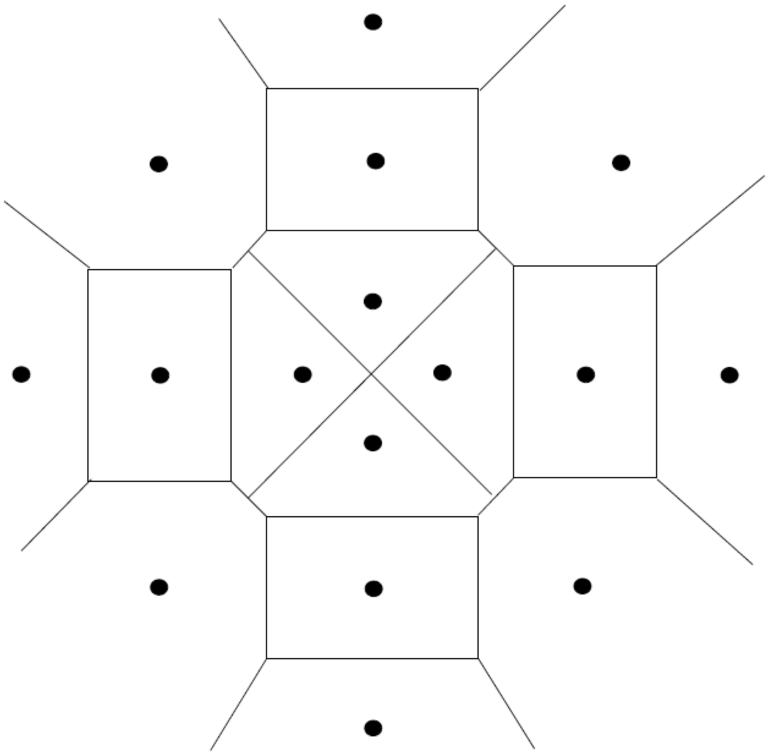
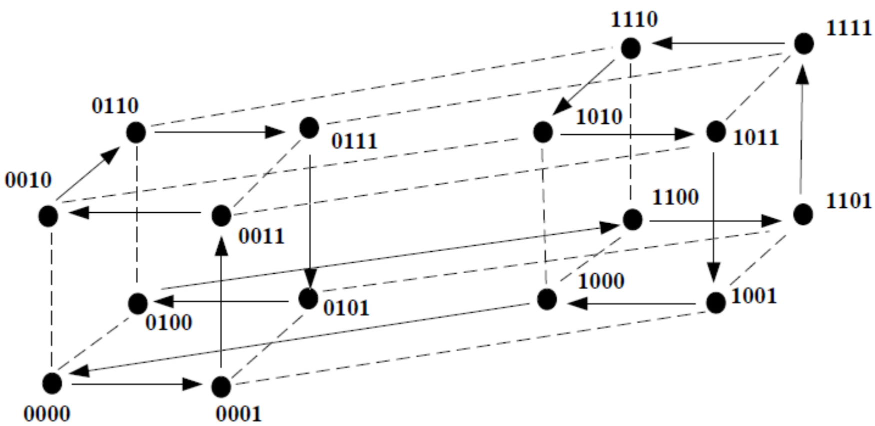
Problem 8.29

The optimum decision boundary of a point is determined by the perpendicular bisectors of each line segment connecting the point with its neighbors. The decision regions for the V.29 constellation are depicted in the next figure.

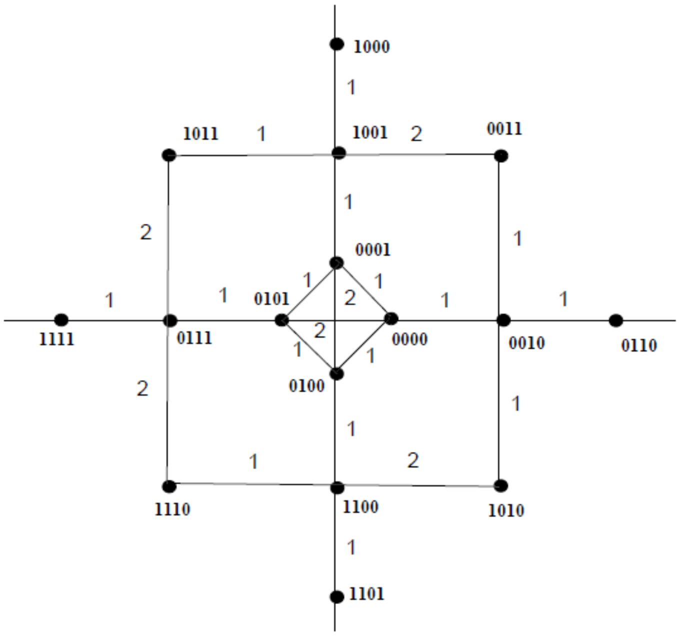


Problem 8.30

The following figure depicts a 4-cube and the way that one can traverse it in Gray-code order (see John F. Wakerly, Digital Design Principles and Practices, Prentice Hall, 1990). Adjacent points are connected with solid or dashed lines.



One way to label the points of the V.29 constellation using the Gray-code is depicted in the next figure. Note that the maximum Hamming distance between points with distance between them as large as 3 is only 2. Having labeled the innermost points, all the adjacent nodes can be found using the previous figure.



Problem 8.32

1) Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, this is not the case for the 8-QAM constellation of Figure P-10.12. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0, 0, \ldots, 0\}$ to point A, then point B and C should have the form

$$B = \{0, \ldots, 0, 1, 0, \ldots, 0\} \qquad C = \{0, \ldots, 0, 1, 0, \ldots, 0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

2) Since each symbol conveys 3 bits of information, the resulted symbol rate is

$$R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec}$$

3) The probability of error for an M-ary PSK signal is

$$P_M = 2Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} \sin\frac{\pi}{M}\right]$$

whereas the probability of error for an M-ary QAM signal is upper bounded by

$$P_M = 4Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

Since, the probability of error is dominated by the argument of the Q function, the two signals will achieve the same probability of error if

$$\sqrt{2\mathrm{SNR}_{\mathrm{PSK}}} \sin\frac{\pi}{M} = \sqrt{\frac{3\mathrm{SNR}_{\mathrm{QAM}}}{M-1}}$$

With $M = 8$ we obtain

$$\sqrt{2\mathrm{SNR}_{\mathrm{PSK}}} \sin\frac{\pi}{8} = \sqrt{\frac{3\mathrm{SNR}_{\mathrm{QAM}}}{7}} \Longrightarrow \frac{\mathrm{SNR}_{\mathrm{PSK}}}{\mathrm{SNR}_{\mathrm{QAM}}} = \frac{3}{7 \times 2 \times 0.3827^2} = 1.4627$$

4) Assuming that the magnitude of the signal points is detected correctly, then the detector for the 8-PSK signal will make an error if the phase error (magnitude) is greater than 22.5^o. In the case of the 8-QAM constellation an error will be made if the magnitude phase error exceeds 45^o. Hence, the QAM constellation is more immune to phase errors.

Problem 8.39

The vector $\mathbf{r} = [r_1, r_2]$ at the output of the integrators is

$$\mathbf{r} = [r_1, r_2] = [\int_0^{1.5} r(t)dt, \; \int_1^2 r(t)dt]$$

If $s_1(t)$ is transmitted, then

$$\begin{aligned}
\int_0^{1.5} r(t)dt &= \int_0^{1.5} [s_1(t) + n(t)]dt = 1 + \int_0^{1.5} n(t)dt \\
&= 1 + n_1 \\
\int_1^2 r(t)dt &= \int_1^2 [s_1(t) + n(t)]dt = \int_1^2 n(t)dt \\
&= n_2
\end{aligned}$$

where n_1 is a zero-mean Gaussian random variable with variance

$$\sigma_{n_1}^2 = E\left[\int_0^{1.5}\int_0^{1.5} n(\tau)n(v)d\tau dv\right] = \frac{N_0}{2}\int_0^{1.5} d\tau = 1.5$$

and n_2 is is a zero-mean Gaussian random variable with variance

$$\sigma_{n_2}^2 = E\left[\int_1^2\int_1^2 n(\tau)n(v)d\tau dv\right] = \frac{N_0}{2}\int_1^2 d\tau = 1$$

Thus, the vector representation of the received signal (at the output of the integrators) is

$$\mathbf{r} = [1 + n_1, \; n_2]$$

Similarly we find that if $s_2(t)$ is transmitted, then

$$\mathbf{r} = [0.5 + n_1, \; 1 + n_2]$$

Suppose now that the detector bases its decisions on the rule

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} T$$

The probability of error $P(e|s_1)$ is obtained as

$$\begin{aligned}
P(e|s_1) &= P(r1 - r_2 < T|s_1) \\
&= P(1 + n_1 - n_2 < T) = P(n_1 - n_2 < T - 1) \\
&= P(n < T)
\end{aligned}$$

where the random variable $n = n_1 - n_2$ is zero-mean Gaussian with variance

$$\begin{aligned}
\sigma_n^2 &= \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2E[n_1 n_2] \\
&= \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2\int_1^{1.5} \frac{N_0}{2} d\tau \\
&= 1.5 + 1 - 2 \times 0.5 = 1.5
\end{aligned}$$

Hence,

$$P(e|s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx$$

Similarly we find that

$$\begin{aligned}
P(e|s_2) &= P(0.5 + n_1 - 1 - n_2 > T) \\
&= P(n_1 - n_2 > T + 0.5) \\
&= \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx
\end{aligned}$$

The average probability of error is

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\
&= \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx + \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx
\end{aligned}$$

To find the value of T that minimizes the probability of error, we set the derivative of $P(e)$ with respect to T equal to zero. Using the Leibnitz rule for the differentiation of definite integrals, we obtain

$$\frac{\vartheta P(e)}{\vartheta T} = \frac{1}{2\sqrt{2\pi\sigma_n^2}} \left[e^{-\frac{(T-1)^2}{2\sigma_n^2}} - e^{-\frac{(T+0.5)^2}{2\sigma_n^2}} \right] = 0$$

or

$$(T-1)^2 = (T+0.5)^2 \Longrightarrow T = 0.25$$

Thus, the optimal decision rule is

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} 0.25$$

9.9.

$$s_1(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t)$$

$$s_2(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t + 2\pi \Delta f t)$$

$$\rho_{12} = \frac{1}{E_s} \int_0^T s_1(t)\, s_2(t)\, dt = \frac{1}{T}\left[\int_0^T \cos(2\pi \Delta f t)\, dt + \int_0^T \cos(4\pi f_c t + 2\pi \Delta f t) \right]$$

$$\because \quad f_c \gg \frac{1}{T}$$

$$\therefore \quad \rho_{12} = \frac{1}{T} \int_0^T \cos(2\pi \Delta f t)\, dt = \frac{\sin(2\pi \Delta f t)}{2\pi \Delta f t}$$

$$\frac{d\rho_{12}}{dt} = \frac{2\pi\Delta f \cdot \cos(2\pi\Delta f t) \cdot 2\pi\Delta f t - \sin(2\pi\Delta f t)\cdot 2\pi\Delta f}{(2\pi\Delta f t)^2} = 0$$

$$\therefore \quad 2\pi \Delta f T = \tan(2\pi \Delta f T)$$

$$\therefore \quad 2\pi \Delta f T = 4.4934$$

$$\therefore \quad \Delta f = \frac{0.7151}{T}$$

$$\cos\theta_{12} = -0.2172$$

$$d_{12}^2 = \|s_1\|^2 + \|s_2\|^2 - 2\|s_1\|\|s_2\| \cos\theta_{12} = 2E_s(1-\cos\theta_{12}).$$

$$P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{1.2172E_s}{N_0}}\right)$$

9.10.

(1) $$r(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t + \phi_m) + n(t)$$

$$r_m = \int_0^T r(t) \cos(2\pi f_c t + 2\pi m \Delta f t + \hat{\phi}_m)\, dt$$

$$= \sqrt{\frac{2E_s}{T}} \cdot \frac{1}{2} \int_0^T \cos(2\pi m \Delta f t + \hat{\phi}_m - \phi_m)\, dt + n \qquad n \sim N\left(0, \frac{N_0}{2}\right)$$

(2) $$\int_0^T \cos(2\pi m \Delta f t + \hat{\phi}_m - \phi_m)\, dt = 0$$

$$m\Delta f = n \cdot \frac{1}{T}$$

$$\therefore \quad (\Delta f)_{min} = \frac{1}{T}$$

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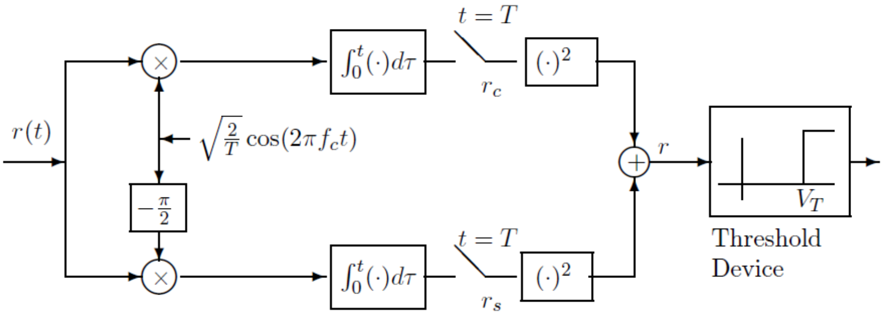
Problem 9.12

1. The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.



2. If $s_0(t)$ is sent, then the received signal is $r(t) = n(t)$ and therefore the sampled outputs r_c, r_s are zero-mean independent Gaussian random variables with variance $\frac{N_0}{2}$. Hence, the random variable $r = \sqrt{r_c^2 + r_s^2}$ is Rayleigh distributed and the PDF is given by :

$$p(r|s_0(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0} e^{-\frac{r^2}{N_0}}$$

If $s_1(t)$ is transmitted, then the received signal is :

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t)$$

Crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}}\cos(2\pi f_c t)$ and sampling the output at $t = T$, results in

$$\begin{aligned}
r_c &= \int_0^T r(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt \\
&= \int_0^T \frac{2\sqrt{\mathcal{E}_b}}{T_b}\cos(2\pi f_c t + \phi)\cos(2\pi f_c t)dt + \int_0^T n(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt \\
&= \frac{2\sqrt{\mathcal{E}_b}}{T_b}\int_0^T \frac{1}{2}\left(\cos(2\pi 2 f_c t + \phi) + \cos(\phi)\right)dt + n_c \\
&= \sqrt{\mathcal{E}_b}\cos(\phi) + n_c
\end{aligned}$$

where n_c is zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. Similarly, for the quadrature component we have :

$$r_s = \sqrt{\mathcal{E}_b}\sin(\phi) + n_s$$

The PDF of the random variable $r = \sqrt{r_c^2 + r_s^2} = \sqrt{\mathcal{E}_b + n_c^2 + n_s^2}$ follows the Rician distibution :

$$p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) = \frac{2r}{N_0} e^{-\frac{r^2+\mathcal{E}_b}{N_0}} I_0\left(\frac{2r\sqrt{\mathcal{E}_b}}{N_0}\right)$$

3. For equiprobable signals the probability of error is given by:

$$P(\text{error}) = \frac{1}{2}\int_{-\infty}^{V_T} p(r|s_1(t))dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t))dr$$

Since $r > 0$ the expression for the probability of error takes the form

$$\begin{aligned} P(\text{error}) &= \frac{1}{2}\int_0^{V_T} p(r|s_1(t))dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t))dr \\ &= \frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr + \frac{1}{2}\int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr \end{aligned}$$

The optimum threshold level is the value of V_T that minimizes the probability of error. However, when $\frac{\mathcal{E}_b}{N_0} \gg 1$ the optimum value is close to: $\frac{\sqrt{\mathcal{E}_b}}{2}$ and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of $I_0(x)$ we will use the approximation :

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$$

which is valid for large x, that is for high SNR. In this case :

$$\frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr \approx \frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr$$

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of $\sqrt{\mathcal{E}_b}$ and therefore, the lower limit can be substituted by $-\infty$. Also

$$\sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}$$

and therefore :

$$\begin{aligned} \frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr &\approx \frac{1}{2}\int_{-\infty}^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr \\ &= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] \end{aligned}$$

Finally :

$$\begin{aligned} P(\text{error}) &= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2}\int_{\frac{\sqrt{\mathcal{E}_b}}{2}}^{\infty} \frac{2r}{N_0} e^{-\frac{r^2}{N_0}} dr \\ &\leq \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2} e^{-\frac{\mathcal{E}_b}{4N_0}} \end{aligned}$$