin{array}{c} m = 1, \ldots, M \\ 0 \le t < T \end{array}$$

  - E_s = Energy per symbol
  - $f_c >> \frac{1}{T}$
- Basis functions

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$$

$$0 \le t < T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

MPSK (cont'd)

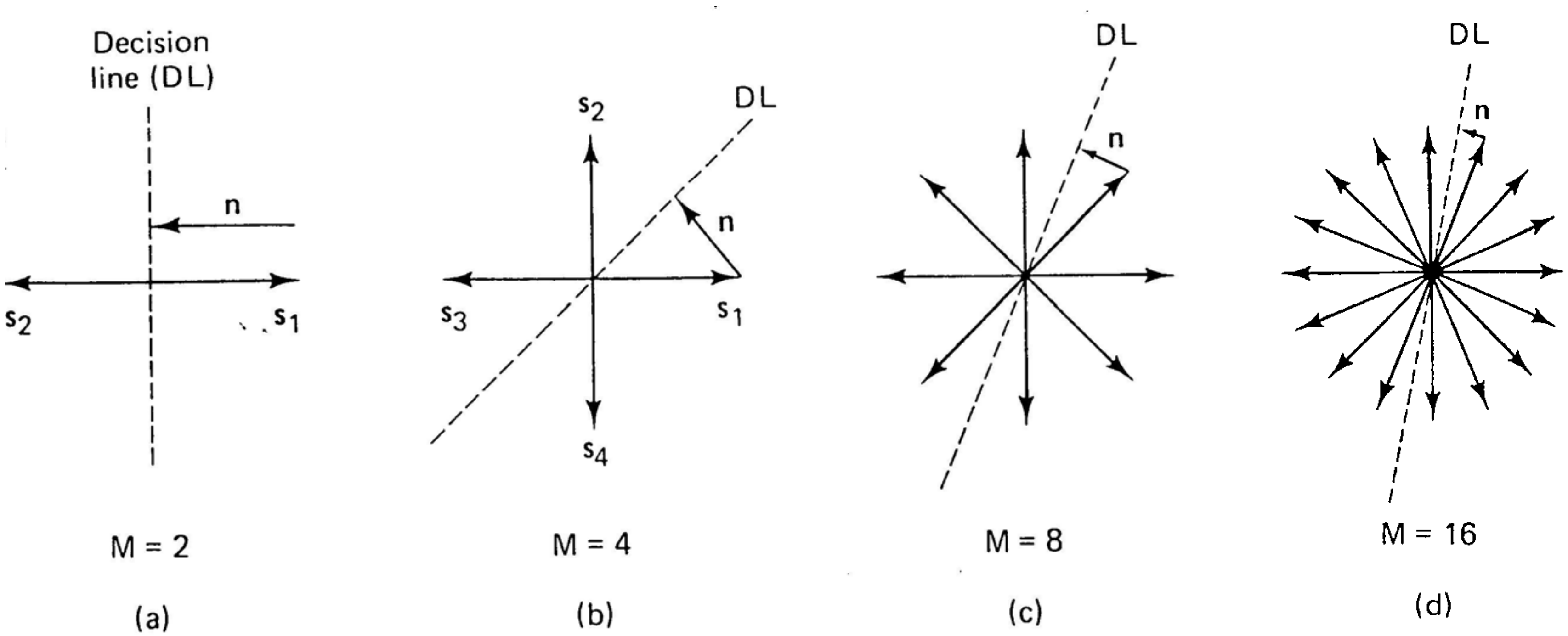
- Signal space representation

$$
\begin{aligned}
s_m(t) &= \sqrt{\frac{2E_s}{T}} \cos\left[2\pi f_c t + \frac{2\pi(m-1)}{M}\right] \\
&= \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t) \cos\left[\frac{2\pi(m-1)}{M}\right] \\
&\qquad - \sqrt{\frac{2E_s}{T}} \sin(2\pi f_c t) \sin\left[\frac{2\pi(m-1)}{M}\right] \\
&= \sqrt{E_s} \cos\left[\frac{2\pi(m-1)}{M}\right] \phi_1(t) - \sqrt{E_s} \sin\left[\frac{2\pi(m-1)}{M}\right] \phi_2(t)
\end{aligned}
$$

$$
\Rightarrow \quad \mathbf{s}_m = \left[\ \sqrt{E_s} \cos\left(\tfrac{2\pi(m-1)}{M}\right) \quad \sqrt{E_s} \sin\left(\tfrac{2\pi(m-1)}{M}\right)\ \right]
$$

$$
m = 1, \ldots, M
$$

MPSK Signal Constellations



BPSK QPSK 8PSK 16PSK

- The Euclidean distance between any two signal points in the constellation is

$$d_{mn} = \|\mathbf{s}_m - \mathbf{s}_n\| = \sqrt{2E_s\left(1-\cos\frac{2\pi(m-n)}{M}\right)}$$

- The minimum Euclidean distance is

$$d_{\min} = \sqrt{2E_s\left(1-\cos\frac{2\pi}{M}\right)} = 2\sqrt{E_s}\sin\frac{\pi}{M}$$

- dmin plays an important role in determining error performance as discussed previously (union bound)
- In the case of PSK modulation, the error probability is dominated by the erroneous selection of either one of the two signal points adjacent to the transmitted signal point.
- Consequently, an approximation to the symbol error probability is

$$P_{MPSK} \approx 2Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}}\right) = 2Q\left(\sqrt{2E_s}\sin\frac{\pi}{M}\right)$$

Exercise

- Consider the M=2, 4, 8 PSK signal constellations. All have the same transmitted signal energy Es.
- Determine the minimum distance $d_{\min}$ between adjacent signal points
- For M=8, determine by how many dB the transmitted signal energy Es must be increased to achieve the same $d_{\min}$ as M =4.

Error Performance of MPSK

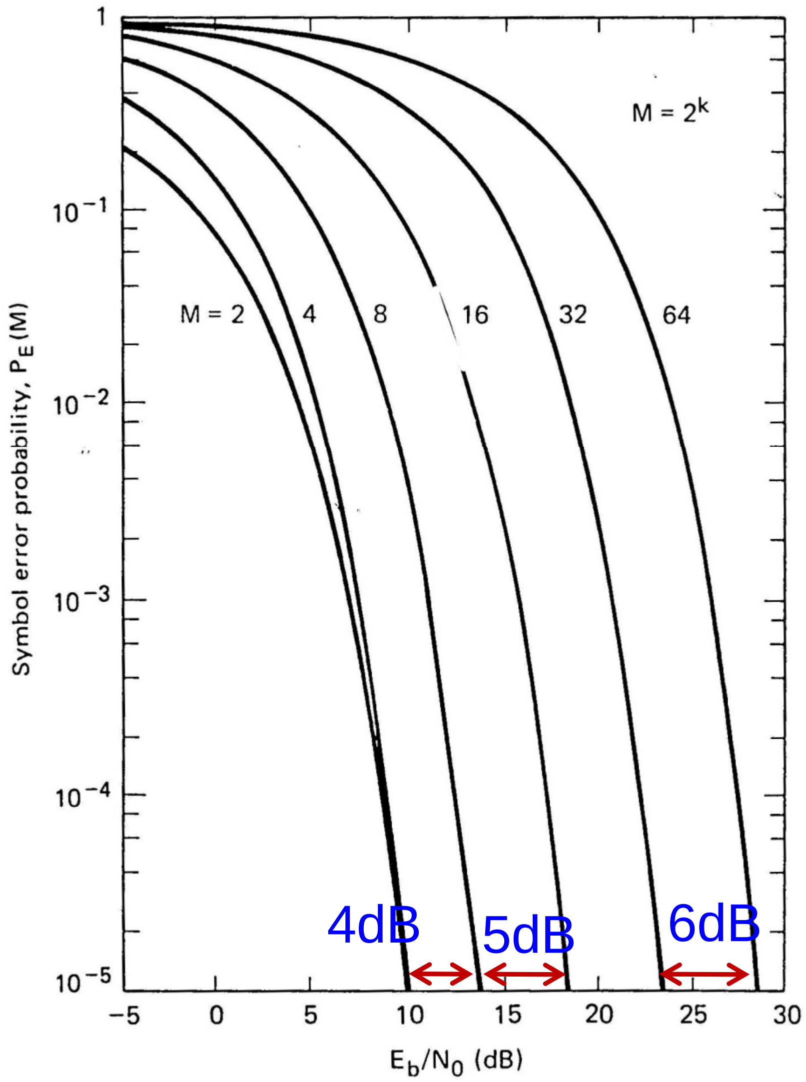


Figure 3.32 Symbol error probability for coherently detected multiple phase signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon.)

- For large M, doubling the number of phases requires an additional 6dB/bit to achieve the same performance

M-ary Quadrature Amplitude Modulation (MQAM)

- In an M-ary PSK system, in-phase and quadrature components are interrelated in such a way that the envelope is constant (circular constellation). If we relax this constraint, we get M-ary QAM.
- Signal set:

$$s_i(t) = \sqrt{\frac{2E_0}{T}} a_i \cos(2\pi f_c t) + \sqrt{\frac{2E_0}{T}} b_i \sin(2\pi f_c t) \quad 0 \le t < T$$

  - E_0 is the energy of the signal with the lowest amplitude
  - a_i, b_i are a pair of independent integers

MQAM (cont'd)

- Basis functions:

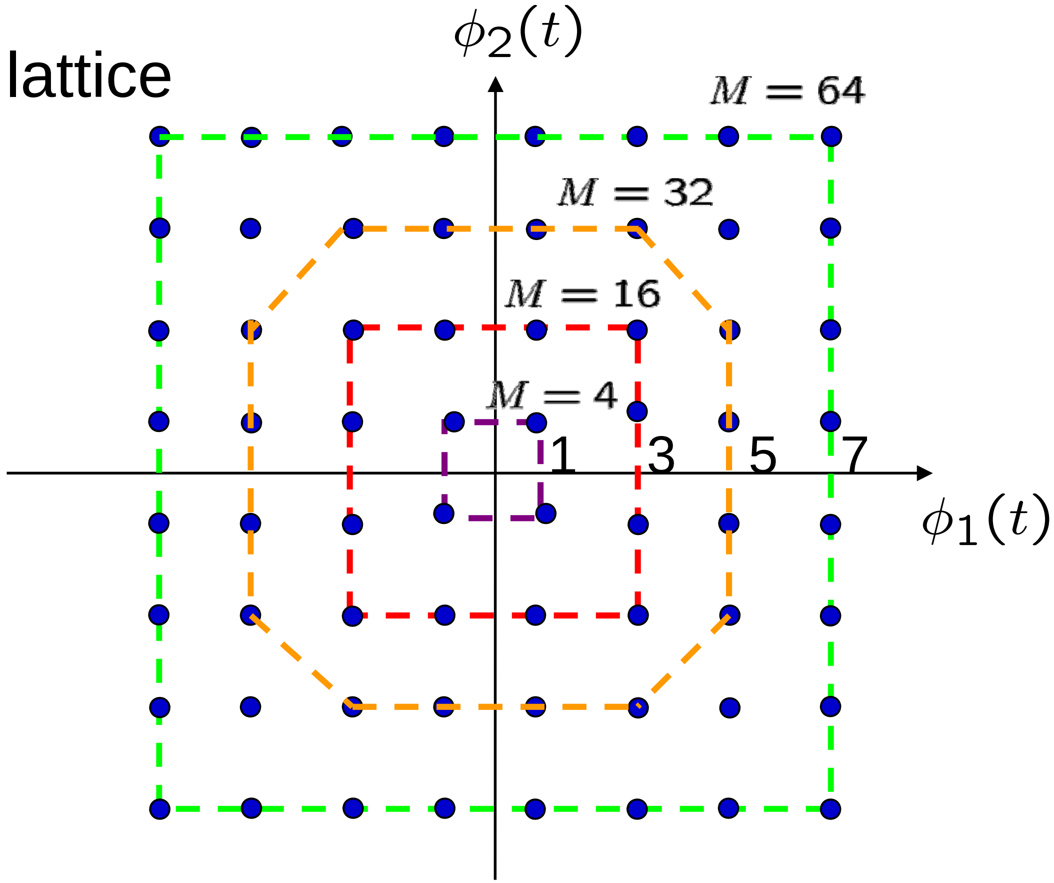
$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \qquad 0 \le t < T$$
$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$$

- Signal space representation

$$\vec{s}_i = [\sqrt{E_0} a_i \quad \sqrt{E_0} b_i]$$

MQAM Signal Constellation

- Square lattice



- Can be related with two L-ary ASK in in-phase and quadrature components, respectively, where $M = L^2$

Error Performance of MQAM

- It can be shown that the symbol error probability of MQAM is tightly upper bounded as

$$P_e \leq 4Q\left(\sqrt{\frac{3kE_b}{(M-1)N_0}}\right) \qquad \text{(for } M = 2^k \text{)}$$

- **Exercise:** From the above expression, determine the increase in the average energy per bit Eb required to maintain the same error performance if the number of bits per symbol is increased from k to k+1, where k is large.

M-ary Frequency-Shift Keying (MFSK) or Multitone Signaling

- Signal set:

$$s_m(t) = \sqrt{\frac{2E_s}{T}} \cos\{2\pi(f_c + (m-1)\triangle f)t\} \qquad \begin{matrix} m = 1, \ldots, M \\ 0 \le t < T \end{matrix}$$

  - where $\triangle f = f_m - f_{m-1}$ with $f_m = f_c + m\triangle f$

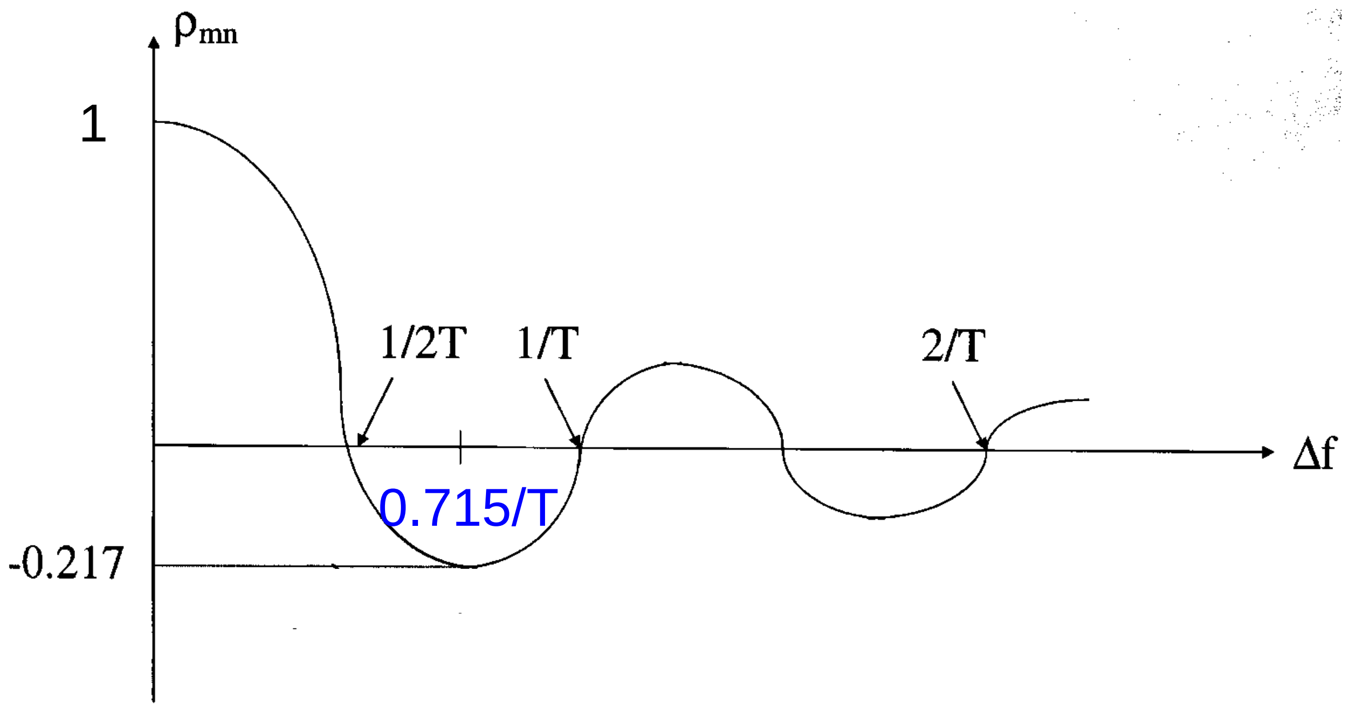
- As a measure of similarity between a pair of signal waveforms, we define the correlation coefficients

$$\rho_{mn} = \frac{1}{E_s}\int_0^T s_m(t)s_n(t)dt$$

$$= \frac{\sin[2\pi(m-n)\triangle fT]}{2\pi(m-n)\triangle fT}$$

$$= \text{sinc}[2(m-n)\triangle fT]$$

MFSK (cont'd)



- For orthogonality, minimum frequency separation between successive frequencies is 1/(2T)

- M-ary orthogonal FSK has a geometric presenation as M M-dim orthogonal vectors, given as

$$\mathbf{s}_0 = \left(\sqrt{E_s}, 0, 0, \cdots, 0\right)$$

$$\mathbf{s}_1 = \left(0, \sqrt{E_s}, 0, \cdots, 0\right)$$

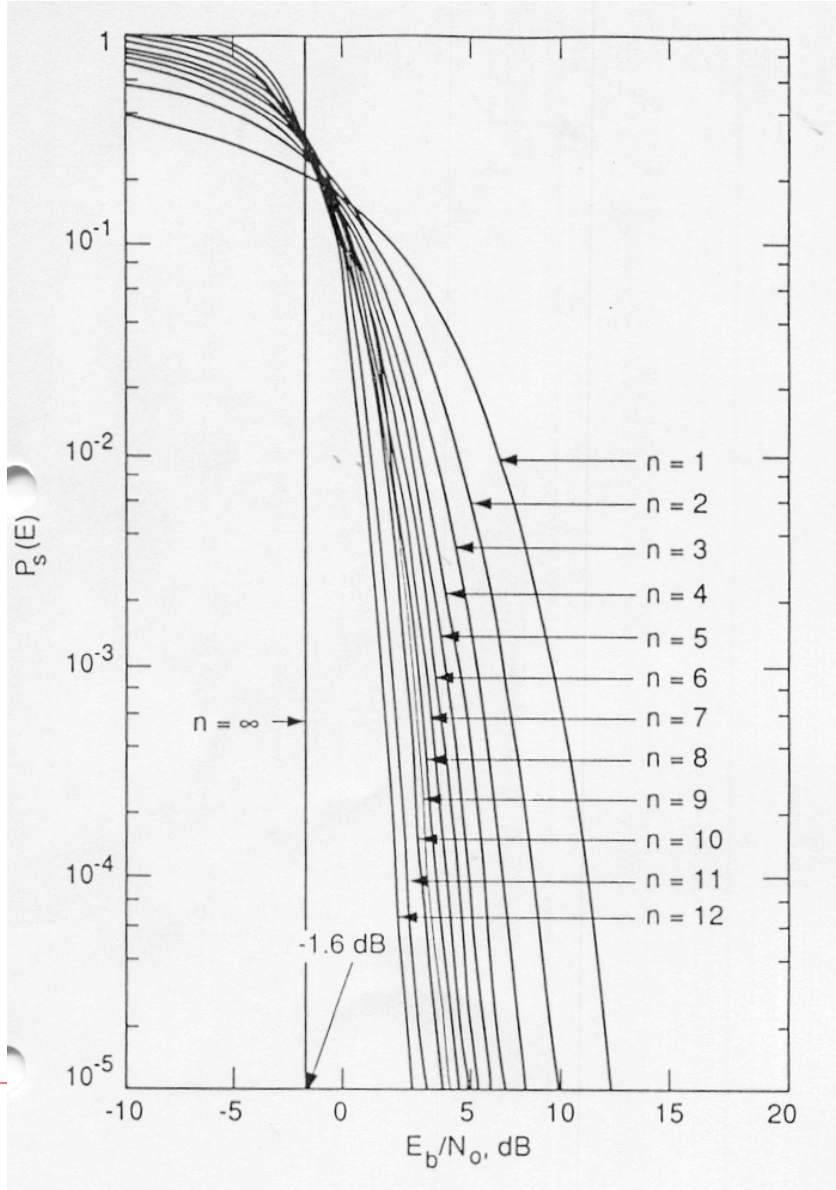
$$\vdots$$

$$\mathbf{s}_{M-1} = \left(0, 0, \cdots, 0, \sqrt{E_s}\right)$$

- The basis functions are

$$\phi_m = \sqrt{\frac{2}{T}} \cos 2\pi \left(f_c + m\Delta f \right) t$$

Error Performance of MFSK



Notes on Error Probability Calculations

- Pe is found by integrating conditional probability of error over the decision region
  - Difficult for multi-dimensions
  - Can be simplified using union bound (see ch04)

- Pe depends only on the distance profile of signal constellation

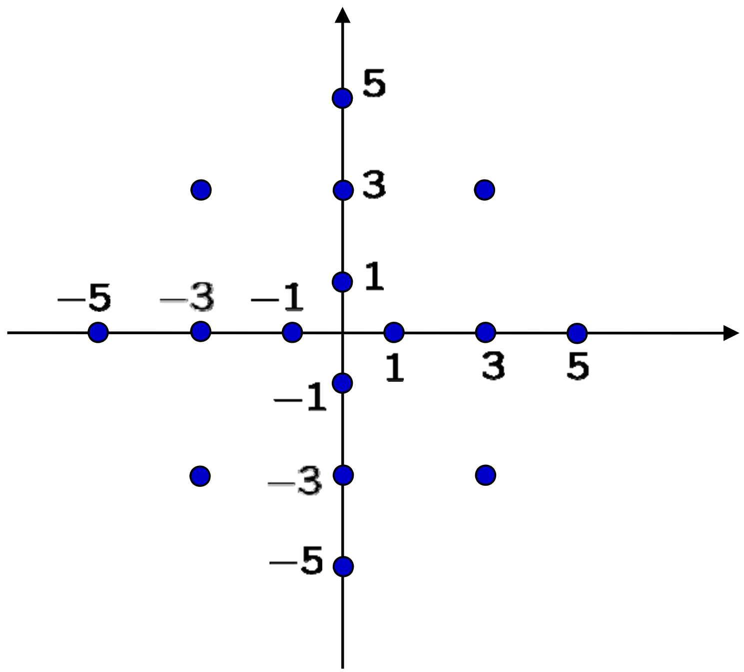
Example #2

The 16-QAM signal constellation shown below is an international standard for telephone-line modems (called V.29).

a) Determine the optimum decision boundaries for the detector

b) Derive the union bound of the probability of symbol error assuming that the SNR is sufficiently high so that errors only occur between adjacent points

c) Specify a Gray code for this 16-QAM V.29 signal constellation



Symbol Error versus Bit Error

- Symbol errors are different from bit errors
- When a symbol error occurs, all $k = \log_2 M$ bits could be in error
- In general, we can find BER using

$$P_b = \sum_{i=1}^{M} P(\vec{s}_i) \sum_{j=1, j\neq i}^{M} \frac{n_{i,j}}{\log_2 M} P\left(\hat{\vec{s}} = \vec{s}_j | \vec{s}_i\right)$$

  - n_{ij} is the number bits which differ between s_i and s_j

Bit Error Rate with Gray Coding

- Gray coding is a bit-to-symbol mapping
- When going from one symbol to an adjacent symbol, only one bit out of the k bits changes
- An error between adjacent symbol pairs results in one and only one bit error.

Example: Gray Code for QPSK

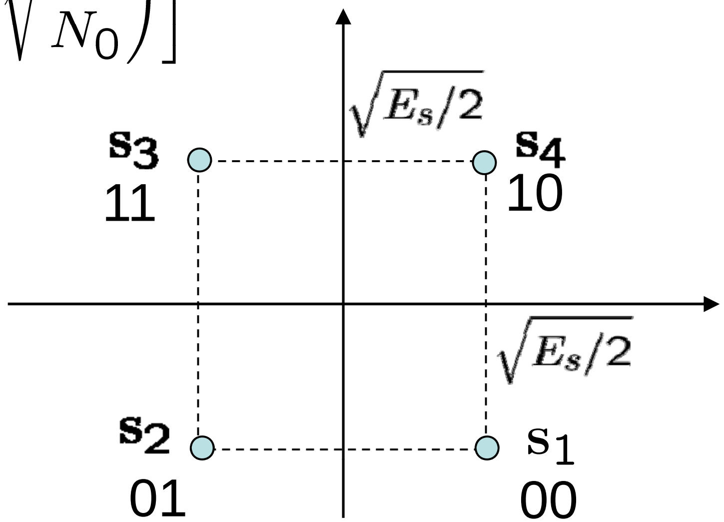
$$P_b = \sum_{i=1}^{M} \frac{1}{4} \sum_{j=1, j \neq i}^{M} \frac{n_{i,j}}{\log_2 M} P\left(\hat{\vec{s}} = \vec{s}_j | \vec{s}_i\right)$$

$$= \frac{1}{2} P\left(\hat{\vec{s}} = \vec{s}_1 | \vec{s}_4\right) + \frac{2}{2} P\left(\hat{\vec{s}} = \vec{s}_2 | \vec{s}_4\right) + \frac{1}{2} P\left(\hat{\vec{s}} = \vec{s}_3 | \vec{s}_4\right)$$

$$= \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right] \cdot Q\left(\sqrt{\frac{E_s}{N_0}}\right) + \left[Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2$$

$$= Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$



Bit Error Rate for MPSK and MFSK

- For MPSK with gray coding
  - An error between adjacent symbol will most likely occur
  - Thus, bit error probability can be approximated by

$$P_b \approx \frac{P_e}{\log_2 M}$$

- For MFSK
  - When an error occurs anyone of the other symbols may result equally likely.
  - On average, therefore, half of the bits will be incorrect. That is k/2 bits every k bits will on average be in error when there is a symbol error
  - Thus, the probability of bit error is approximately half the symbol error

$$P_b \cong \frac{1}{2} P_e$$

7.4 Comparison of M-ary Modulation Techniques

- Channel bandwidth and transmit power are two primary communication resources and have to be used as efficient as possible
  - Power utilization efficiency (energy efficiency): measured by the required E_b/N_o to achieve a certain bit error probability
  - Spectrum utilization efficiency (bandwidth efficiency): measured by the achievable data rate per unit bandwidth R_b/B
- It is always desired to maximize bandwidth efficiency at a minimal required Eb/No

Example # 3

Suppose you are a **system engineer** designing a part of the communication systems. You are required to design three systems as follow:

I. **An ultra-wideband system**. This system can use a large of amount of bandwidth to communicate. But the band it uses is overlaying with the other communication system. The main purpose of deploying this system is to provide high data rates.

II. **A wireless remote control system** designated for controlling devices remotely under unlicensed band.

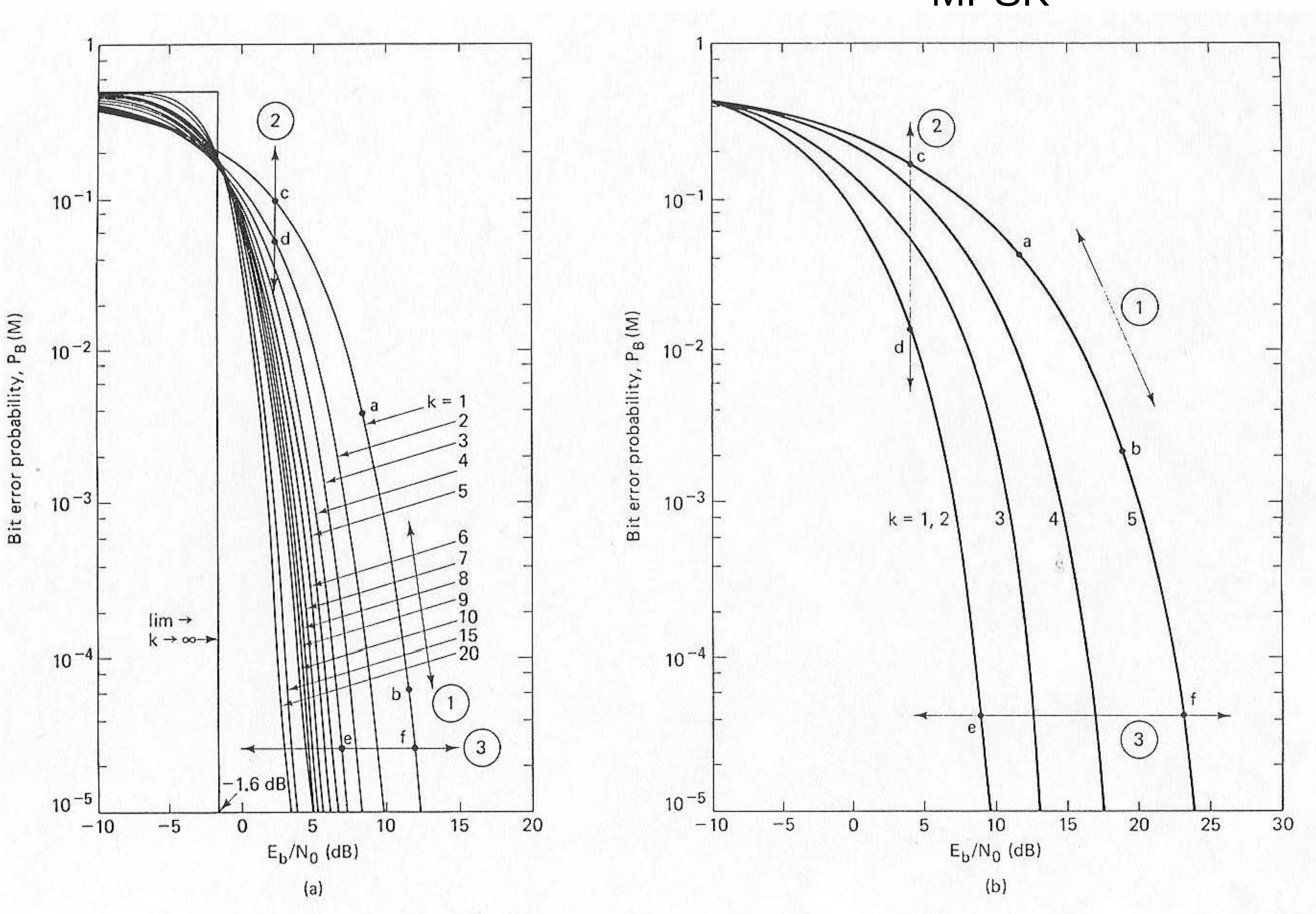
III. **A fixed wireless system**. The transmitters and receivers are mounted in a fixed position with power supply. This system is to support voice and data connections in the rural areas or in developing countries. The main reason to deploy this in such areas is because it is either very difficult or not costeffective to cover the area through wired networks. This system works under licensed band.

You are only required to **design a modulation scheme for each of the above systems**. You are allowed to use MFSK, MPSK and MSK only. If you choose to use MFSK or MPSK, you also need to state the modulation level. For simplicity, the modulation level should be chosen from M=[Low, Medium, High]. Justify your answers. (**Hints**: Federal Communications Commission (FCC) has a power spectral density limit in unlicensed band. It is meant that if your system works under unlicensed band, the power cannot be larger than a limit.)

Energy Efficiency Comparison

MFSK

MPSK



Energy Efficiency Comparison (cont'd)

- MFSK:
  - At fixed E_b/N_o, increase M can provide an improvement on P_b
  - At fixed P_b increase M can provide a reduction in the E_b/N_o requirement
- MPSK
  - BPSK and QPSK have the same energy efficiency
  - At fixed E_b/N_o, increase M degrades Pb
  - At fixed Pb, increase M increases the Eb/No requirement

MFSK is more energy efficient than MPSK

Bandwidth Efficiency Comparison

- To compare bandwidth efficiency, we need to know the power spectral density (power spectra) of a given modulation scheme
- MPSK/MQAM
  - Bandwidth required to pass MPSK/MQAM signal is given by

$$\boxed{B = \frac{1}{T_s}}$$

  - But

$$R_b = \frac{\log_2 M}{T_s} \text{ = bit rate}$$

  - Then bandwidth efficiency may be expressed as

$$\boxed{\rho = \frac{R_b}{B} = \log_2 M \text{ (bits/sec/Hz)}}$$

Bandwidth Efficiency Comparison (cont'd)

- MFSK:
  - Bandwidth required to transmit MFSK signal is

$$\boxed{B = \frac{M}{2T}}$$

(Adjacent frequencies need to be separated by 1/2T to maintain orthogonality)

  - Bandwidth efficiency of MFSK signal

$$\boxed{\rho = \frac{R_b}{B} = \frac{2\log_2 M}{M} \quad \text{(bits/s/Hz)}}$$

- As M increases, bandwidth efficiency of MPSK/MQAM increases, but bandwidth efficiency of MFSK decreases.
- This is a consequence of the fact that the dimension of the signal space is two for MPSK/MQAM and is M for MFSK.

Fundamental Tradeoff :
Bandwidth Efficiency and Energy Efficiency

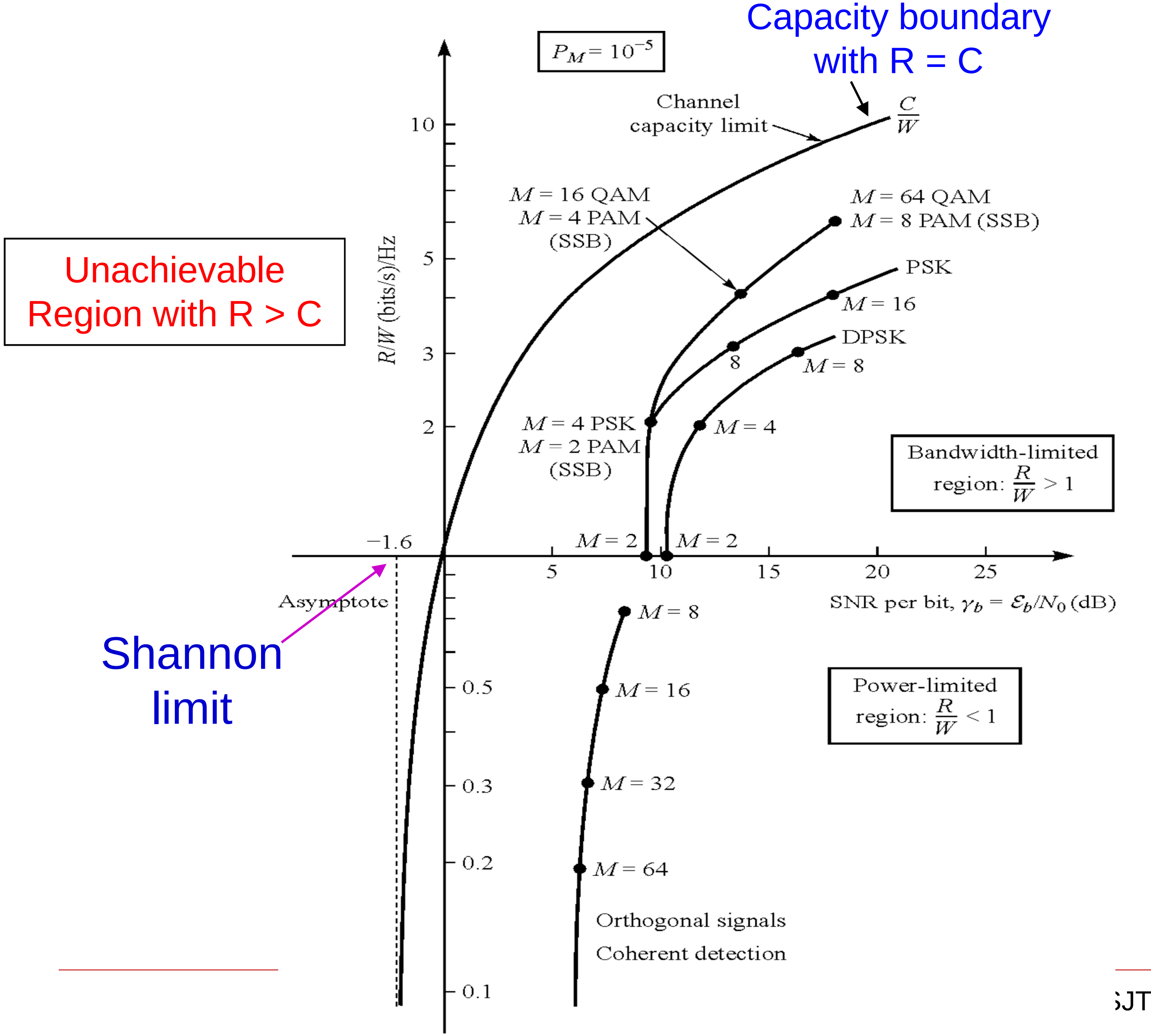
- To see the ultimate power-bandwidth tradeoff, we need to use Shannon's channel capacity theorem:
  - Channel Capacity is the theoretical upper bound for the maximum rate at which information could be transmitted without error *(Shannon 1948)*
  - For a bandlimited channel corrupted by AWGN, the maximum rate achievable is given by

$$R \le C = B\log_2(1+SNR) = B\log_2\left(1+\frac{P_s}{N_0 B}\right)$$

- Note that $$\frac{E_b}{N_0} = \frac{P_s T}{N_0} = \frac{P_s}{RN_0} = \frac{P_s B}{RN_0 B} = SNR\frac{B}{R}$$

- Thus $$\frac{E_b}{N_0} = \frac{B}{R}\left(2^{R/B}-1\right)$$

Power-Bandwidth Tradeoff



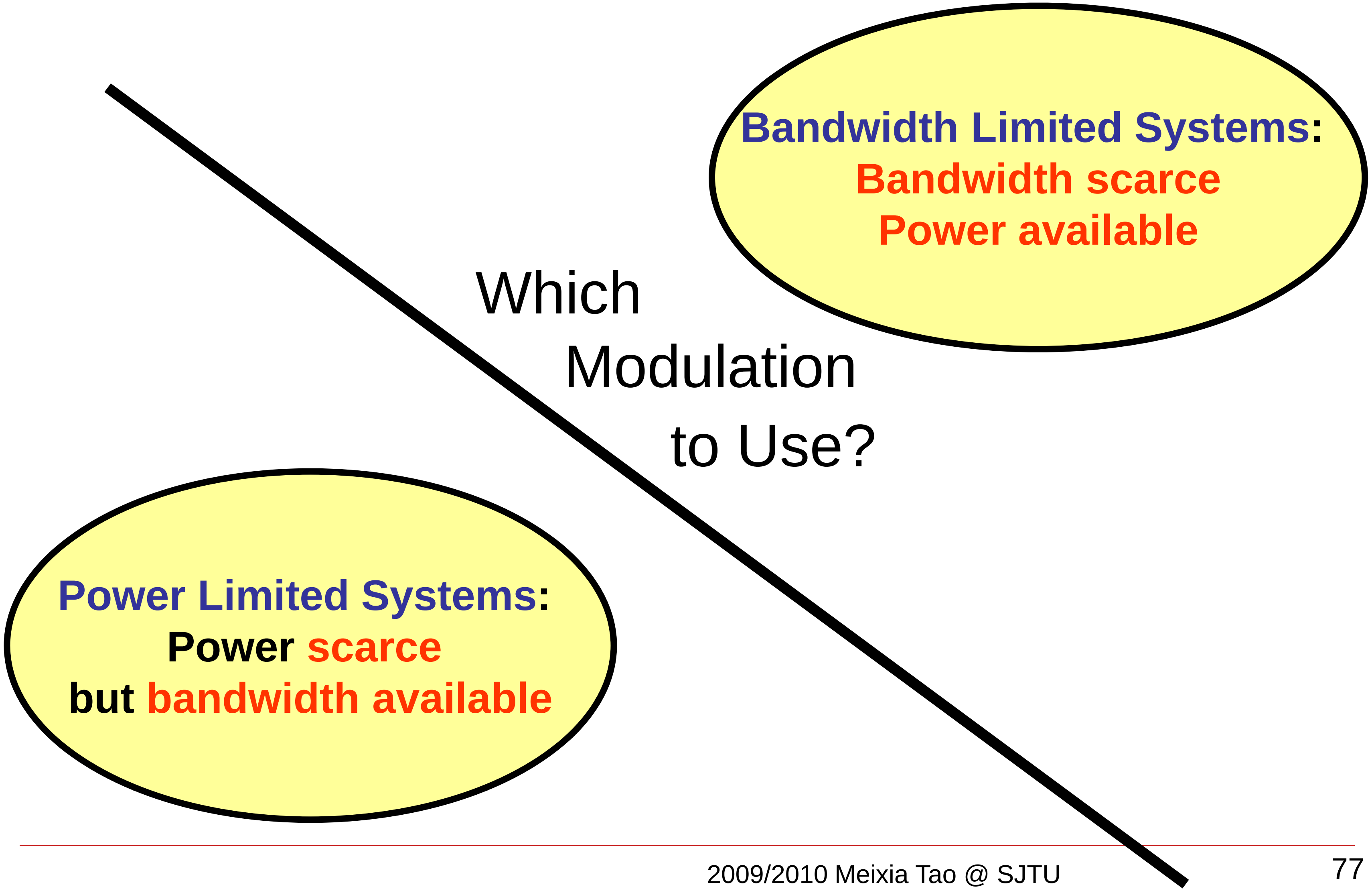
Notes on the Fundamental Tradeoff

- In the limits as R/B goes to 0, we get

$$\frac{E_b}{N_0} = \ln 2 = 0.693 = -1.59 \text{ dB}$$

  - This value is called the Shannon Limit
  - Received E_b/N_0 must be >-1.6dB for reliable communications to be possible
- BPSK and QPSK require the same Eb/N0 of 9.6 dB to achieve $P_e=10^{-5}$. However, QPSK has a better bandwidth efficiency, which is why QPSK is so popular
- MQAM is superior to MPSK
- MPSK/MQAM increases bandwidth efficiency at the cost of lower energy efficiency
- MFSK trades energy efficiency at reduced bandwidth efficiency.

System Design Tradeoff



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Practical Applications

- BPSK:
  - WLAN IEEE802.11b (1 Mbps)
- QPSK:
  - WLAN IEEE802.11b (2 Mbps, 5.5 Mbps, 11 Mbps)
  - 3G WDMA
  - DVB-T (with OFDM)
- QAM
  - Telephone modem (16QAM)
  - Downstream of Cable modem (64QAM, 256QAM)
  - WLAN IEEE802.11a/g (16QAM for 24Mbps, 36Mbps; 64QAM for 38Mbps and 54 Mbps)
  - LTE Cellular Systems
- FSK:
  - Cordless telephone
  - Paging system