Problem 8.3

1) As an orthonormal set of basis functions we consider the set

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$\psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad \psi_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

2) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

3) The distance between the first and the second vector is

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that

$$\begin{aligned} d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19} \end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 8.7

1) The impulse response of the filter matched to $s(t)$ is

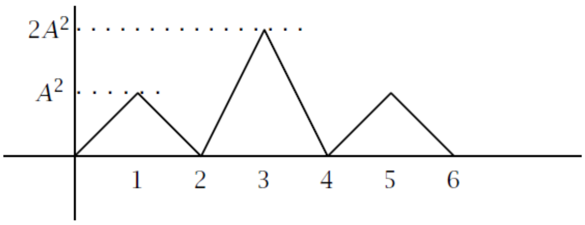
$$h(t) = s(T-t) = s(3-t) = s(t)$$

where we have used the fact that $s(t)$ is even with respect to the $t = \frac{T}{2} = \frac{3}{2}$ axis.

2) The output of the matched filter is

$$\begin{aligned} y(t) &= s(t) \star s(t) = \int_0^t s(\tau)s(t-\tau)d\tau \\ &= \begin{cases} 0 & t < 0 \\ A^2 t & 0 \le t < 1 \\ A^2(2-t) & 1 \le t < 2 \\ 2A^2(t-2) & 2 \le t < 3 \\ 2A^2(4-t) & 3 \le t < 4 \\ A^2(t-4) & 4 \le t < 5 \\ A^2(6-t) & 5 \le t < 6 \\ 0 & 6 \le t \end{cases} \end{aligned}$$

A sketch of $y(t)$ is depicted in the next figure



3) At the output of the matched filter and for $t = T = 3$ the noise is

$$\begin{aligned} n_T &= \int_0^T n(\tau)h(T-\tau)d\tau \\ &= \int_0^T n(\tau)s(T-(T-\tau))d\tau = \int_0^T n(\tau)s(\tau)d\tau \end{aligned}$$

The variance of the noise is

$$\begin{aligned} \sigma_{n_T}^2 &= E\left[\int_0^T \int_0^T n(\tau)n(v)s(\tau)s(v)d\tau dv\right] \\ &= \int_0^T \int_0^T s(\tau)s(v)E[n(\tau)n(v)]d\tau dv \\ &= \frac{N_0}{2}\int_0^T \int_0^T s(\tau)s(v)\delta(\tau - v)d\tau dv \\ &= \frac{N_0}{2}\int_0^T s^2(\tau)d\tau = N_0 A^2 \end{aligned}$$

4) For antipodal equiprobable signals the probability of error is

$$P(e) = Q\left[\sqrt{\left(\frac{S}{N}\right)_o}\right]$$

where $\left(\frac{S}{N}\right)_o$ is the output SNR from the matched filter. Since

$$\left(\frac{S}{N}\right)_o = \frac{y^2(T)}{E[n_T^2]} = \frac{4A^4}{N_0 A^2}$$

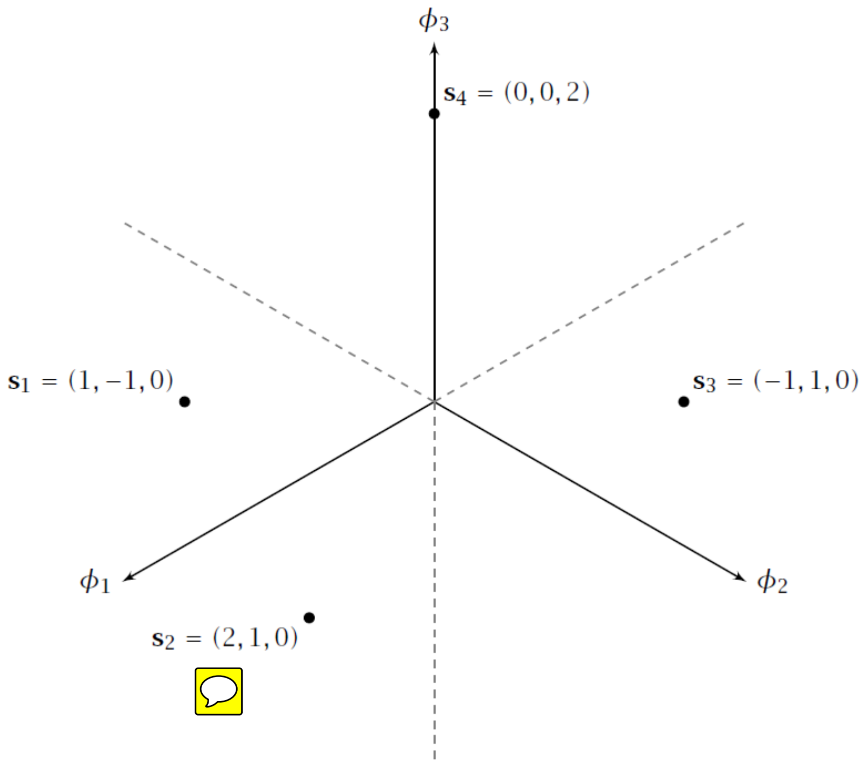
we obtain

$$P(e) = Q\left[\sqrt{\frac{4A^2}{N_0}}\right]$$

Problem 8.8

1. Since $s_3(t) = -s_1(t)$, it is sufficient to consider just $s_1(t)$, $s_2(t)$ and $s_4(t)$. By inspection, we can choose $\phi_1(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$, $\phi_2(t) = \phi_1(t-1)$, and $\phi_3(t) = \phi_1(t-2)$. With this selection $\mathbf{s}_1 = (1, -1, 0)$, $\mathbf{s}_2 = (2, 1, 1)$, $\mathbf{s}_3 = (-1, 1, 0)$, and $\mathbf{s}_4 = (0, 0, 2)$.

2. The constellation is shown below



3. The matrix representation of the four vectors is

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The three columns are clearly linearly independent, hence the rank of the matrix is 3. Therefore the dimensionality of the signal space is 3.

4. We know that in general $E_m = \|\mathbf{s}_m\|^2$, hence, $E_1 = \|\mathbf{s}_1\|^2 = 2$, $E_2 = \|\mathbf{s}_2\|^2 = 5$, $E_3 = \|\mathbf{s}_3\|^2 = 2$, and $E_4 = \|\mathbf{s}_4\|^2 = 4$. Therefore, $E_{\text{avg}} = \frac{1}{4}(2+5+2+4) = \frac{13}{4}$ and $E_{\text{bavg}} = \frac{E_{\text{avg}}}{\log_2 M} = \frac{13}{8}$.

Problem 8.16

1) The optimum threshold is given by

$$\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$$

2) The average probability of error is ($\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$)

$$\begin{aligned} P(e) &= p(a_m = -1) \int_{\alpha^*}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &\quad + p(a_m = 1) \int_{-\infty}^{\alpha^*} \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &= \frac{2}{3} Q\left[\frac{\alpha^* + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right] + \frac{1}{3} Q\left[\frac{\sqrt{\mathcal{E}_b} - \alpha^*}{\sqrt{N_0/2}}\right] \\ &= \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4}\right] \end{aligned}$$

3) Here we have $P_e = \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4}\right]$, substituting $\mathcal{E}_b = 1$ and $N_0 = 0.1$ we obtain

$$P_e = \frac{2}{3} Q\left[\frac{\sqrt{0.2} \times \ln 2}{4} + \sqrt{20}\right] + \frac{1}{3}\left[\sqrt{20} + \frac{\sqrt{0.2} \times \ln 2}{4}\right] = \frac{2}{3} Q(4.5496) - \frac{1}{3} Q(4.3946)$$

The result is $P_e = 3.64 \times 10^{-6}$.

Problem 8.24

1) The impulse response of the matched filter is

$$s(t) = u(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

2) The output of the matched filter at $t = T$ is

$$
\begin{aligned}
g(T) &= u(t) \star s(t)|_{t=T} = \int_0^T u(T-\tau)s(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T (T-\tau)^2 \cos^2(2\pi f_c(T-\tau))d\tau \\
&\stackrel{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2 \cos^2(2\pi f_c v)dv \\
&= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\
&= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]
\end{aligned}
$$

3) The output of the correlator at $t = T$ is

$$
\begin{aligned}
q(T) &= \int_0^T u^2(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T \tau^2\cos^2(2\pi f_c\tau)d\tau
\end{aligned}
$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vise versa.