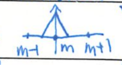
sup ≠ limsup

1. (2.3.14) Let $X_1, X_2, \cdots$ independent. Show that $\sup X_n < \infty$ a.s. $\Leftrightarrow \sum_n P(X_n > A) < \infty$ for some A
(B-C lemma)

2. (3.2.11) Let $X_1, X_2, \cdots$ int-valued. Show that $X_n \xrightarrow{d} X_\infty \Leftrightarrow P(X_n = m) \to P(X_\infty = m)$ $\forall m$.
($Ef(X_n) \to Ef(X_\infty)$ f: )

3. (4.4.5) $\mathcal{F} \subset \mathcal{G}$, then $E(E[Y|\mathcal{G}] - E[Y|\mathcal{F}])^2 = E(E[Y|\mathcal{G}])^2 - E(E[Y|\mathcal{F}])^2$
$(E(E[Y|\mathcal{F}] \cdot E[Y|\mathcal{F}]) = E(Y E[Y|\mathcal{F}]))$.

Solution of 2:

Mtd1. $\exists N > 0$, s.t. $P(|X_\infty| \le N) > 1 - \varepsilon$
$\exists N' > 0$, s.t. $\forall n > N'$, $\left|\sum_{|m|\le N} P(X_n = m) - \sum_{|m|\le N} P(X_\infty = m)\right| < \varepsilon$
and $\left|\sum_{|m|\le N} f(m) P(X_n = m) - \sum_{|m|\le N} f(m) P(X_\infty = m)\right| < \varepsilon$
$\therefore \left|\sum \boxed{n} - \sum \boxed{\infty}\right| \le \left|\sum \boxed{n} - \sum_{|m|\le N} \boxed{n}\right| + \left|\sum_{|m|\le N} \boxed{n} - \sum_{|m|\le N} \boxed{\infty}\right| + \left|\sum_{|m|\le N} \boxed{\infty} - \sum \boxed{\infty}\right| \le 2a\varepsilon + \varepsilon + a\varepsilon$. #

Mtd2. $\forall G$ open, $G_N \triangleq G \cap \{-N, \dots, N\}$, then $\liminf P(X_n \in G) \ge P(X_\infty \in G_N)$
$\Rightarrow \liminf P(X_n \in G) \ge P(X_\infty \in G)$ Apply Thm3.2.11 #

Mtd3. (DCT) 由: $|f_n| \le g_n$, $f_n \to f$, $g_n \to g$, $\mu(g_n) \to \mu(g) < \infty$, then $\mu(f_n) \to \mu(f)$.
此时取 $(\Omega, \mathcal{F}, \mu) = (\mathbb{Z}, 2^{\mathbb{Z}}, \#)$, $g_n(m) = a P(X_n = m)$ $g(m) = a P(X_\infty = m)$ #.

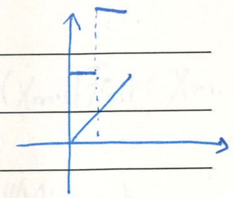
1. Prob. spaces & r.v.s
- $(\Omega, \mathcal{F}, P)$, $\omega \in \Omega$, $A \in \mathcal{F}$, $P(A) \in [0,1]$
- $X: \Omega \to \mathbb{R}$, $\omega \mapsto X(\omega)$, $(\mathbb{R}, \mathcal{B})$, $X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$, $P \circ X^{-1}(B) = P(X^{-1}(B))$
$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}\}$

2. Independence
$\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$, $\mathcal{F}_1 \perp \mathcal{F}_2$: $P(A_1 \cap A_2) = P(A_1) P(A_2)$
$X_1 \perp X_2$: $\sigma(X_1) \perp \sigma(X_2)$: $P(X_1 \in B_1, X_2 \in B_2) = P(X_1 \in B_1) P(X_2 \in B_2)$

3. Kolmogorov's 0-1 Law & Borel-Cantelli lemma.
- $\mathcal{F}_1, \mathcal{F}_2, \cdots$ independent, then $\forall A \in \bigcap_n \bigvee_{k>n} \mathcal{F}_k$, $P(A) = 0$ or 1.
$\bigvee_{k>n} \mathcal{F}_k = \sigma(\mathcal{F}_{n+1}, \mathcal{F}_{n+2}, \cdots)$ the tail σ-field
- ① $A_n \in \mathcal{F}$, $\sum_n P(A_n) < \infty \Rightarrow P(A_n, i.o.) = 0$, 其中 $\{A_n, i.o.\} = \{\omega : \sum_n 1_{A_n}(\omega) = \infty\}$.
② A_n 独立, $\sum_n P(A_n) = \infty \Rightarrow P(A_n, i.o.) = 1$

4. Convergence of r.v.s
a.s. $\longrightarrow$ p $\underset{X\,\text{const.}}{\rightleftarrows}$ d. $Ef(X_n) \xrightarrow{n\to\infty} Ef(X)$, $\forall f \in C_b(\mathbb{R})$
$L^p \longrightarrow$ p
$P(\omega : |X_n(\omega) - X(\omega)| > \varepsilon) \xrightarrow{n\to\infty} 0$

5. Conditional expectation
- $E(X|Y) = E(X|\sigma(Y))$ 例: $\Omega = [0,1]$, $\mathcal{B}$, $P(A)$: Lebesgue measure.
$X(\omega) = \omega$, $EX = \frac{1}{2}$
$Y(\omega) = 1_{[0,\frac{1}{2})}(\omega) + 2 \cdot 1_{[\frac{1}{2},1]}(\omega)$
则 $E(X|Y) = \frac{1}{4} 1_{[0,\frac{1}{2})}(\omega) + \frac{3}{4} 1_{[\frac{1}{2},1]}(\omega)$



6. Random walks & Brownian motion
$\xi, \xi_1, \xi_2, \cdots$ i.i.d. $\sim P(\xi = 1) = P(\xi = -1) = \frac{1}{2}$
$S_n = \xi_1 + \cdots + \xi_n$ 由CLT, $\frac{1}{\sqrt{n}} S_n \xrightarrow[n\to\infty]{d,a.s.} N(0,1)$
$T_n = \frac{1}{\sqrt{n}} S_n$, $T_{3.6} = 0.4 T_3 + 0.6 T_4$.
$T_n^m = \frac{1}{\sqrt{m}} S_n$
$T_t^m \xrightarrow{d} B$ $B_t \sim N(0,t)$.
Now consider $(\frac{1}{\sqrt{n}} S_{nt})_{t\ge0} \xrightarrow{d} (B_t)_{t\ge0}$ as random variables with values in $C([0,\infty), \mathbb{R})$
where $B = (B_t)_{t\ge0}$: the standard Brownian motion ($B_0 = 0$, $B_t \sim N(0,t)$.)

E.g. random walk: ξ_i i.i.d. $S_n=\xi_1+\cdots+\xi_n$, $\mathcal{F}_n=\sigma(\xi_1,\cdots,\xi_n)$, $\forall n\geq 1$, $\mathcal{F}_0=\{\phi,\Omega\}$

$E\xi_1=0$

Check 它为martingale: $E(S_{n+1}|\mathcal{F}_n)=E(S_n+\xi_{n+1}|\mathcal{F}_n)=S_n+E(\xi_{n+1}|\mathcal{F}_n)=S_n$

E.g. (linear) $E\xi_1=\mu$, $(S_n-n\mu)_{n\geq1}$ is a martingale.

E.g. (quadratic) $E\xi_1=0$, $\sigma^2=Var(\xi_1)<\infty$. $(S_n^2-n\sigma^2)$ is a martingale

Check: $E[S_{n+1}^2-(n+1)\sigma^2|\mathcal{F}_n]=E[S_n^2-n\sigma^2+2S_n\xi_{n+1}+\xi_{n+1}^2-\sigma^2|\mathcal{F}_n]=S_n^2-n\sigma^2$

E.g. $X_n=-\frac{1}{n}$ sub, while $X_n^2=\frac{1}{n^2}$ super.

Pf of $H\cdot X$ sup $\Leftarrow$ X sup + H bdd predictable:

$E((H\cdot X)_{n+1}-(H\cdot X)_n|\mathcal{F}_n)$

$=E(H_{n+1}(X_{n+1}-X_n)|\mathcal{F}_n)$

$=H_{n+1}E[X_{n+1}-X_n|\mathcal{F}_n]\leq 0$. $\Rightarrow (H\cdot X)$ is super. #

Chap 4 Martingales

· Martingale $X=(X_n)_{n\geq0}$: is the fortune at time n of a player who is betting on a fair game.

Submartingale / Supermartingale: favorable / unfavorable game

· Two Facts:

① (Thm 4.2.8) $EX_N=X_0$, $\forall$ bdd. stopping time N.

② (Thm 4.2.11) Submartingales: nondecreasing sequences. 上方控制即可收敛.

If $\sup_n X_n^+<\infty$, then X_n a.s. converges

4.2. a.s. convergence

· Filtration $\{\mathcal{F}_n\}$: an increasing seq of σ-fields.

· $(X_n)_{n\geq0}$ is adapted to $(\mathcal{F}_n)_{n\geq0}$: $X_n\in\mathcal{F}_n$, $\forall n$

Def. (Martingales) $X=(X_n)_{n\geq0}$ is a martingale (w.r.t. $\mathcal{F}_n$): $\forall n$,

① $E|X_n|<\infty$

② $X_n\in\mathcal{F}_n$

③ $E(X_{n+1}|\mathcal{F}_n)=X_n$ "$\geq$" sub, "$\leq$" super

Thm 4.2.4/4.2.5 If X super (sub), then $E(X_n|\mathcal{F}_m)\leq(\geq) X_m$, $\forall n>m$.

Pf: $E(X_{m+1}|\mathcal{F}_m)\leq X_m$, $E(X_{m+2}|\mathcal{F}_m)=E(E(X_{m+2}|\mathcal{F}_{m+1})|\mathcal{F}_m)\leq E(X_{m+1}|\mathcal{F}_m)\leq X_m$. #

If X martingale, then $E(X_n|\mathcal{F}_m)=X_m$

Thm 4.2.6/4.2.7 (Jensen) ① X: martingale, φ: convex, $E|\varphi(X_n)|<\infty$, then $\varphi(X)$ is sub.

② X: sub, φ: increasing convex, then $\varphi(X_n)$ is sub.

③ X: super, then $X\wedge a$ super

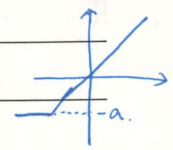
Pf: (Jensen: $E(\varphi(X))\geq\varphi(EX)$, $E(\varphi(X)|\mathcal{F})\geq\varphi(E(X|\mathcal{F}))$.)

① $E(\varphi(X_{n+1})|\mathcal{F}_n)\geq\varphi(E(X_{n+1}|\mathcal{F}_n))=\varphi(X_n)$

② $E(\varphi(X_{n+1})|\mathcal{F}_n)\geq\varphi(E(X_{n+1}|\mathcal{F}_n))\geq\varphi(X_n)$

③ consider $-[(-x)\wedge a]$, which is increasing convex, then use ②

$-X$: sub $(-X)\vee(-a)$: sub $-((-X)\vee(-a))=X\wedge a$: sup



#

· Discrete Stochastic Integral $(H\cdot X)_n=\sum_{1\leq m\leq n}H_m(X_m-X_{m-1})$, $n\geq0$.

Def. $H=(H_n)_{n\geq1}$ is a predictable sequence, if $H_n\in\mathcal{F}_{n-1}$, $\forall n\geq1$.

Thm 4.2.8 X super (sub), if $H\geq0$ predictable and H_n is bdd, then $H\cdot X$ is super (sub)

Remark If X martingale, $H\geq0$ is not needed.

E.g. 设N为停时，$H_n = 1_{\{n \le N\}}$ 为 predictable. $(\because \{N \ge n\} = \{N \le n-1\}^c \in \mathcal{F}_{n-1})$

$(H\cdot X)_n = \sum_{1\le m\le n} H_m(X_m - X_{m-1}) = X_{N\wedge n} - X_0$

Pf of bdd increments:

Let $0<K<\infty$, $N := \inf\{n : X_n \le -K\}$, then $X_{N\wedge n} \overset{\ge -K-M}{}$ is martingale

$\therefore (X_{N\wedge n} + K + M) \ge 0$

在$\{N=\infty\}$上，$\overset{\text{由Cor.}}{} X_{N\wedge n} + K + M = X_n + K + M$ 存在收敛极限 $\Rightarrow \{N=\infty\} \subset C$.

否则，考虑D的余集，

$\because \{\underline{\lim}\, X_n > -\infty\} = \{\inf X_n > -\infty\} = \bigcup_K \{\inf X_n > -K\}$

且在$\{\inf X_n > -K\}$上$N=\infty$ $\quad\therefore \{\underline{\lim}\, X_n > -\infty\} \subset C$

同理$\{\overline{\lim}\, X_n < +\infty\}$中，极限存在. #

C.e.g. (Cor $\not\Rightarrow L^1$) simple random walk with $S_0 = 1$.

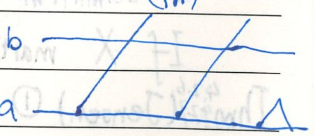
$N = \inf\{n : S_n = 0\}$, then $(S_{n\wedge N}) = (X_n) \to X_\infty \equiv 0$ (若X_∞为其它，$X_\infty \pm 1$距地太大了)

但$EX_n = 1$, $EX_\infty = 0$.

C.e.g. 2. $X_n \xrightarrow{P} 0$ but $X_n \not\to$ a.s. 见 Example 4.2.14.

下面处理 Upcrossing:

$$H_m \triangleq \begin{cases} 1, & N_{2k-1} \le m \le N_{2k} \\ 0, & \text{otherwise} \end{cases}$$



Pf: $Y_m \triangleq a + (X_m - a)^+$ is sub. 且 upcrossing 次数与(X_m)一致

只须证 $(b-a)U_n \underset{\text{显然}}{\le} (H\cdot Y)_n \underset{H\le 1}{\le} EY_n - EY_0 = \text{RHS}$

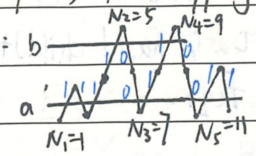
$E(H\cdot Y)_n + E((1-H)\cdot Y)_n = EY_n - EY_0$.

而由Y sub，$(1-H)\ge 0$，$(1-H)\cdot Y$ sub $\ge ((1-H)\cdot Y)_0 = 0$ #

Def. Stopping time: a r.v. $N \in \mathbb{N} \cup \{\infty\}$ is a stopping time, if $\{N=n\} \in \mathcal{F}_n$, $\forall n$.

Thm 4.2.9. X super$^{(sub)}$, N stopping time, then $X_{N\wedge n}$ is super$^{(sub)}$ (H如左构造，且X_0 is super)

• Upcrossing:



$N_{2k-1} = \inf\{m > N_{2k-2} : X_m \le a\}$

$N_{2k} = \inf\{m > N_{2k-1} : X_m \ge b\}$ 令 $N_0 = -1$

$U_n = \sup\{k : N_{2k} \le n\}$.

Thm 4.2.10 (Upcrossing ineq.) If X sub, then $(b-a)EU_n \le E(X_n-a)^+ - E(X_0-a)^+$

▲ Thm 4.2.11 (Martingale convergence). If X is sub with $\sup_n EX_n^+ < \infty$ $\left(= \lim_{n\to\infty} EX_n^+\right)$, Then $X_n \to X$ a.s., $E|X| < \infty$

Pf: $\overset{\forall a,b.}{} EU_n \uparrow EU_\infty \overset{\text{upcrossing}}{\le} \frac{1}{b-a}\left(E(X_n-a)^+ - E(X_0-a)^+\right) \le \frac{1}{b-a}\left(\sup X_n^+ + |a|\right) < \infty$ $(\because (x-a)^+ \le x^+ + |a|)$

$\Rightarrow U_\infty < \infty$, a.s.

$\therefore \bigcup_{a,b\in\mathbb{Q}} \{\underline{\lim}\, X_n < a < b < \overline{\lim}\, X_n\}$ has prob. 0. $\Rightarrow \lim X_n$ exists. a.s.

By Fatou, $EX^+ \le \underline{\lim}\, EX_n^+ < \infty$

$EX^- \le \underline{\lim}\, EX_n^- = \underline{\lim}\,(EX_n^+ - EX_n) \le \underline{\lim}\,(EX_n^+ - EX_0) \le \sup EX_n^+ - EX_0 < \infty$

Thm (Cor). If $X \ge 0$ super, then $X_n \to X$ a.s. & $EX \le EX_0$ (由 $EX \le \underline{\lim}\, EX_n \le EX_0$)

Thm 4.3.1 (bdd increments) X: martingale, $|X_{n+1} - X_n| \le M < \infty$, $\forall n$. Let $C = \{\lim X_n$ 存在有限$\}$,

$D = \{\overline{\lim}\, X_n = +\infty$ && $\underline{\lim}\, X_n = -\infty\}$,

then $P(C\cup D) = 1$.

Rmk. $\exists$ strict sub X, s.t. (X_n^-) not super: $P(1) = \frac{2}{3}$ 的 random walk, $S_0^- = 0 \le S_1^-$.

Thm (Doob's decomposition) Any sub $(X_n)_{n\ge 0}$, $\exists 1$ decomposition $X_n = M_n + A_n$.

where (M_n) is martingale and (A_n) predictable nondecreasing seq with $A_0 = 0$.

$A_n = \sum_{m=1}^{n} E(X_m - X_{m-1} \mid \mathcal{F}_{m-1})$

Thm. (B-C lemma, II) $B_n \in \mathcal{F}_n$, then $\{B_n, \text{i.o.}\} = \{\sum P(B_n \mid \mathcal{F}_{n-1}) = \infty\}$

Pf: $X_n = \sum_{0\le m\le n} 1_{B_m}$ $(X_0 = 0)$, then Doob's decomposition: $A_n = \sum_{m=1}^{n} E(1_{B_m} \mid \mathcal{F}_{m-1})$, $M_n = \sum_{m=1}^{n} \left(1_{B_m} - P(B_m \mid \mathcal{F}_{m-1})\right)$

By bdd increments, on D, $\sum 1_{B_n} = \infty$ and $\sum P(B_n \mid \mathcal{F}_{n-1}) = \infty$

on C, $\sum 1_{B_n} = \infty$ iff $\sum P(B_n \mid \mathcal{F}_{n-1}) = \infty$ #

Rmk. If B_n independent. $P(B_n, \text{i.o.}) = 1 \Leftrightarrow \sum P(B_n) = \infty$

$P(B_n, \text{i.o.}) = 0 \Leftrightarrow \sum P(B_n) < \infty$

4.3.4. 分支过程

• $p_n = P(\xi = n)$, $p_n \ge 0$, $1 = p_0 + p_1 + \cdots$ 表示生孩子的个数等.

T: random tree 比如有:

$P(T = \bullet) = p_0$

$P(T = \vee) = p_2 p_0 p_0$

Pf of branching extinction theorem when $\mu>1$:

$q = P(\tau \text{ is finite}) = \sum_{n\ge0} P(\tau \text{ is finite}, \phi \text{ has } n \text{ children})$

$= P(\phi \text{ no child}) + P(\text{finite } \tau, \phi \text{ has 1 child}) + P(\tau \text{ finite}, \phi\ 2 \text{ ch}$ ...

$= p_0 + p_1 q + p_2 q^2 + \cdots = f(q).$ #

(Branching Processes)

Def. ξ_i^n, $i, n \ge 1$ i.i.d. 非负整值, $Z = (Z_n)_{n\ge0}$, $Z_0 = 1$,

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1}, & Z_n > 0. \\ 0, & Z_n = 0 \end{cases}$$

称 $p = (p_n)_{n\ge0}$ offspring distribution

Lem. If $\mu = E\xi \in (0,\infty)$, then $\frac{Z_n}{\mu^n}$ is martingale.

(Thm If $X=Y$ a.s. on $B \in \mathcal{F}$, then $E(X|\mathcal{F}) = E(Y|\mathcal{F})$ a.s. on B)

Pf: on $\{Z_n = k\}$, $E(Z_{n+1}|\mathcal{F}_n) = k\mu = \mu Z_n$. Then take all k. #.

Thm. If $\mu<1$, then $Z_n = 0$ for n sufficiently large, so $\frac{Z_n}{\mu^n} \to 0$.

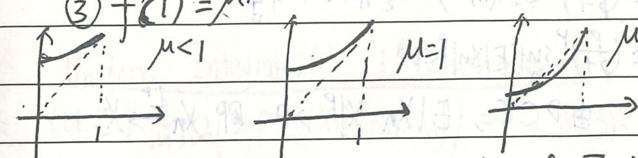
($\Rightarrow E(Z_n) = \mu^n \Rightarrow P(Z_n>0) = P(Z_n \ge 1) \le E(Z_n I(Z_n\ge1)) = EZ_n = \mu^n \to 0.$)

- Generating func.: if $1 = p_0 + p_1 + \cdots > p_0 + p_1$, 称 $f(s) = \sum_{k\ge0} p_k s^k$, $0 \le s \le 1$ 为 Generating func.

prop: ① strictly convex, increasing

② $f(0) = p_0$ $f(1) = 1$

③ $f'(1) = \mu$.

- 

$\mu>1$ 时, $\exists f(s) = s$.

Thm. If $\mu>1$, the extinction prob. of Z is the smallest root q of equation $f(s)=s$.

Pf thm 4.4.1: 1° 由 thm 4.2.9. $EX_0 \le EX_{N\wedge k} = EX_N$

2° $H_n = 1_{\{n > N\}}$ is predictable.

$(H\cdot X)_k = X_k - X_N$ 由4.2.8 它是 sub

$\Rightarrow E(X_k - X_N) \ge 0 \Rightarrow EX_N \le EX_k$. #

Pf Doob's Ineq: Let $N = \inf\{m: X_m \ge \lambda \text{ or } m = n\}$ 由4.4.1, $EX_N \le EX_n$

$\therefore \lambda P(A) \le EX_N 1_A \le EX_n 1_A$. 且 $X_N = X_n$ on A^c #

Pf L^p convergence:

$\because (EX_n^+)^p \le (E|X_n|)^p \le E|X_n|^p \le K < \infty$

$\therefore$ 由4.2.11, $X_n \to X$, a.s.

由4.4.4 $E(\max_{0\le m\le n}|X_m|)^p \le (\frac{p}{p-1})^p E(|X_n|^p)$ 令 $n\to\infty$. 得

$E(\sup_m |X_m|)^p \le (\frac{p}{p-1})^p \sup_n E|X_n|^p < \infty$

$\because |X_n - X|^p \le (2\sup|X_n|)^p \in L^1$ 由DCT, $E|X_n-X|^p\to 0$ 即 $X_n \xrightarrow{L^p} X$. #

Pf Thm 4.6.1:

$\because E(E(X|F)Y) = E(XE(Y|F)) = E(E(X|F)E(Y|F))$

$\therefore E(|E(\xi|F)|; A) \le E(E(|\xi||F); A) = E(|\xi|\cdot E(1_A|F))$

令 $A = \{|E(\xi|F)| > M\}$ 只需证 $P(A) \le \frac1M E(|E(\xi|F)|;A) \le \frac1M E|\xi| \to 0$. By DCT 得证. #

(⇐) 方法同上

Pf (K. Lem 4.10): (⇒) 若 ∃ A_n $P(A_n)\to 0$, $E(|\xi_{t_n}|; A_n) \ge \varepsilon$

$1_{A_n} \xrightarrow{P} 0 \overset{子列}{\Rightarrow} 1_{A_n} \xrightarrow{a.e.} 0$

$\sup E[|\xi_{t_n}|; |\xi_{t_n}| > M] < \frac{\varepsilon}{2}$

$E[\xi_{t_n}1_{A_n}; |\xi_{t_n}| \le M] \ge \frac\varepsilon2 \Rightarrow P(A_n) \ge \frac{\varepsilon}{2M}$ 矛盾! #

Pf Thm 4.6.4: ①⇒②: $\sup E|X_n| < \infty$, 由鞅收敛得 a.s., 由Thm 4.6.3 得 L^1.

③⇒①: Thm 4.6.3. ($L^1 \Rightarrow P$) #

4.2.8 (X_n) sub, $H_n \ge 0$ predictable bdd, then $(H\cdot X)$ is sub. 特别 $H_n = 1_{\{n\le N\}}$ is predictable.

4.2.9 X sub, N stopping time, then $X_{n\wedge N}$ is sub.

4.2.11 X sub, $\sup_n EX_n^+ < \infty$, then $X_n \xrightarrow{a.s.} X$, $E|X| < \infty$.

4.4 Doob's Ineq. convergence in $L^p (p>1)$

Thm 4.4.1 X sub, N stopping time $P(N\le k) = 1$, then $EX_0 \le EX_N \le EX_k$.

c.e.g. $S_0 = 1$ 的 simple random walk. $N = \inf\{n: S_n = 0\}$, then $1 = ES_0 \le ES_N = 0$.

这是因为 N is not bdd.

Thm 4.4.2 (Doob's Ineq) X sub, $\bar X_n = \max_{0\le m\le n} X_m^+$, $\lambda>0$, $A = \{\bar X_n \ge \lambda\}$

Then $\lambda P(A) \le EX_n 1_A \le EX_n^+$

Thm 4.4.4 (L^p maximum Ineq.) X sub, $1<p<\infty$, $E(\bar X_n^p) \le (\frac{p}{p-1})^p E(X_n^+)^p$

Consequently, if Y martingale, $Y_n^* = \max_{0\le m\le n}|Y_m|$, then $E|Y_n^*|^p \le (\frac{p}{p-1})^p E(|Y_n|^p)$

Rmk. There is no L^1 maximal inequality

Thm 4.4.6 (L^p convergence) X martingale with $\sup_n E|X_n|^p < \infty$, $p>1$.

then $X_n \to X$ a.s./L^p.

4.6 Uniform Integrability, L^1 convergence

Def. (U.I.) $\lim_{M\to\infty}(\sup_i E[|X_i| I_{|X_i|>M}]) = 0$, 则称 $\{X_i, i\in I\}$ U.I.

Rmk. $E|X|<\infty \iff \lim_{M\to\infty} E(|X|; |X|>M) = 0$ (∵(⇐) $E|X| \le M+1 < \infty$ (⇒) DCT.)

Rmk. U.I. ⇒ 一致 L^1 有界 (∵ $\sup E|X_i| \le M+1$.)

Thm 4.6.2 (一致L^1有界+条件 ⇒ U.I.) $\varphi \ge 0$, $\frac{\varphi(x)}{x}\to\infty$ $(x\to\infty)$. If $\sup E\varphi(|X_i|) \le C$.

then $\{X_i\}_{i\in I}$ is U.I.

Thm 4.6.3. $E|X_n|<\infty$, $X_n \xrightarrow{P} X$, then

① $\{X_n, n\ge 0\}$ is U.I.

⇔ ② $X_n \xrightarrow{L^1} X$

⇔ ③ $E|X_n| \to E|X| < \infty$.

Thm 4.6.1 (Kallenberg lem 6.5) $\forall \xi \in L^1$, $\{E(\xi|F), F \subset \mathcal{A}\}$ are U.I.

• (Kellenberg lem 4.10) $\{\xi_t, t\in T\}$ are U.I. ⇔ $\begin{cases} \sup_t E|\xi_t| < \infty \\ \lim_{P(A)\to 0}\sup_{t\in T} E(|\xi_t| I_A) = 0 \end{cases}$

• $f_n \xrightarrow{P} f \iff \forall$ subseq, ∃ subseq $f_n \xrightarrow{a.e.} f$.

从而 $|f_n| \le g$, $f_n \xrightarrow{P} f \Rightarrow Ef_n \to Ef$.

Thm 4.6.4. X_n sub, then ① X_n U.I.

⇔ ② a.s. & L^1 收敛

⇔ ③ L^1 收敛.

Pf Thm 4.6.8: $Y_n = E(X|\mathcal{F}_n)$ is martingale. U.I. on $\mathcal{F}_\infty$ (Thm 4.6.1)

$\xRightarrow{\text{Thm 4.6.7}}$ $Y_n \to Y_\infty$ a.s. & L^1

$\xRightarrow{\text{lem 4.6.6.}}$ $E(X|\mathcal{F}_n) = Y_n = E(Y_\infty|\mathcal{F}_n)$, $\forall n$.

∴ $E(X;A) = E(Y_\infty;A)$, $\forall A \in \bigcup \mathcal{F}_n$, 且 $\bigcup \mathcal{F}_n$ 为π类.

由π-λ定理 及 $Y_\infty \in \mathcal{F}_\infty$, $Y_\infty = E(X|\mathcal{F}_\infty)$. #.

Pf (DCT): 由 Thm 4.6.8. $E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty)$ a.s.

AIM: $|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| \le E(|Y_n - Y|\,|\mathcal{F}_n) \overset{?}{\to} 0$, a.s. 【$|Y_n - Y|$不单调!】

考虑 $W_N \triangleq \sup\{|Y_n - Y_m| : n, m \ge N\}$,

则 $W_N \le 2Z \in L^1$, $W_N \downarrow 0$, a.s.

∴ $\varlimsup_n E(|Y_n - Y|\,|\mathcal{F}_n) \le \lim_n E(W_N|\mathcal{F}_n) = E(W_N|\mathcal{F}_\infty) \xrightarrow{a.s.} 0 \ (N \to \infty)$ #.

lem 4.6.5. If $X_n \xrightarrow{L^1} X$, then $E(X_n;A) \to E(X;A)$, $\forall A \in \mathcal{F}$

lem 4.6.6. If martingale $X_n \xrightarrow{L^1} X$, then $X_n = E(X|\mathcal{F}_n)$. ($\forall A \in \mathcal{F}_n$, $E(X_m;A) \xrightarrow{\text{lem 4.6.5}} E(X;A)$ $= E(X_n;A)$)

Thm 4.6.7. For a martingale, ① U.I

⟺ ② a.s. & L^1

⟺ ③ L^1

⟺ ④ $\exists X \in L^1$, $X_n = E(X|\mathcal{F}_n)$.

Thm 4.6.1

lem 4.6.6.

Thm 4.6.8. $\mathcal{F}_n \uparrow \mathcal{F}_\infty = \sigma(\bigcup \mathcal{F}_n)$, then $E(X|\mathcal{F}_n) \to E(X|\mathcal{F}_\infty)$ a.s. & L^1.

Thm 4.6.9. (Lévy's 0-1 Law) $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, $A \in \mathcal{F}_\infty$, then $E(1_A|\mathcal{F}_n) \to 1_A$ a.s.

(Cor.) (Kolmogorov's 0-1 Law) X_n independent, $A \in \mathcal{T}$ (the tail σ-field), then $P(A) = 0$ or 1. ($E(1_A|\mathcal{F}_n) \overset{\perp\!\!\!\perp \mathcal{F}_n}{=} P(A) \to 1_A$ in $\Omega \setminus N$, $P(N) = 0$.)

Thm 4.6.10 (DCT) $Y_n \to Y$, a.s. $|Y_n| \le Z \in L^1$. If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, then $E(Y_n|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty)$, a.s.

4.7 Backwards Martingale

Def. (Backwards Martingale) $X=(X_n)_{n\le 0}$ adopted to $(\mathcal{F}_n)_{n\le 0}$ ($\mathcal{F}_n \subset \mathcal{F}_{n+1}$)

$$E(X_{n+1}|\mathcal{F}_n) = X_n \quad \text{for } n\le -1.$$

Thm 4.7.1 (极限存在) $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. & L^1

Pf: $\{X_{-n}, \cdots, X_0\}$ 中上穿 $[a,b]$ 次数为 U_n. 由 Upcrossing ineq. $(b-a)EU_n \le E(X_0-a)^+$

$\therefore EU_\infty < \infty$. $\therefore$ the limit exists a.s.

$\because X_n = E(X_0|\mathcal{F}_n)$ $\therefore$ 由 Th 4.6.1, X_n u.I., 由 Th.4.6.3 $X_n \xrightarrow{L^1} X_{-\infty}$ #.

Thm 4.7.2 (极限刻画) $X_{-\infty} = \lim_{n\to-\infty} X_n$, $\mathcal{F}_{-\infty} = \bigcap_{n\le 0} \mathcal{F}_n$, then $X_{-\infty} = E(X_0|\mathcal{F}_{-\infty})$

Pf: ① $X_{-\infty} \in \mathcal{F}_{-\infty}$

② $E(X_{-\infty};A) = E(X_0;A)$, $\forall A\in\mathcal{F}_{-\infty}$ $\quad$ ($\because$ $\underset{\underset{\displaystyle E(X_0;A)}{\parallel}}{E(X_n;A)} \to E(X_{-\infty};A)$, $\forall A\in\mathcal{F}_{-\infty}$)

Thm 4.7.3. If $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$ as $n\to-\infty$,

then $E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_{-\infty})$ a.s. & L^1.

Pf: $X_n = E(Y|\mathcal{F}_n)_{n\le 0}$ is a back martingale $\Rightarrow$ $E(Y|\mathcal{F}_n)$ a.s. & L^1 收

$E(X_0|\mathcal{F}_{-\infty}) = E(E(Y|\mathcal{F}_0)|\mathcal{F}_{-\infty}) = E(Y|\mathcal{F}_{-\infty})$

SLLN: ξ_i, i.i.d. $E\xi = \mu$, then $\frac{S_n}{n} \longrightarrow \mu$, a.s. $\quad$ 考虑 $\mathcal{F}_{-n} = \sigma(S_n, S_{n+1}, \cdots)$

$$X_{-n} = \frac{S_n}{n}$$

则 $E(X_{-n}|\mathcal{F}_{-n-1}) = E\left(\frac{S_{n+1}-\xi_{n+1}}{n}\middle|\mathcal{F}_{-n-1}\right)$

$\overset{\text{对称性}}{=} \frac{S_{n+1}}{n} - \frac{S_{n+1}}{n(n+1)} = X_{-n-1}$

$\therefore X_{-n} \to X_{-\infty} = E(X_{-1}|\mathcal{F}_{-\infty}) = E(X_{-1})$ a.s.

Hewitt-Savage 0-1律. $\mathcal{F}_{-\infty} = \{\phi, \Omega\}$

4.8.1 Applications to RW | ξ_i i.i.d. $S_n=\xi_1+\cdots+\xi_n$, $\mathcal{F}_n=\sigma(\xi_1,\cdots,\xi_n)$

Thm 4.8.7 (Symmetric simple RW) $P(\xi=\pm1)=\frac{1}{2}$, $S_0=x$, $N=\min\{n: S_n\notin(a,b)\}$, $a<x<b$.

(a). $N<\infty$ a.s. ($E_xN<\infty$) $P_x(S_N=a)=\frac{b-x}{b-a}$, $P_x(S_N=b)=\frac{x-a}{b-a}$

(b) $E_0N=-ab$，从而 $E_xN=(b-x)(x-a)$

Pf: (a) $P_x(N>m(b-a))\overset{\text{不能有m个连续b-a次+1}}{\le}(1-2^{-(b-a)})^m$ ∴ $E_xN<\infty$.

∵ $E|S_N|<\infty$, $S_n 1_{\{N>n\}}$ U.I. 由Thm4.8.2

$ES_0=x=ES_N=aP_x(S_N=a)+b\cdot(1-P_x(S_N=a))\Rightarrow P_x(S_N=a)=\frac{b-x}{b-a}$

(b) 考虑 $S_0=0$, (S_n^2-n) is martingale $\Rightarrow ES^2_{n\wedge N}=E_0(n\wedge N)$

$\xrightarrow[BCT]{n\to\infty}\underbrace{E_0S_N^2}_{-ab}=E_0N$ #

(仍是symmetric SRW)

Thm 4.8.8 $S_0=0$, $T_1=\min\{n: S_n=1\}$，则 $Es^{T_1}=\frac{1-\sqrt{1-s^2}}{s}$，从而 $P(T_1=2n-1)=\frac{1}{2n-1}\frac{(2n)!}{(n!)^2}2^{-2n}$

(Hint: 为证$T_1<\infty$ a.s., $P_0(T_1<T_{-n})=\frac{n}{n+1}\to1$

$P_0(T_1<\infty)=\lim_{n\to\infty}P_0(T_1<T_{-n})=1$ (∵ $n\le T_{-n}\uparrow\infty$).

但是 $EN_{(1,-n)}=ET_1\wedge T_{-n}\le ET_1$，由Thm4.8.7知 $ET_1=\infty$.)

Thm 4.8.9 (Asymmetric SRW) $p(\xi=1)=p$, $p(\xi=-1)=q=1-p$, $p\neq q$.

(a) If $\varphi(y)=(\frac{1-p}{p})^y$, then $\varphi(S_n)$ is martingale.

(b) $P_x(T_a<T_b)=\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}$, $P_x(T_b<T_a)=\frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}$, $a<x<b$.

若 $\frac{1}{2}<p<1$，则 $S_0=0$

(c) If $a<0$, then $P(\min S_n\le a)=P(T_a<\infty)=(\frac{1-p}{p})^{-a}$

(d) If $b>0$, then $P(T_b<\infty)=1$ and $ET_b=\frac{b}{2p-1}$

Pf. (a)略

(b) $N=T_a\wedge T_b$, $\varphi(S_{N\wedge n})$ is bdd. 由Thm4.8.2

$\varphi(x)=P_x(T_a<T_b)\varphi(a)+(1-P_x(T_a<T_b))\varphi(b)$

(c) $P_x(T_a<T_b)\xrightarrow{b\to\infty}P_x(T_a<\infty)=(\frac{1-p}{p})^{-a}$ (d) $(S_n-(p-q)n)$ is martingale

$P_x(T_b<T_a)\xrightarrow{a\to-\infty}P_x(T_b<\infty)=1$ $ES_{T_b\wedge n}=(p-q)E(T_b\wedge n)$

$\min S_m\le S_{T_b\wedge n}\le b$ #

Ex 4.4.2. If X sub, $M\le N$ stopping times, N bdd, then $EX_M\le EX_N$

4.8 Optimal Stopping

Thm 4.8.1 X: U.I. sub, then ∀ stopping time N, $(X_{N\wedge n})$ is U.I. 由于X_n U.I.,

Pf: $E(|X_{N\wedge n}|;|X_{N\wedge n}|>K)=E(|X_N|;|X_N|>K,N\le n)+E(|X_n|;|X_n|>K,N>n)$. 只须$E|X_N|<\infty$

由于 $N\wedge n\le n$ 为stopping times 且n有界，由Ex 4.4.2. $E(X_{N\wedge n})^+\le E(X_n)^+$

∵ $\sup E(X_{N\wedge n})^+\le\sup E(X_n)^+\overset{U.I.}{<}\infty$

由Martingale Convergence, $X_{N\wedge n}\xrightarrow{a.s.}X_N$，且 $E|X_N|<\infty$. #

Thm 4.8.2. If $E|X_N|<\infty$, $X_n1_{\{N>n\}}$ is U.I., then $(X_{N\wedge n})$ is U.I. (由Th4.8.1证明第一行)

If further $(X_{N\wedge n})$ is sub, then $EX_0\le EX_N$. (由于$EX_0\le EX_{N\wedge n}\xrightarrow{L^1}EX_N$)

Thm 4.8.3. X_n: U.I. sub, then ∀ stopping time $N\le\infty$, $EX_0\le EX_N\le EX_\infty$

(∵ $EX_0\le EX_{N\wedge n}\le EX_n$. 令$n\to\infty$ 得 $X_{N\wedge n}\to X_N$, $X_n\to X_\infty\in L^1$)

Thm 4.8.4. $X_n\ge0$ super, N: stopping time, then $EX_0\ge EX_N$

($EX_0\ge\liminf EX_{N\wedge n}\ge EX_N$)

Thm 4.8.5 X sub, $E(|X_{n+1}-X_n|\,|\mathcal{F}_n)\le B$, a.s. N: stopping time, $EN<\infty$.

Then $X_{N\wedge n}$ is U.I., $EX_N\ge EX_0$.

Pf: $|X_{N\wedge n}|\le|X_0|+\sum_{m=0}^{\infty}|X_{m+1}-X_m|1_{\{N>m\}}\triangleq Y$ 只须 $EY<\infty$，从而有U.I.

∵ $E(|X_{m+1}-X_m|;N>m)=E(E(|X_{m+1}-X_m|\,|\mathcal{F}_m);N>m)\le BP(N>m)$

而 $\sum_{m=0}^{\infty}BP(N>m)=B\cdot EN<\infty$. #

Thm 4.8.6 (Wald's equation) ξ_i i.i.d. $E\xi=\mu$, $S_n=\xi_1+\cdots+\xi_n$. N: stopping time $EN<\infty$

Then $ES_N=\mu EN$ (Lebesgue积分下.)

(∵ $(S_n-n\mu)$ is a martingale, 由Thm4.8.5. $E(|\xi_{n+1}-\mu|\,|\mathcal{F}_n)=E|\xi-\mu|<\infty$)

Eg. ① RW. ξ_i i.i.d. $\sim \mu$. $X_n = X_0 + \xi_1 + \cdots + \xi_n$, 从而 $p(i,j) = \mu(\{j-i\})$, $p(i+m, j+m) = p(i,j)$ (spatially homogeneous)

② Branching Processes. $S = \{0,1,2,\cdots\}$, $p(i,j) = P(\sum_{m=1}^{i} \xi_m = j)$.

若 $\xi_i \overset{i.i.d.}{\sim} \mu$, 则 $p(i,j) = \mu^{*i}(j) = \underbrace{\mu * \cdots * \mu}_{i\uparrow}(j)$

但 $p(i,0) \neq p(i+1,1)$

③ Ehrenfest chain.(气体扩散) $S = \{0,1,\cdots,r\}$, X_n: number of balls in a urn.

$$p(0,j) = \begin{cases} 1, & \text{if } j=1 \\ 0, & \text{otherwise} \end{cases} \qquad p(r,j) = \begin{cases} 1, & \text{if } j=r-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\overset{0<i<r}{p(i,j)} = \begin{cases} 1-\frac{i}{r}, & j=i+1 \\ \frac{i}{r}, & j=i-1 \\ 0, & \text{otherwise} \end{cases}$$

Chap 5 Markov Chains

5.1 Examples | countable state space S

- Markov property $P(X_{n+1} = j \mid X_n = i, \cdots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i)$
- $p(i,j) = P(X_{n+1} = j \mid X_n = i)$, $p = (p(i,j))_{i,j \in S}$ one-step transition probability

$p^n(i,j) = P(X_n = j \mid X_0 = i)$ $\quad P(X_0 = i_0, \cdots, X_n = i_n) = P(X_0 = i_0)\, p(i_0, i_1) \cdots p(i_{n-1}, i_n)$

5.2 Construction, Markov Properties | $(S, \mathcal{S})$ measurable space. $X: (\Omega, \mathcal{F}) \to (S^\infty, \mathcal{S}^\infty)$, $\omega \mapsto (X_n(\omega))_{n\geq 0}$.

Def. transition probability $p: S \times \mathcal{S} \to \mathbb{R}$, if

① each $x \in S$, $A \to p(x,A)$ is a prob. measure on $(S, \mathcal{S})$

② each $A \in \mathcal{S}$, $x \to p(x,A)$ is a measurable func from $(S, \mathcal{S})$ to $(\mathbb{R}, \mathcal{B})$

We say X_n is Markov (w.r.t. $\mathcal{F}_n$) with trans prob. p, if

$$P(X_{n+1} \in B \mid \mathcal{F}_n) = p(X_n, B).$$

(随便)Thm (Kolmogorov's extension theorem) Given prob measure μ_n on $(\mathbb{R}^n, \mathcal{B}^n)$ that are consistent, 即 $\mu_{n+1}(B_1 \times \cdots \times B_n \times \mathbb{R}) = \mu_n(B_1 \times \cdots \times B_n)$

Then there is a unique prob measure P on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}})$ with $P(\omega: \omega_i \in B_i, 1 \leq i \leq n) = \mu_n(B_1 \times \cdots \times B_n)$.

- Given a trans prob. p, prob. measure μ on $(S, \mathcal{S})$, we can define a consistent set of prob measure $(\mu^{\{0,1,\cdots,n\}})_{n\geq 0}$ by

$$\mu^{\{0,1,\cdots,n\}}(B_0 \times \cdots \times B_n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n)$$

- Suppose that $(S, \mathcal{S})$ is nice, there is a prob measure P_μ on $(\Omega_0, \mathcal{F}_\infty) = (S^{\{0,1,\cdots\}}, \mathcal{S}^{\{0,1,\cdots\}})$

$X_n(\omega) \triangleq \omega_n$, 其中 $\omega = (\omega_0, \omega_1, \cdots)$ $\quad P_\mu(X_0 \in B_0, \cdots, X_n \in B_n) = \mu^{\{0,1,\cdots,n\}}(B_0 \times \cdots \times B_n)$

得到恒等映射 $(\Omega_0, \mathcal{F}^\infty, P_\mu) \to (\Omega_0, \mathcal{F}^\infty, P_\mu)$: $X(\omega) \mapsto \omega$ 满足有限维分布.

下面 check:

Thm 5.2.1. X_n is Markov (w.r.t. $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$) with trans prob. p. 即 $P_\mu(X_{n+1} \in B \mid \mathcal{F}_n) = p(X_n, B)$

Pf: 即证 $E_\mu(1_{\{X_{n+1} \in B\}}; A) = E_\mu(p(X_n, B); A)$ for any $A \in \mathcal{F}_n$.

由π-λ定理, 只须对π类 $\{A = \{X_0 \in B_0, \cdots, X_n \in B_n\}\}$ 成立即可.

左式 $= P_\mu\{X_0 \in B_0, \cdots, X_n \in B_n, X_{n+1} \in B\}$

$= \int_{B_0} \mu(dx_0) \cdots \int_{B_n} p(x_{n-1}, dx_n)\, p(x_n, B) \overset{?}{=}$ 右式

① $\int_{B_0} \cdots \int_{B_n} p(x_{n-1}, dx_n)\, 1_C(x_n) = E_\mu(1_C(X_n); A)$, $\forall C \in \mathcal{S}$

$\Rightarrow \int_{B_0} \cdots \int_{B_n} p(x_{n-1}, dx_n)\, f(x_n) = E_\mu(f(X_n); A)$, $\forall f$ 简单

$\Rightarrow$ $\quad \forall$ 有界可测 #

(特别, 取 $f(x) = p(x, B_{n+1})$).

If $\mu = \delta_x$, $P_\mu = P_{\delta_x} \equiv P_x$, then $P_\mu(A) = \int \mu(dx) P_x(A)$

Pf of strong Markov prop:

Let $A \in \mathcal{F}_N$, $\mathcal{F}_n$

$E_\mu(Y_N \circ \theta_N ; A \cap \{N<\infty\}) = \sum_{n=0}^{\infty} E_\mu(Y_n \circ \theta_n ; A \cap \{N=n\})$

$\overset{马性}{=} \sum_{n=0}^{\infty} E_\mu(E_{X_n} Y_n ; A \cap \{N=n\})$

$= E_\mu(E_{X_N} Y_N ; A \cap \{N<\infty\})$ #

Application of 强马:

· $T_y^0 = 0$, $T_y^k \triangleq \inf\{n > T_y^{k-1} : X_n = y\}$, $k \geq 1$: k-th return to y ($T_y = T_y^1$)

$\rho_{xy} \triangleq P_x(T_y < \infty)$

Thm 5.2.6 $P_x(T_y^k < \infty) = \rho_{xy} \rho_{yy}^{k-1}$

Pf: 强马: $E_x(Y \circ \theta_N | \mathcal{F}_N) = E_{X_N} Y$ on $\{N<\infty\}$

令 $Y(\omega) = \begin{cases} 1, & \text{if } \omega_n = y, \exists n \\ 0, & \text{else} \end{cases}$, $N = T_y^{k-1}$ $\Rightarrow Y \circ \theta_N = 1$ if $T_y^k < \infty$

强马左右在$\{N<\infty\}$上积分

$P_x(T_y^k < \infty) = E_x(E_{X_N} Y ; N<\infty)$

$\overset{由 X_N = y \text{ on } \{N<\infty\}}{=} E_x(E_y Y ; N<\infty)$

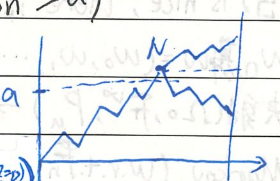
$= E_x(\rho_{yy} ; N<\infty)$

$= \rho_{yy} P_x(N<\infty) = \rho_{yy} P_x(T_y^{k-1} < \infty)$ #

Thm 5.2.7 (Reflection principle). $\xi_i \overset{iid}{\sim}$ symmetric about 0. $S_n = \xi_1 + \cdots + \xi_n$. $\forall a>0$,

$P(\sup_{m \leq n} S_m \geq a) \leq 2P(S_n \geq a)$

(不严格) Pf: $N = \inf\{m \leq n : S_m > a\}$



on $\{N<\infty\}$, $S_n - S_N \perp S_N$

$P(S_n - S_N \geq 0) \geq \frac{1}{2}$ (① 对称 Z $\Rightarrow \frac{1}{2} = P(Z>0) + \frac{1}{2}P(Z=0)$)

$\Rightarrow P(S_n > a) \geq \frac{1}{2} P(N \leq n)$

f, f_m有界可测.

· $E_\mu(f(X_{n+1}) | \mathcal{F}_n) = \int p(X_n, dy) f(y)$, 可导出 $E_\mu\left(\prod_{m=0}^{n} f_m(X_m)\right) \overset{(5.2.3)}{=} \int \mu(dx_0) f_0(x_0) \int p(x_0, dx_1) f_1(x_1) \cdots \int p(x_{n-1}, dx_n) f_n(x_n)$.

$= E_\mu\left(E_\mu\left(\prod_{m=0}^{n} f_m(X_m) \middle| \mathcal{F}_{n-1}\right)\right)$

$= E_\mu\left(\prod_{m=0}^{n-1} f_m(X_m) \cdot E_\mu(f_n(X_n) | \mathcal{F}_{n-1})\right)$ || 归纳

$= E_\mu\left(\prod_{m=0}^{n-1} f_m(X_m) \int p(X_{n-1}, dy) f_n(y)\right) = \cdots$

Next, Two extensions:

① $\{X_{n+1} \in B\}$ is replaced by $h(X_n, X_{n+1}, \cdots)$

② n is replaced by a stopping time N.

Thm 5.2.3 (Markov prop) Let $Y: \Omega_0 \to \mathbb{R}$ be bdd and measurable, then $E_\mu(Y \circ \theta_m | \mathcal{F}_m) = E_{X_m}(Y)$

where $\theta_n(\omega_0, \omega_1, \cdots) = (\omega_n, \omega_{n+1}, \cdots)$

Rmk. Define $\varphi(x) = E_x Y$ 关于x可测, then $E_{X_m}(Y) = \varphi(X_m)$. 两边同取$E(\cdot | X_m)$, 则$E_\mu(Y \circ \theta_m | X_m) = E_{X_m} Y$

Pf: 先考虑 $A = \{\omega : \omega_0 \in A_0, \cdots, \omega_m \in A_m\}$, $g_0, \cdots, g_n$ be bdd and measurable

取 $f_k = 1_{A_k}$ $(k<m)$, $f_m = 1_{A_m} g_0$, $f_k = g_{k-m}$ $(m<k \leq m+n)$, 则(5.2.3)变为.

$E_\mu\left(\prod_{k=0}^{n} g_k(X_{m+k}) ; A\right) \overset{(5.2.3)}{=} \int_{A_0} \mu(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_m} p(x_{m-1}, dx_m) \boxed{g_0(x_m) \cdots \int p(x_{m+n-1}, dx_{m+n}) g_n(x_{m+n})}$

$\boxed{\cdots} \overset{(5.2.3)}{=} E_{X_m}\left(\prod_{k=0}^{n} g_k(X_k)\right)$

$= E_\mu\left(E_{X_m}\left(\prod_{k=0}^{n} g_k(X_k)\right) ; A\right)$

$\therefore E_\mu(Y \circ \theta_m ; A) = E_\mu(E_{X_m} Y ; A)$, 对 $Y(\omega) = \prod_{0 \leq k \leq n} g_k(\omega_k)$ 成立.

Thm 6.22. (Monotone Class). Let $\mathcal{A}$ be a π类 containing Ω, $\mathcal{H}$: collection of 实值函数 that:

① $A \in \mathcal{A} \Rightarrow 1_A \in \mathcal{H}$

② $f, g \in \mathcal{H} \Rightarrow f+g, cf \in \mathcal{H}$

③ $f_n \overset{\in \mathcal{H}}{\geq} 0 \uparrow f$ bdd $\Rightarrow f \in \mathcal{H}$

then $\mathcal{H}$ contains all bdd funcs $\in \sigma(\mathcal{A})$

现在 $\mathcal{A}$ 取 $\{\omega : \omega_0 \in A_0, \cdots, \omega_n \in A_n\}$, 由 Monotone class theorem, 满足目标式的$\mathcal{H}$ contains all bdd measurable funcs w.r.t. $\sigma(\mathcal{A}) = \mathcal{F}_\infty$. #

Thm 5.2.4 (CK equation) $P_x(X_{m+n} = z) = \sum_y P_x(X_m = y) P_y(X_n = z)$ $p^{m+n} = p^m \cdot p^n$

$P_x(X_{m+n} \in B) = \int_S P_x(X_m \in dy) P_y(X_n \in B)$

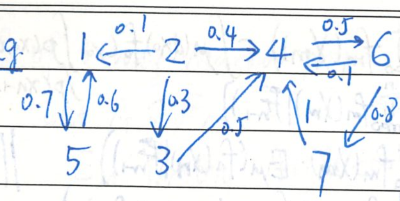
· stopping time $\{N=n\} \in \mathcal{F}_n$. 记 $\mathcal{F}_N = \{A : A \cap \{N=n\} \in \mathcal{F}_n, \forall n\}$. (information known at N)

$\theta_N(\omega) = \begin{cases} \theta_n(\omega) & \text{on } \{N=n\} \\ \Delta & \text{on } \{N=\infty\} \end{cases}$, ($\Delta$ add to Ω_0: $\Omega \cup \{\Delta\}$.)

只关心$\{N<\infty\}$情形, 不用管Δ.

Thm 5.2.5 (Strong Markov Property) $E_\mu(Y \circ \theta_N | \mathcal{F}_N) = E_{X_N} Y$ on $\{N<\infty\}$

$E_\mu(Y_N \circ \theta_N | \mathcal{F}_N) = E_{X_N} Y_N$ on $\{N<\infty\}$ $Y_n : \Omega_0 \to \mathbb{R}$ measurable $|Y_n| \leq M$.

E.g. 

由 Thm 5.3.2., 2 transient ($\rho_{21}>0$)

3 transient ($\rho_{34}>0$)

$\{1,5\}, \{4,6,7\}$ irreducible 由 Thm 5.3.3, recurrent

E.g. (Branching process). If $p_0>0$ (不生的概率), then $\rho_{k0} \geq (p_0)^k > 0$ ($\forall k\geq 1$), $\rho_{0k}=0 \Rightarrow k$ is transient.

$\rho_{00}=1 \Rightarrow 0$ is recurrent & absorbing ($P(0,0)=1$)

E.g. (Birth & death chains) $S=\{0,1,\cdots\}$, $p(i,i+1)=p_i$, $p(i,i-1)=q_i$, $p(i,i)=r_i$, $q_0=0$

记 $T_c=\min\{n\geq 1: X_n=c\}$. If $a<x<b$, 记 $T=T_a\wedge T_b$, 则

CLAIM 1. $T<\infty$, a.s.

∵ 由 [(Ex 5.2.2): X_n: MC, if $P\left(\bigcup_{m=n+1}^{\infty}\{X_m\in B_m\}\,\middle|\,X_n\right)\geq\delta>0$ on $\{X_n\in A_n\}$

then $P(\{X_n\in A_n, \text{i.o.}\}-\{X_n\in B_n, \text{i.o.}\})=0$.]

若 $\varphi(X_{n\wedge T})$ is mar, 则 $\varphi(k)=p_k\varphi(k+1)+r_k\varphi(k)+q_k\varphi(k-1)$

$\Rightarrow \varphi(k+1)-\varphi(k)=\frac{q_k}{p_k}(\varphi(k)-\varphi(k-1))$

$=\prod_{j=1}^{k}\frac{q_j}{p_j}(\varphi(1)-\varphi(0))$

Set $\varphi(0)=0$, $\varphi(1)=1$, 则 $\varphi(n)=\sum_{m=0}^{n-1}\prod_{j=1}^{m}\frac{q_j}{p_j}$ for $n\geq 1$.

CLAIM 2. $\varphi(X_{n\wedge T})$ is mar, 即 $E(\varphi(X_{(n+1)\wedge T})|\mathcal{F}_n)=\varphi(X_{n\wedge T})$

∵ 若 $T\leq n$, (✓); 若 $T>n$, $E(\varphi(X_{n+1})|X_n)=\varphi(X_n)$

Thm 5.3.10. $P_x(T_a<T_b)=\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}$, $P_x(T_b<T_a)=\frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(a)}$

特别, $P_x(T_0>T_M)=\frac{\varphi(x)}{\varphi(M)}$

∵ $\varphi(x)=E_x\varphi(X_T)$, $X_T\in\{a,b\}$ a.s.

$\Rightarrow \varphi(x)=\varphi(a)P_x(T_a<T_b)+\varphi(b)(1-P_x(T_a>T_b))$.

Thm 5.3.11. 0 is recurrent $\Leftrightarrow \varphi(M)\to\infty$ ($M\to\infty$) ∵ 由上一句"特别", 令 $M\to\infty$, $T_M\geq M-x$, P_x a.s.

If $\varphi(\infty)<\infty$, then $P_x(T_0=\infty)=\frac{\varphi(x)}{\varphi(\infty)}$

5.3 Recurrence & Transience

$\rho_{xy}=P_x(T_y<\infty)$

Def. A state y is recurrent, if $\rho_{yy}=1$

transient, if $\rho_{yy}<1$.

- If $\rho_{yy}=1$, then $P_y(T_y^k<\infty)=1, \forall k \Rightarrow P_y(X_n=y, \text{i.o.})=1$

If $\rho_{yy}<1$, define $N(y)\triangleq\sum_{n=1}^{\infty}1_{\{X_n=y\}}$, then $E_xN(y)=\sum_{k=1}^{\infty}P_x(N(y)\geq k)$

$=\sum_{k=1}^{\infty}P_x(T_y^k<\infty)\overset{\text{Thm 5.2.6}}{=}\frac{\rho_{xy}}{1-\rho_{yy}}<\infty$.

Thm 5.3.1 y is recurrent $\Leftrightarrow E_yN(y)=\infty$

Thm 5.3.2. x is recurrent, $\rho_{xy}>0 \Rightarrow y$ is recurrent $\rho_{yx}=1$.

Def. $C\subset S$ is closed, if $x\in C$, $\rho_{xy}>0$ implies $y\in C$.

Rmk. If C is closed, then $P_x(X_n\in C \text{ for all } n)=1$, $\forall x\in C$.

Def. $D\subset S$ is irreducible, if $\forall x,y\in D$, $\rho_{xy}>0$.

Thm 5.3.3. Let C be finite, closed. Then C contains a recurrent state.

If C is irreducible, then all states in C are recurrent. (Thm 5.3.2.)

- When S is finite, $\forall x\in S$, either

① $\exists y$, $\rho_{xy}>0$, $\rho_{yx}=0 \Rightarrow x$ transient

② $\forall y$, 若 $\rho_{xy}>0$, 则 $\rho_{yx}>0 \Rightarrow$ 记 $C_x=\{y:\rho_{xy}>0\}$ 若 $x\notin C_x \Rightarrow x$ transient

若 $x\in C_x$, 则 C_x irreducible & closed $\Rightarrow x$ recurrent.

Thm 5.3.5. (Decomposition theorem).

Let $R=\{x:\rho_{xx}=1\}$ be the recurrent states.

Then $R=\bigcup_i R_i$, where R_i are irreducible & closed, and $R_i\cap R_j\neq\phi$, $\forall i\neq j$.

Sketch Pf: $\forall x\in R$, define $C_x=\{y:\rho_{xy}>0\}$, then $x\in C_x$, $R=\bigcup_x C_x$, C_x irreducible & closed

$C_x=C_y$ or $C_x\cap C_y=\phi$. #

5.4 RW

- SRW on $\mathbb{Z}^d$: $P(X_i=e_j)=P(X_i=-e_j)=\frac{1}{2d}$, $\forall e_j$: d unit vectors.

Thm 5.4.4. SRW is recurrent if $d\leq 2$, and transient if $d\geq 3$.

e.g. (ASRW-asymmetric RW) $S=\mathbb{Z}$, $p(x,x+1)=p$, $p(x,x-1)=q=1-p$.

$\mu(x)\equiv 1$ is a stationary measure.

$\mu(x)=(\frac{p}{q})^x$ is another stationary measure.

e.g. (Ehrenfest chain) $S=\{0,1,\cdots,r\}$, $p(k,k+1)=\frac{r-k}{r}$, $p(k,k-1)=\frac{k}{r}$

则 $\mu(x)=2^{-r}\binom{r}{x}$ is a stationary dist.

且 $\mu(k)p(k,k+1)=\mu(k+1)p(k+1,k)$

5.5 Stationary Measures

Def. A measure μ is a stationary measure, if $\sum_x \mu(x)p(x,y)=\mu(y)$

Rmk $\mu(y)=P_\mu(X_1=y)=P_\mu(X_n=y)$

If μ is a prob. measure, call it a stationary distribution.

Rmk. If $0<\mu(S)<\infty$, then $\mu'(x)=\frac{\mu(x)}{\mu(S)}$ is a stationary distribution.

If $\mu(S)=\infty$, an example:

e.g. (RW): $S=\mathbb{Z}^d$, $p(x,y)=f(y-x)$, where $f(z)\geq 0$, $\sum f(z)=1$.

则 $\mu(x)\equiv 1$ 为 stationary measure. ($\because \sum_x p(x,y)=\sum_x f(y-x)=1$).

且 transition prob is doubly stochastic.

Def. μ satisfies the Detailed Balance Condition, if $\mu(x)p(x,y)=\mu(y)p(y,x)$

Call μ a reversible measure

Rmk. DBC $\Rightarrow \sum_x \mu(x)p(x,y)=\mu(y)$.

Thm 5.5.5 Let μ: stationary measure, X_0 has "distribution" μ,

then $Y_m=X_{n-m}$, $0\leq m\leq n$ is a MC with init measure μ

and t.p. $q(x,y)=\frac{\mu(y)p(y,x)}{\mu(x)}$, q is called the dual t.p.

If μ is reversible, then $q=p$.

Ex(5.2.1) $A\in\sigma(X_0,\cdots,X_n)=\mathcal{F}_n$, 则 ① $P_\mu(A\cap B\mid X_n)=P_\mu(A\mid X_n)P_\mu(B\mid X_n)$

$B\in\sigma(X_n,X_{n+1},\cdots)=\mathcal{G}_n$ ② $P_\mu(B\mid\mathcal{F}_n)=P_\mu(B\mid X_n)$

③ $P_\mu(A\mid\mathcal{G}_n)=P_\mu(A\mid X_n)$

(Construct a stationary measure)

Thm 5.5.7 x: recurrent state, $T=\inf\{n\geq 1, X_n=x\}$, then $\mu_x(y)=E_x(\sum_{n=0}^{T-1}1_{\{X_n=y\}})$ $=\sum_{n=0}^{\infty}P_x(X_n=y,T>n)$

is a stationary measure.

"cycle trick"

"$\mu_x(y)\overset{?}{=}\underbrace{\sum_z\mu_x(z)p(z,y)}_{=\mu_x p(y)}=$ the expected numbers of visits to y in $\{0,\cdots,T-1\}$"

Pf: 记 $\bar{p}_n(x,y)=P_x(X_n=y,T>n)$, 则 $\sum_y\mu_x(y)p(y,z)\overset{\text{Fubini}}{=}\sum_{n=0}^{\infty}\sum_y\bar{p}_n(x,y)p(y,z)$

Case 1. $z\neq x$ 时, $\sum_y\bar{p}_n(x,y)p(y,z)\overset{\text{强M性}}{=}\sum_y P_x(\underbrace{X_n=y,T>n}_{\in\mathcal{F}_n},X_{n+1}=z)=\sum_y P_x(T>n+1,X_{n+1}=z)=\bar{p}_{n+1}(x,z)$

$\Rightarrow\sum_{n=0}^{\infty}\sum_y\bar{p}_n(x,y)p(y,z)=\sum_{n=0}^{\infty}\bar{p}_{n+1}(x,z)=\sum_{n=0}^{\infty}\bar{p}_n(x,z)=\mu_x(z)$

Case 2. $z=x$ 时, $\sum_y\bar{p}_n(x,y)p(y,z)=\cdots\cdots=\sum_y P_x(T=n+1,X_n=y)=P_x(T=n+1)$

$\Rightarrow\sum_{n=0}^{\infty}\sum_y\bar{p}_n(x,y)p(y,z)=\sum_{n=0}^{\infty}P_x(T=n+1)\overset{T\geq 1}{=}\sum_{n=0}^{\infty}P_x(T=n)=1=\mu_x(x)$ #

Rmk 1. $\mu_x p(z)=\mu_x(z)$, $\forall z\neq x$

$\rho_{xx}=\mu_x p(x)\leq\mu_x(x)=1$

2. $\mu_x(y)<\infty$, $\forall y$.

$\because \mu_x p=\mu_x\Rightarrow\mu_x p^n=\mu_x$, $\forall n\geq 0$

$\mu_x(x)=1$, if $p^n(y,x)>0\Rightarrow\mu_x(y)<\infty$

e.g. 5.5.13 (Birth & death chains) $\exists$ stationary dist $\Leftrightarrow \sum_x \prod_{k=1}^{x} \frac{p_{k-1}}{q_k} < \infty$

首先，e.g. 5.5.4.: $\mu(x) = \prod_{k=1}^{x} \frac{p_{k-1}}{q_k}$ (p_k是生，q_k是灭，r_k是不变) is reversible,

即 $\mu(x)\, p(x,x+1) = \mu(x+1)\, p(x+1,x)$

($\Leftarrow$) Th 5.5.9 ($\Rightarrow$) $\mu(0)=1$, Th 5.5.10 + 5.5.9.

Thm 5.5.9 (唯一性) If p is irreducible & recurrent, then the stationary measure is unique up to constant multiples. (Rmk: total mass 要么全是∞，要么全部有限.)

Pf: 设$\nu(\cdot)$为 stationary measure

$$\nu(z) = \sum_y \nu(y) p(y,z) = \nu(a) p(a,z) + \sum_{y\neq a} \nu(y) p(y,z)$$

$$= \nu(a) p(a,z) + \sum_{y \neq a} \nu(a) p(a,y) p(y,z) + \sum_{x,y\neq a} \nu(x) p(x,y) p(y,z)$$

$$= \nu(a) P_a(X_1 = z) + \nu(a) P_a(X_1 \neq a, X_2 = z) + P_\nu(X_0 \neq a, X_1 \neq a, X_2 = z)$$

$$= \cdots = \nu(a) \sum_{m=1}^{n} P_a(X_{1\sim m-1} \neq a, X_m = z) + P_\nu(X_{0\sim n-1} \neq a, X_n = z)$$

$\Rightarrow \nu(z) \geq \nu(a)\mu_a(z)$，并由 $\nu(a) = \sum_x \nu(x) p^n(x,a)$ 得 $\nu(z) = \nu(a)\mu_a(z)$ #.

$\geq \sum_x \nu(a)\mu_a(x) p^n(x,a) = \nu(a)\mu_a(a) = \nu(a)$, ($\mu_a$为5.5.7构造 stationary dist.)

· Stationary measures may exists for transient chains, e.g. RW in $\mathbb{R}^{d\geq 3}$; but

Thm 5.5.10 (stationary dist must be recurrent) If $\exists$ stationary distribution, then $\forall y, \pi(y)>0$, y is recurrent.

Pf: $\pi p^n = \pi \Rightarrow \sum_x \pi(x) \underbrace{\sum_{n=1}^{\infty} p^n(x,y)}_{\text{‖在Th5.3.1前}\ \frac{\rho_{xy}}{1-\rho_{yy}}} = \sum_{n=1}^{\infty} \pi(y) = \infty \Big\} \Rightarrow \infty = \sum_x \pi(x) \frac{\rho_{xy}}{1-\rho_{yy}} \leq \frac{1}{1-\rho_{yy}} \Rightarrow \rho_{yy} = 1$

$\Rightarrow y$ recurrent #.

Thm 5.5.11 (Stationary dist 计算) If p is irreducible and has stationary dist. π, then

$$\pi(x) = \frac{1}{E_x T_x}$$

震惊: $\sum_x \frac{1}{E_x T_x} = 1$!

Pf: Irreducible $\Rightarrow \pi(x) > 0, \forall x \overset{\text{Th5.5.10}}{\Rightarrow} x$ recurrent $\overset{\text{Th5.5.7}}{\Longrightarrow}$ stationary measure: $\mu_x(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)$ with $\mu_x(x) = 1$.

$\because \sum_y \mu_x(y) \overset{\text{Fubini}}{=} \sum_{n=0}^{\infty} P_x(T_x > n) = E_x T_x$. 且 $\begin{cases} \pi(x) > 0 \\ \mu_x(x) = 1 \end{cases}$ 由 5.5.9. 相差有限倍

$\therefore \pi(x) = \frac{\mu_x(x)}{E_x T_x} = \frac{1}{E_x T_x}$. #

Recall:

Def (常返: $T_x < \infty$, a.s.) $\begin{cases} \underline{\text{positive}} \text{ recurrent}: E_x T_x < \infty \\ \underline{\text{null}} \text{ recurrent}: E_x T_x = \infty \end{cases}$

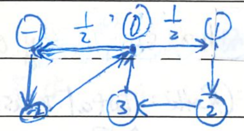
Thm 5.5.12 (Summary: how to 判别 positive recurrent) p: irreducible, then

① some x is positive — 用Th 5.5.11的构造

$\Leftrightarrow$ ② $\exists$ stationary dist. — Th 5.5.10 + Th 5.5.11

$\Leftrightarrow$ ③ all states are positive

Thm 5.5.15. If p: irreducible, has a stationary dist. π, then any other stationary measure is a multiple of π.

e.g.  $3,4\in I_0 \Rightarrow d_0=1$ ...

Pf of Lem 5.6.4: 由于周期定义, x recurrent ⇒ $\rho_{xy}>0$.

∴ ∃ K, L, s.t. $p^K(x,y),\ p^L(y,x)>0$

⇒ $p^{K+L}(y,y)\ge p^L(y,x)\,p^K(x,y)>0$

⇒ $d_y \mid K+L$

若 $p^n(x,x)>0$, 则 $p^{K+n+L}(y,y)>0 \Rightarrow d_y\mid K+n+L \Rightarrow d_y\mid n \Rightarrow d_y\mid d_x$

反之亦然, 从而 $d_x=d_y$. #

Pf of prop: Define $J=\{n\in\mathbb{N}: p^{nd}_{xx}>0\}$, then J is closed under addition (由C-K eq.)

J has gcd 1 ⇒ generated additive group equals $\mathbb{Z}$.

⇒ ∃ $n_1,\dots,n_k\in J$, $z_1,\dots,z_k\in\mathbb{Z}$, s.t. $\sum_{j=1}^k n_j z_j=1$.

取 $m=n_1\sum|z_j|n_j$, 则 ∀$n\ge m$, $n=m+hn_1+r = hn_1+\sum(n_1|z_j|+rz_j)n_j\in J$ #

Pf of Thm (convergence): $S^2=S\times S$, $\bar p$ (on S^2) is defined as $\bar p((x_1,y_1),(x_2,y_2))=p(x_1,x_2)\,p(y_1,y_2)$

step 1. CLAIM: $\bar p$ irreducible

① $p^K(x_1,x_2)>0$, $p^L(y_1,y_2)>0$

由 Lem 5.6.5., M large enough, $p^{L+M}(x_2,x_2),\ p^{K+M}(y_2,y_2)>0$

⇒ $\bar p^{K+L+M}((x_1,y_1),(x_2,y_2))>0$, ∀M large enough.

step 2. $\bar\pi(a,b)=\pi(a)\pi(b)$ is a stationary dist.

⇒ all states for $\bar p$ are recurrent

step 3. (X_n,Y_n): the chain on $S\times S$.

T: first time hit the diagonal

由 $T_{(x,x)}<\infty$, a.s. 得 $T<\infty$, a.s.

on $\{T\le n\}$, $X_n\overset{d}{=}Y_n$.

∴ $P(X_n=y)=P(X_n=y,T\le n)+P(X_n=y,T>n)$

$\le P(Y_n=y)+P(X_n=y,T>n)$

反之, 从而 $|P(X_n=y)-P(Y_n=y)|\le P(X_n=y,T>n)+P(Y_n=y,T>n)$

∴ $\sum_y|P(X_n=y)-P(Y_n=y)|\le 2P(T>n)$

令 $X_0=x$, $Y_0\sim\pi$, 从而 $Y_n\sim\pi$, $X_n\sim p^n(x,\cdot)$, 此时 $\sum_y|p^n(x,y)-\pi(y)|\le 2P(T>n)\to 0$ #

5.6 Asymptotic Behavior

- y transient ⇒ $\sum_n p^n(x,y)<\infty \Rightarrow p^n(x,y)\to 0$
- $N_n(y)=\sum_{m=1}^n 1\{X_m=y\}$

Thm 5.6.1 y: recurrent. ∀$x\in S$, $\frac{N_n(y)}{n}\to\frac{1}{E_yT_y}1_{\{T_y<\infty\}}$ P_x-a.s. (if $T_y=\infty$, $\frac{N_n(y)}{n}\to 0$; if $T_y<\infty$, $\frac{N_n(y)}{n}\to\frac{1}{E_yT_y}$)

Pf: ① Under P_y, 令 $R(k)=\min\{n\ge1, N_n(y)=k\}$: kth return to y.

$t_k=R(k)-R(k-1)$, $R(0)=0$.

$X_0=y$ + y recurrent ⇒ $t_1,t_2,\dots$ i.i.d. (强马氏性)

由SLLN $\frac{R(k)}{k}\to E_yT_y$, P_y-a.s.

∵ $R(N_n(y))\le n< R(N_n(y)+1)$

∴ $\frac{R(N_n(y))}{N_n(y)}\le\frac{n}{N_n(y)}<\frac{R(N_n(y)+1)}{N_n(y)+1}\cdot\frac{N_n(y)+1}{N_n(y)}$

∴ $\frac{n}{N_n(y)}\to E_yT_y$ P_y-a.s.

② $x\ne y$ 时, if $T_y=\infty \Rightarrow N_n(y)=0$ (✓)

Conditioned on $\{T_y<\infty\}$, $t_2,t_3,\dots$ i.i.d. $t_k\overset{d}{=}T_y$ (under P_y)
(Under P_x, $\{T_y<\infty\}$)

∴ $\frac{R(k)}{k}=\frac{t_1}{k}+\frac{t_2+\dots+t_k}{k}\to 0+E_yT_y$ on $\{T_y<\infty\}$, P_x-a.s.

同①操作, 有 $\frac{N_n(y)}{n}\to E_yT_y$ P_x-a.s. on $\{T_y<\infty\}$. #

Rmk. 从而 $E_x\frac{N_n(y)}{n}\to E_x\left(\frac{1}{E_yT_y}1_{\{T_y<\infty\}}\right)\Rightarrow\frac{1}{n}\sum_{m=1}^n p^m(x,y)\to\frac{\rho_{xy}}{E_yT_y}$

Def. x: recurrent, $I_x\triangleq\{n\ge1: p^n(x,x)>0\}$, $d_x\triangleq$ g.c.d. of I_x is period of x

"$d_x=1$ + irreducible" is aperiodic

Lem 5.6.4 $\rho_{xy}>0\Rightarrow d_y=d_x$.

Lem 5.6.5 $d_x=1$, ∃$m_0(x)$, s.t. $p^m(x,x)>0$, ∀$m\ge m_0$.

更一般地 prop (现代概率论基础 7.14)(positivity) If $x\in S$, $d_x<\infty$, then $p^{nd}_{xx}>0$ for all but finitely many n.

Thm 5.6.6 (Convergence) p: irreducible, aperiodic, has stationary dist. π

Then $p^n(x,y)\to\pi(y)$ $(n\to\infty)$

Rmk 除 $p^n(x,y)\to0$ transient, 还 $p^n(x,y)\to0$ null recurrent

* Thm (现代……3.30) (Coupling, Skorohod) Let $\xi,\xi_1,\xi_2,\dots$ be random elements in 可分度量空间 (S,ρ)

若 $\xi_n\overset{d}{\to}\xi$, 则 ∃ prob. space, ∃ $\eta\overset{d}{=}\xi$, $\eta_n\overset{d}{=}\xi_n$, s.t. $\eta_n\overset{a.s.}{\to}\eta$

证法二: $q((x_1,y_1),(x_2,y_2))\triangleq\begin{cases}p(x_1,x_2)p(y_1,y_2), & x_1\ne y_1\\ p(x_1,x_2), & x_1=y_1,\ x_2=y_2\\ 0, & \text{otherwise}\end{cases}$ new chain (X',Y'), $X'_n=Y'_n$ on $\{T'\le n\}$

$\sum_y|P(X'_n=y)-P(Y'_n=y)|\le 2P(X'_n\ne Y'_n)=2P(T'>n)$. 且 $T\overset{d}{=}T'$ #

(续右下Rmk) ① Assume $c=\nu(E)\in(0,\infty)$

Take $\xi_i \overset{i.i.d.}{\sim} \frac{\nu}{c}$, $N\sim Poi(c) \perp (\xi_1,\xi_2,\cdots)$

$M \triangleq \delta_{\xi_1}+\cdots+\delta_{\xi_N}$

② 若 $\nu(E)=\infty$，则由σ-finite，$E=(E_n)_{n\ge1}$, s.t. $\nu(E_n)\in(0,\infty)$

Let M_n be independent Poisson measure with intensity $1_{E_n}\nu$.

Let $M=M_1+M_2+\cdots$ #

~~Def.~~

• (Poisson point process) the product space $E\times[0,\infty)$: a Poisson measure M with intensity $\nu\otimes dx$

a.s. $M(E\times\{t\})=0$ or 1, $\forall t\ge0$.

Define $E\cup\{*\}$: if $M(E\times\{t\})=0$, put $e(t)=*$

if $M(E\times\{t\})=1$, define $e(t)$ to be the point in E, s.t.

$M|_{E\times\{t\}}=\delta_{(e(t),t)}$

∴ $M=\sum_{t\ge0}\delta_{(e(t),t)}\Big|_{E\times[0,\infty)}$

$N=(N_t)_{t\ge0}$ P.P with rate λ:

$N_0=0$, r.c.l.l. $N_t<\infty$ a.s. $t\ge0$

$N_t\to\infty$. ($\frac{N_n}{n}\to\lambda$ a.s.)

记 $\Delta N_t=N_t-N_{t-}$, $(S_n)_{n\ge0}$

$\Delta N_t=0$, a.s. for any $t\ge0$

$\Delta N_t=0$ or 1, $\forall t$, a.s.

Poisson Processes

• Def. (Poisson distribution) $N\sim Poi(\lambda)$, $P(N=k)=e^{-\lambda}\frac{\lambda^k}{k!}$

(Binomial) $X\sim B(n,p)$, $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$

(Generating Function) $ES^X \overset{Bin}{=} (q+pS)^n$, $0\le S\le1$, $\overset{Poi}{=} e^{-\lambda(1-S)}$

(Laplace Transform) $Ee^{-SX}\overset{Bin}{=}(q+pe^{-S})^n$, $S\ge0$. $\overset{Poi}{=}e^{-\lambda(1-e^{-S})}$

• $X_n\sim B(n,p_n)\xrightarrow{d} Poi(\lambda)$ (∵$ES^{X_n}\to ES^N$, $\forall s$)

• (CLT). $\xi_i \overset{i.i.d.}{\sim}$, $E\xi_i=0$, $E\xi_i^2=1$, then $\frac{\xi_1+\cdots+\xi_n}{\sqrt n}\xrightarrow{d}N(0,1)$.

• Def. (Poisson Processes) $X\sim Exp(\lambda)\sim f(x)=\lambda e^{-\lambda x}$ 或 $P(X>x)=e^{-\lambda x}$

$EX=\frac1\lambda$, $P(X>t+s|X>s)=P(X>t)$

若 $S_0=0$, $S_n=T_1+\cdots+T_n$, 其中 $T_i\overset{i.i.d.}{\sim}Exp(\lambda)$, 则 $S_n\sim Gamma(n,\lambda)$

记 $N_t=\sup\{n, S_n\le t\}$, 则 $N_t\sim Poisson(\lambda t)$ (∵ $P(N_t=n)=P(S_n\le t<S_{n+1})$ $=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$)

称 $N=(N_t)_{t\ge0}$ 为 P.P with parameter λ (rate/intensity)

Thm. ① a.s. $(N_t)_{t\ge0}$ is r.c.l.l. (右连续有左极限)

② a.s. $N_0=0$

③ $N_t-N_s\overset{d}{=}N_{t-s}$, $0\le s\le t$

④ $N_t-N_s\perp(N_u)_{0\le u\le s}$

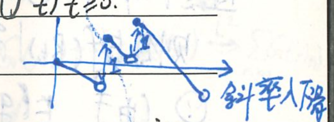
⑤ $N_t\sim Poi(\lambda t)$

Rmk. 1. ③+④+⑤: $0\le t_0<t_1<\cdots<t_n$, then $X_{t_i}-X_{t_{i-1}}$, $i=1,\dots,n$ are independent $\sim Poi(\lambda(t_i-t_{i-1}))$

2. ④: N is a P.P. w.r.t. $(\mathcal F_t)_{t\ge0}$, $N_t-N_s\perp\mathcal F_s$ ($N_s\in\mathcal F_s$).

Lem. $N=(N_t)_{t\ge0}$ is $(\mathcal F_t)$-P.P., then $(N_t-\lambda t)_{t\ge0}$ is mar w.r.t. $(\mathcal F_t)_{t\ge0}$.

(∵ $E(N_t-\lambda t|\mathcal F_s)\overset{③+④}{=}N_s-\lambda s$.) ⇒ $E(N_t-N_s|\mathcal F_s)=\lambda(t-s)$.



• Def. (Compound Poisson Process) $\xi_i\overset{i.i.d.}{\sim}\mu$. $S_t\triangleq\xi_1+\cdots+\xi_{N_t}$

Prop. $Ee^{-uN}=e^{-\lambda(1-e^{-u})}$ ⇒ $Ee^{iuN}=e^{-\lambda(1-e^{iu})}$ ⇒ $Ee^{iuN_t}=e^{-\lambda t(1-e^{iu})}$

⇒ $Ee^{iuS_t}=e^{-\lambda t\int(1-e^{iux})\mu(dx)}$ if $\mu(dx)=\delta_1(dx)$.

* Lévy过程 (a,b,ν): $Ee^{iuX_t}=e^{t\psi_u}$, 其中 $\psi_u=ibu-\frac12au^2+\int(e^{iux}-1-iux1_{\{|x|\le1\}})\nu(dx)$

Lévy measure: $\nu(dx)=\lambda\mu(dx)$ 时 为C.P.P.

• Def. (Poisson Measure) E: Polish space, ν: σ-finite measure on E.

A random measure M on E is a Poisson measure with intensity ν,

if M is a sum of Dirac masses, 即 $M=\sum_{i\in I}\delta_{\xi_i}$, $(\xi_i)_{i\in I}$ is random.

Rmk. P.P. "is" a Poisson Measure: ∵ $E=[0,\infty)$, $\nu(B)=\lambda m(B)$, $N_t=M([0,t])$.