Thm 4.2 的一些证明的一些细节

式(13)的由来：

注意到 $J^{\varepsilon,\delta}_{n+m}(x,y,z) = J^{\varepsilon,\delta}_n(x_1,y_1,z_1)\, J^{\varepsilon,\delta}_m(x_2,y_2,z_2)$

从而 $\left(\int_{\mathbb{R}^{n+m}} \left| \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} J^{\varepsilon,\delta}_{n+m}(x,y,z) g(y) h(z) \, dy\, dz \right|^p dx\right)^{\frac{1}{p}}$

$$= \left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2n}} J^{\varepsilon,\delta}_m(x_2,y_2,z_2)\, J^{\varepsilon,\delta}_n(x_1,y_1,z_1)\, g(y_1,y_2) h(z_1,z_2)\, dy_1 dz_1 dy_2 dz_2 \right|^p dx_1 dx_2\right)^{\frac{1}{p}}$$

(1) 对 $\int_{\mathbb{R}^n}\int_{\mathbb{R}^{2m}}$ 用 Minkowski's inequality 可知

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2n}} J_m^{\varepsilon,\delta}\, J_n^{\varepsilon,\delta}\, g(y_1,y_2) h(z_1,z_2)\, dy_1 dz_1\, dy_2 dz_2 \right|^p dx_1$$

$$\leq \int_{\mathbb{R}^{2m}} \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^{2n}} J_m^{\varepsilon,\delta}\, J_n^{\varepsilon,\delta}\, g(y_1,y_2) h(z_1,z_2)\, dy_1 dz_1 \right|^p dx_1 \right)^{\frac{1}{p}} dy_2 dz_2$$

$$= \int_{\mathbb{R}^{2m}} J_m^{\varepsilon,\delta}(x_2,y_2,z_2) \underline{\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^{2n}} J_n^{\varepsilon,\delta} g(y_1,y_2) h(z_1,z_2)\, dy_1 dz_1 \right|^p dx_1 \right)^{\frac{1}{p}}} dy_2 dz_2$$

(2)

$$\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^{2n}} J_n^{\varepsilon,\delta}\, g(y_1,y_2) h(z_1,z_2)\, dy_1 dz_1 \right|^p dx_1 \right)^{\frac{1}{p}}$$

$$\leq C_n^{\varepsilon,\delta} \|g(\cdot,y_2)\|_q \cdot \|h(\cdot,z_2)\|_r$$

n维情形的 Young 不等式

代入到上式中可知

$$\left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2n}} \overline{J}_m^{\varepsilon,\delta}\,\overline{J}_n^{\varepsilon,\delta}\, g(y_1,y_2)\,h(z_1,z_2)\,dy_1dz_1dy_2dz_2\right|^p dx_1dx_2\right)^{\frac{1}{p}}$$

$$\overset{(1)}{\leq} \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^{2m}} \overline{J}_m^{\varepsilon\delta}(x_2,y_2,z_2)\left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^{2n}} \overline{J}_n^{\varepsilon\delta}\, g(y_1,y_2)\,h(z_1,z_2)\,dy_1dz_1\right|^p dx_1\right)^{\frac{1}{p}} dy_2dz_2\right)^{p} dx_2\Bigg)^{\frac{1}{p}}$$

$$\overset{(2)}{\leq} \left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^{2m}} \overline{J}_m^{\varepsilon\delta}(x_2,y_2,z_2)\, C_n^{\varepsilon\delta}\,\|g(\cdot,y_2)\|_q\cdot\|h(\cdot,z_2)\|_r\,dy_2dz_2\right)^p dx_2\right)^{\frac{1}{p}}$$

$$= C_n^{\varepsilon\delta}\left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^{2m}} \overline{J}_m(x_2,y_2,z_2)\,\|g(\cdot,y_2)\|_q\cdot\|h(\cdot,z_2)\|_r\,dy_2dz_2\right)^p dx_2\right)^{\frac{1}{p}}$$

再由Young不等式

$$\leq C_n^{\varepsilon\delta}\,C_m^{\varepsilon\delta}\,\|g\|_q\cdot\|h\|_r$$

11 班：

In G_g. extremizer exists. which gives the sharp constant.

(所有的Gauss函数) $C = \sup_{(G_g)^3} \dfrac{\langle f, g*h\rangle}{\|f\|_p\|g\|_q\|h\|_r}$

<u>Step1</u>：我们来证明 $C = \sup_{\substack{(G_g)^3 \\ f,g,h \text{ centered} \\ \text{at } 0}} \dfrac{\langle f, g*h\rangle}{\|f\|_p\|g\|_q\|h\|_r}$

这是因为 $|\langle f, g*h\rangle| \leq \langle |f|, |g|*|h|\rangle$

考查 G_g 的子类 $\widehat{G}_g = \{ g(x) = \exp(-ax^2 + bx + c) : a>0, b, c \in \mathbb{R}\}$

从而可知 $C = \sup_{(\widehat{G}_g)^3} \dfrac{\langle f, g*h\rangle}{\|f\|_p\|g\|_q\|h\|_r}$

利用平移不变性 $= \sup_{\substack{(G_g)^3 \\ f,g,h \text{ centered} \\ \text{at } 0}} \dfrac{\langle f, g*h\rangle}{\|f\|_p\|g\|_q\|h\|_r}$

<u>Step2</u>：计算出 C 值

有如下事实 $\begin{cases} \langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \\ \widehat{f*g} = \hat{f}\cdot\hat{g} \end{cases} \Longrightarrow \langle f, g*h\rangle = \langle \hat{f}, \hat{g}\cdot\hat{h}\rangle$

且 $f(x) = e^{-ax^2}$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ax^2} e^{-2\pi i x\xi} dx = e^{-\frac{4\pi^2}{a}\xi^2} \cdot \int_{\mathbb{R}} e^{-ax^2} dx = \frac{C}{\sqrt{a}} e^{-\frac{4\pi^2}{a}\xi^2}$$

从而可知 $C = C \sup_{\alpha,\beta,\gamma>0} \dfrac{(\alpha\beta\gamma)^{-\frac{1}{2}} \cdot \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^{-1/2}}{(\alpha p)^{-1/2p} \cdot (\beta q)^{-1/2q} \cdot (\gamma r)^{\frac{1}{2r}}}$

N=1. optimizers - 9日 ^

$$q^* = \inf_{\alpha,\beta,\gamma>0} \frac{(\alpha\beta\gamma)^{\frac12}\cdot\left(\frac1\alpha+\frac1\beta+\frac1\gamma\right)^{1/2}}{(\alpha p)^{1/2p}\cdot(\beta q)^{\frac{1}{2q}}\cdot(\gamma r)^{1/2r}} \qquad \frac1p+\frac1q+\frac1r=2$$

(求得相应的常数)

从而 $\frac{1}{p'}+\frac{1}{q'}+\frac{1}{r'}=1$

Recall the 基本不等式: $a^{\frac{1}{p'}}b^{\frac{1}{q'}}c^{\frac{1}{r'}} \le \frac{a}{p'}+\frac{b}{q'}+\frac{c}{r'}$ for $a,b,c\ge0$

从而

$$q^* = \inf_{\alpha,\beta,\gamma>0} \frac{(\alpha\beta\gamma)^{\frac12}}{(\alpha p)^{\frac{1}{2p}}(\beta q)^{\frac{1}{2q}}\cdot(\gamma r)^{\frac{1}{2r}}}\cdot\left(\frac1\alpha+\frac1\beta+\frac1\gamma\right)^{\frac12}$$

$$= \inf_{\alpha,\beta,\gamma>0}\left(\frac{p^{1/p}\cdot q^{1/q}\cdot r^{1/r}}{(p')^{1/p'}\cdot(q')^{1/q'}\cdot(r')^{1/r'}}\right)^{-\frac12}\cdot\frac{\left(\frac{p'}{\alpha}\right)^{-\frac{1}{2p'}}\cdot\left(\frac{q'}{\beta}\right)^{-\frac{1}{2q'}}\cdot\left(\frac{r'}{\gamma}\right)^{-\frac{1}{2r'}}}{\left(\frac1\alpha+\frac1\beta+\frac1\gamma\right)^{-\frac12}}$$

$$\ge \left(\frac1\alpha+\frac1\beta+\frac1\gamma\right)^{-\frac12}$$

且当且仅当 $\iff \frac{p'}{\alpha}=\frac{q'}{\beta}=\frac{r'}{\gamma}$

因此便求出了 q, 及 optimizers.