2. Tangent vectors on Manifolds.

Def: Let M be an n-dim. differential manifold, $p \in M$, if $X: C_p^\infty \longrightarrow \mathbb{R}$ satisfies:

(1) $X(af+bg) = aX(f) + bX(g)$. where $a, b \in \mathbb{R}$.

(2) $X(fg) = X(f)g(p) + f(p)X(g)$. $f, g \in C_p^\infty$.

then say X is a <u>tangent vector</u> of M at p.

$T_pM :=$ the linear space of all tangent vectors at p of M.

called, the tangent space.

Sep 19.

Example: Let $(U, \varphi; x^i)$, $(V, \psi; y^i)$ be two coordinate charts of M. $p \in U \cap V$. $\forall f \in C_p^\infty$. define

$$X_i(f) = \frac{\partial}{\partial x^i}(f) \triangleq \left.\frac{\partial (f \circ \varphi^{-1})}{\partial x^i}\right|_{x_0 = \varphi(p)}.$$

$$Y_i(f) = \frac{\partial}{\partial y^i}(f) \triangleq \left.\frac{\partial (f \circ \psi^{-1})}{\partial y^i}\right|_{y_0 = \psi(p)}.$$

$\Rightarrow X_i, Y_i \in T_pM$.

$$X_i(f) = \left.\frac{\partial (f \circ \varphi^{-1})}{\partial x^i}\right|_{x_0} = \left.\frac{\partial (f \circ \psi^{-1} \circ \psi \circ \varphi^{-1})}{\partial x^i}\right|_{x_0}$$

$$= \left.\frac{\partial f(y(x))}{\partial x^i}\right|_{x_0} = \sum_{j=1}^n \left.\frac{\partial f}{\partial y^j}\right|_{y_0} \left.\frac{\partial y^j}{\partial x^i}\right|_{x_0}$$

$$= \sum_{j=1}^n \left.\frac{\partial y^j}{\partial x^i}\right|_{x_0} Y_j(f) = \left(\sum_{j=1}^n \left.\frac{\partial y^j}{\partial x^i}\right|_{x_0} Y_j\right)(f).$$

$\overset{f \text{ is arbitrary}}{\Longrightarrow}$ $X_i = \sum_{j=1}^n \left.\frac{\partial y^j}{\partial x^i}\right|_{x_0} Y_j$; $Y_i = \sum_{j=1}^n \left.\frac{\partial x^j}{\partial y^i}\right|_{y_0} X_j$

§1.4. Tangent Map & Submanifold.

1. Tangent Map.

<u>Question</u>: Given smooth $f: M \to N$, $\overset{?}{\rightsquigarrow}$ $T_pM \xrightarrow{\text{smooth}} T_qN$.
$q = f(p)$.

<u>Def</u>: Let $F: M \to N$ be C^∞, $p \in M$, $q = F(p)$.
$\forall f \in C_q^\infty$, define $F_*(X) \in T_qN$, s.t.
$F_*(X)(f) \triangleq X(f \circ F)$, $f \circ F \in C_p^\infty$
$\Rightarrow F_*: T_pM \to T_qN$ is linear. Say
F_* is the <u>tangent map associated to</u> F. at p.

<u>Rmk</u>: (1) $F: M \to N$, $G: N \to L$ are C^∞.
then $(G \circ F)_* = G_* \circ F_*$.

(2) If $F: M \to N$ is a differential homeomorphism,
then F_* is an isomorphism, and $(F_*)^{-1} = (F^{-1})_*$.

<u>Example</u>: $F: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $\quad x_0 \in \mathbb{R}^m$. $y_0 = F(x_0)$.
$(x^1, \dots, x^m) \longmapsto (y^1, \dots, y^n)$

$$T_{x_0}\mathbb{R}^m = \text{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right\}$$

$$T_{y_0}\mathbb{R}^n = \text{span}\left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right\}$$

$$F_*\left(\frac{\partial}{\partial x^i}\right) \in T_{y_0}\mathbb{R}^n. \Rightarrow \exists\, a_i^\alpha \text{ s.t. } F_*\left(\frac{\partial}{\partial x^i}\right) = \sum_{\alpha=1}^n a_i^\alpha \frac{\partial}{\partial y^\alpha}$$

<u>Question</u>: $a_i^\alpha = ?$

$$\forall f \in C_{y_0}^\infty,\quad F_*\left(\frac{\partial}{\partial x^i}\right)(f) = \frac{\partial}{\partial x^i}(f \circ F) = \frac{\partial}{\partial x^i}(f(y(x)))$$

$$= \left(\sum_{\alpha=1}^n \frac{\partial y^\alpha}{\partial x^i}\Big|_{x_0} \frac{\partial}{\partial y^\alpha}\right)(f) \quad \text{i.e. } a_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}\Big|_{x_0}.$$

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Exercise: Let $F: \mathbb{R}^n \to \mathbb{R}^{n+1}$,

$$u = (u^1, \cdots, u^n) \longmapsto \left(\frac{2u^1}{1+|u|^2}, \cdots, \frac{2u^n}{1+|u|^2}, \frac{1-|u|^2}{1+|u|^2} \right) \triangleq x$$

Q: $F_*\left(\frac{\partial}{\partial u^i}\right) = ?$

$F: M \to N$, $\dim M = m$. $\dim N = n$.

Def: Let $F: M \to N$ be C^∞, $p \in M$. $q = F(p) \in N$. if $F_*: T_pM \to T_qN$ is injective, then say F is an <u>immersion at p</u>. If F is an immersion at any p, say F is an <u>immersion map</u>.

Example: $M = N = \mathbb{R}$. $F: \mathbb{R} \to \mathbb{R}$. $x_0 \in M$. $y_0 = F(x_0) \in N$.

F is an immersion at x_0 $\Longleftrightarrow$ $\left.\frac{dF}{dx}\right|_{x_0} \neq 0$ or $\frac{dy}{dx}(x_0) \neq 0$.

F is monotone at x_0.

$\Rightarrow$ F is <u>injective</u> near x_0.

Thm: Let $F: M \to N$ be C^∞, $p \in M$, if F is an immersion at p (i.e. $F_*|_p$ inj), then F is injective near p.
(i.e. $\exists$ nghd $U \ni p$. s.t. $F|_U$ is injective).

Proof: Choose two charts $(\underset{p}{U \ni}, \varphi; x^i)$, $(\underset{q=F(p)}{V \ni}, \psi; y^i)$

φ, ψ: differential homeomorphisms.

F_* inj $\Longleftrightarrow (\psi \circ F \circ \varphi^{-1})_*$ inj.

F is locally injective $\Longleftrightarrow \psi \circ F \circ \varphi^{-1}$ is locally injective

$(\psi \circ F \circ \varphi^{-1})_*: T_{x_0}\mathbb{R}^m \longrightarrow T_{y_0}\mathbb{R}^n$ $x_0 = \varphi(p)$. $y_0 = \psi(q)$.

$$(\psi \circ F \circ \varphi^{-1})_*\left(\frac{\partial}{\partial x^i}\right) = \sum_{A=1}^{n} \left.\frac{\partial y^A}{\partial x^i}\right|_{x_0} \frac{\partial}{\partial y^A}.$$

$\Rightarrow$ The matrix of linear map $(\varphi \circ F \circ \varphi^{-1})_*$ is :

$$J = \left(\frac{\partial y^j}{\partial x^i}, \frac{\partial y^\alpha}{\partial x^i} \right)\Big|_{x_0}, \quad 1 \le i, j \le m, \quad m+1 \le \alpha.$$

$F_*|_p$ is injective $\Rightarrow$ w.l.o.g. $\det\left(\frac{\partial y^j}{\partial x^i}\Big/_{x_0}\right) \neq 0$.

By the implicit function theorem. $x^i = x^i(y^1, \dots, y^m)$, $1 \le i \le m$

$$\frac{\partial x^i}{\partial u^j} = \frac{\partial x^i}{\partial y^j} \Rightarrow \tilde{\varphi} : (u^1, \dots, u^m) \longrightarrow (x^1, \dots, x^m)$$

is a coordinate transform

Now, $\psi \circ F \circ \varphi^{-1} \circ \tilde{\varphi}$ is :

$$\begin{cases} y^i = u^i & 1 \le i \le m \\ y^\alpha = y^\alpha(x) = y^\alpha(x(u)) & m+1 \le \alpha \le n \\ \qquad \triangleq g^\alpha(u) \end{cases}$$

$$\tilde{\psi} : (v^1, \dots, v^n) \longrightarrow (y^1, \dots, y^n) \qquad \begin{cases} v^i = y^i & 1 \le i \le m \\ v^\alpha = y^\alpha - g^\alpha & m+1 \le \alpha \le n. \end{cases}$$

$$\cancel{J\psi} \quad J\tilde{\psi} = \left(\frac{\partial v^A}{\partial y^B}\right) = \begin{pmatrix} I_m & 0 \\ * & I_n \end{pmatrix} \quad \text{invertible}$$

$\Rightarrow \tilde{\psi}$ is also a coordinate transform.

$$\tilde{\psi}^{-1} \circ \psi \circ F \circ \varphi^{-1} \circ \tilde{\varphi} : \begin{cases} v^i = u^i & 1 \le i \le m \\ v^\alpha = 0 & m+1 \le \alpha \le n \end{cases}$$

Since $\varphi, \tilde{\varphi}, \psi, \tilde{\psi}$ are all differential homeomorphisms.

$\therefore$ F is locally injective near p. $\square$

<u>Rmk</u>: The theorem tells us that the property of F_* reflect ~~the~~ same local property of F.

2. Submanifold.

Def: Let $F: M \to N$ be smooth, if F is inj & immersion, then say (M, F) is an <u>immersed submanifold</u>.

Example: $F: \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$.

tangent vector $(-\sin t, \cos t)$

$(\mathbb{R}, F)$ is not an immersed submanifold. but F is an immersion

$((0, 2\pi), F)$ is an immersed submanifold.

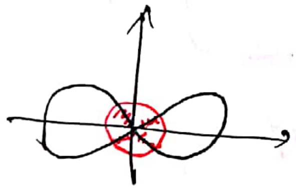
- Topology on $F(M) \subset N$.

(1) Restriction Topology : $\{ V \cap F(M) \mid V \text{ open in } N \}$

(2) Induced Topology : $\{ V \subseteq F(M) \mid F^{-1}(V) \text{ open in } M \}$.

Rmk: Generally, these two topologies are <u>different</u>!

<u>Example</u>:



$(\sin t, \sin 2t)$.