Week 8

EX1: Poisson Summation formula.

Suppose that $f \in \mathcal{S}(\mathbb{R}^n)$, then we can construct a periodic function by setting

$$F_1(x) = \sum_{m \in \mathbb{Z}^n} f(x+m)$$

On the other hand, by restricting the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

with $$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$$

we get another periodic function

$$F_2(x) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i x \cdot m}$$

(1) show that if $f \in \mathcal{S}(\mathbb{R}^n)$, then F_1, F_2 are well defined functions on $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. Moreover $F_1 \equiv F_2$.

(2) if $f \in L^1(\mathbb{T}^n)$, then the series $F_1(x)$ converges in $L^1(\mathbb{T}^n)$ and the resulting function has the Fourier expansion

$$\sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i x \cdot m}$$

(3) Suppose $|f(x)| \le A(1+|x|)^{-n-\delta}$ and $|\hat{f}(\xi)| \le A(1+|\xi|)^{-n-\delta}$, $\delta > 0$ then

$$\sum_{m \in \mathbb{Z}^n} f(x+m) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x}.$$

In particular

$$(*) \qquad \sum_{m \in \mathbb{Z}^n} f(m) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m).$$

(4). Now take $f(x) = |x|^{\alpha - n}$, with $0 < \mathrm{Re}(\alpha) < n$. We have computed $\hat{f}(\xi) = \gamma_\alpha |x|^{-\alpha}$, $\gamma_\alpha = \pi^{-\alpha + \frac{n}{2}} \Gamma(\alpha/2) \Gamma(\frac{n-\alpha}{2})$

if we apply (*), we would get

$$(\#) \quad \gamma_\alpha^{-1} \sum_{m\in\mathbb{Z}^n} |x+m|^{\alpha-n} = \sum_{m\in\mathbb{Z}^n} |m|^{-\alpha} e^{2\pi i m\cdot x}$$

Which is not true, due to lack of convergence

However, we can interprete (#) as:

for $0 < \mathrm{Re}(\alpha) < n$, the series $\sum_{|m|>0} |m|^{-\alpha} e^{2\pi i m\cdot x}$ is the Fourier series of some integrable function on $\mathbb{T}^n$, that is in the ~~cal~~ class $C^\infty(\mathbb{T}^n \setminus \{0\})$. Moreover,

$$\gamma_\alpha^{-1} |x|^{\alpha-n} - \sum_{|m|>0} |m|^{-\alpha} e^{2\pi i m\cdot x} \in C^\infty(\mathbb{T}^n). \text{ that is,}$$

the LHS and RHS of (#) have the same singularity.

**: try to prove this interpretation

(5) apply (*) with $f(x) = e^{-\pi|x|^2}$, what do you find?