E.g. Comparison:

P.P: $E=[0,\infty)$, $m(B)\to M$ = Poisson measure

$\updownarrow$ $\quad N_t = M([0,t])$

B.M: $E=[0,\infty)$, $m(B)\to G$ = Gaussian white noise

$B_t = G([0,t])$ (为 $G(1_{[0,t]})$ 简写)

不同点: $M(\cdot)$ 为测度, $G(\cdot)$ 不是 a.e. 收敛, 而是 L^2 收敛, 见 Rmk.

Pf prop 2.3: ①⇒② trivial

②⇒①: 为证 $X_t - X_s \perp \sigma(X_r, r\le s)$

[Lem: $\mathcal{F}_t, t\in T$: independent σ-fields, disjoint partition $\mathcal{T}$ of T. Then $\mathcal{F}_S = \bigvee_{t\in S}\mathcal{F}_t$, $S\in\mathcal{T}$ are independent]

$\forall 0=s_0<\cdots<s_n=s$, $\sigma(X_{s_i}-X_{s_{i-1}})$ and $\sigma(X_t-X_s)$ are independent. By Lem.,

$\sigma(X_{s_1}-X_{s_0},\cdots,X_{s_n}-X_{s_{n-1}}) \perp \sigma(X_t-X_s)$

$\quad\quad \| $

$\sigma(X_{s_1},\cdots,X_{s_n})$ $\quad$ 由于 $\bigcup_{s_0,\cdots,s_n}\sigma(X_{s_1},\cdots,X_{s_n}) \triangleq C$ 为π-类, 且 $\sigma(C)=\sigma(X_r, r\le s)$

∴ $\sigma(X_t-X_s)\perp\sigma(X_r, r\le s)$ (高概的独立性中有学, 用π-λ定理证的.)

②⇒③: (X_1, X_2-X_1, X_3-X_2) 独立 (centered) Gaussian vector ⇒ (X_1,X_2,X_3) centered Gaussian.

下面 check covariance: $s\le t$, $E(X_s X_t) = E(X_s(X_t-X_s)) + EX_s^2 = s$.

③⇒②: 显然 $X_{t_i}-X_{t_{i-1}}$ Gaussian (centered), 下面 check: $\forall 0\le t_1\le t_2\le t_3$, 有

$E[(X_{t_2}-X_{t_1})(X_{t_3}-X_{t_2})]=0$ 且 $E(X_{t_2}-X_{t_1})^2 = t_2-t_1$ #

Rmk. Prop 2.5 ③ $\sigma(B^{(s)})\perp\sigma(B_r, r\le s)$:

则有 $E(f(B_t)|\mathcal{F}_s) = E(f(B_t)|B_s)$, $0\le s<t$

∵ 由于 $E[g(X,Y)|\mathcal{F}] = E(g(X,Y)|Y)$, 若 $X\perp\mathcal{F}$, $Y\in\mathcal{F}$.

LHS $= E[f(B_t-B_s+B_s)|\mathcal{F}_s] = E[f(B_t-B_s+B_s)|B_s] = E(f(B_t)|B_s)$.

Le Gall

Chap 2. Brownian Motion : pre-B.M. + continuity of sample paths

· Focus: classical Kolmogorov Lemma, S.M.P., reflection principle

· Setting: $(\Omega,\mathcal{F},P)$, indexed by $T=\mathbb{R}_+$, taking values in $\mathbb{R}$.

2.1 Pre BM

Def 2.1 (Pre-Brownian Motion) $X=(X_t)_{t\ge0}$ is called pre-BM, if: ① $X_0=0$ a.s.

② $\forall 0\le s<t$, $X_t-X_s\perp\sigma(X_r, r\le s)$ ③ $X_t-X_s\sim N(0,t-s)$ $X_t\sim N(0,t)$.

Prop 2.3 (等价刻画) ① $(X_t)_{t\ge0}^{\in\mathbb{R}}$ is pre-BM

⇔ ② $X_0=0$ a.s. & $\forall 0=t_0<t_1<\cdots<t_p$, $X_{t_i}-X_{t_{i-1}}$, $1\le i\le p$ are independent

with $X_{t_i}-X_{t_{i-1}}\sim N(0,t_i-t_{i-1})$, $1\le i\le p$

⇔ ③ X is centered Gaussian process with covariance $K(s,t)=s\wedge t$.

*Def 1.6 (Centered Gaussian Process) $(X_t)_{t\in T}$ (real-valued) is called a centered GP

if ∀ finite linear combination of $X_t, t\in T$ is centered Gaussian

i.e. $\forall t_1,\cdots,t_n\in T$, $(X_{t_1},\cdots,X_{t_n})$ is a centered Gaussian vector.

*Def 2.1' (pre-BM 另一种定义). G: Gaussian white noise on $\mathbb{R}^+$ whose intensity is Lebesgue measure. Define $B=(B_t)_{t\ge0}$ by

$$B_t = G(1_{[0,t]}),$$

(where G: an isometry from $L^2(\mathbb{R}_+,\mathcal{B},m)$ into a centered Gaussian space $L^2(\Omega,\mathcal{F},P)$.)

Rmk. $f,g\in L^2(\mathbb{R}_+,\mathcal{B},m)$, $E[G(f)]^2=\int f^2dm$, $E[G(f)G(g)]=\int fg\,dm$

特别 $f=1_A$ with $m(A)<\infty$, $G(1_A)\sim N(0,m(A))$.

若 $A=A_1\cup\cdots\cup A_n$, $m(A)<\infty$, then $G(A)=G(A_1)+\cdots+G(A_n)$ a.s.

$A=\bigcup_{n\ge1}A_n$, $m(A)<\infty$, then $G(A)=\sum_{n\ge1}G(A_n)$, $G(A_1)+\cdots+G(A_n)\xrightarrow{L^2}\sum_{n\ge1}G(A_n)$

Rmk. $X_t-X_s\perp\sigma(X_r, r\le s)$, $X_t-X_s\sim N(0,t-s)$ is called the prop of [stationary and independent increments].

Cor 2.4 (分布). $0=t_0<t_1<\cdots<t_n$, the law of $(B_{t_1},B_{t_2},\cdots,B_{t_n})$ has density

$$p(x_1,\cdots,x_n)=\frac{1}{2\pi^{\frac{n}{2}}\sqrt{t_1(t_2-t_1)\cdots(t_n-t_{n-1})}}\exp\left\{-\sum_{i=1}^n\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right\}$$

Prop 2.5 若 $B=(B_t)_{t\ge0}$ pre-BM, 则

① $-B$ is pre-BM (对称性)

② $\forall\lambda>0$, $B_t^\lambda=\frac{1}{\lambda}B_{\lambda^2t}$ is pre-BM (scaling 不变性)

③ $\forall s\ge0$, $B_t^{(s)}=B_{t+s}-B_s$ is pre-BM and $B^{(s)}\perp\sigma(B_r, r\le s)$ (simple Markov 性)

(check 等价 ②.)

(⇒)：

Pf of prop：$\forall t\in\mathbb{Q}$，设 N_t 为 $\tilde{X}_t\neq X_t$ 的集合，记 $N=\bigcup_{t\in\mathbb{Q}} N_t$，则 $\tilde{X}_t=X_t$ $\forall t\in\mathbb{Q}$，$\omega\in N^c$

由连续，$\tilde{X}_t=X_t$，$\forall t\in T$，$\omega\in N^c$. #.

Pf of Kolmogorov's Lemma：不妨 $I=[0,1]$.

记 $D\triangleq\{\frac{i}{2^n};n\geq1,0\leq i\leq 2^n\}$，目标：在 D 上证明 $d(\cdot,\cdot)\leq C|\cdot-\cdot|^\alpha$，再延至 $[0,1]$ 上.

Lem 2.10. $d(f(\frac{i-1}{2^n}),f(\frac{i}{2^n}))\leq K2^{-n\alpha}$, then $\forall s,t\in D$, $d(f(s),f(t))\leq\frac{2K}{1-2^{-\alpha}}|t-s|^\alpha$，$\alpha>0$

(直观：$\alpha<1$ 时拆小不经济，从而尽量从 s 到 t 走大步.)

下面考虑 $s=\frac{i-1}{2^n}$，$t=\frac{i}{2^n}$，$a=2^{-n\alpha}$

$$P(d(X_{\frac{i-1}{2^n}},X_{\frac{i}{2^n}})\geq2^{-n\alpha})\leq C2^{nq\alpha}\cdot(\tfrac{1}{2^n})^{1+\varepsilon}$$

$$\Rightarrow P(\bigcup_{i=1}^{2^n}\{d(X_{\frac{i-1}{2^n}},X_{\frac{i}{2^n}})\geq2^{-n\alpha}\})\leq C2^{-n(\varepsilon-q\alpha)}$$

$(\varepsilon-q\alpha>0)\Rightarrow\sum_{n=1}^\infty P(\bigcup_{i=1}^{2^n}\{\cdots\})<\infty$

B-C: a.s.

$\therefore\exists n_0(\omega)$ s.t. $\forall n\geq n_0$，$\forall i\in\{1,\cdots,2^n\}$，$d(X_{\frac{i-1}{2^n}},X_{\frac{i}{2^n}})<2^{-n\alpha}$

$\therefore K_\alpha(\omega)\triangleq\sup_{n\geq1}\left(\sup_{1\leq i\leq2^n}d(X_{\frac{i-1}{2^n}},X_{\frac{i}{2^n}})/2^{-n\alpha}\right)<\infty$，a.s.

由 Lem 2.10，$\forall s,t\in D$，$d(X_s,X_t)\leq C_\alpha(\omega)|t-s|^\alpha$ a.s.

$\Rightarrow X_s$ extend to $I=[0,1]$. $\forall s,t\in I$，$d(X_s,X_t)\leq C_\alpha(\omega)|t-s|^\alpha$ a.s.

构造 $\tilde{X}_t(\omega)=\begin{cases}\lim_{s\to t,s\in D}X_s(\omega), & \text{if } K_\alpha(\omega)<\infty\text{，则满足 }\alpha\text{-Hölder continuity}\\ x_0\text{ (随便一个定点)}, & \text{if } K_\alpha(\omega)=\infty\end{cases}$

验证 modification：$\forall t\notin D$，$\left.\begin{array}{l}\tilde{X}_{t_n}\xrightarrow{a.s.}\tilde{X}_t,\ t_n\in D\\ \text{且 } X_{t_n}\xrightarrow{P}X_t\text{ (从假设条件)}\end{array}\right\}\Rightarrow X_t\overset{a.s.}{=}\tilde{X}_t$ #.

More about Rmk 2：

设 $\alpha_1<\cdots<\frac{\varepsilon}{q}$，对应 X 的 modification $\tilde{X}^{(n)}$，$n\geq1$，$\tilde{X}^{(0)}$ 是 Kolmogorov Lem 中构造的.

若 $0<\beta<\alpha$，$C|t-s|^\alpha\leq C|t-s|^\beta L^{\alpha-\beta}$；若 $\beta>\alpha$，$C|t-s|^\alpha\leq C|t-s|^\beta|t-s|^{\alpha-\beta}$

2.2 The Continuity of Sample Paths

- $(E,\mathcal{B}(E))$：E metric space，$\mathcal{B}(E)$ its Borel σ-field.

Def 2.6 (轨道) $(X_t)_{t\in T}$：random process valued in E. The sample paths of X are the mappings $t\in T\longmapsto X_t(\omega)$，fixing $\omega\in\Omega$.

The sample paths of X form a collection of mappings：$T\to E$ indexed by ω.

i.e., $X(\omega)=(X_n(\omega))_{n\geq0}$ 或 $X(\omega)=(X_t(\omega))_{t\geq0}\in\mathbb{R}^{\mathbb{R}_+}(\mathbb{R}^d)$

- Existence of pre-Brownian motion $B=(B_t)_{t\geq0}$.

由① invariance principle (scaled R.W. $\xrightarrow{d}B$)

或② Kolmogorov extension $\begin{cases}\text{Gaussian (white noise)}\\ \text{Markov process.}\end{cases}$

At the cost of "slightly" modifying B, we ensure that sample paths are continuous

Def 2.7 (Modification) $(X_t)_{t\in T}$，$(\tilde{X}_t)_{t\in T}$ valued in E. We say $\tilde{X}$ is a modification of X if $\forall t\in T$，$P(\tilde{X}_t=X_t)=1$.

Rmk 1. $\tilde{X}$ and X have the same finite-dim distributions：$(X_1,\cdots,X_n)\overset{d}{=}(\tilde{X}_1,\cdots,\tilde{X}_n)$

2. 无穷情况可能很糟！e.g. P.P. $\forall t\geq0$，$\Delta N_t=0$，a.s.，but $(\Delta N_t)_{t\geq0}\neq0$！

Def 2.8 (Indistinguishable) $\tilde{X}$ is indistinguishable from X, if $\exists$ null set $N\subset\Omega$, s.t.

$$\tilde{X}_t(\omega)=X_t(\omega),\forall\omega\in N^c,\forall t\in T.$$

$\Leftrightarrow\tilde{X}$ is indistinguishable if $P(\forall t\in T,X_t=\tilde{X}_t)=1$

- If $P(\forall t\in T,\tilde{X}_t=X_t)=1$, then $\forall t\in T$，$P(\tilde{X}_t=X_t)=1$. (即 indistinguishable ⇒ modification.)
- Two distinguishable processes have a.s. the same sample paths.

Prop. Let $T=I$：an interval of $\mathbb{R}$，$X,\tilde{X}$：continuous，then $\forall t\in T$，$P(\tilde{X}_t=X_t)=1$

(也可 right/left-continuous) $\Leftrightarrow P(\forall t\in T,\tilde{X}_t=X_t)=1$.

Thm 2.9 (Kolmogorov's Lemma). $X=(X_t)_{t\in I}$，I：bounded interval of $\mathbb{R}$. (E,d) 完备度量空间.

若 $\exists q,\varepsilon,C>0$，s.t. $\forall s,t\in I$，$E[d(X_s,X_t)^q]\leq C|t-s|^{1+\varepsilon}$

则 $\exists$ modification $\tilde{X}$ of X，whose sample paths are α-Hölder continuous，$\forall\alpha\in(0,\frac{\varepsilon}{q})$

i.e. $d(\tilde{X}_s(\omega),\tilde{X}_t(\omega))\leq C_\alpha(\omega)\cdot|t-s|^\alpha$.

Rmk. 1. 若 $I=\mathbb{R}_+$，则取 $[0,1],[1,2],\cdots$，则 sample paths are locally α-Hölder continuous.

i.e. $\forall0\leq s,t\leq n$，$d(\tilde{X}_s(\omega),\tilde{X}_t(\omega))\leq C_{n,\alpha}(\omega)|t-s|^\alpha\leq\max_{i\leq n}C_{[i-1,i],\alpha}(\omega)|t-s|^\alpha$

2. $\forall$ fixed $\alpha\in(0,\frac{\varepsilon}{q})$，$X$ has a modification whose sample paths are α-Hölder continuous.

Cor 2.11 $B=(B_t)_{t\geq0}$：pre-BM，then B has a modification whose sample paths are $(\frac12-\delta)$-Hölder，$\forall\delta\in(0,\frac12)$.

$\because\forall q>0$. $E|B_t-B_s|^q\overset{B_t-B_s\overset{d}{=}(t-s)^{\frac12}N(0,1)}{=}|t-s|^{\frac{q}{2}}E|N(0,1)|^q=C_q|t-s|^{\frac{q}{2}}$，其中 $\varepsilon=\frac{q}{2}-1$，$\frac{\varepsilon}{q}=\frac12-\frac1q$，

$\Rightarrow(0,\frac12-\frac1q)\nearrow(0,\frac12)$.

Def 2.12 (BM) $B=(B_t)_{t\geq 0}$ is <u>Brownian motion</u>, if ① B is pre B-M
② All sample paths are continuous.

Ex 2.29 a.s., $\varlimsup_{t\downarrow 0}\frac{B_t}{\sqrt{t}}=\infty$, $\varliminf_{t\downarrow 0}\frac{B_t}{\sqrt{t}}=-\infty$.

<u>Wiener measure</u> (the distribution of $B=(B_t)_{t\geq 0}$)

- $C([0,\infty),\mathbb{R})$：记其上的最小σ-域 $\mathscr{C}$，for which the coordinate mapping $w\mapsto w(t)$ are measurable, $\forall t\geq 0$

  ($\mathscr{C}$ = Borel σ-field on $C([0,\infty),\mathbb{R})$ associated with the topology of uniform convergence on every compact set) Arzela-Ascoli.

- Given BM (B_t), consider $B:\Omega\to C([0,\infty),\mathbb{R})$

$$w\mapsto (t\mapsto B_t(w))$$

$$\forall A\in\mathscr{C},\ W(A)=P(B_\cdot\in A)$$

- For "cylinder set" $A=\{w\in C([0,\infty),\mathbb{R}): w(t_0)\in A_0,\cdots,w(t_n)\in A_n\}$, $0=t_0<\cdots<t_n$,

$$W(A)=P(B_{t_0}\in A_0,\cdots,B_{t_n}\in A_n)\overset{\text{set } x_0=0}{=\!=}1_{A_0}(0)\cdot\int_{A_1\times\cdots\times A_n}\frac{dx_1\cdots dx_n}{(2\pi)^{\frac{n}{2}}\sqrt{t_1(t_2-t_1)\cdots}}\exp\left(-\sum_{i=1}^{n}\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)$$

- The law of BM is unique defined! i.e. B': another BM, $\forall A\in\mathscr{C}$, $P(B'_\cdot\in A)=P(B_\cdot\in A)=W(A)$.

  从而只考虑一个特殊选择：$\Omega=C([0,\infty),\mathbb{R})$, $\mathcal{F}=\mathscr{C}$, $P(dw)=W(dw)$, $X_t(w)=w(t)$

  称为 <u>canonical construction</u>.

Pf of Thm 2.13: $0<t_1<\cdots<t_k$, $g:\mathbb{R}^k\to\mathbb{R}\in C_b$（连续有界）. Fix $A\in\mathcal{F}_{0+}$

（想证 $P(A)=P(A\cap A)=P(A)P(A)$，即它与自身独立）

$$E[1_A\, g(B_{t_1},\cdots,B_{t_k})]\overset{想证}{=}P(A)\,E[g(B_{t_1},\cdots,B_{t_k})]$$

$$\overset{连续}{=}\lim_{\varepsilon\downarrow 0}E[1_A\, g(B_{t_1}-B_\varepsilon,\cdots,B_{t_k}-B_\varepsilon)]\quad（由 (B_{t_1}-B_\varepsilon,\cdots,B_{t_k}-B_\varepsilon)\perp\mathcal{F}_\varepsilon\supset\mathcal{F}_{0+}）$$

$$=\lim_{\varepsilon\downarrow 0}P(A)\,E[g(B_{t_1}-B_\varepsilon,\cdots,B_{t_k}-B_\varepsilon)]$$

$$\overset{BCT}{=}P(A)\,E[g(B_{t_1},\cdots,B_{t_k})]$$

$\therefore\ \mathcal{F}_{0+}\perp\sigma(B_s,s>0)=\sigma(B_s,s\geq 0)$ （由 $B_0=\lim_{t\downarrow 0}B_t\in\sigma(B_s,s>0)$）

$\therefore\ \mathcal{F}_{0+}\perp\mathcal{F}_{0+}$ #.

Pf of Prop 2.14:

① 令 $A=\bigcap_n\{\sup_{0\leq s\leq\frac{1}{n}}B_s>0\}\in\mathcal{F}_{0+}$，则由 Thm 2.13 (0-1律)，$P(A)=0$或$1$.

又由 $P(A)=\lim_{n\to\infty}P(\sup_{0\leq s\leq\frac{1}{n}}B_s>0)$

固定 n，$P(\sup_{0\leq s\leq\frac{1}{n}}B_s>0)\geq P(B_{\frac{1}{n}}>0)=\frac{1}{2}$ $\Rightarrow P(A)\geq\frac{1}{2}\Rightarrow P(A)=1$

② $1=P(\sup_{0\leq s\leq 1}B_s>0)=\lim_{\delta\downarrow 0}\uparrow P(\sup_{0\leq s\leq 1}B_s>\delta)$

$P(\sup_{s\geq 0}B_s>M)\overset{?}{=}1$

由 $P(\sup_{0\leq s\leq 1}B_s>\delta)=P(\sup_{0\leq s\leq\frac{M}{\delta^2}}B_s>M)\overset{\delta\downarrow 0}{\Rightarrow}P(\sup_{s\geq 0}B_s>M)=1$ #

Pf of Cor 2.15: 由 Prop 2.14①,

a.s. $\forall q\in\mathbb{Q}_+$, $\forall\varepsilon>0$, $\sup_{q\leq t\leq q+\varepsilon}B_t>B_q$, $\inf_{q\leq t\leq q+\varepsilon}B_t<B_q$

由于抠去可数个零测仍零测，故 a.s. B_t 不单调. #

Pf of Prop 2.16: $E\left(\sum_{i=1}^{p_n}(\Delta B_{t_i^n})^2-t\right)^2=Var\left(\sum_{i=1}^{p_n}(\Delta B_{t_i^n})^2\right)$

$$=\sum_{i=1}^{p_n}Var(\Delta B_{t_i^n})^2$$

$$\overset{Var(N(0,\sigma^2)^2)=2\sigma^4}{=}2\sum(t_i^n-t_{i-1}^n)^2\xrightarrow{mesh\to 0\ \sum(t_i^n-t_{i-1}^n)=t}0.$$ #

Pf of Cor 2.17: 考虑 $[0,t]$, $t>0$ 即可. 用 prop 2.16.

$$\underbrace{\sum_{i=1}^{p_n}(B_{t_i^n}-B_{t_{i-1}^n})^2}_{\downarrow L^2,\ prop\ 2.16\ \ t}\leq\underbrace{\sup_{1\leq i\leq p_n}|B_{t_i^n}-B_{t_{i-1}^n}|}_{\downarrow mesh\to 0\ \ 0}\times\underbrace{\sum_{i=1}^{p_n}|B_{t_i^n}-B_{t_{i-1}^n}|}_{\infty}$$

$\therefore\ \sum_{i=1}^{p_n}|B_{t_i^n}-B_{t_{i-1}^n}|\to\infty\ \Rightarrow$ infinite variation. #

2.3 Properties of Brownian sample paths

- $B=(B_t)_{t\geq 0}$, $\mathcal{F}_t=\sigma(B_s,s\leq t)$, $t\geq 0$. , $\mathcal{F}_{0+}=\bigcap_{s>0}\mathcal{F}_s\supset\mathcal{F}_0$.

Thm 2.13 (Blumenthal's 0-1 law) $\mathcal{F}_{0+}$ is trivial, that is, $P(A)=0$或1, $\forall A\in\mathcal{F}_{0+}$.

Prop 2.14 ① a.s. $\forall\varepsilon>0$, $\sup_{0\leq s\leq\varepsilon}B_s>0$, $\inf_{0\leq s<\varepsilon}B_s<0$.

② let $T_a=\inf\{t\geq 0: B_t=a\}$, then a.s. $\forall a\in\mathbb{R}$, $T_a<\infty$.

Cor. 2.15 a.s. $t\mapsto B_t$ is not monotone on any nontrivial interval.

Prop 2.16 Let $0=t_0^n<\cdots<t_{p_n}^n=t$ be a sequence of $[0,t]$ 划分 whose mesh $\to 0$ (即 $\sup_{1\leq i\leq p_n}(t_i^n-t_{i-1}^n)\to 0$.), then $\sum_{i=1}^{p_n}(B_{t_i^n}-B_{t_{i-1}^n})^2\xrightarrow{L^2}t$.

- 称 f on $[a,b]$ has infinite <u>variation</u>, if $\sup\sum_{i=1}^{p}|f(t_i)-f(t_{i-1})|=\infty$.

Cor 2.17 a.s. $t\mapsto B_t$ has infinite variation on any interval.

Pf of S.M.P. Thm 2.20 step1. $T<\infty$, a.s. 而 $\bigcup \sigma(B_{t_1},\cdots,B_{t_n})$ 为 π-类.

下证 $\forall A\in\mathcal{F}_T$, $0\le t_1<\cdots<t_p$, $F:\mathbb{R}^p\to\mathbb{R}_+$ bounded continuous,

都有 $E[1_A F(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_p})] = P(A)\, E[F(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_p})]$ ······ $(*)$

特别, 取 $A=\Omega$, 知 $B^{(T)}_t$ 为 BM. 且 上式 $\Rightarrow \mathcal{F}_T \perp \sigma(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_p})$

$\Rightarrow \mathcal{F}_T\perp\sigma(B^{(T)})$

(考虑π类 $\{A: A\in\sigma(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_p}), \forall 0\le t_1<\cdots<t_p, \forall p\}$)

证明$(*)$式: $\forall n\ge1$, $t\ge0$, $[t]_n$: the smallest real of form $\frac{k}{2^n}\ge t$ $(k\in\mathbb{Z})$

Rmk. $F(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_n}) = \lim_{n\to\infty} F(B^{[T]_n}_{t_1},\cdots,B^{[T]_n}_{t_n})$

则 $F(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_p}) \longleftarrow F(B^{([T]_n)}_{t_1},\cdots,B^{([T]_n)}_{t_p})$

$(*)$ LHS $= \lim_{n\to\infty} E[1_A F(B^{([T]_n)}_{t_1},\cdots,B^{([T]_n)}_{t_p})]$

$= \lim_{n\to\infty}\sum_{k=0}^{\infty} E[1_A 1_{\{\frac{k-1}{2^n}<T\le\frac{k}{2^n}\}} F(B_{\frac{k}{2^n}+t_1}-B_{\frac{k}{2^n}},\cdots,B_{\frac{k}{2^n}+t_p}-B_{\frac{k}{2^n}})]$

$(\because A\cap\{\frac{k-1}{2^n}<T\le\frac{k}{2^n}\}\in\mathcal{F}_{\frac{k}{2^n}})$

$\overset{M.P.}{=} \lim_{n\to\infty}\sum_{k=0}^{\infty} P(A\cap\{\frac{k-1}{2^n}<T\le\frac{k}{2^n}\})\, E[F(B_{t_1},\cdots,B_{t_p})]$

$= P(A)\, E[F(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_p})]$.

Step2. $P(T<\infty)>0$. $(*)$式变为

$E[1_{A\cap\{T<\infty\}} F(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_p})] = P(A\cap\{T<\infty\})\, E[F(B^{(T)}_{t_1},\cdots,B^{(T)}_{t_p})]$.

$E[F(B^{(T)})1_A;\ T<\infty] = E[F(B)]\, E[1_A;\ T<\infty]$ #

Why $T=\sup\{s\le1, B_s=0\}$ 不是停时? ∵ on $\{T=1\}$, $B_1=0$ $\Rightarrow T<1$, a.s.

$(B_{T+t}-B_T)_{t\ge0}$ $B_t=0$, $t\in[T,1)$ has infinitely many roots.

Pf of (Reflection) Th.2.21: $T_a=\inf\{t\ge0, B_t=a\}$, then $T_a<\infty$, a.s.

记 $B'=B^{(T_a)}$, 则 $B'\perp\mathcal{F}_{T_a}$, B' 为 BM. $\Rightarrow (T_a,B')\overset{d}{=}(T_a,-B')$

固定 t,a,b. 定义 $H=\{(s,w)\in\mathbb{R}_+\times C(\mathbb{R}_+,\mathbb{R}): s\le t, w(t-s)\le b-a\}$

$P(S_t\ge a, B_t\le b) = P(T_a\le t, B_t\le b) = P(T_a\le t, B^{(T_a)}_{t-T_a}\le b-a)$

$= P((T_a,B')\in H) = P((T_a,-B')\in H)$

$= P(T_a\le t, -B^{(T_a)}_{t-T_a}\le b-a)$

$= P(T_a\le t, B_t\ge 2a-b) \overset{b\le a}{=} P(B_t\ge 2a-b)$ #

· $C([0,\infty],\mathbb{R}^d)$: Wiener measure in d-dim is the product measure of Wiener measure in 1-dim?

· Def 2.24' ① $B_0=(0,\cdots,0)$
② $0\le s\le t$, $B_t-B_s\perp\sigma(B_r, 0\le r\le s)$
③ $0\le s\le t$, $B_t-B_s\sim N(0,(t-s)I_d)$

2.4 Strong Markov Property

· $(B_t)_{t\ge0}$, $\mathcal{F}_t$, $\mathcal{F}_\infty=\sigma(B_s, s\ge0)$.

Def (Stopping time) A r.v. T with values in $[0,\infty]$ is a stopping time, if $\forall t\ge0$, $\{T\le t\}\in\mathcal{F}_t$.

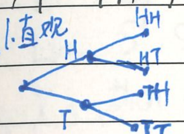
Rmk. $\{T<t\}=\bigcup_{n\ge m}\{T\le t-\frac{1}{n}\}\in\mathcal{F}_t$. $\Rightarrow$ $\{T=t\}\in\mathcal{F}_t$, 但常是零测.

E.g. ① $T=t$
② $T=T_a=\inf\{t\ge0, B_t=a\}$ ∵ $\{T_a\le t\}=\{\inf_{0\le s\le t}|B_s-a|=0\}$
③ $T=\sup\{s\le1: B_s=0\}$ 不是停时, 但 $\{T\le\frac12\}\in\mathcal{F}_{\frac12}$.

Rmk. $\forall t\ge0$, $T+t$ is also 停时.

Def. $\mathcal{F}_T\triangleq\{A\in\mathcal{F}_\infty: \forall t\ge0, A\cap\{T\le t\}\in\mathcal{F}_t\}$.

Rmk. 1.直观



$\mathcal{F}_T\cap\{T=1\}=\mathcal{F}_1\cap\{T=1\}$ $(\mathcal{F}_T\cap\{T=1\}=\{B\cap\{T=1\}: B\in\mathcal{F}_T\})$

$\mathcal{F}_T\cap\{T=1,0\}=\mathcal{F}_{1,0}\cap\{T=1,0\}$

2. $T\in\mathcal{F}_T$, $\mathcal{F}_T$ 是σ-field.

3. $1_{\{T<\infty\}}B_T\in\mathcal{F}_T$ 定义: $1_{\{T<\infty\}}B_T(\omega)=\begin{cases}B_{T(\omega)}(\omega), & \text{if } T(\omega)<\infty\\ 0, & \text{if } T(\omega)=\infty\end{cases}$

∵ $1_{\{T<\infty\}}B_T=\lim_{n\to\infty}\sum_{i=0}^{\infty}1_{\{\frac{i}{n}\le T<\frac{i+1}{n}\}}B_{\frac{i}{n}}$

$=\lim_{n\to\infty}\sum_{i=0}^{\infty}1_{\{T<\frac{i+1}{n}\}}1_{\{T\ge\frac{i}{n}\}}B_{\frac{i}{n}}$

Thm 2.20 (S.M.P.) 停时 $T<\infty$, a.s. $\forall t\ge0$, set $B^{(T)}_t=B_{T+t}-B_T$, then $(B^{(T)}_t)_{t\ge0}$ is a BM independent of $\mathcal{F}_T$.

If $P(T<\infty)>0$, $\forall t\ge0$, set $B^{(T)}_t=(B_{T+t}-B_T)1_{\{T<\infty\}}$.

Then under $P(\cdot|T<\infty)$, $B^{(T)}$ is a BM independent of $\mathcal{F}_T$

Thm 2.21 (Reflection) Let $S_t=\sup_{s\le t}B_s$, if $a\ge0$, $b\in(-\infty,a]$, then

$P(S_t\ge a, B_t\le b)=P(B_t\ge 2a-b)$. 特别, $S_t\overset{d}{=}|B_t|$

∵ $P(S_t\ge a)=P(S_t\ge a, B_t\le a)+P(S_t\ge a, B_t>a)$
$=P(B_t\ge 2a-a=a)+P(B_t>a)$
$=P(|B_t|\ge a)$

Rmk. $(S_t)_{t\ge0}\overset{d}{\neq}(|B_t|)_{t\ge0}$.

Cor. (S_t,B_t) has density $g(a,b)=f(a,b)1_{\{a\ge0,b<a\}}$
$=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(2a-b)^2}{2t}}\frac{2(2a-b)}{t}$

Cor 2.22. $a>0$ $T_a\overset{d}{=}\frac{a^2}{B_1^2}$, density $f(t)=\frac{a}{\sqrt{2\pi t^3}}e^{-\frac{a^2}{2t}}1_{\{t>0\}}$ ∵ $P(T_a\le t)=P(S_t\ge a)$
$=P(B_t^2\ge a^2)$
$=P(\frac{a^2}{B_1^2}\le t^2)$

Rmk. $ET_a=\infty$

Def 2.23 (不从0出发的BM) $(X_t)_{t\ge0}$ is a real BM started from Z, if $X_t=Z+B_t$, where B is a real BM started from 0 and $B\perp Z$.

Def 2.24 (多维BM) $B_t=(B^1_t,\cdots,B^d_t)$ is a d-dimensional BM started from 0, if B^i 独立BM started from 0.

$X_t=Z+B_t$, $B\perp Z$ is a d-dim BM started from Z. Rmk X的分量可能不独立.

- If X is a d-dim BM and Φ is an isometry of $\mathbb{R}^d$, then $(\Phi(X_t))_{t\geq 0}$ is a d-dim BM.

$\because$ $\Phi(X_t) - \Phi(X_s) \overset{A正交}{=} A(X_t - X_s) \sim N(0, (t-s)I_d)$

Functional CLT, Donsker

<u>Th1</u> Let $\xi_1, \xi_2, \cdots$ i.i.d. $E\xi_i = 0$, $E\xi_i^2 = 1$. Define $X_t^n = \frac{1}{\sqrt{n}}\left[\sum_{k\leq nt}\xi_k + (nt - [nt])\xi_{[nt]+1}\right]$, $t\geq 0, n\in\mathbb{N}$

Then $X^n \xrightarrow{d} B$ in $C([0,\infty), \mathbb{R})$

<u>Th2</u> $X_t^n = \frac{1}{\sqrt{n}}\sum_{k\leq nt}\xi_k$, then $X^n \xrightarrow{d} B$ in $D([0,\infty), \mathbb{R})$ (r.c.l.l 空间)

CLT: $X_1^n \xrightarrow{d} B_1$

<u>Th3</u>. Let $S^1, S^2, \cdots$ be random walks in $\mathbb{R}^d$, s.t. $S^n_{m_n} \xrightarrow{d} X_1$ for some Lévy process X, then the processes $X_t^n = S^n_{[m_n t]}$ satisfy $X^n \xrightarrow{d} X$ in $D([0,\infty), \mathbb{R}^d)$.

E.g. ① $\mathcal{F}_t^X=\sigma(X_s, 0\le s\le t)$, $\mathcal{F}_\infty^X=\sigma(X_s, s\ge 0)$ → canonical filtration

② "右极限" $\mathcal{F}_{t+}=\bigcap_{s>t}\mathcal{F}_s$, $\mathcal{F}_{\infty+}=\mathcal{F}_\infty$. → $(\mathcal{F}_{t+})_{0\le t\le\infty}$ right-continuous

(注：右连续, if $\mathcal{F}_{t+}=\mathcal{F}_t, \forall t\ge 0$.)

Pf of right-continuity: $\mathcal{F}_{t+}=\bigcap_{s>t}\mathcal{F}_{s+}$

即证 $\bigcap_{s>t}\mathcal{F}_s=\bigcap_{s>t}\bigcap_{s'>s}\mathcal{F}_{s'}$. #

Pf of prop 3.4:

$\forall t\ge 0$, 为证 $X_s(\omega)$ on $\Omega\times[0,t]$ 可测.

$\forall n\ge 1$, $\forall s\in[0,t]$, 令 $X_s^n=X_{k\cdot\frac{t}{n}}$, $(s\in[(k-1)\frac{t}{n}, k\cdot\frac{t}{n}))$, $X_t^n=X_t$

右连续 ⇒ $X_s(\omega)=\lim_{n\to\infty}X_s^n(\omega)$.

∵ $\{X_s^n(\omega)\in A\}=\bigcup_{k=1}^{n}\{X_{k\cdot\frac{t}{n}}\in A\}\times[(k-1)\frac{t}{n}, k\cdot\frac{t}{n})\cup(\{X_t\in A\}\times\{t\})\in\mathcal{F}_t\otimes\mathcal{B}([0,t])$

∴ $(X_s^n(\omega))$ on $\Omega\times[0,t]$ is measurable. ⇒ $(X_s(\omega))$ measurable ⇒ X progressive #

Pf of prop 3.6:

(i) "⇒": $\{T<t\}=\bigcup_{q<t,q\in\mathbb{Q}}\{T\le q\}\in\mathcal{F}_t$;

"⇐": $\{T\le t\}=\bigcap_{t<q<s,q\in\mathbb{Q}}\{T<q\}\in\mathcal{F}_s$, $\forall s>t$. ⇒ $\{T\le t\}\in\mathcal{F}_{t+}=\mathcal{G}_t$.

$\forall t$, $T\wedge t$ is $\mathcal{F}_t$-measurable ⇔ $\forall s<t$, $\{T\le s\}\in\mathcal{F}_t$ ⇒ $\{T<t\}\in\mathcal{F}_t$ ⇒ T stopping time w.r.t. $(\mathcal{G}_t)$.

T is a stopping time w.r.t. $(\mathcal{G}_t)$ ⇒ $\{T\le s\}\in\mathcal{G}_s\subset\mathcal{F}_t$, $\forall s<t$ ⇒ $T\wedge t$ is $\mathcal{F}_t$-measurable.

(ii) "⇒": $\forall A\in\mathcal{G}_T$, $A\cap\{T\le t\}\in\mathcal{G}_t$ ⇒ $A\cap\{T<t\}=\bigcup_{q<t,q\in\mathbb{Q}}(A\cap\{T\le q\})\in\mathcal{F}_t$

"⇐": 若 $A\cap\{T<t\}\in\mathcal{F}_t, \forall t>0$, 则 $A\cap\{T\le t\}=\bigcap_{t<q<s,q\in\mathbb{Q}}(A\cap\{T<q\})\in\mathcal{F}_s$, $\forall s>t$

⇒ $A\cap\{T\le t\}\in\mathcal{F}_{t+}=\mathcal{G}_t$. #

E.g. $T=3$ 或 10, 则 $\mathcal{F}_T=\sigma((\{T=3\}\cap\mathcal{F}_3)\cup(\{T=10\}\cap\mathcal{F}_{10}))$

3.1 Filtrations & Processes

Def. (Filtration) $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$: $\mathcal{F}_s\subset\mathcal{F}_t$, $\forall s\le t<\infty$

Def. (Complete) Let $\mathcal{N}$: the class of $(\mathcal{F}_\infty,P)$-negligible sets

($A\in\mathcal{N}$, if $\exists A'\in\mathcal{F}_\infty$, s.t. $A\subset A'$ and $P(A')=0$.)

We say $(\mathcal{F}_t)$ is complete, if $\mathcal{N}\subset\mathcal{F}_0$ ($\subset\mathcal{F}_t,\forall t$).

$\mathcal{F}_t'=\mathcal{F}_t\vee\sigma(\mathcal{N})$ is complete.

称 $((\mathcal{F}_t^X)')_{0\le t\le\infty}$ 为 complete canonical filtration of X

Def. (measurable): $(E,\mathcal{E})$: measurable space, $(X_t)_{t\ge 0}$ is measurable, if $(\omega,t)\mapsto X_t(\omega)$ on $(\Omega\times\mathbb{R}_+,\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+))$ is measurable

Rmk. "(X_t) measurable" is stronger than "X_t is $\mathcal{F}$-measurable, $\forall t\ge 0$."

Def. $(X_t)_{t\ge 0}$ on $(E,\mathcal{E})$ is adapted, if $\forall t\ge 0$, X_t is $\mathcal{F}_t$-measurable.

progressive, if $\forall t\ge 0$, $(\omega,s)\mapsto X_s(\omega)$ on $\Omega\times[0,t]$ is measurable for $\mathcal{F}_t\otimes\mathcal{B}([0,t])$.

progressive ⇒ measurable + adapted

(adapted ⇏ progressive)

Prop 3.4. $(X_t)_{t\ge 0}$ on (E,d). adapted, sample paths are right continuous ($X_t(\omega)$ 右连, $\forall\omega$). Then X is progressive. (or left-continuous).

· the progressive σ-field: $\mathcal{P}\triangleq\{A\in\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+): (X_t(\omega)=1_A(\omega,t))$ is progressive$\}$ which is a σ-field on $\Omega\times\mathbb{R}_+$.

· $A\in\mathcal{P}$ ⇔ $A\cap(\Omega\times[0,t])\in\mathcal{F}_t\otimes\mathcal{B}([0,t])$, $\forall t\ge 0$.

· X: progressive ⇔ $(\omega,t)\mapsto X_t(\omega)$ measurable on $\Omega\times\mathbb{R}_+$ with σ-field $\mathcal{P}$ (原来是 with $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)$).

3.2 Stopping Times & Associated σ-Fields

Def. stopping time: $T:\Omega\to[0,\infty]$, $\{T\le t\}\in\mathcal{F}_t, \forall t\ge 0$.

σ-field of the past before T: $\mathcal{F}_T=\{A\in\mathcal{F}_\infty: A\cap\{T\le t\}\in\mathcal{F}_t,\forall t\ge 0\}$.

· $\forall t>0$, $\{T<t\}=\bigcup_n\{T\le t-\frac{1}{n}\}\in\mathcal{F}_t$. , $\{T=\infty\}=(\bigcup_{n=1}^{\infty}\{T\le n\})^c\in\mathcal{F}_\infty$

· if T is a stopping time w.r.t. $\mathcal{F}_t$, then is a stopping time for $\mathcal{F}_{t+}$ (∵ $\{T\le t\}\in\mathcal{F}_t\subset\mathcal{F}_{t+}$)

Prop 3.6. $\mathcal{G}_t=\mathcal{F}_{t+}$, $t\in[0,\infty]$. (i) T stopping time w.r.t. $(\mathcal{G}_t)$ ⇔ $\{T<t\}\in\mathcal{F}_t$, $\forall t>0$.

⇔ $T\wedge t$ is $\mathcal{F}_t$-measurable, $\forall t>0$

(ii) T stopping time w.r.t. $(\mathcal{G}_t)$, then $\mathcal{G}_T=\{A\in\mathcal{F}_\infty: \forall t>0, A\cap\{T<t\}\in\mathcal{F}_t\}$

记为 $\boxed{\mathcal{F}_{T+}}$.

Pf of properties: (abc 回略)

(d): $\{T^A\le t\} = A\cap\{T\le t\}$

(e): If $A\in\mathcal{F}_S$, then $A\cap\{T\le t\} = (A\cap\{S\le t\})\cap\{T\le t\}\in\mathcal{F}_t$.

(f): $\{S\vee T\le t\} = \{S\le t\}\cap\{T\le t\}\in\mathcal{F}_t$. $\Rightarrow$ $S\vee T$ 为停时.

由(e)知 $\mathcal{F}_{S\wedge T}\subset\mathcal{F}_S\cap\mathcal{F}_T$.

若 $A\in\mathcal{F}_S\cap\mathcal{F}_T$, $A\cap\{S\wedge T\le t\} = (A\cap\{S\le t\})\cap(A\cap\{T\le t\})\in\mathcal{F}_t$

$\Rightarrow A\in\mathcal{F}_{S\wedge T}$.

prop3.6: $S\wedge t$: $\mathcal{F}_t$-measurable

further, $\{S\le T\}\cap\{T\le t\} = \{S\le t\}\cap\{T\le t\}\cap\{S\wedge t\le T\wedge t\}\in\mathcal{F}_t$

$\{S=T\} = \{S\le T\}\cap\{T\le S\}$.

(g) $\{S\le t\} = \bigcap\{S_n\le t\}\in\mathcal{F}_t$

(h) $\{S<t\} = \bigcup\{S_n<t\}\in\mathcal{F}_t$. 且 $\mathcal{F}_{S+}\subset\bigcap_n\mathcal{F}_{S_n+}$

另一方面, 若 $A\in\bigcap_n\mathcal{F}_{S_n+}$, $A\cap\{S<t\} = \bigcup_n(A\cap\{S_n<t\})\in\mathcal{F}_t\Rightarrow A\in$

(i) $\{S\le t\} = \bigcup_n\{S_n\le t\}\in\mathcal{F}_t$ 且 $\mathcal{F}_S\subset\bigcap_n\mathcal{F}_{S_n}$

另一方面, 若 $A\in\bigcap_n\mathcal{F}_{S_n}$, $A\cap\{S\le t\} = \bigcup_n(A\cap\{S_n\le t\})\in\mathcal{F}_t\Rightarrow A\in\mathcal{F}_S$.

(j) "$\Leftarrow$": $\forall$ measurable $A\in E$, $\{Y\in A\}\cap\{T\le t\}\in\mathcal{F}_t \overset{t\to\infty}{\Rightarrow} \{Y\in A\}\in\mathcal{F}_\infty$ 且 $\{Y\in A\}\in\mathcal{F}_T$

"$\Rightarrow$": $\{Y\in A\}\in\mathcal{F}_T\Rightarrow\{Y\in A\}\cap\{T\le t\}\in\mathcal{F}_t$. #.

Pf of Thm3.7: use prop (j), $\forall t\ge 0$, X_T 限制在 $\{T\le t\}$ 上:

$$\omega\in\{T\le t\}\longmapsto(\omega, T(\omega)\wedge t)\longmapsto X_{T(\omega)}(\omega)$$

$$\mathcal{F}_t\longrightarrow\mathcal{F}_t\otimes\mathcal{B}([0,t])\longrightarrow\mathcal{E}$$

$\because$ 由 prop 3.6 (i), $T\wedge t$ is $\mathcal{F}_T$-measurable
由progressive, $(\omega, s)\mapsto X_s(\omega)$ is measurable
$\Big\}\Rightarrow\omega\mapsto X_T(\omega)$ is $\mathcal{F}_T$-measurable

(S为$\mathcal{F}_T$-measurable) $\Rightarrow\{S\le t\}$

Pf of prop3.8: $\{S\le t\} = \{S\le t\}\cap\{T\le t\}\in\mathcal{F}_t$

特别 T_n 为 T的函数 $\Rightarrow T_n$: $\mathcal{F}_T$-measurable, 且 $T_n\downarrow T$. #.

Pf of prop 3.9 (i) $\forall t>0$, $\{T_O<t\}\overset{右连}{=}\bigcup_{s\in[0,t)\cap\mathbb{Q}}\{X_s\in O\}\in\mathcal{F}_t$

(ii) $\forall t\ge 0$, $\{T_F\le t\}\overset{连续}{=}\{\inf_{0\le s\le t} d(X_s,F)=0\} = \{\inf_{s\in[0,t]\cap\mathbb{Q}} d(X_s,F)=0\}\in\mathcal{F}_t$. #

Properties:

(a). $\mathcal{F}_T\subset\mathcal{F}_{T+}$. 且 $\mathcal{F}_T = \mathcal{F}_{T+}$, 若 $(\mathcal{F}_t)$ 右连续

(b) $T=t$, 则 $\mathcal{F}_T = \mathcal{F}_t$, $\mathcal{F}_{T+} = \mathcal{F}_{t+}$.

(c) T is $\mathcal{F}_T$-measurable

(d) $A\in\mathcal{F}_\infty$, $T^A(\omega) = \begin{cases}T(\omega), & \omega\in A\\ +\infty, & \omega\notin A\end{cases}$, 则 $A\in\mathcal{F}_T\Leftrightarrow T^A$ is a stopping time.

(e) $S\le T$. 则 $\mathcal{F}_S\subset\mathcal{F}_T$, $\mathcal{F}_{S+}\subset\mathcal{F}_{T+}$

(f). $S\vee T$. $S\wedge T$ 为 stopping times 且 $\mathcal{F}_{S\wedge T} = \mathcal{F}_S\cap\mathcal{F}_T$, $\{S\le T\}\in\mathcal{F}_{S\wedge T}$, $\{S=T\}\in\mathcal{F}_{S\wedge T}$.

(g) $S_n\uparrow S$ 为 stopping time (h) $S_n\downarrow S$ 为 stopping time w.r.t. $(\mathcal{F}_{t+})$. $\mathcal{F}_{S+} = \bigcap_n\mathcal{F}_{S_n+}$

(i) $S_n\downarrow S$ 且 stationary ($\forall\omega$, $\exists N(\omega)$, s.t. $S_n(\omega) = S(\omega)$, $\forall n\ge N(\omega)$.), 则 S 为 stopping time,
且 $\mathcal{F}_S = \bigcap_n\mathcal{F}_{S_n}$.

(j) $Y(\omega)\in(E,\mathcal{E})$ on $\{T<\infty\}$ is $\mathcal{F}_T$-measurable $\Leftrightarrow\forall t\ge 0$, $Y|_{\{T\le t\}}$ is $\mathcal{F}_t$-measurable.

Thm 3.7. $(X_t)_{t\ge 0}$: progressive in $(E,\mathcal{E})$, T = stopping time, then $X_T(\omega) := X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable on $\{T<\infty\}$.

Prop 3.8. T = stopping time, S: $\mathcal{F}_T$-measurable r.v. in $[0,\infty]$, $S\ge T$. Then S is 停时.
特别, $T_n = \sum_{k=0}^{\infty}\frac{k+1}{2^n}1_{\{\frac{k}{2^n}<T\le\frac{k+1}{2^n}\}} + \infty\cdot 1_{\{T=\infty\}}$ 为一列停时, $T_n\downarrow T$.

Prop 3.9. $(X_t)_{t\ge 0}$: adapted, values in (E,d).
(i) 若 X: 右连, $O\overset{开}{\subset}E$, 则 $T_O = \inf\{t\ge 0: X_t\in O\}$ 为 stopping time w.r.t. $(\mathcal{F}_{t+})$
(ii) 若 X: 连续, $F\overset{闭}{\subset}E$, 则 $T_F = \inf\{t\ge 0: X_t\in F\}$ 为 stopping time w.r.t. $(\mathcal{F}_t)$.

Appendix A Discrete-time Martingales $\quad\mathcal{F}_\infty = \bigvee_{t\ge 0}\mathcal{F}_t$.

① Thm (Maximal ineq). $(Y_n)_{n\ge 0}$ sup, then $\forall\lambda>0$, $k\in\mathbb{N}$,
$$\lambda P(\sup_{n\le k}|Y_n|\ge\lambda)\le E|Y_0| + 2E|Y_k|.$$

(Recall thm 4.4.2: X sub, $\lambda P(\sup_{0\le m\le n}X_m^+\ge\lambda)\le E|X_n|$.)

Pf: $\lambda P(\sup|Y_n|\ge\lambda)\le\lambda P(\sup_{n\le k}Y_n\ge\lambda) + \lambda P(\inf_{n\le k}Y_n\le-\lambda)$ $(\le E|Y_k|)$
$(\le E|Y_0| + E|Y_k|.)$ #

② Thm. (Doob's ineq in L^p) (Y_n) mar, $E[\sup_{n\le k}|Y_n|^p]\le(\frac{p}{p-1})^p E|Y_k|^p$, $\forall p>1$.

③ Thm (Doob's upcrossing ineq.) (Y_n) sup, $\forall a<b$, $n\in\mathbb{N}$, 有 $E[M_{ab}^Y(n)]\le\frac{1}{b-a}E[(Y_n-a)^-]$.

Recall: $(b-a)EU_n\le E(X_n-a)^+ - E(X_0-a)^+$, (X_n) sub.
$\Rightarrow[-a-(-b)]EU_n\le E[-Y_n-(-b)]^+ - E[-Y_0-(-b)]^+$, (Y_n) sup
$\Rightarrow(b-a)EU_n\le E(Y_n-b)^- - E(Y_0-b)^-$, (Y_n) sup, U_n downcrossing

注意 $|M_n - U_n|\le 1$.

Pf of super Convergence:

我们学过 sub, $\sup E X_n^+ < \infty$ 的 a.s. 收敛

注意 $\sup E X_n^+ < \infty \Leftrightarrow \sup E|X_n| < \infty$ $\left(\because E|X_n| = 2EX_n^+ - EX_n \le 2EX_n^+ - EX_0 \right)$. #

E.g. Say $Z = (Z_t)_{t\ge 0}$ with values in $\mathbb{R}$ or $\mathbb{R}^d$ has 独立增量 w.r.t. $(\mathcal{F}_t)$, if Z is adapted and $\forall 0 \le s < t$, $Z_t - Z_s \perp \mathcal{F}_s$

(不一定有 $Z_t - Z_s \overset{d}{=} Z_{t-s}$).

则 ① If $Z_t \in L^1$, then $\tilde{Z}_t = Z_t - E(Z_t)$ mar

② If $Z_t \in L^2$, then $Y_t = \tilde{Z}_t^2 - E\tilde{Z}_t^2$ mar

③ If $\exists \theta \in \mathbb{R}$, $E[e^{\theta Z_t}] < \infty$, then $\frac{e^{\theta Z_t}}{Ee^{\theta Z_t}}$ mar.

e.g. $\begin{cases} (B_t)_{t\ge 0} \text{ mar} \\ (B_t^2 - t)_{t\ge 0} \text{ mar} \\ (e^{\theta B_t - \frac{\theta^2}{2}t})_{t\ge 0} \text{ mar.} \end{cases}$

$\begin{cases} (N_t - \lambda t)_{t\ge 0} \text{ mar} \\ ((N_t - \lambda t)^2 - \lambda t)_{t\ge 0} \text{ mar} \\ (e^{\theta N_t - \lambda t(e^\theta - 1)})_{t\ge 0} \text{ mar} \end{cases}$

Pf of Prop 3.13:

If X sub, $X^+ = (X_t^+)$ sub $\Rightarrow EX_s^+ \le EX_t^+$, $\forall 0 \le s \le t$.

$\Rightarrow E|X_s| = 2EX_s^+ - EX_s \le 2EX_t^+ - EX_0 < \infty$ #

Pf of prop 3.14:

只须证 $E[(M_t - M_s)^2 | \mathcal{F}_s] = E[M_t^2 - M_s^2 | \mathcal{F}_s]$

$= E[M_t^2 + M_s^2 - 2M_tM_s | \mathcal{F}_s]$

$= E[M_t^2 | \mathcal{F}_s] + M_s^2 - 2M_s E(M_t | \mathcal{F}_s) = E(M_t^2 - M_s^2 | \mathcal{F}_s)$ #

Pf of prop 3.15:

① 在 countable dense subset D of $[0,t]$ 上考虑. $D_m = \{t_0^m, \cdots, t_m^m\}$, 其中 $0 = t_0^m < \cdots < t_m^m = t$

由离散的 maximal ineq, $\lambda P(\sup_{s\in D_m} |X_s| > \lambda) \le E|X_0| + 2E|X_t|$.

$m \to \infty$ 得 $\overset{(MCT)}{\lambda P(\sup_{s\in D} |X_s| > \lambda)} \le E|X_0| + 2E|X_t|$, 由 r.c. 得证.

② $E[\sup_{s\in D_m} |X_s|^p] \le$ RHS.

$\downarrow m\to\infty$ (MCT)

$E[\sup_{s\in D} |X_s|^p]$ 由 r.c. 得证 #

Rmk. 若无 r.c. 条件, $P(\sup_{s\in D} |X_s| > \lambda) \le \frac{1}{\lambda}(E|X_0| + 2E|X_t|) \Rightarrow \sup_{s\in D} |X_s| < \infty$, a.s.

④ Thm (Convergence, super) Y_n: super, $\sup E|Y_n| < \infty \Rightarrow \exists Y_\infty \in L^1$, s.t. $Y_n \to Y_\infty$ a.s.

⑤ Thm. (Convergence, u.i. mar) (1) (Y_n) is closed ($\exists Z$, s.t. $Y_n = E(Z|\mathcal{F}_n)$.)

$\Leftrightarrow$ (2) (Y_n) a.s. & L^1

$\Leftrightarrow$ (3) (Y_n) u.i.

then, $Y_n \xrightarrow{a.s.} Y_\infty = E(Z | \mathcal{F}_\infty)$.

⑥ Thm. (Optional stopping, u.i. mar) $S \le T$, then $Y_S = E[Y_T | \mathcal{F}_S]$ (recall Th 4.8.1 & Ex 4.8.1)

⑦ Thm. (Optional stopping, super bdd case) $S \le T$ bdd, then $Y_S \ge E[Y_T | \mathcal{F}_S]$

⑧ Thm (Convergence, backward, super) $E(Y_{n+1} | \mathcal{F}_n) \le Y_n$, $\forall n \le -1$ 为 backward super.

$(Y_n)_{n\le 0}$ super, $\sup E|Y_n| < \infty$, then $Y_n \to Y_\infty$ a.s. & L^1

(a.s. 不需条件; $Y_\infty \in L^1$ 需 $\sup E|Y_n| < \infty$, 且自动变为 L^1 收敛.)

3.3 Continuous Time Martingales & Supermartingales

Def. $X = (X_t)_{t\ge 0}$, $X_t \in L^1$, $\forall t \ge 0$.

① mar: $\forall 0 \le s \le t$, $E(X_t | \mathcal{F}_s) = X_s$.

② super $\le$

③ sub $\ge$

E.g. $Z \in L^1$, $X_t = E(Z | \mathcal{F}_t)$

Def ($(\mathcal{F}_t)$-BM): If B is adapted, 独立增量 ($B_t - B_s \perp \mathcal{F}_s \supset \sigma(B_r, 0 \le r \le s)$.)

Prop 3.12 $(X_t)_{t\ge 0}$ adapted, $f: \mathbb{R} \to \mathbb{R}_+$ convex, $E(f(X_t)) < \infty$.

① X mar $\Rightarrow f(X)$ sub

② X sub $+ f$ nondecreasing $\Rightarrow f(X)$ sub   Rmk. X mar, $E|X_t|^p$ nondecreasing, $p \ge 1$.

Prop 3.13 X sub/super, then $\forall t > 0$, $\sup_{0\le s\le t} E|X_s| < \infty$

Prop 3.14 (M_t) square integrable ($M_t \in L^2, \forall t \ge 0$). $0 \le s \le t$, $s = t_0 < \cdots < t_p = t$, then

$$E\left[\sum_{i=1}^{p} (M_{t_i} - M_{t_{i-1}})^2 \Big| \mathcal{F}_s\right] = E[M_t^2 - M_s^2 | \mathcal{F}_s] = E[(M_t - M_s)^2 | \mathcal{F}_s]$$

Rmk. Quadratic Variation process L_t: $E(M_t^2 - L_t | \mathcal{F}_s) = M_s^2 - L_s$

· the regularity properties of sample paths.

Prop 3.15 ① (Maximal ineq) X: super, r.c. sample paths, then $\forall t > 0$, $\lambda > 0$,

$$\lambda P(\sup_{0\le s\le t} |X_s| > \lambda) \le E|X_0| + 2E|X_t|.$$

② (Doob's ineq in L^p) X: mar, r.c. sample paths, then $\forall t > 0$, $p > 1$,

$$E(\sup_{0\le s\le t} |X_s|^p) \le \left(\frac{p}{p-1}\right)^p E|X_t|^p$$

由Thm 3.15 证明后的 Rmk

Pf of Thm 3.17: ① $\sup_{s\in D\cap[0,T]} |X_s| < \infty$, a.s.

$0, T \in D_m$, $D_m \uparrow D\cap[0,T]$

则由上节, $E[M_{ab}^X(D_m)] \le \frac{1}{b-a} E[(X_T-a)^-] < \infty$

令 $m\to\infty$ 得 $E M_{ab}^X(D\cap[0,T]) \le \frac{1}{b-a} E[(X_T-a)^-] < \infty$

$M_{ab}^X(D\cap[0,T]) < \infty$, a.s.

Set $N = \bigcup_{T\in D}\left\{\sup_{t\in D\cap[0,T]} |X_t| = \infty\right\} \cup \left(\bigcup_{\substack{a<b,\\ a,b\in\mathbb{Q}}} \{M_{ab}^X(D\cap[0,T]) = \infty\}\right)$

then $P(N) = 0$.

$\omega \notin N$ 时 由Lem 3.16 得 ①

② 令 $X_{t+}(\omega) = \begin{cases} \lim_{s\downarrow\downarrow t, s\in D} X_s(\omega), & \text{if 极限存在} \\ 0, & \text{else} \end{cases}$，则 $X_{t+} \in \mathcal{F}_{t+}$
(由于 {极限存在} $\in \mathcal{F}_{t+}$.)

Fix t, $\exists t_n \in D \downarrow t$. $X_{t+} = \lim_{n\to\infty} X_{t_n}$

Set $Y_k = X_{t_k}$, $k\le 0$ 倒向鞅

由 prop.3.13 $\sup_{k\le 0} E|Y_k| < \infty \Rightarrow X_{t_n} \xrightarrow{L^1} X_{t+} \in L^1$

$X_t \ge E(X_{t_n}|\mathcal{F}_t) \overset{L^1\text{收}}{\Rightarrow} X_t \ge E(X_{t+}|\mathcal{F}_t)$

If r.c., $EX_t = EX_{t+} = E(E(X_{t+}|\mathcal{F}_t)) \Rightarrow$ a.s. $X_t = E(X_{t+}|\mathcal{F}_t)$

$\because X_{t+} \in \mathcal{F}_{t+}$ 且 $\forall s<t$, $E[X_{t+}|\mathcal{F}_{s+}] \le X_{s+} \Rightarrow (X_{t+})$ super

$\because$ $s_n \in D \downarrow s$, $s_n \le t_n$

$\forall A \in \mathcal{F}_{s+}$
$\Rightarrow E(X_{s+} 1_A) = \lim E(X_{s_n} 1_A) \ge \lim E(X_{t_n} 1_A) = E(X_{t+} 1_A)$ #

Pf of Thm 3.18: D countable dense, N in Thm 3.17 证明.

$\forall t\ge 0$, $Y_t(\omega) = \begin{cases} X_{t+}(\omega), & \omega\notin N \\ 0, & \omega\in N \end{cases}$，则 $(Y_t)_{t\ge 0}$ r.c.l.l.

$Y_t \in \mathcal{F}_{t+} = \mathcal{F}_t$　Y 为所求 modification!

$\forall s<t$, $E(Y_t|\mathcal{F}_s) \le Y_s$. 且 $X_t = E(X_{t+}|\mathcal{F}_t) = X_{t+} = Y_t$ a.s. #

• Upcrossing numbers: $f: I \to \mathbb{R}$, $a<b$. 记 the upcrossing number of f along $[a,b]$ 为 $M_{ab}^f(I)$.

即 $M_{ab}^f(I) = \sup\{k: \exists s_1<t_1<\cdots<s_k<t_k, \text{ s.t. } f(s_i)\le a, f(t_i)\ge b\}$.

• 记 $\lim_{s\downarrow\downarrow t} f(s) \triangleq \lim_{s\downarrow t, s>t} f(s)$

• 称 $g: \mathbb{R}_+ \to \mathbb{R}$ 为 càdlàg，即 g r.c.l.l. 

~~metric & topology~~

① $(\Omega, \mathcal{F}, P)$, $X = (X_t)_{t\ge 0}$, $X_t: (\Omega, \mathcal{F}) \to (S, \mathcal{S})$. $\forall \omega\in\Omega$, 令 $X(\omega) = (X_t(\omega))_{t\ge 0} \in S^{[0,\infty)}$

$\pi_t(f) = f(t)$, $\mathcal{S}^{[0,\infty)} = \sigma(\pi_t, t\ge 0)$, $\sigma(X_t, t\ge 0) \perp \sigma(Y_t, t\ge 0)$

② $P\circ X^{-1}(B) = P(X\in B)$, $B\in \mathcal{S}^{[0,\infty)}$.　$P\circ X^{-1}(B) = P((X_t)_{t\ge 0}\in B)$　$B\in\mathcal{S}^{[0,\infty)}$

③ $X \overset{d}{=} Y \Leftrightarrow X \overset{fd}{=} Y$

④ $C([0,\infty),\mathbb{R}) \cap \sigma(\pi_t, t\ge 0) = \sigma(\pi_t|_C, t\ge 0)$.

⑤ $\sigma(\pi_t|_C, t\ge 0) = \mathcal{B}(C([0,\infty),\mathbb{R}))$, $f, g\in C$, $d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\max_{0\le t\le n} |f(t)-g(t)| \wedge 1\right)$

Lem 3.16　D: countable dense subset of $\mathbb{R}_+$, f defined on D. Assume $\forall T\in D$,

① f bdd on $[0,T]\cap D$

② $\forall a<b$ rationals, $M_{a,b}^f(D\cap[0,T]) < \infty$

Then $f(t+) = \lim_{s\downarrow\downarrow t, t\in D} f(s)$ exist, $f(t-) = \lim_{s\uparrow\uparrow t, s\in D} f(s)$ exists.

Further, $g(t) = f(t+)$ is rcll.

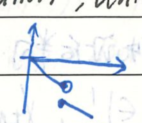
Thm 3.17　$(X_t)_{t\ge 0}$ super 则 ① $X_{t+}(\omega) \triangleq \lim_{s\downarrow\downarrow t, s\in D} X_s(\omega)$ and $X_{t-}(\omega) = \lim_{s\uparrow\uparrow t, s\in D} X_s(\omega)$ a.s. ∃

② $\forall t\ge 0$, $X_{t+} \in L^1$, $X_t \ge E(X_{t+}|\mathcal{F}_t)$

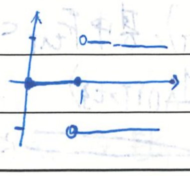
且 equality if $t\mapsto E(X_t)$ is r.c.

$(X_{t+})_{t\ge 0}$ is super w.r.t. $(\mathcal{F}_{t+})$, (X_{t+}) is a martingale. if (X_t) mar.

Thm 3.18　$(\mathcal{F}_t)$ r.c. & complete, (X_t) super, $t\mapsto E(X_t)$ r.c.,
then X has an rcll modification, which is super.

c.e.g. ① $X_t = f(t)$, f nonincreasing, not r.c.  $\mathcal{F}_t = \{\phi, \Omega\}$. 必须改断点的值

② $\Omega = \{-1, 1\}$, $P(\{1\}) = P(\{-1\}) = \frac{1}{2}$, $\varepsilon(\omega) = \omega$, $X_t = \begin{cases} 0, & \text{if } t\in[0,1] \\ \varepsilon, & \text{if } t>1 \end{cases}$



$\mathcal{F}_1^X = \{\phi, \Omega\}$, $\mathcal{F}_{1+}^X = \{\phi, \{-1\}, \{1\}, \Omega\}$ 不r.c.

Pf of Th 3.19: D: countable dense subset of $\mathbb{R}_+$

$E[M_{ab}^X(D\cap[0,T])] \le \frac{1}{b-a} E[(X_T-a)^-]$

$E[M_{ab}^X(D)] \le \frac{1}{b-a}\sup_t E[(X_t-a)^-]$

$\therefore M_{ab}^X(D)<\infty,\ \forall a,b\in\mathbb{Q}$

$\therefore X_\infty = \lim_{D\ni t\to\infty} X_t$ exists in $[-\infty,+\infty]$

$\because E|X_\infty| \le \varliminf_{D\ni t\to\infty} E|X_t| <\infty \quad \therefore X_\infty\in L'$

由r.c. $\lim X_t = X_\infty$ a.s. (若条件右连也对). #

Pf of Th 3.21: ①⇒② trivial

②⇒③: Th 3.19: $X_t \xrightarrow{a.s.} X_\infty \overset{u.i}{\Longrightarrow} X_t \xrightarrow{a.s.\&L'} X_\infty$

③⇒①: $X_t \xrightarrow{a.s.} X_\infty$, $X_s = E(X_\infty|\mathcal{F}_s) \xleftarrow{a.s.} E(X_t|\mathcal{F}_s)$ $(t\to\infty)$

$\propto X_n \to X_\infty \in L'$, then $E(X_n|\mathcal{F}_n)\uparrow E(X|\mathcal{F}$

Pf of Th 3.22: $T_n = \sum_{k=0}^{\infty}\frac{k+1}{2^n}1_{\{\frac{k}{2^n}<T\le\frac{k+1}{2^n}\}} + \infty 1_{\{T=\infty\}}$

则 $T_n\downarrow T$, $S_n\downarrow S$, $S_n\le T_n$.

$X_{S_n} = E(X_{T_n}|\mathcal{F}_{S_n})$,

$\forall A\in\mathcal{F}_S$,

$E(X_{S_n}1_A) = E(X_{T_n}1_A)$

$\because X_{S_n}\xrightarrow{a.s.} X_S$ 且由 $X_{S_n}=E(X_\infty|\mathcal{F}_{S_n})$, $X_{S_n}\xrightarrow{a.s.\&L'} X_S$

$\therefore EX_{S_n}1_A \to EX_S1_A$ #

Pf of Cor 3.23: $S\le T\le a$, $(X_{t\wedge a})$ is closed by X_a. $E(X_a|\mathcal{F}_t) = X_{t\wedge a}$ #

Pf of Cor 3.24: 和3.23一样先证后半句.

②: $X_{t\wedge T}, X_T\in L'$, $X_{t\wedge T}\in\mathcal{F}_{t\wedge T}\subset\mathcal{F}_t$. 下证 $\forall A\in\mathcal{F}_t$

$E(X_{t\wedge T}1_A) = E(X_T1_A)\cdots(*)$

Th 3.22 给的是 $X_{t\wedge T} = E(X_T|\mathcal{F}_{t\wedge T})$, 其中 $\mathcal{F}_{t\wedge T}\subset\mathcal{F}_t$.

(*) 只须 $E(X_t1_{A\cap\{T>t\}}) = E(X_T1_{A\cap\{T>t\}})$ $A\cap\{T>t\}\in\mathcal{F}_t$. 由定义 $A\cap\{T$

$\therefore A\cap\{T>t\}\in\mathcal{F}_{t\wedge T}$ #

3.4 Optional Stopping Theorems

Th 3.19 X: super, r.c. $\sup E|X_t|<\infty$; Then $\exists X_\infty\in L'$, s.t. $\lim_{t\to\infty} X_t = X_\infty$ a.s.

Def (Close) mar $(X_t)_{t\ge0}$ is closed, if $\exists Z\in L'$, s.t. $X_t = E(Z|\mathcal{F}_t)$, $\forall t$.

Th 3.21 X: mar, r.c. ① X is closed

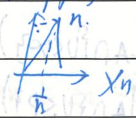
⇔② (X_t) u.i

⇔③ $(X_t)\xrightarrow{a.s.\&L'} X_\infty$, $(t\to\infty)$.

Then $X_t = E(X_\infty|\mathcal{F}_t)$, where $X_\infty\in L'$ is the a.s. limit of X_t as $t\to\infty$.

Rmk ① $X_t \xrightarrow{L'\&a.s.} X_\infty$, then $E(X_t|\mathcal{F}) \to E(X_\infty|\mathcal{F})$

② $X_n\xrightarrow{a.s.} X_\infty$ 但 $EX_n \not\to EX_\infty$, 会被



Th 3.22 (Optional Stopping thm for mar) $(X_t)_{t\ge0}$: u.i mar & r.c., $S\le T$ stopping times.

Then $X_S\in L'$, $X_T\in L'$, $X_S = E(X_T|\mathcal{F}_S)$

特别, $X_S = E(X_\infty|\mathcal{F}_S)$, $EX_S = EX_\infty = EX_0$. 常用.

Cor 3.23 (X_t): mar & r.c. $S\le T$ bdd, Then $X_S\in L'$, $X_T\in L'$, $X_S = E(X_T|\mathcal{F}_S)$

Cor 3.24 (X_t): mar & r.c. ① $X_{t\wedge T}$ mar ② If (X_t) u.i., then $(X_{t\wedge T})$ is u.i.

that is, $X_{t\wedge T} = E(X_T|\mathcal{F}_t)$

the conditional expectation is continuous for the L'-norm

$X_s = E(X_t|\mathcal{F}_s)$ $E(X_s1_A) = E(X_t1_A) \to E(X_\infty1_A)$

Applications:

① $T_a = \inf\{t\ge0, B_t=a\}$, $T_a<\infty$, a.s. (前面已证)

$a<0<-b$, $P(T_a<T_b) = \frac{b}{b-a}$. $\because T = T_a\wedge T_b$, then $(B_{t\wedge T})$ mar $\Rightarrow 0 = EB_{0\wedge T} = EB_T$

$= bP(T_b<T_a) + aP(T_a<T_b)$

Rmk. 对 $\forall$ continuous sample path 的mar都成立.

[Durrett 例证]: $P(T>n\cdot\lceil b-a\rceil) \le (1-[P(N(0,1)>1)]^{b-a})^n \Rightarrow ET<\infty \Rightarrow T<\infty$, a.s.

(连续 $B_{k+1},\cdots,B_{k+(b-a)}$ 都增长1)

Cor: $P(T_a<T_b) = \frac{b}{b-a} \xrightarrow{b\to\infty} P(T_a<\infty)=1$

② $U_a = \inf\{t\ge0, |B_t|=a\}$, Then $EU_a = a^2$ $\because M_t = B_t^2 - t$ 为mar.

$EB_{t\wedge U_n}^2 = E(t\wedge U_n)$

$\downarrow$ BCT $\quad \downarrow$ MCT

$a^2 \quad EU_n$

· Wald identity: $T<\infty$, a.s. $\Rightarrow EB_T = 0$, $EB_T^2 = ET$.

Pf of Th3.25: ① $U.V$ bdd 时，证明 $EZ_U \geq EZ_V$.

由二分构造 $U_n \downarrow U$, $V_n \downarrow V$, $EZ_{U_n} \geq EZ_{V_n}$

Let $Y_n = Z_{U_{-n}}$, $n \leq 0$, then (Y_n) backward super

($\because$ $Z_{U_{n+1}} \geq E(Z_{U_n} | \mathcal{F}_{U_{n+1}})$)

$\therefore EZ_{U_n} \leq EZ_0 \Rightarrow Y_n$ bdd in L^1 $\Rightarrow Y_n \xrightarrow[n\to-\infty]{L^1} Y_{-\infty}$

$Z_{U_n} \xrightarrow{L^1} Z_U$

$\therefore EZ_U \geq EZ_V$.

② $U.V$ general, $\forall A \in \mathcal{F}_U$, 证明 $E(Z_U 1_A) \geq E(Z_V 1_A)$

即证 $E(Z_U 1_{A\cap\{U<\infty\}}) \geq E(Z_V 1_{A\cap\{V<\infty\}})$

即证 $E(Z_U 1_{A\cap\{U\leq p\}}) \geq E(Z_V 1_{A\cap\{V\leq p\}})$, $Z_U, Z_V \in L^1$.

$\because A \in \mathcal{F}_U \subset \mathcal{F}_V$

令 $U^A = \begin{cases} U & \text{on } A \\ \infty & \text{on } A^c \end{cases}$ $V^A = \begin{cases} V & \text{on } A \\ \infty & \text{on } A^c \end{cases}$

则 $E(Z_{U^A\wedge p}) \geq E(Z_{V^A \wedge p}) \Leftarrow$ 由①

全集 $= A^c + A\cap\{U\leq p\} + A\cap\{U>p\}$

on $A\cap\{U>p\}$, $U^A\wedge p = p$, $V^A \wedge p = p$

$\Rightarrow E[Z_U 1_{A\cap\{U\leq p\}}] \geq E[Z_V 1_{A\cap\{U\leq p\}}]$ #.

$\lambda>0$ 时.

③ $Ee^{-\lambda T_a}$, $a>0$ $\quad$ $\because$ $N_t^\lambda = e^{\lambda B_t - \frac{\lambda^2}{2}t}$ 为 mar $\quad$ (也可以用密度 T_a 求)

$Ee^{\lambda B_{T_a} - \frac{\lambda^2}{2}T_a} = 1 \quad \Rightarrow Ee^{-\frac{\lambda^2}{2}T_a} = e^{-\lambda a}$

$\Rightarrow Ee^{-\lambda T_a} = e^{\sqrt{2\lambda}a}$

$\lambda<0$ 时，$Ee^{-\frac{\lambda^2}{2}T_a} \leq 1 < e^{-\lambda a}$! 因为 $N^\lambda_{t\wedge T_a}$ 不一致可积!

④ $Ee^{-\lambda U_a} = \frac{1}{\cosh(a\sqrt{2\lambda})}$, $U_a = \inf\{t\geq0, |B_t| = a\}$, $a>0$, $\lambda>0$.

$\because$ $N_t^\lambda = e^{\lambda B_t - \frac{\lambda^2}{2}t}$, $U_a \perp B_{U_a} \Leftarrow E(1_{\{B_{U_a}=a\}}e^{-\lambda U_a}) = E(1_{\{B_{U_a}=-a\}}e^{-\lambda U_a})$ $= \frac{1}{2}Ee^{-\lambda U_a}$

$\Leftarrow$ 由 $Ef(B) = Ef(-B)$

$EN^\lambda_{U_a} = EN^\lambda_0 = 1$

$= \frac{1}{2}(e^{\lambda a} + e^{-\lambda a}) \cdot Ee^{-\frac{\lambda^2}{2}U_a}$

<u>Th 3.25</u> $(Z_t)_{t\geq0} \geq 0$ super & r.c. (then $EZ_t \leq EZ_0$, $Z_t \xrightarrow{a.s.} Z_\infty \in L^1$)

$U \leq V$, then $Z_U, Z_V \in L^1$, $Z_U \geq E(Z_V | \mathcal{F}_U)$

从而 $EZ_U \geq EZ_V$, $E1_{\{U<\infty\}}Z_U \geq E1_{\{U<\infty\}}Z_V \geq E1_{\{V<\infty\}}Z_V$.