Chapter 4 参考答案

1. 略

2. 我以前的笔记上写过了

3. $$\int_0^\infty v(a)^{1-\frac{r}{n}} \int_{a^{p/r}}^\infty w(b)\,db\,da = \int_0^\infty\int_0^\infty v(a)^{1-\frac{r}{n}} w(b)\cdot \chi_{\{b\geq a^{p/r}\}}\,da\,db$$
$$= \int_0^\infty \int_0^{b^{r/p}} v(a)^{1-\frac{r}{n}}\,da\cdot w(b)\,db$$

4. a) $$\int_{-\infty}^{\infty} \exp(-\lambda x^2)\,dx = \sqrt{\int_{\mathbb{R}^2} \exp(-\lambda(x^2+y^2))\,dx\,dy}$$
$$= \sqrt{2\pi\int_0^\infty \exp(-\lambda r^2)\,r\,dr} = \sqrt{\frac{\pi}{\lambda}}$$

b) $A=B+Ci$ B对称正定 C对称，故存在可逆矩阵P st

~~PAB~~ $B=P^TP$. $C=P^T\Sigma P$ 其中Σ=diag$\begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0\end{pmatrix}$

$$\int_{\mathbb{R}^n} \exp(-(x\cdot Ax))\,dx \overset{y=Px}{=} \frac{1}{\det P}\int_{\mathbb{R}^n} \exp\Big(\sum_{k=1}^{p}(-1+i)x_k^2+\sum_{k=p+1}^{p+q}(-1-i)x_k^2\Big)dx$$

利用围道积分易知

$$\int_{\mathbb{R}} \exp((-1+i)x^2)\,dx = \frac{1}{\sqrt{1-i}}\int_{\mathbb{R}}\exp(-x^2)\,dx = \sqrt{\frac{\pi}{1-i}}$$

$$\int_{\mathbb{R}} \exp((-1-i)x^2)\,dx = \frac{1}{\sqrt{1+i}}\int_{\mathbb{R}}\exp(-x^2)\,dx = \sqrt{\frac{\pi}{1+i}}$$

$$\Longrightarrow \int_{\mathbb{R}^n}\exp(-(x\cdot Ax))\,dx = \frac{\sqrt{\pi}}{\det P}\cdot\frac{1}{(\sqrt{1-i})^p(\sqrt{1+i})^q} = \frac{\pi^{n/2}}{\sqrt{\det A}}$$

c) $$\int_{\mathbb{R}^n}\exp(-(x\cdot Ax)+2(v\cdot x))\,dx = \int_{\mathbb{R}^n}\exp(-(x+A^{-1}v, A(x+A^{-1}v)))\cdot\exp(v\cdot A^{-1}v)\,dx$$

$$= \exp((v\cdot A^{-1}v))\cdot\int_{\mathbb{R}^n}\exp(-(x\cdot Ax))\,dx$$

$$= \exp((v\cdot A^{-1}v))\cdot \pi^{n/2}/\sqrt{\det A}$$

$\int e^{-yA^Ty} = \int e^{-(y+i\delta)^TA(y+i\delta)}$ $\forall \delta\in\mathbb{R}^n$ 也是将A相合到对角

$\Leftarrow$ ~~p,q,r,n Minkowski不等式~~

只需考虑 $n=1$ 情形

5. 由4.2节所的论证可知 $C_q^{\varepsilon,0} = \sup_{g,h\in\mathcal{G}} \frac{\|K_g^{\varepsilon_0} h\|_p}{\|g\|_q\cdot\|h\|_r} = \sup_{g,h\in\mathcal{G}} \frac{\|j_\varepsilon * g * h\|_p}{\|g\|_q\cdot\|h\|_r}$

recall: $j_\varepsilon = \frac{1}{(\pi\varepsilon)^{\frac{1}{2}}}\exp\left(-\frac{|x|^2}{\varepsilon}\right)$

$\boxed{\text{Fact}}$: $e^{-\alpha x^2} * e^{-\beta x^2} = \left(\frac{\pi}{\alpha+\beta}\right)^{\frac{1}{2}} e^{-\frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}x^2}$

从而 $j_\varepsilon * e^{-\alpha x^2} = \frac{1}{(1+\varepsilon\alpha)^{\frac{1}{2}}}\cdot e^{-\frac{1}{\varepsilon+\frac{1}{\alpha}}x^2}$

$$j_\varepsilon * e^{-\alpha x^2} * e^{-\beta x^2} = \frac{1}{(1+\varepsilon\alpha)^{\frac{1}{2}}}\cdot\frac{(\pi)^{\frac{1}{2}}}{\left(\frac{1}{\varepsilon+\frac{1}{\alpha}}+\beta\right)^{\frac{1}{2}}}\cdot\exp\left(-\frac{1}{\varepsilon+\frac{1}{\alpha}+\frac{1}{\beta}}x^2\right)$$

$$= \frac{\pi^{\frac{1}{2}}}{(\alpha+(1+\varepsilon\alpha)\beta)^{\frac{1}{2}}}\cdot\exp\left(-\frac{1}{\varepsilon+\frac{1}{\alpha}+\frac{1}{\beta}}x^2\right)$$

$\boxed{\text{Fact}}$: $\|e^{-\alpha x^2}\|_p = C\cdot\frac{1}{(\alpha p)^{1/2p}}$

从而 $\|j_\varepsilon * e^{-\alpha x^2} * e^{-\beta x^2}\|_p = C\cdot\frac{1}{(\alpha+(1+\varepsilon\alpha)\beta)^{\frac{1}{2}}}\cdot\left(\frac{\varepsilon+\frac{1}{\alpha}+\frac{1}{\beta}}{p}\right)^{\frac{1}{2p}}$

$\|e^{-\alpha x^2}\|_q = C\frac{1}{(\alpha q)^{\frac{1}{2q}}}$

$\|e^{-\beta x^2}\|_r = C\frac{1}{(\beta r)^{\frac{1}{2r}}}$

回 ... 的关系 ReF. v2.1

归结成

$$\sup_{\alpha,\beta} \frac{1}{(\alpha+(1+\varepsilon\alpha)\beta)^{\frac{1}{2}}} \cdot \left(\frac{\varepsilon+\frac{1}{\alpha}+\frac{1}{\beta}}{p}\right)^{\frac{1}{2p}} \cdot \left(\frac{1}{\alpha q}\right)^{-\frac{1}{2q}} \cdot \left(\frac{1}{\beta r}\right)^{-\frac{1}{2r}}$$

$$= \sup_{\alpha,\beta} \frac{1}{(\alpha\beta)^{\frac{1}{2}}} \cdot \left(\frac{1}{p}\right)^{\frac{1}{2p}} \cdot \left(\frac{1}{q}\right)^{-\frac{1}{2q}} \cdot \left(\frac{1}{r}\right)^{-\frac{1}{2r}} \cdot \left(\varepsilon+\frac{1}{\alpha}+\frac{1}{\beta}\right)^{2\left(\frac{1}{p}-1\right)} \cdot \left(\frac{1}{\alpha}\right)^{-\frac{1}{2q}} \left(\frac{1}{\beta}\right)^{-\frac{1}{2r}}$$

不依赖于 α,β 的常数

$$\sim \sup_{\alpha,\beta} \left(\frac{\frac{1}{\alpha}}{\varepsilon+\frac{1}{\alpha}+\frac{1}{\beta}}\right)^{\frac{1}{q'}} \cdot \left(\frac{\frac{1}{\beta}}{\varepsilon+\frac{1}{\alpha}+\frac{1}{\beta}}\right)^{\frac{1}{r'}}$$

$$\sim \sup_{x,y\geq 0} \left(\frac{x}{\varepsilon+x+y}\right)^{\frac{1}{q'}} \cdot \left(\frac{y}{\varepsilon+x+y}\right)^{\frac{1}{r'}}$$

$$\sim \sup_{x,y\geq 0} \left(\frac{x}{x+y}\right)^{\frac{1}{q'}} \cdot \left(\frac{y}{x+y}\right)^{\frac{1}{r'}} = \cdots$$

对 x,y 做拉伸可得

6. 逐点收敛 + DCT

Lemma: $f_i \to f$ in L^p, $g_i \to g$ in L^p, $1 \le p < \infty$ then

$f_i(x)g_i(y) \to f(x)g(y)$ in L^p.

Pf. 只需证明 $\forall\, \varphi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$

$$\cancel{\int f_i g} \quad \int f_i(x)g_i(y)\varphi(x,y)\,dxdy \longrightarrow \int f(x)g(y)\varphi(x,y)\,dxdy$$

当 φ 有形式 $= \sum_{i=1}^{n} \varphi_i(x)\psi_i(y)$ 时 这是显然成立

而一般的 这种形式的函数在 $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ 中稠密

$\square$

7. a) ~~只需对~~ 用与Thm4.6类似讨论可知 只需对紧支集有界函数证明结论

· R 与 U_2 都正算子 $\Rightarrow \exists\, C > 0$ s.t. $f_k^* \le C h_f$ a.e. (还是与书上讨论类似)

f_k 一致有界 对称递减 $\xrightarrow{\text{Helly}}$ $\exists$ 子列 $f_{k_i} \to g$ a.e. $\Rightarrow g \le h_f$ a.e.

~~$g = h_f$ a.e.~~ $\Rightarrow g \in L^p$ 对称递减

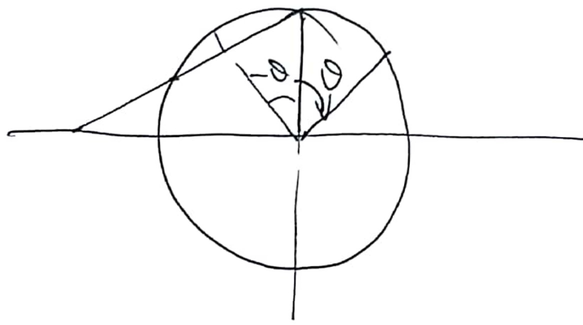
因为 $\|g - h_f\|_p \ge \|RU_2 g - h_f\|_p = \lim_{i \to \infty} \|f_{k_i} - h_f\|_p \underset{\text{Fatou}}{\ge} \|g - h_f\|_p$

$\Longrightarrow \|g - h_f\|_p = \|RU_2 g - h_f\|_p$

$\Longrightarrow \|g - h_f\|_p = \|U_2 g - U_2 h_f\| = \|RU_2 g - RU_2 h_f\|$

Riesz 重排不等式知 $RU_2 g = U_2 g$

7.(b) 用θ代表S^1上的参数 $\theta\in[0,2\pi)$



g 对称 $\Longrightarrow$ $G(\theta)=G(-\theta)$

$U_\alpha g$ 对称 $\Longrightarrow$ $G(\theta-\alpha)= G(-\theta-\alpha)$

$\Longrightarrow$ $G(\theta-2\alpha) = G((\theta-\alpha)-\alpha) = G(\alpha-\theta-\alpha) = G(-\theta) = G(\theta)$

$\Longrightarrow$ $\cancel{U_{2\alpha} g = g}$ $G(\theta - 2n\alpha) = G(\theta)$ $\forall n\in\mathbb{N}$

$\Longrightarrow$ $G(\theta)$ 为常数 $\Longrightarrow$ $g = h_f$

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