HW2 Solution

Problem 5.35

1) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0 < f(1/4f_0) = 1$. Thus the maximum absolute value of $f(\tau)$ is not achieved at the origin $\tau = 0$.

2) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0$ whereas $f(\tau) \neq 0$ for $\tau \neq 0$. The maximum absolute value of $f(\tau)$ is not achieved at the origin.

3) $f(0) = 1$ whereas $f(\tau) > f(0)$ for $|\tau| > 1$. Thus $f(\tau)$ cannot be the autocorrelation function of a random process.

4) $f(\tau)$ is even and the maximum is achieved at the origin ($\tau = 0$). We can write $f(\tau)$ as

$$f(\tau) = 1.2\Lambda(\tau) - \Lambda(\tau - 1) - \Lambda(\tau + 1)$$

Taking the Fourier transform of both sides we obtain

$$S(f) = 1.2\text{sinc}^2(f) - \text{sinc}^2(f)\left(e^{-j2\pi f} + e^{j2\pi f}\right) = \text{sinc}^2(f)(1.2 - 2\cos(2\pi f))$$

As we observe the power spectrum $S(f)$ can take negative values, i.e. for $f = 0$. Thus $f(\tau)$ can not be the autocorrelation function of a random process.

Problem 5.38

1)

$$\begin{aligned}m_X(t) &= E[X(t)] = E[X\cos(2\pi f_0 t)] + E[Y\sin(2\pi f_0 t)] \\ &= E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) \\ &= 0\end{aligned}$$

where the last equality follows from the fact that $E[X] = E[Y] = 0$.

2)

$$\begin{aligned}R_X(t+\tau,t) &= E[(X\cos(2\pi f_0(t+\tau)) + Y\sin(2\pi f_0(t+\tau))) \\ &\quad (X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t))] \\ &= E[X^2\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] + \\ &\quad E[XY\cos(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] + \\ &\quad E[YX\sin(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] + \\ &\quad E[Y^2\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] \\ &= \frac{\sigma^2}{2}[\cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0\tau)] + \\ &\quad \frac{\sigma^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))] \\ &= \sigma^2\cos(2\pi f_0\tau)\end{aligned}$$

where we have used the fact that $E[XY] = 0$. Thus the process is stationary for $R_X(t+\tau,t)$ depends only on τ.

3) The power spectral density is the Fourier transform of the autocorrelation function, hence

$$S_X(f) = \frac{\sigma^2}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right].$$

4) If $\sigma_X^2 \neq \sigma_Y^2$, then

$$m_X(t) = E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) = 0$$

and

$$\begin{aligned}R_X(t+\tau,t) &= E[X^2]\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t) + \\ &\quad E[Y^2]\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t) \\ &= \frac{\sigma_X^2}{2}[\cos(2\pi f_0(2t+\tau)) - \cos(2\pi f_0\tau)] + \\ &\quad \frac{\sigma_Y^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))] \\ &= \frac{\sigma_X^2 - \sigma_Y^2}{2}\cos(2\pi f_0(2t+\tau) + \\ &\quad \frac{\sigma_X^2 + \sigma_Y^2}{2}\cos(2\pi f_0\tau)\end{aligned}$$

The process is not stationary for $R_X(t+\tau,t)$ does not depend only on τ but on t as well. However the process is cyclostationary with period $T_0 = \frac{1}{2f_0}$. Note that if X or Y is not of zero mean then the period of the cyclostationary process is $T_0 = \frac{1}{f_0}$.

Problem 5.40

1) $S_X(f) = \frac{N_0}{2}$, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$. The autocorrelation function and the power spectral density of the output are given by

$$R_Y(t) = R_X(\tau) \star h(\tau) \star h(-\tau), \qquad S_Y(f) = S_X(f)|H(f)|^2$$

With $H(f) = \Pi(\frac{f}{2B})$ we have $|H(f)|^2 = \Pi^2(\frac{f}{2B}) = \Pi(\frac{f}{2B})$ so that

$$S_Y(f) = \frac{N_0}{2}\Pi(\frac{f}{2B})$$

Taking the inverse Fourier transform of the previous we obtain the autocorrelation function of the output

$$R_Y(\tau) = 2B\frac{N_0}{2}\text{sinc}(2B\tau) = BN_0\text{sinc}(2B\tau)$$

2) The output random process $Y(t)$ is a zero mean Gaussian process with variance

$$\sigma^2_{Y(t)} = E[Y^2(t)] = E[Y^2(t+\tau)] = R_Y(0) = BN_0$$

The correlation coefficient of the jointly Gaussian processes $Y(t+\tau)$, $Y(t)$ is

$$\rho_{Y(t+\tau)Y(t)} = \frac{COV(Y(t+\tau)Y(t))}{\sigma_{Y(t+\tau)}\sigma_{Y(t)}} = \frac{E[Y(t+\tau)Y(t)]}{BN_0} = \frac{R_Y(\tau)}{BN_0}$$

With $\tau = \frac{1}{2B}$, we have $R_Y(\frac{1}{2B}) = \text{sinc}(1) = 0$ so that $\rho_{Y(t+\tau)Y(t)} = 0$. Hence the joint probability density function of $Y(t)$ and $Y(t+\tau)$ is

$$f_{Y(t+\tau)Y(t)} = \frac{1}{2\pi BN_0}e^{-\frac{Y^2(t+\tau)+Y^2(t)}{2BN_0}}$$

Since the processes are Gaussian and uncorrelated they are also independent.

Problem 5.49

1. The power is the area under the power spectral density, which has a triangular shape with a base of 2×10^5 and height of 4×10^{-5}. Therefore

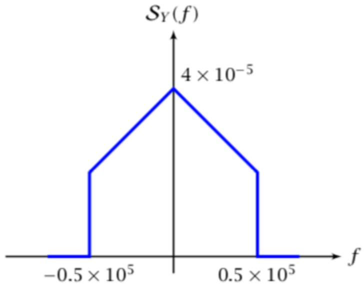
$$P_x = \int_{-\infty}^{\infty} S_X(f)\,df = \frac{1}{2} \times 2 \times 10^5 \times 4 \times 10^{-5} = 4 \text{ W}$$

2. The range of frequencies are $[-10^5, 10^5]$, hence the bandwidth is 10^5 Hz or 100 kHz.

3. The transfer function of the ideal lowpass filter is $H(f) = \Pi\left(\frac{f}{10^5}\right)$, therefore,

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = 4 \times 10^{-5}\Lambda\left(\frac{f}{10^5}\right)\Pi\left(\frac{f}{10^5}\right) = \begin{cases} 4 \times 10^{-5}\Lambda\left(\frac{f}{10^5}\right), & |f| < 0.5 \times 10^5 \\ 0, & \text{otherwise} \end{cases}$$

and the total power is the area under $\mathcal{S}_Y(f)$. Plot of $\mathcal{S}_Y(f)$ is shown below



Therefore,

$$P_Y = \int_{-\infty}^{\infty} \mathcal{S}_Y(f)\, df = 10^5 \times 2 \times 10^{-5} + \frac{1}{2} \times 10^5 \times 2 \times 10^{-5} = 3 \text{ W}$$

4. Since $X(t)$ is Gaussian, $X(0)$ is a Gaussian random variable. Since $X(t)$ is zero-mean, $X(0)$ has mean equal to zero. The variance in $X(0)$ is $\mathsf{E}\left[X^2(0)\right]$ which is equal to $R_X(0)$, i.e, the power in $X(t)$ which is 4. Therefore, $X(0)$ is a Gaussian random variable with mean $m = 0$ and variance $\sigma^2 = 4$. The desired PDF is

$$f_{X(0)}(x) = \frac{1}{\sqrt{8\pi}} e^{-x^2/8}$$

5. Since for Gaussian random variables independence means uncorrelated, we need to find the smallest t_0 such that $R_X(t_0) = 0$. But

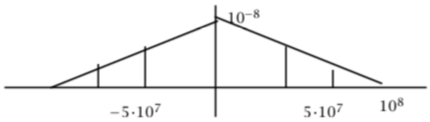
$$R_X(\tau) = \mathscr{F}^{-1}\left[\mathcal{S}_X(f)\right] = 4\,\text{sinc}^2(10^5\tau)$$

and its first zero occurs at $10^5 t_0 = 1$ or $t_0 = 10^{-5}$.

Problem 5.62

1) The power spectral density $S_n(f)$ is depicted in the following figure. The output bandpass process has non-zero power content for frequencies in the band $49 \times 10^6 \leq |f| \leq 51 \times 10^6$. The power content is

$$
\begin{aligned}
P &= \int_{-51\times 10^6}^{-49\times 10^6} 10^{-8}\left(1+\frac{f}{10^8}\right)df + \int_{49\times 10^6}^{51\times 10^6} 10^{-8}\left(1-\frac{f}{10^8}\right)df \\
&= 10^{-8}x\Big|_{-51\times 10^6}^{-49\times 10^6} + 10^{-16}\frac{1}{2}x^2\Big|_{-51\times 10^6}^{-49\times 10^6} + 10^{-8}x\Big|_{49\times 10^6}^{51\times 10^6} - 10^{-16}\frac{1}{2}x^2\Big|_{49\times 10^6}^{51\times 10^6} \\
&= 2 \times 10^{-2}
\end{aligned}
$$



2) The output process $N(t)$ can be written as

$$N(t) = N_c(t)\cos(2\pi 50 \times 10^6 t) - N_s(t)\sin(2\pi 50 \times 10^6 t)$$

where $N_c(t)$ and $N_s(t)$ are the in-phase and quadrature components respectively, given by

$$
\begin{aligned}
N_c(t) &= N(t)\cos(2\pi 50 \times 10^6 t) + \hat{N}(t)\sin(2\pi 50 \times 10^6 t) \\
N_s(t) &= \hat{N}(t)\cos(2\pi 50 \times 10^6 t) - N(t)\sin(2\pi 50 \times 10^6 t)
\end{aligned}
$$

The power content of the in-phase component is given by

$$
\begin{aligned}
E[|N_c(t)|^2] &= E[|N(t)|^2]\cos^2(2\pi 50 \times 10^6 t) + E[|\hat{N}(t)|^2]\sin^2(2\pi 50 \times 10^6 t) \\
&= E[|N(t)|^2] = 2 \times 10^{-2}
\end{aligned}
$$

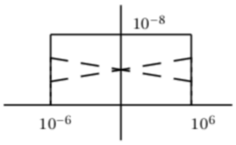
where we have used the fact that $E[|N(t)|^2] = E[|\hat{N}(t)|^2]$. Similarly we find that $E[|N_s(t)|^2] = 2 \times 10^{-2}$.

3) The power spectral density of $N_c(t)$ and $N_s(t)$ is

$$S_{N_c}(f) = S_{N_s}(f) = \begin{cases} S_N(f - 50 \times 10^6) + S_N(f + 50 \times 10^6) & |f| \leq 50 \times 10^6 \\ 0 & \text{otherwise} \end{cases}$$

$S_{N_c}(f)$ is depicted in the next figure. The power content of $S_{N_c}(f)$ can now be found easily as

$$P_{N_c} = P_{N_s} = \int_{-10^6}^{10^6} 10^{-8} df = 2 \times 10^{-2}$$



4) The power spectral density of the output is given by

$$S_Y(f) = S_X(f)|H(f)|^2 = (|f| - 49\times 10^6)(10^{-8} - 10^{-16}|f|) \quad \text{for } 49\times 10^6 \le |f| \le 51\times 10^6$$

Hence, the power content of the output is

$$\begin{aligned} P_Y &= \int_{-51\times 10^6}^{-49\times 10^6} (-f - 49\times 10^6)(10^{-8} + 10^{-16} f)df \\ &\quad + \int_{49\times 10^6}^{51\times 10^6} (f - 49\times 10^6)(10^{-8} - 10^{-16} f)df \\ &= 2\times 10^4 - \frac{4}{3}10^2 \end{aligned}$$

The power spectral density of the in-phase and quadrature components of the output process is given by

$$\begin{aligned} S_{Y_c}(f) = S_{Y_s}(f) &= \left((f + 50\times 10^6) - 49\times 10^6\right)\left(10^{-8} - 10^{-16}(f + 50\times 10^6)\right) \\ &\quad + \left(-(f - 50\times 10^6) - 49\times 10^6\right)\left(10^{-8} + 10^{-16}(f - 50\times 10^6)\right) \\ &= -2\times 10^{-16} f^2 + 10^{-2} \end{aligned}$$

for $|f| \le 10^6$ and zero otherwise. The power content of the in-phase and quadrature component is

$$\begin{aligned} P_{Y_c} = P_{Y_s} &= \int_{-10^6}^{10^6} (-2\times 10^{-16} f^2 + 10^{-2})df \\ &= -2\times 10^{-16}\frac{1}{3}f^3\Big|_{-10^6}^{10^6} + 10^{-2} f\Big|_{-10^6}^{10^6} \\ &= 2\times 10^4 - \frac{4}{3}10^2 = P_Y \end{aligned}$$