Week 9

for $a \in (0,n)$, define the fractional integral operator

$$I_a \phi(x) = \frac{1}{\gamma_a} \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-a}}\, dy$$

wher

$$\gamma_a = \pi^{\frac{n}{2}-a} \frac{\Gamma(\frac{a}{2})}{\Gamma(\frac{n-a}{2})},$$

(1) show that in the ~~se~~ sense of distributions, one has

$$(I_a \phi)^\wedge(\xi) = |\xi|^{-a} \hat{\phi}(\xi).$$

Hint use previous calculation of Fourier transform of $|x|^{-a}$, $a \in (0,n)$.

(2) by a scaling argument, show that a necessary condition for the inequality

(*1) $$\|I_a f\|_{L^q} \leq C \|f\|_{L^p}$$

to hold is

(*) $$\frac{1}{q} = \frac{1}{p} - \frac{a}{n}$$

Rmk: indeed for $1 < p < \frac{n}{a}$, (*) is also sufficient for (*1) to hold.

(3) define the fractional maximal function.

$$M_a f(x) = \sup_{r>0} \frac{1}{|B_r|^{1-a/n}} \int_{B_r} |f(x-y)|\, dy, \quad 0 < a < n.$$

then for positive funcs f, show

$$M_a f(x) \leq C\, I_a f(x)$$

the converse of which does not hold in general. but we have the equivalence of the two quantities in Norm:

(*) <u>thm</u>, $1<p<\infty$, and $0<a<n$, there exists a constant $C_{a,p}$ such that

$$\|I_a f\|_{L^p} \leq C_{a,p} \|M_a f\|_{L^p}.$$