Ans, to Week 10.

(1)

(a) is easy.

(b). By the assumption, we need to consider the asymptotic behavior of

$$K_{a,b}(x) = \int \phi(\xi) \frac{e^{i|\xi|^a}}{|\xi|^b} e^{ix\cdot\xi} d\xi$$

- $\phi(\xi)$ vanish around zero. $\Longrightarrow$ the amplitude function

$$\phi(\xi)\cdot / |\xi|^b \text{ is indeed smooth.}$$

- let's next consider the phase function fix x, small, around the origin

$$f_x(\xi) = |\xi|^a + x\cdot\xi$$

the critical point is the point ξ_0 satisfying

$$\nabla_\xi f_x(\xi) = a|\xi|^{a-1}\frac{\xi}{|\xi|} + x = 0.$$

from this, we can solve

$$\xi_0 = -x\cdot|x|^{-1-\frac{1}{1-a}} = -x|x|^{-\frac{2-a}{1-a}}$$

or. $x = -\dfrac{a\xi_0}{|\xi_0|^{2-a}}$

the Hessian of f at $\xi = \xi_0$ is

$$\nabla^2_\xi f_x(\xi_0) = \frac{a\, Id}{|\xi|^{2-a}} - a(2-a)\frac{1}{|\xi|^{2-a}} \frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}$$

$$= \frac{a}{|\xi|^{2-a}}\left(Id - (2-a)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)$$

at ξ_0

claim: $a \in (0,1) \Rightarrow$
the matrix

$$\left(Id - (2-a)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right) \text{ is}$$

non-degenerate (at least for ξ_0)
and the determinant has its absolute value strictly bigger than zero, say bigger than some constant $c_0 > 0$.

- I deduce this claim via an "geometric" consideration, by rotating the coordin. (NOT change the Hessian) so that $\frac{\xi_1}{|\xi|} \neq 0$ and reduce the problem to lower dimensional case. Interested student can try to prove this by techiques from linear algebra.

thus, we can apply the "square root" law ~~from~~ from the theory of oscillatory integrals.

- the contribution to the size of $K_{a,b}(x)$ from $\phi(\xi)/|\xi|^b$ is
$$\frac{1}{|\xi_0|^b} \approx |x|^{\frac{b}{1-a}}$$

- the contribution from the non-degeneracy of the phase function at the critical point ξ_0 is
$$\left(\frac{1}{|\xi_0|^{2-a}}\right)^{-\frac{n}{2}}$$
$$= \left(|x|^{\frac{1}{1-a}\cdot(2-a)}\right)^{-\frac{n}{2}}$$
$$= \frac{1}{|x|^n \cdot |x|^{\frac{\frac{na}{2}}{1-a}}}$$

Combing these two points gives the ~~decay~~ asymptotic behavior.

(c) this can be checked as in (b), but with some gradient...

(2) this follows from a contradiction argument, since the boundedness of the Fourier transform of the kernel is naturally true.

(3). by substituting the kernel asy. (1).(b). to verify the condition in the theorem.