1.5em 0pt

COMS W4705–Spring, 2021
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HW0

1. **Solution to Problem 1**

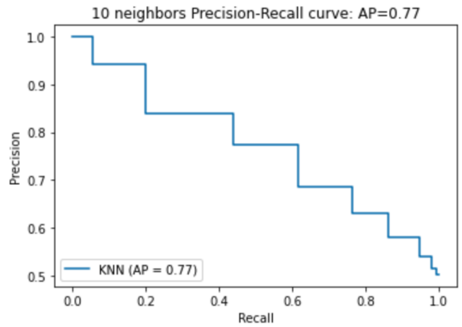

Figure 1: Figure for 10 neighbors knn

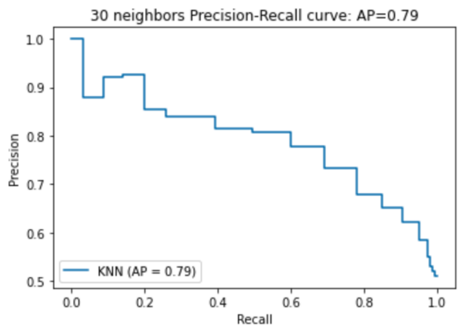

Figure 2: Figure for 30 neighbors knn

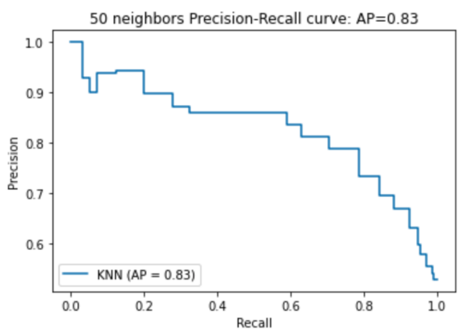

Figure 3: Figure for 50 neighbors knn

2. **Solution to Problem 2.1**

   We have:

   $f(x, y) = ye^{-x} - g(\sin x, y)$

   $g(x, y) = \cos x + x^2 e^y$

   First, we can get:

$$\begin{aligned} \frac{\partial g(x,y)}{\partial x} &= -\sin x + 2xe^y \\ \frac{\partial g(x,y)}{\partial y} &= x^2 e^y \end{aligned} \tag{1}$$

   Then we can substitute g(x,y) into f(x,y).

$$f(x, y) = ye^{-x} - \cos(\sin(x)) - \sin^2(x)e^y \tag{2}$$

   With (2) we are able to get:

$$\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= -ye^{-x} + \sin(\sin(x))\cos(x) - 2\sin(x)\cos(x)e^y \\ \frac{\partial f(x,y)}{\partial y} &= e^{-x} - \sin^2(x)e^y \end{aligned} \tag{3}$$

   Then we can use (1) and (3) to get:

$$\begin{aligned} &\frac{\partial f(x,y)}{\partial g(x,y)} \\ &= \frac{\partial f(x,y)}{\partial x}\frac{\partial x}{\partial g(x,y)} + \frac{\partial f(x,y)}{\partial y}\frac{\partial y}{\partial g(x,y)} \\ &= \frac{-ye^{-x} + \sin(\sin(x))\cos(x) - 2\sin(x)\cos(x)e^y}{-\sin(x) + 2xe^y} + \frac{e^{-x} - \sin^2(x)e^y}{x^2 e^y} \end{aligned} \tag{4}$$

3. **Solution to Problem 2.2**

   Critical points are points where the gradient of the function is 0. We first get the derivative of the function, then we find the point where the derivative is 0.

$$\begin{aligned} &\frac{\partial f}{\partial x} = -\sin(x) - \frac{1}{2} = 0 \\ &\sin x = -\frac{1}{2} \\ &x = \frac{\pi n}{2} - \frac{1}{6}\pi, n \in Z \\ &or \\ &x = \frac{\pi n}{2} + \frac{7}{6}\pi, n \in Z \end{aligned}$$

(5)

For y:

$$\begin{aligned}\frac{\partial f}{\partial y} &= 4y = 0 \\ y &= 0\end{aligned} \tag{6}$$

Therefore we have two critical points:$(\frac{\pi n}{2} - \frac{1}{6}\pi,\, 0)$ and $(\frac{\pi n}{2} + \frac{7}{6}\pi,\, 0)$, where $n \in Z$.
To classify critical points, we need to calculate:

$$\begin{aligned}&D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 \\ &and \\ &f_{xx}(a,b)\end{aligned} \tag{7}$$

From equation (5), we have:

$$\begin{aligned}f_{xx} &= -\cos{(x)} \\ f_{yy} &= 4 \\ f_{xy} &= 0\end{aligned} \tag{8}$$

We substitute two critical points into (7).
For point $(\frac{\pi n}{2} - \frac{1}{6}\pi,\, 0)$, $D < 0$ which means this point is a saddle point.
For point $(\frac{\pi n}{2} + \frac{7}{6}\pi,\, 0)$, $D > 0$ and $f_{xx}(a,b) > 0$ which means it is a minimum point.

4. **Solution to Problem 2.3**

For $\partial f/\partial x$, we are taking derivative with respect to a column vector. We are converting x from a m dimension vector to a n*m dimension vector. Thus the Jacobian matrix should have n*m dimension.

$$\begin{aligned}(\frac{\partial f}{\partial x})_{ij} &= \frac{\partial f_i}{\partial x_j} \\ &= \sum A_{ik}\frac{\partial}{\partial x_j}x_k \\ &= A_{ij}\end{aligned} \tag{9}$$

Thus we can conclude:

$$\frac{\partial f}{\partial x} = A \tag{10}$$

For the same reason:

$$\frac{\partial f}{\partial A} = x \tag{11}$$

5. **Solution to Problem 3.1**

   P(x)=$\frac{1}{3}$

6. **Solution to Problem 3.2**

   From the question, we have:
   $P(BoxA) = P(BoxB)$=$\frac{1}{2}$
   $P(BlueBall|BoxA) = \frac{1}{2}$
   $P(BlueBall|BoxB) = \frac{7}{10}$
   According to Baye's rule, we have:
   $P(BoxA|BlueBall)$=$\frac{P(BlueBall|BoxA)P(BoxA)}{P(BlueBall)}$=$\frac{P(BlueBall|BoxA)P(BoxA)}{P(BlueBall|BoxA)*P(BoxA)+P(BlueBall|BoxB)*P(BoxB)}$
   $= \frac{0.5*0.5}{0.5*0.5+\frac{7}{10}*0.5} = \frac{5}{12} = 0.417$

7. **Solution to Problem 4.1.1**

$$\begin{bmatrix} 1 & 5 & 0 \\ 3 & 4 & 1 \\ 2 & 3 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 20 \\ 21 & 28 \\ 26 & 25 \end{bmatrix}$$

   (1,1)=1*1+5*4+0*2=21
   (1,2)=1*5+5*3+0*1=20
   (2,1)=3*1+4*4+1*2=21
   (2,2)=3*5+4*3+1*1=28
   (3,1)=2*1+3*4+6*2=26
   (3,2)=2*5+3*3+6*1=25

8. **Solution to Problem 4.1.2**

   Corr(X,Y)=$\frac{COV(X,Y)}{\sigma X \sigma Y}$=$\frac{\frac{1}{n}*\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\frac{1}{n}\sum(x_i-\bar{x})^2}\sqrt{\frac{1}{n}\sum(y_i-\bar{y})^2}}$=$\frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2}\sqrt{\sum(y_i-\bar{y})^2}}$
   Then we can write the summation in vector form. Let's assume x_i and y_i are elements in u and v respectively. We have:
   $\frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2}\sqrt{\sum(y_i-\bar{y})^2}} = \frac{<u-\bar{u}>*<v-\bar{v}>}{||u||*||v||}$
   Let $u_{new} =< u - \bar{u} >$ and $v_{new} =< v - \bar{v} >$, we have $u_{new} * v_{new} = ||u_{new}|| * ||v_{new}|| * \cos\theta$.
   And $\cos\theta = \frac{u_{new}*v_{new}}{||u_{new}||*||v_{new}||}$
   To have Corr(u_{new}, v_{new})=$\cos\theta$, we need both u and v to be centered which means u and v have 0 mean. Since $\bar{u} = 0$ and $\bar{v} = 0$, we have Corr(u_{new}, v_{new})=$\frac{<u-\bar{u}>*<v-\bar{v}>}{||u||*||v||} = \frac{<u>*<v>}{||u||*||v||}$=$\cos\theta$

9. **Solution to Problem 4.2.1**

   The dimension of U is: $m * m$. The dimension of $\sum$ is $m * n$. The dimension of V is $n * n$. $\sum$ is the diagonal matrix that contains singular value corresponding to each column in M. The columns of U and V are called the left singular vectors and right singular vectors which correspond to rows and column of M respectively.

10. **Solution to Problem 4.2.2**

We know in SVD, U and V are orthogonal matrices. By the property of orthogonal matrices, their transposes equal to their inverses. We know that $U^{-1} = U^T$ and $U^{-1} = U^T$.
$M^{-1} = (U\sum V^T)^{-1}$=$(V^T)^{-1}\sum^{-1}U^{-1}$=$(V\sum^{-1}U^T)$. It is easy to compute V and U^T. We know that $\sum$ is a diagonal matrix, thus $\sum^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix}$

To prove our correctness: $M^{-1}M = (V\sum^{-1}U^T)*(U\sum V^T)$=$VS^{-1}(U^TU)SV^T = V(S^{-1}S)V^T = VV^T = I$

We have computed inverse of M using SVD.

11. **Solution to Problem 5**

I think there are two possibilities for translation.
First one is:

Skri	English
1.leebork	B. mountain fire
2.zonaga	D. tree river
3.borknaga	A. Fire river
4.minzo	C. Cherry tree

Then the pairing should be $lee- > mountain$, $naga- > river$, $bork- > fire$, $zo- > tree$ and $min- > cherry$.

Another possibility is that the corresponding words for tree and fire are interchanged.

Skri	English
1.leebork	C. Cherry tree
2.zonaga	A. Fire river
3.borknaga	D. tree river
4.minzo	B. mountain fire

Then the pairing should be $lee->cherry$, $naga->river$, $bork->tree$, $zo->fire$ and $min->mountain$.