FRACTIONAL HOMOGENEOUS DISTRIBUTION

Definition 1. *For $s \in \mathbb{R}$, we try to define a homogeneous tempered distribution u_s of order s, roughly by (setting $u_1 = u\chi_B, u_2 = u\chi_{B^c}$)*

$$u_s(\varphi) := \int_{\mathbb{R}^n} \frac{\pi^{(s+n)/2}}{\Gamma((s+n)/2)} |x|^s \varphi(x)\, dx = L_{u_1}(\varphi) + L_{u_2}(\varphi), \qquad \varphi \in \mathcal{S}\,.$$

By "roughly", we mean it is only well-defined when $s > -n$ in the sense of the r.h.s is finite, i.e., $L^1_{loc+poly} \subset \mathcal{S}'$. But we can extend the definition to $s > -n-N-1$ for any $N \in \mathbb{N}$, by defining

$$u_s(\varphi) := L_{u_2}(\varphi) + L_{u_1}\left(\varphi - \sum_{|\alpha|\le N} \frac{D^\alpha \varphi(0)}{\alpha!} x^\alpha\right) + \sum_{|\alpha|\le N} c_{n,s,\alpha,B}\, D^\alpha \delta(\varphi)\,,$$

where $c_{n,s,\alpha,B}$ is the constant to make the definition admissible with the case for $z > -n$, given by

$$\begin{aligned} c_{n,s,\alpha,B} = \frac{\pi^{(s+n)/2}}{\Gamma((s+n)/2)} \frac{1}{\alpha!} \int_B x^\alpha |x|^s\, dx &= \frac{\pi^{(s+n)/2}}{\Gamma((s+n)/2)} \frac{1}{\alpha!} \frac{R_B^{s+n+|\alpha|} \int_{\mathbb{S}^{n-1}} \omega^\alpha\, d\omega}{s+n+|\alpha|} \\ &\overset{B=B(0,1)}{=} \frac{\pi^{(s+n)/2} \int_{\mathbb{S}^{n-1}} \omega^\alpha\, d\omega}{\alpha!\, \Gamma((s+n)/2)\, (s+n+|\alpha|)}\,. \end{aligned}$$

The constant is well-defined for $s > -n-N-1$ as, in the case α_i is even for all i (otherwise $c=0$), the simple poles $\{s = -n-2j\}$ of the Gamma function in the denominator will be cancelled by the other term $(s+n+|\alpha|)$ at $2j = |\alpha|$. The second term in the definition is finite, and controlled by the the seminorm $\rho_{0,\alpha}(\varphi)$, as $\varphi - \sum_{|\alpha|\le N} \frac{D^\alpha\varphi(0)}{\alpha!} x^\alpha \lesssim_N \sum_{|\alpha|=N+1} \rho_{0,\alpha}(\varphi)\, |x|^{N+1}$.

Remark 1. *Similarly, the distribution can be extended to all complex number $s \in \mathbb{C}$, being an entire function of s with values of tempered distribution. In this sense, we have furthermore that $(u_s)\hat{} = u_{-n-s}$, $\forall\, s \in \mathbb{C}$, which generalizes the cases $s \in (-n, 0)$.*

Remark 2. *Obviously, δ is homogeneous of order $-n$. Besides, we have another homogeneous distribution of order $-n$: Let $\Omega \in L^1(\mathbb{S}^{n-1})$, $\int_{\mathbb{S}^{n-1}} \Omega = 0$. Then a principle value integral gives a distribution by*

$$p.v.\ \Omega(x/|x|)/|x|^n\, (\varphi) := \lim_{\epsilon\to 0} \int_{|x|\ge\epsilon} \Omega(x/|x|)/|x|^n\, \varphi(x)\, dx\,.$$

A typical property is that all homogeneous distributions of order $-n$ that coincide with a smooth function away from the origin arise in this way, up to a constant multiple of standard Dirac.

Date: December 28, 2018.