Problem 3

Recall the proof of Brezis-Lieb lemma ~~$\lim_{j\to\infty}\int |f-f_j|^p$~~

$$\lim_{j\to\infty}\int \left| |f_j|^p - |f-f_j|^p - |f|^p \right| = 0 \qquad 0<p<\infty$$

令 $g_j = f - f_j$ $\Updownarrow$

$$\lim_{j\to\infty}\int \underline{\left| |f+g_j|^p - |g_j|^p - |f|^p \right|} = 0$$

$$\leq \int \underline{\varepsilon|g_j|^p} + \underline{\left(\left||f+g_j|^p - |g_j|^p - |f|^p\right| - \varepsilon|g_j|^p\right)_+}$$

(2) $\|g_j\|_p$ uniformly bonded

$$\leq C|f|^p$$

$$\left|\int f+g_j|^p - |g_j|^p - |f|^p\right| \leq |f|^p + \varepsilon|g_j|^p + C_\varepsilon|f|^p$$

(1) $\underline{\left| |f+g_j|^p - |g_j|^p \right| \leq \varepsilon|g_j| + C_\varepsilon|f|}$ (2)

(1) For any $\varepsilon \in (0, \frac{1}{2})$, take $k = \lceil \varepsilon^{-1} \rceil$

Let $\phi_\varepsilon = J(kt) - kJ(t)$, $\psi_\varepsilon = \left|J\left(\frac{t}{\varepsilon(k-1)}\right)\right| + \left|J\left(-\frac{t}{\varepsilon(k-1)}\right)\right|$

Note that J convex, $J(0)=0$ $\Rightarrow$ $\phi_\varepsilon = J(kt) - kJ(t) \geq 0$

$$a+b = (1-k\varepsilon)a + \varepsilon(ka) + (k-1)\varepsilon \frac{b}{(k-1)\varepsilon}$$

By convexity $J(a+b) \leq (1-k\varepsilon)J(a) + \varepsilon J(ka) + \overset{<1}{(k-1)\varepsilon}\, J\left(\frac{b}{(k-1)\varepsilon}\right)$

$$\Longrightarrow J(a+b) - J(a) \leq \varepsilon \phi_\varepsilon + \psi_\varepsilon$$

$$a = \frac{1}{1+k\varepsilon}(a+b) + \frac{\varepsilon}{1+k\varepsilon}(ka) + \frac{\varepsilon(k-1)}{1+k\varepsilon}\left(\frac{-b}{\varepsilon(k-1)}\right)$$

$$J(a) \leq \frac{1}{1+k\varepsilon} J(a+b) + \frac{\varepsilon}{1+k\varepsilon} J(ka) + \frac{\varepsilon(k-1)}{1+k\varepsilon} J\left(-\frac{b}{\varepsilon(k-1)}\right)$$

~~$\Rightarrow J(a) - J(a+b) \leq$~~

$$\Rightarrow (1+k\varepsilon)J(a) \leq J(a+b) + \varepsilon J(ka) + \varepsilon(k-1) J\left(-\frac{b}{\varepsilon(k-1)}\right)$$

$$\Rightarrow J(a) - J(a+b) \leq \varepsilon J(ka) - k\varepsilon J(a) + \varepsilon(k-1) J\left(-\frac{b}{\varepsilon(k-1)}\right)$$

$$\leq \varepsilon \phi_\varepsilon + \psi_\varepsilon$$

(2)

$|J(f)| + |J(f+g_j) - J(g_j)|$

$$\int |J(f+g_j) - J(g_j) - J(f)| \leq \cancel{\varepsilon|J(g_j)| + |J(f+g_j) - J(g_j) - J(f) - \varepsilon J(g_j)|}$$

$$|J(f+g_j) - J(g_j)| \leq \cancel{\varepsilon\left(J(kg_j) - kJ(g_j)\right) + \psi_\varepsilon(f)}$$

$$\varepsilon\phi_\varepsilon(g_j) + \psi_\varepsilon(f)$$

$$\int \left(J(f+g_j) - J(g_j) - \varepsilon\phi_\varepsilon(g_j)\right)_+ \leq \psi_\varepsilon(f)$$

$$\overset{DCT}{\Longrightarrow} \int \left(J(f+g_j) - J(g_j) - \varepsilon\phi_\varepsilon(g_j)\right)_+ \to 0 \text{ as } j \to \infty$$

$$\Rightarrow \int |J(f+g_j) - J(g_j) - J(f)| \leq \varepsilon \int |\phi_\varepsilon(g_j)| + \quad \leq C\varepsilon$$

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We need $\begin{cases} \|\phi_\varepsilon(f_j)\|_{L^1} \le C \leftarrow \text{independent of } j \\ \psi_\varepsilon(f_j) \in L^1 \end{cases}$

(3). $J = |x|^2$ $\quad f_j = 1 + j\,\chi_{[0,\frac{1}{j}]}$ then $f_j \xrightarrow{a.e} f$

$$\int \left| \underbrace{f_j^2 - (f_j - f)^2}_{=2j\chi_{[0,\frac{1}{j}]}} - f^2 + f_j^2 \right| = 2 \not\to 0$$

$\cdot\ J = e^{|x|} - 1 \qquad f_j = \cancel{1+}\log(1+j)\,\chi_{[0,\frac{1}{j}]} \qquad f_j \xrightarrow{a.e} \cancel{0}\ 1 \quad = 0$

~~e^{1+}~~

~~$J(f_j) = (1+j)\chi_{[0,\frac{1}{j}]}$, $\sup_j \|J(f_j)\|_{L^1} \le \infty$~~

~~$\int \left| J(f_j) - J(f) - J(f_j - f) \right| =$~~

~~$= \int$~~

$J(f_j) = e(1+j)\cdot\chi_{[0,\frac{1}{j}]} + e\,\chi_{[\frac{1}{j},1)} - 1$

$J(f_j - f) = j\,\chi_{[0,\frac{1}{j}]}$

$\int J(f_j) = 2e - 1 \quad \int J(f_j - f) = 1 \quad \int J(f) = e - 1$