Ans. to week 7:

(1). the key point is to show the convergent of the series. They are indeed uniformly convergent in $x \in \mathbb{T}$?

- the $1^{st}$ one follows from

$$\left| f(x+m) \right| \leq C_N \, (1+|m|)^{-N} \quad \forall N.$$

- the second one follows from

$$\left| \hat{f}(m) \right| \leq C_N \, (1+|m|)^{-N} \quad \forall N.$$

Moreover

$$F_2(x) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i x \cdot m}$$

$$= \cancel{\sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} f(y) e^{-2\pi i y \cdot m} dy \, e^{2\pi i x \cdot m}}$$

$$= \cancel{\int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} f(y) e^{-2\pi i y \cdot m + 2\pi i x \cdot m}} dy$$

$\Downarrow$ Fourier transform on $\mathbb{T}^n$

$$\widehat{F_2}(m) = \hat{f}(m) := \int_{\mathbb{T}^n} F_2(x) e^{-2\pi i m x} dx$$

while the fourier transform of $F_1$ on $\mathbb{T}^n$ is

$$\widehat{F_1}(m) = \int_{\mathbb{T}^n} \sum_{k\in\mathbb{Z}^n} f(x+k)\, e^{-2\pi i m\cdot x}\, dx$$

$$= \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{T}^n} f(x+k)\, e^{-2\pi i m\cdot x}\, dx$$

$$= \sum_{k\in\mathbb{Z}^n} \int_{k+\mathbb{T}^n} f(x)\, e^{-2\pi i m\cdot(x-k)}\, dx$$

$$= \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i m x}\, dx$$

$$= \hat{f}(m).$$

then, $F_1 = F_2$ follows from the uniqueness.

(2). the $1^{st}$ assertion is indeed the restatement of absolute condi. of integration (at $\infty$) while the $2^{nd}$ one follows from the $1^{st}$ one by exchanging the order of summation and the integration.

(3). As can be done in the same vein in (1), where the condition that guarantees the exchange of order of summation and integration is replaced by the almost sharp decay condition.

(5) If you do this, you can find some properties of Jacobi $\theta$-function, which plays a role in the proof of ~~ana~~ meromorphic extention of Riemann zeta-function.

(4)

See, thm 2.17, chapter VII, in Stein, Weiss, Introduction to Fourier Analysis on Euclidean spaces.