Littlewood-Paley theory; via vector-valued inequalities

I) Vector-valued version of Calderon Zygmund operators

Let B be a separable Banach space. For a measurable function f from $\mathbb{R}^n$ to B, define the norm

$$\|f\|_{L^p(\mathbb{R}^n;B)} = \left(\int_{\mathbb{R}^n} \|F(x)\|_B^p dx\right)^{1/p}, \quad 1 \le p < \infty$$

&

$$\|f\|_{L^\infty(\mathbb{R}^n;B)} = \text{ess}\sup\{\|F(x)\|_B : x \in \mathbb{R}^n\}$$

the space $L^p(\mathbb{R}^n;B)$ of measurable function consists of measurable funcs s.t. the corresponding norms are finite.

Given $F = \sum_j f_j \cdot b_j \in L^1 \otimes B$, define it integral to be an element of B by

$$\int_{\mathbb{R}^n} F(x)dx = \sum_j \left(\int_{\mathbb{R}^n} f_j(x)dx\right) b_j.$$

for $F \in L^p(\mathbb{R}^n, B)$, $G \in L^{p'}(\mathbb{R}^n, B^*)$, then

$$\langle F, G\rangle_{B,B^*}(x) = \langle F(x), G(x)\rangle_{B,B^*}$$

is integrable. Furthermore

$$\|G\|_{L^{p'}(\mathbb{R}^n,B^*)} = \sup\left\{\left|\int_{\mathbb{R}^n}\langle F(x),G(x)\rangle dx\right| : \|F\|_{L^p(\mathbb{R}^n,B)} \le 1\right\}$$

if $1 \le p < \infty$, and B is reflexible, then

$$L^{p'}(\mathbb{R}^n, B^*) = \left(L^p(\mathbb{R}^n, B)\right)^*$$

let A and B be Banach spaces, and let $\mathcal{L}(A,B) := \{ P: A \to B \text{ bdd linear} \}$. let

$$K: \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \longrightarrow \mathcal{L}(A,B)$$

and T be an operator which has K as its kernel:

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) \cdot f(y)\, dy. \quad x \notin \operatorname{supp} f. \; f \in C_c \cap L^\infty$$

then, the vector-valued C.-Z. theorem asserts that

thm 1:

let T be a bounded linear operator from $L^r(\mathbb{R}^n, A)$ to $L^r(\mathbb{R}^n, B)$ for some $r \in (1,\infty)$. with K as its associated kernel. K. if K satisfies

$$(1) \qquad \int_{|x-y|>2|y-z|} \| K(x,y) - K(x,z) \|_{\mathcal{L}(A,B)}\, dx \leq C$$

$$(2) \qquad \int_{|x-y|>2|x-w|} \| K(x,y) - K(w,y) \|_{\mathcal{L}(A,B)}\, dy \leq C$$

then T is bounded from $L^p(\mathbb{R}^n, A)$ to $L^p(\mathbb{R}^n, B)$. and

$$\left| \{ x \in \mathbb{R}^n \,/\, \| Tf(x) \|_B > \lambda \} \right| \leq \frac{C}{\lambda} \| f \|_{L^1(\mathbb{R}^n, A)}$$

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I will not present the proof. I believe that it is of no problem to follow the routine in the proof of C.-Z. theory (generalized), as long as you keep the concept clear (in particular, the concept of duality). We next with draw some conclusion from this.

<u>Cor 2</u> If T is a convolution operator bounded on $L^2(\mathbb{R}^n)$, with its associated kernel K satisfying the Hörmander condition, then for any $r \in (1,\infty)$ and $p \in (1,\infty)$,

$$(3) \quad \left\| \left(\sum_j |Tf_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)} \lesssim_{r,p} \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)}$$

and ~~given~~ in the case $p=1$,

$$(4) \quad \left| \left\{ x \in \mathbb{R}^n \,\middle|\, \left(\sum_j |Tf_j(x)|^r \right)^{1/r} > \lambda \right\} \right| \leq \frac{C_r}{\lambda} \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_{L^1(\mathbb{R}^n)}$$

<u>Proof.</u>

- App. thm 1 with $A = B = \ell^r$.
- in the case $p=r$, (3) & (4) is obviously valid thanks to the fact that T is bounded from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, $r \in (1,\infty)$
- Consider the vector-valued operator

$$\{f_j\} \longmapsto \{Tf_j\}$$

$$\ell^r \longrightarrow \ell^r$$

with its associated kernel $K(x)\,Id.$, where Id is the identity operator acting on ℓ^r. thus in view of thm 1, it remains to check the Hörmander condition. For this, the required cond.

note

is indeed.

$$\int_{|x|>2|y|} \underbrace{\|(K(x-y)-K(x))\,Id\|_{\ell^r}}_{\displaystyle \| \atop \displaystyle |K(x-y)-K(x)|} dx \le C$$

$\leftarrow$ exactly the Hörmander cond.

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Now, the question arising naturally is to generalize Cor 1 or (3) to an eq. of the form

$$\Big\| \Big(\sum_j |T_j f_j|^r\Big)^{1/r} \Big\|_{L^p(\mathbb{R}^n)} \lesssim_{p,r} \Big\| \Big(\sum_j |f_j|^r\Big)^{1/r} \Big\|_{L^p(\mathbb{R}^n)}$$

with a seq. of operators $\{T_j\}$, rather than the fixed operator $\{T\}$.

Cor 2: Let $\{I_j\}$ be a seq. of intervals in $\mathbb{R}$, finite or infinite, define

$$(S_j f)^\wedge(\xi) = \chi_{I_j}(\xi)\hat{f}(\xi).$$

then for $1<r,p<\infty$,

$$\Big\| \Big(\sum_j |S_j f_j|^r\Big)^{1/r} \Big\|_{L^p} \lesssim_{p,r} \Big\| \Big(\sum_j |f_j|^r\Big)^{1/r} \Big\|_{L^p}$$

proof: this follows immediately from the unf. bddness of Hilbert transform and Cor. 1 by losing some operator constant the following theorem is indeed the common feature for proving vector-valued ineq: involving weighted norm inequalities.

thm 4: let $\{T_j\}$ be a seq. of operators bounded on $L^2(w)$, with uniform const., for any $w \in A_2$. then for all $p \in (1,\infty)$,

$$\Big\| \Big(\sum_j |T_j f_j|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)} \leq C \Big\| \Big(\sum_j |f_j|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)}$$

Proof:

- $p=2$, immediate.
- $p>2$. then $\exists\, u \in L^{(p/2)'}$ with norm 1 s.t.

$$\Big\| \Big(\sum_j |T_j f_j|^2\Big)^{1/2} \Big\|_{L^p}^2 = \Big\| \Big(\sum_j |T_j f_j|^2\Big)^{\frac{p}{2}} \Big\|_{L^{\frac{p}{2}}}$$

$$= \int_{\mathbb{R}^n} \Big(\sum_j |T_j f_j|^2\Big) u$$

claim: $\left[\begin{array}{l} \delta \in (0,1) \Rightarrow M(u^{1/\delta})^{\delta} \in A_1 \subset A_2 \\ u(x) \leq M(u^{1/\delta})^{\delta}(x), \text{ a.e.} \end{array}\right]$

pointwise ineq.

$$\leq \int_{\mathbb{R}^n} \Big(\sum_j |T_j f_j|^2\Big) M(u^{1/\delta})^{\delta}(x)\,dx$$

$$= \sum_j \int_{\mathbb{R}^n} |T_j f_j|^2 M(u^{1/\delta})^{\delta}(x)\,dx$$

$$\overset{A_2}{\leq} C \sum_j \int_{\mathbb{R}^n} |f_j|^2 M(u^{1/\delta})^{\delta}(x)\,dx$$

$$= C \int_{\mathbb{R}^n} \sum_j |f_j|^2 M(u^{1/\delta})^{\delta}(x)\,dx$$

$$\leq C\left(\int_{\mathbb{R}^n}\Big(\sum_j |f_j|^2\Big)^{\frac{p}{2}}\right)^{2/p}\cdot\left(\int_{\mathbb{R}^n} M(u^{1/s})^{s(\frac{p}{2})'}\right)^{\frac{1}{(\frac{p}{2})'}}$$

$$s\cdot(p/2)' > 1$$

$$\leq \| u^{1/s\cdot s(p/2)'} \|_{L^1}$$

$$= \| u^{(p/2)'} \|_{L^1}$$

$$= \| u \|_{L^{(p/2)'}} = 1$$

$$\leq C\,\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}.$$

Here we have admitted some facts about A_2-weight: which means that. if $w \in A_2$, ~~then~~ (relative to some operator, for instance T), then

$$\int_{\mathbb{R}^n} |Tf|^2 w\,dx \leq C\int_{\mathbb{R}^n} |f|^2 w\,dx$$

the claim is a deep result concerning weight theory. which can not be explained clear in one page. if we wish, you can admit it as a blackbox..

L.-P. theory

In the 1-D case, let

$$\Delta_j := (-2^{j+1}, 2^j] \cup [2^j, 2^{j+1})$$

define the operator S_j by setting

$$(S_j f)^\wedge(\xi) = \chi_{\Delta_j}(\xi) \hat{f}(\xi), \quad j \in \mathbb{Z}$$

thm 5 for $\boxed{f \in L^p(\mathbb{R})}$, $p \in (1, \infty)$,

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} \sim \|f\|_{L^p(\mathbb{R})}$$

To prove thm 5, we consider instead the smooth version. let $\psi \in \mathcal{S}(\mathbb{R})$ /

$0^\le$ · $\operatorname{supp}\psi \subset \{ 1/2 \le |\xi| \le 4 \}$

· $\psi|_{1 \le |\xi| \le 2} \equiv 1$.

define

$$\psi_j(\xi) = \psi(\xi/2^j) \;\&\; (\tilde{S}_j f)^\wedge(\xi) = \psi_j(\xi) \hat{f}(\xi).$$

then $S_j \tilde{S}_j = S_j$

thm 6

$$\left\| \left(\sum_j |\tilde{S}_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

Proof. • $\widehat{\Psi} := \psi$, & $\Psi_j(x) = 2^j \Psi(2^j x)$

then $\widehat{\Psi_j} = \psi_j$, $\tilde{S}_j f = \Psi_j * f$.

• Consider the vector-valued map, mapping f to $\{\tilde{S}_j f\}$, it suffices to show:

$\boxed{\text{the bddness from } L^p(\mathbb{R}) \text{ to } L^p(\mathbb{R}; l^2)}$

• the case $p=2$ is ok, following the finite overlapping property.

• thus to apply ~~theory~~ thm 1 or Cor. 2, it suffices to note the associated kernel is $\{\Psi_j\}$ ~~ma~~ and to show it satisfies the Hörmander condition. by the Gradient condition it suffices ~~the~~ to show

$$\|\Psi_j'(x)\|_{l^2_j} \leq C\, |x|^{-2}.$$

this indeed follows from

$$\left(\sum_j |\Psi_j'(x)|^2\right)^{1/2} \leq \sum_j |\Psi_j'(x)| = \sum_j 2^{2j} |\Psi'(2^j x)|$$

$$\leq \sum_j 2^{2j} \min(1, (2^j|x|)^{-3})$$

$$= \sum_{2^j |x| < 1} 2^{2j} + C|x|^{-3} \sum_{2^j |x| \geq 1} 2^{2j} \cdot 2^{-3j}$$

$$\sim |x|^{-2}$$

#

Proof of thm 5: by Cor. 3 and the identity $S_j \tilde{S}_j = S_j$

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} = \left\| \left(\sum_j |S_j \tilde{S}_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}$$

$$\leq C \left\| \left(\sum_j |\tilde{S}_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}.$$

$$\overset{\text{thm 6}}{\leq} C \left\| f \right\|_{L^p(\mathbb{R})}$$

Noting that.

$$\int_{\mathbb{R}} f \bar{g} = \int_{\mathbb{R}} \sum_j S_j f \sum_k \overline{S_k g}$$

we have

$$\|f\|_{L^p} = \sup \left\{ \left| \int_{\mathbb{R}} f \bar{g} \right| : \|g\|_{L^{p'}} \leq 1 \right\}$$

$$= \sup \left\{ \left| \int_{\mathbb{R}} \sum_j S_j f \cdot \sum_k \overline{S_k g} \right| \cdot \|g\|_{L^{p'}} \leq 1 \right\}$$

finite overlapp. $S_j f \cdot S_k g$

$$\leq C \cdot \sup \left\{ \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_{L^p} \cdot \left\| \left(\sum_j |S_j g|^2 \right)^{1/2} \right\|_{L^{p'}}, \dots \right\}$$

$$\leq C_p \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}$$

$\blacksquare$

thm 5 has been generalized to higher Dim.
in two directions:

- One is the dyadic decomposi into annuli.
- the other one is to the produer of dyadic intervals...

the details are indeed in the Same spirit of the proof of thm 5 and 6. So we omitt them here.