为证明Thm 3.8. 我们先做一些准备工作

Prop. A.B.C are L^1-measurable subset of $\mathbb{R}^1$. $C \supset A+B$ then $|C| \geq |A|+|B|$

pf: It's suffice to show this inequality holds when A.B are compact (why?)

$C \supseteq \underbrace{(\sup A + B) \cup (\inf B + A)}_{\text{"almost disjoint"}}$

$\Longrightarrow |C| \geq |B| + |A|$

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Prékopa-Leindler inequality

$f, g, h: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ L^1-measurable. if $h(x+y) \geq f(x)^{1-\theta} g(y)^{\theta}$, $\forall x, y \in \mathbb{R}^n$

for some $\theta \in [0,1]$, then $\int h \geq \frac{1}{(1-\theta)^{n(1-\theta)} \theta^{n\theta}} (\int f)^{1-\theta} (\int g)^{\theta}$

pf. 只需证n=1的情形 $\int_{\mathbb{R}} h \geq \frac{1}{(1-\theta)^{1-\theta} \theta^{\theta}} (\int f)^{1-\theta} (\int g)^{\theta}$

(这是因为对一般的n 令 $\tilde{h}(x_2, \cdots x_n) = \int h(x_1, \cdots x_n) dx_1$, $\tilde{g}$, $\tilde{f}$ 这么定义

则 $\tilde{h}(x+y) \geq \frac{1}{(1-\theta)^{1-\theta} \theta^{\theta}} \tilde{f}(x)^{1-\theta} \tilde{g}(y)^{\theta}$ 便化到n-1情形)

利用齐次性质 只需证 $\int f = \int g = 1$ 时的情形

$\forall \lambda > 0$, $\{h > \lambda\} \supset \{f > \lambda\} + \{g > \lambda\}$

由LCR $\int h = \int_0^{\infty} |\{h > \lambda\}| d\lambda \underset{\text{Prop}}{\geq} \int_0^{\infty} |\{f > \lambda\}| + |\{g > \lambda\}| d\lambda$

$= 2$

而 $(1-\theta)^{1-\theta} \theta^{\theta} \geq \frac{1}{2}$, $\forall \theta \in [0,1]$

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$\Phi: U\times\mathbb{R} \to V\times\mathbb{R}$

$(\cdot,\cdot) \to (G(x), v\cdot E_1|_x, -v\cdot E_n|_x)$

Brunn-Minkowski inequality

设 $A, B, A+B \subset \mathbb{R}^n$ 为Lebesgue可测集 Then the following inequality holds:

$$(\mu(A+B))^{\frac{1}{n}} \geq (\mu(A))^{\frac{1}{n}} + (\mu(B))^{\frac{1}{n}}$$

Pf: Take $h = \chi_{A+B}$, $f = \chi_A$, $g = \chi_B$ 则显然 f, g, h 满足 ~~Prekopa~~

Prekopa-Leindler不等式的条件 $\forall \theta \in [0,1]$ 从而有

$$\mu(A+B)^{\frac{1}{n}} \geq \frac{1}{(1-\theta)^{1-\theta}\theta^{\theta}} \mu(A)^{\frac{1-\theta}{n}} \mu(B)^{\frac{\theta}{n}} \quad \forall \theta \in [0,1] 成立$$

右侧关于 $\theta \in [0,1]$ 取最大值便得 $\mu(A+B)^{\frac{1}{n}} \geq \mu(A)^{\frac{1}{n}} + \mu(B)^{\frac{1}{n}}$ #

Rmk 1: For convex sets A and B of positive measure, the inequality is strict ~~for~~ unless A and B are homothetic, ie are equal up to dilation and translation.

~~Rmk 2~~.

Cor 1: $C \subset \mathbb{R}^{n+1}$ convex, $e \in \mathbb{R}^{n+1}$. Set $S_e(t) = \{x \in C \cap \mathbb{R}^{n+1} : x\cdot e = t\}$

then $|S_e(t)|^{\frac{1}{n}}$ is concave

Pf ~~由 $e \in C$ 及 e convex 有 $S_e(t)$~~

由 C 凸可知 $\lambda_1 S_e(t_1) + \lambda_2 S_e(t_2) \subset S_e(\lambda_1 t_1 + \lambda_2 t_2)$ $(\lambda_i \geq 0, \lambda_1+\lambda_2=1)$

再利用上述不等式便得 $|S_e(\lambda_1 t_1 + \lambda_2 t_2)|^{\frac{1}{n}} \geq \lambda_1 |S_e(t_1)|^{\frac{1}{n}} + \lambda_2 |S_e(t_2)|^{\frac{1}{n}}$ #

Cor 2: $C \subset \mathbb{R}^{n+1}$ convex, antipodality (即中心对称 $A = -A$) then

$\underline{|S_e(t)| = |S_e(-t)|}$ 且 $\underline{|S_e(t)| \text{ is nonincreasing as } t \text{ varies from } 1 \text{ to } \infty}$

Pf 显然! 　　　上面可以 $|S_e(0)| \geq |S_e(t)|$ 利用 $|S_e(t)|^{\frac{1}{n}}$ concave 及对称性

auxiliary equations (2.15)

Pf of Thm 3.8.

1维的情形：由 LER 及 Lebesgue 测度的正则性可知

只需记 $f_j(x) = \sum_i f_{ji}(x - a_{ji})$ $f_{ji}(x)$ is the characteristic function of an bounded interval I_{ji} centered at the origin.

Let ~~then~~ $I_t(f_1 \dots f_m) = \underbrace{\sum_{i_1 \dots i_m} \int_{(\mathbb{R})^k} \prod_{j=1}^{m} f_{j i_j}\left(\sum_{i=1}^{k} b_{ij} x_i - t a_{ij}\right) dx_1 \dots dx_k}_{(*)}$

只需证明 $(*)$ is nondecreasing as t varies from 1 to 0

Let $A = \bigcup_{i_1 \dots i_m} \bigcap \left\{ (x,t) \in \mathbb{R}^k \times [0,1] : \sum_{j=1}^{k} b_{ij} x_i - t a_{ij} \in I_{j i_j} \right\}$

则 A convex. antipodality (仍何对称 $A = -A$)

Let $e_{n+1} = (0, \dots, 0, 1)$ 则

$(*) = |S_{e_{n+1}}(t)|$ is nondecreasing as t varies from 1 to 0

高维的情形用归纳.

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