HOMEWORK FOR "MARTINGALE THEORY AND STOCHASTIC CALCULUS"

WEIYU LI
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1. Problem

Suppose that there exists a "unique solution" of the following equation

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t G(s, X_s)dB_s, \quad t \in [0, T], \tag{1}$$

where $x_0 \in \mathbb{R}$ is given, both $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $G : [0,T] \times \mathbb{R} \to \mathbb{R}$ are Borel measurable, uniform bounded and continuous, and $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ satisfies the usual conditions, on which $B = \{B_t\}_{t\in[0,T]}$ is the standard Brownian motion. Prove that for any $h \in L^2([0,T], \mathbb{R})$, there exists $\{Y_t\}_{t\in[0,T]}$, such that it is the "solution" of the equation

$$Y_t = x_0 + \int_0^t b(s, Y_s)ds + \int_0^t G(s, Y_s)dB_s + \int_0^t G(s, Y_s)h_s ds, \quad t \in [0, T]. \tag{2}$$

2. Preliminary definitions

The first question is: "how to define *the solution of an equation*?" We consider the definitions of *strong solution* and *weak solution*.

Definition 1 (Strong solution). *Given $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ satisfying the usual conditions and $B = \{B_t\}_{t\in[0,T]}$ the standard Brownian motion on it, we say the $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted $X = \{X_t\}_{t\in[0,T]}$ is the **strong solution of** (1), if*

(i) $X \in C([0,T], \mathbb{R})$, $\mathbb{P}$-a.s., and
(ii) For any $t \in [0,T]$,

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t G(s, X_s)dB_s, \quad \mathbb{P}\text{-a.s.}.$$

*Analogously, we say the $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted $Y = \{Y_t\}_{t\in[0,T]}$ is the **strong solution of** (2), if*

(i) $Y \in C([0,T], \mathbb{R})$, $\mathbb{P}$-a.s., and
(ii) For any $t \in [0,T]$,

$$Y_t = x_0 + \int_0^t b(s, Y_s)ds + \int_0^t G(s, Y_s)dB_s + \int_0^t G(s, Y_s)h_s ds, \quad \mathbb{P}\text{-a.s.}.$$

Definition 2 (Weak solution). *We say $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}, \tilde{B}, \tilde{X} = \{\tilde{X}_t\}_{t\in[0,T]}\right)$ is the **weak solution of** (1), if*

(i) $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}})$ satisfies the usual conditions,

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Date: 2019/11/27.
liweiyu@mail.ustc.edu.cn.

(ii) $\tilde{B} = \{\tilde{B}_t\}_{t\in[0,T]}$ *is a standard Brownian motion on* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}})$,
(iii) $\tilde{X}$ *is* $\{\mathcal{F}_t\}_{t\in[0,T]}$*-adapted,*
(iv) $\tilde{X} \in C([0,T], \mathbb{R})$, $\quad \tilde{\mathbb{P}}$*-a.s., and*
(v) For any $t \in [0,T]$,

$$\tilde{X}_t = x_0 + \int_0^t b(s, \tilde{X}_s)ds + \int_0^t G(s, \tilde{X}_s)d\tilde{B}_s, \quad \tilde{\mathbb{P}}\text{-a.s.}.$$

Analogously, we can define the weak solution of (2).

In spite of the existence of solutions as we define above, the uniqueness of the equations is still not determined.

Definition 3 (Strong/pathwise uniqueness). *Given* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, B = \{B_t\}_{t\in[0,T]}\right)$ *as in Definition 1, we say the solution of* (1) *is pathwise unique, if any two strong solutions* X_1, X_2 *of* (1) *satisfy* $\mathbb{P}(X_1 = X_2) = 1$.

Definition 4 (Uniqueness/weak uniqueness). *We say the solution of* (2) *is unique, if any two weak solutions* $\left(\tilde{\Omega}^1, \tilde{\mathcal{F}}^1, \{\tilde{\mathcal{F}}^1_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}^1, \tilde{B}^1, \tilde{X}^1\right)$, $\left(\tilde{\Omega}^2, \tilde{\mathcal{F}}^2, \{\tilde{\mathcal{F}}^2_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}^2, \tilde{B}^2, \tilde{X}^2\right)$ *of* (2) *satisfy* $\tilde{\mathbb{P}}^1 \circ (\tilde{X}^1)^{-1} = \tilde{\mathbb{P}}^2 \circ (\tilde{X}^2)^{-1}$, *i.e.,*

$$\text{Law of } \tilde{X}^1 \text{ under } \tilde{\mathbb{P}}^1 = \text{Law of } \tilde{X}^2 \text{ under } \tilde{\mathbb{P}}^2.$$

3. Problem Restatement

With the above definitions, we can now restate our problem in a formal way.

Problem 1. *Suppose that there exists a weak solution of the equation* (1) *and the pathwise uniqueness holds. Prove that for any* $h \in L^2([0,T], \mathbb{R})$, *there exists* $\{Y_t\}_{t\in[0,T]}$, *such that it is the strong solution of the equation* (2).

4. Useful theorems

To prove the result, we first give some existing theorems, including Girsanov Theorem, and Yamada-Watanabe Theorem.

Theorem 1 (Girsanov Theorem). *Given* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, B = \{B_t\}_{t\in[0,T]})$ *as in Definition 1, the uniform integrable exponential martingale*

$$\exp\left[\int_0^t V_s dB_s - \frac{1}{2}\int_0^t |V_s|^2 ds\right], \quad t \in [0,T],$$

where V *is a progressively measurable process, defines probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F}_T)$ *such that the Radon-Nikodym derivative*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left[\int_0^t V_s dB_s - \frac{1}{2}\int_0^t |V_s|^2 ds\right], \quad t \in [0,T].$$

Then the Brownian motion with a drift

$$X_t = B_t - \int_0^t V_s ds, \quad t \in [0,T]$$

is a Brownian motion under $\mathbb{Q}$.

Theorem 2 (Yamada-Watanabe Theorem). *If there exists a weak solution of* (1) *and the pathwise uniqueness holds, then there exists a unique mapping* $\Gamma : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$, *such that for any given* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}, \tilde{B}\right)$, $\Gamma(\tilde{B}) = \{\Gamma(\tilde{B})_t\}_{t\in[0,T]}$ *is the strong solution of* (1), *i.e.,*

(i) $\Gamma(\tilde{B})$ *is* $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$*-adapted,*
(ii) $\Gamma(\tilde{B}) \in C([0,T],\mathbb{R})$, $\quad \tilde{\mathbb{P}}$*-a.s., and*
(iii) For any $t \in [0,T]$,

$$\Gamma(\tilde{B})_t = x_0 + \int_0^t b(s,\Gamma(\tilde{B})_s)ds + \int_0^t G(s,\Gamma(\tilde{B})_s)d\tilde{B}_s, \quad \tilde{\mathbb{P}}\text{-}a.s..$$

5. Main proof

Now we turn back to the proof of our problem.

Using Yamada-Watanabe Theorem on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, B\right)$ and from our problem's assumptions, we know that (1) exists unique strong solution.

Further plugging in $V_s = -h_s$, $\mathbb{Q} = \tilde{\mathbb{P}}$ in Girsanov Theorem, we know that

$$\tilde{B}_t = B_t + \int_0^t h_s ds, \quad t \in [0,T] \tag{3}$$

is a Brownian motion under $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}\right)$, where

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left[\int_0^t -h_s dB_s - \frac{1}{2}\int_0^t |h_s|^2 ds\right], \quad t \in [0,T].$$

Using Yamada-Watanabe Theorem again on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}, \tilde{B}\right)$, we have that

(a) $\Gamma(\tilde{B})$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted,
(b) $\Gamma(\tilde{B}) \in C([0,T],\mathbb{R})$, $\tilde{\mathbb{P}}$-*a.s.*, and
(c) For any $t \in [0,T]$,

$$\Gamma(\tilde{B})_t = x_0 + \int_0^t b(s,\Gamma(\tilde{B})_s)ds + \int_0^t G(s,\Gamma(\tilde{B})_s)d\tilde{B}_s, \quad \tilde{\mathbb{P}}\text{-}a.s.. \tag{4}$$

We are going to prove that $\Gamma(\tilde{B})$ is exactly the strong solution of (2) given $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, B\right)$. Equivalently, to show that

(a′) $\Gamma(\tilde{B})$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted,
(b′) $\Gamma(\tilde{B}) \in C([0,T],\mathbb{R})$, $\quad \mathbb{P}$-*a.s.*, and
(c′) For any $t \in [0,T]$,

$$\Gamma(\tilde{B})_t = x_0 + \int_0^t b(s,\Gamma(\tilde{B})_s)ds + \int_0^t G(s,\Gamma(\tilde{B})_s)dB_s + \int_0^t G(s,\Gamma(\tilde{B})_s)h_s ds, \quad \mathbb{P}\text{-}a.s.. \tag{5}$$

Checking (a′) It is exactly (a).
Checking (b′) From (b), it suffices to show that $\mathbb{P} \ll \tilde{\mathbb{P}}$, that is, if $\tilde{\mathbb{P}}(A) = 0$ for some $A \in \mathcal{F}_T$, then $\mathbb{P}(A) = 0$. Let $A_m = A \cap \{\omega : \int_0^T -h_s dB_s(\omega) > -m\}$. Since $h \in L^2([0,T],\mathbb{R})$, $\int_0^t -h_s dB_s$ is a continuous martingale, and $\int_0^T -h_s dB_s > -\infty, \mathbb{P}$-*a.s.*. Therefore, $\mathbb{P}(A - \lim_{m\to\infty} A_m) = 0$. Note that

$$\begin{aligned}
0 =&\tilde{\mathbb{P}}(A) = \int \mathbb{1}_A d\tilde{\mathbb{P}} \\
=& \int \mathbb{1}_A \exp\left[\int_0^T -h_s dB_s - \frac{1}{2}\int_0^T |h_s|^2 ds\right] d\mathbb{P} \\
=& c\int (\mathbb{1}_{A_m} + \mathbb{1}_{A-A_m}) \exp\left[\int_0^T -h_s dB_s\right] d\mathbb{P} \\
\geq& ce^{-m}\mathbb{P}(A_m),
\end{aligned}$$

where $c = \exp\left[-\frac{1}{2}\int_0^T |h_s|^2 ds\right]$ is a constant. Then we have $\mathbb{P}(A_m) = 0$ for all m, thus $\mathbb{P}(A) = \lim \mathbb{P}(A_m) + \mathbb{P}(A - \lim A_m) = 0$.
Checking (c′) Note that with any $\omega \in \Omega$ fixed, the terms $\int_0^t b(s, \Gamma(\tilde{B}(\omega))_s)ds$ in (4) and (5) are equal and finite, since they're normal integrals with the same bounded integrated function $b(s, \Gamma(\tilde{B}(\omega))_s)$. It remains to show that $\int_0^t G(s, \Gamma(\tilde{B})_s)d\tilde{B}_s$ on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}\right)$ is the same as $\int_0^t G(s,\Gamma(\tilde{B})_s)dB_s + \int_0^t G(s,\Gamma(B)_s)h_s ds$ on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$.

From the definition (3), if G is a step process, then

$$\int_0^t G(s,\Gamma(\tilde{B})_s)d\tilde{B}_s = \int_0^t G(s,\Gamma(\tilde{B})_s)dB_s + \int_0^t G(s,\Gamma(\tilde{B})_s)h_s ds, \forall t \in [0,T]. \tag{6}$$

More generally, any continuous and uniform bounded G is progressively measurable, and can be the $\mathcal{L}_T^2$-limit of a series of step processes $\{G_n(s,\Gamma(\tilde{B})_s)\}$. Then (6) holds *a.s.* for our supposed G in the problem.

6. Some remarks

In literature[1], h_s can be relaxed to $h(s, X_s)$ that is a continuous, bounded variation process with

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |h(s,X_s)|^2 ds\right)\right] < \infty, \quad t \in [0,T].$$

It also shows that if (weak) uniqueness holds for equation (1), then the solution we find above is the (weak) unique one. We state the sketch proof here. Note that for any weak solution $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, B, Y\right)$ of (2), $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}}, \tilde{B}, Y\right)$ is the weak solution of (1) (to prove this claim, note that $\tilde{\mathbb{P}} \ll \mathbb{P}$), where $\tilde{\mathbb{P}} = M \circ \mathbb{P}$ with exponential martingale transform $M(t) = \exp\left[\int_0^t -h_s dB_s - \frac{1}{2}\int_0^t |-h_s|^2 ds\right]$. Therefore, the (weak) uniqueness of (1) suggests that any two weak solutions of (2) can be transformed to a unique weak solution of (1), which concludes the uniqueness of (2).

[1]Nobuyuki Ikeda, Shinzo Watanabe (1988). *Stochastic Differential Equations and Diffusion Processes (2nd ed.)* Elsevier.