AMS 40. Week 5

Ex1

(1) ~~decompose R into disjoint cubes of side length h then the total number of such cubes is $(Nh/h)^{n-1}$ $= N^{n-1}$~~

enlarge the rectangle R to be the cube $\tilde{R}$ of side length Nh, with volume $(Nh)^n$

$$K_N f(x) = \sup_{x \in R \in \mathcal{R}_N} \frac{1}{|R|} \int_R |f|$$

$$\leq \sup_{x \in R \in \mathcal{R}_N} \frac{1}{|R|} \int_{\tilde{R}} |f|$$

$$= \sup_{x \in R \in \mathcal{R}_N} \frac{|\tilde{R}|}{|R|} \cdot \left(\frac{1}{|\tilde{R}|} \int_{\tilde{R}} |f| \right)$$

$$\leq C N^{n-1} M f(x).$$

. by defin $\|K_N f\|_{L^\infty} \leq C \|f\|_{L^\infty}$

&

the weak boundedness with constant

$\sim N^{n-1}$

$\Downarrow$ Marcinkiwicz Interpolation

OK.

(2) We take $f(x) = \mathbb{1}_B(x)$. and try to calculate $K_N f(x)$ for some given N. (Assume N is large)

We next try to calculate $K_N f(x)$ by considering the size of $|x|$ separately.

- $|x| \le 1$,

$$K_N f(x) = 1.$$

- $|x| > 1$,

$h \to 0$

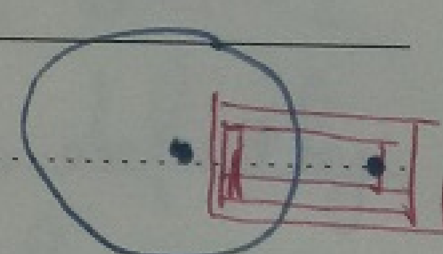
- for $h \ll 1$ so that

$$h < \frac{|x|}{N},$$

we would have

$$K_N f(x) = 0.$$

- $|h| \sim \frac{|x|}{N}$

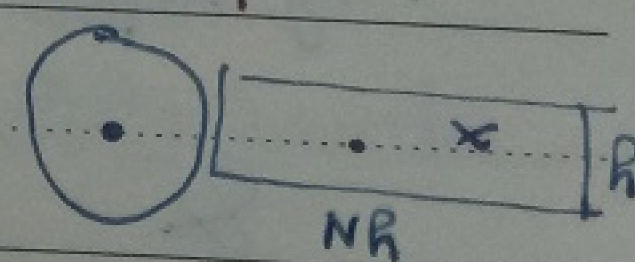


$$K_N f(x) \sim \frac{h^{n-1}}{Nh \cdot h^{n-1}}$$

$$= \frac{1}{Nh} \approx \frac{1}{|x|}$$

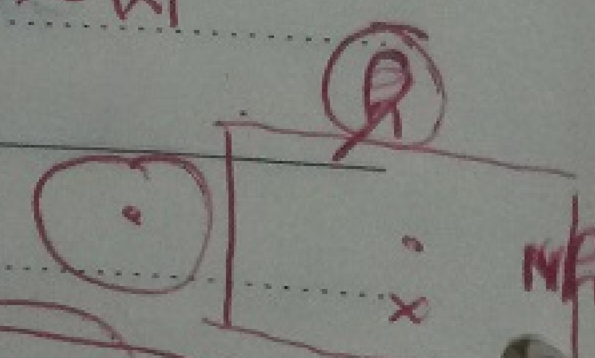
$h \to \infty$

- $1 \ll h < N$



$$K_N f(x) = \frac{1}{Nh \cdot h^{n-1}} \Leftarrow Nh \sim |x|$$

$$\sim \frac{N^{n-1}}{|x|^n}$$



- ~~$h \gg N$~~ $K_N f(x) = \frac{1}{Nh \cdot h^{n-1}}$

In Summary, we set

$$K_N f(x) = \begin{cases} 1 & |x| \le 1 \\ \sim \max\left(\frac{1}{|x|}, \frac{N^{n-1}}{|x|^n}\right) & |x| > 1 \end{cases}$$

$$= \begin{cases} 1 & |x| \le 1 \\ \frac{N^{n-1}}{|x|^n} & 1 < |x| \le N \\ \frac{1}{|x|} & N < |x| \end{cases}$$

from this, one can calculate the L^p-norms ~~$\in (1,\infty) \setminus \{n\}$~~ of $K_N f$. for different values of p:

$$\|K_N f\|_{L^p(|x| \le 1)} \sim 1 \qquad \forall p.$$

$$\|K_N f\|_{L^p(1 \le |x| \le N)} \sim N^{\frac{n}{p}-1} \qquad \forall p > 1$$

$$N^{\frac{n-1}{p}} (\log N)^{\frac{1}{p}} \qquad p = 1$$

$$\|K_N f\|_{L^p(|x| \ge N)} \sim N^{\frac{n-p}{p}} \qquad p > n$$

$p=n$ weak-type considerations

then one should compare diff L^p-norms according to diff values of p's.

⚠!!: the L.-P. theory, Hilbert-transform, singular integral, are valid for all $p \in (1, \infty)$, and hence it is naturally to expect that these theories are not so effective to attack this last conjecture, which depends on the values of p.

Ex2.:

(1). $P_r * f$ takes values at $re^{2\pi i\theta}$:

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{2\pi i k\theta}.$$

note that $r^{|k|} e^{2\pi i k\theta}$ is a harmonic func for each $k \in \mathbb{Z}$, then the harmonicity of $P_r * f$ follows from the uniform convergence.....

(2) for $f \in L^1(\mathbb{T})$, and the uniform boundedness of Q_r, P_r (for fixed $r<1$), it suffices to check the analyticity of

$$P_r(t) + i Q_r(t) \cdot \text{ in } re^{i2\pi t} =: z$$

which is exactly

?? $$\sum r^{|k|} e^{2\pi i k t} e^{2\pi i |k| t}$$

$$= \sum z^{|k|} e^{2\pi i k t}.$$

(3) note first that

$$\frac{1}{\tan t} - \frac{1}{t} = \frac{\cos t}{\sin t} - \frac{1}{t}$$

$= O(1)$ around $t=0$.

it suffices to show the existence of the limit

$$\lim_{\varepsilon \to 0} \int_{\varepsilon < t < 1/2} \frac{f(x-t) - f(x+t)}{t} dt,$$

this can be deduced as we did in the case of real line.

for the first limit, we can reduce to the 2nd one, by the computation of the kernel Q_r.

(4) this is the convergence result.

for more information on the Poisson kernel and Hilbert transform see:

S. Krantz: Geometric function theory

Page 207, Page 171