Problem 12.5

$$\begin{aligned} H(X,Y) &= H(X,g(X)) = H(X) + H(g(X)|X) \\ &= H(g(X)) + H(X|g(X)) \end{aligned}$$

But, $H(g(X)|X) = 0$, since $g(\cdot)$ is deterministic. Therefore,

$$H(X) = H(g(X)) + H(X|g(X))$$

Since each term in the previous equation is non-negative we obtain

$$H(X) \geq H(g(X))$$

Equality holds when $H(X|g(X)) = 0$. This means that the values $g(X)$ uniquely determine X, or that $g(\cdot)$ is a one to one mapping.

Problem 12.13

1. The marginal distribution $P(x)$ is given by $P(x) = \sum_y P(x,y)$. Hence,

$$\begin{aligned} H(X) &= -\sum_x P(x)\log P(x) = -\sum_x \sum_y P(x,y)\log P(x) \\ &= -\sum_{x,y} P(x,y)\log P(x) \end{aligned}$$

Similarly it is proved that $H(Y) = -\sum_{x,y} P(x,y)\log P(y)$.

2. Using the inequality $\ln w \leq w - 1$ with $w = \frac{P(x)P(y)}{P(x,y)}$, we obtain

$$\ln \frac{P(x)P(y)}{P(x,y)} \leq \frac{P(x)P(y)}{P(x,y)} - 1$$

Multiplying the previous by $P(x,y)$ and adding over x, y, we obtain

$$\sum_{x,y} P(x,y)\ln P(x)P(y) - \sum_{x,y} P(x,y)\ln P(x,y) \leq \sum_{x,y} P(x)P(y) - \sum_{x,y} P(x,y) = 0$$

Hence,

$$\begin{aligned} H(X,Y) &\leq -\sum_{x,y} P(x,y)\ln P(x)P(y) = -\sum_{x,y} P(x,y)(\ln P(x) + \ln P(y)) \\ &= -\sum_{x,y} P(x,y)\ln P(x) - \sum_{x,y} P(x,y)\ln P(y) = H(X) + H(Y) \end{aligned}$$

Equality holds when $\frac{P(x)P(y)}{P(x,y)} = 1$, i.e when X, Y are independent.

Problem 12.14

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

Also, from part 2., $H(X,Y) \leq H(X) + H(Y)$. Combining the two relations, we obtain

$$H(Y) + H(X|Y) \leq H(X) + H(Y) \Longrightarrow H(X|Y) \leq H(X)$$

Suppose now that the previous relation holds with equality. Then,

$$-\sum_x P(x)\log P(x|y) = -\sum_x P(x)\log P(x) \Rightarrow \sum_x P(x)\log(\frac{P(x)}{P(x|y)}) = 0$$

However, $P(x)$ is always greater or equal to $P(x|y)$, so that $\log(P(x)/P(x|y))$ is non-negative. Since $P(x) > 0$, the above equality holds if and only if $\log(P(x)/P(x|y)) = 0$ or equivalently if and only if $P(x)/P(x|y) = 1$. This implies that $P(x|y) = P(x)$ meaning that X and Y are independent.

Problem 12.46

The capacity of the additive white Gaussian channel is :

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N_0 W} \right)$$

For the nonwhite Gaussian noise channel, although the noise power is equal to the noise power in the white Gaussian noise channel, the capacity is higher, The reason is that since noise samples

are correlated, knowledge of the previous noise samples provides partial information on the future noise samples and therefore reduces their effective variance.

12.6.

$$H(X) = \sum_{i=1}^{6} P_i \log_2 \frac{1}{P_i} = 2.4087 \text{ bits/sample}$$

$$f = 2\times 6000 + 2000 = 14000 \text{ Hz}$$

$$H(x) = 2.4087 \times 14000 = 33721.8 \text{ bits/sec}$$

12.32.

(1) $$H(X) = p\log_2\frac{1}{p} + (1-p)\log_2\frac{1}{1-p}$$

$$H(Y) = (1+2\varepsilon p-\varepsilon-p)\log_2\frac{1}{1+2\varepsilon p-\varepsilon-p} + (\varepsilon+p-2\varepsilon p)\log_2\frac{1}{\varepsilon+p-2\varepsilon p}$$

$$H(Y|X) = \varepsilon\log_2\frac{1}{\varepsilon} + (1-\varepsilon)\log_2\frac{1}{1-\varepsilon}$$

$$H(X,Y) = p\log_2\frac{1}{p} + (1-p)\log_2\frac{1}{1-p} + \varepsilon\log_2\frac{1}{\varepsilon} + (1-\varepsilon)\log_2\frac{1}{1-\varepsilon}$$

$$H(X|Y) = H(X,Y) - H(Y)$$

$$I(X;Y) = H(Y) - H(Y|X) = (1+2\varepsilon p-\varepsilon-p)\log_2\frac{1}{1+2\varepsilon p-\varepsilon-p} + (\varepsilon+p-2\varepsilon p)\log_2\frac{1}{\varepsilon+p-2\varepsilon p} - \varepsilon\log_2\frac{1}{\varepsilon} - (1-\varepsilon)\log_2\frac{1}{1-\varepsilon}$$

(2) $$\max I(X;Y) = \max H(Y).$$

$$p(Y=1) = p(Y=0) = \frac{1}{2}$$

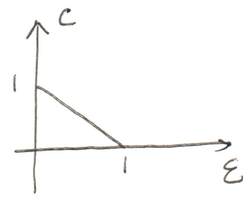
$$\therefore\ p = \frac{1}{2}$$

(3) $\min I(X;Y) \Rightarrow$ X and Y are independent.

$$\begin{cases} p(X=0,Y=0) = p(X=0)p(Y=0) \\ p(X=0,Y=1) = p(X=0)p(Y=1) \\ p(X=1,Y=0) = p(X=1)p(Y=0) \\ p(X=1,Y=1) = p(X=1)p(Y=1) \end{cases} \qquad \therefore\ \varepsilon = \frac{1}{2}$$

12.36.

$$p(X) = \begin{cases} p & X=0 \\ 1-p & X=1. \end{cases}$$



$$C = \max\{I(X;Y)\} = \max\{H(Y) - H(Y|X)\}$$

$$H(Y) = p(1-\varepsilon)\log_2\frac{1}{p(1-\varepsilon)} + (1-p)(1-\varepsilon)\log_2\frac{1}{(1-p)(1-\varepsilon)} + \varepsilon\log_2\frac{1}{\varepsilon}$$

$$H(Y|X) = pH(Y|X=0) + (1-p)H(Y|X=1) = pH(\varepsilon) + (1-p)H(\varepsilon)$$

$$C = \max\{H(Y) - H(Y|X)\} = (1-\varepsilon)H(p) = 1-\varepsilon$$

13.5.

Let the parity check matrix $\vec{H} = [\vec{h}_1, \cdots, \vec{h}_n]$ and the linear block codeword $\vec{c} = [c_1, \cdots, c_n]$. with nonzero elements $c_{i1}, \cdots, c_{il}$

$$\therefore \quad \vec{c}\vec{H} = c_1\vec{h}_1 + \cdots + c_n\vec{h}_n = c_{i1}\vec{h}_{i1} + \cdots + c_{il}\vec{h}_{il} = 0$$

$$\therefore \quad \vec{h}_{i1} + \cdots + \vec{h}_{il} = 0$$

$\therefore$ the minimum number of columns of H that are dependent $= l_{min}$

Let the minimum number of columns of H that are dependent is d_{min}.

$$\vec{c}\vec{H} = c_1\vec{h}_1 + \cdots + c_n\vec{h}_n = c_1\vec{h}_{i1} + \cdots + c_{id_{min}}\vec{h}_{id_{min}} = 0$$

$\therefore$ the minimum Haming distance $= d_{min}$.

For Hamming code, $\forall\, h_i + h_j = h_m \in H \Rightarrow h_i + h_j + h_m = 0$

$\therefore \quad d_{min} = 3$.

13.9.

$c_1, \cdots, c_{16}$ 為 Hamming codewords in P_{15} of lec 10.

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_{16} \\ e_1 & e_1 \oplus c_2 & \cdots & e_1 \oplus c_{16} \\ \vdots & \vdots & & \vdots \\ e_7 & e_7 \oplus c_2 & \cdots & e_7 \oplus c_{16} \end{bmatrix} \quad \begin{matrix} 0 \\ e_1 H^T \\ \vdots \\ e_7 H^T \end{matrix} \quad \text{syndrome}$$

$r = [1110100]$

$$H^T = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$s = rH^T = [1\ 1\ 0]$

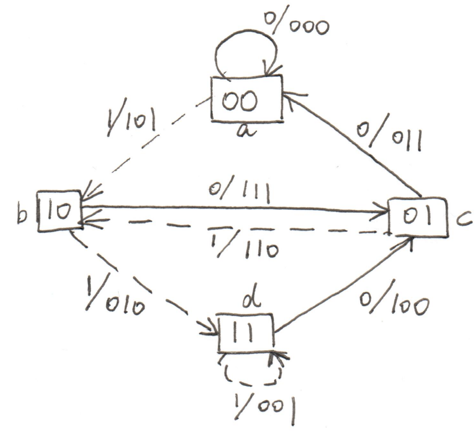
$e = [1000000]$

$\therefore c = [0110100]$

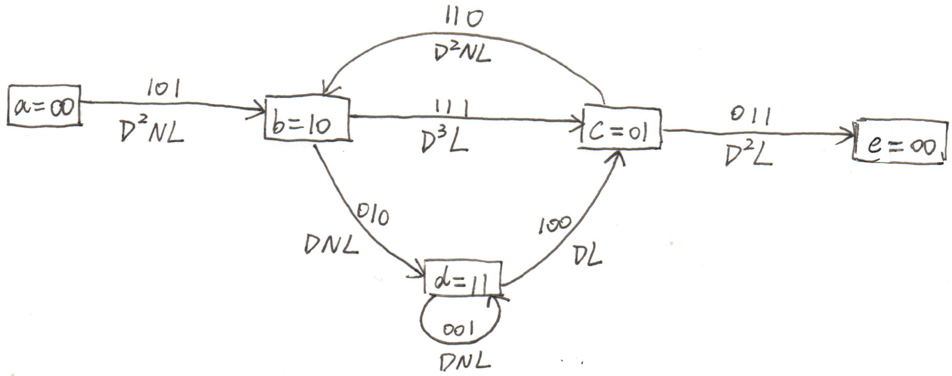
$\therefore$ the message is 0110.

13.13.

(1)



(2)

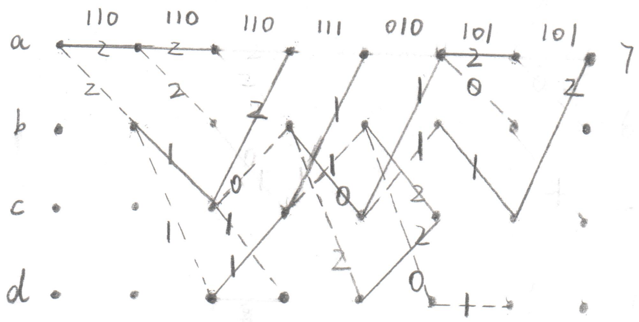


$$\begin{cases} X_b = D^2NL\,X_a + D^2NL\,X_c \\ X_c = D^3L\,X_b + DL\,X_d \\ X_d = DNL\,X_b + DNL\,X_d \\ X_e = D^2L\,X_c \end{cases}$$

$$T(D) = \frac{X_e}{X_a} = \frac{D^6N^2L^4 + D^7NL^3 - D^8N^2L^4}{1 - DNL - D^5NL^2 + D^6N^2L^3 - D^4N^2L^3}$$

(3) $d_{free} = 6$.

(4)



∴ decoding sequence is $\{101, 111, 110, 111, 110, 111, 011\}$

information sequence is $\{10101\}$

(5) $$P_b \leq \left.\frac{\partial T_2(D,N)}{\partial N}\right|_{N=1,\ D=\sqrt{4p(1-p)}} \qquad T_2(D,N) = T(D,N)\big|_{L=1} \approx D^6N^2$$

$$\approx \left.\frac{\partial D^6N^2}{\partial N}\right|_{N=1,\ D=\sqrt{4p(1-p)}} = 2(4p(1-p))^3$$

$\therefore P_b \leq 1.28 \times 10^{-13}$