Ans to Week 9:

(1). denote, $\Gamma(x) = \frac{1}{\gamma_a} \frac{1}{|x|^{n-a}}$.

then for $a \in (0,n)$, $\Gamma(x)$ is a locally integrable function. thus, we can interpret $I_a \phi(x)$ as

$$I_a \phi(x) = \Gamma * \phi(x)$$

at least in the sense of distributions. One can also see the decay of $I_a \phi(x)$. and hence ~~we~~ - can also say it is indeed a distribution. We can ~~then~~ take its distributional fourier transform.

$$\langle \widehat{I_a \phi}, \psi \rangle = \langle I_a \phi, \hat{\psi} \rangle$$

for some $\psi \in \mathcal{S}(\mathbb{R}^n)$.

$$= \langle \Gamma * \phi, \hat{\psi} \rangle$$

$$= \langle \Gamma, \phi * \hat{\psi} \rangle$$

~~Some~~

$$= \langle \hat{\Gamma}, \hat{\phi} \cdot \psi \rangle$$

$$= \langle \hat{\Gamma} \hat{\phi}, \psi \rangle$$

i.e.

$$\widehat{I_a \phi}(\xi) = \hat{\Gamma}(\xi) \hat{\phi}(\xi).$$

(2) fix a, and given $f \in \mathcal{S}(\mathbb{R}^n)$. we then have

(*1) $\|I_a f\|_{L^q} \leq C \|f\|_{L^p}$.

furthermore, for each $\lambda > 0$, $f_\lambda(x) := f(\lambda x)$ satisfies

(*2) $\|I_a f_\lambda\|_{L^q} \leq C \|f_\lambda\|_{L^p}$

using scaling argument one can show

$$\|I_a f_\lambda\|_{L^q} = \lambda^{a-\frac{n}{q}} \|I_a f\|_{L^q}$$

$$\|f_\lambda\|_{L^p} = \lambda^{-\frac{n}{p}} \|f\|_{L^p}$$

and hence, we have from (*2)

(*3) $\|I_a f\|_{L^q} \leq \lambda^{\frac{n}{q} - \frac{n}{p} + a} \|f\|_{L^p}$

Combining (*1) and (*3), we conclude

$$\frac{n}{q} - \frac{n}{p} + a = 0$$

(thanks also to the arbitrary choice of $f \in \mathcal{S}(\mathbb{R}^n)$)

(3) for each $a \in (0, n)$, fix $r > 0$. then

$$\frac{1}{|B_r|^{1-a/n}} \int_{B_r} |f(x-y)| \, dy$$

$$= \frac{1}{|B_r|^{1-a/n}} \sum_{j=0}^{\infty} \int_{B_{2^{-j}r} \setminus B_{2^{-(j+1)}r}} |f(x-y)|\, dy$$

$$= \sum_{j=0}^{\infty} \frac{}{|B_r|^{1-\frac{a}{n}}} \int_{B_{2^{-j}r} \setminus B_{2^{-(j+1)}r}} \frac{|y|^{n-a} |f(x-y)|}{|y|^{n-a}}\, dy$$

$$\leq \sum_{j=0}^{\infty} \frac{(2^{-j}r)^{n-a}}{|B_r|^{1-a/n}} \int_{B_{2^{-j}r} \setminus B_{2^{-(j+1)}r}} \frac{|f(x-y)|}{|y|^{n-a}}\, dy$$

$$\leq c \cdot \sum_{j=0}^{\infty} (2^{-j})^{n-a} \underbrace{I_a f(x)}$$

$$\leq c\ I_a f(x).$$

- for the proof of thm, see Muckenhoupt – Wheeden 1974. Trans. AMS.
- for the proof of inequality in (2), see Sogge: Fourier integrals in classical analysis, Thm 0.3.2.