MIDTERM EXAM — ARA 2018 FALL 45

TIME: 15:55 — 18:00, NOVEMBER 9TH, 2018 LECTURE ROOM 5306 CLOSED-BOOK

Write your answers by order in the independent answer sheet, which you have to hand back together with this exam sheet in the end. For any questions, you need to state the reason rigorously, except with particular indication.

"i.e." means "id est" or "in another word/that is". "s.t." means "such that". "a.e." means almost everywhere. By "$A \lesssim B$", we mean $A \le CB$, for some C, which we do not care in the context. All functions are complex-valued, measurable, and are finite, a.e. in a measure space, except with particular indication. $\|\cdot\|_p := (\int_\Omega |\cdot|^p d\mu)^{1/p}$ denotes the standard scale in L^p $(p \neq 0)$ space, on a measurable subset Ω of $\mathbb{R}^n$. $f_j \xrightarrow{L^p} f$ means the strong convergence. For any nonzero exponent p, the exponent p' always denote the conjugate exponent, such that $1/p + 1/p' = 1$.

PROBLEMS

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Problem 1 (10'). (1) *Let Ω be a set, $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a subalgebra, and $\Sigma = \sigma(\mathcal{A})$, the smallest sigma-algebra, generated by $\mathcal{A}$. If μ is a $\mathcal{A}-$strong sigma-finite measure, and ν is another measure s.t. $\nu(A) = \mu(A), \forall A \in \mathcal{A}$, then prove that $\mu = \nu$.*

(2) *State the definition of **product measure** and prove the **Fubini theorem**.*

(3) *Is it necessary for the measures to be sigma-finite in (1) and (2)?*

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Problem 2 (12'). (1) *Let $f_j \to f$, a.e. (pointwise a.e. convergent). If there exists a strong convergent dominating sequence F_j, s.t. $|f_j| \le F_j \xrightarrow{L^1} F \in L^1$, then prove that, $f, f_j \in L^1$, and*

$$\int f = \lim_j \int f_j .$$

(2) *Is there any uniform bounded, integrable, pointwise a.e. convergent sequence, i.e. $f_j \to f$, a.e., with $\sup_j \|f_j\|_1 + \sup_{j,x} |f_j| < \infty$, which*

(a) *however does NOT satisfy the interchangement of integral and limit?*

(b) *satisfies the interchangement of integral and limit, but has NO dominating sequence?*

(c) *has a dominating sequence, but has NO (nonnegative, integrable) **dominating function**?*

Problem 3 (14'). (1) *Let $J : \mathbb{C} \to \mathbb{R}$, continuous, convex, and $J(0) = 0$. Prove that, for any small $\epsilon > 0$, there exists two nonnegative, continuous functions ϕ_ϵ and ψ_ϵ, s.t., for any $a, b \in \mathbb{C}$,*

$$|J(a+b) - J(a)| \le \epsilon\, \phi_\epsilon(a) + \psi_\epsilon(b) .$$

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(2) *Let $f_j \to f$, a.e.. If $J(cf) \in L^1$ for any $c \in \mathbb{R}$, and there exists some $k \in \mathbb{N}$, $k \geq 2$, s.t., $\sup_j \|[J(k\cdot) - kJ(\cdot)] \circ (f_j - f)\|_1 < \infty$, then prove that*

$$\int |J(f_j) - J(f) - J(f_j - f)| \to 0.$$

(3) *Can we eliminate the uniform assumption in (2), or weaken it to $\sup_j \|J(f_j - f)\|_1 < \infty$?*

(4) *Let $\{f_j\} \subset L^p$, $p \in (0, \infty)$, $f_j \to f$, a.e., and $\{\|f_j\|_p\}$ converges, then does this imply $f_j \xrightarrow{L^p} f$?*

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Problem 4 (18'). (1) (a) *State the definition of **convergence by measure**, denoted by $f_j \xrightarrow{\mu} f$, and (pointwise) **almost uniform convergence**, denoted by $f_j \rightrightarrows_{alm} f$.*

(b) *Prove the **Egorov theorem**. Is there any sequence, s.t. $f_j \to f$, a.e., and $f_j \xrightarrow{\mu} f$, but f_j is not almost uniformly convergent ?*

(c) *Prove the **Riesz subsequence lemma** (i.e., if $f_j \xrightarrow{\mu} f$, then there exists a subsequence, still denoted by $\{f_j\}$, s.t., $f_j \to f$, a.e.). Is there any sequence, $f_j \xrightarrow{\mu} f$, but f_j is not pointwise convergent at any point?*

(2) *Let $1 < p < \infty$. Prove that, if $f_j \rightharpoonup f$ (**weak convergent** in L^p), and $\|f_j\|_p \to \|f\|_p$, then $f_j \xrightarrow{L^p} f$. Is there any weak convergent sequence, which*

(a) *is NOT strong convergent, and also is NOT pointwise convergent at any point?*

(b) *is NOT strong convergent, but is pointwise a.e. convergent, and $\lim_j \|f_j\|_p$ exists?*

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Problem 5 (12'). *Let $p \in (0,1)$. Is L^p a **normed linear space**? Is it a **complete** linear space? Is there any **logarithmic convexity** for the scale $\|\cdot\|_p$?*

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Problem 6 (12'). (1) *For $1 \leq p \leq 2$, prove that*

$$\|f+g\|_p^{p'} + \|f-g\|_p^{p'} \leq 2\,(\|f\|_p^p + \|g\|_p^p)^{p'/p}.$$

Using Hanner inequality directly, you might only get partial points.

(2) *A normed linear space $(X, \|\cdot\|)$ is called **uniformly convex** if*

$$\delta(\epsilon) := \inf\{1 - \|(x+y)/2\| : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\} > 0.$$

Furthermore, it is called $p-$order uniformly convex, if $\delta(\epsilon) \gtrsim \epsilon^p$.

Prove that L^p $(1 < p < \infty)$ is $(\max\{p, p'\})-$order uniformly convex, using the inequality above.

(3) *Show that, for some p at least, L^p is not $(2-\epsilon)$-order uniformly convex, for any small $\epsilon > 0$.*

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Problem 7 (12'). (1) *State the **projection theorem** on a closed convex subset in L^p for any $p \in (1, \infty)$ (without proof). Is the projection unique?*

(2) *Using the projection theorem to prove the **Ascoli theorem** in real L^p: for any closed convex proper subset $K \subsetneq L^p$, and any $f \notin K$, there is a (real) bounded linear functional $L \in (L^p)^*$, strictly separating f and K, i.e., there exists a constant $\alpha \in \mathbb{R}$, s.t., $L(f) > \alpha > L(g)$, $\forall\, g \in K$.*

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Problem 8 (10'). *Let $p \in [1, \infty)$. State and prove the theorem of **identity approximation** of L^p functions by smooth functions with compact support, using convolution (i.e., $\overline{C_c^\infty} = L^p$).*

Step1. μ有限

1. (1) $\mathcal{M} \triangleq \{A \in \Sigma : \mu(A) = \nu(A)\}$

- $\Omega \subset \mathcal{M}$ (σ-finite)
- $\mathcal{M}$：单调类

Step2. 一般

$\forall A \in \mathcal{A}, B \in \Sigma, \mu(A \cap B) = \nu(A \cap B)$.

定义 $\mu_0 = \mu|_{\mathcal{A}}, \nu_0 = \nu|_{\mathcal{A}}$,

$\mu_0 = \mu(A \cap C)$

由 $\mathcal{A}$ 强 σ-finite, $\forall B \in \Sigma, \mu(B) = \lim \mu(B \cap A_i) = \lim \nu(B \cap A_i) = \nu(B)$.

(2) $(\Omega_i, \mu_i, \Sigma_i) \Rightarrow (\Omega_1 \times \Omega_2, \tilde{\Sigma}, \tilde{\mu})$

$\Omega_1 \times \Omega_2 \Rightarrow \{A \times B : A \in \Sigma_1, B \in \Sigma_2\}$ 构成alg.

$\tilde{\Sigma} = \sigma(\{A \times B\})$.

$\tilde{\mu} : \tilde{\mu}(A \times B) \triangleq \mu_1(A) \times \mu_2(B)$

$\forall A \subset \tilde{\Sigma}, \forall y \in \Omega_2, A_1(y) = \{x \in \Omega_1 : (x,y) \in A\}$

Prop 1. $A_1(y) \in \Sigma_1, \forall y \in \Omega_2$.

2. $\int_{\Omega_2} \mu_1(A_1(y)) d\mu_2(y) = \int_{\Omega_1} \mu_2(A_2(x)) d\mu_1(x) \triangleq \tilde{\mu}(A)$

乘积测度结合/交换性证Fubini

(3) Fubini : $\{(x,x)\}$，一个 Lebesgue 测度，一个 counting 测度.

(c.e.g. for A 不可测：$(\mathbb{R}, \ll)$，其中偏序关系满足，$\forall y \in \mathbb{R}, \{x \in \mathbb{R} : x \ll y\}$ 可数.
考虑 $A = \{(x,y) : x \ll y\} \subset I \times I$
但 $\mu_1(A_1(y)) = 0, \mu_2(A_2(x)) = 1, \forall x, y \in I$，$\mu_i$ 为 Lebesgue measure.)

2. (1) 对 $F + F_j - |f - f_j|$ 用 Fatou. (Check 它为正，由 Riesz Lemma.)

$$\int \varliminf (F + F_j - |f - f_j|) \le \varliminf \int F + F_j - |f - f_j|$$

$$\downarrow$$

$$2\int F - \varlimsup \int |f - f_j|$$

仅证：$\varlimsup \int |f - f_j| = a > 0$.

且 $F_{k_j} \xrightarrow[a.e]{L^1} F$.

则 $2\int F \le 2\int F - a$. 矛盾.

由 扫描全能王 扫描创建

(2) $f_k = \sin kx$ on $[0,1]$

$f_k = \chi_{[k,k+1]} \longrightarrow$ (a).

$f_k = \frac{1}{k}\chi_{[0,k]}$

$f_k = k\chi_{[0,\frac{1}{k}]}$

(b) $f_k = \chi_{[k,k+1]} - \chi_{[k-1,k]}$.

若 $|f_k(x)| \le F_k(x) \xrightarrow{L^1} F \in L^1$

$$\int F = \sum_{j=0}^{\infty}\int_j^{j+1} F = \sum_{j=0}^{\infty}\int_j^{j+1} F_j + \int_j^{j+1}(F-F_j)$$

$$= \sum_{j=0}^{\infty} 1 + o(1) = \infty.\quad 矛盾.$$

(c) $f_j = \frac{1}{j}\chi_{[j,j+1]}$.

若 $\exists F$, s.t. $F \ge f_j \Rightarrow \int F \ge \sum_{j\ge 1}\frac{1}{j} = \infty$.

4. (1) (a) $f_j \xrightarrow{\mu} f$: $\forall \varepsilon > 0$, $\mu(\{|f_j - f| \ge \varepsilon\}) \to 0$

$f_j \xrightarrow{a.e.} f$: $\forall \varepsilon > 0$, $\mu\left(\bigcap_{n=1}^{\infty}\bigcup_{j\ge n}\{|f_j - f| \ge \varepsilon\}\right) = 0$

$f_j \rightrightarrows_{alm} f$: $\forall \varepsilon > 0$, $\lim_{n\to\infty}\mu\left(\bigcup_{j\ge n}\{|f_j - f| \ge \varepsilon\}\right) = 0$.

(b) 略. c.e.g. $f_j = \chi_{[j, j+\frac{1}{j}]}$.

(c) $\exists f_{k_i}$, s.t. $\mu(\{|f_{k_i} - f| > \frac{1}{2^i}\}) < \frac{1}{2^i}$ $\Rightarrow f_{k_i} \xrightarrow{a.e.} f$.

c.e.g. $f_{2^k+j} = \chi_{[\frac{j}{2^k}, \frac{j+1}{2^k}]}$, $0 \le j < 2^k$

(2) <u>Thm</u>. X: 一致凸 Banach, $f_j \rightharpoonup f$, $\|f_j\| \to \|f\|$, 则 $f_j \xrightarrow{\|\cdot\|} f$.

Pf: 不妨 $f \ne 0$, $g_j = \frac{f_j}{\|f_j\|} \rightharpoonup \frac{f}{\|f\|} = g$.

$\therefore \left\|\frac{g_j + g}{2}\right\| \to 1$ (范数下半连续性).

由一致凸, $\|g_j - g\| \to 0$

c.e.g. (a) $f_k = \sin kx$ on $[0,1]$.

(b) $f_k = \chi_{[k,k+1]}$.

c.e.g <u>$p=1$ 不对</u>: $f_k = 1+\sin kx$. (?)

$f_j \rightharpoonup f \not\Rightarrow f_j \xrightarrow{a.e.} f$: $\sin jx$

$f_j \xrightarrow{a.e.} f \not\Rightarrow f_j \rightharpoonup f$: $\chi_{[0,j]}$.

<u>Thm</u>. $u_n \in L^p(\mathbb{R}^n)$, $1<p<\infty$. converging a.e. ~~in~~ in measure to u. with $\sup\|u_n\|_p < C$ then $u \in L^p$, $u_n \rightharpoonup u$. (Hint: $\int_{A, \mu(A)<\infty}(u_n v - uv) = \left(\int_{A_\delta (绝)} + \int_{A\setminus A_\delta (一致收)}\right) < \varepsilon$.)

5. **prop** 1. $\|fg\|_1 \geq \|f\|_p \|g\|_{p'}$, $p\in(0,1)$.

pf: $\int |f|^p = \int |fg|^p |g|^{-p}$

$$\overset{\text{Hölder}}{\leq} \left(\int |fg|\right)^p \left(\int |g|^{p'}\right)^{1-p} \quad \#$$

2. $\left(\int |f|^p\right)^{\frac{1}{p}} + \left(\int |g|^p\right)^{\frac{1}{p}} \leq \left(\int |f+g|^p\right)^{\frac{1}{p}}$, $f,g\geq 0$.

pf: $\int |f+g|^p = \int |f+g|^{p-1}(|f|+|g|)$

$$\overset{\text{反Hölder}}{\leq} \left(\left(\int |f|^p\right)^{\frac{1}{p}} + \left(\int |g|^p\right)^{\frac{1}{p}}\right)\left(\int |f+g|^p\right)^{\frac{p-1}{p}} \quad \#$$

$\Rightarrow \|f+g\|_p \geq \|f\|_p + \|g\|_p$.

$d(f,g)=\int |f-g|^p$：正定，交换，三角

$(a+b)^p \leq a^p + b^p$, $p\in(0,1)$, $a,b\geq 0$.

$(\because\ 1 \leq \left(\frac{a}{a+b}\right)^p + \left(\frac{b}{a+b}\right)^p$.

log-凸：是否有 $0<p,q,r<1$, $\frac{1}{r}=\frac{\theta}{p}+\frac{1-\theta}{q}$, $\|f\|_r \geq \|f\|_p^\theta \|f\|_q^{1-\theta}$?

pf: $\int |f|^r = \int |f|^{r\theta} |f|^{r(1-\theta)} \overset{\text{反Hölder}}{\geq} \left(\int |f|^p\right)^{\frac{r\theta}{p}} \left(\int |f|^q\right)^{\frac{r(1-\theta)}{q}} \quad \#$

prop. $L^p(\mathbb{R}^n)$ $(0<p<1)$ contains no convex open set other than ϕ and $\mathbb{R}^n$.

pf: 设 $\phi \neq V \subset L^p$ 开凸. 不妨 $0\in V$, $\exists r>0$, s.t. $B_r \subset V$.

$\forall f\in L^p$, $\exists n\geq 1$, s.t. $n^{p-1}\int |f|^p < r$. (由 $p<1$).

$\exists E_i$, $\mathbb{R}^n = \bigsqcup_{i=1}^n E_i$, s.t. $\int_{E_i} |f|^p = \frac{1}{n}\int_{\mathbb{R}^n} |f|^p$

令 $g_i \triangleq n\chi_{E_i} f$，则 $f=\sum \frac{1}{n} g_i$, $g_i\in B_r$. $(\because \int |g_i|^p = n^{p-1}\int |f|^p < r)$. #.

Cor. $0<p<1$. $f\in (L^p)^*$，则 $f\equiv 0$.

pf: $\forall \varepsilon>0$, $f^{-1}((-\varepsilon,\varepsilon))$ 开凸.非空 $\Rightarrow f^{-1}((-\varepsilon,\varepsilon)) = L^p \overset{\forall\varepsilon}{\Rightarrow} f\equiv 0$. #.

6. **prop**. $|A+B|^p + |A-B|^p \geq \alpha(r)|A|^p + \beta(r)|B|^p$, $\forall r\in[0,1]$, $1\leq p\leq 2$.

其中 $\alpha(r)=(1+r)^{p-1}+(1-r)^{p-1}$ $\quad \beta(r)=[(1+r)^{p-1}-(1-r)^{p-1}]\, r^{1-p}$.

积分得 $\|f+g\|_p^p + \|f-g\|_p^p \geq \alpha(r)\|f\|_p^p + \beta(r)\|g\|_p^p$

(1) 不妨 $\|g\|_p \leq \|f\|_p$, $r=\left(\|g\|_p / \|f\|_p\right)^{p'}$ 得证.

(2) $1\leq p\leq 2$ 时, $\|f\|=\|g\|=1$, $\|f-g\|_p=\varepsilon$.

$\left\|\frac{f+g}{2}\right\|_p^{p'} + \left\|\frac{f-g}{2}\right\|_p^{p'} \leq 1 \Rightarrow \left\|\frac{f+g}{2}\right\|_p^{p'} \leq (1-C\varepsilon^p)^{\frac{1}{p'}} \leq 1-\tilde{C}\varepsilon^{p'}$

$2\leq p\leq\infty$ 时, (1) 结论反

$f\leftarrow \frac{f+g}{2}$, $g\leftarrow \frac{f-g}{2}$ 时

$\Rightarrow \|f\|_p^{p'} + \|g\|_p^{p'} \geq 2\left(\left\|\frac{f+g}{2}\right\|_p^{p'} + \left\|\frac{f-g}{2}\right\|_p^{p'}\right)$

Prop. L^2: $\left\|\frac{f+g}{2}\right\|_2^2+\left\|\frac{f-g}{2}\right\|_2^2=\frac{1}{2}\left(\|f\|_2^2+\|g\|_2^2\right)$

7.(1) 一致凸 $\Rightarrow$ 投影唯一

i.e. 闭凸 C, $0\notin C$, $\exists 1\, x_0\in C$, $\|x_0\|=\inf\limits_{x\in C}\|x\|$.

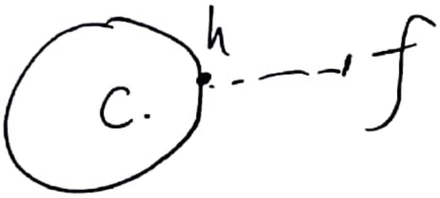
已有∃性，下证唯一性.

$\exists\ x_0, x_1\in C$, s.t. $\|x_0\|=\|x_1\|=\inf\limits_{x\in C}\|x\|=d>0$.

若 $\|x_0-x_1\|\geq\varepsilon$, 则: $1-\left\|\frac{\frac{x_0}{d}+\frac{x_1}{d}}{2}\right\|>0$ (一致凸)

$\Rightarrow \frac{x_0+x_1}{2}\in C$, $\left\|\frac{x_0+x_1}{2}\right\|<d$ 矛盾.

(2)



$L(g)\triangleq \mathrm{Re}\int g\,|f-h|^{p-2}\,\overline{(f-h)}$

$L(g-h)\leq 0.$

$L(f-h)=\int|f-h|^p>0.$

8. 略.