Week 3

Ex1. given a function f. call the set $\overline{\{x : f(x) \neq 0\}}$ its support.

a) We see that the Dirichlet kernel $D_N(t)$, Fejér kernel $F_N(t)$ and Poisson kernel $P_r(t)$ are function on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then specify the supports of $\widehat{D_N}$, $\widehat{F_N}$ and $\widehat{P_r}$ (as functions ~~de~~ defined on $\mathbb{Z}$) respectively. and Picture out their graphs.

b) For Fourier transform.

(i) on $\mathbb{R}^1$, try to picture out the graphs of $\widehat{D_R}$ and $\widehat{F_R}$ respectively.

(ii) on $\mathbb{R}^n$. try to picture out the graphs of $\widehat{P_t}$ and $\widehat{W_t}$

Ex2: density problems

a) $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$, in the topology generated by the family $\{P_{\alpha,\beta}\}$ of semi-norms.

b) $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$.

Hints: given $f \in L^p$, one can reduce to show this for f a Lebesgue step function; linearity allows to assume $f = \mathbb{1}_A$ where A is of finite measure.

- Outer regularity of Lebesgue measure $\rightsquigarrow$ Ass. A is open
- Inner regularity of Lebesgue measure $\rightsquigarrow$ ~~Ass.~~ $\exists\ K_n \subset K_{n+1} \subset A$ s.t. $\|\mathbb{1}_A - \mathbb{1}_{K_n}\|_{L^p} \longrightarrow 0$.
- Urysohn lemma, $\forall n$, $\exists\ \varphi_n \in C_c(\mathbb{R}^n)$ with $\mathbb{1}_{K_n} \le \varphi_n \le \mathbb{1}_A$

c). $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$

Conclude that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$.

Ex3. use complex analytic method to show

$$\widehat{f}(\xi) = e^{-\pi|\xi|^2}$$

where $f(x) = e^{-\pi|x|^2}$. one possible reference for this is the book « lectures on harmonic analysis », by. T. Wolff. (Chapter 3)

Ex4. let $h_a(x) := \frac{\Gamma(a/2)}{\pi^{a/2}} |x|^{-a}$, $x \in \mathbb{R}^n$

a) if $\frac{n}{2} < a < n$, then $h_a \in L^1 + L^2$, and

$$\widehat{h_a} = h_{n-a}$$

b) using complex analytic theory, ~~to~~ show

$$\widehat{h_a} \underset{T}{=} h_{n-a}. \quad \text{if } \frac{n}{2} < \mathrm{Re}(a) < n.$$

c). in the sense of ~~distribution~~ (as function), there holds

$$\widehat{h_a} = h_{n-a}. \quad \text{if } 0 < \mathrm{Re}(a) < n.$$

hints: functions of the form $f(x) = c|x|^{-a}$ with constant may be characterized by

(1) f is radial

(2). f is homogeneous of degree $-a$.

Ex5. verify the exact formula of Poisson kernel.

Ex6. We have seen in the class time the chain of inclusions

$$C^\infty \subset C^k \subset C^0 \overset{(*)}{\subset} L^p \subset \mathcal{S}'.$$

Here (*) means $C^0 \subset L^p_{loc}$

a) show that ~~the~~ each inclusion is strict.

b) given an integrable func. f, then $|f(x)|dx$ defines a finite Borel measure. prove this. the reverse of this is Radon-Nikodym thm. More generally, a finite Borel measure can define a tempered distribution, ~~try to~~ to make this clear.

c) show that their exists a tempered distribution that is not a finite Borel measure.