2.2.1. 由Riesz表示. $\exists\, e_k \in H$, s.t. $\langle f_k, x\rangle = \langle e_k, x\rangle$

若 ~~$\exists$~~ $x_0 \in H$. y_0 为 x_0 在 M 上正交投影. 则 $x_0 - y_0 := z_0 \in M^\perp$.

下证 $z_0 = \sum_{k=1}^{n} \alpha_k e_k$. 反设 $z_0 \notin \mathrm{span}\{e_k\}_{k=1}^n$. 由 H-B 定理,

$\exists\, f \in H^*$. $\langle f, \mathrm{span}\{e_k\}_{k=1}^n\rangle < \alpha < \langle f, z_0\rangle \Rightarrow \langle f, e_k\rangle = 0\ (\forall k)$

故 $\langle f, z_0\rangle > 0$. 但由 Riesz, $\exists\, e_0$. $\langle f, x\rangle = \langle e_0, x\rangle \Rightarrow e_0 \in M$. 但 $\langle e_0, z_0\rangle > 0$ 矛盾!

(Δ. $\mathbb{R}^1$-线性. 若 $\mathbb{C}$ 上, 考虑 $\mathrm{span}\{e_k, ie_k\}_{k=1}^n$) (P108)

2.2.5. 若 P 投影: $H \longrightarrow M$. 则由定义 $P^2x = Px$. $\langle Px, y\rangle = \langle Px, Py\rangle = \langle x, Py\rangle$

反之, 若 $P^2 = P$, $P = P^*$. 令 $M = (\mathrm{Ker}\,P)^\perp$. 则 $y - Py \in \mathrm{Ker}\,P$, $\forall y \in H$

$\therefore 0 = \langle x, y - Py\rangle = \langle x, y\rangle - \langle Px, y\rangle = \langle x - Px, y\rangle$, $\forall y \in H$. $\therefore x = Px \in \mathrm{Im}\,P$.

$\therefore M \subseteq \mathrm{Im}\,P$. if $x \in \mathrm{Im}\,P$, $x = Pz$. 若 $Py = 0$. $\langle x, y\rangle = \langle Pz, y\rangle = \langle z, Py\rangle = 0$

$\therefore \mathrm{Im}\,P \subseteq M$. $\therefore M = \mathrm{Im}\,P$. $\forall x \in H$. $x = Px + (x - Px) \in M + M^\perp$.

(3) $\langle P_L P_M x, y\rangle = \langle P_{L\cap M} x, y\rangle = \langle x, P_{L\cap M} y\rangle = \langle x, P_M P_L y\rangle$.

$\therefore P_L P_M = P_{L\cap M} = P_M P_L$.

反之. 令 $K = P_L P_M = P_M P_L$. $\langle x, Ky\rangle = \langle x, P_L P_M y\rangle = \langle P_M P_L x, y\rangle = \langle Kx, y\rangle$.

$K^2 = P_L P_M P_M P_L = P_L P_M P_L$. ~~$\langle x, K^2 y\rangle = \langle x, P_L P_M P_L y\rangle$~~ $= P_L P_L P_M = P_L P_M = K$

2.3.3. $R(A)^\perp = \{0\} \not\Rightarrow R(A) = \{0\}$.

2.3.5. $C[0,1]$ 赋 L^2 范数 ??

2.3.6. 令 $F_n = \{x \in X : p(x) \le n\}$. 则 F_n 闭. $X = \bigcup_{n\ge1} F_n$. $\therefore \exists\, n_0$, $B_r(x_0) \subseteq F_{n_0}$.

$\therefore p(x_0 + rz) \le n_0 \Rightarrow p(z) \le \frac{1}{r}(n_0 + p(-x_0))$. $\forall x \in X$. $\frac{x}{2\|x\|} \in B_1(0)$.

$\therefore p(x) = p\left(\frac{x}{2\|x\|}\right)\cdot 2\|x\| \le \frac{2}{r}(n_0 + p(-x_0))\cdot\|x\|$ $\left(\sup_{|\alpha|=1}\|\alpha x\| < +\infty\ ?\right)$

2.3.8 同 $\underline{2.3.7}$.

2.3.11. $\because A$ 满 $\therefore \exists\, C_0 > 0$, $B_Y(0,1) \subseteq A(B_X(0, C_0))$. $\therefore$ 取 $N \gg 1$. 使 $\|y_n - y_0\| \le \frac{1}{2}\|y_0\|$ $(n > N)$

对 $\{y_k\}_{k=1}^N$. 取 x_k 使 $Ax_k = y_k$, $\|x_k\| \le C_0\|y_k\|$. if $k > N$ or $k=0$ 取 x_k, $Ax_k = y_k$,

$\|x_k - x_0\| \le C_0\|y_k - y_0\|$. $\therefore \{x_k\} \to x_0$. $\|x_k\| \le \|x_0\| + \|x_k - x_0\| \le C_0\|y_0\| +$

$C_0\|y_k - y_0\| \le 3C_0\|y_k\|$. $\therefore$ 取 $C = 3C_0$.

2.3.12 (2) 在 $D(T)$ 上, 定义 $\|x\|_2 = \|x\| + \|Tx\|$. 则 $(D(T), \|\cdot\|_2)$ 是B空间.

则 $\|Tx\| \le \|x\|_2$, $\forall x \in D(T)$. if $N(T) = 0$, $R(T)$ 闭.

对 $T: (D(T), \|\cdot\|_2) \longrightarrow (R(T), \|\cdot\|_Y)$ 用逆算子...

中国科学技术大学
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
地址:中国安徽省合肥市 电话:0551-63602184 传真:0551-63631760 网址:http://www.ustc.edu.cn

(3). 若 $R(T)$ 闭. 由 $(D(T), \|\cdot\|_G) \longrightarrow (R(T), \|\cdot\|_Y)$ 满. $\exists z$. $Tz = Tx$.
且 $\|z\|_G \le C\|Tx\|$. $\therefore$ $d(x, N(T)) \le \|z\|_X \le \|z\|_G \le C\|Tx\|$.

反之. 设 $\{y_n\}$ 是 $R(T)$ 中 Cauchy. 取 $\{y_{k_n}\} \subseteq \{y_n\}$. $\|y_{k_{n+1}} - y_{k_n}\| \le \frac{1}{2^n}$.
取 $x_1 \in D(T)$. $\|x_1\| \le 2a\|y_{k_1}\|$, $y_{k_1} = Tx_1$. 取 $x_k \in D(T)$, $\|x_n - x_{n-1}\| \le 2a\|y_{k_n} - y_{k_{n-1}}\|$, $y_{k_n} = Tx_n$, $n>$
$\therefore$ $\sum_{n=2}^{\infty} \|x_n - x_{n-1}\| < \infty$ $\therefore$ $\{x_n\}$ 收敛 $(\longrightarrow x_0)$. 而由 $y_{k_n} \to y_0$ 及 T 闭.
$\therefore$ $x_0 \in D(T)$, $Tx_0 = y_0$. $\therefore$ $y_0 \in R(T)$. 闭 #

(Δ. 关于 $N(T)^\perp$. 对一般 B 空间如何定义?).

X 是B空间. $N \subseteq X$ 线性子空间. $N^\perp := \{f \in X^* : \langle f, x\rangle = 0, \forall x \in N\}$.

若 X^* 是 X 对偶. $M \subseteq X^*$ 线性子空间. $M^\perp := \{x \in X : \langle f, x\rangle = 0, \forall f \in M\}$
有 $(N^\perp)^\perp = \overline{N}$. 首先 $(N^\perp)^\perp$ 闭. $\forall x \in N, \forall f \in N^\perp$. $\langle f, x\rangle = 0$ $\therefore x \in (N^\perp)^\perp$.
$\therefore$ $N \subseteq (N^\perp)^\perp \Rightarrow \overline{N} \subseteq (N^\perp)^\perp$.
反之, 若 $\overline{N} \subsetneq (N^\perp)^\perp$. $\exists x_0 \in (N^\perp)^\perp \setminus \overline{N}$. 由 H-B 定理. $\exists f \in X^*$.
$\langle f, x_0\rangle > 0 = \langle f, x\rangle$, $\forall x \in N$. $\Rightarrow$ $f \in N^\perp$. 而 $\langle f, x_0\rangle \ne 0$, 矛盾!

(Δ. 仍是实的; 1.6.5)
(以下均为实)

例1: E, F 是B空间. $a: E \times F \to \mathbb{R}^1$. 1) fix x. $y \longmapsto a(x,y)$ conti. (2) fix y ...
证: $|a(x,y)| \le C\|x\| \cdot \|y\|$.
由 1), $\exists M_x > 0$. $\|a(x,y)\| \le M_x \cdot \|y\|$. 再由共鸣. $\sup_{x \in X} M_x < \infty$.

例2: E 是B空间. $T: E \to E^*$ linear. $\left\{\begin{array}{l} \langle Tx, x\rangle \ge 0 \\ \text{or } \langle Tx, y\rangle = \langle Ty, x\rangle \end{array}\right.$ $\Rightarrow$ T bounded.