Ans. to Ex. Week 2

Ex1

(i). Given $f \in C^k(\mathbb{T})$, then by integrati by parts.

$$c_n(f) = \int_{\mathbb{T}} e^{-2\pi i n \cdot x} f(x)\,dx = \cdots$$

$$= c \cdot \frac{1}{n^k} \int_{\mathbb{T}} e^{-2\pi i n \cdot x} f^{(k)}(x)\,dx$$

Since $f^{(k)}$ is contin. $\int_{\mathbb{T}} e^{-2\pi i n \cdot x} f^{(k)}(x) = o(1)$ as $n \to \infty$, by R.-L. lemma. Indeed, we have more: suppose $f^{(k)}(x)$ is differentiable, except ~~for~~ for a finitely many number of points, say $0 < t_1 < t_2 < \cdots < t_N < 1$. then.

$$c_n(f) = c \cdot \frac{1}{n^k} \Big(e^{-2\pi i n \cdot x} f^{(k+1)}(x) \Big|_{t_1^-}^{t_1^+} + \cdots + e^{-2\pi i n \cdot x} f^{(k+1)}(x) \Big|_{t_N^-}^{t_N^+}$$

$$- \frac{c}{n} \int_{\mathbb{T}} e^{-2\pi i n \cdot x} f^{(k+1)}(x)\,dx \Big)$$

if $e^{-2\pi i n \cdot x} f^{(k+1)}(x) \Big|_{t_1^-}^{t_1^+} + \cdots + e^{-2\pi i n \cdot x} f^{(k+1)}(x) \Big|_{t_N^-}^{t_N^+} = 0$

we then ~~sh~~ have

$$c_n(f) = c \cdot \frac{1}{n^{k+1}}, \text{ or even better.}$$

if this quantity does not vanish, we can not improve the small "o" quantitatively.

(ii) it follows from the diff. construction:

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n \cdot x} dx = -\int_0^1 f(x) e^{-2\pi i n (x + \frac{1}{2n})}$$

$\int_0^1 = \int_{\mathbb{T}}$ $$= -\int_0^1 f(x - \tfrac{1}{2n}) e^{-2\pi i n x} dx.$$

is much more convenient

$$\hat{f}(n) = \frac{1}{2}\int_0^1 \left(f(x) - f(x - \tfrac{1}{2n})\right) e^{-2\pi i n x} dx$$

$\in$ Hölder Condition here.

$=$?

(iii). say, the restriction of hol. $\frac{1}{1 - z/2}$ on $|z| < 2$ onto the circle, works.

- say the restriction of hol. $\frac{1}{1 - \frac{1}{2z}}$, on $|z| > 1/2$ on to the circle, works as well.
- say any combination of these,

Conclusion: there are so many, and we can not even tell the type of the singularity at the origin or ∞.

Ex 2:

(i) • it follows from Yong's Ineq. that

$$\|T_N f\|_{L^1} \leq \underbrace{\|D_N\|_{L^1}} \cdot \|f\|_{L^1}$$
$$\leq L_N \|f\|_{L^1}$$

• to show the equality, one can do as in the indication in the notes.

(ii) • it follows for Yonges Ineq. again

$$|\tilde{T}_N f(0)| \leq \|\tilde{T}_N f\|_{L^\infty} \leq \|D_N\|_{L^1} \cdot \|f\|_{L^\infty}$$
$$= L_N \|f\|_{L^\infty}.$$

• the ~~con~~ "=" follows from the similar argument as in (i)

(iii). as in (ii). Yong inequality.

+ Some specific "test" functions.

Problem I:

(i) $f_n \xrightarrow{n\to\infty} f$ in $L^p([0.1])$

$$\not\Longrightarrow f_n \xrightarrow{n\to\infty} f \quad \text{a.e. } x \in [0.1]$$

take the example against the convergence in measure.

(ii) $f_n \to f$ a.e. $[0,1]$ $\not\Rightarrow$ $f_n \to f$ in $L^p([0,1])$

take the example, against D.C.T. that is the seq should be unbdd.

(iii). see L. Grafakos. GTM 249. 3rd edition thm 2.1.14.