UNIFORM SMOOTHNESS AND UNIFORM CONVEXITY

Theorem 1. *L^p $(1 < p < \infty)$ is uniformly convex and uniformly smooth, respectively with order max$\{2, p\}$, and min$\{2, p\}$.*

Proof. We say X is $p-$(US), if

$$\rho(t) := \sup\{(\|f + tg| + \|f - tg\| - 2)/2 : \|f\| = \|g\| = 1\} \lesssim t^p.$$

So, $1-$(US) does not give smoothness, of the original definition. Similarly, $p-$(US) only means $(p - \epsilon)-$order strong smoothness (higher order infinitesimal).

$p-$(US) for $p \in (1, 2]$. Actually, after taking

$$a = \|f + tg|, b = \|f - tg|, \tilde{f} = f + tg, \tilde{g} = f - tg,$$

in the[Hanner inequality (1)],

$$(0 \leq)\ 2\rho(t) = a + b - 2 \leq (a + b)^p - 2^p \leq 2^p + (2t)^p - |a - b|^p - 2^p \lesssim t^p$$

We use the following inequality for the first "$\leq$", for any $a, b \geq 0$

$$pb^{p-1}(a - b) \leq a^p - b^p, \quad p \geq 1,$$

which reduces to

$$a - b \leq (p + o(1))(a - b) \leq a^p - b^p, \quad \text{if } a \geq b \geq 1 + o(1) \quad (< \quad \text{for} \quad p > 1)$$

as

$$b \geq (1/p)^{1/(p-1)} = e^{-\frac{\log p}{p-1}} \uparrow \frac{1}{e^{-1}}(\quad < 1)$$

Pay attention, this only given the smoothness, when $p > 1$.

$2-$(US) for $p \geq 2$. Using [Hanner 1] with $\tilde{f} = f, \tilde{g} = tg$, and the same notation a, b as above,

$$(0 \leq) a + b - 2 \leq a^p + b^p - 2 \leq (1 + t)^p + (1 - t)^p - 2 \lesssim t^2,$$

where for the first "$\leq$", we use the inequality: if $a, b \geq 0, a + b \geq 2$, then

$$a + b \leq 2^{1-p}(a + b)^p \leq a^p + b^p, \quad p \geq 1$$

($a + b \geq 2$ is necessary, if we choose $a = b$, then $a \geq 1$).

We say X is $p-$(UC), if

$$\delta(\epsilon) := \inf\{1 - \|(f + g)/2\| : \|f\| = \|g\| = 1, \|f - g\| = \epsilon\} \gtrsim \epsilon^p.$$

We can similarly, or use the following duality directly:

Proposition 2. *X is $p-$(UC) iff X^* is $p'-$(US).*

$\square$

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