Chapter IV: the Hilbert Transform

The conjugate Poisson kernel

Given ~~a~~ $f \in \mathcal{S}(\mathbb{R})$, we can extend it harmonically to $\mathbb{R} \times \mathbb{R}_+$ by $u(x,t) = P_t * f(x)$; P_t is the Poisson kernel. By setting $z = x+it$, we have also

$$u(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z \xi} d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi$$

If we define

$$iv(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z \xi} d\xi - \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i \bar{z} \xi} d\xi$$

then v is also harmonic in $\mathbb{R} \times \mathbb{R}_+$. If f is real, then both u and v are real. Furthermore, $u+iv$ is hol. so v is the harmonic conjugate of u.

Rewrite v as

$$v(z) = \int_{\mathbb{R}} -i\,\mathrm{sgn}(\xi) e^{-2\pi t|\xi|} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

or equivalently

(1) $$v(x,t) = Q_t * f(x)$$

where

(2) Ex1 $$\hat{Q}_t(\xi) = -i\,\mathrm{sgn}(\xi) e^{-2\pi t|\xi|}$$

Inversing the Fourier transform, yields

(3) $$Q_t(x) = \frac{1}{\pi} \frac{x}{t^2+x^2}$$

which is the conjugate Poisson kernel. Then

$$P_t(x) + iQ_t(x) = \frac{1}{\pi} \frac{t+ix}{t^2+x^2} = \frac{i}{\pi z}$$

is hol. in $\mathbb{R} \times \mathbb{R}_+$.

We have studies the behavior of $u(x,t)$ as $t \to 0^+$ via the fact that $\{P_t\}_{t>0}$ is an approximation of id.

Here we would like to do the same for $v(x,t)$. But we run into an obstacle: $\{Q_t\}$ is NOT an approximation of the identity. $\Leftarrow$ Q_t not integrable for any $t>0$.

Even worse, the formal limit

$$\lim_{t\to 0^+} Q_t(x) = \frac{1}{\pi x}$$

is NOT locally integrable, and hence we cannot define its convolution with smooth functions.

the principal value of $\frac{1}{x}$

We define the principal value of $1/x$, (abbr. $p.v.\frac{1}{x}$), by

$$p.v.\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\phi(x)}{x}\,dx\,,\quad \phi\in\mathcal{S}.$$

Claim: $p.v.\frac{1}{x}$ is indeed a tempered distribution.

given $\phi\in\mathcal{S}$, then $\int_{\varepsilon<|x|\le 1}\frac{\phi(0)}{x}\,dx=0$, which allows us to rewrite

$$p.v.\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|\le 1}\frac{\phi(x)-\phi(0)}{x}\,dx + \int_{|x|>1}\frac{\phi(x)}{x}\,dx.$$

It follows immediately that

$$\left|p.v.\frac{1}{x}(\phi)\right| \le C\left(\|\phi'\|_\infty + \|x\phi\|_\infty\right)$$

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We have further

proposition 1. In $\mathcal{S}'$, $\lim Q_t = \frac{1}{\pi}\,p.v.\frac{1}{x}$

Proof. - for each $\varepsilon>0$, the functions $\psi_\varepsilon(x)=x^{-1}\chi_{|x|>\varepsilon}$ are bounded and hence define tempered distributions. It then follows from the definition that in $\mathcal{S}'$

$$\lim_{\varepsilon\to 0}\psi_\varepsilon = p.v.\frac{1}{x}.$$

Therefore, it suffices to show that in $\mathcal{S}'$

$$\lim_{t\to0^+}\left(Q_t - \frac{1}{\pi}\psi_t\right) = 0.$$

Given $\phi\in\mathcal{S}$, we calculate

$$\begin{aligned}
(\pi Q_t - \psi_t)(\phi) &= \int_{\mathbb{R}} \frac{x\phi(x)}{t^2+x^2}\,dx - \int_{|x|>t}\frac{\phi(x)}{x}\,dx \\
&= \int_{|x|\le t}\frac{x\phi(x)}{t^2+x^2}\,dx + \int_{|x|>t}\left(\frac{x}{t^2+x^2}-\frac{1}{x}\right)\phi(x)\,dx \\
&= \int_{|x|\le1}\frac{x\phi(tx)}{1+x^2}\,dx - \int_{|x|>1}\frac{\phi(tx)}{x(1+x^2)}\,dx \\
&\xrightarrow[DCT]{t\to0^+} \int_{|x|\le1}\frac{x\phi(0)}{1+x^2}\,dx - \int_{|x|>1}\frac{\phi(0)}{x(1+x^2)}\,dx \\
&= 0.
\end{aligned}$$

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Consequently, we get

$$\lim_{t\to0} Q_t * f(x) = \frac{1}{\pi}\lim_{\varepsilon\to0}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}\,dy.$$

By continuity of the Fourier transform on $\mathcal{S}'$ and (2), we get

$$\left(\frac{1}{\pi}\,\mathrm{p.v.}\frac{1}{x}\right)^\wedge(\xi) = -i\,\mathrm{sgn}(\xi).$$

Finally, for $f\in\mathcal{S}(\mathbb{R})$, we define its Hilbert transform by any one of the following equivalent expressions:

$$Hf = \lim_{t\to0} Q_t * f$$

$$Hf = \frac{1}{\pi}\,\mathrm{p.v.}\frac{1}{x} * f$$

$$(Hf)^\wedge(\xi) = -i\,\mathrm{sgn}(\xi)\,\hat{f}(\xi).$$

The last expression allows us to define Hilbert transform of functions in $L^2(\mathbb{R})$. In this case, it satisfies

(4) $$\|Hf\|_{L^2(\mathbb{R})} = \|f\|_{L^2}$$

(5) $$H(Hf) = -f$$

(6) $$\int Hf \cdot g = -\int f\, Hg.$$

<u>Calderon-Zygmund decomposition</u>

<u>thm 2</u> let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Then we can decompose

$$f = g + b \quad \text{with}$$

(7)
- $|g| \le \lambda$
- $b = \sum_Q \chi_Q f$, where the sum runs over a collection $\mathcal{B} = \{Q\}$ of disjoint cubes such that for each such Q we have

(8) $$\lambda < \frac{1}{|Q|}\int_Q |f|\,dx \le 2^n \lambda$$

Furthermore,

(9). $$\Big| \bigcup_{Q \in \mathcal{B}} Q \Big| < \frac{1}{\lambda}\|f\|_{L^1}$$

Proof. For each $l \in \mathbb{Z}$, define a collection $\mathcal{D}_l$ of dyadic cubes by

$$\mathcal{D}_l := \Big\{ \prod_{i=1}^{n} [2^l m_i, 2^l(m_i+1)) \,/\, m_1, \dots, m_n \in \mathbb{Z} \Big\}$$

Note that, if $Q \in \mathcal{D}_l$ and $Q' \in \mathcal{D}_{l'}$, then $Q \cap Q' = \emptyset$, or $Q \subset Q'$ or $Q' \subset Q$.

Since $f \in L^1(\mathbb{R}^n)$, we can pick l_0 large enough so that

$$\frac{1}{|Q|}\int_Q |f|\,dx \le \lambda$$

for every $Q \in \mathcal{D}_{l_0}$. For each such cube, consider its

2^n sub-cubes of size 2^{l_0-1}. Each such subcube Q' will then have the property that

either $\frac{1}{|Q'|}\int_{Q'}|f(x)|dx \leq \lambda$ or $\frac{1}{|Q'|}\int_{Q'}|f(x)|dx > \lambda$.

In the lator case, we stopp dividing the sub-cube Q' and include it into the family B. Note also that in this case we have

$$\frac{1}{|Q'|}\int_{Q'}|f|dx \leq \frac{1}{|Q|}\int_{Q}|f(x)|dx \leq 2^n\lambda \overset{\text{gives}}{\rightsquigarrow} (8)$$

If, however, the 1^{st} case holds, we then ~~sub~~ divide Q' into its subcubes, of size 2^{l_0-2}. Continuing in this manner produces a collection of disjoint dyadic cubes B satisfying (8). Consequently

$$\Big|\bigcup_{B\ni Q} Q\Big| \leq \sum_{Q\in B}|Q| \leq \sum_{Q\in B}\frac{1}{\lambda}\int_Q|f(x)|dx \leq \frac{1}{\lambda}\int_{\mathbb{R}^n}|f(x)|dx.$$

Now let $x_0\in\mathbb{R}^n\setminus\bigcup_B \bar{Q}$. Then x_0 is contained in a decreasing sequence $\{Q_j\}$ of dyadic cubes, each of which satisfies

$$\frac{1}{|Q_j|}\int_{Q_j}|f(x)|dx \leq \lambda.$$

By Lebesgue differentiation theorem, we then have $|f(x_0)|\leq\lambda$ for a.e. such x_0. Then the function

$$g := f - \sum_{Q\in B}\chi_Q f$$

satisfies the requirement, thanks to the fact that $\mathbb{R}^n\setminus\bigcup_B Q$ and $\mathbb{R}^n\setminus\bigcup_B \bar{Q}$ differs by a set of measure zero. #

thms of M. Riesz and Kolmogrov

As in the case of Fourier transform, we use interpolating techniques to extend the Hilbert transforms to be defined on L^p, for all $1 \le p < \infty$.

thm 3. For $f \in \mathcal{S}(\mathbb{R}^n)$, there hold the following assertions.

(1) (Kolmogrov). H is weak (1,1).

$$\left|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}\right| \leqslant \frac{C}{\lambda} \|f\|_{L^1}$$

(2) (M. Riess). H is strong (p,p). $1 < p < \infty$

$$\|Hf\|_{L^p} \leqslant C_p \|f\|_{L^p}.$$

Proof. (1). Fix $\lambda > 0$ and assume f is non-negative. Performing C.-Z. decomposition of f at height λ, yields a sequence of disjoint intervals $\{I_j\}$ such that.

$$f(x) \le \lambda \qquad \text{a.e. } x \notin \Omega = \bigcup_j I_j$$

$$|\Omega| \le \frac{1}{\lambda} \|f\|_{L^1}$$

$$\lambda < \frac{1}{|I_j|} \int_{I_j} f \le 2^1 \lambda$$

Instead of using the decomposition in C.-Z., we use the following decomposition (to capture the cancellation)

$$g(x) = \begin{cases} f(x) & x \in \Omega \\ \frac{1}{|I_j|} \int_{I_j} f & , x \in I_j \end{cases}$$

$$b(x) = \sum_j b_j(x)$$

where

$$b_j(x) = \left(f(x) - \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x).$$

then $|g(x)| \le 2\lambda$ a.e. and b_j is supported on I_j and has mean zero. Since $Hf = Hg + Hb$

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}|$$
$$\le |\{x \in \mathbb{R} : |Hg(x)| > \lambda/2\}| + |\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}|$$

• Using (4) to estimate the 1^{st} term

$$|\{x \in \mathbb{R} \mid |Hg(x)| > \lambda/2\}| \le \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}} |Hg(x)|^2 dx$$
$$= \frac{4}{\lambda^2} \int_{\mathbb{R}} |g(x)|^2 dx$$
$$\le \frac{8}{\lambda} \int_{\mathbb{R}} |g(x)| dx = \frac{8}{\lambda} \int_{\mathbb{R}} |f(x)| dx$$

• to control the 2^{nd} term, we denote by $2I_j$ the interval with the same center as I_j and twice the length. let $\Omega^* := \bigcup 2I_j$ then. $|\Omega^*| \le 2|\Omega|$. and.

$$\left|\{x \in \mathbb{R} : |Hb(x)| > \lambda/2\}\right| \le |\Omega^*| + |\{x \notin \Omega^* : |Hb(x)| > \lambda/2\}|$$
$$\le \frac{2}{\lambda}\|f\|_{L^1} + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |Hb(x)| dx.$$
$$\le \frac{2}{\lambda}\|f\|_{L^1} + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} \sum_j |Hb_j(x)| dx$$
$$\le \frac{2}{\lambda}\|f\|_{L^1} + \frac{2}{\lambda}\sum_j \int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)| dx$$

Where in the 3^{rd} ineq. we have used the fact

Ex. $\quad |Hb(x)| \le \sum_j |Hb_j(x)|$. a.e. x.

Thus to complete the proof of the weak (1,1) inequality. it suffices to show

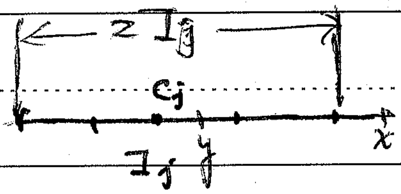
$$\sum_j \int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)| dx \le C\|f\|_1.$$

for this, we note that, even though $b_j \notin \mathcal{S}$, the formula

$$Hb_j(x) = \int_{I_j} \frac{b_j(y)}{x-y}\,dy$$

still holds for $x \notin 2I_j$. Since $\int b_j = 0$, we have

$$\int_{\mathbb{R}\setminus 2I_j} |Hb_j(x)|\,dx = \int_{\mathbb{R}\setminus 2I_j}\left|\int_{I_j}\frac{b_j(y)}{x-y}\,dy\right|dx$$



$$= \int_{\mathbb{R}\setminus 2I_j}\left|\int_{I_j}\left(\frac{1}{x-y}-\frac{1}{x-c_j(I_j)}\right)b_j(y)\,dy\right|dx$$

$$\le \int_{I_j}|b_j(y)|\cdot\left(\int_{\mathbb{R}\setminus 2I_j}\frac{|y-c_j(I_j)|}{|x-y|\cdot|x-c_j(I_j)|}\,dx\right)dy$$

$$\le \cancel{A}\int_{I_j}|b_j(y)|\cdot\left(\int_{\mathbb{R}\setminus 2I_j}\frac{|I_j|}{|x-c_j(I_j)|^2}\,dx\right)dy$$

where in the last inequality, we used $|y-c_j| < |I_j|/2$ and $|x-y| > |x-c_j|/2$. the inner integral is 2, so.

$$\sum_j \int_{\mathbb{R}\setminus 2I_j}|Hb_j(x)|\,dx \le 2\sum_j\int_{I_j}|b_j(y)|\,dy \le 4\|f\|_{L^1}.$$

- Weak (1,1), ok for non-negative f.

  $\Downarrow$

  real-valued function f.

  $\Downarrow$

  complex-valued function.

(2) Since H is weak (1,1) and strong (2,2). By Marcinkiewicz interpolation theorem we have strong (p,p) inequality for $1<p<2$. if $p>2$, we apply (6) and the result for $p<2$

$$\|Hf\|_{L^p} = \sup\{\,|\textstyle\int_{\mathbb{R}} Hf\cdot g| ;\ \|g\|_{L^{p'}} \le 1\}$$

$$= \sup\{\,|\textstyle\int_{\mathbb{R}} f\cdot Hg| :\ \|g\|_{L^{p'}} \le 1\}$$

$$\overset{\text{Hölder}}{\le} \|f\|_{L^p}\sup\{\|Hg\|_{L^{p'}} :\ \|g\|_{L^{p'}} \le 1\}$$

$$\le C_{p'}\|f\|_{L^p}.$$

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by using Inequalities in thm 3, we can extend the Hilbert transform to functions in L^p, $1 \le p < \infty$.

- if $f \in L^1(\mathbb{R}^d)$, ~~then~~ and $\{f_n\} \subset \mathcal{S}$ s.t. $f_n \to f$ in L^1 then by weak (1,1) inequality $\{Hf_n\}$ is a Cauchy sequence in measure. Therefore, it converges in measure to a measurable function, which we define to be the Hilbert transform of f
- if $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and $\{f_n\} \in \mathcal{S}$ s.t. $f_n \to f$ in L^p. then by strong (p,p) Inequality $\{Hf_n\}$ is a Cauchy sequence in ~~measure~~ L^p, so it converges ~~to~~ a function in $L^p(\mathbb{R})$, which is said to be the Hilbert transform of f.

the strong (p,p) inequality is false if $p=1$ or $p=\infty$. This can be seen

Ex. $H\chi_{[0,1]}(x) = \frac{1}{\pi} \log\left|\frac{x}{x-1}\right|$

For Schwartz function, ~~we~~ to require its Hilbert transform to belong to L^1, we have the following characterization:

Ex for $\phi \in \mathcal{S}$, $H\phi \in L^1$ iff $\int \phi = 0$

Truncated integrals and pointwise convergence.

for $\varepsilon > 0$, $y^{-1}\chi_{|y|>\varepsilon} \in L^q(\mathbb{R})$, $1 < q \le \infty$. so

$$H_\varepsilon f(x) = \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy$$

is well-defined for $f \in L^p$, $p \ge 1$. Moreover, H_ε is weak (1,1) and strong (p,p) as in thm 3. with constant $\perp\!\!\!\perp$ ε.

to see this

$$\left(\frac{1}{y}\chi_{|y|>\varepsilon}\right)^{\wedge}(\xi) = \lim_{N\to\infty}\int_{\varepsilon<|y|<N} \frac{e^{-2\pi i y \xi}}{y}\,dy$$

$$= \cancel{\pm}\, 2i\,\mathrm{sgn}(\xi) \lim_{N\to\infty}\int_{2\pi\varepsilon|\xi|}^{2\pi N|\xi|} \frac{\sin t}{t}\,dt$$

- this is uniformly bdd. so strong (2,2) ineq. holds. with constant independent of ε.

$\boxed{\text{Ex}}$

- one can show weak (1,1) as that in thm 3.
- then it follows from interpolation and duality that the strong (p,p) inequalities hold.

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for fixed $f \in L^p$, $1 \le p < \infty$, then the sequence $\{H_\varepsilon f\}$ converges to Hf in $\begin{cases} L^p & \text{if } p>1 \\ \text{mesure} & \text{if } p=1. \end{cases}$

to see this, take $\{f_n\} \subset \mathcal{S}$ s.t. $f_n \to f$ in L^p. then

$$Hf \underset{\substack{\uparrow \\ (p,p)\leftarrow H}}{=} \lim_{n\to\infty} Hf_n = \lim_{n\to\infty}\lim_{\varepsilon\to 0} H_\varepsilon f_n = \lim_{\varepsilon\to 0}\lim_{n\to\infty} H_\varepsilon f_n \underset{\substack{\uparrow \\ (p,p)-H_\varepsilon}}{=} \lim_{\varepsilon\to 0} H_\varepsilon f.$$

both.

This convergence does indeed hold a.e. pointwise.

Thm 4 Given $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then

(10). $$Hf(x) = \lim_{\varepsilon\to 0} H_\varepsilon f(x). \quad \text{a.e. } x \in \mathbb{R}$$

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by the convergence in L^p-norm, we know that (10) holds for some subsequence $\{H_{\varepsilon_k} f\}$. Thus we only need to show that $\lim H_\varepsilon f(x)$ exists for almost every x. Since this is the case for Shwartz functions, by thm 2 in chap. 3. it suffices to show the maximal operator

$$H^*f(x) = \sup_{\varepsilon>0} |H_\varepsilon f(x)|$$

is weak (p,p), which follows from.

thm 5 H^* is strong (p,p), $1<p<\infty$, and weak $(1,1)$.

to prove this we need a lemma which is referred to as

Lemma 6 (Cotlar's Inequality)

if $f \in \mathcal{S}$, then $H^*f(x) \le M(Hf)(x) + C\,Mf(x)$.

Proof. It suffices to show the ineq. with LHS replaced by $H_\varepsilon f(x)$ holds with constant independent of ε.

- Fix $\phi \in \mathcal{S}(\mathbb{R})$, that is non-negative, even, decreasing on $(0,\infty)$ and supported on $\{x \in \mathbb{R};\ |x| < 1/2\}$, $\int \phi = 1$. Let $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(\frac{x}{\varepsilon})$.

Then

$$\frac{1}{y}\chi_{|y|>\varepsilon} = \left(\phi_\varepsilon * p.v.\frac{1}{x}\right)(y) + \left[\frac{1}{y}\chi_{|y|>\varepsilon} - \left(\phi_\varepsilon * p.v.\frac{1}{x}\right)(y)\right]$$

- 1^{st} term $* f(x) \le M(Hf)(x)$.
- for the 2^{nd} term, we only NEED to bound it for $\varepsilon = 1$ by dilation

→ for $|y|>1$,

$$\left| \frac{1}{y} - \int_{|x|<1/2} \frac{\phi(x)}{y-x}\,dx \right| = \left| \int_{|x|<1/2} \phi(x)\left(\frac{1}{y} - \frac{1}{y-x}\right)dx \right|$$

$$\le \int_{|x|<1/2} \frac{|\phi(x)|\,|x|}{|y|\cdot|y-x|}\,dx \le \frac{C}{y^2}$$

→ $|y|<1$,

$$\left| -\lim_{\delta\to0}\int_{|x|>\delta} \frac{\phi(y-x)}{x}\,dx \right| \le \left| \lim_{\delta\to0}\int_{\delta<|x|<2} \frac{\phi(y-x)-\phi(y)}{x}\,dx \right| \le C$$

$\Downarrow$

$$\left| \frac{1}{y}\chi_{\{|y|>\varepsilon\}} - \left(\phi_\varepsilon * p.v.\frac{1}{x}\right)(y) \right| \le \frac{C}{1+y^2}$$

$\Rightarrow$ $2^{nd} * f(x) \le C\,Mf(x)$

#.

thm 5

Proof of ~~Cotlar's Ineq.~~

.. Both the maximal function and the Hilbert transform are strong (p,p), $1<p<\infty$, it then follows from Cotlar's Inequality that H^* is strong (p,p).

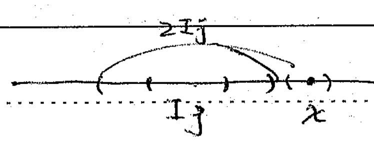
.. To show that H^* is weak $(1,1)$, we argue in the same spirit as in thm 3. 1st assume $f \geq 0$. Fix $\lambda > 0$ and perform C.-Z. decomposition of f at height λ. Then we can write f as

$$f = g + b = \sum b_j + g.$$

- H^* acting on g is strong $(2,2)$.
- Next to show

$$\left|\{x \notin \Omega^*,\ H^*b(x) > \lambda\}\right| \leq \frac{C}{\lambda}\|b\|_{L^1}$$

Fix $x \notin \Omega^*$, $\varepsilon > 0$ and b_j ~~is~~ supported on I_j. Then one of the following holds



(1) $(x-\varepsilon, x+\varepsilon) \cap I_j = I_j$

(2) $(x-\varepsilon, x+\varepsilon) \cap I_j = \phi$

(3) $x-\varepsilon \in I_j$ or $x+\varepsilon \in I_j$

In the 1st case, $H_\varepsilon b_j(x) = 0$.

In the 2nd case, $H_\varepsilon b_j(x) = Hb_j(x)$. Hence,

$$|H_\varepsilon b_j(x)| \underset{\boxed{\int_{I_j} b_j = 0}}{\leq} \int_{I_j}\left|\frac{1}{x-y} - \frac{1}{x-c(I_j)}\right||b_j(y)|\,dy \leq \frac{|I_j|}{|x-c_j|^2}\|b_j\|_{L^1}$$

In the 3rd case, since $x \notin \Omega^*$, $I_j \subset (x-3\varepsilon, x+3\varepsilon)$, and for $y \in I_j$, $|x-y| > \varepsilon/3$. Therefore

$$|H_\varepsilon b_j(x)| \leq \int_{I_j}\frac{|b_j(y)|}{|x-y|}\,dy \leq \frac{3}{\varepsilon}\int_{x-3\varepsilon}^{x+3\varepsilon}|b_j(y)|\,dy$$

Summing over j, yields

$$|H_\varepsilon b(x)| \leq \sum_j \frac{|I_j|}{|x-c_j|^2}\|b_j\|_{L^1} + \frac{3}{\varepsilon}\int_{x-3\varepsilon}^{x+3\varepsilon}|b(y)|dy$$
$$\leq \sum_j \frac{|I_j|}{|x-c_j|^2}\|b_j\|_{L^1} + cMb(x)$$

It then follows from this that

$$\left|\{x\notin\Omega^*: H^*b(x)>\lambda\}\right| \leq \left|\{x\notin\Omega^*: \sum_j \frac{|I_j|}{|x-c_j|^2}\|b_j\|_{L^1} > \lambda/2\}\right|$$
$$+ \left|\{x\in\mathbb{R}: Mb(x)>\lambda/2c\}\right|$$
$$\leq \frac{2}{\lambda}\|b_j\|_{L^1}\sum_j\int_{\mathbb{R}\setminus 2I_j}\frac{|I_j|}{|x-c_j|^2}dx + \frac{c'}{\lambda}\|b\|_{L^1}$$
$$\leq \frac{c''}{\lambda}\|b\|_{L^1}.$$

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In conclusion, we have shown (10). and the strategy is summarized as follows:

(1) $H_\varepsilon f \longrightarrow Hf$ in $\begin{cases} L^p, & 1<p<\infty \\ \text{measure} & p=1 \end{cases}$

(2) $H_\varepsilon f(x) \longrightarrow Hf(x)$ up to subsequence.

(3) $\lim_{\varepsilon\to 0} H_\varepsilon f(x)$ Exists a.e. x.

closedness argument $\boxed{\text{OK}}$

<u>Multipliers</u>

Given a func. $m\in L^\infty(\mathbb{R}^n)$, we define a bounded operator T_m on $L^2(\mathbb{R}^n)$ by

$$(T_m f)\hat{}(\xi) = m(\xi)\hat{f}(\xi) \tag{11}$$

By Plancherel's theorem, for $f\in L^2$, we have

$$\|T_m f\|_{L^2} \leq \|m\|_{L^\infty}\|f\|_{L^2}.$$

claim $\|T_m\|_{L^2\to L^2} = \|m\|_{L^\infty}$

In this case, we say that m is the multiplier of T_m, or we say directly that T_m is a multiplier. When T_m can be extended to a bounded operator on L^p, we say that m is a multiplier on L^p.

In the case of Hilbert transform, $m(\xi) = -i\,\mathrm{sgn}(\xi)$ is a multiplier on L^p. more generally, given $a, b \in \mathbb{R}$, $a<b$, define $m_{a,b}(\xi) = \chi_{(a,b)}(\xi)$. Associate to this multiplier the operator $S_{a,b}$:

$$(S_{a,b}f)^\wedge(\xi) = \chi_{(a,b)}(\xi)\hat{f}(\xi).$$

equivalently,

$$S_{a,b} = \frac{i}{2}\left(M_a H M_{-a} - M_b H M_{-b}\right) \tag{12}$$

where M_a is the operator defined by

$$M_a f(x) = e^{2\pi i a x} f(x)$$

This follows from the calculations

$$\widehat{iM_aHM_{-a}f}(\xi) = i(HM_{-a}f)^\wedge(\xi-a)$$
$$= \mathrm{sgn}(\xi-a)(M_{-a}f)^\wedge(\xi-a)$$
$$= \mathrm{sgn}(\xi-a)\hat{f}(\xi).$$

M_a is clearly bdd on L^p, $1\le p\le\infty$. therefore, from (12) and strong (p,p) inequality for H, we conclude that $S_{a,b}$ is bounded on L^p, $1<p<\infty$. Or more generally,

__Prop 7.__ there exists constant C_p, for $1<p<\infty$, such that for all $a, b \in [-\infty, \infty]$ with $a<b$,

$$\|S_{a,b}f\|_{L^p} \le C_p\|f\|_{L^p}$$

For application, let $a=-R$, $b=R$, then $S_{a,b}f = S_R f = D_R * f$, where D_R is the Dirichlet kernel. We thus have

$$\|S_R f\|_{L^p} \le C_p \|f\|_{L^p}.$$

By U.B.P. we get

Cor. 8 For $f \in L^p(\mathbb{R})$, $1<p<\infty$, there holds

$$\lim_{R\to\infty} \|S_R f - f\|_{L^p} = 0$$

However, for $p=1$, we only have convergence in measure:

$$\lim_{R\to\infty} \left|\{x\in\mathbb{R} \,/\, |S_R f(x) - f(x)| > \varepsilon\}\right| = 0.$$

Consequently, we have

$$S_R f(x) \longrightarrow f \quad a.e.\ x,$$

up to subsequence. (depending on f).

Given a family of uniformly bdd operators on L^p, any convex combination of them is also bounded. Hence from proposition 7, we can prove

Cor 9: if m is a function of bounded variation on $\mathbb{R}$, then m is a multiplier on L^p, $1<p<\infty$.

Proof:

- Since m is of bounded variation, $\lim_{t\to-\infty} m(t)$ exists. By adding a constant to m if necessary, we may assume this limit equals 0.
- By normalization, we can assume m is right continuous at each $x \in \mathbb{R}$.

let dm denote the Lebesgue-Stieltjes measure associated with m.
then,

$$m(\xi)=\int_{-\infty}^{\xi} dm(t) = \int_{\mathbb{R}} \chi_{(-\infty,\xi)}(t)\,dm(t) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi)\,dm(t)$$

Therefore

$$\widehat{T_m f}(\xi) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi)\,\hat{f}(\xi)\,dm(t)$$

and hence

$$T_m f(x) = \int_{\mathbb{R}} S_{t,\infty} f(x)\,dm(t)$$

$\Downarrow$ Minkowski's Ineq.

$$\|T_m f\|_{L^p} \le \int_{\mathbb{R}} \|S_{t,\infty} f\|_{L^p} |dm|(t) \le C_p \|f\|_{L^p} \int_{\mathbb{R}} |dm|(t).$$

#.

Given a multiplier, we can contruct others by translato dilation and rotation.

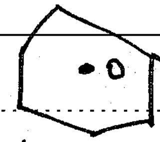
Prop 10 let m be a multiplier on $L^p(\mathbb{R}^n)$, then the functions defined by $m(\xi+a)$, $a\in\mathbb{R}^n$, $m(\lambda\xi)$, $\lambda>0$, and $m(\rho\xi)$, $\rho\in O(n)$, are multiplers of bdd operators on L^p with the same norm as T_m.

EX

- If m is a multiplier on $L^p(\mathbb{R})$, then the function on $\mathbb{R}^n$ defined by $\tilde{m}(\xi)=m(\xi_1)$ is a multiplier on $L^p(\mathbb{R}^n)$.

if we take $m=\chi_{(0,\infty)}$, then by Prop. 7, m is a multiplier on $L^p(\mathbb{R})$, $1<p<\infty$. As a consequence of Prop. 10, $\chi_{\{\xi\in\mathbb{R}^n:\,\xi_1>0\}}$ will be a multiplier on $L^p(\mathbb{R}^n)$.

~~and~~, its rotato dilation and translato.



We finally can state

Cor 11 If $P \subset \mathbb{R}^n$ is a convex polyhedron that contains the origin, then

$$\lim_{\lambda\to\infty} \|S_{\lambda P} f - f\|_{L^p} = 0, \quad 1<p<\infty.$$

where $S_{\lambda P}$ is the operator whose multiplier is the characteristic function of $\lambda P = \{\lambda x : x \in P\}$.