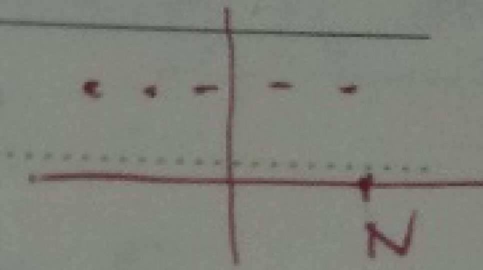
Ans. to Week 3

Ex1:

a) $\cdot\ \widehat{D_N}(n) = \begin{cases} 1, & |n| \le N \\ 0 & |n| > N. \end{cases}$



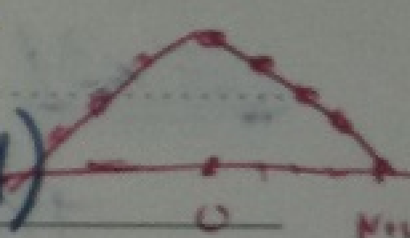
$\cdot\ F_N(t) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(t)$

$\Downarrow$

$$\widehat{F_N}(n) = \frac{1}{N+1} \sum_{k=0}^{N} \widehat{D_k}(n)$$

$$= \frac{1}{N+1} \sum_{k=0}^{N} \mathbb{1}_{|n| \le k}$$

$$= \frac{1}{N+1} (N-n+1)$$

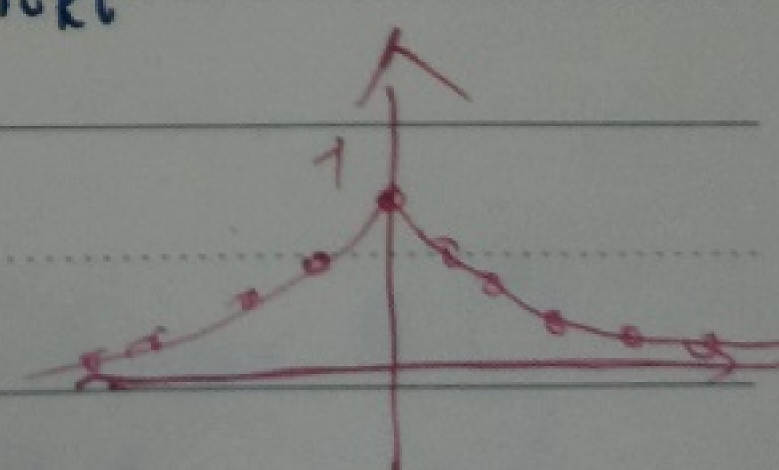


$$= 1 - \frac{|n|}{N+1}. \qquad |n| \le \underline{N+1}$$

$\cdot\ P_r(t) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k t}$

$\Downarrow$

$\widehat{P_r}(k) = r^{|k|}, \ \ldots$



b) (i). on $\mathbb{R}^1$

$\cdot\ \widehat{D_R}(\xi) = \mathbb{1}_{|\xi| \le R}$

$\cdot\ \widehat{F_R}(\xi) = \frac{1}{R} \int_0^R \mathbb{1}_{|\xi| \le t}\, dt$

$$= \frac{1}{R}(R - |\xi|)$$

(ii) on $\mathbb{R}^n$.

- $\widehat{D_R}(\xi) = \mathbb{1}_{|\xi| \le R}$
- $\widehat{F_R}(\xi) = \frac{1}{R} \int_0^R \mathbb{1}_{|\xi| \le t}\, dt$

$$= \frac{1}{R}(R - |\xi|).$$

Ex2: the indication is indeed the program for the proof.

Ex3: As in the class time, we only need to show the 1D case.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} e^{-\pi x^2}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2 + 2ix\xi)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi((x+i\xi)^2 + \xi^2)}\, dx$$

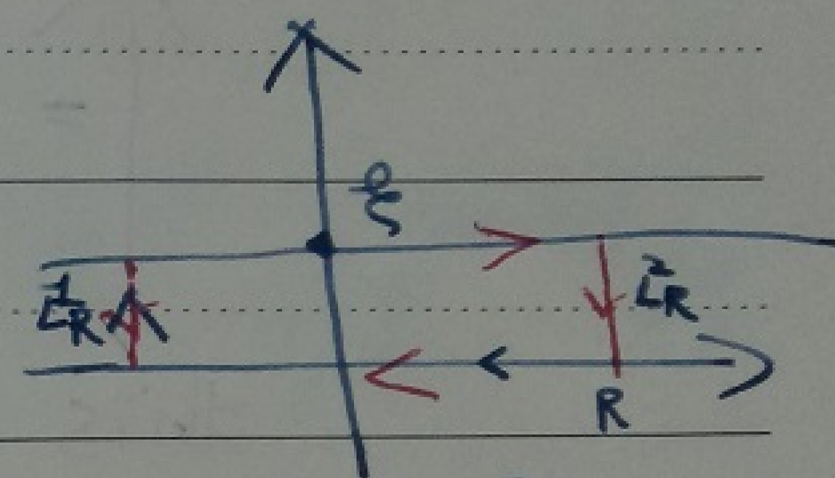
$$= e^{-\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\, dx$$

now it suffices to show

$$\int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\,dx = 1$$

- $\xi = 0$. nothing to prove. this is exactly $\boxed{\int e^{-\pi x^2}dx = 1}$
- $\xi \neq 0$. $z = x + i\xi$, $\leftarrow L$ in $\mathbb{C}$

$$\int_L e^{-\pi z^2}\,dz$$



$$\overset{R\to\infty}{=} -\int_{L_R^1} e^{-\pi z^2}\,dz + -\int_{L_R^2} e^{-\pi z^2}\,dz$$

$$+\int_{-R}^{R} e^{-\pi x^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$$

$\boxed{\text{Ex4}}$

a). Since $\frac{n}{2} < a < n$, $|x|^{-a}\mathbb{1}_{|x|>1} \in L^2$

& $|x|^{-a}\mathbb{1}_{|x|<1} \in L^1$

i.e. $h_a \in L^1 + L^2$.

therefore, we can consider $\hat{h}_a$ as a function.

- furthermore, it is a func, homogeneous of degree $n-a$. by a scaling argument.

$$\widehat{h_a}(x) = c\, h_{n-a}(x).$$

(*)
$$= c \cdot \frac{\Gamma(\frac{n-a}{2})}{\pi^{\frac{n-a}{2}}} |x|^{-(n-a)}$$

while
$$LHS = \int \frac{\Gamma(a)}{\pi^{a/2}} \cdot |\xi|^{-a} \cdot e^{-2\pi i x \cdot \xi} d\xi$$

$$= \frac{\Gamma(a)}{\pi^{a/2}} \cdot \int_0^\infty r^{-a} \cdot r^{n-1} \underbrace{\int_{S^{n-1}} e^{-2\pi i x \cdot r\omega} d\sigma_0(\omega)\, dr}$$

this involves the Bessel funcs. diff

~~to evaluate~~. turn to duality argument.

$$\int \widehat{h_a}(x) e^{-\pi|x|^2} dx = \int c \frac{\Gamma(\frac{n-a}{2})}{\pi^{\frac{n-a}{2}}} |x|^{-(n-a)} e^{-\pi|x|^2} dx$$

|| plancherel

$$\int h_a(x) \widehat{e^{-\pi|x|^2}} dx$$

$$\boxed{c = ?}$$

(b). ~~on~~ take $\phi \in \mathcal{S}(\mathbb{R}^n)$.

$$A(\gamma) = \int h_\gamma \hat{\phi}$$

$$B(\gamma) = \int h_{n-\gamma} \hat{\phi}$$

it suffices to show $\int |x|^{-\gamma} \phi(x)\, dx$ is analytic.

which is a consequence of DCT.

Ex5 :

$$P_t(x) = \int_{\mathbb{R}^n} e^{-2\pi t|\xi|} e^{2\pi i x\cdot\xi}\, d\xi$$

$$= \int_{\mathbb{R}^n} \left(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{\pi^2 t^2 |\xi|^2}{u}} du \right) e^{2\pi i x\cdot\xi}\, d\xi$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \underbrace{\left(\int_{\mathbb{R}^n} e^{-\frac{\pi^2 t^2}{u}|\xi|^2 + 2\pi i x\cdot\xi}\, d\xi \right)}_{\text{previous exercise}} \frac{e^{-u}}{\sqrt{u}}\, du$$

$\downarrow$

Gamma funcs

ok