Ans to Week 6:

Ex1

Fix $\varepsilon > 0$,

$$\left(\frac{1}{y}\chi_{|y|>\varepsilon}\right)^\wedge(\xi) = 2i\, \mathrm{sgn}(\xi) \lim_{N\to\infty} \int_{2\pi\varepsilon|\xi|}^{2N|\xi|} \frac{\sin t}{t}\, dt$$

is bounded. then by Plancherel's theorem. H_ε is bounded from L^2 to L^2.

~~(2)~~ • duality argument. also works

(1) weak (1,1): given $f \in L^1$, $f \geq 0$

Play the C.-Z. decomp of f at height λ.

$$f = g + \sum_j b_j \left(f\chi_{I_j} - f_{I_j}\chi_{I_j}\right)$$

• $|g| \leq 2\lambda$. a.e.

then

$$\left|\{x\in\mathbb{R} / |H_\varepsilon f(x)| > \lambda\}\right|$$

$$\leq \left|\{x\in\mathbb{R} / |H_\varepsilon g(x)| \geq \lambda/2\}\right| + \left|\{x\in\mathbb{R} / |H_\varepsilon b(x)| \geq \lambda/2\}\right|$$

$= I + II$ $\qquad$ L^2-bddness

• $I \leq \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}} |H_\varepsilon g(x)|^2 dx \lesssim \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}} |g(x)|^2 dx$

$\leq \left(\frac{2}{\lambda}\right)^2 \cdot 2\lambda \cdot \int_{\mathbb{R}} |g(x)| dx$

$\leq \frac{C}{\lambda} \|f\|_{L^1}$

$$\bigcup I_j = \Omega \rightsquigarrow \Omega^*$$

• $$\left|\{x\in\mathbb{R} : |H_\varepsilon b(x)| > \tfrac{\lambda}{2}\}\right| \le |\Omega^*| + \left|\{x\notin\Omega^* : |H_\varepsilon b(x)| > \lambda/2\}\right|$$

$$\le \frac{2}{\lambda}\|f\|_{L^1} + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |H_\varepsilon b(x)|\,dx$$

$$\le \frac{2}{\lambda}\|f\|_{L^1} + \frac{2}{\lambda}\sum_j \int_{\mathbb{R}\setminus\Omega^*} |H_\varepsilon b_j(x)|\,dx$$

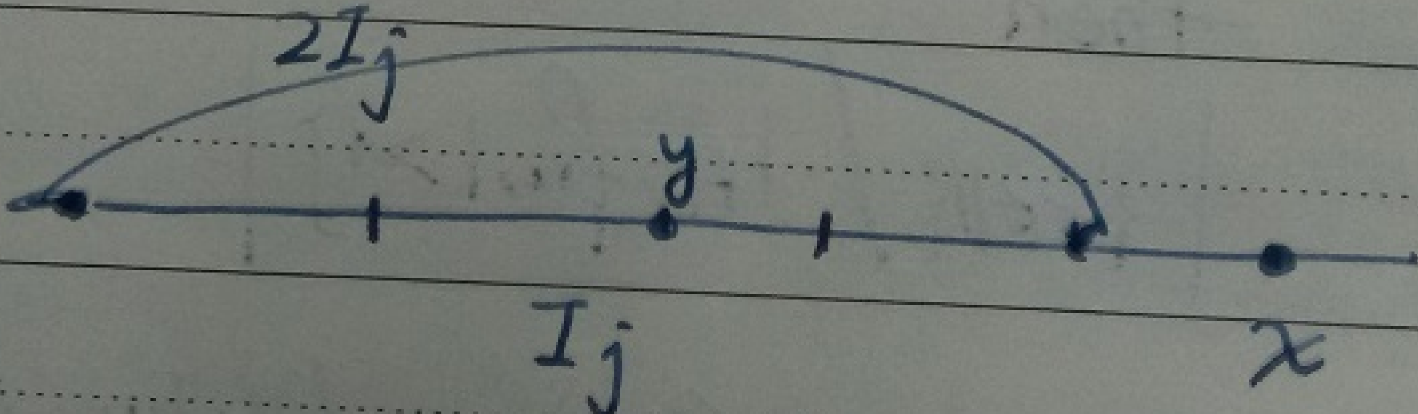
$$\le \frac{2}{\lambda}\|f\|_{L^1} + \frac{2}{\lambda}\sum_j \int_{\mathbb{R}\setminus 2I_j} |H_\varepsilon b_j(x)|\,dx$$

the key point is the formula

$$H_\varepsilon b_j(x) = \int_{I_j : |y-x|\ge\varepsilon} \frac{b_j(y)}{x-y}\,dy \qquad x\notin 2I_j$$

$$\Downarrow$$

$$\int_{\mathbb{R}\setminus 2I_j} |H_\varepsilon b_j(x)|\,dx = \int_{\mathbb{R}\setminus 2I_j} \left| \int_{I_j : |y-x|>\varepsilon} \frac{b_j(x)}{x-y}\,dy \right| dx$$



then one needs to tell the relative posi. of x, y & the sizes of $|x-y|$, ε to estimate the last integral, to conclude the proof.

• Strong (p,p) $\longrightarrow$ OK

(2) · $\underline{1 < p < \infty}$, given $f \in L^p$.

$$Hf \in L^p.$$

&

$$H_\varepsilon f \in L^p.$$

~~furthermore~~, we have $H_\varepsilon f \to Hf$ a.e.

~~then~~, by ~~DCT~~, $\boxed{\text{~~Fatou~~ DCT}}$ we have $\|H_\varepsilon f\|_{L^p} \to \|Hf\|_{L^p}$.

Eaton

$$H_\varepsilon f \to Hf \text{ in } L^p.$$

$\boxed{\text{~~Lieb~~ Loss}}$ or, as we did in the class time

· $\underline{p = 1}$

given $f \in L^1$, ~~take~~ $\{f_n\} \in \mathcal{S}$. ~~then~~

$f_n \to f$ in L^1

~~then~~, by definition

$$Hf_n \longrightarrow Hf \text{ in measure.}$$

One can also show (NOT definition)

$$H_\varepsilon f_n \longrightarrow H_\varepsilon f \text{ in measure.}$$

In order to show the convergence in measure $H_\varepsilon f \to Hf$, we write

$$\begin{aligned} H_\varepsilon f - Hf = {} & H_\varepsilon f - H_\varepsilon f_n \\ & + H_\varepsilon f_n - Hf_n \\ & + Hf_n - Hf \end{aligned}$$

thus, we only need to control $H_\varepsilon f_n - Hf_n$ small, $\boxed{\text{uniformly in } n}$

fix $\varepsilon > 0$,

$$H_\varepsilon f_n(x) - Hf_n(x)$$

$$= \lim_{\delta \to 0} \int_{\delta < |y-x| \le \varepsilon} \frac{f_n(y)}{y-x}\, dy$$

$$= \int_{|y-x| \le \varepsilon} \frac{f_n(y) - f_n(x)}{y-x}\, dy.$$

$$\xrightarrow{\varepsilon \to 0} 0 \qquad \forall x.$$

~~to get the convergence in measure~~

we need to choose $f_n \in \mathcal{S}$ as

$$f_n(x) = f * \boxed{\phi_n}(x), \quad \phi_n(x) = \frac{1}{2^n}\phi\left(\frac{x}{2^n}\right)$$

where ϕ is a molifiedl, compactly supported

Ex2: is a direct calculation

Ex3:

- One can get this by interpolation between $L^\infty \to L^\infty$ and $L^1 \to L^1$. Here the last two boundedness property follows directly from the definition.
- Or, one can apply Hölder ineq. and Fubini successively

$$\int |Kf(x)|^p dx = \int \left| \int k(x,y) f(y) dy \right|^p dx$$

$$\leq \int \left[\int |k(x,y)|^{\frac{1}{p'}} |k(x,y)|^{\frac{1}{p}} |f(y)| dy \right]^p dx$$

$$\leq \int \left(\int k(x,y) dy \right)^{\frac{p}{p'}} \cdot \left(\int |k(x,y)| |f(y)|^p dy \right) dx$$

$$\leq C_1^{\frac{p}{p'}} \cdot \int |f(y)|^p \cdot \left(\int |k(x,y)| dx \right) dy$$

$$\leq C_1^{p/p'} C_2 \cdot \|f\|_{L^p}^p$$

Ex 4: this is easy.