chapter3

1. J_r', J_e' existence trivial

note J is convex $\Rightarrow$ J is locally Lip $\Rightarrow$ $J \in AC[t_0, t_1] \Rightarrow J_r' = J_e' = J'$ a.e. and NL formula holds

3. $\operatorname{supp} \chi_A * \chi_B \subseteq \operatorname{supp} \chi_{J_A} * \chi_{J_B}$ ~~and I supp~~

$\Rightarrow$ smallest interval contains $\operatorname{supp} \chi_A * \chi_B$ has length $\leq |J_A| + |J_B|$.

On the other hand, let $J_A = [a,b]$, $J_B = [c,d]$.

$$\chi_A * \chi_B(x) \neq 0 \Leftrightarrow \mathcal{L}^1(A \cap \{x - B\}) > 0.$$

we will prove $\forall \varepsilon > 0$, $\mathcal{L}^1(\{\chi_A * \chi_B > 0\} \cap (a+c, a+c+\varepsilon)) > 0$.

$$\Leftrightarrow \int_{(a+c,a+c+\varepsilon)} \chi_A * \chi_B(x)\, dx > 0$$

$$= \int_{(a+c,a+c+\varepsilon)} \chi_A(y) \chi_B(x-y)\, dy\, dx$$

$$\geq \int_{(a+c,a+c+\varepsilon)} \chi_{A \cap [a,a+\varepsilon]}(y)\, \chi_{B \cap [c,c+\varepsilon]}(x-y)\, dy\, dx$$

$$= \mathcal{L}^1(A \cap [a,a+\varepsilon]) \cdot \mathcal{L}^1(B \cap [c,c+\varepsilon]) > 0.$$

同理可证 $\mathcal{L}^1(\{\chi_A * \chi_B > 0\} \cap (d+b-\varepsilon, d+b)) > 0$.

$\Rightarrow$ length $\geq |J_A| + |J_B|$.

4. 注意 $r(t)$ 的存在性依赖于 f^* 的下半连续性

6. ~~若$l_1 \cap$~~ 设 A 关于 l_1, l_2 两直线旋转对称.

若 $l_1 \cap l_2 = \phi$. $\Rightarrow \exists$ 超平面 P_1, P_2, $P_1 \parallel P_2$. $l_1 \in P_1$, $l_2 \subset P_2$

P_1, P_2 将 A 分为3部分. 依次记为 A_1, A_2, A_3, 其中 A_2 为 $P_1 P_2$ 之间的那部分

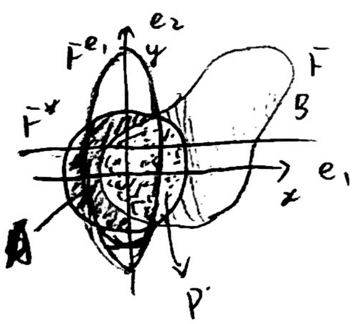
$\Rightarrow \mathcal{L}^n(A_1) = \mathcal{L}^n(A_2) + \mathcal{L}^n(A_3)$ $\qquad \Rightarrow \mathcal{L}^n(A_2) = 0$.

$\mathcal{L}^n(A_3) = \mathcal{L}^n(A_1) + \mathcal{L}^n(A_2)$

$\Rightarrow \mathcal{L}^n(A_1) = \mathcal{L}^n(A_3) = 0$. $\qquad \Rightarrow \mathcal{L}^n(A) = 0$.

7. take $f = g = h = \chi_{\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\}}(x, y)$. $\quad |a| \ne |b|$.

8.



不妨令 e_1 即为 x 轴的方向

~~令~~ $\forall A \subseteq \mathbb{R}^2$, $A(y) := \{x \mid (x, y) \in A\}$

则有 $\delta = \mathcal{L}^2(F^{e_1} \cap F^*) - \mathcal{L}^2(F \cap F^*)$

$= \int_{\mathbb{R}} \mathcal{L}^1(F^{e_1} \cap F(y)) - \mathcal{L}^1(F \cap F(y))\, dy$

$= \int_{\mathbb{R}} \min\{\mathcal{L}^1(F^{e_1}(t)), \mathcal{L}^1(F^*(y))\} - \mathcal{L}^1(F \cap F^*(y))\, dy$

$= \int_{\mathbb{R}} \min\{\mathcal{L}^1(F(t)), \mathcal{L}^1(F^*(y))\} - \mathcal{L}^1(F \cap F^*(y))\, dy$.

$= \int_{\mathbb{R}} \min\{\mathcal{L}^1(\underbrace{F(t) \setminus F^*(t)}_{B(t)}), \mathcal{L}^1(\underbrace{F^*(t) \setminus F(t)}_{A(t)})\}\, dy$.

$\ge \int_{\mathbb{R}} \mathcal{L}^1(B(t) \cap A(t))\, dy$.

$= C(x)$.

9. 只须证 $\{f > t\}^* = \{f^* > t\}$

$\{f^* > t\}^* := B_{r(t)}(0)$, 则 $f^*(x) = \int_0^\infty \chi_{\{f > t\}^*}(x)\, dt = \sup\{t \mid r(t) > |x|\}$

$f^*(x) > t \Leftrightarrow \sup\{s \mid r(s) > |x|\} > t$

$r(t)$ 左连续 $\Leftrightarrow \{f(x) > t\}^* = B_{r(t)}(0) = \{f^*(x) > t\}$.

$\because |\{f > t + \delta\}| - |\{f > t\}| = |\{t < f \le t + \delta\}| \to 0$ as $\delta \to 0$