SECOND HOMEWORK FOR "MARTINGALE THEORY AND STOCHASTIC CALCULUS"

WEIYU LI
PB16001713

1. Problem

Given $(\Omega, \mathcal{F}, \mathcal{F}_*, \mathbb{P})$, $B = \{B_t, t \geq 0\}$ is the standard Brownian motion on it. Consider the following equation

$$\begin{cases} dX_t = f(X_t)dt + \int_{-T_0}^{0} g(r)X_{t+r}drdt + \sigma(t, X_t)dB_t, & t \geq 0, \\ X_0 = x_0 \in \mathbb{R}, \quad \{X_t, t \in [-T_0, 0]\} \in L^2([-T_0, 0], \mathbb{R}), \end{cases} \tag{1}$$

where $T_0 > 0$ is given, f is a polynomial $f(y) = a_n y^n + \ldots + a_0$ with $a_i \in \mathbb{R}$, σ is global Lipschitz and $g : \mathbb{R} \to \mathbb{R}$ is a given function. Analyze the "existence" and "uniqueness" of the "solution" of the above SDE.

2. Preliminary definitions

The first question is: "how to define *the solution of an equation* and its *uniqueness*?" We consider the definitions in the *global* and *local* sense, which is analogous to those in the textbook.

Definition 1 (Solutions). *Given $(\Omega, \mathcal{F}, \mathcal{F}_*, \mathbb{P})$ satisfying the usual conditions and B the standard Brownian motion on it, we say the $\mathcal{F}_*$-adapted $X = \{X_t\}_{t \geq 0}$ is the* ***global solution of*** (1), *if*

(i) X has continuous paths, $\mathbb{P}$-a.s.,
(ii) the initial condition $X_0 = x_0, \{X_t, t \in [-T_0, 0]\}$ is satisfied, and
(iii) for any $T \geq 0$,

$$X_t = x_0 + \int_0^T f(X_t)dt + \int_0^T \int_{-T_0}^{0} g(r)X_{t+r}drdt + \int_0^T \sigma(t, X_t)dB_t, \quad \mathbb{P}\text{-}a.s..$$

We say the pair of $\{\mathcal{F}_t\}_{t \in [0,T]}$-adapted $X = \{X_t\}_{t \in [0,T]}$ and the stopping time τ is the ***local solution of*** (1), *if*

(i') X has continuous paths, $\mathbb{P}$-a.s.,
(ii') the initial condition $X_0 = x_0, \{X_t, t \in [-T_0, 0]\}$ is satisfied, and
(iii') for any $T \geq 0$,

$$X_{T\wedge\tau} = x_0 + \int_0^{T\wedge\tau} f(X_t)dt + \int_0^{T\wedge\tau} \int_{-T_0}^{0} g(r)X_{t+r}drdt + \int_0^{T\wedge\tau} \sigma(t, X_t)dB_t, \quad \mathbb{P}\text{-}a.s..$$

Further, (X, τ) is the **maximal local solution**, *if there exists stopping times $\tau^n \uparrow \tau$, such that (X, τ^n) are local solutions and τ is the explosion time.*

Date: 2019/12/30.
liweiyu@mail.ustc.edu.cn.

Definition 2 (Uniqueness). *Given $(\Omega, \mathcal{F}, \mathcal{F}_*, \mathbb{P}, B)$ as in Definition 1, we say the global/local/maximal local solution of* (1) *is* **unique**, *if any two solutions X, Y satisfy* $\mathbb{P}(X = Y) = 1$.

3. Main Results

With the above definitions, we give our result in a formal way.

Result 1. *Suppose that $g \in L^2([-T_0, 0])$. If f is a linear function $f(y) = a_0 + a_1 y$, then there exists a unique global solution of* (1). *Otherwise, in the more general case, if f is with higher degree, there exists a unique maximal local solution of* (1).

We first state an important global result, and its proof helps proving other results.

Theorem 1. *If f is a global Lipschitz function, not essentially a polynomial, then there exists a unique global solution of* (1).

Proof. **Step 1. Uniqueness.** Suppose that X and Y are two global solutions. Then

$$\begin{cases} d(X_t - Y_t) = \big(f(X_t) - f(Y_t)\big)dt + \int_{-T_0}^{0} g(r)(X_{t+r} - Y_{t+r})drdt + \big(\sigma(t, X_t) - \sigma(t, Y_t)\big)dB_t, \quad t \geq 0, \\ X_t - Y_t = 0, \quad t \in [-T_0, 0], \end{cases}$$

From Ito's formula,

$$\mathbb{E}(X_T - Y_T)^2 = \mathbb{E}\int_0^T 2(X_t - Y_t)d(X_t - Y_t) + \mathbb{E}\int_0^T \big(\sigma(t, X_t) - \sigma(t, Y_t)\big)^2 dt. \tag{2}$$

Denote the stopping time $\tau_M = \inf\{t \geq 0 : |X_t| \vee |Y_t| > M\}$ and the Lipschitz constants for σ, f as L_σ, L_f, then we have the upper-bound that

$$\begin{aligned} \mathbb{E}(X_{T\wedge\tau_M} - Y_{T\wedge\tau_M})^2 \leq & 2L_f \int_0^{T\wedge\tau_M} \mathbb{E}(X_t - Y_t)^2 dt \\ & + 2\mathbb{E}\left|\int_0^{T\wedge\tau_M} (X_t - Y_t)\int_{-T_0}^0 g(r)(X_{t+r} - Y_{t+r})drdt\right| \\ & + \int_0^{T\wedge\tau_M} L_\sigma^2 \mathbb{E}(X_t - Y_t)^2 dt. \end{aligned}$$

Note that the middle term can be further bounded as

$$\begin{aligned} & \left|\int_0^{T\wedge\tau_M} (X_t - Y_t)\int_{-T_0}^0 g(r)(X_{t+r} - Y_{t+r})drdt\right| \\ \leq & \left|\int_0^{T\wedge\tau_M} (X_t - Y_t)\|g\|_{L^2([-T_0,0])}\|X - Y\|_{L^2\left([(t-T_0)\vee 0, t]\right)} dt\right| \\ \leq & \|g\|_{L^2([-T_0,0])}\|X - Y\|_{L^2([0,T\wedge\tau_M])}\int_0^{T\wedge\tau_M} |X_t - Y_t| dt \\ \leq & \|g\|_{L^2([-T_0,0])} T \|X - Y\|^2_{L^2([0,T\wedge\tau_M])}. \end{aligned}$$

Consequently for $Z_t = \mathbb{E}(X_{t\wedge\tau_M} - Y_{t\wedge\tau_M})^2 \geq 0$, we have that for any $\bar{T}$,

$$Z_T \leq c_{M,\bar{T}} \int_0^T Z_t dt, \quad \forall T \in [0, \bar{T}].$$

From the Gronwall's inequality and that $Z_0 = 0 \leq Z_T$, it holds that

$$Z_T = 0, \forall T \in [0, \bar{T}].$$

Since $\bar{T}$ is arbitrarily chosen, and $T \wedge \tau_M \uparrow T$ as $M \to \infty$, we conclude from Fatou's lemma that

$$\mathbb{E}(X_T - Y_T)^2 = 0, \forall T$$

which completes the proof of the uniqueness of global solution.

Step 2. Existence. We construct a solution via Picard iteration. In specific, let $X_T^0 = x_0, T \in [0, \bar{T}]$, and

$$X_T^n = x_0 + \int_0^T f(X_t^{n-1})dt + \int_0^T \int_{-T_0}^0 g(r) X_{t+r}^{n-1} dr dt + \int_0^T \sigma(t, X_t^{n-1}) dB_t, \quad n \geq 1. \tag{3}$$

Then

$$\mathbb{E} \max_{T \in [0,\bar{T}]} (X_T^n - X_T^{n-1})^2 \leq c_{\|g\|_2} \bar{T} \mathbb{E} \max_{T \in [0,\bar{T}]} (X_T^{n-1} - X_T^{n-2})^2$$

implies that if we choose $\bar{T}$ small enough (depending only on the constant $c_{\|g\|_2}$), then (3) converges to the unique fixed-point characterized by (1). Induction of doing such an iterative process on $[0, \bar{T}]$, $[\bar{T}, 2\bar{T}]$ and so on gives a solution of (1). □

Corollary 1. *If f is a linear function, then there exists a unique global solution of* (1).

Now we turn to the general situation that f is polynomial, which is local Lipschitz. Note that what differs linear functions from higher-order polynomials is that the latter ones increase too quickly (not in the same order) when the variable goes to infinity. Therefore, stopping times prevent the end increasing in a catastrophic way.

Theorem 2. *If f is local Lipschitz, then there exists a unique maximal local solution of* (1).

Proof. **Step 1. Local solution.** Let $f_N(x)$ be a truncated (or smoothed) function of f, such that

$$f_N(x) = \begin{cases} f(x), & |x| \leq N \\ 0, & |x| \geq N+1 \end{cases}, \quad \text{and } f_N \text{ is global Lipschitz with constant } L_N.$$

Then **Theorem 1** tells us that there exists a unique global solution X^N of the equation

$$\begin{cases} dX_t^N = f_N(X_t^N)dt + \int_{-T_0}^0 g(r) X_{t+r}^N dr dt + \sigma(t, X_t^N) dB_t, & t \geq 0, \\ X_0^N = x_0 \in \mathbb{R}, \quad \{X_t^N, t \in [-T_0, 0]\} \in L^2([-T_0, 0], \mathbb{R}). \end{cases}$$

Further letting $\tau_N = \inf\{t \geq 0 : |X_t^N| \geq N\}$ gives the local solution (X^N, τ_N) of (1).

Step 2. Maximal local solution. From the uniqueness in **Step 1** and that $f_N(x) = f_{N+1}(x), \forall |x| \leq N$, we have

$$X_t^N = X_t^{N+1}, \quad \forall t \in [0, \tau_N \wedge \tau_{N+1}] = [0, \tau_N].$$

As τ_N increasing, the limit $\tau = \lim_N \tau_N$ is well-defined in $[0, \infty]$. In specific when $\tau < \infty$,

$$\lim_{t \uparrow \tau} \max_{s \in [0,t]} |X_s| \geq \lim_N |X_{\tau_N}| = \infty, \quad \mathbb{P} - a.s.,$$

which implies that τ is an explosion time and (X, τ) is the unique maximal local solution. (Uniqueness is from truncating the solution at N and comparing it with X^N.) □

4. Some remarks

The L_2 assumption on g is not necessary. In fact, we can give a weaker assumption that $g \in L_1$.

Result 2. *If $g \in L^1([-T_0, 0])$, all the statements in* **Result 1** *hold.*

The proof follows almost the same as above. When tackling the term including g, instead of using Holder's inequality that $\int_{-T_0}^{0} g(r)(\cdot)dr \leq \|g\|_{L^2([-T_0,0])}\| \cdot \|_{L^2([-T_0,0])}$, we consider $\int_{-T_0}^{0} g(r)(\cdot)dr \leq \|g\|_{L^1([-T_0,0])} \max_{[-T_0,0]} \cdot$, where BDG inequality or Doob's inequality can be applied to upper-bound the "$\cdot$" term.