Ex. 1 in the class time, we showed that the coefficients

$$c_k(g) = \int_0^1 g(x) e^{-2\pi i k x} dx$$

decays (arbitrarily) rapidly (only polynomial) if the function g is of smooth class $C^\infty(\mathbb{R}/\mathbb{Z})$. this exercise is designed to see ~~the~~ how the regularity affects the decay rate.

(i) if g is exactly from $C^k(\mathbb{T})$, then what's the decay rate of its fourier coefficients?

Using Integration by parts, try to find out the obstacles?

(ii). Say a continuous function f is of α-Holder, $\alpha \in (0,1)$ if

$$\sup_{\substack{x,y \in \mathbb{T} \\ x \neq y}} \frac{|f(x) - f(y)|}{\mathrm{dist}(x,y)^\alpha} < \infty, \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

$\mathrm{dist}(x,y) := \min(|x-y|, |x+1-y|)$

Then. if $f \in \alpha$-Hölder$(\mathbb{R}/\mathbb{Z})$,

whats the possible decay rate of $c_k(f)$??

(*) (iii) Is there some nontrivial func. f, defined on $\mathbb{R}/\mathbb{Z}$, s.t. $\{c_k(f)\}$ decays exponentially?? other than trigonometric polynomials??

Ex. 2 Recall from the class that $D_N(t)$ is the Dirichlet kernel. Define (at least formally) the operator

$$T_N f(x) := D_N * f(x).$$

where f is a function defined on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

(i) show that T_N, ~~when~~ is a bounded operator from $L^1(\mathbb{T})$ to $L^1(\mathbb{T})$, with the operator norm given by $\int_0^1 |D_N(t)| dt =: L_N$

(ii). let $X := (C(\mathbb{T}), \|\cdot\|_\infty)$, $Y = \mathbb{C}$, define an operator

$$\tilde{T}_N : X \longrightarrow Y$$
$$f \longmapsto T_N f(0).$$

Show that $\|\tilde{T}_N\|_{X \to Y} = L_N$

(iii). is T_N, as an operator from $L^\infty(\mathbb{T})$ to $L^\infty(\mathbb{T})$, bounded?. if it is bdd, what's its operator norm?

(*) **Pb. I**: In order to understand the difference between the question (III) and (VI), please give examples, that tell the differences between the convergence in norm and pointwise (a.e.) convergence??

Is there some "operator norm" characterisation of question (VI)??

I there any linkage between (III) and (VI)??

(*): means I DO NOT KNOW THE ANS.