Week 4

Ex 1 Heisenberg's uncertainty principle.

1) $n=1$.

a) Show that the commutator $[\partial_x, x] := \partial_x(x\cdot) - x\partial_x \cdot$ satisfies

$$[\partial_x, x]f = f \quad \text{for } f \in \mathcal{S}(\mathbb{R}).$$

b) conclude from a) that

$$\|xf\|_{L^2} \cdot \|\partial_x f\|_{L^2} \geq c\|f\|_{L^2}^2, \quad f \in \mathcal{S}(\mathbb{R})$$

for some $c>0$.

2) $n>1$, first show the commutar $[\nabla_x, x]$ defined by $\nabla_x \cdot (x\cdot) - x\cdot\nabla_x$ satisfies

$$[\nabla_x, x]f = nf \qquad f \in \mathcal{S}(\mathbb{R}^n)$$

and conclude from this that

$$\text{(I)} \qquad \|xf\|_{L^2} \cdot \|\nabla_x f\|_{L^2} \geq c\|f\|_{L^2}^2, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

3) Now suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain. let f be a function on Ω such that there exists a function $G \in C_c^\infty(\mathbb{R}^n)$ with $G|_{\Omega^c} = 0$ and $G|_\Omega = f$. Conclude from (I) that

$$\text{(II)} \qquad \|f\|_{L^2(\Omega)} \leq \frac{c}{\operatorname{dist}(0,\Omega) + \operatorname{diam}(\Omega)} \|\nabla_x G\|_{L^2(\mathbb{R}^n)}$$

for some $c>0$.

Problem: under what conditions can we replace the term $\|\nabla_x G\|_{L^2(\mathbb{R}^n)}$ in (II) by $\|\nabla_x f\|_{L^2(\Omega)}$??

(4) If we apply Plancherel's theorem. We then have

$$\|\nabla_x f\|_{L^2} = \|2\pi i \xi \hat{f}\|_{L^2}.$$

And hence (I) reads

$$\|x f\|_{L^2} \cdot \|\xi \hat{f}\|_{L^2} \geq \frac{c}{2\pi} \|f\|_{L^2}.$$

This is Heisenberg's uncertainty principle

(5). In the same spirit, one can show, with $r = |x|$

$$\left[\tfrac{1}{r}\nabla_x\left(\tfrac{1}{r}\,\cdot\right), x\cdot\right] f = n\, r^{-2} f \quad . \quad f \in \mathcal{S}(\mathbb{R}^n)$$

Consequently, we have

$$n \cdot \|r^{-1} f\|_{L^2}^2 = 2 \left\langle \tfrac{1}{r}\nabla_x\left(\tfrac{1}{r} f\right), x f \right\rangle . \quad f \in \mathcal{S}(\mathbb{R}^n).$$

show that

$$\nabla_x\left(\frac{1}{|x|} f\right) = \frac{1}{|x|}\nabla_x f + \frac{x}{|x|^3} f.$$

From this, we conclude

$$(d-2)\|r^{-1} f\|_2^2 = 2 \langle \nabla_x f, x \cdot r^{-2} f \rangle . \quad f \in \mathcal{S}(\mathbb{R}^n).$$

It then follows from Cauchy-Schwarz inequality that

$$\||x|^{-1} f\|_{L^2} \leq \frac{2}{d-2} \|\nabla f\|_{L^2}.$$

This is Hardy's ineq.

Ex5 Covering lemmas

for $B = B(x,r)$, denote $tB = B(x,tr)$.

(1) Wiener's Vitali-type lemma.

let $\{B_j\}_{j\in J}$ be a collection of balls in $\mathbb{R}^n$. Then there exists an at most countable subcollection of disjoint balls $\{B_k\}$ such that

$$\bigcup_{j\in J} B_j \subset \bigcup_k B_k$$

Use this lemma to show weak (1,1) boundedness of Hardy-Littlewood Maximal operator.

(2) Besicovitch-Morse covering lemma.

let A be a bounded set in $\mathbb{R}^n$, and suppose that $\{B_x\}_{x\in A}$ is a collection of balls s.t. $B_x = B(x, r_x)$, $r_x > 0$. Then there exists an at most countable subcollection of balls $\{B_j\}$ and a constant C_n, depending only on the dimension, such that

$$A \subset \bigcup_j B_j \quad \text{and} \quad \underbrace{\sum_j \chi_{B_j}(x) \leq C_n}_{\text{finitely overlap}}.$$

Ex6. by the same technique as in thm2. and assuming T^* is weak (p,q) show that the set

$$\{f \in L^p(X,\mu) : \lim_{t\to t_0} T_t f(x) \text{ exists a.e.}\}$$

is closed in $L^p(X,\mu)$.

Ex2 Let (X,μ), (Y,ν) be two measure spaces. let $f: X\times Y \to \mathbb{C}$ be a measurable function. show that for $0<p<q<\infty$ we have

$$\| \| f \|_{L^p(d\mu)} \|_{L^q(d\nu)} \leq \| \| f \|_{L^q(d\nu)} \|_{L^p(d\mu)}.$$

'a priority.

(1). Let $(X,\mu) = (Y,\nu) = (\mathbb{R}^n, \text{Leb.})$, show ~~the~~ Young's Ineq.

(2) Use Riesz-Thorin interpolation to show Yong's Ineq.

$$\| f*g \|_{L^r} \leq \| f \|_{L^p} \, \| g \|_{L^q}$$

where

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \qquad r>0, p>0, q>0.$$

(3) let $0<s<n$, then one can see $|x|^{-s} \notin L^{\frac{n}{s}}(\mathbb{R}^n)$. Nevertheless, we still have, for some constant $C>0$,

$$\| |\cdot|^{-s} * g \|_{L^r} \leq C \| g \|_{L^q}$$

where

$$1 + \frac{1}{r} = \frac{s}{n} + \frac{1}{q}$$

Ex.3 prove Riesz-Thorin interpolation theorem.

Ref: T. Ransford, Potential theory in the complex plane, Page 162.

Ex4. Show, for $f \in L^{loc}(\mathbb{R}^n)$, that

$$\lim_{r\to 0^+} \frac{1}{|B_r|} \int_{B_r} |f(x-y) - f(x)| dy = 0, \quad a.e.$$