D ... continuity and boundaness

Test function spaces (on $\mathbb{R}^n$)

$\mathcal{D}=C_c^\infty$, $\mathcal{S}$, C^∞ 也记作 $\mathcal{E}$, 我们关心这3个函数空间上"收敛的性质"（更一般是关心它们的拓扑）但它们不是Banach空间. ~~我们可以看到~~在一组半范数下它们构成一个 Fréchet space 具体的细节可见 Rudin Functional Analysis. Chapter1 + Chapter6.

我们在这里只满足于给出收敛性的刻画:

- $\varphi_k \longrightarrow \varphi$ in $\mathcal{E}$ $:=$ $\tilde{p}_{\alpha N}(\varphi_k-\varphi)\to 0$. $\tilde{p}_{\alpha N}(\varphi) := \sup\limits_{|x|\le N} |D^\alpha(\varphi)|$ $\forall \alpha, N$
- $\varphi_k \longrightarrow \varphi$ in $\mathcal{S}$ $:=$ $p_{\alpha\beta}(\varphi_k-\varphi)\to 0$ $\quad p_{\alpha\beta}(\varphi)=\sup\limits_{x\in\mathbb{R}^n}|x^\alpha D^\beta(\varphi)|$
- $\varphi_k \longrightarrow \varphi$ in $\mathcal{D}$ $:=$ ~~sup~~ $\exists K$ compact st $\operatorname{supp}\varphi_k \subseteq K$ $\sup\limits_{x\in K}|D^\alpha(\varphi_k-\varphi)|\to 0$ $\forall \alpha$

自然有 $\varphi_k\to\varphi$ in $\mathcal{D} \Longrightarrow \varphi_k\to\varphi$ in $\mathcal{S} \Longrightarrow \varphi_k\to\varphi$ in $\mathcal{E}$

Example: 取 $\varphi\in\mathcal{D}$, $\varphi\neq 0$ Let $\varphi_k=\frac{1}{k}\tau_k\varphi$ then

$\varphi_k\to 0$ in $\mathcal{E}$ $\quad \sup\limits_{|x|\le N}|D^\alpha\varphi_k| \le \frac{M_{\alpha N}}{k} \longrightarrow 0 \quad \forall \alpha, N$

$\varphi_k \not\to 0$ in $\mathcal{D}$ $\quad$ φ_k 不一致紧支

$\varphi_k \not\to 0$ in $\mathcal{S}$ $\quad p_{\alpha\beta}(\varphi_k)=k^{-1}\sup\limits_{x\in\mathbb{R}^n}|x^\alpha D^\beta\varphi(x-k)|$

$$=k^{-1}\sup_{x\in K}(x+k)^\alpha D^\beta\varphi(x) \longrightarrow \infty \text{ as } k\to\infty$$

（此处k表示各分量都是k的向量）

接下来定义连续线性泛函

$u\in\mathcal{D}'$ (or $\mathcal{E}'$ or $\mathcal{S}'$) is called a continuous linear functional if

① u is linear ie. $u(\lambda\varphi+\mu\psi)=\lambda u(\varphi)+\mu u(\psi)$

② u is continuous ie 若 $\varphi_k\to\varphi$ in $\mathcal{D}$ (or $\mathcal{E}$, $\mathcal{S}$) then $u(\varphi_k)\to u(\varphi)$

则有自然的包含关系. $\underset{\text{distribution}}{\mathcal{D}'} \supset \underset{\text{tempered distribution}}{\mathcal{S}'} \supset \mathcal{E}' \longrightarrow$ compactly supported distribution

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The relation between continuity and boundness

Prop:

1. $u\in\mathcal{D}' \iff \forall$ compact K $\exists C>0, m\in\mathbb{N}$ s.t. $|u(\varphi)|\le C\sum_{|\alpha|\le N}\|D^\alpha\varphi\|_{L^\infty}$ $\forall\varphi\in C^\infty$ $\operatorname{supp}\varphi\subset K$

2. $u\in\mathcal{S}' \iff \exists C>0$ $m,l\in\mathbb{N}$ s.t. $|u(\varphi)|\le C\sum_{\substack{|\alpha|\le m\\|\beta|\le l}}\rho_{\alpha\beta}(\varphi)$ $\forall\varphi\in\mathcal{S}$

3. $u\in\mathcal{E}' \iff \exists C>0$ $N, m\in\mathbb{N}$ s.t. $|u(\varphi)|\le C\sum_{|\alpha|\le m}\rho_{\alpha,N}(\varphi)$ $\forall\varphi\in\mathcal{E}$.

Rmk:

1. 这个命题是自然的 如果我们理解了 $\mathcal{D}$. $\mathcal{S}$. $\mathcal{E}$ 上的拓扑 i.e 它们每点处的拓扑 均是由
$V(x_0, p, n) = \{x : p(x-x_0) < \frac{1}{n}\}$ 给出. p 为半范数. $n\in\mathbb{N}$.

2. 证明我跳过了 但是要会用 即用它来 check 一个泛函是否连续!

Order and Supports

Recall that $u\in\mathcal{D}' \Leftrightarrow \forall \text{compact } K\ \exists\, C>0,\ m\in\mathbb{N}$ s.t. $|u(\varphi)|\le C\sum_{|\alpha|\le N}\|D^\alpha\varphi\|_{L^\infty}$

$\forall\varphi\in C^\infty,\ \operatorname{supp}\varphi\subset K$

Def: If u is such that one N will do for all K (but not necessarily with the same C), then the smallest such N is called the order of u. If no N will do for all K then u is said to have infinite order.

Prop: $u\in\mathcal{S}' \Rightarrow u$ has finite order

Pf: $u\in\mathcal{S}' \Rightarrow \exists\, C>0,\ m\in\mathbb{N}$ st $|u(\varphi)|\le C\sum_{|\alpha|,|\beta|\le m}\|x^\alpha D^\beta\varphi\|_{L^\infty}\ \forall\varphi\in\mathcal{S}$

$\Rightarrow |u(\varphi)|\le C\sum_{|\alpha|,|\beta|\le m}\|x^\beta D^\alpha\varphi\|_{L^\infty}\quad \le$.

$\Rightarrow$ ~~and~~ u has order no more than m.

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$u\in\mathcal{D}'$

Def: $\operatorname{supp}u=\bigcap K$ where K is closed and $u(\varphi)$ for all $\varphi\in C_c^\infty(\mathbb{R}^n\setminus K)$

Prop: $u(\varphi)=0\quad\forall\varphi\in C_c^\infty(\mathbb{R}^n\setminus\operatorname{supp}u)$

Pf. Partition of unity.

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Prop: $\operatorname{supp}u$ compact $\Leftrightarrow u\in\mathcal{E}'$

Pf: $\Rightarrow$) Recall $u\in\mathcal{E}'\Leftrightarrow\exists\, C>0,\ N,m\in\mathbb{N}$ st

$|u(\varphi)|\le C\sum_{|\alpha|\le m}P_{\alpha,N}(\varphi)\quad\forall\varphi\in\mathcal{E}$

$\Rightarrow$) 此时取 $N=\max\{|x|: x\in K\}$ 可得知 $u\in\mathcal{E}'$

$\Leftarrow$) 易知成立 $\operatorname{supp}u\subset\{x: |x|\le N+1\}$ #

Thm: $u \in \mathcal{D}'$, $\operatorname{supp} u = \{0\}$, u has order N. Then there are constant C_α s.t.

$$u = \sum_{|\alpha| \le N} C_\alpha D^\alpha \delta$$

Rmk: 此结论对于 $u \in \mathcal{S}'$ 成立

Lemma: Suppose $u_1 \cdots u_n$ and u are linear functionals on a vector space X.

$\ker u \supset \bigcap_{i=1}^{n} \ker u_i \implies u = \sum_{i=1}^{n} c_i u_i$ for ~~some constant~~ c_i

pf: WLOG $\{u_i\}_{i=1}^{n}$ are linearly independent.

$\implies U: X \longrightarrow \mathbb{C}^n$ is surjective

$\varphi \longrightarrow (u_1(\varphi) \cdots u_n(\varphi))$

~~$\implies \exists \{\varphi_i\}_{i=1}^{n} \subset X$ s.t. $U(\varphi) = \sum_{i=1}^{n} c_i \varphi_i$~~

$\ker u \supset \bigcap_{i=1}^{n} \ker u_i \implies \tilde{u} = \mathbb{C}^n \longrightarrow \mathbb{C}$ is well-defined

$(u_1(\varphi) \cdots u_n(\varphi)) \longrightarrow u(\varphi)$

$\implies \tilde{u} = \sum_{i=1}^{n} c_i u_i(\varphi)$ c_i are constant

$\implies u(\varphi) = \sum_{i=1}^{n} c_i u_i(\varphi)$

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~~Pf of the thm~~: 借助上述引理来证明:

若 $\phi \in \mathcal{D}$ 满足 $D^\alpha\phi(0) = 0$ $\forall |\alpha| \le N \implies u(\phi) = 0$

$\forall \eta > 0$ $\exists$ compact ball K s.t. $|D^\alpha \phi| \le \eta$ in K, if $|\alpha| = N$

~~Exe~~ Claim: $|D^\alpha \phi(x)| \le \eta\, n^{N-|\alpha|} \cdot |x|^{N-|\alpha|}$ $\forall x \in K$ $|\alpha| \le N$

(Left as exercise ; Hint: Induction + Mean value thm)

choose an auxiliary function $\psi \in \mathcal{D}$ s.t. $\operatorname{supp}\psi \subset B(0,1)$, $\psi \equiv 1$ in 0 邻域

Let $\psi_r(x) = \psi(\frac{x}{r})$

Then $D^\alpha(\psi_r\phi)(x) = \sum_{\beta\le\alpha} C_{\alpha\beta}\,(D^{\alpha-\beta}\psi)(\frac{x}{r})\,(D^\beta\phi)(x)\cdot r^{|\beta|-|\alpha|}$

由上述 claim

$\Longrightarrow \sum_{|\alpha|\le N}\|D^\alpha\psi_r\phi\|_\infty \le C\cdot\sum_{|\alpha|\le N}\|D^\alpha\phi\|_\infty\cdot\eta$ for $r \ll 1$

$\Longrightarrow |u(\phi)| \underset{\text{supp}\,\phi=\{0\}}{=} |u(\psi_r\phi)| \underset{u \text{ has order } N}{\le} C\|\psi_r\phi\|_N \le C_1\eta\|\psi\|_N$

由 η 任意的 便知 $u(\phi)=0$

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