lecture notes Dec. 01. ARA

- some facts about rearrangement. $f \geq 0$, finite at ∞.
  - f^* semicontinuous (def of A^*).

    $\{f>t\}^* = \{f^*>t\}$

    $\int_{\mathbb{R}^n} \phi(|f|) = \int_{\mathbb{R}^n} \phi(f^*)$ for "good" ϕ
  - take $\phi(t) = \begin{cases} Id & x > t_0 \\ 0 & x \leq t_0 \end{cases}$

    $\Rightarrow \int_{\{f>t\}} f(x)\,dx = \int_{\{f^*>t\}} f^*(x)\,dx = \int_{\{f>t\}^*} f^*(x)\,dx$

    let $C = \{f>t\}$, i.e. $\int_{\mathbb{R}^n} \chi_C f = \int \chi_{C^*} f^*$

    on the other hand, by theorem 3.4

    $$\int_{\mathbb{R}^n} \chi_C f \leq \int_{\mathbb{R}^n} \chi_C^* f^* = \int_{\mathbb{R}^n} \chi_{C^*} f^*$$

    $$\Rightarrow \sup_{\substack{M \subseteq \mathbb{R}^n \\ \mathcal{L}^n(M) = \mathcal{L}^n(C)}} \int_M f = \int_{M^*} f^*\,dx.$$
  - consider Riesz's Ineq.

    $$\int f(x)\, g(x-y)\, h(y)\,dx\,dy \leq \int f^*(x)\, g^*(x-y)\, h^*(y)\,dx\,dy$$

    take $f = \chi_C$ with $C \subseteq \mathbb{R}^n$ measurable

    $$\Rightarrow \int_C g * h(x)\,dx \leq \int_{C^*} g^* * h^*(x)\,dx$$

    for $\mathcal{L}^n(C)$ fixed, take sup on the left

    $$\Rightarrow \int_{C^*} (g*h)^*(x)\,dx \leq \int_{C^*} g^* * h^*(x)\,dx$$
  - rearrangement decrease the L^p distance

    i.e. $\|f-g\|_{L^p} \geq \|f^* - g^*\|_{L^p}$

- Pólya-Szegö Ineq: $f \in W^{1,p}, f \geq 0 \Rightarrow \|\nabla f\|_{L^p} \geq \|\nabla f^*\|_p$.
  - $p=1$: isoperimetric ineq $\|\nabla f\|_1 \geq \|\nabla f^*\|_p$
  - Coarea formula: $f: \mathbb{R}^n \to \mathbb{R}$, Lipschitz.

$$\int_{\mathbb{R}^n} g |\nabla f| \, dx = \int_0^\infty \int_{\{f^{-1}(t)\}} g(x) \, d\mathcal{H}^{n-1}(x) \, dt$$

- $g=1 \Rightarrow$

$$\int |\nabla f| = \int_0^\infty \partial \{f>t\} \, dt$$

$$\geq \int_0^\infty \partial \{f>t\}^* \, dt$$

$$= \int_0^\infty \partial \{f^* > t\} \, dt$$

$$= \int |\nabla f^*|$$

i.e. $f \in \text{Lipschitz}(\mathbb{R})$, we have

$$\|\nabla f\|_1 \geq \|\nabla f^*\|_1$$

- 思考题：对于 $f \in C_c^\infty(\mathbb{R}^n)$，证明 $\|\nabla f\|_2 \geq \|\nabla f^*\|_2$

(Hint: $u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy$

$u_t = \Delta u$

$u(0,x) = f(x)$

consider $\frac{d}{dt} \left(\int u(t,x) f(x) \, dx \right)$)

- Brunn-Minkowski Ineq: $\mathcal{L}^n(A)^{\frac{1}{n}} + \mathcal{L}^n(B)^{\frac{1}{n}} \leq \mathcal{L}^n(A+B)^{\frac{1}{n}}$
  - The relations of B-M Ineq and isoperimetric ineq.

consider arbitrary domain D in $\mathbb{R}^n$.

$D_r = D + B_r$ where $B_r = B(0,r)$

$$\Rightarrow \mathcal{L}^n(D_r) \geq \left(\mathcal{L}^n(D)^{\frac{1}{n}} + \mathcal{L}^n(B_r)^{\frac{1}{n}} \right)^n$$

$$= \left(\mathcal{L}^n(D)^{\frac{1}{n}} + \omega_n^{\frac{1}{n}} r\right)^n$$

$$\geq \mathcal{L}^n(D) + n\,\mathcal{L}^n(D)^{\frac{n-1}{n}}\,\omega_n^{\frac{1}{n}}\, r$$

$$\Rightarrow \frac{\mathcal{L}^n(D_r) - \mathcal{L}^n(D)}{r} \geq n\,\omega_n^{\frac{1}{n}}\,\mathcal{L}^n(D)^{\frac{n-1}{n}}$$

let $r \to 0$, we get

$$M_{n-1}(\partial D) \geq n\,\omega_n^{\frac{1}{n}}\,\mathcal{L}^n(D)^{\frac{n-1}{n}}$$

where $M_{n-1}(\partial D)$ means the Minkowski content.

(for the properties of $M_{n-1}(\partial D)$, you can refer to H. Federer, "Geometric measure theory".)

• **Some Application of Rearrangement**

1. • comparision of principal eigenvalue of Δ

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$

proof: $\lambda_1(\Omega) = \inf\limits_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^2} = 1}} \int_\Omega |\nabla u|^2$

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2. • Minimizer for Sobolev constant S

$$S = \inf_{\substack{\|u\|_{p^*} = 1 \\ u \in W^{1,p}(\mathbb{R}^n)}} \|\nabla u\|_p \quad \text{where } p^* = \frac{np}{n-p}$$

if $\{f_k\}$ is a sequence of minimizer for S.

we can replace f_k with f_k^*

$\{f_k^*\}$ is still a sequence of Minimizer.

this idea can be applied to many problem involving energy variation

If we can prove the convergence of existence and uniqueness of extremal.

$\Rightarrow$ extremal is symmetric descreasing.

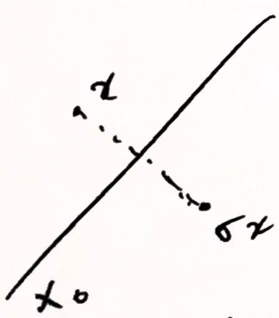
- 比书上 Schwarz 对称与 Steiner 对称化更简单的对称化

Polarziation or two-point rearrangement.

以下简称 TPR.

定义如下: X_0 为 $\mathbb{R}^n$ 中一不经过原点的超平面, 将 $\mathbb{R}^n$ 分成 2 部分. 含原点的部分记作 X_+, 另一部分记作 X_-

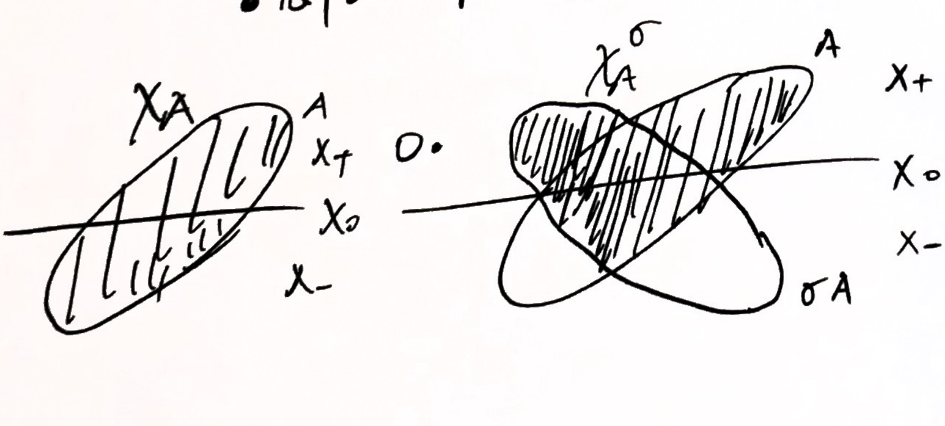
$$f^\sigma(x) = \begin{cases} \max\{f(x), f(\sigma x)\} & x \in X_+ \\ \min\{f(x), f(\sigma x)\} & x \in X_- \\ f(x) & x \in X_0 \end{cases}$$



where σ 是关于 X_0 超平面的反射.

(X_0 过原点时, 随意指定一侧取 max).

- 例子: $f = \chi_A(x)$ 时.



- Properties of TPR:

① (TPR improves the modulus of continuity)

i.e Supose f uniformly continuous on $\mathbb{R}^n$, then, f^σ is uniformly continuous with the same modulus of continuity

i.e $[\forall \varepsilon, |x-y| < \delta(\varepsilon) \Rightarrow |f(x)-f(y)| < \varepsilon]$

$$\Downarrow$$

$[\forall \varepsilon, |x-y| < \delta(\varepsilon) \Rightarrow |f^\sigma(x) - f^\sigma(y)| < \varepsilon]$

(same as).

proof: $\forall x, y \in X_+$, $|f^\sigma(x) - f^\sigma(y)| = |\max\{f(x), f(\sigma x)\} - \max\{\cdots\}|$

$\leq \max\{|f(x)-f(y)|, |f(\sigma x) - f(\sigma y)|\}$

$< \varepsilon$

similarly $x, y \in X_-$ ok.

if $x \in X_+$, $y \in X_-$,

$|f^\sigma(x) - f^\sigma(y)| \leq \max\{|f(x)-f(y)|, |f(\sigma x)-f(y)|, |f(x)-f(\sigma y)|, |f(\sigma x)-f(\sigma y)|\}$

$< \varepsilon$

($\because$ $|\sigma x - y| = |x - \sigma y| \leq |x-y| < \varepsilon$.)

② $\dfrac{|f(x)-f(y)|}{|x-y|^\alpha} \leq M \Rightarrow \dfrac{|f^\sigma(x) - f^\sigma(y)|}{|x-y|^\alpha} \leq M.$

i.e. TPR maps $C^\alpha(\mathbb{R}^n) \to C^\alpha(\mathbb{R}^n)$, $\alpha \in (0, 1]$

③ Ex: $\int fg \le \int f^\sigma g^\sigma$

"=" 取得 $\Leftrightarrow$ $(f(x)-f(\sigma x))(g(x)-g(\sigma x))\ge 0$.

④ Ex: TDS decreases $\mathcal{L}^p$ norm

(*) ⑤ **[compactness]**. Suppose $f \in C_c^0(\mathbb{R}^n)$

$TPR_f := \{ f^{\sigma_1,\cdots\sigma_k} \mid k\ge 0,\ \sigma_1,\cdots\sigma_k \text{ reflecting} \}$

Then, $\exists \{g_k\} \subseteq TPR_f$. s.t.

$g_k \to f^*$ uniformly

Proof: let H be a fixed, strickly decreasing bounded function on $\mathbb{R}^+$, and $\lim_{t\to\infty} H(t) = 0$

define $I[u] := \int u(x) H(|x|)\, dx$

by property ②. we know TPR_f is satisfies A-A lemma's condition

$\Rightarrow \forall g \in \overline{TPR_f}$, $\exists \{g_k\} \subseteq TPR_f$. s.t.

$g_k \rightrightarrows g\ \overline{TPR_f}$

suppose $I[g] = \sup_{u\in TPR_f} I[u]$.

claim: $f^* = g$.

$g_k \rightrightarrows g \Rightarrow g_k^\sigma \rightrightarrows g^\sigma$ $\forall\sigma$ 反射.

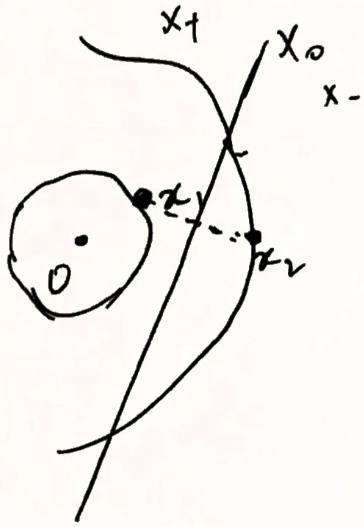
$\Rightarrow I[g] \ge I[g^\sigma]$

on the other hand, by property ③

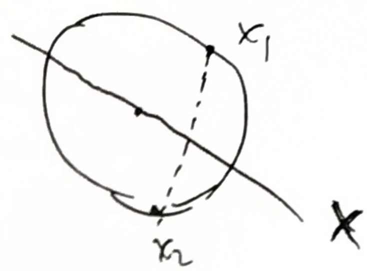
$I[g] \le I[g^\sigma]$

"=" 成立 $\Leftrightarrow$ $g = g^\sigma$.

claim : $\forall \sigma$, $g = g^\sigma \Rightarrow g = g^*$



$\sigma(x_1) = x_2$



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$\boxed{\text{Ex:}}$ if $f \in L^p$, can you choose sequence in TPR_f s.t. $g_k \xrightarrow{L^p} f^*$?

if $f \in W^{1,p}$?

How about convergence in measure.

or weak convergence ?

Problems in Homework:

① Banach-Saks for $L^2(\Omega)$

$f_k \xrightarrow{L^2} f$, want to show

$\exists$ subsequence, s.t.

$$\sum_{k=1}^{m} \frac{1}{m} f_{k_n} \to f$$

只须 $\|\sum \frac{1}{m} f_{k_n}\|^2 \to \|f\|^2$ since $\sum \frac{1}{m} f_{k_n} \xrightarrow{L^2} f$)

$\|$

$\sum \frac{1}{m^2}\|f_{k_n}\|^2 + \frac{1}{m^2}\sum_{i\neq j}\langle f_{k_i}, f_{k_j}\rangle$

① $\to 0$

② how to choose subsequence

f_1 ok

take $\hat{f}_2$ $\langle f_2, f_1\rangle < \frac{1}{2^1}$

take $\hat{f}_3$ s.t. $\langle f_3, f_1\rangle < \frac{1}{2^2(3-1)}$ $\langle f_3, \hat{f}_2\rangle < \frac{1}{2^2(3-1)}$

$\vdots$

$\hat{f}_n$. s.t. $\langle \hat{f}_n, \hat{f}_i\rangle < \frac{1}{(n-1)}\cdot\frac{1}{2^n}$

$\Rightarrow$ ② $\le \frac{1}{m^2}\left(\sum_{k=1}^{m}\frac{1}{2^k}\right) \le \frac{1}{m^2} \to 0$.

done.