Interpolation:

(X,μ) (Y,μ) be a pair of measure spaces.

Riesz-Thorin interpolation Thm:

Let T be a linear operator with domain $L^{p_0}+L^{p_1}$ $1\le p_0<p_1\le\infty$, satisfying

$$\begin{cases} \|T\|_{(p_0\to q_0)} \le A_0 \\ \|T\|_{(p_1\to q_1)} \le A_1 \end{cases}$$

$\forall\theta\in(0,1)$ Let $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$ then

$$\|T\|_{(p,q)} \le A_0^{1-\theta}A_1^{\theta}$$

证明可见《实分析》Stein 2.2节

Some applications

1. (Hausdorff-Young) If $1\le p\le 2$ then $\|\hat{f}\|_{p'}\le\|f\|_p$. 此处 $\mathcal{F}(f) = \hat{f}(\xi)=\int f(x)e^{-2\pi i x\xi}dx$ for $f\in\mathcal{S}$ 可唯一延拓到 L^2 函数

Pf: $\begin{cases} \|\mathcal{F}(f)\|_\infty \le \|f\|_1 \\ \|\mathcal{F}(f)\|_2 = \|f\|_2 \end{cases} \implies \|\mathcal{F}(f)\|_q\le\|f\|_p$ for $\frac{1}{p}=1-\theta+\frac{\theta}{2}$, $\frac{1}{q}=0+\frac{\theta}{2}$

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2. (Young's inequality) Let $\phi\in L^p$, $\psi\in L^q$ $1\le p,q\le\infty$ then

$$\|\phi*\psi\|_r\le\|\phi\|_p\|\psi\|_q \quad \text{where } 1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$$

Pf: Fix $\phi\in L^p$. Consider the operator $T\psi=\phi*\psi$ then

$\|T\psi\|_p\le\|\phi\|_p\|\psi\|_1$ (Minkowski inequality)

$\|T\psi\|_\infty\le\|\phi\|_p\|\psi\|_{p'}$ (Hölder inequality)

It's easy to check ~~$\theta = 1 - p/q$~~ $\theta = 1 - \frac{p}{n} \in [0,1]$ satisfying

$$\begin{cases} \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{\infty} \\ q = \frac{1-\theta}{1} + \frac{\theta}{p'} \end{cases}$$

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3. $\left([0,2\pi], \frac{1}{2\pi}d\theta\right)$, $(\mathbb{Z}$, counting measure$)$

$T(f) = \{a_n\}$ $\quad a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta} d\theta$

Prop $1 \le p \le 2$ then $\|Tf\|_{L^{p'}(\mathbb{Z})} \le \|f\|_{L^p([0,2\pi])}$

Pf. $\begin{cases} \|T\|_{(1\to\infty)} \le 1 \\ \|T\|_{(2\to 2)} = 1 \end{cases} \Longrightarrow \|T\|_{p'\to p} \le 1.$

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4. $1 \le p \le 2$, $\left(\frac{|a+b|^{p'} + |a-b|^{p'}}{2}\right)^{\frac{1}{p'}} \le 2^{1/p}\left(\frac{|a|^p + |b|^p}{2}\right)^{\frac{1}{p}}$, $\forall a, b \in \mathbb{C}$

Rmk. when $p=1$, $p' = \infty$ LHS $= \max\{|a+b|, |a-b|\}$

Pf: Let l^p denotes the normalized two point counting measure space

$T(a,b) = (a+b, a-b)$ We need to show $\|T\|_{p\to p'} \le 2^{1/p}$

$$\begin{cases} \|T\|_{1\to\infty} \le 2 \\ \|T\|_{(2\to 2)} = 1 \end{cases} \xrightarrow[\text{Riesz-Thorin}]{} \|T\|_{p\to p'} \le 2^{1/p}.$$

Rmk. $2 \le p \le \infty$ 使不等号反向

5. Use interpolation to prove the Clarkson's inequality.

$1 \le p \le 2$. $\|f+g\|_p^{p'} + \|f-g\|_p^{p'} \le 2(\|f\|_p^p + \|g\|_p^p)^{p'/p}$

Which is equivalent to $\left(\frac{\|f+g\|_p^{p'} + \|f-g\|_p^{p'}}{2}\right)^{\frac{1}{p'}} \le 2^{1/p}\left(\frac{\|f\|_p^p + \|g\|_p^p}{2}\right)^{\frac{1}{p}}$

~~Note that~~ $\cancel{\left(\frac{|a|^{p'}+|b|^{p'}}{2}\right)^{\frac{1}{p'}} \le \left(\frac{|a|^p+|b|^p}{2}\right)^{\frac{1}{p}}}$ ~~(Recall~~

Minkowski inequality.

$$\left(\frac{\|f+g\|_p^{p'} + \|f-g\|_p^{p'}}{2}\right)^{\frac{1}{p'}} \le \left(\int \left(\frac{|f+g|^{p'} + |f-g|^{p'}}{2}\right)^{\frac{p}{p'}}\right)^{\frac{1}{p}}$$

$\underline{\ell^{p'} L^p \ge L^p \ell^{p'}}$!

Therefore it's suffice to show

$$\int \left(\frac{|f+g|^{p'} + |f-g|^{p'}}{2}\right)^{\frac{p}{p'}} \le \int |f|^p + |g|^p$$

$$\Longleftarrow \quad \left(\frac{|f+g|^{p'} + |f-g|^{p'}}{2}\right)^{\frac{1}{p'}} \le 2^{1/p}\cdot\left(\frac{|f|^p + |g|^p}{2}\right)^{\frac{1}{p}} \quad \boxed{\text{pointwise estimate}}$$

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