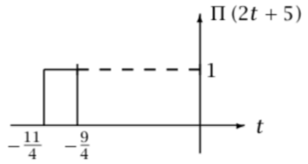
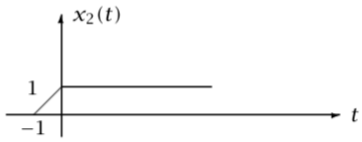
HW1 Solution

Problem 2.1

1. $\Pi(2t+5) = \Pi\left(2\left(t+\frac{5}{2}\right)\right)$. This indicates first we have to plot $\Pi(2t)$ and then shift it to left by $\frac{5}{2}$. A plot is shown below:

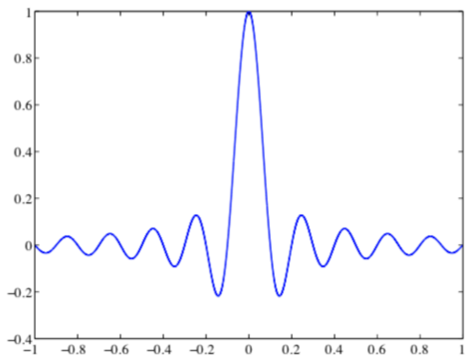


2. $\sum_{n=0}^{\infty} \Lambda(t-n)$ is a sum of shifted triangular pulses. Note that the sum of the left and right side of triangular pulses that are displaced by one unit of time is equal to 1, The plot is given below



3. It is obvious from the definition of $\text{sgn}(t)$ that $\text{sgn}(2t) = \text{sgn}(t)$. Therefore $x_3(t) = 0$.

4. $x_4(t)$ is $\text{sinc}(t)$ contracted by a factor of 10.



Problem 2.9

1)

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \left|e^{j(2\pi f_0 t+\theta)}\right|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty}\frac{1}{T}A^2 T = A^2$$

Thus $x(t) = Ae^{j(2\pi f_0 t+\theta)}$ is a power-type signal and its power content is A^2.

2)

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f_0 t+\theta)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{A^2}{2}dt + \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{A^2}{2}\cos(4\pi f_0 t+2\theta)\,dt$$

As $T\to\infty$, the there will be no contribution by the second integral. Thus the signal is a power-type signal and its power content is $\frac{A^2}{2}$.

3)

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} u_{-1}^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{\frac{T}{2}} dt = \lim_{T\to\infty}\frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Thus the unit step signal is a power-type signal and its power content is $1/2$

4)

$$\begin{aligned} E_x &= \lim_{T\to\infty}\int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty}\int_0^{\frac{T}{2}} K^2 t^{-\frac{1}{2}}dt = \lim_{T\to\infty} 2K^2 t^{\frac{1}{2}}\Big|_0^{T/2} \\ &= \lim_{T\to\infty}\sqrt{2}K^2 T^{\frac{1}{2}} = \infty \end{aligned}$$

Thus the signal is not an energy-type signal.

$$\begin{aligned} P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{\frac{T}{2}} K^2 t^{-\frac{1}{2}}dt \\ &= \lim_{T\to\infty}\frac{1}{T}2K^2 t^{\frac{1}{2}}\Big|_0^{T/2} = \lim_{T\to\infty}\frac{1}{T}2K^2(T/2)^{\frac{1}{2}} = \lim_{T\to\infty}\sqrt{2}K^2 T^{-\frac{1}{2}} = 0 \end{aligned}$$

Since P_x is not bounded away from zero it follows by definition that the signal is not of the power-type (recall that power-type signals should satisfy $0 < P_x < \infty$).

Problem 2.37

1) Since $(a-b)^2 \geq 0$ we have that

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

with equality if $a = b$. Let

$$A = \left[\sum_{i=1}^{n}\alpha_i^2\right]^{\frac{1}{2}}, \qquad B = \left[\sum_{i=1}^{n}\beta_i^2\right]^{\frac{1}{2}}$$

Then substituting α_i/A for a and β_i/B for b in the previous inequality we obtain

$$\frac{\alpha_i}{A}\frac{\beta_i}{B} \leq \frac{1}{2}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\frac{\beta_i^2}{B^2}$$

with equality if $\frac{\alpha_i}{\beta_i} = \frac{A}{B} = k$ or $\alpha_i = k\beta_i$ for all i. Summing both sides from $i = 1$ to n we obtain

$$\begin{aligned} \sum_{i=1}^{n}\frac{\alpha_i\beta_i}{AB} &\leq \frac{1}{2}\sum_{i=1}^{n}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\sum_{i=1}^{n}\frac{\beta_i^2}{B^2} \\ &= \frac{1}{2A^2}\sum_{i=1}^{n}\alpha_i^2 + \frac{1}{2B^2}\sum_{i=1}^{n}\beta_i^2 = \frac{1}{2A^2}A^2 + \frac{1}{2B^2}B^2 = 1 \end{aligned}$$

Thus,

$$\frac{1}{AB}\sum_{i=1}^{n}\alpha_i\beta_i \leq 1 \Rightarrow \sum_{i=1}^{n}\alpha_i\beta_i \leq \left[\sum_{i=1}^{n}\alpha_i^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\beta_i^2\right]^{\frac{1}{2}}$$

Equality holds if $\alpha_i = k\beta_i$, for $i = 1, \ldots, n$.

2) The second equation is trivial since $|x_i y_i^*| = |x_i||y_i^*|$. To see this write x_i and y_i in polar coordinates as $x_i = \rho_{x_i}e^{j\theta_{x_i}}$ and $y_i = \rho_{y_i}e^{j\theta_{y_i}}$. Then, $|x_i y_i^*| = |\rho_{x_i}\rho_{y_i}e^{j(\theta_{x_i}-\theta_{y_i})}| = \rho_{x_i}\rho_{y_i} = |x_i||y_i| = |x_i||y_i^*|$. We turn now to prove the first inequality. Let z_i be any complex with real and imaginary components $z_{i,R}$ and $z_{i,I}$ respectively. Then,

$$\begin{aligned} \left|\sum_{i=1}^{n} z_i\right|^2 &= \left|\sum_{i=1}^{n} z_{i,R} + j\sum_{i=1}^{n} z_{i,I}\right|^2 = \left(\sum_{i=1}^{n} z_{i,R}\right)^2 + \left(\sum_{i=1}^{n} z_{i,I}\right)^2 \\ &= \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I}) \end{aligned}$$

Since $(z_{i,R}z_{m,I} - z_{m,R}z_{i,I})^2 \geq 0$ we obtain

$$(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})^2 \leq (z_{i,R}^2 + z_{i,I}^2)(z_{m,R}^2 + z_{m,I}^2)$$

Using this inequality in the previous equation we get

$$\begin{aligned}
\left|\sum_{i=1}^{n} z_i\right|^2 &= \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I}) \\
&\leq \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)\left(\sum_{m=1}^{n}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}}\right) = \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2
\end{aligned}$$

Thus

$$\left|\sum_{i=1}^{n} z_i\right|^2 \leq \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2 \quad \text{or} \quad \left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n}|z_i|$$

The inequality now follows if we substitute $z_i = x_i y_i^*$. Equality is obtained if $\frac{z_{i,R}}{z_{i,I}} = \frac{z_{m,R}}{z_{m,I}} = k_1$ or $\angle z_i = \angle z_m = \theta$.

3) From 2) we obtain

$$\left|\sum_{i=1}^{n} x_i y_i^*\right|^2 \leq \sum_{i=1}^{n}|x_i||y_i|$$

But $|x_i|$, $|y_i|$ are real positive numbers so from 1)

$$\sum_{i=1}^{n}|x_i||y_i| \leq \left[\sum_{i=1}^{n}|x_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|y_i|^2\right]^{\frac{1}{2}}$$

Combining the two inequalities we get

$$\left|\sum_{i=1}^{n} x_i y_i^*\right|^2 \leq \left[\sum_{i=1}^{n}|x_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|y_i|^2\right]^{\frac{1}{2}}$$

From part 1) equality holds if $\alpha_i = k\beta_i$ or $|x_i| = k|y_i|$ and from part 2) $x_i y_i^* = |x_i y_i^*|e^{j\theta}$. Therefore, the two conditions are

$$\begin{cases} |x_i| = k|y_i| \\ \angle x_i - \angle y_i = \theta \end{cases}$$

which imply that for all i, $x_i = Ky_i$ for some complex constant K.

4) The same procedure can be used to prove the Cauchy-Schwartz inequality for integrals. An easier approach is obtained if one considers the inequality

$$|x(t) + \alpha y(t)| \geq 0, \qquad \text{for all } \alpha$$

Then

$$\begin{aligned}
0 &\leq \int_{-\infty}^{\infty}|x(t) + \alpha y(t)|^2 dt = \int_{-\infty}^{\infty}(x(t) + \alpha y(t))(x^*(t) + \alpha^* y^*(t))dt \\
&= \int_{-\infty}^{\infty}|x(t)|^2 dt + \alpha\int_{-\infty}^{\infty}x^*(t)y(t)dt + \alpha^*\int_{-\infty}^{\infty}x(t)y^*(t)dt + |a|^2\int_{-\infty}^{\infty}|y(t)|^2 dt
\end{aligned}$$

The inequality is true for $\int_{-\infty}^{\infty}x^*(t)y(t)dt = 0$. Suppose that $\int_{-\infty}^{\infty}x^*(t)y(t)dt \neq 0$ and set

$$\alpha = -\frac{\int_{-\infty}^{\infty}|x(t)|^2 dt}{\int_{-\infty}^{\infty}x^*(t)y(t)dt}$$

Then,

$$0 \leq -\int_{-\infty}^{\infty}|x(t)|^2 dt + \frac{[\int_{-\infty}^{\infty}|x(t)|^2 dt]^2\int_{-\infty}^{\infty}|y(t)|^2 dt}{|\int_{-\infty}^{\infty}x(t)y^*(t)dt|^2}$$

and

$$\left|\int_{-\infty}^{\infty}x(t)y^*(t)dt\right| \leq \left[\int_{-\infty}^{\infty}|x(t)|^2 dt\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}|y(t)|^2 dt\right]^{\frac{1}{2}}$$

Equality holds if $x(t) = -\alpha y(t)$ a.e. for some complex α.

Problem 2.46

1) Using the Fourier transform pair

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2} \frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

and the duality property of the Fourier transform: $X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$ we obtain

$$\left(\frac{2\alpha}{4\pi^2}\right) \mathcal{F}\left[\frac{1}{\frac{\alpha^2}{4\pi^2} + t^2}\right] = e^{-\alpha|f|}$$

With $\alpha = 2\pi$ we get the desired result

$$\mathcal{F}\left[\frac{1}{1+t^2}\right] = \pi e^{-2\pi|f|}$$

2)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3) + \Pi(t+3)] \\
&= \mathrm{sinc}(f)e^{-j2\pi f3} + \mathrm{sinc}(f)e^{j2\pi f3} \\
&= 2\mathrm{sinc}(f)\cos(2\pi 3f)
\end{aligned}$$

3) $\mathcal{F}[\Pi(t/4)] = 4\,\mathrm{sinc}(4f)$, hence $\mathcal{F}[4\Pi(t/4)] = 16\,\mathrm{sinc}(4f)$. Using modulation property of FT we have $\mathcal{F}[4\Pi(t/4)\cos(2\pi f_0 t)] = 8\,\mathrm{sinc}(4(f-f_0)) + 8\,\mathrm{sinc}(4(f+f_0))$.

4)

$$\mathcal{F}[t\mathrm{sinc}(t)] = \frac{1}{\pi}\mathcal{F}[\sin(\pi t)] = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2}) - \delta(f-\frac{1}{2})\right]$$

The same result is obtain if we recognize that multiplication by t results in differentiation in the frequency domain. Thus

$$\mathcal{F}[t\mathrm{sinc}] = \frac{j}{2\pi}\frac{d}{df}\Pi(f) = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2}) - \delta(f-\frac{1}{2})\right]$$

5)

$$\begin{aligned}
\mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right) \\
&= \frac{j}{4\pi}\left(\delta'(f-f_0) + \delta'(f+f_0)\right)
\end{aligned}$$

Problem 2.48

$$\begin{aligned}
\mathcal{F}[\frac{1}{2}(\delta(t+\frac{1}{2}) + \delta(t-\frac{1}{2}))] &= \int_{-\infty}^{\infty} \frac{1}{2}(\delta(t+\frac{1}{2}) + \delta(t-\frac{1}{2}))e^{-j2\pi ft}dt \\
&= \frac{1}{2}(e^{-j\pi f} + e^{-j\pi f}) = \cos(\pi f)
\end{aligned}$$

Using the duality property of the Fourier transform:

$$X(f) = \mathcal{F}[x(t)] \Longrightarrow x(f) = \mathcal{F}[X(-t)]$$

we obtain

$$\mathcal{F}[\cos(-\pi t)] = \mathcal{F}[\cos(\pi t)] = \frac{1}{2}(\delta(f+\frac{1}{2}) + \delta(f-\frac{1}{2}))$$

Note that $\sin(\pi t) = \cos(\pi t + \frac{\pi}{2})$. Thus

$$\begin{aligned}
\mathcal{F}[\sin(\pi t)] &= \mathcal{F}[\cos(\pi(t+\frac{1}{2}))] = \frac{1}{2}(\delta(f+\frac{1}{2}) + \delta(f-\frac{1}{2}))e^{j\pi f} \\
&= \frac{1}{2}e^{j\pi\frac{1}{2}}\delta(f+\frac{1}{2}) + \frac{1}{2}e^{-j\pi\frac{1}{2}}\delta(f-\frac{1}{2}) \\
&= \frac{j}{2}\delta(f+\frac{1}{2}) - \frac{j}{2}\delta(f-\frac{1}{2})
\end{aligned}$$

Problem 5.5
Let us denote by nS the event that n was produced by the source and sent over the channel, and by nC the event that n was observed at the output of the channel. Then
1)

$$\begin{aligned} P(1C) &= P(1C|1S)P(1S) + P(1C|0C)P(0C) \\ &= .8 \cdot .7 + .2 \cdot .3 = .62 \end{aligned}$$

where we have used the fact that $P(1S) = .7$, $P(0C) = .3$, $P(1C|0C) = .2$ and $P(1C|1S) = 1 - .2 = .8$
2)

$$P(1S|1C) = \frac{P(1C, 1S)}{P(1C)} = \frac{P(1C|1S)P(1S)}{P(1C)} = \frac{.8 \cdot .7}{.62} = .9032$$

Problem 5.6
1) X can take four different values. 0, if no head shows up, 1, if only one head shows up in the four flips of the coin, 2, for two heads and 3 if the outcome of each flip is head.
2) X follows the binomial distribution with $n = 3$. Thus

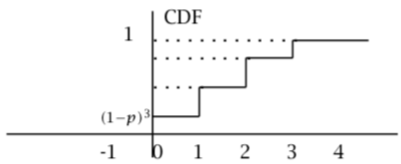
$$P(X = k) = \begin{cases} \begin{pmatrix} 3 \\ k \end{pmatrix} p^k (1-p)^{3-k} & \text{for } 0 \le k \le 3 \\ 0 & \text{otherwise} \end{cases}$$

3)

$$F_X(k) = \sum_{m=0}^{k} \begin{pmatrix} 3 \\ m \end{pmatrix} p^m (1-p)^{3-m}$$

Hence

$$F_X(k) = \begin{cases} 0 & k < 0 \\ (1-p)^3 & k = 0 \\ (1-p)^3 + 3p(1-p)^2 & k = 1 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) & k = 2 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) + p^3 = 1 & k = 3 \\ 1 & k > 3 \end{cases}$$



4)

$$P(X > 1) = \sum_{k=2}^{3} \begin{pmatrix} 3 \\ k \end{pmatrix} p^k (1-p)^{3-k} = 3p^2(1-p) + (1-p)^3$$

Problem 5.28
1) Z and W are linear combinations of jointly Gaussian RV's, therefore they are jointly Gaussian too.

2) Since Z and W are jointly Gaussian with zero-mean, they are independent if they are uncorrelated. This implies that they are independent if $E[ZW] = 0$. But $E[ZW] = E[XY](\cos^2\theta - \sin^2\theta)$ where we have used the fact that since X and Y are zero-mean and have the same variance we have $E[X^2] = E[Y^2]$, and therefore, $(E(Y^2) - E(X^2)) \sim \theta \cos\theta = 0$. From above, in order for Z and W to be independent we must have

$$\cos^2\theta - \sin^2\theta = 0 \Longrightarrow \theta = \frac{\pi}{4} + k\frac{\pi}{2}, \quad k \in Z$$

Note also that if X and Y are independent, then $E[XY] = 0$ and any rotation will produce independent random variables again.