Week 8

ex1: this exercise is designed to prove

$$\frac{\partial}{\partial x_j}|x|^{-n+1} = (1-n)\ \text{p.v.}\ \frac{x_j}{|x|^{n+1}}$$

in the sense of distn

(1) by definition, we have for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

$$(*) \qquad \langle \frac{\partial}{\partial x_j}|x|^{-n+1}, \varphi \rangle = -\langle |x|^{-n+1}, \frac{\partial}{\partial x_j}\varphi \rangle$$

(2) thanks to the local integrability and the decay of $|x|^{-n+1}$ at infinity.

$$(*1)\quad RHS(*) = -\int |x|^{-n+1}\frac{\partial}{\partial x_j}\varphi(x)\,dx$$

(3) for each $\varepsilon>0$, split RHS(*1) as

$$-\int |x|^{-n+1}\frac{\partial}{\partial x_j}\varphi(x) = -\int_{|x|>\varepsilon}|x|^{-n+1}\frac{\partial}{\partial x_j}\varphi(x)\,dx$$
$$-\int_{|x|<\varepsilon}|x|^{-n+1}\frac{\partial}{\partial x_j}\varphi(x)\,dx$$

then use ~~divergence~~ divergence theorem to show

$$-\int_{|x|>\varepsilon}|x|^{-n+1}\frac{\partial}{\partial x_j}\varphi(x) = \int_{|x|>\varepsilon}\frac{\partial}{\partial x_j}\frac{1}{|x|^{n-1}}\varphi + \int_{|x|=\varepsilon}\frac{x_j}{|x|}\frac{\varphi}{|x|^{n-1}}\,d\sigma_\varepsilon$$

(4) show that as $\varepsilon\to 0$, we have

$$\int_{|x|=\varepsilon}\frac{x_j}{|x|}\frac{\varphi}{|x|^{n-1}}\,d\sigma_\varepsilon \longrightarrow 0$$

and

$$\int_{|x|<\varepsilon}|x|^{-n+1}\frac{\partial}{\partial x_j}\varphi(x)\,dx \longrightarrow 0.$$

Combing all these steps, we finish the proof. Use the strategy above, show that in the sense of distribution, we have

$$\Delta \frac{1}{|x|^{n-2}} = c_n \delta \qquad n \geq 3$$

$$\Delta \log \frac{1}{|x|} = c_n \delta \qquad n = 2$$

$$\Delta |x| = c_n \delta \qquad n = 1$$

for some constant C_n, depending only on the dimension.

(*) Ex2: this exercise is to exploit further properties of the distribution p.v. $\frac{1}{x}$.

(1) p.v. $\frac{1}{x}$ is not a Radon measure.

(2) show that in the sense of distribution

$$x \cdot \text{p.v.} \frac{1}{x} = 1.$$

from this, we conclude that

$$(\text{p.v.} \tfrac{1}{x} \cdot x) \cdot \delta_0 = 1 \cdot \delta_0 = \delta_0$$

while $\text{p.v.} \frac{1}{x} (x \cdot \delta_0) = \text{p.v.} \frac{1}{x} \cdot 0 = 0$.

(3). ~~observe~~ check that

$$\frac{x}{x^2+\varepsilon^2} \xrightarrow{\varepsilon \to 0} \text{p.v.} \frac{1}{x}$$

and $\frac{\varepsilon}{x^2+\varepsilon^2} \xrightarrow{\varepsilon \to 0} \delta_0 \cdot \pi$.

in the sense of distributions. from this we get

$$S_\varepsilon := \frac{x-\varepsilon}{x^2+\varepsilon^2} \xrightarrow{\varepsilon \to 0} S_0 := \text{p.v.} \frac{1}{x} - \pi \delta_0$$

$$T_\varepsilon := \frac{x+\varepsilon}{x^2+\varepsilon^2} \xrightarrow{\varepsilon \to 0} T_0 := \text{p.v.} \frac{1}{x} + \pi \delta_0$$

While the product

$$S_\varepsilon T_\varepsilon = \frac{x^2-\varepsilon^2}{(x^2+\varepsilon^2)^2} = \frac{d}{dx}\left(\frac{-x}{x^2+\varepsilon^2}\right)$$

$$\xrightarrow{\varepsilon \to 0} -\frac{d}{dx}\left(\text{p.v.}\frac{1}{x}\right) =: U_0$$

Thus, in the sense of distributions, we get

$$U_0 = S_0 T_0 \ \underset{\text{formally}}{=} \ \left(\text{p.v.}\frac{1}{x}\right)\cdot\left(\text{p.v.}\frac{1}{x}\right) - (\pi\delta_0)\cdot(\pi\delta_0).$$

Comments

Therefore, from (2), we conclude that we cannot define distributions in an associative manner. While from (3), we see that the product of distribution is a troublesome problem. For the products of distribution, one can refer to Hörmander's "wave front" theory.