Ans to Week 8

Ex1:

the steps listed there are just the Strategies for the proof.

Ex2:

(1) a key feature of Radon measure is locally finite. then by showing

$$p.v.\frac{1}{x}([0,1]) = \lim_{\varepsilon\to 0} \log\frac{1}{\varepsilon} = \infty.$$

One can conclude, ...

(2)

$$x \cdot p.v.\frac{1}{x}(\phi) = p.v.\frac{1}{x}(x\phi)$$

$$= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{x\phi(x)}{x}\,dx$$

$$= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \phi(x)\,dx = \int \phi(x)\,dx$$

(3) the key is to show:

$$\delta_0(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\,\frac{\varepsilon}{x^2+\varepsilon^2}$$

$$\delta_0 * f(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int \frac{\varepsilon}{(x-y)^2+\varepsilon^2} f(y)\,dy$$

$$\| $$

$$f(x)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int \frac{f(\varepsilon y)}{(\frac{x}{\varepsilon} - y)^2 + 1} dy$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int \frac{f(x + \varepsilon y)}{1 + y^2} dy$$

$$\overset{DCT}{=} f(x) \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+y^2} dy$$

$$= f(x).$$

Next, to show $\frac{x}{x^2+\varepsilon^2} \xrightarrow{\varepsilon \to 0+} \text{p.v.} \frac{1}{x}$.

Fix $\phi \in \mathcal{S}(\mathbb{R})$. Suffices to show

$$\int \frac{x}{x^2+\varepsilon^2} \phi(x) \xrightarrow{\varepsilon \to 0} \text{p.v.} \frac{1}{x}(\phi)$$

$$\text{p.v.} \frac{1}{x}(\phi) = \lim_{\delta \to 0} \int_{|x| > \delta} \frac{\phi(x)}{x} dx$$

$\Longleftrightarrow$

$$\int \frac{x}{\varepsilon^2 + x^2} \phi(x) - \int_{|x| > \varepsilon} \frac{\phi(x)}{x} dx \xrightarrow{\varepsilon \to 0+} 0$$

$$\| $$

$$\int_{|x| \le \varepsilon} \frac{x}{\varepsilon^2 + x^2} \phi(x) + \int_{|x| > \varepsilon} \left(\frac{x}{x^2 + \varepsilon^2} - \frac{1}{x} \right) \phi(x) \, dx$$

$$=: I + II.$$

$$\mathrm{I} = \int_{|\frac{x}{\varepsilon}|<1} \frac{x/\varepsilon}{(\frac{x}{\varepsilon})^2+1}\,\phi(\varepsilon\,\tfrac{x}{\varepsilon})\,d\tfrac{x}{\varepsilon}$$

$$= \int_{|x|<1} \frac{x}{1+x^2}\,\phi(\varepsilon x)\,dx$$

$$\xrightarrow[DCT]{\varepsilon\to 0^+} \int_{|x|<1} \underbrace{\frac{x}{1+x^2}}_{\text{odd.}}\,\phi(0)\,dx$$

$$= 0.$$

$$\mathrm{II} = \int_{|x|>\varepsilon} \left(\frac{x}{x^2+\varepsilon^2} - \frac{1}{x}\right)\phi(x)$$

$$= \int_{|x|>\varepsilon} \frac{x^2-x^2-\varepsilon^2}{x(x^2+\varepsilon^2)}\,\phi(x)\,dx$$

$$= -\int_{|\frac{x}{\varepsilon}|>1} \frac{\cancel{\phi}\,\cancel{\varepsilon^2}\,1}{\frac{x}{\varepsilon}\left((\frac{x}{\varepsilon})^2+1\right)}\,\phi(x)\,d\tfrac{x}{\varepsilon}$$

$$\xrightarrow{\varepsilon\to 0^+} -\int_{|x|>1} \frac{\phi(0)}{x(1+x^2)}\,dx = 0.$$

the remaining part is the formal calculation. #