Durrett. Probability: Theory and Examples

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Petrov (1995). Limit theorems of Probability Theory - Sequences of Independent R.V.s

期末: WLLN、SLLN. CLT. Donsker. Martingale

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WLLN Pf: $P(|\frac{S_n}{n}-EX|>\varepsilon) \le \frac{1}{\varepsilon^2}E\frac{(S_n-nEX)^2}{n^2} = \frac{Var(X)}{n\varepsilon^2} \to 0\ (n\to\infty)$，若 $EX^2<\infty$

截尾：$S_n'=\sum X_i I(|X_i|\le N)$，$S_n''=\sum X_i I(|X_i|>N)$

$P(|\frac{S_n-nEX}{n}|>2\varepsilon) \le P(|\frac{S_n'-ES_n'}{n}|>\varepsilon) + P(|\frac{S_n''-ES_n''}{n}|>\varepsilon)$

$\le \frac{Var(S_n')}{n^2\varepsilon^2} + \frac{1}{\varepsilon}E|X''-EX''|$

$\le \frac{N^2}{n^2\varepsilon^2} + \frac{2E|X|I(|X|>N)}{\varepsilon}$

$\overline{\lim_{n\to\infty}} P(|\frac{S_n-nEX}{n}|>2\varepsilon) \le \frac{2E|X|I(|X|>N)}{\varepsilon} \to 0\ (N\to\infty)$ $\Rightarrow \lim_{n\to\infty} P(|\frac{S_n-nEX}{n}|>2\varepsilon)=0$ #

Pf2: $Y_n = X_n I(|X_n|\le n)$，$T_n=\sum_{i=1}^{n} Y_i$

$\sum P(X_n\ne Y_n) = \sum P(|X_n|>n) = \sum E I(|X|>n) = E\sum I(|X|>n) \le E|X| <\infty$

由B-C引理，$P(X_n\ne Y_n, \text{i.o.})=0 \Rightarrow \frac{\sum_{k=1}^{n}X_k-\sum_{k=1}^{n}Y_k}{n}\to 0$ a.s.

$\frac{S_n}{n}\xrightarrow{P}EX$，只须 $\frac{T_n}{n}\xrightarrow{P}EX$.

$\because P(|\frac{T_n-ET_n}{n}|>\varepsilon) \le \frac{1}{n^2\varepsilon^2}\sum_{k=1}^{n}Var(Y_k) \le \frac{1}{n^2\varepsilon^2}\sum_{k=1}^{n}EX^2I(|X|\le k) \xrightarrow{(*)} 0\ (n\to\infty)$

$\therefore \frac{T_n-ET_n}{n}\xrightarrow{P}0$ 又 $\because \frac{ET_n}{n}=\frac{1}{n}\sum_{k=1}^{n}EXI(|X|\le k)\to EX$ (数列收敛)

$\therefore$ 由Slusky，$\frac{T_n}{n}\xrightarrow{P}EX$ #

(*)：欲证 $\frac{\sum^n EX^2I(|X|\le k)}{n^2}\to 0$ ① $\lim_{n\to\infty}\frac{EX^2I(|X|\le n)}{n^2-(n-1)^2} = \lim_{n\to\infty}\frac{1}{2}E\frac{X^2}{n}I(|X|\le n)$，(根据Stolz)

$\le \frac{1}{2}\lim_{n\to\infty} E\frac{X^2I(|X|\le N)}{n} + E\frac{X^2I(N<|X|\le n)}{n}$

$\le \frac{1}{2}\lim_{n\to\infty}$ $E|X|\cdot I(|X|>N) \to 0\ (N\to\infty)$

② LHS $= \frac{E\sum X^2I(|X|\le k)}{n^2}$

$= \frac{EX^2\cdot(n-[|X|])I(|X|\le n)}{n^2}$

$\le E\frac{X^2(n+1-|X|)I(|X|\le n)}{n^2} \le 2E\frac{X^2}{n}I(|X|\le n)\to 0\ (n\to\infty)$

③ LHS $= \frac{\sum_{k=1}^{n}\sum_{l=1}^{k}EX^2I(l-1<|X|\le l)}{n^2}$

$\le \frac{\sum_{k=1}^{n}\sum_{l=1}^{k} l\,E|X|I(l-1<|X|\le l)}{n^2} = \frac{\sum\sum(l-1)E|X|I(l-1<|X|\le l)}{n^2} + \frac{\sum_{k\le n}E|X|I(|X|\le k)}{n^2}$

换序 $= \frac{\sum_{l=1}^{n}\sum_{k=l}^{n} l\,E|X|I(l-1<|X|\le l)}{n^2}$

??? $\le \frac{\sum_{k\le n}\sum_{l\le k}(l-1)E|X|I(l-1<|X|\le l)}{n^2} + \frac{E|X|I(|X|\le n)}{n}$

$\le \frac{\sum_{l\le n}(n-(l-1))(l-1)E|X|I(l-1<|X|\le l)}{n^2} + \frac{E|X|}{n} \to 0$

$\le \sum_{l\le n}\frac{E|X|^2I(l-1<|X|\le l)}{n}$

$= \frac{E|X|^2I(|X|\le n)}{n}\to 0$

·大数定律

Date .

· $X, X_1, X_2, \cdots$ i.i.d. r.v.s $S_n \triangleq \sum_{i=1}^{n}X_i$

Thm WLLN: $E|X|<\infty$ 时，$\frac{S_n}{n}\xrightarrow{P}EX$ (Rmk：非必要，即若 $\frac{S_n}{n}\xrightarrow{P}a$ 推 $E|X|<\infty$)

SLLN: $E|X|<\infty$ 时，$\frac{S_n}{n}\xrightarrow{a.s.}EX$

Thm "∃常数序列 $\{b_n\}$，s.t. $\frac{S_n}{n}-b_n\xrightarrow{P}0$" $\Leftrightarrow nP(|X|>n)\to 0$.，此时 $b_n=EXI(|X|\le n)+o(1)$

($\Rightarrow E|X|^{1-\varepsilon}<\infty$)

C.e.g. $X_1, X_2, \ldots \overset{i.i.d}{\sim} P(X_i\le x)=\int_{-\infty}^{x}\frac{dt}{\pi(1+t^2)}$，有 $E|X|^r<\infty, r<1$

$P(|X|>x)\sim\frac{2}{\pi x}$ 不满足上述充要条件

$\frac{S_n}{n}\overset{d}{=}X_1$

Pf: $X_{nk}\triangleq X_k I(|X_k|\le n)$，$T_n\triangleq\sum_{k=1}^{n}X_{nk}$，$\mu_n\triangleq EX_{n1}$

(⇐): $P(|\frac{S_n}{n}-EXI(|X|\le n)|>\varepsilon) = P(S_n=T_n, |\frac{S_n}{n}-EXI(|X|\le n)|>\varepsilon)$

$+P(S_n\ne T_n, |\frac{S_n}{n}-EXI(|X|\le n)|>\varepsilon)$

$\le P(|\frac{T_n-ET_n}{n}|>\varepsilon)+P(S_n\ne T_n)$ $(+o(1))$

$\because P(S_n\ne T_n)\le P(\exists k, X_{nk}\ne X_k)\le\sum_{k=1}^{n}P(X_{nk}\ne X_k)=nP(|X|>n)\to 0$

$P(|\frac{T_n-ET_n}{n}|>\varepsilon)\le\frac{1}{n^2\varepsilon^2}\sum_{k=1}^{n}Var(X_{nk})$

$\le\frac{1}{n\varepsilon^2}EX^2I(|X|\le n)\to 0.$ ① $= \frac{\int_0^\infty P(X^2I(|X|\le n)>t^2)dt^2}{n}$

$\therefore \frac{S_n}{n}-EXI(|X|\le n)\xrightarrow{P}0$

$= \frac{1}{n}\int_0^n 2t\,P(|X|>t)\,dt$

$= \frac{1}{n}(\int_0^{x_0}+\int_{x_0}^{n})2tP(|X|>t)dt\to 0.$

(⇒) 对称化，见群文件

Eg. $\{X_n\}$ 独立 r.v $b_n\to\infty$ 若 (1) $\sum_{i=1}^n P(|X_i|>b_n)\to 0$ (2) $\frac{1}{b_n^2}\sum_{i=1}^n EX_i^2 I(|X_i|\le b_n)\to 0$

则 $a_n \triangleq \sum_{i=1}^n EX_i I(|X_i|\le b_n)$, $\frac{1}{b_n}(S_n-a_n)\xrightarrow{P}0$.

Pf: $X_{in}\triangleq X_i I(|X_i|\le b_n)$, $T_n=\sum_{i=1}^n X_{in}$

则 $P(|\frac{S_n-a_n}{b_n}|>\varepsilon)\le P(|\frac{T_n-a_n}{b_n}|>\varepsilon)+P(S_n\ne T_n)$

$\le \frac{1}{\varepsilon^2 b_n^2}\sum_{i=1}^n EX_i^2 I(|X_i|\le b_n)+\sum_{i=1}^n P(|X_i|>b_n)\to 0$ #

Eg. $X_1,X_2,\dots \overset{i.i.d.}{\sim} U(\{1,2,\dots,n\})$, $\tau_k^n \triangleq \inf\{m: |\{X_1,\dots,X_m\}|=k\}$, $T_n\triangleq \tau_n^n$

Sol: $P(X_{nk}=m)=P(X_{nk}>m-1)-P(X_{nk}>m)=\left(\frac{k-1}{n}\right)^{m-1}\left(1-\frac{k-1}{n}\right)$, $k=1,\dots,n$ 为几何分布且 $X_{nk}, k=1,\dots,n$ 独立 (马氏性)

$\therefore T_n=\sum_{k=1}^n X_{nk}$

$\therefore ET_n=\sum\left(1-\frac{k-1}{n}\right)^{-1}=n\sum_{m=1}^n\frac{1}{m}\sim n\log n$

$\mathrm{Var}T_n=\sum\left(1-\frac{k-1}{n}\right)^{-2}=n^2\sum_{m=1}^n\frac{1}{m^2}\sim Cn^2$

$\frac{T_n-ET_n}{ET_n}\xrightarrow{P}0$ $\frac{T_n}{ET_n}\xrightarrow{P}1$, 即 $\frac{T_n}{n\log n}\xrightarrow{P}1$ #

$P(|\frac{T_n-ET_n}{ET_n}|>\varepsilon)\le\frac{\mathrm{Var}(T_n)}{\varepsilon^2(ET_n)^2}\sim\frac{Cn^2}{\varepsilon^2 n^2\log^2 n}\to 0$

~~(Glivenko-Cantelli)~~

Eg. 设 $f\in C[0,1]$, $f_n(x)=\sum_{m=0}^n\binom{n}{m}x^m(1-x)^{n-m}f(\frac{m}{n})$, 则 $\lim_{n\to\infty}\sup_{x\in[0,1]}|f_n(x)-f(x)|=0$.

Pf: $X_1(x),\dots$ i.i.d. $P(X_1(x)=1)=x$, $P(X_1(x)=0)=1-x$

则 $f_n(x)=Ef(\frac{S_n}{n})$

Fix x, $\frac{S_n}{n}\xrightarrow{P}x$ (WLLN)

$\Rightarrow f(\frac{S_n}{n})\xrightarrow{P}f(x)$

$\underset{\text{(Skorohod)}}{\Rightarrow} Ef(\frac{S_n}{n})\to f(x)$ 即 $f_n(x)\to f(x)$

由 f 一致连续 on $[0,1]$: $\forall\varepsilon>0,\exists\delta>0$, s.t. $|f(x)-f(y)|<\varepsilon$, $\forall|x-y|<\delta$

$\sup_{x\in[0,1]}|f_n(x)-f(x)|=\sup_{x\in[0,1]}|E(f(\frac{S_n}{n})-f(x))|$

$\le\sup_{x\in[0,1]}E|f(\frac{S_n}{n})-f(x)|I(|\frac{S_n}{n}-x|<\delta)+\sup_{x\in[0,1]}E|f(\frac{S_n}{n})-f(x)|I(|\frac{S_n}{n}-x|\ge\delta)$

$\le\varepsilon+\max f\cdot\sup_{x\in[0,1]}P(|\frac{S_n}{n}-x|\ge\delta)$

$\le\varepsilon+\max f\cdot\sup_{x\in[0,1]}\underbrace{\frac{\mathrm{Var}(\frac{S_n}{n})}{\delta^2}}_{=\frac{x(1-x)}{n\delta^2}}\to\varepsilon\ (n\to\infty)$

由 ε 任意性, $\sup_{x\in[0,1]}|f_n(x)-f(x)|\to 0$ #

Thm 三角组列: $X_{11},\dots,X_{1k_1}$ (独立)

$X_{21},\dots,X_{2k_2}$

$\dots\dots$

记 $S_n=\sum_{k=1}^{k_n}X_{nk}$, $b_n>0\to\infty$. 若 (1) $\sum_{k=1}^{k_n}P(|X_{nk}|>b_n)\to 0$ (2) $\frac{1}{b_n^2}\sum_{k=1}^{k_n}EX_{nk}^2I(|X_{nk}|\le b_n)\to 0$

则 $\frac{S_n-\sum_{k=1}^{k_n}EX_{nk}I(|X_{nk}|\le b_n)}{b_n}\xrightarrow{P}0$.

WLLN 套路: chebyshev ; SLLN 套路: B-C lemma

- $X_n\xrightarrow{a.s.}X \Leftrightarrow \forall\varepsilon>0$, $P(|X_n-X|>\varepsilon, i.o.)=0$.
- B-C: $\begin{cases}\sum P(A_n)<\infty\Rightarrow P(A_n, i.o.)=0\\ \sum P(A_n)=\infty, A_n \text{独立}\Rightarrow P(A_n, i.o.)=1.\end{cases}$

Eg. (B-C 在 random walk 中应用) $S_n=X_1+\dots+X_n$, $X_i\overset{i.i.d.}{\sim}U(\{(a_1,\dots,a_d): a_i=\pm1\})$, $\in\mathbb{Z}^d$

则 $P(S_n=0, i.o.)=\begin{cases}1, & d=1,2\\ 0, & d=3\end{cases}$

Pf: $n=1$ 时, $P(S_{2n}=0)=\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\sim\frac{C}{\sqrt n}$ $\left(n!\sim\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)$

$\therefore n\ge3$ 时 $P(S_{2n}=0)\sim C/n^{\frac{d}{2}}$ $\therefore\sum P(S_{2n}=0)<\infty\Rightarrow P(S_n=0, i.o.)=0$. #

Thm (SLLN) $X_1,X_2,\dots$ i.i.d, $E|X|<\infty$, 则 $\frac{S_n}{n}\xrightarrow{a.s.}EX$

- 若 $EX=0$, $EX^2<\infty$, 证明如下:

$\sum_{n=1}^\infty P(|\frac{S_{n^2}}{n^2}-EX|>\varepsilon)\le\sum_{n=1}^\infty\frac{\mathrm{Var}X}{n^2\varepsilon^2}<\infty$. $\therefore\frac{S_{n^2}}{n^2}\xrightarrow{a.s.}EX$, 由 B-C. (1)

下证 $\frac{\sup_{(n-1)^2<k\le n^2}|S_k-S_{(n-1)^2}|}{(n-1)^2}\xrightarrow{a.s.}0$. (若 $|X|\le c$ a.s. 易证)

记 $D_n=\sup_{n^2<k\le(n+1)^2}|S_k-S_{n^2}|$, 则 $\sum_{n=1}^\infty P(\frac{D_n}{n^2}>\varepsilon)\le\sum_{n=1}^\infty\frac{ED_n^2}{n^4\varepsilon^2}$

$\le\sum_{n=1}^\infty\frac{E(\sum_{n^2<k\le(n+1)^2}|S_k-S_{n^2}|)^2}{n^4\varepsilon^2}$

$\lesssim\sum_{n=1}^\infty\frac{1}{n^4}\cdot n\cdot E(S_{(n+1)^2}-S_{n^2})^2$

$\lesssim\sum_{n=1}^\infty\frac{1}{n^2}<\infty$ #

$\therefore\frac{D_n}{n^2}\xrightarrow{a.s.}0$, 由 B-C. (2)

由 (1), (2) 得证.

- 若 $EX=0$, $EX^4<\infty$, 则: $ES_n^4\lesssim nEX^4+n^2(EX^2)^2\lesssim n^2$

$\therefore\sum P(|\frac{S_n}{n}|>\varepsilon)\lesssim\sum\frac{n^2}{n^4\varepsilon^4}<\infty\xrightarrow{B-C}\frac{S_n}{n}\xrightarrow{a.s.}EX$. #

- 子序列方法: $\frac{S_n}{n}\xrightarrow{a.s.}0\Leftrightarrow P(|\frac{S_n}{n}|>\varepsilon, i.o.)=0, \forall\varepsilon>0\overset{B-C}{\Leftarrow}\sum P(|\frac{S_n}{n}|>\varepsilon)<\infty$

$\sum P(|\frac{S_{n_k}}{n_k}|>\varepsilon)<\infty\Rightarrow\frac{S_{n_k}}{n_k}\to 0$, a.s. ($n_k$ 常取 $n^2, \alpha^n, e^{\sqrt n},\dots$)

$\Rightarrow\frac{S_n}{n}\xrightarrow{a.s.}0$, 用其它条件.

Rmk: $\frac{S_n}{n}\xrightarrow{a.s.}a\Leftrightarrow E|X|<\infty$ 的 ($\Rightarrow$) 方向: $\frac{X_n}{n}=\frac{S_n}{n}-\frac{S_{n-1}}{n}\to 0$ a.s. $\Leftrightarrow\sum P(|\frac{X_n}{n}|>\varepsilon)<\infty, \forall\varepsilon>0$.

$\Leftrightarrow\sum P(|\frac{X_n}{n}|>1)<\infty$

$\Leftrightarrow\sum P(|X|>n)<\infty\Leftrightarrow E|X|<\infty$

E.g. 设 A_1, A_2 两两独立，$\sum P(A_n)=\infty$，则当 $n\to\infty$，$\dfrac{\sum_{k=1}^{n} I_{A_k}}{\sum_{k=1}^{n} P(A_k)} \to 1$, a.s.

证 记 $S_n=\sum_{k=1}^{n} I_{A_k}$

$P\left(\left|\frac{S_n-ES_n}{ES_n}\right|>\varepsilon\right) \overset{\text{两两独立}}{\le} \frac{1}{\varepsilon^2(ES_n)^2}\sum Var(I_{A_k}) \le \frac{1}{\varepsilon^2 ES_n}$

取 $k_n=\inf\{k: ES_k \ge n^2\}$，则 $\sum_{n=1}^{\infty} P\left(\left|\frac{S_{k_n}-ES_{k_n}}{ES_{k_n}}\right|>\varepsilon\right)<\infty$

由 B-C，$\frac{S_{k_n}}{ES_{k_n}} \xrightarrow{a.s.} 1$

$\because S_{k_{n-1}} \le S_m \le S_{k_n}$，$\forall m\in(k_{n-1},k_n)$

$\frac{S_m}{ES_m} < \frac{S_{k_n}}{ES_{k_{n-1}}} = \frac{S_{k_n}}{ES_{k_n}}\cdot\frac{ES_{k_n}}{ES_{k_{n-1}}}$　　且 $\frac{ES_{k_n}}{ES_{k_{n-1}}} \le \frac{n^2+1}{(n-1)^2} \to 1$.

$\Rightarrow \overline{\lim}\frac{S_m}{ES_m} \overset{a.s.}{\le} 1$　　同理 $\underline{\lim}\frac{S_m}{ES_m} \overset{a.s.}{\ge} 1$　　$\therefore \frac{S_m}{ES_m} \xrightarrow{a.s.} 1 \quad (m\to\infty)$　　#.

Pf of SLLN：设 $X\ge 0$，$Y_n \triangleq X_n I(|X_n|\le n)$，$T_n=\sum_{k=1}^{n} Y_k$

$\because \sum P(X_n\ne Y_n)=\sum P(|X|>n)<\infty$.

$\therefore P(X_n\ne Y_n, \text{i.o.})=0$　$\therefore \frac{S_n-T_n}{n} \xrightarrow{a.s.} 0$

为证 $\frac{T_n}{n} \xrightarrow{a.s.} EX$，先证 $\frac{T_{n_k}-ET_{n_k}}{n_k} \xrightarrow{a.s.} 0$，再由 $\frac{ET_{n_k}}{n_k} \xrightarrow{\text{控制收敛}} EX$ 知 $\frac{T_{n_k}}{n} \xrightarrow{a.s.} EX$.

① Fix $\alpha>1$，令 $n_k=[\alpha^k]$

$\sum_{k=1}^{\infty} P(|T_{n_k}-ET_{n_k}|>\varepsilon n_k) \le \sum_{k=1}^{\infty}\frac{1}{\varepsilon^2 n_k^2}\sum_{m=1}^{n_k} EY_m^2$

$\overset{\text{换序}}{=} \sum_{m=1}^{\infty} EY_m^2 \sum_{\{k: m\le n_k\}} \frac{1}{\varepsilon^2 n_k^2}$

$\le C_{\alpha,\varepsilon}\sum_{m=1}^{\infty}\frac{1}{m^2}EY_m^2$

$=C_{\alpha,\varepsilon}\sum_{m=1}^{\infty}\frac{1}{m^2}EX^2 I(|X|\le m)$

$<\infty$ [CLAIM].　　由 B-C 可得 $\frac{T_{n_k}-ET_{n_k}}{n_k} \xrightarrow{a.s.} 0$

$\sum\frac{1}{n_k^2} \simeq \sum\frac{1}{(\alpha^k)^2} \simeq C_\alpha\cdot\frac{1}{(\alpha^k)^2} \le C_\alpha\cdot\frac{1}{m^2}$

当 $n_k<n<n_{k+1}$ 时，$\frac{T_n}{n} \le \frac{T_{n_{k+1}}}{n_k} \to EX\cdot\alpha \Rightarrow \overline{\lim}\frac{T_n}{n} \le \alpha EX$　同理 $\underline{\lim}\frac{T_n}{n} \ge \frac{1}{\alpha}EX$

令 $\alpha\to 1$ 得 $\frac{T_n}{n} \xrightarrow{a.s.} EX \Rightarrow \frac{S_n}{n} \xrightarrow{a.s.} EX$.

一般地，$\frac{S_n}{n}=\frac{\sum X_k^+ - \sum X_k^-}{n} \to EX^+ - EX^- = EX$. a.s.　　#

[CLAIM] $\sum_m EX^2 I(|X|\le m)\frac{1}{m^2}<\infty$

(mtd1) $= E\sum_{m=1}^{\infty} X^2 I(|X|\le m)\frac{1}{m^2}$

$\le EX^2\cdot\frac{1}{|X|}<\infty$

(mtd2) $=\sum_m\sum_{n\le m} EX^2 I(n-1<|X|\le n)\cdot\frac{1}{m^2}$

$\le \sum_m\sum_{n\le m} P(n-1<|X|\le n)\cdot n^2\cdot\frac{1}{m^2}$

$=\sum_n\sum_{m\ge n}\frac{1}{m^2}\, n^2 P(n-1<|X|\le n) \simeq E|X|<\infty$

Thm. $X_1, X_2, \cdots$ i.i.d. 若 $EX^+=\infty$，$EX^-<\infty$，则 $\frac{S_n}{n} \xrightarrow{a.s.} +\infty$

Rmk. $E|X|=\infty \Rightarrow \overline{\lim}\frac{S_n}{n}=\infty$, a.s.

Pf：$\forall M>0$，$X_i^M \triangleq \min(X_i, M)$　$\therefore \frac{\sum_{i=1}^{n} X_i^M}{n} \xrightarrow{a.s.} E(X_i^M)$

$\therefore \underline{\lim}\frac{S_n}{n} \ge E(X\wedge M)$ a.s.　令 $M\to\infty$ 得 $\underline{\lim}\frac{S_n}{n}\ge\infty$.　　#

(Thm)例. $X_i \overset{i.i.d.}{\sim} F$，$F_n(x)=\frac{1}{n}\sum I(X_i\le x)$，则 $\sup_x |F_n(x)-F(x)| \to 0$ a.s.

(Glivenko-Cantelli).

Pf：$F_n(x) \xrightarrow{a.s.} F(x)$　By SLLN.

① F 连续

$\forall\varepsilon>0$，$\exists -\infty=x_0<\cdots<x_{m+1}=\infty$，s.t. $F(x_i)-F(x_{i-1})\le\varepsilon$，$\forall i$

$\exists n_0\in\mathbb{N}$，s.t. $\forall n>n_0$，$|F_n(x_i)-F(x_i)|\le\varepsilon$，$\forall i$

对 $\forall x\in\mathbb{R}$，$x\in[x_{i-1},x_i]$，$F_n(x)-F(x) \le F_n(x_i)-F(x_{i-1}) \le 2\varepsilon$.

同理 $F_n(x)-F(x)\ge 2\varepsilon$

② 一般的 F，

找 F 的所有跳跃 $\ge\varepsilon$ 的点 $x_i^0<\cdots<x_l^0$，则 $F(x)-\sum\Delta F(x_i^0) := G(x)$ 可用 ①

而 x_i^0 可由 SLLN 逼近.　　#

Thm. (Kolmogorov ineq.) X_k 独立，$EX_k=0$，$EX_k^2<\infty$. 则 $\forall\varepsilon>0$，$P(\max_{k\le n}|S_k|\ge\varepsilon) \le \frac{1}{\varepsilon^2}\sum_{k\le n} EX_k^2$

进一步，若 $|X_k|\le C<\infty$，则 $1-\frac{(\varepsilon+C)^2}{\sum_{k\le n} EX_k^2} \le P(\max_{k\le n}|S_k|\ge\varepsilon)$.

Pf：$T\triangleq\inf\{k: |S_k|\ge\varepsilon\}$，则 $\{\max|S_k|\ge\varepsilon\}=\{T\le n\}=\bigcup_{k=1}^{n}\{T=k\}$

$ES_n^2 \ge ES_n^2 I(T\le n)=\sum_{k=1}^{n} ES_n^2 I(T=k)=\sum_{k=1}^{n}\left[ES_k^2 I(T=k)+E(S_n-S_k)^2 I(T=k)+2\underbrace{E(S_n-S_k)S_k I(T=k)}_{=0\,(\text{独立})}\right]$

$\ge \varepsilon^2\sum_{k=1}^{n} P(T=k)=\varepsilon^2 P(T\le n)$

另一方面，$ES_n^2 I(T\le n)=\sum_{k=1}^{n}\Big[E\underbrace{S_k^2}_{\le(|S_{k-1}|+|X_k|)^2} I(T=k)+\underbrace{E(S_n-S_k)^2 I(T=k)}_{\text{独立}\ E(S_n-S_k)^2 P(T=k)}\Big]$

$\le (\varepsilon+C)^2 P(T\le n) + ES_n^2 P(T\le n)$

$ES_n^2 I(T>n) \le \varepsilon^2 P(T>n)$

$\therefore ES_n^2 \le ((\varepsilon+C)^2+ES_n^2)P(T\le n)+\varepsilon^2(1-P(T\le n))$

$\Rightarrow P(T\le n) \ge \frac{ES_n^2-\varepsilon^2}{(\varepsilon+C)^2+ES_n^2-\varepsilon^2} = 1-\frac{(\varepsilon+C)^2}{(\varepsilon+C)^2-\varepsilon^2+ES_n^2} \ge 1-\frac{(\varepsilon+C)^2}{ES_n^2}$　　#

Thm (Kolmogorov 0-1 Law) X_k 独立，$\mathcal{Y}_n=\sigma\{X_m, m\ge n\}$，$\mathcal{Y}\triangleq\bigcap_{n=1}^{\infty}\mathcal{Y}_n$，则 $\forall A\in\mathcal{Y}$，$P(A)=0$ 或 1

(尾事件 σ 域).

Pf：由 $\mathcal{Y}$ 与 $\sigma\{X_1,\cdots,X_n\}$ 独立 知 $\mathcal{Y}$ 与 $\sigma\{X_1,\cdots\}$ 独立.

$\because \mathcal{Y}\in\sigma\{X_1,\cdots\}$ $\therefore \mathcal{Y}$ 与 $\mathcal{Y}$ 独立. $(P(A)=P(A\cap A)=P(A)^2)$　#.

Date .

Pf of 二级数定理

2° $P(\max\limits_{1\le k\le m}|X_{n+1}+\cdots+X_{n+k}|\geq\varepsilon)\overset{(\text{Kolmogorov})}{\geqslant} 1-\dfrac{(\varepsilon+c)^2}{E(S_{n+m}-S_n)^2}\to 1\quad(m\to\infty)$.

∴ S_n a.s. 发散.

1° 对称化方法 ($X-X'$ 为 X 的对称化，$\frac{1}{2}P(|X-m|\geq x)\leq P(|X-X'|\geq x)\leq 2P(|X-a|\geq\frac{x}{2}),\forall a$)

(从而，$E|X|^p<\infty \Leftrightarrow E|X-\cancel{X'}|^p<\infty$.)

取独立复制 $\{X_n'\}$，$\tilde{X}_n\triangleq X_n-X_n'$，则

$|\tilde{X}_n|\leq 2C$，$E\tilde{X}_n=0$，$Var(\tilde{X}_n)=2Var(X_n)$

$\sum X_n$ a.s. 收 $\Rightarrow \sum X_n'$ a.s. 收 $\Rightarrow \sum(X_n-X_n')$ a.s. 收 $\overset{\text{CLAIM: 1°在}EX_n=0\text{时成立}}{\Longrightarrow}\sum Var(X_n-X_n')<\infty\Rightarrow\sum Var(X_n)<\infty$

$\Downarrow$

$\sum(X_n-EX_n)$ a.s. 收 #

$\Downarrow$

$\sum EX_n$ a.s. 收

Pf of 三级数：记 $Y_n=X_nI(|X_n|\leq c)$

($\Leftarrow$): $\overset{由1°}{\sum}P(X_n\neq Y_n)<\infty \Rightarrow P(X_n\neq Y_n, i.o.)=0$ ∴ $\sum X_n$ a.s. 收敛 $\Leftrightarrow \sum Y_n$ a.s. 收敛

由 $\sum Var(Y_n)<\infty$，$\sum EY_n$ 收敛 知 (二级数) $\sum Y_n$ a.s. 收敛.

($\Rightarrow$) CLAIM: 1°成立 $\forall c$. 从而 $\sum X_n$ a.s. 收敛 $\Rightarrow \sum Y_n$ a.s. 收敛 $\overset{\text{二级数,II}}{\Longrightarrow}\sum EY_n$, $\sum Var Y_n<\infty$.

下证 CLAIM.

∵ $\sum P(|X_n|>C)<\infty \overset{\text{独立, B-C}}{\Longleftrightarrow} P(|X_n|>C, i.o.)=0$, $\forall C \Leftrightarrow X_n\xrightarrow{a.s.}0 \Leftarrow \sum X_n<\infty$.

#.

Pf of Kronecker: 记 $b_n=\sum\limits_{k=1}^{n}\dfrac{x_k}{a_k}\to b$. , $b_0=0$

则 $x_{n+1}=(b_{n+1}-b_n)a_{n+1}$

∴ $\dfrac{1}{a_n}\sum\limits_{k=1}^{n}x_k=\dfrac{1}{a_n}\sum\limits_{k=1}^{n}a_k(b_k-b_{k-1})=\dfrac{1}{a_n}[b_1(a_1-a_2)+\cdots+b_{n-1}(a_{n-1}-a_n)+b_na_n]$

$=b_n-\dfrac{1}{a_n}\sum\limits_{k=1}^{n-1}b_k(a_{k+1}-a_k)$

$\to b-b=0$ #.

Pf of SLLN: ($\Rightarrow$) $\dfrac{X_n}{n}=\dfrac{S_n}{n}-\dfrac{S_{n-1}}{n}\to 0$, a.s. $\Rightarrow \sum P(|X_n|>n)<\infty \Rightarrow E|X|<\infty$

由 SLLN 充分性，$\frac{S_n}{n}\to EX$, a.s. 则 $EX=a$.

(X) ($\Leftarrow$) 若 $EX=0$，由 Kronecker，只须 $\sum\frac{X_n}{n}$ 收敛

由三级数，$\Leftrightarrow\begin{cases}1°\sum P(|X_n|>n)<\infty\\2°\sum E\frac{X_n}{n}I(|X_n|\leq n)\text{ 收敛}\\3°\sum Var(\frac{X_n}{n}I(|X_n|\leq n))<\infty\end{cases}$

1° $\sum P(|X_n|>n)\lesssim E|X|<\infty$

3° $\sum Var(\frac{X_n}{n}I(|X_n|\leq n))\leq\sum\limits_n E\frac{X^2}{n^2}I(|X|\leq n)\lesssim EX^2\frac{1}{|X|}<\infty$ 或 $\sum Var(\frac{X_n}{n}I(|X_n|\leq n))$

$\leq\sum E\frac{X^2}{n^2}\sum\limits_{k\leq n}I(k-1<|X|\leq k)$ $=\sum\limits_{k=1}^{\infty}\sum\limits_{n\geq k}E\frac{k^2}{n^2}I(k-1<|X|\leq k)\lesssim\sum\limits_{k=1}^{\infty}Ek\cdot I_{(k-1,k]}(|X|)\leq E|X|<\infty$.

但2°过不去.

Date .

- $\frac{S_n}{n}\xrightarrow{a.s.}0 \Leftrightarrow \forall\varepsilon>0$, $P(|\frac{S_n}{n}|>\varepsilon, i.o.)=0 \Leftarrow \sum\limits_n P(|\frac{S_n}{n}|>\varepsilon)<\infty$

常先证 $\frac{S_{n_k}}{n_k}\xrightarrow{a.s.}0$，$\frac{S_n}{n}\to\frac{S_{n_k}}{n_k}$

- $\frac{S_n}{n}\xrightarrow{a.s.}0 \Leftarrow \sum\limits_k\frac{X_k}{n}$ a.s. 收敛.

Def $\sum\limits_{n=1}^{\infty}X_n$ a.s. 收敛: $P(\sum\limits_{n=1}^{\infty}X_n(\omega)\text{ 收敛})=1$

则 $\sum\limits_{k=1}^{n}X_k\xrightarrow{a.s.}S \Leftrightarrow S_n$ a.s. 收敛.

Thm. X_n 独立，$EX_n=0$，$\sum\limits_{k=1}^{\infty}EX_k^2<\infty$，则 $\sum X_n$ a.s. 收敛.

Pf: 由 Kolmogorov ineq. $P(\max\limits_{M\leq n\leq N}|S_n-S_M|\geq\varepsilon)\leq\frac{1}{\varepsilon^2}\sum\limits_{m=M}^{N}EX_m^2$

令 $N\to\infty$，$P(\max\limits_{n\geq M}|S_n-S_M|\geq\varepsilon)\leq\frac{1}{\varepsilon^2}\sum\limits_{m=M}^{\infty}EX_m^2$

∴ $\lim\limits_{M\to\infty}P(\max\limits_{n\geq M}|S_n-S_M|\geq\varepsilon)=0$

∴ S_n a.s. 收敛. (由 S_n a.s. Cauchy 列) #.

Cor (一级数定理) $\sum Var(X_n)<\infty \Rightarrow \sum(X_n-EX_n)$ a.s. 收敛 (二级数) $\left.\begin{array}{l}\sum Var(X_n)<\infty\\ \sum EX_n\text{ 收敛}\end{array}\right\}\Rightarrow\sum X_n$ a.s. 收敛.

Thm (二级数, II). X_n 独立，$\exists C>0$, s.t. $|X_n|\leq C$, a.s.

则 1° 若 $\sum X_n$ a.s. 收敛，则 $\sum EX_n$，$\sum Var(X_n)$ 收敛

2° 若 $EX_n=0$，$\sum Var(X_n)=\infty$，则 $\sum X_n$ a.s. 发散.

Thm (Kolmogorov 三级数定理) X_n 独立，$\sum X_n$ a.s. 收敛 $\Leftarrow \exists C>0$, s.t.

1° $\sum P(|X_n|>C)<\infty$

2° $\sum EX_nI(|X_n|\leq C)$ 收敛

3° $\sum Var(X_nI(|X_n|\leq C))<\infty$

"$\Rightarrow$": $\forall C>0$, 1°2°3° 收敛.

Lem. (Kronecker) 实数列 $\{x_n\}$，正数列 $\{a_n\}\uparrow\infty$，则当级数 $\sum\frac{x_n}{a_n}$ 收敛，有 $\frac{1}{a_n}\sum\limits_{k=1}^{n}x_k\to 0$

Thm (SLLN) X_n, i.i.d.，则 $\exists a$, s.t. $\frac{S_n}{n}\xrightarrow{a.s.}a \Leftrightarrow E|X|<\infty$, $EX=a$.

($\Leftarrow$): 记 $Y_n=X_nI(|X_n|<n)$，由1°知 $P(X_n\neq Y_n, i.o.)=0$，下证 $\frac{T_n}{n}\xrightarrow{a.s.}EX$.

由于3°仍成立，$\sum Var(\frac{Y_n}{n})<\infty \overset{\text{一级数}}{\Longrightarrow}\sum\frac{(Y_n-EY_n)}{n}$ a.s. 收敛，由 Kronecker，$\frac{1}{n}(T_n-ET_n)\to 0$ a.s.

∵ $\frac{1}{n}ET_n=\frac{1}{n}\sum\limits_{k=1}^{n}EX_kI(|X_k|\leq k)\to EX$ ∴ $\frac{T_n}{n}\xrightarrow{a.s.}EX$. #

Date .

Pf of M-Z SLLN: 设 $r\neq 1$. 不妨 $EX=0$, 只证 $\frac{S_n}{n^{r}}\xrightarrow{a.s.}0$,

由 Kronecker, 只须 $\sum\frac{X_n}{n^{r}}$ a.s. 收敛

由三级数, 只须 $\begin{cases}1^\circ\ \sum P(|X_n|>n^{\frac{1}{r}})<\infty\\ 2^\circ\ \sum E\frac{X_n}{n^{r}}I(|X_n|\le n^{\frac{1}{r}})\ \text{收敛}.\\ 3^\circ\ \sum Var(\frac{X_n}{n^{r}}I(|X_n|\le n^{\frac{1}{r}}))<\infty\end{cases}$

$1^\circ\ \sum P(|X_n|>n^{\frac{1}{r}})=\sum P(|X|^{r}>n)<\infty$

$3^\circ\ \sum Var(\frac{X_n}{n^{r}}I(|X_n|\le n^{\frac{1}{r}}))\le\sum E\frac{X^2}{n^{\frac{2}{r}}}I(|X|\le n^{\frac{1}{r}})$

$\lesssim EX^2\sum_{n\ge|X|^r}\frac{1}{n^{\frac{2}{r}}}$

$\overset{r<2}{\lesssim} EX^2(|X|^r)^{1-\frac{2}{r}}=E|X|^r<\infty$

$2^\circ\ \sum|E\frac{X_n}{n^{r}}I(|X_n|\le n^{\frac{1}{r}})|\begin{cases}\le\sum E\frac{|X_n|}{n^{\frac{1}{r}}}I(|X_n|\le n^{\frac{1}{r}})<\infty,\ \text{if } 0<r<1, \text{方法同}3^\circ\\ =\sum|E\frac{X_n}{n^{r}}I(|X_n|>n^{\frac{1}{r}})|\le\sum E\frac{|X|}{n^{r}}I(|X|>n^{\frac{1}{r}})\lesssim E|X|(|X|^r)^{1-\frac{1}{r}}<\infty.\end{cases}$

#

Pf of Thm (用 $g_n(x)$ 控制的 SLLN)

由 Kronecker, 只须 $\sum\frac{X_n}{a_n}$ a.s. 收敛 $\overset{\text{三级数}}{\Longleftrightarrow}\begin{cases}1^\circ\sum P(|X_n|>a_n)<\infty\\ 2^\circ\sum E\frac{X_n}{a_n}I(|X_n|\le a_n)\ \text{收敛}\\ 3^\circ\sum Var(\frac{X_n}{a_n}I(|X_n|\le a_n))<\infty\end{cases}$

$1^\circ\ \sum P(|X_n|>a_n)\le\sum E\frac{g_n(X_n)}{g_n(a_n)}I(|X_n|>a_n)\le\sum E\frac{g_n(X_n)}{g_n(a_n)}<\infty.$

$3^\circ\ \sum Var(\frac{X_n}{a_n}I(|X_n|\le a_n))\le\sum E(\frac{X_n}{a_n})^2I(|X_n|\le a_n)$

$\le\sum E(\frac{g_n(X_n)}{g_n(a_n)}I(|X_n|\le a_n))$　　∵ $|x|\le a_n$ 时 若有(ii), $\frac{x^2}{a_n^2}\le\frac{g_n(x)}{g_n(a_n)}$; 若有(i) $\frac{x^2}{a_n^2}\le(\frac{g_n(x)}{g_n(a_n)})^2\le\frac{g_n(x)}{g_n(a_n)}$

$\le\sum E\frac{g_n(X_n)}{g_n(a_n)}<\infty$

$2^\circ\ \sum|E\frac{X_n}{a_n}I(|X_n|\le a_n)|\le\sum E\frac{|X_n|}{a_n}I(|X_n|\le a_n)\overset{(i)}{\le}\sum E\frac{g_n(|X_n|)}{g_n(a_n)}I(|X_n|\le a_n)<\infty$

或 $\sum|E\frac{X_n}{a_n}I(|X_n|\le a_n)|=\sum|E\frac{X_n}{a_n}I(|X_n|>a_n)|\le\sum E\frac{|X_n|}{a_n}I(|X_n|>a_n)\overset{(ii)}{\le}\sum E\frac{g_n(X_n)}{g_n(a_n)}<\infty$　#

Date .

Thm (Marcinkiewicz-Zygmund SLLN) X_n, i.i.d. $\exists a$, s.t. $\frac{S_n-na}{n^{\frac{1}{r}}}\xrightarrow{a.s.}0\Longleftrightarrow E|X|^r<\infty$　$a=\begin{cases}EX, & 1\le r<2\\ \text{任意实数}, & 0<r<1\end{cases}$

应用: $E|X|<\infty$ 推 $\max_{1\le k\le n}\frac{|X_k|}{n}\xrightarrow{a.s.}0\Longleftrightarrow\max\frac{|X_k|^2}{n^2}\xrightarrow{a.s.}0$.

Pf: 令 $Y=X^2$, $\max\frac{|X_k|^2}{n^2}\le\sum_{k\le n}\frac{Y_k}{n^2}\xrightarrow[a.s.]{M-Z}0\ (r=\frac{1}{2})$　#.

- X_n, i.i.d. $EX=0$, $EX^2<\infty\Rightarrow\frac{S_n}{n}\xrightarrow{a.s.}0$. $\frac{S_n}{n^r}\xrightarrow{a.s.}0$, $\forall 0.5<r<1$, 但 $\frac{S_n}{\sqrt{n}}\not\xrightarrow{a.s.}0$.

Thm. $\frac{S_n}{\sqrt{n}(\log n)^{\frac{1}{2}+\varepsilon}}\xrightarrow{a.s.}0$. Pf: 由 Kronecker, 只须 $\sum\frac{X_n}{\sqrt{n}(\log n)^{\frac{1}{2}+\varepsilon}}$ a.s. 收敛

由 $EX=0$ 和二级数, 只须 $\sum Var(\frac{X_n}{\sqrt{n}(\log n)^{\frac{1}{2}+\varepsilon}})\asymp\sum\frac{1}{n(\log n)^{1+2\varepsilon}}<\infty$　#

Thm. (Hartman-Wintner 重对数律) X_n, i.i.d., $EX=0$, $EX^2=\sigma^2<\infty$, 则 $\overline{\lim_{n\to\infty}}\frac{S_n}{\sqrt{2n\log\log n}}=\sigma$, a.s.

Cor. $\underline{\lim}\frac{S_n}{\sqrt{2n\log\log n}}=-\sigma$, a.s.; $\overline{\lim}\frac{|S_n|}{\sqrt{2n\log\log n}}=\sigma$, a.s.; $\underline{\lim}\frac{|S_n|}{\sqrt{2n\log\log n}}=0$, a.s. (因为 $\underline{\lim}\frac{|S_n|}{\sqrt{n}}=0$, a.s.)

Thm. X_n 独立, $g_n(x)>0$ 偶, 在 $x>0$ 非降 且满足

(i) $\frac{g_n(x)}{x}$ 非升 in $x>0$.

或 (ii) $\frac{x}{g_n(x)}$ 与 $\frac{g_n(x)}{x^2}$ 非升 in $x>0$.

若 $\exists\{a_n\}\uparrow\infty$, $\sum\frac{Eg_n(X_n)}{g_n(a_n)}<\infty$, (若满足(ii), 还要有 $EX_n=0$), 则 $\frac{S_n}{a_n}\to 0$, a.s.

Rmk. 常取 $g_n(x)=|x|^r$, $0<r\le 2$

- X_n, i.i.d. $\frac{\log n}{n}\sum_{k=1}^{n}\frac{X_k}{\log k}\to 0$, a.s. $\Longleftrightarrow E|X|<\infty$, $EX=0$
- 若 $EX=0$, $E|X|\log^r|X|<\infty$, 则 $\frac{(\log n)^r}{n}\sum_{k=1}^{n}X_k\to 0$, a.s. (自行练习!) (截尾)

(约定, $|X|<1$ 时, $\log|X|=\log(|X|\vee e)=1$.)

重要不等式: • $\forall x>0$ $\frac{x}{1+x^2}e^{-\frac{x^2}{2}}\le\int_x^\infty e^{-\frac{t^2}{2}}dt\le\frac{1}{x}e^{-\frac{x^2}{2}}$ 特别 $1-\Phi(x)\approx\frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$, $x\to\infty$

Pf: $\int_x^\infty e^{-\frac{t^2}{2}}dt\le\int_x^\infty\frac{t}{x}e^{-\frac{t^2}{2}}dt=RHS$.

$\int_x^\infty e^{-\frac{t^2}{2}}dt\ge\int_x^\infty\frac{\frac{1+t^2}{t^2}}{\frac{1+x^2}{x^2}}e^{-\frac{t^2}{2}}dt=\frac{x^2}{1+x^2}\cdot-\frac{e^{-\frac{t^2}{2}}}{t}\Big|_{t=x}^{\infty}=LHS$.　#

- (Lévy 不等式) X_n 独立, 则 $\forall x>0$, $P(\max_{1\le k\le n}(S_k+med(S_n-S_k))>x)\le 2P(S_n>x)$

$P(\max_{1\le k\le n}|S_k+med(S_n-S_k)|>x)\le 2P(|S_n|>x)$.

Pf: 记 $T=\inf\{k: S_k+med(S_n-S_k)>x\}$

$LHS=P(T\le n)=\sum_{k=1}^{n}P(T=k)\le\sum_{k=1}^{n}P(T=k)\cdot 2P(S_n-S_k\ge med(S_n-S_k))$

$\overset{T\text{停时}}{=}2\sum_{k=1}^{n}P(T=k, S_n-S_k\ge med(S_n-S_k))$

$\le 2\sum P(T=k, S_n>x)=RHS$.

∴ (中位数估计) $|med(S_n-S_k)|\le\sqrt{2Var(S_n)}$　∵ $P(|S_n-S_k|\le\sqrt{2Var(S_n)})\overset{\text{chebyshev}}{\ge}1-\frac{1}{2}=\frac{1}{2}$.

特别, $EX_i=0$, $EX_i^2<\infty$, 则 $P(\max_{1\le i\le n}S_i\ge x)\le 2P(S_n\ge x-\sqrt{2\sum_{i=1}^{n}EX_i^2})$

Date .

Pf of S_n 依P收敛 $\Leftrightarrow$ S_n 依 a.s. 收敛?

记 $S_n=\sum_{i=1}^{n}X_i$. 若 $S_n\xrightarrow{P}S$. 下证 $S_n\xrightarrow{a.s.}S$.

$\exists$ 子列 $\{n_k\}$, s.t. $\sum P(|S_{n_{k+1}}-S_{n_k}|>2^{-k})<\infty \Rightarrow S_{n_k}\xrightarrow{a.s.}S$

$\sum_{k=1}^{\infty}P\left(\max_{n_k\le n\le n_{k+1}}|S_n-S_{n_k}+med(S_{n_{k+1}}-S_n)|>2^{-k}\right)\overset{Lévy}{\le}\sum_k 2P(|S_{n_{k+1}}-S_{n_k}|>2^{-k})<\infty$

(*) $\therefore \max_{n_k\le n\le n_{k+1}}|S_n-S_{n_k}+med(S_{n_{k+1}}-S_n)|\to 0$, a.s.

$\because \forall\varepsilon>0, \max_{n_k\le n\le n_{k+1}}P(|S_{n_{k+1}}-S_n|>\varepsilon)\to 0$

(**) $\therefore \max_{n_k<n\le n_{k+1}}|med(S_{n_{k+1}}-S_n)|\to 0$. (对上式取 k 充分大, 使 $\max P(\cdot)<\frac{1}{2}$, 则 $\max|med(\cdot)|\le\varepsilon$)

由 (*) 和 (**) 得证. #

Pf of Hoeffding: 不妨 $EX_i=0$.

$\forall t>0$, $P(S_n\ge nx)\le e^{-tnx}\prod_{k=1}^{n}Ee^{tX_k}$

$\because e^{tx}\le\frac{x-a_i}{b_i-a_i}e^{tb_i}+\frac{b_i-x}{b_i-a_i}e^{ta_i}$, $\forall x\in[a_i,b_i]$

$\therefore Ee^{tX_i}\le-\frac{a_i}{b_i-a_i}e^{tb_i}+\frac{b_i}{b_i-a_i}e^{ta_i}=e^{g(u)}$,

其中 $g(u)=\ln(1-\theta+\theta e^{u})-\theta u$, $\theta=-\frac{a_i}{b_i-a_i}$, $u=t(b_i-a_i)$

$\because g'(0)=g(0)=0$, $g''(u)\le\frac{1}{4}$

$\therefore g(u)\le\frac{u^2}{8}=\frac{t^2(b_i-a_i)^2}{8}$

$\therefore P(S_n\ge nx)\le e^{-tnx+\frac{\sum(b_i-a_i)^2}{8}t^2}$ (minimizor $t=\frac{4n}{\sum(b_i-a_i)^2}$) $=RHS$. #

Eg. $U[-a,a]$ 的 ch.f. : $\frac{\sin at}{at}$

三角分布 $p(x)=\frac{a-|x|}{a^2}$ in $[-a,a]$ 的 ch.f. : $\left(\frac{\sin\frac{at}{2}}{\frac{at}{2}}\right)^2$

(拨杯) 设 g 有界连续, $Eg(X)=\lim_{\sigma\to0}Eg(X+\sigma Z)=\lim_{\sigma\to0}\int g(x)P_{X+\sigma Z}(x)dx$

思路 $f_X(t)\Rightarrow P_{X+\sigma Z}(x)\Rightarrow Eg(X)\Rightarrow X$ 的 d.f.

若 $|f_X(t)|$ 可积, 记 $p(x)=\lim_{\sigma\to0}P_{X+\sigma Z}(x)=\frac{1}{2\pi}\int f_X(t)e^{-itx}dt$, 下证 $p(x)$ 为 X 的 d.f.

$\because$ $\forall$ 有界区间 I, $P(X\in I)=\lim_{\sigma\to0}P(X+\sigma Z\in I)=\lim_{\sigma\to0}\int_I P_{X+\sigma Z}(x)dx=\int_I p(x)dx$ #

Date .

Thm. X_n 独立, 则 $\sum X_n$ 依P收敛 $\Leftrightarrow\sum X_n$ a.s. 收敛 $\Leftrightarrow\sum X_n$ 依d收敛.

Thm (Hoeffding) X_i 独立, $a_i\le X_i\le b_i$, 则 $\forall x>0$, $P(S_n-ES_n\ge nx)\le\exp\left\{-\frac{2n^2x^2}{\sum(b_i-a_i)^2}\right\}$.

- 指数 ineq
- Fuk-Nagaev ineq
- Talagrand ineq
- Concentration ineq.

(矩不等式)

Thm. X_j 独立 $(j=1,\cdots,n)$. (1) 若 $0<r\le1$. $E|X_j|^r<\infty$, 则 $E|S_n|^r\le\sum E|X_j|^r$

(2) $1\le r\le2$, $E|X_j|^r<\infty$, $EX_j=0$, 则 $E|S_n|^r\le2^{2-r}\sum E|X_j|^r$

(Rosenthal)

(3) $r\ge2$, $E|X_j|^r<\infty$, $EX_j=0$, 则 $\exists C(r)>0$, s.t.

$E|S_n|^r\le C(r)\left(\sum E|X_j|^r+(\sum EX_j^2)^{\frac{r}{2}}\right)$

- 特征函数 $f(t)=Ee^{itX}=E\cos(tX)+iE\sin(tX)$

prop. (i) $|f(t)|\le1$

(ii) $f(-t)=\overline{f(t)}$

(iii) $f(t)$ 一致连续. $\because \sup_t|f(t+h)-f(t)|=\sup_t|Ee^{i(t+h)X}-Ee^{itX}|\le\sup_t|Ee^{ihX}-1|\xrightarrow{D.C.T.}0$ $(h\to0)$.

(iv) $f(t)$ 非负定. (判定 f 为特征的充要条件).

i.e., $\forall t_1,\cdots,t_n$, $\lambda_1,\cdots,\lambda_n\in\mathbb{C}$, $\sum_{k,j=1}^{n}f(t_k-t_j)\lambda_k\bar{\lambda}_j\ge0$

$\because$ $=E\left(\sum_k f(t_k)\lambda_k\right)\left(\sum_j\overline{f(t_j)}\bar{\lambda}_j\right)$

(v) $f_{aX+b}(t)=e^{itb}f_X(at)$.

(vi) f_1,f_2 为特征, 则 f_1f_2 为特征 $=f_{X_1+X_2}$

f 为 ch.f., 则 $|f|^2$ 为 ch.f. $=f_{X-\tilde{X}}$.

f_i 为 ch.f., $a_i\in[0,1]$, $\sum a_i=1$, 则 $\sum a_if_i$ 为 ch.f. 分布为 $\sum a_iF_i(x)$.

Thm. (Parseval) 设 X,Y 分布 F_X,F_Y, 特征 f_X,f_Y, 则

$$\int f_X(t)dF_Y(t)=\int f_Y(t)dF_X(t).$$

- 若 X,Y 独立, $Ee^{iXY}=E_XE_{Y|X}e^{iXY}=\int f_Y(x)dF_X(x)$
- 若 $Y\sim N(0,\sigma^2)$,

$LHS=\int f_X(t)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}}dt$, $RHS=\int e^{-\frac{\sigma^2}{2}x^2}dF_X(x)$.

$=\iint e^{itx}dF_X(x)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}}dt$ $\qquad=\sqrt{\frac{2\pi}{\sigma^2}}\int\frac{1}{\sqrt{2\pi}/\sigma}e^{-\frac{\sigma^2}{2}x^2}dF_X(x)$

$=\frac{\sqrt{2\pi}}{\sigma}P_{X+\frac{Z}{\sigma}}(0)$, $Z\sim N(0,1)$ 与 X 独立

用 σ 代替 $\frac{1}{\sigma}$

$\Rightarrow P_{X+\sigma Z}(0)=\frac{1}{2\pi}\int f_X(t)e^{-\frac{\sigma^2}{2}t^2}dt$

$\Rightarrow P_{X+\sigma Z}(x)=P_{X+\sigma Z-x}(0)=\frac{1}{2\pi}\int f_X(t)e^{-itx-\frac{\sigma^2t^2}{2}}dt$

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Pf of tail prob: 由 Parseval $\int f_X(t)\,dF_Y(t)=\int f_Y(t)\,dF_X(t)$.

取 $Y\sim U(-u,u)$，则 $\frac{1}{2u}\int_{-u}^{u} f_X(t)\,dt=\int \frac{\sin ut}{ut}\,dF_X(t)$

$$\le \int_{|t|\le\frac{2}{u}} dF_X(t)+\int_{|t|>\frac{2}{u}}\frac{1}{2}\,dF_X(t)$$

$$=1-\frac{1}{2}P(|X|\ge\tfrac{2}{u})$$

$\therefore P(|X|\ge\frac{2}{u})\le\frac{1}{u}\int_{-u}^{u}(1-f_X(t))\,dt.$

用 u 代替 $\frac{2}{u}$ 得证. #

Pf of (特征.矩.导):

$$\Big|f(t)-\sum_{j=0}^{k}\frac{i^j}{j!}t^jEX^j\Big|\le E\Big|e^{itX}-\sum_{j=0}^{k}\frac{i^j}{j!}t^jX^j\Big|$$

$$\overset{\text{Lem}}{\le}\frac{|t|^k}{(k+1)!}\underbrace{E|X|^k(t|X|\wedge 2(k+1))}_{\downarrow\ \text{DCT}\ (t\to0)\ \ 0}=o(|t|^k)$$

用归纳，$k=1$ 时，$\frac{f(t+h)-f(t)}{h}=\int\frac{e^{itx}(e^{ixh}-1)}{h}\,dF(x)\xrightarrow{\text{DCT}}\int(ix)\cdot e^{itx}\,dF(x)$

被积项 $\le\frac{|xh|}{h}=|x|$

若 $(k-1)$ 时成立. $f^{(k-1)}(t)=\int_{-\infty}^{+\infty}(ix)^{k-1}e^{itx}\,dF(x)$，且有 $E|X|^k<\infty$.

$$\frac{f^{(k-1)}(t+h)-f^{(k-1)}(t)}{h}=\int\frac{(ix)^{k-1}e^{itx}(e^{ixh}-1)}{h}\,dF(x)\longrightarrow\int(ix)^ke^{itx}\,dF(x)$$

被积 $|\le|x|^{k-1}\cdot|x|$, DCT #

Pf of k(偶)阶导 $\Rightarrow$ k 阶矩:

① $k=2$ $\quad f''(0)=\lim_{h\to0}\frac{f(h)-2f(0)+f(-h)}{h^2}$

$$=\lim_{h\to0}\int\frac{e^{ihx}+e^{-ihx}-2}{h^2}\,dF$$

$$=-2\lim_{h\to0}\int\frac{1-\cos hx}{h^2}\,dF$$

$\because \int x^2dF\overset{\text{Fatou}}{\le}\underline{\lim}\int\frac{2(1-\cos hx)}{h^2}\,dF=-f''(0)<\infty$ $\quad$ (进一步, $f''(0)=-\int x^2dF$)

② 设 $2k-2$ 时成立，f 在0点附近有 $2k$ 阶导，则 $E|X|^{2k-2}<\infty$

$f^{(2k-2)}(x)=(-1)^{k-1}\int x^{(2k-2)}e^{itx}\,dF$

令 $G(x)=\int_{-\infty}^{x}y^{2k-2}\,dF(y)$. 若 $G(\infty)\ne0$，则 $\frac{G(x)}{G(\infty)}$ 为 d.f.

其 ch.f. $\varphi(t)=\frac{1}{G(\infty)}\int e^{itx}x^{2k-2}\,dF=\frac{(-1)^{k-1}f^{(2k-2)}(t)}{G(\infty)}$ $\quad$ 在0附近有2阶导数

由①知 $\int x^2dG(x)<\infty\Rightarrow\int x^{2k}dF<\infty$

从而得到:

Date .

Thm. 若 $\int|f_X(t)|\,dt<\infty$，则 X 的密度 $p_X(x)=\frac{1}{2\pi}\int f_X(t)e^{-itx}\,dt$

Thm (反演公式) $\forall x_1<x_2$，$P((x_1,x_2))+\frac{1}{2}P(\{x_1\})+\frac{1}{2}P(\{x_2\})$

$$=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f_X(t)\,dt.$$

Thm (tail prob.). $P(|X|\ge u)\le\frac{u}{2}\int_{-\frac{2}{u}}^{\frac{2}{u}}(1-f_X(t))\,dt$

- 若 $|f_X(t)|\equiv1$，则 X 退化. ($\because$ $|f_X(t)|^2\equiv1$ 为 $X-\tilde{X}$ 的 ch.f. $\overset{\text{由tail. prob.}}{\Rightarrow}P(|X-\tilde{X}|\ge u)=0,\forall u.$ $\Rightarrow X-\tilde{X}\overset{a.s.}{=}a\Rightarrow X\overset{a.s.}{=}a$)
- 特征与矩:

c.e.g. Ch.f. 的导数不存在: $P(X=5^k)=2^{-(k+1)}$ $k=0,1,\cdots$

Lem. $\forall n\in\mathbb{N}$，$\Big|e^{ix}-\sum_{m=0}^{n}\frac{(ix)^m}{m!}\Big|\le\min\Big\{\frac{|x|^{n+1}}{(n+1)!},\frac{2|x|^n}{n!}\Big\}$

特别. $|e^{ix}-1|\le|x|$

$|e^{ix}-1-ix|\le\frac{|x|^2}{2}$

Thm (特征.矩.导). 若 $E|X|^k<\infty$，则 $f(t)=\sum_{j=0}^{k}\frac{i^j}{j!}t^jEX^j+o(|t|^k)$

且 $f(t)$ k 阶可导，则 $f^{(k)}(t)=\int_{-\infty}^{+\infty}(ix)^ke^{itx}\,dF(x)$.

$f^{(k)}(0)=i^kEX^k$

Thm. k 为偶，若 f 在0点附近有 k 阶导，则 X 有 k 阶矩.

c.e.g. f 有 $2k+1$ 阶导，$E|X|^{2k}<\infty$ 但 $E|X|^{2k+1}<\infty$ 不成立.

$P(X=\pm j)=\frac{1}{2Cj^2\log j}$

- 若 $f(t)=\sum_{k=0}^{\infty}\frac{(it)^kEX^k}{k!}$，则若 $EX^k=EY^k$，则 $X\overset{d}{=}Y$ (X 的矩唯一确定分布.)

Thm. (矩确定分布) 设 X 的矩能唯一确定分布. $EX_n^k\to EX^k,\forall k$，则 $X_n\xrightarrow{d}X$.

- $X_n\xrightarrow{d}X\Leftrightarrow\forall\{n_k\},\exists\{n_k'\}\subset\{n_k\},X_{n_k'}\xrightarrow{d}X$
- Helly 定理: $\forall\{n_k\},\exists\{n_k'\}\subset\{n_k\}$, s.t. $X_{n_k'}\xrightarrow{\text{vague}}G$

当 X_n 胎紧，G is d.f.

Thm (无穷Taylor). 若 $\forall n\ge1$，$\nu_n=E|X|^n<\infty$，且 $\limsup\frac{(\nu_n)^{\frac{1}{n}}}{n}=r<\infty$，

则当 $|t|<\frac{1}{er}$ 时，$f(t+\theta)=\sum_{n=0}^{\infty}\frac{t^n}{n!}f^{(n)}(\theta)$，$\forall\theta\in\mathbb{R}$.

Pf: $\Big|f(t+\theta)-\sum_{n=0}^{k-1}\frac{t^n}{n!}f^{(n)}(\theta)\Big|\le E\Big|e^{i\theta X}\Big(e^{itX}-\sum_{n=0}^{k-1}\frac{(itX)^n}{n!}\Big)\Big|\le\frac{|t|^k}{k!}E|X|^k$

$\le(|t|(r+\varepsilon))^k\frac{k^k}{k!}$

$\le(e|t|(r+\varepsilon))^k$ $\quad$ 取 ε, s.t. $e|t|(r+\varepsilon)<1$ 即可

($\because\forall\varepsilon>0$，当 k 充分大，$(\nu_k)^{\frac{1}{k}}\le k(r+\varepsilon)$) #

e.g. $X\sim N(0,1)$，若 $EX^n=EY^n$，$\forall n$，则由下页 Thm，$X\overset{d}{=}Y$.

从而 $EX_n^m\to EX^m\overset{\text{Thm(矩定分布)}}{\Rightarrow}X_n\xrightarrow{d}X\Rightarrow\frac{S_n}{\sqrt{n}}\xrightarrow{d}N(0,1)$.

Pf of 连续性定理:

(1) $Ee^{itX_n} \to Ee^{itX_\infty}$

(2) $P(|X_n|>r) \le \frac{r}{2}\int_{-\frac{2}{r}}^{\frac{2}{r}} (1-f_n(t))\,dt$

$\overline{\lim_{n\to\infty}} P(|X_n|>r) \le \frac{r}{2}\int_{-\frac{2}{r}}^{\frac{2}{r}} (1-f_\infty(t))\,dt \to 0\ (r\to\infty)$ $\because f_\infty(t) \xrightarrow{t\to 0} f_\infty(0) = \lim_{n\to\infty} f_n(0) = 1$

$\therefore \overline{\lim_{r\to\infty}}\ \overline{\lim_{n\to\infty}} P(|X_n|>r) = 0$ $\therefore X_n$ 胎紧

由Helly定理 $\forall \{n_k\}$, $\exists \{n_k'\}$, r.v. X, s.t. $X_{n_k'} \xrightarrow{d} X$

由 (1) $f_{n_k'}(t) \to Ee^{itX}$ $\Rightarrow f_\infty(t) = Ee^{itX}$ $\Rightarrow X$分布与子列无关, 完全由$f_\infty(t)$决定

$\therefore X_n \Rightarrow X$ 且 X 的 ch.f. 为 f_∞. #

Pf of Lem: CLAIM1: $|b|\le 1$ 时, $|e^b-1-b| \le |b|^2$

$\because$ LHS $= \left|\sum_{k=2}^{\infty}\frac{b^k}{k!}\right| \le |b|^2 \sum_{k=2}^{\infty}\frac{1}{2^{k-1}} = |b|^2$.

CLAIM2: $|z_m|, |w_m| \le \theta$ 时, $\left|\prod_{m=1}^{n} z_m - \prod_{m=1}^{n} w_m\right| \le \theta^{n-1}\sum_{m=1}^{n}|z_m - w_m|$

$z_m \triangleq (1+\frac{c_n}{n})$, $w_m = e^{c_n/n}$, $|c|<\gamma$

则当n充分大时 $\left|\frac{c_n}{n}\right| < \frac{\gamma}{n} < 1$. $\Rightarrow |z_m|, |w_m| \le e^{\frac{\gamma}{n}}$

$\therefore$ 由CLAIM2, $\left|(1+\frac{c_n}{n})^n - e^{c_n}\right| \le (e^{\frac{\gamma}{n}})^{n-1}\sum_{k=1}^{n}\left|1+\frac{c_n}{n} - e^{\frac{c_n}{n}}\right|$

$\overset{\text{CLAIM1.}}{\le} e^{\gamma}\cdot n\cdot\left|\frac{c_n}{n}\right|^2 \le e^{\gamma}\cdot\frac{\gamma^2}{n} \to 0\ (n\to\infty)$

$\therefore \lim_{n\to\infty}(1+\frac{c_n}{n})^n = \lim_{n\to\infty} e^{c_n} = e^c$ #

Pf of CLT: WLOG, $\mu=0$. $f_X(t) = 1-\frac{\sigma^2 t^2}{2}+o(t^2)$

$Ee^{it(\frac{S_n}{\sqrt{n}\sigma})} = (1-\frac{t^2}{2n}+o(\frac{1}{n}))^n \xrightarrow{\text{lem.}} e^{-\frac{1}{2}t^2}$ #

Pf of Lindeberg CLT. [粗暴分析法: $\prod_{k=1}^{k_n} Ee^{itX_{nk}} = \prod_{k=1}^{k_n}(1-E\frac{t^2X_{nk}^2}{2}+\Delta_k) = \prod_{k=1}^{k_n}(e^{-t^2\frac{EX_{nk}^2}{2}}+\Delta_k+\Delta_k')$

$\to e^{-\frac{t^2}{2}}$, 但Δ须bound住]

设 X_{nk} 的 ch.f. $f_{nk}(t)$

设 $\sigma_{nk}^2 = EX_{nk}^2$, $G_{nk}\sim N(0,\sigma_{nk}^2)$ with ch.f $\varphi_{nk}(t) = e^{-\frac{\sigma_{nk}^2 t^2}{2}}$

$\left|Ee^{it(X_{n1}+\cdots+X_{nk_n})} - e^{-\frac{t^2}{2}}\right| = \left|\prod_{k=1}^{k_n} f_{nk}(t) - \prod_{k=1}^{k_n}\varphi_{nk}(t)\right|$

$\le \sum_{k=1}^{k_n}|f_{nk}(t)-\varphi_{nk}(t)|$

$\le \sum_{k=1}^{k_n}\left(|f_{nk}(t)-1+\frac{1}{2}t^2\sigma_{nk}^2| + |\varphi_{nk}(t)-1+\frac{1}{2}t^2\sigma_{nk}^2|\right)$

$\le C_t\sum_{k=1}^{k_n}\left(E[X_{nk}^2\wedge|X_{nk}|^3] + E[G_{nk}^2\wedge|G_{nk}|^3]\right)$

$\sum_{k=1}^{k_n}E[X_{nk}^2\wedge|X_{nk}|^3] \le \sum_{k=1}^{k_n}\{\underbrace{E|X_{nk}|^3 I(|X_{nk}|\le\varepsilon)}_{\le E|X_{nk}|^2\cdot\varepsilon.} + E|X_{nk}|^2 I(|X_{nk}|>\varepsilon)\} \le \varepsilon\ (n\to\infty)$

由ε任意性知 $\sum_{k=1}^{k_n}E[X_{nk}^2\wedge|X_{nk}|^3]\to 0$.

Thm. 若$\{\mu_n\}$有 $\limsup_{n\to\infty}\frac{(\mu_{2n})^{\frac{1}{2n}}}{2n} = r<\infty$, 则至多$\exists$一个 d.f. F, s.t. $\mu_n = \int x^n dF$, $n\ge 1$.

从而, 若$\{EX^n\} = \{EY^n\}$满足上式, 则 $X \overset{d}{=} Y$

(太碗)Pf: 首先, 由 $E|X|^{2n+1} \le \sqrt{E|X|^{2n+2}E|X|^{2n}}$ 知 $\limsup\frac{(\mu_n)^{\frac{1}{n}}}{n} = r<\infty$, 其中 $\mu_n = EX^n = EY^n$

由上一定理, $|t|<\frac{1}{er}$时, $f_X(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\mu_k = f_Y(t)$ #.

递推知 当 $|t|<\frac{k}{er}$时, $f_X(t) = f_Y(t)$ $\Rightarrow X\overset{d}{=}Y$.

Thm. (连续性定理) (1) $X_n\xrightarrow{d}X_\infty$, 则 $\forall t\in\mathbb{R}$, $f_n(t)\to f_\infty(t)$. (且在有限区间上一致收敛)

(2) 若$\exists f(t)$, s.t. $f_n(t)\to f_\infty(t)$, $\forall t\in\mathbb{R}$, 且$f(t)$在0点连续,

(律与型, type & Law) 则 $\exists$ r.v. X, s.t. $\underset{(X_n\xrightarrow{d}X)}{X_n\Rightarrow X}$ 且 X的ch.f. 为 f_∞

Thm. 常数列$\{a_n\}$, $\{b_n\}$, $F_n\xrightarrow{w}F$非退化, 则

(1) 若 $F_n(a_nx+b_n)\xrightarrow{d}G(x)$非退化, 则 $\exists a,b$, s.t. $G(x) = F(ax+b)$, $a_n\to a$, $b_n\to b$.

(2) 若 $a_n\to a$, $b_n\to b$, 则 $F_n(a_nx+b_n)\xrightarrow{d}F(ax+b)$.

- $X_n\xrightarrow{d}X \Leftarrow f_{X_n}(t)\to f(t)$
- $\frac{S_n}{n}\xrightarrow{P}0 \Leftrightarrow \frac{S_n}{n}\xrightarrow{d}0 \Leftarrow Ee^{it\frac{S_n}{n}}\to 1$ 但 $f_{X_n}(t)\to f_X(t) \not\Rightarrow X_n\xrightarrow{P}X$.

证明 X_i, i.i.d. $EX_i=0$ 的WLLN:

$Ee^{it\frac{S_n}{n}} = (Ee^{i\frac{t}{n}X})^n = (1+\frac{it}{n}EX+o(\frac{1}{n}))^n \to e^{itEX} = 1$. $\Rightarrow \frac{S_n}{n}\xrightarrow{P}0$. #

Lem 设 $c_n\to c\in\mathbb{C}$, 则 $(1+\frac{c_n}{n})^n\to e^c$.

Thm (CLT) X_i, i.i.d. $EX=\mu$, $Var(X)=\sigma^2$, 则 $\frac{S_n-n\mu}{\sqrt{n}\sigma}\xrightarrow{d}N(0,1)$

- 三角阵列: $X_{11}\cdots X_{1k_1}$ ① $X_{n1},\cdots,X_{nk_n}$独立

  $X_{21}\cdots X_{2k_2}$ ② $EX_{nk}=0$

  ③ $\sum_k EX_{nk}^2 = 1$.

Thm (Lindeberg CLT). 上述setting, 且$\forall\varepsilon>0$, $\sum_{k=1}^{k_n}EX_{nk}^2 I(|X_{nk}|\ge\varepsilon)\to 0$.

则 $\sum_{k=1}^{k_n}X_{nk}\xrightarrow{d}N(0,1)$

Rmk ① Feller条件: $\lim_{n\to\infty}\max_k X_{nk}^2 = 0$ ② 无穷小条件: $\lim_{n\to\infty}\max_k P(|X_{nk}|>\varepsilon) = 0$, $\forall\varepsilon>0$.

Thm (Lindeberg-Feller) 在Lindeberg CLT setting下, 以下等价.

(1) $\sum_{k=1}^{k_n}X_{nk}\xrightarrow{d}N(0,1)$ 且 $\lim_{n\to\infty}\max_k X_{nk}^2 = 0$

(2) $\forall\varepsilon>0$, $\sum_{k=1}^{k_n}EX_{nk}^2 I(|X_{nk}|\ge\varepsilon)\to 0$.

Cor. (Lyapunov) 若 $\sum_{k=1}^{k_n}E|X_{nk}|^3\to 0$, 则 $\sum_{k=1}^{k_n}X_{nk}\xrightarrow{d}N(0,1)$.

$\sum_{k=1}^{k_n}E[G_{nk}^2\wedge|G_{nk}|^3] \le \sum_{k=1}^{k_n}E|G_{nk}|^3 = C\cdot\sum_k\sigma_{nk}^3 \le C\cdot\max_k\sigma_{nk}\xrightarrow{\because}0$

$\because \max_k\sigma_{nk}^2 \le \max_k EX_{nk}^2 I(|X_{nk}|\le\varepsilon) + \max_k EX_{nk}^2 I(|X_{nk}|>\varepsilon) \le \varepsilon\ (n\to\infty)$, $\forall\varepsilon>0$. #

Pf of lemma:

step1 设 F_1、F_2 均值为 0，ch.f $f_1 \cdot f_2$ 可积，则 $F_1(x)-F_2(x) = -\frac{1}{2\pi}\int e^{-itx}\frac{f_1(t)-f_2(t)}{it}dt$

∵ 密度 $p_j(x) = \frac{1}{2\pi}\int e^{-itx} f_j(t)dt,\ j=1,2$ （由 f_j 可积知有密度）

$F_1(x)-F_1(a)-(F_2(x)-F_2(a))$

$= \int_a^x (p_1(y)-p_2(y))dy$

$= \frac{1}{2\pi}\int_a^x\int e^{-ity}(f_1(t)-f_2(t))dt\,dy$

$= \frac{1}{2\pi}\int dt\,(f_1(t)-f_2(t))\frac{e^{-ita}-e^{-itx}}{it}$

下证 $\lim_{a\to-\infty}\int e^{-ita}\frac{f_1(t)-f_2(t)}{it}dt = 0$，由 R-L 引理，只须 $\frac{f_1(t)-f_2(t)}{it}$ 可积.

（在 0 附近，$f_1(t) = 1+o(t)$，$\frac{f_1(t)-1}{t}\to 0 \Rightarrow \frac{(f_1(t)-1)-(f_2(t)-1)}{t}\to 0 \Rightarrow$ 在 $[0,1]$ 上 bdd.）
又在 $[1,\infty)$ 上可积得证

step2. $h_L(x) \triangleq \frac{1-\cos\pi x}{\pi L x^2}$，$x\in\mathbb{R}$ 为密度 且 $f_L(t) = (1-\frac{|t|}{L})^+$，记分布函数为 H_L.

取 $F_L = F * H_L$，$G_L = G * H_L$（均值仍为 0），则 $|F_L(x)-G_L(x)| \overset{\text{step1}}{\le} \frac{1}{2\pi}\int |f_F(t)f_L(t)-f_G(t)f_L(t)|\frac{1}{|t|}dt$

$\le \frac{1}{2\pi}\int_{-L}^{L}|f_F(t)-f_G(t)|/|t|\,dt$

step3. 设 $G'(x)\le\lambda<\infty$，则 $\sup_x|F(x)-G(x)| \le 2\sup_x|F_L(x)-G_L(x)| + \frac{24\lambda}{\pi L}$ //从而由 step2.3. 得 lemma.

∵ 记 $\Delta(x) = F(x)-G(x)$，$\eta = \sup_x|\Delta(x)|$，由于 $\Delta(x)\to 0$，$|x|\to\infty$ 且 G 连续，WLOG $\Delta(x_0)=\eta$

（或 $\Delta(x_0-) = -\eta$）

∴ $\sup_x|F_L(x)-G_L(x)| \ge F_L(t_0)-G_L(t_0)$ for some t_0 （否则，用 $-X$，$-Y$ 代之即可.）

$= P(X+L\le t_0) - P(Y+L\le t_0)$

$= \int (P(X\le t_0-x)-P(Y\le t_0-x))h_L(x)dx.$

$\Delta(x_0+s) \ge F(x_0)-G(x_0+s) \ge \eta - s\lambda$

记 $\delta = \eta/2\lambda$，$t_0 = x_0+\delta$，则当 $|x|\le\delta$ 时，$\Delta(t_0-x) \ge \eta-\lambda(\delta-x) = \frac{\eta}{2}+\lambda x.$

当 $|x|>\delta$ 时，$\Delta(t_0-x)\ge -\eta$

∴ $\sup_x|F_L(x)-G_L(x)| \ge \int_{|x|\le\delta}(\frac{\eta}{2}+\lambda x)h_L(x)dx - \eta\int_{|x|>\delta}h_L(x)dx$

$= \frac{\eta}{2}\int_{|x|\le\delta}h_L(x) - \eta\int_{|x|>\delta}h_L(x)dx.$

$= \frac{\eta}{2} - \frac{3}{2}\eta\cdot\int_{|x|>\delta}\frac{2}{\pi L x^2}dx$

$= \frac{\eta}{2} - \frac{6\eta}{\pi L\delta} = \frac{\eta}{2} - \frac{12\lambda}{\pi L}$ #.

Pf of Lemma2: $|w_k| = |Ee^{it\frac{X_k}{\sqrt{B_n}}}| \overset{t<1\ (t>1\text{时 }\exp\{\frac{1}{2}\})}{\le} 1-\frac{\sigma_k^2t^2}{2B_n}+\frac{E|X_k|^3|t|^3}{6B_n^{\frac{3}{2}}} \le \exp\{-\frac{\sigma_k^2t^2}{2B_n}+\frac{E|X_k|^3|t|^3}{6B_n^{\frac{3}{2}}}\}.$

∴ $\prod_{m=1}^{k-1}|w_m|\cdot\prod_{m=k+1}^{n}|\gamma_m| \le \exp\{-\sum_{m\ne k}\frac{\sigma_m^2t^2}{2B_n}+\sum_{m=1}^{k-1}\frac{E|X_m|^3|t|^3}{6B_n^{\frac{3}{2}}}\}$

$\le \exp\{-\frac{t^2}{2}+\frac{\sigma_k^2t^2}{2B_n}+\frac{|t|^3}{6}L_n\}$ ∵ $\frac{\sigma_k^2}{B_n} \le \frac{(E(X_k)^3)^{\frac{2}{3}}}{B_n} \le L_n^{\frac{2}{3}} \le \frac{1}{4}$

$\le \exp\{-\frac{t^2}{4}\}$

$|\gamma_k - 1+\frac{\sigma_k^2t^2}{2B_n}| \le \frac{\sigma_k^4t^4}{8B_n^2}$ $\Rightarrow |w_k-\gamma_k| \le \frac{E|X_k|^3|t|^3}{6B_n^{\frac{3}{2}}}+\frac{\sigma_k^4t^4}{8B_n^2}$ ∵ $\sigma_k^4 \le \sigma_k\cdot(EX_k^2)^{\frac{3}{2}} \le \sigma_k\cdot E|X_k|^3 \le B_n^{\frac{1}{2}}E|X_k|^3$ #

Thm. (CLT 充要) 设 $\{X_{nk}\}$ 满足无穷小条件 ($\forall\varepsilon>0$, $\lim_{n\to\infty}\max_k P(|X_{nk}|>\varepsilon)=0$), $b\in\mathbb{R}$, $C\ge 0$,

则 $\sum_{k=1}^{k_n}X_{nk} \xrightarrow{d} N(b,c) \iff \begin{cases} 1.\ \forall\varepsilon>0,\ \sum_k P(|X_{nk}|>\varepsilon)\to 0 \\ 2.\ \sum_k E(X_{nk}I(|X_{nk}|\le 1))\to b. \\ 3.\ \sum_k Var(X_{nk}I(|X_{nk}|\le 1))\to C. \end{cases}$

Thm (Berry-Esseen ineq) X_i i.i.d. $EX=0$, $EX^2=\sigma^2$, $E|X|^3=\rho<\infty$，则

$$\sup_x|P(\frac{S_n}{\sqrt{n}\sigma}\le x)-\Phi(x)| \le \frac{C\rho}{\sqrt{n}\sigma^3}.$$

Lem 若 F、G 均值为 0，$\lambda = \sup_x G'(x)$，则 $\forall L>0$，

$$\sup_x|F(x)-G(x)| \le \frac{1}{\pi}\int_{-L}^{L}\frac{|f_F(t)-f_G(t)|}{|t|}dt + \frac{24\lambda}{\pi L}$$

Pf of B-E: $\lambda = \sup_x\Phi'(x) = \frac{1}{\sqrt{2\pi}} < \frac{2}{5}$，WLOG $\sigma=1$.

∴ $\sup_x|P(\frac{S_n}{\sqrt{n}\sigma}\le x)-\Phi(x)| \overset{\text{lem.}}{\le} C\cdot\int_{-L}^{L}\frac{|(f_X(\frac{t}{\sqrt{n}}))^n-e^{-\frac{t^2}{2}}|}{|t|}dt+\frac{C}{L}$

$(\gamma_n(t) \ge |f_X(\frac{t}{\sqrt{n}})| \vee |e^{-\frac{t^2}{2n}}|)$

$\le C\cdot\int_{-L}^{L}\frac{n\cdot|f_X(\frac{t}{\sqrt{n}})-e^{-\frac{t^2}{2n}}|\gamma_n^{n-1}(t)}{|t|}dt+\frac{C}{L}$

∵ $|f_X(\frac{t}{\sqrt{n}})-1+\frac{t^2}{2n}| \le \frac{|t|^3\rho}{6n^{\frac{3}{2}}}$，$|e^{-\frac{t^2}{2n}}-1+\frac{t^2}{2n}| \le \frac{t^4}{8n^2}$

∴ $|f_X(\frac{t}{\sqrt{n}})| \le 1-\frac{t^2}{2n}+\frac{|t|^3\rho}{6n^{\frac{3}{2}}} \le \exp\{-\frac{t^2}{2n}+\frac{|t|^3\rho}{6n^{\frac{3}{2}}}\}$，$t^2\le 2n$ 时成立.

$\le \exp\{-\frac{5t^2}{18n}\}$，$\frac{|t|\rho}{\sqrt{n}}\le\frac{4}{3}$ 时成立.

∴ $\sup_x|P(\frac{S_n}{\sqrt{n}\sigma}\le x)-\Phi(x)| \le C\cdot\int_{-L}^{L}\frac{n\cdot(\exp\{-\frac{5t^2}{18n}\})^{n-1}\cdot(\frac{|t|^3\rho}{6n^{\frac{3}{2}}}+\frac{t^4}{8n^2})}{|t|}dt+\frac{C}{L}$

$\overset{\text{取}L=\frac{3\sqrt{n}}{2\rho}}{\le} \frac{C\rho}{\sqrt{n}}+\frac{C}{n} \le \frac{C\rho}{\sqrt{n}}$ #

Thm (Berry-Esseen, II). X_n 独立，$EX_n=0$，$EX_n^2=\sigma_n^2<\infty$，$B_n=\sum_{k=1}^{n}\sigma_k^2$，则

$$\sup_x|P(\frac{S_n}{\sqrt{B_n}}\le x)-\Phi(x)| \le A\cdot\frac{\sum_{k=1}^{n}E|X_k|^3}{B_n^{\frac{3}{2}}}$$

Pf: 记 $f_{\frac{S_n}{\sqrt{B_n}}}(t) = \prod_{k=1}^{n}w_k$，$w_k \triangleq Ee^{it\frac{X_k}{\sqrt{B_n}}}$，

$\varphi(t) = e^{-\frac{t^2}{2}} = \prod_{k=1}^{n}\gamma_k$，$\gamma_k \triangleq Ee^{-\frac{\sigma_k^2}{2B_n}t^2}$

∴ $|f_{\frac{S_n}{\sqrt{B_n}}}(t)-\varphi(t)| \le \sum_{k=1}^{n}(\prod_{m=1}^{k-1}|w_m|\prod_{m=k+1}^{n}|\gamma_m|\cdot|w_k-\gamma_k|)$

Lem2 $L_n \triangleq B_n^{-\frac{3}{2}}\sum_{k=1}^{n}E|X_k|^3$，若 $|t|\le\frac{1}{4L_n}$，$|L_n|\le\frac{1}{8}$，则 $\forall k$，有 $\prod_{m=1}^{k-1}|w_m|\cdot\prod_{m=k+1}^{n}|\gamma_m| \le e^{-\frac{t^2}{4}}$，

由 lem2，$|f_{\frac{S_n}{\sqrt{B_n}}}(t)-\varphi(t)| \le e^{-\frac{t^2}{4}}L_n\cdot(\frac{|t|^3}{6}+\frac{t^4}{8})$，当 $|t|\le\frac{1}{4L_n}$，$|L_n|\le\frac{1}{8}$. $|w_k-\gamma_k| \le \frac{E|X_k|^3}{B_n^{\frac{3}{2}}}(\frac{|t|^3}{6}+\frac{t^4}{8})$

下设 $L_n\le\frac{1}{8}$，否则令 $A=8$ 则得证.

令 $L=\frac{1}{4L_n}$，由 B-E 的 lem，$\sup_x|P(\frac{S_n}{\sqrt{B_n}}\le x)-\Phi(x)| \le \frac{1}{\pi}\int_{-L}^{L}\frac{e^{-\frac{t^2}{4}}L_n\cdot(\frac{|t|^3}{6}+\frac{t^4}{8})}{|t|}dt+\frac{C}{L}$

$\le CL_n.$ #

Date .

Pf of lem: step1: $x^{2-p}E|X|^pI(|X|>x) = x^{2-p}\sum_{n\ge0}E|X|^pI(\frac{|X|}{x}\in(2^n,2^{n+1}])$

$\le \sum_{n\ge0}EX^2I(\frac{|X|}{x}\in(2^n,2^{n+1}])\,2^{n(p-2)}$

$= \sum_{n\ge0}2^{(p-2)n}[L(2^{n+1}x)-L(2^nx)]$

($\forall r\in(1,2^{2-p}),\ \exists x_0,\ s.t.\ \forall x>x_0,\ \frac{L(2x)}{L(x)}\le r$)

$\le \sum_{n\ge0}2^{(p-2)n}(r-1)r^nL(x)$

$= \frac{(r-1)L(x)}{1-2^{p-2}\cdot r}$

令 $r\to1$，从而 $\forall p\in[0,2)$, $\lim_{x\to\infty}\frac{x^{2-p}E|X|^pI(|X|>x)}{L(x)}=0$

step2: 若 $EX^2<\infty$，则 $E(X-m)^2<\infty$，$L_m(x)\nearrow E(X-m)^2$ 为缓变化函数

若 $EX^2=\infty$，$L_m(x)=\underbrace{EX^2I(|X-m|\le x)}_{\text{在 }L(x\pm m)\text{ 之间}}+\underbrace{mE(m-2X)I(|X-m|\le x)}_{\text{有界}}$

(写好一点：$\frac{L(x\pm m)}{L(x)}\to1$　$\frac{mE(m-2X)I(|X-m|\le x)}{L(x)}\to0$

$\therefore \frac{L_m(x)}{L(x)}\to1$　$\therefore\frac{L_m(cx)}{L_m(x)}\to1$.)

#

Pf of Thm (正态吸引): (2) ⇒ (3): 由 Lem.

(3) ⇒ (2): $0\le\frac{L(2x)}{L(x)}-1=\frac{EX^2I(x<|X|\le2x)}{L(x)}\le\frac{(2x)^2P(|X|>x)}{L(x)}\to0$.

(2) ⇒ (1): 取 $m_k=EX\overset{\text{不妨}}{=}0$

$a_n=1\vee\sup\{x>0:nL(x)\ge x^2\}$，则 $a_n\nearrow\infty$ 且 $\boxed{nL(a_n)\sim a_n^2}$ ☆

记 $X_{nk}=\frac{X_k}{a_n}$，即证 $\sum_{k\le n}X_{nk}\xrightarrow{d}N(0,1)$.

$\Leftrightarrow\begin{cases}(a)\ \forall\varepsilon>0,\ \sum_kP(|X_{nk}|>\varepsilon)\to0\\(b)\ \sum_kE(X_{nk}I(|X_{nk}|\le1))\to0\\(c)\ \sum_kVar(X_{nk}I(|X_{nk}|\le1))\to1.\end{cases}$

(a). $\sum_kP(|X_{nk}|>\varepsilon)=nP(|X_k|>\varepsilon a_n)\sim\frac{a_n^2P(|X_k|>\varepsilon a_n)}{L(a_n)}\xrightarrow{(3)}0$

(b). $|LHS|=\frac{n}{a_n}|EX\cdot I(|X|\le a_n)|$

$=\frac{n}{a_n}|EXI(|X|>a_n)|$

$\le\frac{n}{a_n}E|X|\cdot I(|X|>a_n)\sim\frac{a_n}{L(a_n)}E|X|I(|X|>a_n)\xrightarrow{lem}0$

(c). $|LHS|=\frac{n}{a_n^2}EX^2I(|X|\le a_n)-\frac{n}{a_n^2}(EXI(|X|\le a_n))^2$

$=\underbrace{\frac{nL(a_n)}{a_n^2}}_{\to1}-\underbrace{\frac{n}{a_n^2}(EXI(|X|>a_n))^2}_{\xrightarrow{(b)}0}$

#

B-E: $\sup_x|P(\frac{S_n}{\sqrt n\sigma}\le x)-\Phi(x)|\le C\cdot\frac{\rho}{\sqrt n\sigma^3}$,

Date .

Rmk. 1. B-E 中 $E|X|^3<\infty$ 可改为 $E|X|^{2+\delta}<\infty$，收敛速度 $\frac{1}{n^{\frac{\delta}{2}}}$

☺ $\sup_x|P(\frac{S_n}{\sqrt n\sigma}\le x)-\Phi(x)|\le C\cdot(\frac{1}{\sigma^3\sqrt n}E|X|^3I(|X|\le\sqrt n)+\frac{1}{\sigma^2}EX^2I(|X|>\sqrt n))$

可改进.

2. 阶 $\frac{1}{\sqrt n}$ in B-E 不可改进.

c.e.g. $X_i\overset{i.i.d.}{\sim}$ sym Bernoulli　$P(S_{2n}=0)=\binom{2n}{n}(\frac12)^{2n}=\frac{1}{\sqrt{\pi n}}(1+o(1))$

$P(S_{2n}\le0)=\frac12(1+\frac{1}{\sqrt{\pi n}}(1+o(1)))$

$\therefore P(S_{2n}\le0)-\Phi(0)=\frac{1}{2\sqrt{\pi n}}(1+o(1))$

3. Edgeworth 展开：$|P(\frac{S_n}{\sqrt n\sigma}\le x)-\Phi(x)-\Sigma\cdots|\le\frac{1}{n^{\Delta}}$

Thm. 设 $X_1,\cdots$ 独立，$EX_i=0$, $E|X_i|^3<\infty$, $Var(X_i)=\sigma_i^2$, $B_n=\sum_{i=1}^n\sigma_i^2$, $L_n=B_n^{-\frac32}\sum_{i=1}^nE|X_i|^3$

则 $|P(\frac{S_n}{\sqrt{B_n}}\le x)-\Phi(x)|\le\frac{A\cdot L_n}{(1+|x|^3)}$

$1-\Phi(x)\approx\frac{1}{x}e^{-\frac12x^2}$

$\left|\frac{P(\frac{S_n}{\sqrt{B_n}}>x)}{1-\Phi(x)}-1\right|\overset{B-E}{\le}\frac{C\rho/(\sqrt n\sigma^3)}{1-\Phi(x)}$ 当 $0\le x\le(1-\varepsilon)\sqrt{\log n}$ 时，$\le\frac{C\rho/(\sqrt n\sigma^3)}{e^{-\frac12(1-\varepsilon)^2\log n}/(1-\varepsilon)\sqrt{\log n}}\lesssim\frac{\sqrt{\log n}}{n^{\frac12-\frac12(1-\varepsilon)^2}}\to0$.

从而，$\frac{P(\frac{S_n}{\sqrt{B_n}}>x)}{1-\Phi(x)}\to1$ 在 $[0,(1-\varepsilon)\sqrt{\log n}]$ 一致成立.

• 考虑"大偏差"：$x_n\to\infty$, $P(\frac{S_n}{\sqrt n}>x_n)$ 的值.

Thm (Cramer 大偏差) 若 $0\le x=o(\sqrt n)$, $EX=0$, $EX^2=1$，则 F_n 为 $\frac{S_n}{\sqrt n}$ d.f.

$\frac{1-F_n(x)}{1-\Phi(x)}=\exp\{\frac{x^3}{\sqrt n}\lambda(\frac{x}{\sqrt n})\}[1+o(\frac{x+1}{\sqrt n})]$，其中 $\lambda(t)=\sum_{k=0}^\infty c_kt^k$ 为 Cramer 级数

· Def. (缓变化函数) L 在 $[0,\infty)$ 上正值可测. 若 $\forall c>0$, $\frac{L(cx)}{L(x)}\to1\ (x\to\infty)$，则称 L 是 (∞ 处的) 缓变化函数. e.g. 常值, $\log x$, $\log\log x$　$\lim L(x)\nearrow c$

Thm (正态吸引). 设 $X_i\overset{i.i.d.}{\sim}$ 非退化，则

(1) $\exists a_n, m_n$, s.t. $\frac{\sum_{k\le n}(X_k-m_k)}{a_n}\xrightarrow{d}N(0,1)$　＊称 F 属于正态吸引场.

$\Leftrightarrow$ (2) $L(x)=EX^2I(|X|\le x)$ 是 (∞ 处) 的缓变化函数.

$\Leftrightarrow$ (3) $\lim_{x\to\infty}\frac{x^2P(|X|>x)}{EX^2I(|X|\le x)}=0$

lem 设 X 非退化，$L(x)$ 缓变化函数，则 $\forall m\in\mathbb R$, $L_m(x)=E(X-m)^2I(|X-m|\le x)$ 也缓变化

且 $\forall p\in[0,2)$. $\lim_{x\to\infty}\frac{x^{2-p}E|X|^pI(|X|>x)}{L(x)}=0$

E.g. $X_i\overset{i.i.d.}{\sim}P(|X|>x)=\frac{1}{x^2}$, ($x\ge1$), 对称

则 $EX^2=\infty$, $EX^2I(|X|\le x)=2\log x$　$nL(a_n)\sim a_n^2\Rightarrow a_n\sim\sqrt{n\log n}$

$\therefore\frac{S_n}{\sqrt{n\log n}}\xrightarrow{d}N(0,1)$. 也可直接 Ch.f. 证之.

Pf of Thm(Ber-Poi 收敛): $Ee^{itS_n}=\prod_{k=1}^{k_n}Ee^{itX_{nk}}=\prod_{k=1}^{k_n}(1+p_{nk}(e^{it}-1))$

$\left|\prod_{k=1}^{k_n}(1+p_{nk}(e^{it}-1))-\prod_{k=1}^{k_n}e^{p_{nk}(e^{it}-1)}\right|\le\sum_{k=1}^{k_n}\left|1+p_{nk}(e^{it}-1)-e^{p_{nk}(e^{it}-1)}\right|$

$\because |e^{p(e^{it}-1)}|=e^{\mathrm{Re}\,p(e^{it}-1)}\le 1$, $|1+p(e^{it}-1)|\le|1-p|+|p|=1$.

由 $|e^b-1-b|\le|b^2|$ $(|b|<1)$ 知: $\le\sum_{k=1}^{k_n}p_{nk}^2|e^{it}-1|^2$ (当n充分大,由条件2, $|b|<1$ 满足)

$\le\sum_{k=1}^{k_n}4\cdot p_{nk}^2\longrightarrow 0.$

$\therefore \lim_{n\to\infty}Ee^{itS_n}=\lim_{n\to\infty}e^{\sum p_{nk}(e^{it}-1)}=e^{\lambda(e^{it}-1)}$ #

Pf of Lem1(全变差刻画): 记 $B=\{i:\mu_i>\nu_i\}$, $\forall A\in\Omega$,

$$\mu(A)-\nu(A)=\mu(AB)-\nu(AB)+\sum_{i\in AB^c}(\mu_i-\nu_i)$$
$$\le\mu(AB)-\nu(AB)$$
$$\le\mu(AB-\nu(AB)+\sum_{i\in A^cB}(\mu_i-\nu_i)=\mu(B)-\nu(B)$$

类似, $\nu(A)-\mu(A)\le\mu(B)-\nu(B)$. $\Rightarrow|\mu(A)-\nu(A)|\le\mu(B)-\nu(B)$, $\forall A\in\Omega$.

$\therefore\|\mu-\nu\|=\mu(B)-\nu(B)=\frac12(\nu(B^c)-\mu(B^c)+\mu(B)-\nu(B))=\frac12\sum|\mu_i-\nu_i|$ #.

Pf of Lem2: $\mathrm{LHS}\overset{\text{lem1}}{=}\frac12\sum_x|\mu_1*\mu_2(x)-\nu_1*\nu_2(x)|$

$$=\frac12\sum_x\left|\sum_y(\mu_1(y)\mu_2(x-y)-\nu_1(y)\nu_2(x-y))\right|$$
$$\le\frac12\sum_{x,y}|\mu_1(y)\mu_2(x-y)-\nu_1(y)\nu_2(x-y)|$$
$$=\frac12\sum_{x,y}|\mu_1(y)\mu_2(x)-\nu_1(y)\nu_2(x)|$$
$$\le\frac12\sum_{x,y}|\mu_1(y)(\mu_2(x)-\nu_2(x))|+|(\mu_1(y)-\nu_1(y))\nu_2(x)|$$
$$=\frac12\sum_x|\mu_2(x)-\nu_2(x)|+\frac12\sum_y|\mu_1(y)-\nu_1(y)|\overset{\text{lem1}}{=}\mathrm{RHS}.$$ #

Pf of Lem3: $2\|\mu-\nu\|\overset{\text{lem1}}{=}|\mu(0)-\nu(0)|+|\mu(1)-\nu(1)|+\sum_{i\ge2}\nu(i)$

$$=|1-p-e^{-p}|+|p-pe^{-p}|+1-e^{-p}-pe^{-p}$$
$$=e^{-p}-(1-p)+p-pe^{-p}+1-e^{-p}(1+p)$$
$$=2p(1-e^{-p})\le 2p^2.$$ #

别证 Ber-Poi 收敛 Thm: 设 $X_{nk}\sim\mu_{nk}$, $S_n\sim\mu_n$, $\nu_{nk},\nu_n,\nu\sim Poi(p_{nk}),Poi(\sum_k p_{nk}),Poi(\lambda)$

则 $\mu_n=\mu_{n1}*\cdots*\mu_{nk_n}$, $\nu_n=\nu_{n1}*\cdots*\nu_{nk_n}$

$\therefore\|\mu_n-\nu_n\|\overset{\text{lem2}}{\le}\sum_{k=1}^{k_n}\|\mu_{nk}-\nu_{nk}\|\overset{\text{lem3}}{\le}\sum_{k=1}^{k_n}p_{nk}^2\le\max_{1\le k\le k_n}p_{nk}\cdot\sum_{k=1}^{k_n}p_{nk}\to 0$

由 $\nu_n\xrightarrow{d}\nu$ 知 $\mu_n\xrightarrow{d}\nu$ #

- Poisson 收敛:

考虑 $B(n,p)$. 若 p fixed, 则 $\frac{B(n,p)-np}{\sqrt{np(1-p)}}\xrightarrow{d}N(0,1)$.

若 $np_n\to\lambda\in(0,\infty)$, $P(S_n=k)=\binom{n}{k}p_n^k(1-p_n)^{n-k}\sim\frac{1}{k!}n^kp_n^k(1-p_n)^{n-k}\to\frac{\lambda^k}{k!}e^{-\lambda}$

$Ee^{itS_n}\to e^{\lambda(e^{it}-1)}$

Thm (Ber-Poi 收敛). 设 X_{nk}, $1\le k\le k_n$ 独立. $P(X_{nk}=1)=p_{nk}=1-P(X_{nk}=0)$.

若 ① $\sum_{k=1}^{k_n}p_{nk}\to\lambda\in(0,\infty)$

② $\max_{1\le k\le k_n}p_{nk}\to 0$,

则 $S_n\xrightarrow{d}Poi(\lambda)$

def 全变差距离: prob. measure μ,ν on measurable space $(\Omega,\mathcal{F})$,

$$\|\mu-\nu\|\triangleq\max_{A\in\mathcal{F}}|\mu(A)-\nu(A)|$$

Lem1 (全变差刻画) 离散空间 Ω, $\|\mu-\nu\|=\frac12\sum_{i\in\Omega}|\mu_i-\nu_i|$

Lem2. (离散空间 Ω) $\|\mu_1*\mu_2-\nu_1*\nu_2\|\le\|\mu_1-\nu_1\|+\|\mu_2-\nu_2\|$.

Lem3. 设 $\mu(1)=p=1-\mu(0)$, $\nu\sim Poi(p)$, 则 $\|\mu-\nu\|\le p^2$

Thm. (Poisson 收敛定理-一般) 独立非负整数值 r.v. 列 $\{X_{nk}\}$ 满足无穷小条件,

$$\sum_{k=1}^{k_n}X_{nk}\xrightarrow{d}Poi(\lambda)\iff\begin{cases}(i)\ \sum_{k=1}^{n_k}P(X_{nk}>1)\to 0\\(ii)\ \sum_k P(X_{nk}=1)\to\lambda\end{cases}$$

Pf: ($\Leftarrow$) 不妨 $X'_{nk}=X_{nk}I(X_{nk}=1)$, 则由 Ber-Poi 收敛 + 无穷小条件, $S'_n\xrightarrow{d}Poi(\lambda)$.

又 $P(S_n\ne S'_n)\le\sum P(X_{nk}>1)\to 0$ $\Rightarrow S_n\xrightarrow{d}Poi(\lambda)$. #

e.g. $X_i \overset{i.i.d}{\sim}$ Cauchy：$p(x)=\frac{a}{\pi(x^2+a^2)}, a>0$，则 $\frac{S_n}{n}\overset{d}{=}X$

Pf：$\int_{-\infty}^{+\infty}\frac{a}{2}e^{-a|x|+itx}dx=\int_0^{\infty}ae^{-ax}\frac{e^{itx}+e^{-itx}}{2}dx=\frac{a}{2}\left(\frac{1}{a-it}+\frac{1}{a+it}\right)=\frac{a^2}{a^2+t^2}$

$\overset{反演}{\Rightarrow}\frac{a}{2}e^{-a|x|}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{a^2}{a^2+t^2}e^{-itx}dt$

$\therefore f_X(t)=\int_{-\infty}^{+\infty}\frac{a}{\pi(a^2+x^2)}e^{itx}dx=e^{-a|t|}$

$\therefore f_{\frac{S_n}{n}}(t)=Ee^{it\frac{S_n}{n}}=(f_X(\frac{t}{n}))^n=e^{-a|t|}=f_X(t)$. #.

e.g. $X_i \overset{i.i.d.}{\sim}$ 对称，$P(|X|>x)=x^{-\alpha}$，$x\geq 1$，$0<\alpha<2$. 则 $\exists$ r.v. Y, s.t. $\frac{S_n}{n^{\frac{1}{\alpha}}}\xrightarrow{d}Y$.

Pf：$1-f_X(t)=\int_{|x|\geq 1}(1-e^{itx})\frac{\alpha}{2|x|^{\alpha+1}}dx$

$=\alpha\int_1^{\infty}(1-\cos(tx))\frac{1}{x^{\alpha+1}}dx$

$\overset{y=tx}{=}\alpha\int_t^{\infty}\frac{1-\cos y}{y^{\alpha+1}}dy\, t^{\alpha}\sim C_\alpha t^\alpha\ (t\to 0)$ （$\because \int_0^\infty\frac{1-\cos y}{y^{\alpha+1}}dy<\infty$）

$Ee^{it\frac{S_n}{n^{1/\alpha}}}=(f(\frac{t}{n^{1/\alpha}}))^n\sim\left[1-\frac{C_\alpha t^\alpha}{n}\right]^n\sim e^{-C_\alpha t^\alpha}$ #

Rmk. 若 $f_X(t)=e^{-c|t|^\alpha}$，则 $f_{\frac{S_n}{n^{1/\alpha}}}(t)=(f_X(\frac{t}{n^{1/\alpha}}))^n=f_X(t)$，即 $\frac{S_n}{n^{1/\alpha}}\overset{d}{=}X$.

Pf of Thm(稳定刻画)：($\Leftarrow$) 令 $X_i\sim F$，由 F 稳定，$\exists\{a_n\},\{b_n\}$, s.t. $(f(t))^n=e^{ib_nt}f(a_nt)$

即 $S_n\overset{d}{=}b_n+a_nX \Rightarrow \frac{S_n-b_n}{a_n}\overset{d}{=}X \Rightarrow \frac{S_n-b_n}{a_n}\xrightarrow{d}F$.

($\Rightarrow$) 若 F 退化，则 F 稳定

若 F 非退化，记 $Z_n=\frac{S_n-b_n}{a_n}$，则

$Z_{kn}=\frac{(X_1+\cdots+X_n)+(X_{n+1}+\cdots+X_{2n})+\cdots+(X_{(k-1)n+1}+\cdots+X_{kn})-b_{kn}}{a_{kn}}$

$\triangleq\frac{S_n^1+\cdots+S_n^k-b_{kn}}{a_{kn}}$

$\therefore \frac{a_{kn}Z_{kn}+b_{kn}-kb_n}{a_n}=\frac{S_n^1-b_n}{a_n}+\cdots+\frac{S_n^k-b_n}{a_n}\xrightarrow{d}F^{*k}(x)$

（$\because Ee^{it(Z_n^1+\cdots+Z_n^k)}=(Ee^{itZ_n})^k=(f_{Z_n})^k\to f_{F^{*k}}$）

又 $\because Z_{kn}\xrightarrow{d}F$

$\therefore$ 由律与型，$\exists a_k,b_k$, s.t. $F^{*k}(x)=F(a_kx+b_k)$ $\therefore F$ 为稳定的 #

e.g. Lévy 过程（独立平稳增量过程）

$0\ \frac{1}{n}\cdots\ 1$ $\forall n, X_1=\sum_{i=1}^n(X_{\frac{i}{n}}-X_{\frac{i-1}{n}})$，其中 $\{X_{\frac{i}{n}}-X_{\frac{i-1}{n}}\}$ i.i.d. 从而为无穷可分分布.

e.g. 1. 平稳分布
2. Poisson 分布 $f(t)=e^{\lambda(e^{it}-1)}=(e^{\frac{\lambda}{n}(e^{it}-1)})^n$
3. Gamma分布

• 稳定分布：$\frac{S_n-a_n}{b_n}\xrightarrow{d}$? 若 $EX^2<\infty$，则CLT $\Rightarrow N(0,1)$
若缓变化 $EX^2I(|X|<x)$.

Def. 称 $F(x)$ 的 ch.f. $f(t)$ 是稳定的，若 $\forall k\in N$，$\exists C_k>0$，ν_k，s.t. $(f(t))^k=e^{i\nu_kt}f(C_kt)$

Thm.（稳定刻画）$\exists X_i$ i.i.d. $\exists\{a_n\},\{b_n\}$, s.t. $\frac{S_n-b_n}{a_n}\xrightarrow{d}F \Leftrightarrow F$ 是稳定的.

Thm. 稳定分布的 ch.f. 为

$$f(t)=\exp\{itc-b|t|^\alpha(1+ik\,\mathrm{sgn}(t)\omega_\alpha(t))\}$$ 称具有指数 α 的稳定分布 f.

其中 $-1\leq k\leq 1$，指数 $\alpha\in[0,2]$，$\omega_\alpha(t)=\begin{cases}\tan(\frac{\pi\alpha}{2}), & \alpha\neq 1\\ \frac{2}{\pi}\log|t|, & \alpha=1\end{cases}$

Rmk. 1. $\alpha=2$，即 $N(c,2b)$.
2. $\alpha=1$，$c=k=0$，即 Cauchy 分布
3. $p(x)=(2\pi x^3)^{-\frac{1}{2}}e^{-\frac{1}{2x}}$，$x>0$ 为稳定分布（$\alpha=\frac{1}{2}$，$k=1$，$c=0$，$b=1$）
4. $\int|f|dt<\infty$ 时有密度，但 1~3 是现在可直接写出显式表示的.

Thm. X 的分布 G 属于某个具有指数 $\alpha\in(0,2)$ 稳定分布的吸引场 $\Leftrightarrow$

$\begin{cases}(1)\ P(|X|>x)=x^{-\alpha}L(x)，L 为缓变化函数，0<\alpha<2.\\ (2)\ \lim\limits_{x\to\infty}\frac{P(X>x)}{P(|X|>x)}=\theta\in[0,1]\end{cases}$

进一步，(1)(2)成立时，记 $S_n=X_1+\cdots+X_n$（X_i i.i.d.），$a_n=\inf\{x: P(|X|>x)\leq\frac{1}{n}\}$，$b_n=nEXI(|X|\leq a_n)$

则 $\exists$ 指数为 α 的稳定分布 F, s.t. $\frac{S_n-b_n}{a_n}\xrightarrow{d}F$.

F 具有 $K=2\theta-1$ 的 ch.f.

Def.（无穷可分分布 i.d.）若分布的 ch.f. $f(t)$ 满足 $\forall n$，$\exists f_n(t)$，s.t. $f(t)=(f_n(t))^n$.

Thm. 分布 F，则 $\exists\{X_{nk}\}_{k\leq n}$，s.t. (1) $\forall n$，X_{nk}，$k=1\sim n$，i.i.d. (2) $X_{n1}+\cdots+X_{nn}\xrightarrow{d}F$.

$\Leftrightarrow F$ 无穷可分的.

Pf：($\Leftarrow$) trivial ($\Rightarrow$) 只证 $n=2$，即 F 可写成两个 i.i.d. r.v. 之和的分布.

$\because S_{2n}=(X_1+\cdots+X_{2n,n})+(X_{2n,n+1}+\cdots+X_{2n,2n})\triangleq Y_n+Y_n'$

且 $\left.\begin{array}{l}P^2(Y_n>x)=P(Y_n>x, Y_n'>x)\leq P(S_{2n}>2x)\\ P^2(Y_n<-x)\leq P(S_{2n}<-2x)\end{array}\right\}$ + S_{2n} 胎紧 $\Rightarrow Y_n$ 胎紧

$\therefore$ 由 Helly，$\exists\{n_k\}$, s.t. $Y_{n_k}\xrightarrow[\exists]{d}Y$，$Y$ 为一个 r.v.，从而 $F\overset{d}{=}Y+Y'$

对 $\forall n\in N$ 可类似处理 $\Rightarrow F$ 无穷可分. #

Thm. 独立 r.v. 组列 $\{X_{nk}\}$ 满足无穷小条件，则 $\sum_{k=1}^{k_n}X_{nk}$ 的极限分布族与无穷可分分布族相重合

<u>Prop.</u> $f(t)$ i.d.，则 $\forall t\in\mathbb{R}$，$f(t)\neq 0$.

<u>Thm</u> (Lévy-Kahintchin表示) $f(t)$ i.d. $\Leftrightarrow f(t)=\exp\{ict-\frac{\sigma^2t^2}{2}+\int(e^{itx}-1-\frac{itx}{1+x^2})\nu(dx)\}$.

其中测度 ν 满足 $\nu(\{0\})=0$ 且 $\int(1\wedge x^2)\nu(dx)<\infty$，称 ν 为 <u>Lévy测度</u>.

(注意 poisson过程 $e^{\int(e^{itx}-1)\nu(dx)}=e^{a(e^{ita}-1)}$，这定理可看作分布可拆成normal + 一些poisson跳)

Date .

lem1 证明：① $Ef'(Z) = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{t^2}{2}} f'(t)dt$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f'(t)dt\int_t^{\infty} we^{-\frac{w^2}{2}}dw - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 f'(t)dt\int_{-\infty}^t we^{-\frac{w^2}{2}}dw$$

$E|f'(Z)|<\infty$，可用Fubini

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} we^{-\frac{w^2}{2}}dw\int_0^w f'(t)dt - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 we^{-\frac{w^2}{2}}\int_w^0 f'(t)dt$$

$$= EZf(Z)$$ #

lem2. 证明：$\because (e^{-\frac{w^2}{2}}f(w))' = e^{-\frac{w^2}{2}}(f'(w) - wf(w)) \overset{方程}{=} e^{-\frac{w^2}{2}}(I(w\le x) - \phi(x))$

而 $\int_t^{\infty}(e^{-\frac{w^2}{2}}f(w))'dw = -e^{-\frac{t^2}{2}}f(t)$，若f有界.

$\therefore$ $e^{-\frac{w^2}{2}}f(w) = -\int_w^{\infty} e^{-\frac{t^2}{2}}(I(w\le t) - \phi(x))dt$

由 $\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}}(\phi(x) - I(t\le x))dt = \phi(x)\cdot\sqrt{2\pi} - \int_{-\infty}^x e^{-\frac{t^2}{2}}dt = 0$ 得第二个等号.

Check 这个解有界：

若 $w\le x$，$f(w) = -e^{\frac{w^2}{2}}\int_{-\infty}^w e^{-\frac{t^2}{2}}(\phi(x)-1)dt = e^{\frac{w^2}{2}}(1-\phi(x))\sqrt{2\pi}\,\phi(w)$

若 $w>x$，$f(w) = e^{\frac{w^2}{2}}\int_w^{\infty} e^{-\frac{t^2}{2}}\phi(x)dt = e^{\frac{w^2}{2}}\phi(x)\sqrt{2\pi}(1-\phi(w))$

下证 $|f|\le\sqrt{\frac{\pi}{2}}$

由于 $w>0$ 时，$1-\phi(w) \le \min\{\frac{1}{2}, \frac{1}{\sqrt{2\pi}w}\}e^{-\frac{w^2}{2}}$ [CLAIM].

从而 $w>0$ 时，若 $w>x$，$f(w) \le e^{\frac{w^2}{2}}\sqrt{2\pi}(1-\phi(w)) \le \sqrt{\frac{\pi}{2}}$

$w\le x$，$f(w) \le e^{\frac{w^2}{2}}(1-\Phi(w))\sqrt{2\pi} \le \sqrt{\frac{\pi}{2}}$

$w<0$ 时类似可证 #

lem 1 证明：② 由Stein方程，$\forall x$，$f_x'(Z) - Zf_x(Z) = I(Z\le x) - \phi(x)$ 且 $Ef_x'(Z) = EZf_x(Z)$

$\therefore$ $EI(Z\le x) = \phi(x) \Rightarrow Z\sim N(0,1)$. #

References:

Δ Fundamentals of Stein's Method, Ross. N. Probability Surveys, 210-293, 2011

Barbour, Holst and Janson, Poisson Approximation. (1992)

Chen. Goldstein and Shao, Normal Approximation by Stein's Method. (2011)

Date .

• Stein方法（证明极限分布，常考虑 ch.f.，矩方法，Stein方法）

lem1 $Z\sim N(0,1)$，则 $\forall$ AC函数 $f: \mathbb{R}\to\mathbb{R}$ 且 $E|f'(Z)|<\infty$，有 $Ef'(Z) = EZf(Z)$

反之，若 $\forall$ 有界连续、逐段可微、$E|f'(Z)|<\infty$ 的函数 f 都有 $Ef'(Z) = EZf(Z)$，则 $Z\sim N(0,1)$

lem2. $\forall x\in\mathbb{R}$，Stein方程：

$$f'(w) - wf(w) = I(w\le x) - \phi(x)$$

有唯一有界解.

$$f(w) = e^{\frac{w^2}{2}}\int_w^{\infty} e^{-\frac{t^2}{2}}(\phi(x) - I(t\le x))dt$$

$$= -e^{\frac{w^2}{2}}\int_{-\infty}^w e^{-\frac{t^2}{2}}(\phi(x) - I(t\le x))dt$$

[CLAIM] $w>0$，$1-\phi(w) \le \min\{\frac{1}{2}, \frac{1}{\sqrt{2\pi}w}\}e^{-\frac{w^2}{2}}$

Pf：$\frac{1}{\sqrt{2\pi}}\int_w^{\infty} e^{-\frac{1}{2}t^2}dt = \frac{1}{\sqrt{2\pi}}\int_w^{\infty} e^{-\frac{1}{2}w^2}e^{-\frac{1}{2}(t-w)^2}e^{-w(t-w)}dt$

$$\le e^{-\frac{w^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}\int_w^{\infty} e^{-\frac{1}{2}(t-w)^2}dt = \frac{1}{2}e^{-\frac{w^2}{2}}$$

另一部分：$\int_w^{\infty} e^{-\frac{t^2}{2}}dt \le \int_w^{\infty}\frac{t}{w}e^{-\frac{t^2}{2}}dt = \frac{1}{w}e^{-\frac{w^2}{2}}$ #

lem 3. (General) Stein方程 $f'(w) - wf(w) = h(w) - Nh$，其中 $Nh = Eh(Z)$，$Z\sim N(0,1)$.

有唯一有界解.

$$f_h(w) = e^{\frac{w^2}{2}}\int_w^{\infty} e^{-\frac{t^2}{2}}(Nh - h(t))dt$$

$$= -e^{\frac{w^2}{2}}\int_{-\infty}^w e^{-\frac{t^2}{2}}(Nh - h(t))dt$$

若 h 有界，$\|f_h\| \le \sqrt{\frac{\pi}{2}}\|h(\cdot) - Nh\|$

$\|f_h'\| \le 2\|h(\cdot) - Nh\|$

若 h AC，$\|f_h\| \le 2\|h'\|$，$\|f_h'\| \le \sqrt{\frac{2}{\pi}}\|h'\|$，$\|f_h''\| \le 2\|h'\|$

Def. 距离 $d_{\mathcal{H}}(\mu,\nu) = \sup_{h\in\mathcal{H}}\left|\int h(x)d\mu(x) - \int h(x)d\nu(x)\right|$

e.g. ① $\mathcal{H} = \{I(\cdot\le x): x\in\mathbb{R}\}$，Kolmogorov 距离

② $\mathcal{H} = \{I(\cdot\in A): A\in\mathcal{B}(\mathbb{R})\}$，全变差距离

③ $\mathcal{H} = \{h: |h(x) - h(y)|\le|x-y|\}$，Wasserstein 距离

$$d_W(W, N(0,1)) = \sup_{h\in\mathcal{H}}|Eh(W) - Eh(Z)|$$

$$\overset{\text{Stein}}{=} \sup_{h\in\mathcal{H}}|E(f_h'(W) - Wf_h(W))|$$

$$\le \sup_{\substack{\|f\|\le 2\\ \|f'\|\le\sqrt{\frac{2}{\pi}}\\ \|f''\|\le 2}}|E(f'(W) - Wf(W))|$$

Prop. ① $d_K(W,Z) \le d_{TV}(W,Z)$

② $d_K(W,Z) \le \sqrt{2Cd_W(W,Z)}$，若 Z 的密度有界 C.

Recall:
$Z\sim N(0,1) \overset{\text{条件}}{\Longleftrightarrow} Ef'(Z)=EZf(Z)$

Stein方程 $f'(x)-xf(x)=I(x\le z)-\Phi(z) \overset{\text{推广}}{\Rightarrow} f'(x)-xf(x)=h(x)-Eh(Z)$ Date.

我们常考虑 Wasserstein 度量: $d_W(X,Y)=\sup_{h:|h(x)-h(y)|\le|x-y|}|Eh(X)-Eh(Y)|$

则有 $d_W(W,N(0,1))\le \sup_{\|f\|,\|f''\|\le 2,\ \|f'\|\le\sqrt{2/\pi}} |E[f'(W)-Wf(W)]|$ ← 关心这个

·考虑 $X_{1\sim n}$ 独立, $EX_i=0$, $EX_i^2=1$, 记 $W\triangleq \frac{\sum X_i}{\sqrt n}$

则 $EWf(W)=\frac{1}{\sqrt n}\sum_i EX_if(W)=\frac1{\sqrt n}\sum_i EX_i(f(W)-f(W_i))$, $W_i=\frac1{\sqrt n}\sum_{j\ne i}X_j$

$=\frac1{\sqrt n}\sum_{i=1}^n EX_i(f(W)-f(W_i)-(W-W_i)f'(W))+\frac1n\sum_{i=1}^n EX_i^2 f(W)$

$\Rightarrow |E[f'(W)-Wf(W)]|\le \frac1{\sqrt n}|E\sum_i X_i(f(W)-f(W_i)-(W-W_i)f'(W))| + |Ef'(W)(1-\frac1n\sum EX_i^2)|$

$\triangleq |I|+|II|$.

其中 $|I|\le\frac1{\sqrt n}|E\sum_i X_i\cdot\frac{(W-W_i)^2}{2}f''(W_i^*)|$, W_i^*在W与W_i之间

$\le \frac{1}{2n^{3/2}}E\sum_i|X_i|^3\|f''\| = \frac{1}{n^{3/2}}\sum_i E|X_i|^3$

$|II|\le\|f'\|\cdot E|1-\frac1n\sum X_i^2|\le\sqrt{\frac2\pi}\cdot\sqrt{E(1-\frac1n\sum X_i^2)^2}=\sqrt{\frac2\pi}\cdot\frac1n\sqrt{Var(\sum X_i^2)}\le\sqrt{\frac2\pi}\cdot\frac1n\cdot\sqrt{\sum EX_i^4}$

$\Rightarrow$ Thm $d_W(W,N(0,1))\le\frac1{n^{3/2}}\sum_i E|X_i|^3+\sqrt{\frac2\pi}\cdot\frac1n\sqrt{\sum_i EX_i^4}$

Mtd 2: $EWf(W)\overset{\xi_i=X_i/\sqrt n}{=}\sum_i E\xi_i(f(W)-f(W_i))$, $W_i=\sum_{j\ne i}\xi_j$

$=\sum_i E(\xi_i\int_0^{\xi_i}f'(W_i+t)dt)$

$=\sum_i E\xi_i\int_{-\infty}^{+\infty}(I_{(0\le t\le\xi_i)}-I_{(0>t\ge\xi_i)})f'(W_i+t)dt$

$=\sum_i\int_{-\infty}^{+\infty}(Ef'(W_i+t))K_i(t)dt$, 其中 $K_i(t)=E\xi_i(I_{(0\le t\le\xi_i)}-I_{(0>t\ge\xi_i)})$

$\because \sum_i\int_{-\infty}^{+\infty}K_i(t)dt=\sum_i E\xi_i^2=1$

$\therefore |E(f'(W)-Wf(W))|=|\sum_{i=1}^n\int_{-\infty}^{+\infty}E(f'(W)-f'(W_i+t))\cdot K_i(t)dt|$

$\underset{\|f''\|\le2}{\le}2\sum_{i=1}^n\int_{-\infty}^{+\infty}(E|\xi_i|+|t|)K_i(t)dt$

$\overset{\text{代入}K_i}{=}2\sum_{i=1}^n(E\xi_i\int_{-\infty}^{+\infty}|t|(I_{(0\le t\le\xi_i)}-I_{(0>t\ge\xi_i)})dt+E|\xi_i|\cdot E\xi_i^2)$

$\le 2\sum_{i=1}^n(\frac12E|\xi_i|^3+E|\xi_i|^3)=3\sum E|\xi_i|^3=\frac{3}{n^{3/2}}\sum E|X_i|^3$

Mtd 3: $\because |f'(W)-f'(W_i+t)|\overset{\|f'\|\le\sqrt{2/\pi},\ \|f''\|\le2}{\le}\min\{2,2(|t|+|\xi_i|)\}$

$\le 2(|t|\wedge1+|\xi_i|\wedge1)$

$\therefore |E[f'(W)-Wf(W)]|\le2\sum_{i=1}^n E\int_{-\infty}^{+\infty}(|t|\wedge1+|\xi_i|\wedge1)K_i(t)dt$

$=2\sum_i(E\xi_i^2I(|\xi_i|>1)-\frac12E|\xi_i|I(|\xi_i|>1)+\frac12E|\xi_i|^3I_{(|\xi_i|\le1)}+E\xi_i^2E(|\xi_i|\wedge1))$

$\le\sum_i(4E\xi_i^2I(|\xi_i|>1)+3E|\xi_i|^3I_{(|\xi_i|\le1)})$

($\because E\xi_i^2E(|\xi_i|\wedge1)\le E\xi_i^2I(|\xi_i|>1)+E|\xi_i|^3I_{(|\xi_i|\le1)}$)

$d_W(W,N(0,1))\le\frac4n\sum_{i=1}^nEX_i^2I_{(|X_i|>\sqrt n)}+\frac3{n^{3/2}}\sum_{i=1}^nE|X_i|^3I(|X_i|\le\sqrt n)$

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Pf of $\sup_z |P(\frac{S_n}{\sqrt{n}} \le z) - \Phi(z)| \le \frac{6.5\,E|X|^3}{\sqrt{n}}$. 记 $\xi_i = \frac{X_i}{\sqrt{n}}$，$W_k = \sum_{i=1}^{k} \xi_i$

则 $E W_n f(W_n) = \sum E \xi_i f(W_n)$

$= n E \xi_n f(W_{n-1} + \xi_n)$

$= n E \xi_n (f(W_{n-1} + \xi_n) - f(W_{n-1}))$

$= n E \xi_n \int_0^{\xi_n} f'(W_{n-1} + t)\, dt$

$= E \int_{-\infty}^{+\infty} f'(W_{n-1} + t)\, K(t)\, dt$，其中 $K(t) = n E \xi_n (I_{(0<t<\xi_n)} - I_{(\xi_n < t < 0)})$

$\because K(t) \ge 0$, $K(\pm\infty) = 0$, $\int K(t)\, dt = 1$，$\int |t| K(t) = \frac{E|X|^3}{2\sqrt{n}}$ (*)

$\therefore E(f'(W_n) - W_n f(W_n)) = E \int_{-\infty}^{+\infty} (f'(W_{n-1} + \xi_n) - f'(W_{n-1} + t))\, K(t)\, dt$

「Pf lem. 令 $g_x(w) = \begin{cases} -\frac{1}{2}(b-a) - x, & w \le a - x \\ w - \frac{1}{2}(b+a), & a - x \le w \le b + x \\ \frac{1}{2}(b-a) + x & w \ge b + x \end{cases}$

则 $g_x \in AC$ 且 $g_x'(w) = I(a - x \le w \le b + x)$

将 $g_{E|X|^3/\sqrt{n}}$ 代入 $E W_n f(W_n) = E \int_{-\infty}^{+\infty} f'(W_{n-1} + t) K(t)\, dt$ 得.

$E W_n g_{E|X|^3/\sqrt{n}}(W_n) = E \int I(a - \frac{E|X|^3}{\sqrt{n}} \le W_{n-1} + t \le b + \frac{E|X|^3}{\sqrt{n}})\, K(t)\, dt$

$\ge E\, I(a \le W_{n-1} \le b) \int I(|t| \le \frac{E|X|^3}{\sqrt{n}})\, K(t)\, dt$

$\ge P(a \le W_{n-1} \le b) \cdot \frac{1}{2}$

且 $LHS \le (\frac{1}{2}(b-a) + \frac{E|X|^3}{\sqrt{n}}) E|W_n| \le \frac{1}{2}(b-a) + \frac{E|X|^3}{\sqrt{n}}$. #

L

对于 f_z，$|E f_z'(W_n) - W_n f_z'(W_n))| = |E \int ((W_{n-1} + \xi_n) f_z(W_{n-1} + \xi_n) - (W_{n-1} + t) f_z(W_{n-1} + t)$
$+ I(W_{n-1} + \xi_n \le z) - I(W_{n-1} + t \le z))\, K(t)\, dt|$

$\overset{\text{lem 2}}{\le} \int E(|W_{n-1}| + \frac{\sqrt{2\pi}}{4})(E|\xi_n| + |t|)\, K(t)\, dt$

$+ \int P(z - \xi_n \vee t \le W_{n-1} \le z - \xi_n \wedge t)\, K(t)\, dt$

$\therefore \quad |P(\frac{S_n}{\sqrt{n}} \le z) - \Phi(z)| \underset{\text{lem.}}{\overset{E|W_{n-1}| \le 1}{\le}} 2 \int E(|\xi_n| + |t|)\, K(t)\, dt$

$+ E \int (|\xi_n - t| + 2\frac{E|X|^3}{\sqrt{n}})\, K(t)\, dt$

$\le 3 \int (E|\xi_n| + |t|)\, K(t)\, dt + 2\frac{E|X|^3}{\sqrt{n}}$

$\overset{(*)}{\le} \frac{9}{2\sqrt{n}} E|X|^3 + \frac{2E|X|^3}{\sqrt{n}} = \frac{6.5\, E|X|^3}{\sqrt{n}}$ #.

(Berry-Esseen)
(~~Kolmogorov~~ $d_K(W, N(0,1))$).

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Thm $X_{1 \sim n}$ i.i.d. $EX = 0$, $EX^2 = 1$，则 $\sup_z |P(S_n \le \sqrt{n} z) - \Phi(z)| \le \frac{6.5\, E|X|^3}{\sqrt{n}}$

lem. $a < b$，则 $P(a \le \underset{\frac{S_{n-1}}{\sqrt{n}}}{\underset{\|}{W_{n-1}}} \le b) \le b - a + \frac{2E|X|^3}{\sqrt{n}}$.

lem2. 设 f_z 为 $f'(w) - w f(w) = I(w \le z) - \Phi(z)$ 的解，则
$|f_z'(w+s) - f_z'(w+t)| \le (|t| + |s|)(|w| + 1) + I(z - t \le w \le z - s) I(s \le t)$
$+ I(z - s \le w \le z - t) I(s > t)$

$|(w+u) f_z(w+u) - (w+v) f_z(w+v)| \le (|w| + \frac{\sqrt{2\pi}}{4})(|u| + |v|)$

• Def. 可交换对：$(W, W') \overset{d}{=} (W', W)$

α-Stein可交换对：$E(W'|W) = (1-\alpha) W$，$0 < \alpha \le 1$. 且 (W, W') 可交换对.

Prop. 1. $EW = 0$

2. 若 $Var(W) = \sigma^2$，则 $E(W' - W)^2 = 2\alpha\sigma^2$

• 考虑 $F(w) = \int_0^w f(t)\, dt$，$f \in C^2$

$0 = E(F(W') - F(W)) \overset{Taylor}{=} E[(W' - W) f(W) + \frac{1}{2}(W' - W)^2 f'(W) + \frac{1}{6}(W' - W)^3 f''(W^*)]$

又 $E(W' - W) f(W) = E\, E[(W' - W) f(W) | W] = -\alpha E W f(W)$

$\Rightarrow E W f(W) = \frac{1}{2\alpha} E(W' - W)^2 f'(W) + \frac{1}{6\alpha} E(W' - W)^3 f''(W^*)$

$\Rightarrow |E(f'(W) - W f(W))| \le \|f'\|\, E|1 - \frac{1}{2\alpha} E[(W' - W)^2 | W]| + \|f''\| \cdot \frac{E|W' - W|^3}{6\alpha}$

• 若 $Var W = 1$，则 $E|1 - \frac{1}{2\alpha} E[(W' - W)^2 | W]| = \frac{1}{2\alpha} E| E[(W' - W)^2 | W] - E(W' - W)^2| \le \frac{1}{2\alpha} \sqrt{Var(E[(W' - W)^2 | W])}$

$\Rightarrow$ Thm. 若 (W, W') 为 α-Stein交换对，$Var(W) = 1$，则 $d_W(W, Z) \le \frac{\sqrt{Var(E[(W' - W)^2 | W])}}{\sqrt{2\pi}\, \alpha} + \frac{E|W' - W|^3}{3\alpha}$

e.g. 设 $X_{1,2,\cdots}$ 独立. $EX_i^4 < \infty$，$EX_i = 0$，$Var X_i = 1$，$W = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. 估计 $d_W(W, Z)$.

Sol. 设 $I \sim Unif\{1, \cdots, n\}$，$(X_1', \cdots, X_n')$ 独立复制 $(X_1, \cdots, X_n)$，$W' \triangleq W - \frac{X_I}{\sqrt{n}} + \frac{X_I'}{\sqrt{n}}$

则 (W, W') 可交换，$E(W' - W) = E(\frac{X_I' - X_I}{\sqrt{n}} | W) = \frac{1}{\sqrt{n}} \cdot E(\frac{1}{n} \sum (X_i' - X_i) | W) = -\frac{1}{n} W$

$\therefore (W, W')$ 为 $\frac{1}{n}$-Stein 可交换对

由 Thm. $d_W(W, Z) \le \frac{\sqrt{Var(E[(W' - W)^2 | W])}}{\sqrt{2\pi}\, \alpha} + \frac{E|W' - W|^3}{3\alpha}$

其中 $\frac{E|W' - W|^3}{\alpha} = \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} E|X_i' - X_i|^3 \le \frac{8}{n^{\frac{3}{2}}} \sum_{i=1}^{n} E|X_i|^3$

$E[(W' - W)^2 | X_1, \cdots, X_n] = E[\frac{(X_I' - X_I)^2}{n} | X_1, \cdots, X_n] = \frac{1}{n^2} E[\sum_{i=1}^{n} (X_i' - X_i)^2 | X_1, \cdots, X_n] = \frac{1}{n^2} \sum_{i=1}^{n} (1 + X_i^2)$

$\Rightarrow \frac{\sqrt{Var(E[(W' - W)^2 | W])}}{\alpha} = n \sqrt{\frac{1}{n^4} \sum_{i=1}^{n} Var(X_i^2)} \le \frac{1}{n} \sqrt{\sum_{i=1}^{n} EX_i^4}$

$\therefore d_W(W, Z) \le \frac{1}{\sqrt{2\pi}\, n} \sqrt{\sum EX_i^4} + \frac{8}{3n^{\frac{3}{2}}} \sum E|X_i|^3$ #.

Date .

Pf of 离散情形: (⇒) $Ef(X^s)=\sum_k f(k)P(X^s=k)=\frac{EXf(X)}{\mu}$ #

(⇐) 取 $f=\chi_{\{k\}}$.

Pf of 构造的 X^s 为 X 的 size-bias 分布: $h(x)=E(f(X)|X_i=x)=E(f(\sum_{j\neq i}X_j^{(i)}+X_i^s)|X_i^s=x)$

$\because E(X_i f(X))=E(X_i h(X_i))=\mu_i E(h(X_i^s))=\mu_i E(f(\sum_{j\neq i}X_j^{(i)}+X_i^s))=\mu E(f(X^s)\chi_{\{I=i\}})$

$\therefore E(Xf(X))=\mu Ef(X^s)$ #.

Eg. (Erdos图) $G(n,p)$, X: 图中孤立点数, $X=\sum X_i$, $X_i=\chi_{\{v_i\text{是孤立点}\}}$.

由构造方法, $I\sim Unif\{1,\cdots,n\}$ 与其它 r.v. 独立.

取 $i=1$, 构造 $X_j^{(1)}$, s.t. $(X_j^{(1)}, j\neq 1)\overset{d}{=}(X_j, j\neq 1|X_1=1)$

⇒ $X_j^{(1)}=\chi\{$去掉与 v_1 相连的边后 v_j 孤立$\}=\chi\{v_j$ 至多与 v_1 相连$\}$

⇒ $X^s=$ 去掉与 v_I 相关的边后孤立点个数.

- size-bias coupling

Def. $X\geq 0$, $EX=\mu<\infty$, 称 X^s 具有关于 X 的 size-bias 分布, 若 $\forall f$, $EXf(X)<\infty$, 有

$$EXf(X)=\mu Ef(X^s)$$

Thm. (离散到位) 若 X 离散非负整值, 则 $P(X^s=k)=\frac{kP(X=k)}{\mu}$ ⟺ X^s 有关于 X 的 size-bias 分布.

Thm (一般到位) 若 $X\sim F$, 则 $dF^s(x)=\frac{x}{\mu}dF$.

- 若 $X\geq 0$, $EX=\mu$, $Var(X)=\sigma^2$, 则令 $W=\frac{X-\mu}{\sigma}$

$$EWf(W)=E\frac{X-\mu}{\sigma}f(\frac{X-\mu}{\sigma})=\frac{1}{\sigma}EXf(\frac{X-\mu}{\sigma})-\frac{\mu}{\sigma}Ef(\frac{X-\mu}{\sigma})$$

$$\overset{X^s\text{ size-bias}}{=}\frac{\mu}{\sigma}E[f(\frac{X^s-\mu}{\sigma})-f(\frac{X-\mu}{\sigma})]$$

$$\overset{\text{Taylor}}{=}\frac{\mu}{\sigma}E[\frac{X^s-X}{\sigma}f'(\frac{X-\mu}{\sigma})+\frac{(X^s-X)^2}{2\sigma^2}f''(\frac{X^*-\mu}{\sigma})]$$

$$\Rightarrow |E[f'(W)-Wf(W)]|\leq|Ef'(W)(1-\frac{\mu}{\sigma^2}(X^s-X))|+\frac{\mu}{2\sigma^3}|Ef''(\frac{X^*-\mu}{\sigma})(X^s-X)^2|$$

$$\Rightarrow d_W(W,Z)\leq\sqrt{\frac{2}{\pi}}E|1-\frac{\mu}{\sigma^2}E(X^s-X|X)|+\frac{\mu}{\sigma^3}E(X^s-X)^2$$

$$\leq\sqrt{\frac{2}{\pi}}\frac{\mu}{\sigma^2}\sqrt{Var(E(X^s-X|X))}+\frac{\mu}{\sigma^3}E(X^s-X)^2 \quad (\because E(X^s-X)=\frac{EX^2}{\mu}-EX=\frac{\sigma^2}{\mu})$$

⇒ Thm $X\geq 0$, $EX=\mu$, $VarX=\sigma^2$, X^s 为 X 的 size-bias 分布. $W=\frac{X-\mu}{\sigma}$, 则

$$d_W(W,Z)\leq\sqrt{\frac{2}{\pi}}\frac{\mu}{\sigma^2}\sqrt{Var(E(X^s-X|X))}+\frac{\mu}{\sigma^3}E(X^s-X)^2.$$

- 如何构造 X^s? 考虑 $X=\sum_{i=1}^n X_i$, $X_i\geq 0$, $EX_i=\mu$.
  (1) 设 X_i^s 为 X_i 的 size-bias 分布, 独立于 $X_{(-i)}$, $X_{(-i)}^s$. 定义 $(X_j^{(i)})_{j\neq i}$, s.t. $(X_j^{(i)}, j\neq i|X_i^s=x)\overset{d}{=}(X_j, j\neq i|X_i=x)$
  (2) 取 I, s.t. $P(I=i)=\frac{\mu_i}{\mu=\sum\mu_i}$ 与其他 r.v. 独立.
  (3) $X^s\triangleq\sum_{j\neq I}X_j^{(I)}+X_I^s$

E.g. (特殊构造) (1) X_i 独立, X_i^s 与 $X_{(-i)}$, $X_{(-i)}^s$ 独立, 则 $X^s=X-X_I+X_I^s$

(2) X_i 为 0-1 r.v., $X_i^s=1$, $(X_j^{(i)}, j\neq i)\overset{d}{=}(X_j, j\neq i|X_i=1)$, 则 $X^s=\sum_{j\neq I}X_j^{(I)}+1$.

- zero-bias coupling.

Def. $EW=0$, $VarW=\sigma^2$, 称 W^z 为关于 W 的 zero-bias coupling, 若 $\forall f\in AC$, $E|Wf(W)|<\infty$, 有

$$EWf(W)=\sigma^2Ef'(W^z)$$

Thm. 若 $\sigma^2=1$, 则 $d_W(W,Z)\leq 2E|W-W^z|$ $\quad \because |E(f'(W)-Wf(W))|=|E(f'(W)-f'(W^z))|\leq\|f''\|\cdot E|W-W^z|$

Thm (W^z ∃1性) $W^z\sim p^z(w)=\sigma^{-2}E(WI(W>w))=-\sigma^{-2}E(WI(W\leq w))$

- 如何构造 W^z? 考虑 $W=\sum_{i=1}^n X_i$, X_i 独立, $EX_i=0$, $VarX_i=\sigma_i^2$, $\sum\sigma_i^2=1$
  (1) $\forall i$, X_i^z 为 X_i 的 zero-bias coupling, 独立于 $X_{(-i)}$, $X_{(-i)}^z$
  (2) 选取 I, s.t. $P(I=i)=\sigma_i^2$ 且与其他 r.v. 独立
  (3) $W^z=W-X_I+X_I^z=\sum_{i\neq I}X_i+X_I^z$.

Date .

Pf of Lem (1). 即证：$\lambda E f(Z+1) = E(Z f(Z))$

$$LHS = \sum_{k=0}^{\infty} f(k+1) e^{-\lambda} \frac{\lambda^{k+1}}{k!} = \sum_{k=0}^{\infty} f(k+1) e^{-\lambda} \frac{\lambda^{k+1}}{(k+1)!} \cdot (k+1) = E(Z f(Z)).$$

(2) 取 $f(k) = I(k=j)$，由 $\lambda E f(W+1) = E(W f(W))$ 得

$$\lambda P(W=j-1) = j P(Z=j)$$

$$\Rightarrow P(W=j) = \frac{\lambda^j}{j!} P(W=0) \overset{\sum P(W=j)=1}{\Rightarrow} P(W=0) = e^{-\lambda} \Rightarrow W \sim P(\lambda).$$ #.

Pf of Thm$^{(独立和性)}$: $d_{TV}(W,Z) \le \sup_{f\in\mathcal{F}} |E(\lambda f(W+1) - W f(W))|$

$\because EWf(W)$

$= \sum E X_i f(X_i + (W - X_i))$

$\overset{取条件期望}{=} \sum p_i E f(W_i+1)$, $W_i \triangleq W - X_i = \sum_{j\ne i} X_j$

且 $E\lambda f(W+1) = \sum_{i=1}^{n} p_i E f(W+1)$

$\therefore |E(\lambda f(W+1) - W f(W))| \le \sum_{i=1}^{n} p_i |E f(W+1) - f(W_i+1)|$

$\le \sum_{i=1}^{n} p_i \cdot E X_i \cdot \|\Delta f\| \le (1 \wedge \frac{1}{\lambda}) \sum p_i^2$ #

Pf of Thm(不独立和性): $\because EWf(W) = \sum E X_i f(W_i+1)$, $W_i = W - X_i$

$= \sum E X_i (f(W_i+1) - f(V_i+1))$, $V_i = \sum_{j\notin N_i} X_j$

$+ \sum E X_i f(V_i+1)$

$= \sum E X_i (f(W_i+1) - f(V_i+1)) + \sum p_i E f(V_i+1)$

且 $\lambda E f(W+1) = \sum p_i E f(W+1)$

$\therefore |E(\lambda E f(W+1) - W f(W))| \le |\sum_{i=1}^{n} E X_i (f(W_i+1) - f(V_i+1))| + \sum_{i=1}^{n} p_i |E f(V_i+1) - f(W+1)|$

$\le \|\Delta f\| \cdot (\sum E X_i (W_i - V_i) + \sum p_i E(W - V_i))$ #

E.g. n次掷硬币，正面pr为p，X：不少于k个连续正面出现的次数

设 $Y_i \overset{i.i.d.}{\sim} Ber(p)$ 为第i次出现正面

$$\begin{cases} X_1 = \prod_{i=1}^{k} Y_i \\ X_i = (1-Y_{i-1}) \prod_{j=0}^{k-1} Y_{i+j}, \quad i = 2,3,\cdots,n-k+1 \end{cases} \quad 则 \ X = \sum_{i=1}^{n-k+1} X_i$$

记 $\lambda = EX = p^k((n-k)(1-p)+1)$，则 $d_{TV}(X, P(\lambda)) \le \lambda^2 \frac{2k+1}{n-k+1} + 2\lambda p^k$.

Pf: 由 Thm(不独立和性)，$d_{TV}(X, P(\lambda)) \le (1 \wedge \frac{1}{\lambda}) (\sum_{i=1}^{n} \sum_{j\in N_i} p_i p_j + \sum_{i=1}^{n} \sum_{j \in N_i \setminus \{i\}} p_{ij})$

$N_i = \{|i-j| \le k\}$，而 $j \in N_i \setminus \{i\}$ 时 $p_{ij} = 0$，$p_1 = p^k$，$p_i = (1-p)p^k$ #.

Rmk. $k_n = \frac{\log(n(1-p))}{\log(1/p)} + C$ 时，$\lambda_n \to p^c \Rightarrow d_{TV}(X, P(\lambda)) \le O(\frac{\log n}{n})$.

Date .

· Poisson 逼近 Stein-Chen 方法 $\lambda f(k+1) - k f(k) = h(k) - E h(Z)$，考虑 TV：$h(k) = I(k\in A)$
$= I(k\in A) - P_\lambda(A)$

Lem $\lambda > 0$，$\mathcal{A} f(k) = \lambda f(k+1) - k f(k)$.

(1) $Z \sim P(\lambda)$，则 $\forall$ 有界 f，有 $E\mathcal{A}f(Z) = 0$.

(2) 若 $\exists$ 非负整值 W，s.t. $E\mathcal{A}f(W) = 0$，$\forall f$ 有界，则 $W \sim P(\lambda)$.

Thm Stein 方程 $\begin{cases} \lambda f_A(k+1) - k f(k) = I(k\in A) - P_\lambda(A) \\ f(0) = 0 \end{cases}$

有唯一解 $f_A(k) = \lambda^{-k} e^{\lambda} (k-1)! (P_\lambda(A \cup U_k) - P_\lambda(A) P_\lambda(U_k))$. $U_k = \{0,1,\cdots,k-1\}$

Prop. $\|f_A\| \le \min\{1, \frac{1}{\sqrt{\lambda}}\}$，$\|\Delta f_A\| \le \frac{1-e^{-\lambda}}{\lambda} \le \min\{1, \frac{1}{\lambda}\}$，其中 $\Delta f_A(k) = f_A(k+1) - f_A(k)$，$\|\cdot\| = \|\cdot\|_{\ell^\infty}$

· 记 $Z \sim P(\lambda)$，关心 $|P(W\in A) - P(Z\in A)| = |E(\lambda f_A(W+1) - W f_A(W))|$

· $d_{TV}(W,Z) = \sup_A |P(W\in A) - P(Z\in A)| \le \sup_{f\in\mathcal{F}} |E(\lambda f(W+1) - W f(W))|$,

其中 $\mathcal{F} = \{f : \mathbb{Z}_{\ge 0} \to \mathbb{R} \mid \|f\| \le \min\{1, \frac{1}{\sqrt{\lambda}}\}, \|\Delta f\| \le \frac{1-e^{-\lambda}}{\lambda}\}$.

2 cases：① $W = \sum I_i$，I_i 独立 ② size-bias coupling：$EWf(W) = \lambda E f(W^s)$

Thm$^{(独立和性)}$ $X_i \overset{独立}{\sim} Ber(p_i)$，$W = \sum_{i=1}^{n} X_i$，$\lambda = EW = \sum_{i=1}^{n} p_i$. 若 $Z \sim P(\lambda)$，则 $d_{TV}(W,Z) \le (1\wedge\frac{1}{\lambda}) \sum p_i^2$

(小数定律)

Cor ① 若 $p_i = p$，$\lambda = np \equiv C$ 与n无关，则 $d_{TV}(W_n, Z) \le \frac{\lambda^2}{n} \to 0$. $W_n \xrightarrow{d} P(\lambda)$

② $p = n^{-\frac{3}{4}}$，$np \to \infty$，但 $d_{TV}(W_n, Z_n) \le n^{-\frac{1}{2}} \to 0$，其中 $Z_n \overset{d}{=} X_1 + \cdots + X_{EW_n}$，$X_i \overset{i.i.d.}{\sim} P(1)$

又由 $\frac{Z_n - EZ_n}{\sqrt{Var(Z_n)}} \xrightarrow{d} N(0,1)$ 得 $\frac{W_n - \lambda_n}{\sqrt{\lambda_n}} \xrightarrow{d} N(0,1)$.

$\left(\because P(\frac{W_n - EZ_n}{\sqrt{Var(Z_n)}} \le x) = P(\frac{Z_n - EZ_n}{\sqrt{Var(Z_n)}} \le x) + \underbrace{[P(\frac{W_n - EZ_n}{\sqrt{Var(Z_n)}} \le x) - P(\frac{Z_n - EZ_n}{\sqrt{Var(Z_n)}} \le x)]}_{绝对值 \le d_{TV}(W_n, Z_n) \to 0} \right)$

Thm$^{(不独立和性)}$ $X_i \overset{(不独立)}{\sim} Ber(p_i)$，$W = \sum_{i=1}^{n} X_i$，$\lambda = EW = \sum_{i=1}^{n} p_i$. 设 $N_i \subset \{1,\cdots,n\}$，s.t. $i \in N_i$ 且 i 与 $\{X_j : j \notin N_i\}$ 独立.

记 $p_{ij} = E X_i X_j$，若 $Z \sim P(\lambda)$，则

$$d_{TV}(W,Z) \le (1\wedge\frac{1}{\lambda}) (\sum_{i=1}^{n} \sum_{j\in N_i} p_i p_j + \sum_{i=1}^{n} \sum_{j\in N_i \setminus \{i\}} p_{ij})$$

E.g.2 续上例 Rmk，R_n：最长连续正面长度 $P(R_n < k) = P(X(k个连续正) = 0)$

$|P(X(k) = 0) - P(P(\lambda) = 0)| = O(\frac{\log n}{n})$

$\Rightarrow |P(R_n - \frac{\log(n(1-p))}{\log\frac{1}{p}} < x) - e^{-p^x}| \le C \cdot \frac{\log n}{n}$.

$\Rightarrow \frac{R_n}{\frac{\log(n(1-p))}{\log\frac{1}{p}}} \xrightarrow{P} 1 \Rightarrow \frac{R_n}{\log n} \xrightarrow{P} \frac{1}{\log\frac{1}{p}}$ 进一步还可证 a.s. 收.

Date .

- size-bias coupling : $EWf(W) = \lambda Ef(W^s)$.

<u>Thm</u>. $d_{TV}(W, P(\lambda)) \le \min\{1, \lambda\} E|W+1-W^s|$

$\le \sup_{f\in\mathcal{F}} |EWf(W) - \lambda Ef(W+1)|$ $\quad \|$

$\le \sup_{f\in\mathcal{F}} \lambda \|\Delta f\| \cdot E|W^s - (W+1)| \le \lambda \cdot \min(1, \frac{1}{\lambda}) \cdot E|W+1-W^s|$.

- W^s 的构造：

0. 设 X_i 示性独立，$W = \sum_{i=1}^n X_i$

1. 定义 I 与其它独立，$P(I=i) = \frac{p_i}{\lambda}$

2. $W^s = \sum_{j\ne I} X_j + 1$. $\quad$ Check: $\lambda Ef(W^s) = \lambda \sum_{i=1}^n Ef(W^s)\cancel{\chi}_{\{I=i\}} = \sum_{i=1}^n p_i Ef(\sum_{j\ne i} X_j + 1) = \sum_{i=1}^n EX_i f(W) = EWf(W)$

$\Rightarrow$ <u>Cor</u>. $d_{TV}(W, P(\lambda)) \le \min\{1, \lambda\} E|X_I| = \min\{1, \frac{1}{\lambda}\} \sum p_i^2$ 与前文一致！

Date .

Pf of Portmanteau. (1) ⇒ (2), (3) ⇒ (4) trivial.

(2) ⇒ (3): $\forall F$, $F^{\varepsilon} \triangleq \{x: \rho(x,F) < \varepsilon\}$

∃一致连续 f, s.t. $I_F(x) \le f_\varepsilon(x) \le I_{F^\varepsilon}(x)$ （取 $f = \left(1 - \frac{\rho(x,F)}{\varepsilon}\right)^+$）

$\therefore \overline{\lim} P(X_n \in F) \le \overline{\lim} E f_\varepsilon(X_n) = E f_\varepsilon(X) \le P(X \in F^\varepsilon)$ 令 $\varepsilon \to 0$. 得证.

(3)+(4) ⇒ (5): $P_X(E) = P_X(\overline{E}) = P_X(E^\circ)$

$P(X \in E) \le \underline{\lim} P(X_n \in E^\circ) \le \overline{\lim} P(X_n \in \overline{E}) = P(X \in E) \Rightarrow \lim P(X_n \in E) = P(X \in E)$

$\le \underline{\lim} P(X_n \in E) \le \overline{\lim} P(X_n \in E)$

(5) ⇒ (1): WLOG, $f \in [0,1]$.

$E f(X_n) = \int_0^1 P(f(X_n) > t)\,dt$, $E f(X) = \int_0^1 P(f(X) > t)\,dt$.

考虑 $E_t = \{f(x) > t\}$, 记 $T = \{t: P(f = t) > 0\}$, 则 T 可数, 且

$\forall t \in T^c$, E_t 为 X-连续集 $\Rightarrow P(f(X_n) > t) \to P(f(X) > t)$. #

Pf of 连续映射: 由 Portmanteau, 只须 $\forall G$ 开, $P(f(X) \in G) \le \underline{\lim} P(f_n(X_n) \in G)$

注意 $f^{-1}(G) \cap D \subset \bigcup_m \left(\bigcap_{k \ge m} f_k^{-1}(G)\right)^\circ$

$P(f(X) \in G) \le P\left(X \in \bigcup_m \left(\bigcap_{k \ge m} f_k^{-1}(G)\right)^\circ\right)$

$= \sup_m P\left(X \in \left(\bigcap_{k \ge m} f_k^{-1}(G)\right)^\circ\right)$

$\le \sup_m \underline{\lim} P\left(X_n \in \bigcap_{k \ge m} f_k^{-1}(G)\right)$

$\le \sup_m \underline{\lim} P(X_n \in f_n^{-1}(G)) = \underline{\lim} P(f_n(X_n) \in G)$. #

Pf of $\xrightarrow{P}$ 与 $\xrightarrow{d}$ 关系:

① 由 Portmanteau, $\forall X$ 连续集 E, $P(\{X_n \in E\} \,\Delta\, \{X \in E\})$

$= P(X_n \in E, X \in E^c) + P(X_n \in E^c, X \in E)$

$\le 2P(\rho(X_n, X) \ge \varepsilon) + P(\rho(X_n,X) < \varepsilon, X_n \in E, X \in E^c)$

$+ P(\rho(X_n,X) < \varepsilon, X_n \in E^c, X \in E)$

$\le \underset{\downarrow n\to\infty \atop 0}{2P(\rho(X_n,X) \ge \varepsilon)} + \underbrace{P(X \in E^c, \rho(X,E) < \varepsilon) + P(X \in E, \rho(X,E^c) < \varepsilon)}$

$\xrightarrow{\varepsilon \to 0} P(X \in E^c, \rho(X,E) = 0) + P(X \in E, \rho(X,E^c) = 0)$

$\xrightarrow{n\to\infty, \varepsilon \to 0} P(X \in \partial E) = 0$

② (⇐): $P(\rho(X_n, a) \ge \varepsilon) = P(X_n \in A)$, 其中 $A = \{x \in S: \rho(x,a) \ge \varepsilon\}$, 由 $\partial A = \{x \in S, \rho(x,a) = \varepsilon\}$ 且 A 为 a-连续集

由 Portmanteau, $P(X_n \in A) \to P(a \in A) = 0$.

$\therefore X_n \xrightarrow{P} a$. #

[Reference：林正炎] Date .

度量空间的弱收敛.

How to extend $X_n \xrightarrow{d} X$ to $(X_n^1, \dots, X_n^k) \xrightarrow{d} (X^1, \dots, X^k)$?（可由 $E e^{i\sum_{j=1}^k c_j X_n^j} \to E e^{i\sum c_j X^j}$ 证）

$(X_n^1, X_n^2, \dots) \xrightarrow{d} X$

$(X_n^t, t \ge 0) \xrightarrow{d} (X_t, t \ge 0)$

· (S, ρ): 度量空间, $\mathcal{B} = \mathcal{B}(S)$ = 由 S 中所有开集生成的 Borel σ-域. $X: (\Omega, \mathcal{F}) \to (S, \mathcal{B}(S))$

Def. $(\Omega, \mathcal{F}, P)$: prob. space, X: mapping $\Omega \to S$. 称 X 可测, 且称为 (S-值) 随机元,

若 $\forall B \in \mathcal{B}(S)$, 有 $X^{-1}(B) \triangleq \{\omega \in \Omega: X(\omega) \in B\} \in \mathcal{F}$.

Def. 随机元 X 的 (prob) dist: $P_X(B) = P(X^{-1}(B)) = P(X \in B)$

Def. (依分布收敛). $C = C(S) \triangleq \{S$ 上有界连续函数全体$\}$.

若 $\forall f \in C(S)$, $E f(X_n) \to f(X)$,

则称 $X_n \xrightarrow{d} X$, X_n 依分布收敛于 X.

$P_{X_n} \Rightarrow P_X$: $\int_S f(x)\,dP_{X_n}(x) \to \int_S f\,dP_X$, $\forall f \in C(S)$

$\nu_n \Rightarrow \nu$: $\int_S f\,d\nu_n \to \int f\,d\nu$, $\forall f \in C(S)$.

(收敛唯一性)

Prop. 若 $X_n \xrightarrow{d} X$, $X_n \xrightarrow{d} Y$, 则 $X \overset{d}{=} Y$. ($\forall f \in C(S)$, $E f(X_n) \to E f(X)$, $E f(X_n) \to E f(Y) \Rightarrow E f(X) = E f(Y) \Rightarrow X \overset{d}{=} Y$.)

Thm (Portmanteau) $\{X_n\}$, X 为 S-值随机元, 则

(1) $X_n \xrightarrow{d} X$ ($\forall f$ 有界连续, $E f(X_n) \to E f(X)$.)

⇔ (2) $\forall f$ 有界一致连续, $E f(X_n) \to E f(X)$

⇔ (3) $\forall F$ 闭, $P(X \in F) \ge \overline{\lim} P(X_n \in F)$

⇔ (4) $\forall G$ 开, $P(X \in G) \le \underline{\lim} P(X_n \in G)$

⇔ (5) $\forall X$ 连续集 E, $\lim P(X_n \in E) = P(X \in E)$, X 连续集 E: $P_X(\partial E) = 0$, $\partial E = \overline{E} - E^\circ$

Thm. (连续映射) 设度量空间 S, T, 随机元 $X_n \xrightarrow{d} X$, $\{f_n\}$, $f: S \to T$ 可测映射

若 $\exists D \subset S$, $P(X \in D) = 1$, 且 $s_n \to s \in D$ 时 $f_n(s_n) \to f(s)$

则 $f_n(X_n) \xrightarrow{d} f(X)$.

（f a.s. 连续, $f_n(X_n) \xrightarrow{d} f(X)$）

Cor. $X_n \xrightarrow{d} X$ 且 $f: S \to T$ 在 X 上 a.s. 连续, 则 $f(X_n) \xrightarrow{d} f(X)$

Def. $X_n \xrightarrow{P} X$: 若 $\forall \varepsilon > 0$, $P(\rho(X_n, X) \ge \varepsilon) \xrightarrow{n\to\infty} 0$, 则称 X_n 依概率收敛于 X.

Thm. ($\xrightarrow{P}$ 与 $\xrightarrow{d}$ 关系) ① $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$

② $X_n \xrightarrow{P} a \Leftrightarrow X_n \xrightarrow{d} a$.

Pf of thm: $\forall F$ 闭 $\subset S$，记 $F_\varepsilon = \{x: \rho(x,F)\le\varepsilon\}$

$P(X_n\in F) \le P(\rho(Y_{nk},X_n)\ge\varepsilon) + P(Y_{nk}\in F_\varepsilon)$

$\overline{\lim_n} P(X_n\in F) \le \overline{\lim_n} P(\rho(Y_{nk},X_n)\ge\varepsilon) + P(Y_k\in F_\varepsilon)$ (Portmanteau)

令 $k\to\infty$，$\overline{\lim_n} P(X_n\in F) \le \overline{\lim_k} P(Y_k\in F_\varepsilon) \le P(X\in F_\varepsilon)$ (Portmanteau).

令 $\varepsilon\downarrow 0$，则 $F_\varepsilon\downarrow F$，则 $\overline{\lim_n} P(X_n\in F)\le P(X\in F)$ 由 Portmanteau，$X_n\xrightarrow{d}X$. #

Pf of lem1: 设 $X_n\sim P_n$，$X\sim P$. $\forall A_i\in\mathcal{F}$，

$P_n(\bigcup_{i=1}^r A_i) = \sum_i P_n(A_i) - \sum_{i<j} P_n(A_iA_j) + \sum_{i<j<k} P_n(A_iA_jA_k) - \cdots$

$\to P(\bigcup_{i=1}^r A_i)$

$\forall G$ 开，$G=\bigcup_{i=1}^\infty A_i$，$A_i\in\mathcal{F}$. $\forall\varepsilon>0$，$\exists r$，s.t. $P(\bigcup_{i=1}^r A_i) > P(G)-\varepsilon$.

$\therefore P(G)-\varepsilon < \lim P_n(\bigcup_{i=1}^r A_i) \le \underline{\lim} P_n(G)$

由 ε 任意 $P(G)\le \underline{\lim} P_n(G)$，由 Portmanteau，$X_n\xrightarrow{d}X$. #

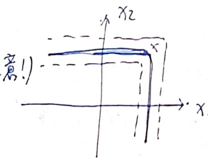
Pf of Lem2: 只须证 $\forall G$ 开，可写为 $\mathcal{F}$ 中可数并，从而用 lem1 即证.

$\forall x\in G$，$\exists\varepsilon>0$，s.t. $B(x,\varepsilon)\subset G$.

$\exists A_x\in\mathcal{F}$，s.t. $x\in A_x^\circ\subset A_x\subset B(x,\varepsilon)\subset G$.

$\therefore G=\bigcup_{x\in G} A_x^\circ$. 由 S 可分，G 有可数覆盖，即 $G=\bigcup_{i=1}^\infty A_{x_i}^\circ = \bigcup_{i=1}^\infty A_{x_i}$. #

Pf of Thm ($\mathbb{R}^k$ 中 $X_n\xrightarrow{d}X \Leftrightarrow \forall x\in C_F$，$F_n(x)\to F(x)$):



($\Rightarrow$) $\forall x\in C_F$，有 $F(x_n)\to F(x)$，$\forall x_n\to x$ (高维，注意!)

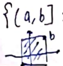
记 $A_x=(-\infty,x] \triangleq \{y\in\mathbb{R}^k, y\le x\}$

$P(X\in\partial A_x)\le P(X\in A_{x+\varepsilon}-A_{x-\varepsilon})$

$\le F(x+\varepsilon)-F(x-\varepsilon)\to 0$ ($\varepsilon\to 0$) $\Rightarrow$ A_x 为 X-连续集.

$\therefore F_n(x)=P(X_n\in A_x) \xrightarrow{X_n\xrightarrow{d}X} P(X\in A_x)=F(x)$.

($\Leftarrow$) 记 $\mathcal{F}=\{(a,b]$：$2k$ 个包含 $(a,b]$ 表面的 $k-1$ 维超平面的 P_X-测度为 0.$\}$



Check lem2：1°. 有限交封闭　2° $\forall x\in\mathbb{R}^k$，$\forall\varepsilon>0$，$\exists x\in(a,b)\subset(a,b]\subset B(x,\varepsilon)$ 且

$2k$ 个包含 $(a,b]$ 表面的 $k-1$ 维超平面的 P_X-测度为 0.

$B_x(\varepsilon)$ (因为测度 $>\frac{1}{n}$ 的超平面至多 n 个，把 $1+2+3+\cdots+n+\cdots$ 个超平面去掉，剩下的测度为 0) 对 $\forall A\in\mathcal{F}$，check $P(X_n\in A)\to P(X\in A)$

(可用 F_n 容斥得) #

Thm. 设 $Y_{nk}\xrightarrow[(n\to\infty)]{d}Y_k\xrightarrow[(k\to\infty)]{d}X$. 若 $\forall\varepsilon>0$，$\overline{\lim_k}\,\overline{\lim_n} P(\rho(Y_{nk},X_n)\ge\varepsilon)=0$，则 $X_n\xrightarrow{d}X$.

Cor (Slusky) $Y_n\xrightarrow{d}X$ 且 $\rho(X_n,Y_n)\xrightarrow{P}0$，则 $X_n\xrightarrow{d}X$.

Thm. (Skorohod 表示定理) 若 S 可分，$X_n\xrightarrow{d}X$，则 $\exists(\tilde\Omega,\tilde{\mathcal{F}},\tilde P)$ 上 S-值随机元 $(Y,Y_n,\ldots)$

s.t. $Y\overset{d}{=}X$，$Y_n\overset{d}{=}X_n$ 且 $Y_n\xrightarrow{a.s.}Y$.

lem1 设可测集类 $\mathcal{F}\subset\mathcal{B}$ 满足：1° $\mathcal{F}$ 有限交封闭　2° S 中 $\forall$ 开集为 $\mathcal{F}$ 中可数并.

若 $\forall A\in\mathcal{F}$，$P(X_n\in A)\to P(X\in A)$，则 $X_n\xrightarrow{d}X$

lem2. 设 S 可分，$\mathcal{F}\subset\mathcal{B}$ 满足：1° $\mathcal{F}$ 有限交封闭　2° $\forall x\in S$，$\forall\varepsilon>0$，$\exists A\in\mathcal{F}$，s.t. $x\in A^\circ\subset A\subset B(x,\varepsilon)$

若 $\forall A\in\mathcal{F}$，$P(X_n\in A)\to P(X\in A)$，则 $X_n\xrightarrow{d}X$.

下面考虑 $\mathbb{R}^k$ 情形.

Thm $X_n\xrightarrow{d}X \Leftrightarrow$ 对 F 的 $\forall$ 连续点 $x\in\mathbb{R}^k$，有 $F_n(x)\to F(x)$

磅 乘积空间 $(\prod_{k=1}^\infty S_k,\rho)$：$\rho(x,y)=\sum_k \frac{1}{2^k}(\rho_k(x^k,y^k)\wedge 1)$ 由 (S_k,ρ_k) 生成.

Prop. $\rho(x,y)\to 0 \Leftrightarrow \rho_k(x^k,y^k)\to 0$，$\forall k$. 与欧氏度量等价 $(\sqrt{\sum(x^k-y^k)^2})$

Thm. 若 S_k 可分，则 $\mathcal{B}(S)=\prod_{k\in T}\mathcal{B}(S_k)\triangleq\sigma(B_1\times\cdots\times B_k\times\prod_{n>k}S_n)$

Pf: 令 $\pi_k: S\to S_k$，$x\mapsto x^k$，则 π_k 连续 $\Rightarrow$ 可测 $\Rightarrow$ $\forall B_k\in\mathcal{B}(S_k)$，$\pi_k^{-1}(B_k)\in\mathcal{B}(S)\Rightarrow B_k\times\prod_{n\ne k}S_n\in\mathcal{B}(S)$

$\therefore\prod_{k\in T}\mathcal{B}(S_k)\subset\mathcal{B}(S)$

设 $\forall G\overset{开}{\subset}(S,\rho)$，$\forall x\in G$，$\exists r$，s.t. $B(x,r)\subset G$

记 $A_x=B(x^1,\frac{r}{2})\times\cdots\times B(x^{n_0},\frac{r}{2})\times\prod_{k>n_0}S_k$，其中 $\frac{1}{2^{n_0}}\le\frac{r}{2}$

则 $\forall y\in A_x$，$\rho(y,x)\le\sum_{k\le n_0}\frac{1}{2^k}\cdot\frac{r}{2}+\frac{r}{2}<r$ $\Rightarrow A_x\subset B(x,r)\subset G$

由 S_k 可分 $\Rightarrow S$ 可分，$G\subset\bigcup_{x\in G}A_x\Rightarrow G\subset\bigcup_{x\in 可数集}A_x$　$\therefore\mathcal{B}(S)\subset\prod_{k\in T}\mathcal{B}(S_k)$. #

(因 $\mathcal{B}(S)=\sigma(\{G\})$，而 $\{G\}$ 为 π-类，$\{G\}\subset\prod_{k\in T}\mathcal{B}(S_k)$.)

Thm. $\forall S_k$ 可分，X，X_n：$\prod_{k=1}^\infty S_k$-值随机元 ($\Leftrightarrow\forall k$，X^k，X_n^k 是 S_k-值随机元)

则 $X_n\xrightarrow{d}X \Leftrightarrow (X_n^1,X_n^2,\cdots,X_n^m)\xrightarrow{d}(X^1,X^2,\cdots,X^m)$，$\forall m\in\mathbb{N}$.

Pf: ($\Rightarrow$) 令 $\pi_{1\cdots m}: x\mapsto(x^1,\cdots,x^m)$，则它连续，由连续映射定理得证

($\Leftarrow$) $\forall a_k\in S_k$，$\underbrace{(X_n^1,\cdots,X_n^m,a_{m+1},\cdots)}_{=Y_{n,m}}\xrightarrow{d}\underbrace{(X^1,\cdots,X^m,a_{m+1},\cdots)}_{Y_m}$

则 $\rho(X,Y_m)\le\frac{1}{2^m}$，$\rho(X_n,Y_{n,m})\le\frac{1}{2^m}$，$Y_m\xrightarrow{P}X$

$\forall\varepsilon>0$，$\overline{\lim_k}\,\overline{\lim_n}P(\rho(Y_{nk},X_n)\ge\varepsilon)=0$，由最上面的 Thm，$X_n\xrightarrow{d}X$. #

Pf of Donsker: 由上页定理，只须证如下两点

① 有限维分布收敛：$\forall 0\le t_1<\cdots<t_k\le 1$,

$$\frac{1}{\sqrt{n}\sigma}\left(S_{[nt_1]},\cdots,S_{[nt_k]}\right)$$

$$=\frac{1}{\sqrt{n}\sigma}\left(S_{[nt_1]},S_{[nt_2]}-S_{[nt_1]},\cdots,S_{[nt_k]}-S_{[nt_{k-1}]}\right)\begin{pmatrix}1&\cdots&1\\&\ddots&\vdots\\0&&1\end{pmatrix}$$

$$\xrightarrow{d}\left(B_{t_1},B_{t_2}-B_{t_1},\cdots,B_{t_k}-B_{t_{k-1}}\right)\begin{pmatrix}1&\cdots&1\\&\ddots&\vdots\\0&&1\end{pmatrix}=(B_{t_1},\cdots,B_{t_k}).$$

($\because \frac{S_{[nt_1]}}{\sqrt{[nt_1]}\sigma}\cdot\frac{\sqrt{[nt_1]}}{\sqrt{n}}\xrightarrow{d}N(0,1)\sqrt{t_1}\overset{d}{=}B_{t_1}$，且有

Lem. $X_n^1,\cdots,X_n^k$ 独立，$X_n^i\xrightarrow{d}X_\infty^i$，则 $(X_n^1,\cdots,X_n^k)\xrightarrow{d}(X_\infty^1,\cdots,X_\infty^k)$)

(证明可用 ch.f.)

② 胎紧性.

(Levy): $\forall\varepsilon>0$, $P\left(\max\limits_{1\le k\le n}|S_k+\mathrm{med}(S_n-S_k)|\ge\varepsilon\right)\le 2P(|S_n|\ge\varepsilon)$

$\because P\left(|S_n-S_k|\ge\sqrt{2Var(S_n-S_k)}\right)\le\frac{1}{2}$

$\therefore |\mathrm{med}(S_n-S_k)|\le\sqrt{2Var(S_n-S_k)}\le\sqrt{2n}\sigma$

$\therefore \forall\lambda>2$, $P\left(\max\limits_{1\le k\le n}|S_k|\ge\lambda\sqrt{n}\sigma\right)\le 2P\left(|S_n|\ge(\lambda-\sqrt{2})\sqrt{n}\sigma\right)$

$$\to 2P\left(|N(0,1)|\ge\lambda-\sqrt{2}\right)\le C\frac{1}{\lambda-\sqrt{2}}e^{-\frac{(\lambda-\sqrt{2})^2}{2}}$$

$|X_n(t)-X_n(s)|\le\frac{1}{\sqrt{n}\sigma}\max\{|S_{[nt]}-S_{[ns]}|,|S_{[nt]}-S_{[ns]+1}|,|S_{[nt]+1}-S_{[ns]}|,|S_{[nt]+1}-S_{[ns]+1}|\}$

$\therefore P\left(\sup\limits_{t\le s\le t+\delta}|X_n(t)-X_n(s)|\ge\varepsilon\right)\le P\left(2\sup\limits_{[nt]\le k\le[n(t+\delta)]+1}|S_k-S_{[nt]}|\ge\sqrt{n}\sigma\varepsilon\right)\le P\left(2\sup\limits_{[0,[n\delta]+2]}|S_k|\ge\sqrt{n}\sigma\varepsilon\right)$

$\therefore \frac{1}{\delta}\varlimsup\limits_{n\to\infty}\sup\limits_t P\left(\sup\limits_{t\le s\le t+\delta}|X_n(t)-X_n(s)|\ge\varepsilon\right)$

$\le\frac{1}{\delta}\varlimsup\limits_{n\to\infty}P\left(2\sup\limits_{0\le k\le[n\delta]+2}|S_k|\ge\sqrt{n}\sigma\varepsilon\right)$

$\le\frac{1}{\delta}C\cdot e^{-\frac{1}{2}\left(\frac{\sigma\varepsilon}{2\sqrt{\delta}}-\sqrt{2}\right)^2}\to 0\ (\delta\to 0)$ #

Pf of $\frac{\max S_i}{\sqrt{n}\sigma}\xrightarrow{d}|N|$:

记 $h: C[0,1]\to\mathbb{R}$, $x\mapsto\sup\limits_{t\in[0,1]}x(t)$，则 h 连续

$\therefore \frac{\max S_i}{\sqrt{n}\sigma}=h(X_n)\xrightarrow[\text{Donsker}]{d}h(B)=\sup\limits_{t\in[0,1]}B_t$

下证 $\sup\limits_{[0,1]}B_t\overset{d}{=}|N|$ 即可.

设 $\xi_i\sim$ sym Ber, $\tau=\inf\{i\ge 1: S_i\ge k\}$，求 $\max\limits_{1\le i\le n}S_i$ 分布

则 $P\left(\max\limits_{1\le i\le n}S_i\ge k\right)=P(\tau\le n)=P(S_n\ge k)+\sum\limits_{l=1}^{n}P(S_n<k,\tau=l)$

$\overset{\text{reflection}}{\underset{(*)}{=}}P(S_n\ge k)+\sum\limits_{l=1}^{n}P(S_n>k,\tau=l)=P(S_n\ge k)+P(S_n>k)$

$\therefore P\left(\max\limits_{1\le i\le n}S_i\ge x\sqrt{n}\sigma\right)=2P(S_n>x\sqrt{n}\sigma)+P(S_n=x\sqrt{n}\sigma)\longrightarrow 2P(N(0,1)>x)=P(|N|>x)$ #

(*)的数学证明: $P(S_n<k,\tau=l)=P(S_l=k,S_n-S_l<0,\tau=l)=P(S_l=k,\tau=l)P(S_n-S_l<0)=P(S_l=k,\tau=l)P(S_n-S_l>0)=P(S_n>k,\tau=l)$.

Thm $X,X_1,X_2,\cdots$: $C[0,1]$-值随机元，则 $X_n\xrightarrow{d}X\Leftrightarrow\begin{cases}X_n\xrightarrow{fd}X\\ \lim\limits_{h\to 0}\varlimsup\limits_{n\to\infty}E(w(X_n,h)\wedge 1)=0\quad(*)\end{cases}$

其中 $w(x,h)$ 为 x 的连续模，

$w(x,h)\triangleq\sup\limits_{|s-t|\le h}|x(s)-x(t)|$, $\forall x\in C[0,1]$, $h>0$.

Rmk. 1. "$\varlimsup$"可换成"sup"

2. $w(X_n,h)$ 为 r.v. ($\because w(x,h)$ 关于 x 连续)

3. $(*)\Leftrightarrow\forall\varepsilon>0,\forall\eta>0,\exists\delta(\varepsilon,\eta),\exists n_0(\varepsilon,\eta)$, s.t. $P(w(X_n,\delta)\ge\varepsilon)\le\eta$, $\forall n\ge n_0$.

$\Leftarrow\ \forall\varepsilon>0,\eta>0,\exists\delta(\varepsilon,\eta),n_0(\varepsilon,\eta)$, s.t. $\forall 0\le t\le 1$, $\forall n\ge n_0$, $P\left(\sup\limits_{t\le s\le t+\delta}|X_n(t)-X_n(s)|\ge\varepsilon\right)\le\delta\eta$

$\Leftrightarrow\forall\varepsilon>0$, $\lim\limits_{\delta\to 0}\frac{1}{\delta}\varlimsup\limits_{n\to\infty}\sup\limits_t P\left(\sup\limits_{t\le s\le t+\delta}|X_n(t)-X_n(s)|\ge\varepsilon\right)=0$.

· BM

Def. (标准Brown运动). $[0,1]$ 上随机过程 $\{B_t\}_{[0,1]}$，若

① $B_0=0$, B_t 独立平稳增量

② $B_t\sim N(0,t)$

③ B_t 连续样本路径

称 $\{B_t\}$ 为标准BM.

Def. (Gauss过程) $\forall$ 有限维分布都是Gaussian，分布由均值、Cov 决定.

Rmk 1. BM 是均值0，$Cov(X_s,X_t)=s\wedge t$ 的 Gauss 过程

2. $\{B_{t+s}-B_s\}_{t\ge 0}$, $\{\frac{1}{c}B_{c^2t}\}_{t\ge 0}$ 都为BM

· 设 $\xi_1,\xi_2,\cdots$ i.i.d. $E\xi=0$, $E\xi^2=\sigma^2$, $X_n(t)\triangleq\frac{1}{\sqrt{n}\sigma}\left(\sum\limits_{k=1}^{[nt]}\xi_k+(nt-[nt])\xi_{[nt]+1}\right)=\frac{1}{\sqrt{n}\sigma}\left(S_{[nt]}+(nt-[nt])\xi_{[nt]+1}\right)$

Thm. (Donsker) $X_n\xrightarrow{d}B$，其中 B 是 $[0,1]$ 上标准BM.

Thm $\frac{\max\limits_{1\le i\le n}S_i}{\sqrt{n}\sigma}\xrightarrow{d}|N(0,1)|$

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Pf of lem：令 $\tau=\sup\{t: [0,t]$ 可分成有限个满足 $w_x([t_{i-1},t_i))$ 的小区间$\}$，只须证 $\tau=1$.

首先，$\tau>0$. 因为在 $t=0$ 处右连续.

若 $\tau<1$，则 $[0,\tau]$ 可分成有限个满足的小区间，由 $x(\tau)$ 右连续，则 $\exists\delta$，$[0,\tau+\delta]$ 也可分成 …… 的小区间，矛盾！

#

$\therefore \tau=1$.

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· $D[0,1]$：r.c.l.l 函数全体，即 $\forall x\in D[0,1]$，$\forall t\in[0,1]$，$t_n\downarrow t$，$x(t_n)\to x(t)$

$t_n\uparrow t$，$\lim x(t_n)=x(t-)$ 存在有限

· $x\in C[0,1] \Leftrightarrow w(x,h)=\sup\limits_{|s-t|\le h}|x(s)-x(t)|\to 0$，$h\to 0$.

Lem. $\forall x\in D$，$\forall\varepsilon>0$，$\exists\, 0=t_0<\cdots<t_r=1$，s.t. $w_x([t_{i-1},t_i))=\sup\limits_{s,t\in[t_{i-1},t_i)}|x(s)-x(t)|<\varepsilon$，$\forall i=1,\cdots,r$

$\Rightarrow$ (1) 至多可数个 t，s.t. $\Delta t=|x(t)-x(t-)|>\varepsilon$

(2) 至多可数个 t，s.t. $\Delta t>0$ $\Rightarrow$ 至多可数个间断点

(3) $\sup\limits_{0\le t\le 1}|x(t)|<\infty$

(4) x 是 Borel 可测的 (阶梯逼近：)

· 另一刻画：定义 $w_x'(\delta)\triangleq\inf\limits_{\{t_i\}}\max\limits_{1\le i\le r} w_x([t_{i-1},t_i))$，其中 $\{t_i\}$ 满足 $0=t_0<\cdots<t_r=1$，$t_i-t_{i-1}>\delta$.

则 $x\in D[0,1]\Leftrightarrow w_x'(\delta)\to 0\ (\delta\to 0)$

Def 称 $\lambda:[0,1]\to[0,1]$ 为时间变换，若它单增. 双射，$\lambda(0)=0$，$\lambda(1)=1$，连续.

Thm. (Skorohod J_1 拓扑) $\exists$ $D[0,1]$ 上的完备可分度量 d，s.t. $d(x_n,x)\to 0$

$\Leftrightarrow \exists$ $[0,1]$ 上时间变换 λ_n，s.t. $\sup\limits_{0\le s\le 1}|\lambda_n(s)-s|+\sup\limits_{0\le s\le 1}|x_n(\lambda_n(s))-x(s)|\to 0$

Rmk. $\mathcal{J}(x_n,x)=\sup|\lambda_n(s)-s|+\sup|x_n(\lambda_n(s))-x(s)|$ 定义的度量不完备可分.

(从而定义距离：$\Lambda=\{\lambda$ 为时间变换：$\|\lambda\|=\sup\limits_{s\ne t\in[0,1]}\left|\log\frac{\lambda(s)-\lambda(t)}{s-t}\right|<\infty\}$

$d(x,y)=\inf\{\varepsilon: \exists\lambda\in\Lambda$，s.t. $\|\lambda\|\le\varepsilon$，$\sup\limits_{0\le t\le 1}|x(t)-y(\lambda(t))|\le\varepsilon\}$.).

Rmk. $d(x,y)\le\rho(x,y)$，反之不对.

且 $x\in C[0,1]$ 时，$d(x_n,x)\to 0\Leftrightarrow\rho(x_n,x)\to 0$.

$\because$：($\Rightarrow$) $\sup\limits_{t\in[0,1]}|x_n(t)-x(t)|=\sup\limits_{[0,1]}|x_n(\lambda_n(t))-x(\lambda_n(t))|\le\sup|x_n(\lambda_n(t))-x(t)|\ (\le d(x_n,x))$

$+\sup|x(t)-x(\lambda_n(t))|\ (\to 0)$

Proposition 设 X_n, X 是 $C[0,1]$-值随机元，则在 $(D[0,1],d)$ 上 $X_n\xrightarrow{d}X$，

$\Leftrightarrow$ 在 $(D[0,1],\rho)$ 上 $X_n\xrightarrow{d}X$，其中 d 为 $D[0,1]$ 上 Skorohod J_1 拓扑

ρ 为 $D[0,1]$ 上一致拓扑.

Proposition. 投影 π_t 不一定连续 (在 Skorohod J_1 拓扑下)，但 π_0. π_1 在 $D[0,1]$ 上连续.

$\forall 0<t<1$，π_t 连续 $\Leftrightarrow$ x 在 t 连续.

但 $\forall t_1,\cdots,t_n$，$\pi_{t_1,\cdots,t_n}$ 可测.

Thm 设 $X, X_1, X_2,\cdots$ 为 $(D[0,1],d)$ 随机元，则 $X_n\xrightarrow{d}X \Leftrightarrow$ ① 在稠密 $T=\{t\ge 0: \Delta X(t)=|X(t-)-X(t)|=0, a.s.\}$ 上有 $X_n\xrightarrow{fd}X$，且

+② $\lim\limits_{\delta\to 0}\varlimsup\limits_{n\to\infty}E(w'(X_n,\delta)\wedge 1)=0$. $(*)$

②：$\forall\varepsilon>0$，$\eta>0$，$\exists\,\delta(\varepsilon,\eta)\in(0,1)$. $n_0(\varepsilon,\eta)\in\mathbb{N}$，s.t. $\forall n\ge n_0$，$P(w'(X_n,\delta)\ge\varepsilon)\le\eta$

($\Leftarrow$) $\underbrace{\lim\limits_{\delta\to 0}\varlimsup\limits_{n\to\infty}E(w(X_n,\delta)\wedge 1)=0}_{(**)}$，由 $w(X_n,\delta)\ge w'(X_n,\delta)$

若 $(**)$ 成立且 $X_n\xrightarrow{d}X$，则 $P(X\in C)=1$

Pf of Thm: $X_t^n \triangleq \frac{S_{[nt]} + (nt-[nt])\xi_{[nt]+1}}{\sqrt{n}\sigma}$,

由 Donsker, $X_n \xrightarrow{d} B$ on $(C[0,1], \rho)$

$\Rightarrow X_n \xrightarrow{d} B$ on $(D[0,1], d)$

又 $d(X^n, \frac{S_{[nt]}}{\sqrt{n}\sigma}) \le \rho(X^n, \frac{S_{[nt]}}{\sqrt{n}\sigma}) \le \frac{\max|\xi_i|}{\sqrt{n}\sigma} \xrightarrow{a.s.} 0$

由 Slusky, $\frac{S_{[nt]}}{\sqrt{n}\sigma} \xrightarrow{d} B$. #.

Pf of Rmk 1. $\{S\vee T \le t\} = \{S\le t\}\cap\{T\le t\} \in \mathcal{F}_t$

$\{S\wedge T\le t\} = \{S\le t\}\cup\{T\le t\}\in\mathcal{F}_t$

$\{S+T>t\} = \{T=0, S>t\}\cup\{0<T\le t, S>t-T\}\cup\{T>t, S=0\}\cup\{T>t, S>0\} \in \mathcal{F}_t$ #

Pf of Prop: (i) 由定义

(ii) $\forall A\in\mathcal{F}_S$, $A\cap\{T\le t\} = \underbrace{A\cap\{S\le t\}}_{\in\mathcal{F}_t}\cap\underbrace{\{T\le t\}}_{\in\mathcal{F}_t} \in\mathcal{F}_t$.

(iii) 显然 $\mathcal{F}_S\cap\mathcal{F}_T \supset \mathcal{F}_{S\wedge T}$ (由(ii).)

另一方面, $\forall A\in\mathcal{F}_S\cap\mathcal{F}_T$, $A\cap\{S\wedge T\le t\} = (A\cap\{S\le t\})\cup(A\cap\{T\le t\})\in\mathcal{F}_t$.

(iv) 易证

(v) ① $\{S<T\}\cap\{S\le t\} = \bigcup_{r\in\mathbb{Q}, r\le t}\underbrace{\{S\le r<T\}}_{\in\mathcal{F}_r} \in\mathcal{F}_t$. $\Rightarrow \{S<T\}\in\mathcal{F}_S$.

且 $\{S<T\}\cap\{T\le t\} = \bigcup_{r\in\mathbb{Q}, r<t}\{S\le r<T\le t\}\in\mathcal{F}_t$ $\Rightarrow \{S<T\}\in\mathcal{F}_T$ $\Big\}\Rightarrow \{S<T\}\in\mathcal{F}_S\cap\mathcal{F}_T = \mathcal{F}_{S\wedge T}$.

② $\{S\le T\} = \{T<S\}^c\in\mathcal{F}_{S\wedge T}$ ③ $\{S=T\} = \{S\le T\}\setminus\{S<T\}\in\mathcal{F}_{S\wedge T}$

"且" ① 只须证 "$\subset$":

$\forall C\in\mathcal{F}_{S\vee T}$, $C = A\cup B$, $A\in\mathcal{F}_S$, $B\in\mathcal{F}_T$, $A\cap B=\phi$

$\therefore C\cap\{S\le T\} = (A\cap\{S\le T\})\cup\underbrace{(B\cap\{S\le T\})}_{\in\mathcal{F}_T\cap\{S\le T\}}$

而 (*) $A\cap\{S<T\}\cap\{T\le t\} = A\cap(\bigcup_{r\in\mathbb{Q}, r<t}\{S\le r<T\le t\})\in\mathcal{F}_t \Rightarrow A\cap\{S<T\}\in\mathcal{F}_t$)

$A\cap\{S\le T\}\cap\{T\le t\} = \underbrace{A\cap\{S\le t\}}_{\in\mathcal{F}_t}\cap\underbrace{\{S\le T\}\cap\{T\le t\}}_{\in\mathcal{F}_t\ (*)}\in\mathcal{F}_t \Rightarrow A\cap\{S\le T\}\in\mathcal{F}_t$

$\Rightarrow C\cap\{S\le T\}\in\mathcal{F}_T\cap\{S\le T\}$.

~~(*) 用在平移? 等的证明中, 同理.~~ #

Eg. 设 S, T 为停时, $X\in\mathcal{F}_T$ 且 $X\in L^1$, 则 $E(X|\mathcal{F}_S) = E(X|\mathcal{F}_{S\wedge T})$, a.s.

Pf: on $\{S\le T\}$, 显然.

on $\{S>T\}$, $E(X|\mathcal{F}_S) = E(X|\mathcal{F}_{S\vee T}) = X$ a.s. 而 $E(X|\mathcal{F}_{S\wedge T}) = E(X|\mathcal{F}_T) = X$ a.s. #

Cor. $E(E(Y|\mathcal{F}_T)|\mathcal{F}_S) = E(E(Y|\mathcal{F}_T)|\mathcal{F}_{S\wedge T}) = E(Y|\mathcal{F}_{S\wedge T})$ $\Rightarrow E(E(Y|\mathcal{F}_T)|\mathcal{F}_S) = E(E(Y|\mathcal{F}_S)|\mathcal{F}_T)$.

(Rmk: 当 $\mathcal{G}$ 与 $\mathcal{H}$ 无包含关系, $E(E(X|\mathcal{G})|\mathcal{H}) \ne E(E(X|\mathcal{H})|\mathcal{G})$, 但停时却可以.)

Thm. ξ_1, ξ_2 i.i.d. $E\xi_i = 0$, $E\xi_i^2 = \sigma^2$, B: $[0,1]$ 上标准 BM, 则在 $(D[0,1], d)$ 上有

$\frac{S_{[nt]}}{\sqrt{n}\sigma} \xrightarrow{d} B$, d 是 Skorohod J_1 拓扑.

Proposition X_i i.i.d. 若有稳定分布 $\frac{S_n - a_n}{b_n} \xrightarrow{d} Y$, 则 $\frac{S_{[nt]} - a_{[nt]}}{b_n} \xrightarrow{d} Y(t)$ (稳定过程)

$Y(t)$ 为独立平稳增量过程, $Y(1) \overset{d}{=} Y$.

[Reference] Chris Heyde & Hall Petar, Martingale Limit Theorem

· 鞅: $\{X_n, n\ge 0\}$ $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in T})$ $T = \{0,1,2,\cdots\}$

$\{X_t, t\ge 0\}$ $T = [0,\infty)$ $\mathcal{F}_s\subset\mathcal{F}_t$ $(s\le t)$: σ-域流

Def. (停时) 称 $T: \Omega\to T\cup\{\infty\}$ 为 $(\mathcal{F}_t)$ 的停时, 若 $\forall t\in T$, $\{T\le t\}\in\mathcal{F}_t$.

Rmk 1. 若 S, T 为停时, 则 $S\vee T$, $S\wedge T$, $S+T$ 均为停时.

2. 若 T_n 为停时, 则 $\bigvee_n T_n$ 为停时, 但 $\bigwedge_n T_n$ 不一定.

Def. ($\mathcal{F}_T$) 设 T 为停时, 则 T 前事件 σ-域 $\mathcal{F}_T = \{A\in\mathcal{F}_\infty : A\cap\{T\le t\}\in\mathcal{F}_t, \forall t\in T\}$.

其中 $\mathcal{F}_\infty = \bigvee_t \mathcal{F}_t$ $(=\sigma\{\mathcal{F}_t, t\in T\})$

Prop. 设 S, T 为停时, 则

(i) $T\in\mathcal{F}_T$

(ii) $S\le T \Rightarrow \mathcal{F}_S\subset\mathcal{F}_T$

(iii) $\mathcal{F}_S\cap\mathcal{F}_T = \mathcal{F}_{S\wedge T}$

(iv) $\mathcal{F}_S\vee\mathcal{F}_T = \mathcal{F}_{S\vee T} \triangleq \{A\cup B: A\in\mathcal{F}_S, B\in\mathcal{F}_T, A\cap B = \phi\}$.

(v) $\{S\le T\}, \{S<T\}, \{S=T\}\in\mathcal{F}_{S\wedge T}$, 且

$\mathcal{F}_{S\vee T}\cap\{S\le T\} = \mathcal{F}_T\cap\{S\le T\}$

$\mathcal{F}_{S\vee T}\cap\{S<T\} = \mathcal{F}_T\cap\{S<T\}$.

$\mathcal{F}_{S\wedge T}\cap\{S\le T\} = \mathcal{F}_S\cap\{S\le T\}$

$\mathcal{F}_{S\wedge T}\cap\{S<T\} = \mathcal{F}_S\cap\{S<T\}$

$\mathcal{F}_{S\vee T}\cap\{S=T\} = \mathcal{F}_{S\wedge T}\cap\{S=T\} = \mathcal{F}_S\cap\{S=T\} = \mathcal{F}_T\cap\{S=T\}$.

Thm. (条件期望) 设 σ-域 $\mathcal{G}, \mathcal{H}\subset\mathcal{F}$, X, Y 为可积 r.v., $A\in\mathcal{G}\cap\mathcal{H}$.

若 $\mathcal{G}\cap A = \mathcal{H}\cap A$ 且 $X=Y$, a.s. on A, 则 $E(X|\mathcal{G}) = E(Y|\mathcal{H})$, a.s. on A.

Eg. 记 $X_n = E(X|\mathcal{F}_n)$, 则 $\forall$ 有限停时 T, $X_T = E(X|\mathcal{F}_T)$

Pf: $\forall n\in T$, 记 $A = \{T=n\}$, 则在 $\{T=n\}$ 上, a.s. 有 $X_T = X_n \overset{由上一Thm.}{=} E(X|\mathcal{F}_T) = E(X|\mathcal{F}_n)$

$\therefore$ 在 $\bigcup_n\{T=n\}$ 上 a.s. 有 $X_T = E(X|\mathcal{F}_T)$. #

Date.

Eg. ① X_1, X_2 独立, $EX_i=0$, 则 $(S_n=\sum_{i=1}^n X_i, \mathcal{F}^0_{X,n})$ 为鞅. 若 $EX_i^2=\sigma^2<\infty$, 则 $(S_n^2-n\sigma^2, \mathcal{F}^0_{X,n})$ 为鞅

② $X_1, X_2, \cdots$ 独立正值 $EX_i=1$, 则 $(\prod_{i=1}^n X_i, \mathcal{F}^0_{X,n})$ 为鞅.

$X_1, X_2, \cdots$ 独立, 且 $\exists \lambda>0$, s.t. $Ee^{\lambda X_i}<\infty$, 则 $\left(\frac{e^{\lambda S_n}}{Ee^{\lambda S_n}}, \mathcal{F}^0_{X,n}\right)$ 为鞅.

③ $Y_0, Y_1, \cdots$ i.i.d., f_0, f_1 为密度函数, 则 $Y_n\sim f_0$ 时, $\left(\frac{f_1(Y_0)\cdots f_1(Y_n)}{f_0(Y_0)\cdots f_0(Y_n)}, \mathcal{F}^0_{Y,n}\right)$ 为鞅

④ (罐子模型) $n=0$ 时 一红一绿, 每次抽一个, 放回两个同色的球,

X_n: n 时刻红球占比, 则 X_n 为鞅.

(证明设 $Y_n=(n+2)X_n$: n 时刻红球数)

⑤ $\forall(\mathcal{F}_n)$ 适应过程 (Y_n) (即 $Y_n\in\mathcal{F}_n$). $\left(\sum_{i=1}^n (Y_i-E(Y_i|\mathcal{F}_{i-1})), \mathcal{F}_n\right)$ 为鞅.

$\left(\sum_{i=1}^n (Y_i-E(Y_i|\mathcal{F}_{i-1}))\, a_i(Y_1,\cdots,Y_{i-1}), \mathcal{F}_n\right)$ 也为鞅

Pf of prop: 即证 $\forall\varepsilon>0$, $\exists M$, s.t. $E\left(E(Y|\mathcal{G})1_{\{|E(Y|\mathcal{G})|>M\}}\right)<\varepsilon$, $\forall\mathcal{G}$

$\because \forall\varepsilon>0, \exists\delta>0$, s.t. $\forall P(A)\le\delta$, $E|Y|1_A<\varepsilon$.

且 $P(|E(Y|\mathcal{G})|>M)\le\frac{E|Y|}{M}$ $\qquad$ aim = LHS.

$\therefore$ 取 $M=\frac{E|Y|}{\delta}$ 时, $E\left(|Y|1_{\{|E(Y|\mathcal{G})|>M\}}\right)<\varepsilon$ $\qquad$ #

Pf of Doob 分解: $A_0=0$, $A_n=\sum_{i=1}^n E(X_i-X_{i-1}|\mathcal{F}_{i-1})\in\mathcal{F}_{n-1}$ 可预报.

$M_n=X_n-A_n=X_0+\sum_{i=1}^n (X_i-E(X_i|\mathcal{F}_{i-1}))$ 为鞅.

X 为下鞅 $\Leftrightarrow A_{n+1}-A_n=E(X_{n+1}|\mathcal{F}_n)-X_n\ge 0$, a.s. $\qquad$ #

Eg 对 $X_n=\sum_{k=1}^n 1_{B_k}$, $B_k\in\mathcal{F}_k$ 作 Doob 分解:

$A_n=\sum_{k=1}^n P(B_k|\mathcal{F}_{k-1})$, $M_n=\sum_{k=1}^n (1_{B_k}-P(B_k|\mathcal{F}_{k-1}))$ $\qquad$ #.

Rmk. $\{\sum_{k=1}^\infty I_{B_k}=\infty\}=\{B_n, \text{i.o.}\}\overset{a.s.}{=}\{\sum_{k=1}^\infty P(B_k|\mathcal{F}_{k-1})=\infty\}$.

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Def. 设 $(X_t, t\in T)$ 关于流 $(\mathcal{F}_t, t\in T)$ 适应, 定义 $D_A=\inf\{t\ge 0, X_t\in A\}$ 进入时. (均为停时.)

$T_A=\inf\{t>0, X_t\in A\}$ 首击时

Rmk. T 离散时, D_A, T_A 为停时

连续时, 若 A 开, X 右连, $D_A=T_A$ 不一定为停时

A 闭, X 连续, D_A 为停时.

Def. (鞅) 实值随机过程 $(X_n)_{n\ge0}$, σ-域流 $(\mathcal{F}_n)_{n\ge0}$. 若 $\forall n\ge 0$,

① $E|X_n|<\infty$ $\quad$ ② $X_n\in\mathcal{F}_n$ $\quad$ ③ $E(X_{n+1}|\mathcal{F}_n)=X_n$, a.s.

则称 $(X_n, \mathcal{F}_n, n\ge 0)$ 为鞅. 若③中"="变成"≥"/"≤", 则称为 下/上鞅

- 自然 σ-域流 $\mathcal{F}^0_n=\sigma(X_1,\cdots,X_n)$, 则 $(X_n,\mathcal{F}_n)$ 为鞅 $\Rightarrow (X_n,\mathcal{F}^0_n)$ 为鞅. $\because E(X_{n+1}|\mathcal{F}^0_n)=E[\underbrace{E(X_{n+1}|\mathcal{F}_n)}_{=X_n}|\mathcal{F}^0_n]$
- $\forall Y\in\mathcal{F}_n$, $E(X_{n+1}Y)=E(X_nY)$
- $(X_n,\mathcal{F}_n)$ 下鞅 $\Leftrightarrow (-X_n,\mathcal{F}_n)$ 上鞅.
- ③ $\Leftrightarrow \forall m>n$, $E(X_m|\mathcal{F}_n)=X_n$
- $(X_n,\mathcal{F}_n)$ 下鞅, 则 $EX_n\nearrow$ 且 下鞅 $(X_n,\mathcal{F}_n)$ 为鞅 $\Leftrightarrow EX_n\equiv\text{const}$.
- $Y\in L^1$, $(\mathcal{F}_n)$ σ-域流, 则 $(X_n=E(Y|\mathcal{F}_n), \mathcal{F}_n)$ 为一致可积鞅.

Prop. $\{E(Y|\mathcal{G}), \mathcal{G}$ 为子 σ-域$\}$ 一致可积., 若 $Y\in L^1$.

Thm 若 $(X_n,\mathcal{F}_n)$ 鞅, ϕ: 凸, $E|\phi(X_n)|<\infty$, 则 $(\phi(X_n),\mathcal{F}_n)$ 为下鞅.

若 $(X_n,\mathcal{F}_n)$ 下鞅, ϕ: 递增凸, 且 $|\phi(X_n)|<\infty$, 则 $(\phi(X_n),\mathcal{F}_n)$ 为下鞅. $E(\phi(X_{n+1})|\mathcal{F}_n)\overset{\text{Jensen}}{\ge}\phi(E(X_{n+1}|\mathcal{F}_n))\overset{(③)}{\ge}\phi(X_n)$.

Def (X_n) 关于 $(\mathcal{F}_n)$ 适应, $H=(H_n)$ 可预报 (即 $H_n\in\mathcal{F}_{n-1}$, $n\ge1$.)

$\underset{\text{(也称为鞅变换)}}{\underline{H\text{ 关于 }X\text{ 的随机积分}}}$ $H\cdot X$: $(H\cdot X)_0=H_0X_0$

$(H\cdot X)_n=\sum_{i=1}^n H_i(X_i-X_{i-1})+H_0X_0$, $n\ge1$

Thm. 设 H, X 可积. 若 X 为鞅, 则 $H\cdot X$ 为鞅; $\qquad (H\cdot X)_n\in\mathcal{F}_n$

若 X 为下鞅, H 非负, 则 $H\cdot X$ 为下鞅. $\quad E((H\cdot X)_{n+1}-(H\cdot X)_n|\mathcal{F}_n)=E(H_{n+1}(X_{n+1}-X_n)|\mathcal{F}_n)\underset{(\ge)}{=}0$

Rmk 1. N 为停时, $H_n=1_{\{N\ge n\}}\in\mathcal{F}_{n-1}$ 为可预报过程 $\Rightarrow (X_n)$ 为 (下) 鞅时, $(X_{N\wedge n})$ 为 (下) 鞅.

2. 取 $N=\inf\{n\ge1: |X_n|>M\}$, 则若 $|X_k-X_{k-1}|\le M$, 则 $|X_{N\wedge n}|\le M+\Delta$ 仍为鞅, 且性质较好.

Thm (Doob 分解) 设 (X_n) 是 $(\mathcal{F}_n)$ 适应的, 则 $\exists 1$ (a.s.) 分解: $X_n=M_n+A_n$, s.t.

$(M_n,\mathcal{F}_n)$ 为鞅, (A_n) 可预报且 $A_0=0$.

特别, X 为下鞅 $\Leftrightarrow A$ a.s. 非降.

Pf of upcrossing：令 $Y_n = a + (X_n - a)^+$，则 Y_n 为下鞅，U_n 为 Y_n upcrossing time.

$N_0 = 0$，$N_{2k-1} = \inf\{n \geq N_{2k-2}: Y_n = a\}$，$N_{2k} = \inf\{n \geq N_{2k}; Y_n \geq b\}$

则 N_n 为停时 且 $Y_{N_{2k-1}} = a$，$Y_{N_{2k}} \geq b$.

令 $H_n = \begin{cases} 1, & \exists k, \text{ s.t. } N_{2k-1} < n \leq N_{2k} \\ 0, & \text{else} \end{cases}$，则 $\{H_n = 1\} = \bigcup \{N_{2k-1} < n \leq N_{2k}\} \in \mathcal{F}_{n-1}$ 可预报

$\Rightarrow (H \cdot Y)_n \geq (b-a) U_n$

记 $U_n = 1 - H_n$，则 $Y_n - Y_0 = (H \cdot Y)_n + (U \cdot Y)_n$

$\because (U \cdot Y)_n$ 为下鞅

$\therefore E(U \cdot Y)_n \geq E(U \cdot Y)_0 = 0$

$\therefore (b-a) E U_n \leq E(H \cdot Y)_n \leq E(Y_n - Y_0) = E(X_n - a)^+ - E(X_0 - a)^+$. #

Pf of 鞅收敛：$\forall a < b$，U 表示上穿 $[a,b]$ 次数，则 $U_n \uparrow U$

$$E U_n \leq \frac{E(X_n - a)^+ - E(X_0 - a)^+}{b-a} \leq \frac{E X_n^+ + |a|}{b-a}$$

$\Rightarrow EU \leq \dfrac{\sup E X_n^+ + |a|}{b-a} < \infty \quad \Rightarrow U < \infty$, a.s.

$\therefore P(\underline{\lim}\, X_n < \overline{\lim}\, X_n)$

$= P(\bigcup_{a,b \in \mathbb{Q}} \{\underline{\lim}\, X_n < a < b < \overline{\lim}\, X_n\})$

$\because U_{a,b} < \infty$, a.s. $\therefore P(\underline{\lim}\, X_n < a < b < \overline{\lim}\, X_n) = 0$. $\therefore X_n$ a.s. 有极限 X

且 $E|X| = E \lim |X_n| \leq \underline{\lim}\, E|X_n| \leq \sup E|X_n| < \infty$. #

Pf of Thm(一致可积)：(1) $\Rightarrow$ (2)：$\sup E|X_n| < \infty \Rightarrow X_n \xrightarrow{a.s.} X$ + 一致可积 $\Rightarrow X_n \xrightarrow{L^1} X$

(2) $\Rightarrow$ (3) (✓)

(3) $\Rightarrow$ (1)：自证.

若 (X_n) 为鞅，(4) $\Rightarrow$ (1) (✓)

(3) $\Rightarrow$ (4)：设 $X_n \xrightarrow{L^1} X$，下证 $X_n = E(X|\mathcal{F}_n)$，即 $\forall A \in \mathcal{F}_n$，$E(X_n 1_A) = E(X 1_A)$.

由 X_n 鞅，$E(X_n 1_A) = E(X_m 1_A)$，$\forall m > n$

$\therefore E(X_n 1_A) = E(X 1_A) \leftarrow m \to \infty$ #

Pf of Thm：$X_n = E(X|\mathcal{F}_n)$ 一致可积鞅 $\Rightarrow \exists \eta$, s.t. $E(X|\mathcal{F}_n) \xrightarrow{a.s. \& L^1} \eta$ 且 $\eta \in L^1$，$\eta \in \mathcal{F}_\infty$.

下证 $\eta = E(X|\mathcal{F}_\infty)$, a.s. 即 $\int_A \eta = \int_A X$，$\forall A \in \mathcal{F}_\infty$.

由条件期望，$\forall A \in \mathcal{F}_n$，$\int_A X = \int_A E(X|\mathcal{F}_m) dp$，$\forall m \geq n$，而右边 $\xrightarrow{DCT} \int_A \eta\, dp$ 由 π-λ 定理得证. #

鞅收敛定理.

Thm (下鞅上穿不等式) $-\infty < a < b < +\infty$，$(X_n)$ 下鞅 (sub). 记 U_n 为 $X_0, \cdots, X_n$ 从下往上穿过 $[a,b]$ 次数，

则 $(b-a) E U_n \leq E(X_n - a)^+ - E(X_0 - a)^+$.

Thm (鞅收敛) (X_n)：下鞅，$\sup_n E X_n^+ < \infty$，则 X_n a.s. 收敛 $\to X$，且 $E|X| < \infty$.

(在下鞅条件下，$\sup E X_n^+ < \infty \Leftrightarrow \sup E|X_n| < \infty$)

Cor (X_n) 非负上鞅，则 $X_n \to X$ a.s. 且 $EX \leq EX_0$

c.e.g.1. $\sup E X_n^+ < \infty \not\Rightarrow X_n \xrightarrow{L^1} X$:

SRW，$S_0 = 1$，$T_0 < \infty$, a.s. 则 $S_{T_0 \wedge n}$ 是鞅 $\xrightarrow{a.s.} 0$. 但 $E S_{T_0 \wedge n} = 1 \not\xrightarrow{L^1} 0$.

- $X_n \xrightarrow{a.s.} X$ 加 X_n 一致可积，即有 L^1.

又 X_n 一致可积 $\Rightarrow \sup E|X_n| < \infty \Rightarrow X_n \xrightarrow{a.s.} X$ & L^1

e.g.2. ξ_i 独立，$P(\xi_n = 1) = 1 - \frac{1}{n^2}$，$P(\xi_n = -n^2+1) = \frac{1}{n^2}$，则 S_n 为鞅

$\because \sum P(\xi_n = -n^2+1) < \infty \quad \therefore P(\xi_n = -n^2+1, \text{ i.o.}) = 0$

$\therefore$ a.s. $\xi_n = 1 \quad \therefore X_n \to \infty$, a.s.

e.g.3. 存在鞅 X_n，$E|X_n| \to \infty$ 但有有限收敛极限. (鞅收敛定理条件充分不必要)

(X_n) 为 MC，$P(X_1 = n+1 | X_0 = n) = \frac{2n+1}{2n+2}$，$P(X_1 = -(n+1) | X_0 = n) = \frac{1}{2n+2}$，$P(X_1 = -k | X_0 = -k) = 1$

设 $X_0 = 0$，则 X_n 为鞅，

$P(X_n \text{不收敛}) = P(X_0 = 0, X_1 = 1, X_2 = 2, \cdots) = \prod_{n=0}^{\infty} \frac{2n+1}{2n+2} = 0 \quad \therefore X_n$ a.s. 有有限极限.

但 $P(X_n = -k) = P(X_0 = 0, \cdots, X_{k-1} = k-1, X_k = -k) \geq \frac{1}{2k(2k-2)}$

$\therefore E|X_n| \geq \sum_{k=1}^{n} \frac{k}{2k(2k-2)} \to \infty$

一致可积鞅.

Thm. (一致可积) (X_n)：下鞅，则 (1) X_n 一致可积

$\Leftrightarrow$ (2) X_n a.s. & L^1 收敛

$\Leftrightarrow$ (3) X_n L^1 收敛.

若 (X_n)：鞅，则 $\Leftrightarrow$ (4) $\exists X \in L^1$, s.t. $X_n = E(X|\mathcal{F}_n)$.

Thm. $X \in L^1$，$(\mathcal{F}_n)$ 递增 σ-域流，则 $E(X|\mathcal{F}_n) \xrightarrow{a.s. \& L^1} E(X|\mathcal{F}_\infty)$，$\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$.

Cor (Lévy 0-1律) $A \in \mathcal{F}_\infty$，$P(A|\mathcal{F}_n) \to 1_A$, a.s.

Cor (Kolmogorov) 设 $\mathcal{T}$ 为独立 r.v. $\{X_1, X_2, \cdots\}$ 尾 σ-域 ($\mathcal{T} = \bigcap_n \sigma\{X_n, X_{n+1}, \cdots\}$) 记 $\mathcal{F}_n = \sigma\{X_1, \cdots, X_n\}$，则

$P(A|\mathcal{F}_n) = P(A)$，$\forall A \in \mathcal{T} \Rightarrow P(A) = 1_A$ a.s. $\Rightarrow P(A) = 0$ 或 1

Pf of Thm: $\because E(Y|\mathcal{F}_n) \xrightarrow{a.s.\&L^1} E(Y|\mathcal{F}_\infty)$

又$\because E|E(Y_n|\mathcal{F}_n)-E(Y|\mathcal{F}_n)| \le E|Y_n-Y| \xrightarrow{DCT} 0$

$\therefore E(Y_n|\mathcal{F}_n) \xrightarrow{L^1} E(Y|\mathcal{F}_\infty)$.

对a.s.收敛，$\forall n\ge m$，$E(\inf_{k\ge m} Y_k|\mathcal{F}_n) \le E(Y_n|\mathcal{F}_n) \le E(\sup_{k\ge m} Y_k|\mathcal{F}_n)$

固定m，令$n\to\infty$，$E(\inf_{k\ge m} Y_k|\mathcal{F}_\infty) \le \underline{\lim} E(Y_n|\mathcal{F}_n) \le \overline{\lim} E(Y_n|\mathcal{F}_n) \le E(\sup_{k\ge m} Y_k|\mathcal{F}_\infty)$

令$m\to\infty$+MCT，$E(Y|\mathcal{F}_\infty) \le \underline{\lim} E(Y_n|\mathcal{F}_n) \le \overline{\lim} E(Y_n|\mathcal{F}_n) \le E(Y|\mathcal{F}_\infty)$ a.s. #.

Pf of L^p收敛：$\because \sup E|X_n|^p<\infty$ $\therefore (X_n)$一致可积 $\Rightarrow X_n \xrightarrow{a.s.\&L^1} X_\infty$

下证 $X_n \xrightarrow{L^p} X_\infty$，只须$|X_n|^p$一致可积.

① X_n鞅，则(X_n)一致可积鞅且$X_n=E(X_\infty|\mathcal{F}_n)$ a.s.

$|X_n|^p=|E(X_\infty|\mathcal{F}_n)|^p \le E(|X_\infty|^p|\mathcal{F}_n)$. 只须$X_\infty\in L^p$

由$E|X_\infty|^p \le \underline{\lim} E|X_n|^p \le \sup E|X_n|^p<\infty$. 得证. $X_\infty\in L^p \Rightarrow |X_n|^p$一致可积$\Rightarrow X_n\xrightarrow{L^p}X_\infty$

② X_n非负下鞅，则$0\le X_n\le E(X_\infty|\mathcal{F}_n)$

$|X_n|^p \le |E(X_\infty|\mathcal{F}_n)|^p \le E(|X_\infty|^p|\mathcal{F}_n)$ 由①中类似可得$|X_n|^p$一致可积. #

Pf of Thm(逆鞅)：$\because \{X_{-N},\cdots,X_0\}$为下鞅，记$U_{a,b,N}$为其上穿$[a,b]$次数，

则 $EU_{a,b,N} \le \frac{E(X_0-a)^+}{b-a}<\infty$

$\xRightarrow{N\to\infty} EU_{a,b,\infty} \le \frac{E(X_0-a)^+}{b-a}<\infty \Rightarrow U_{a,b,\infty}<\infty$, a.s. $\Rightarrow X_n$ a.s.收敛. #

· 若X_n为鞅，则$X_n=E(X_0|\mathcal{F}_n)$, $\forall n\le 0$

· 若X_n下鞅，则$X_n\overset{(Doob)}{=}M_n+A_n$，其中$E(X_n|\mathcal{F}_{n-1})=M_{n-1}+E(A_n|\mathcal{F}_{n-1})$

记$\alpha_n=A_n-A_{n-1}=E(X_n|\mathcal{F}_{n-1})-X_{n-1}\ge 0$，则$A_n=\sum_{k\le n}\alpha_k$. 正向鞅

下证$\inf EX_n>-\infty$时(X_n)一致可积

记$\alpha_n=E(X_n-X_{n-1}|\mathcal{F}_n)\ge 0$, $\forall n\le 0$，则$E\sum_{n\le 0}\alpha_n=\sum_{n\le 0}E\alpha_n=\lim_{k\to\infty}\sum_{n=k}^{0}E\alpha_n$

$=EX_0-\lim_{k\to\infty}EX_k<\infty$ $(\inf EX_n>-\infty)$

$\therefore \sum_{k\le 0}\alpha_n<\infty$, a.s.

定义$A_n=\sum_{k\le n}\alpha_k$，$M_n=X_n-A_n$，则$\{M_n\}$为鞅 且 $X_n=M_n+A_n$.

$\because \{M_n\}$一致可积，$0\le A_n\le\sum_{k\le 0}\alpha_k$ 且 $E|\sum_{k\le 0}\alpha_k|<\infty$

$\therefore \{A_n\}$一致可积 $\therefore \{X_n\}$一致可积. #.

Thm. $Y_n\xrightarrow{a.s.}Y$, $|Y_n|\le Z$, $Y,Z\in L^1$，则$E(Y_n|\mathcal{F}_n)\xrightarrow{a.s.\&L^1}E(Y|\mathcal{F}_\infty)$.

· $L^p(p>1)$收敛.

Thm(L^p收敛). X_n鞅/非负下鞅，且$\sup E|X_n|^p<\infty$, $p>1$，则$X_n\xrightarrow{a.s.\&L^p}X_\infty$

· 逆鞅:

$Y_{-1}, Y_{-2},\cdots$. $\mathcal{F}_{-m}$，$E(Y_{-m}|\mathcal{F}_{-m-1})=Y_{-m-1}$

Thm. $(X_n,\mathcal{F}_n)_{n\le 0}$：下鞅，则$\exists X_{-\infty}$, s.t. $X_n\xrightarrow{a.s.}X_{-\infty}$ $(n\to-\infty)$

且若$\inf EX_n>-\infty$，则X_n一致可积，$X_n\xrightarrow{L^1}X_{-\infty}\in L^1$. 此时，$\forall n$, $E(X_n|\mathcal{F}_{-\infty})\ge X_{-\infty}$.

Rmk (Cor1) $(X_n)_{n\le 0}$鞅，则$X_n\xrightarrow{a.s.\&L^1}X_{-\infty}$ 且$E(X_n|\mathcal{F}_{-\infty})=X_{-\infty}$. 其中$\mathcal{F}_{-\infty}=\bigcap_{n\le 0}\mathcal{F}_n$.

(Cor2) 对$\mathcal{F}_n\downarrow\mathcal{F}_\infty$, $\forall X\in L^1$，有$E(X|\mathcal{F}_n)\xrightarrow{a.s.\&L^1}E(X|\mathcal{F}_\infty)$ $(n\to\infty)$

(☺ $(X_n=E(X|\mathcal{F}_{-n}), \mathcal{G}_n=\mathcal{F}_{-n})_{n\le 0}$为逆鞅，则$\exists X_{-\infty}$, s.t. $X_n\xrightarrow{a.s.\&L^1}X_{-\infty}$ $(n\to-\infty)$,

又$\mathcal{G}_{-\infty}=\bigcap_{n\le 0}\mathcal{G}_n=\bigcap_{n\ge 0}\mathcal{F}_n=\mathcal{F}_\infty$

$X_{-\infty}=E(X_n|\mathcal{G}_{-\infty})=E(E(X|\mathcal{G}_n)|\mathcal{G}_{-\infty})=E(X|\mathcal{G}_{-\infty})=E(X|\mathcal{F}_\infty)$ a.s.)

(Cor3) $\Rightarrow$ $Y_n\to Y$, $|Y_n|\le Z$时，则$E(Y_n|\mathcal{F}_n)\xrightarrow{a.s.\&L^1}E(Y|\mathcal{F}_\infty)$. $(n\to\infty)$

应用:

Thm(SLLN) $X, X_i\sim$ i.i.d. $E|X|<\infty$，则$\frac{S_n}{n}\xrightarrow{a.s.}EX$.

Pf: $\mathcal{F}_n\triangleq\sigma\{S_n,S_{n+1},\cdots\}\downarrow$. 且$E(X_1|\mathcal{F}_n)=E(X_k|\mathcal{F}_n)$, $\forall k\le n$.

$\therefore E(X_1|\mathcal{F}_n)=E(\frac{S_n}{n}|\mathcal{F}_n)=\frac{S_n}{n}$. (☺ $\Leftrightarrow \int_A X_1=\int_A X_k$, $\forall A\in\sigma\{S_n,\cdots\}$ $\Leftrightarrow EX_1 1_A=EX_k 1_A$

$\therefore \frac{S_n}{n}=E(X_1|\mathcal{F}_n)\xrightarrow{a.s.}E(X_1|\mathcal{F}_\infty)$，其中$\mathcal{F}_\infty=\bigcap\mathcal{F}_n=\mathcal{T}$尾事件σ-域. $\Leftrightarrow (X_1,S_n,\cdots)\overset{d}{=}(X_k,S_n,\cdots)$)

$\because E(X_1|\mathcal{F}_\infty)\in\mathcal{T}$

$\therefore \exists C$, s.t. $E(X_1|\mathcal{F}_\infty)=C$ a.s. 且$C=E(E(X_1|\mathcal{F}_\infty))=EX_1$.

$\therefore \frac{S_n}{n}\xrightarrow{a.s.}EX$. #.

E.g. 设$\{M_n\}_{n\ge 0}$为鞅且$\exists C<\infty$, s.t. $|\Delta M|\le C$. ($\Delta M = \sup|M_n-M_{n-1}|$)

记$C=\{\lim M_n\ \exists$有限$\}$，$D=\{\overline{\lim} M_n=\infty$ 且 $\underline{\lim} M_n=-\infty\}$，则$P(C\cup D)=1$.

Pf: 只须$\overline{\lim} M_n<\infty$时 C a.s.成立.

(对mar，$\sup EM_n^+<\infty$或$\sup EM_n^-<\infty$或$\sup E|M_n|<\infty$，就有a.s.收)

考虑$\tau_m=\inf\{n: M_n\ge m\}$

则$M_{n\wedge\tau_m}\le m+C$ $\Rightarrow M_{n\wedge\tau_m}$ a.s.收 $\Rightarrow \{\sup M_n<m\}$时，$M_n$ a.s.收$\Rightarrow\{\sup M_n<\infty\}$时 M_n a.s.收

$\Rightarrow \overline{\lim} M_n<\infty$时，$M_n$ a.s.收 $\Rightarrow D^c$上 C a.s.成立 #

Pf of de Finelli: $\forall$ 有界可测 $\varphi: \mathbb{R}^k \to \mathbb{R}$，$A_n(\varphi) \triangleq \frac{(n-k)!}{n!} \sum\limits_{1\le i_1,\cdots,i_k\le n \text{互不相同}} \varphi(X_{i_1}, \cdots, X_{i_k})$，则 $A_n(\varphi) \in \mathcal{E}_n$

$\therefore A_n(\varphi) = E(A_n(\varphi)|\mathcal{E}_n) = \frac{(n-k)!}{n!} \sum E(\varphi(X_{i_1}\cdots X_{i_k})|\mathcal{E}_n)$

$= E(\varphi(X_1,\cdots,X_k)|\mathcal{E}_n) \to E(\varphi(X_1,\cdots,X_k)|\mathcal{E})$

下证 $X_1, X_2, \cdots$ 在 $\mathcal{E}$ 下条件 i.i.d.，即

$E(f_1(X_1),\cdots,f_k(X_k)|\mathcal{E}) = E(f_1(X_1)|\mathcal{E}) \cdots E(f_k(X_k)|\mathcal{E}) \quad \cdots ①$

且 $E(f_1(X_i)|\mathcal{E}) = E(f_1(X_1)|\mathcal{E}) \quad \cdots ②$

令 $\varphi(X_1,\cdots,X_k) = f(X_1,\cdots,X_{k-1})\, g(X_k)$ 有界，其中 f, g 可测

$A_n(f)A_n(g) = \frac{(n-k+1)!}{n!} \sum f(X_{i_1},\cdots,X_{i_{k-1}}) \cdot \frac{1}{n} \sum g(X_m)$

$= \frac{(n-k+1)!}{n!} \frac{1}{n} \sum \varphi(X_{i_1},\cdots,X_{i_k}) + \frac{(n-k+1)!}{n!} \cdot \frac{1}{n} \sum f(X_{i_1},\cdots,X_{i_{k-1}}) \sum\limits_{j=1}^{k-1} g(X_{i_j})$

$= \frac{n-k+1}{n} A_n(\varphi) + \frac{1}{n} \sum\limits_{j=1}^{k-1} A_n(\varphi_j)$，其中 $\varphi_j(X_1,\cdots,X_{k-1}) = f(X_1,\cdots,X_{k-1})\varphi(X_j)$

令 $n\to\infty$，$E(\varphi(X_1,\cdots,X_k)|\mathcal{E}) = E(f(X_1,\cdots,X_{k-1})|\mathcal{E}) \cdot E(g(X_m)|\mathcal{E})$

$\Rightarrow ①$

对②，若证 $\begin{cases} 1.\ \overset{\text{尾事件域}}{\mathcal{T}} \subset \mathcal{E} \\ 2.\ E(\varphi(X_1,\cdots,X_k)|\mathcal{E}) \in \mathcal{T} \end{cases}$，则 $E(\varphi(X_1,\cdots,X_k)|\mathcal{E})$
$= E(E(\varphi(X_1,\cdots,X_k)|\mathcal{E})|\mathcal{T})$
$= E(\varphi(X_1,\cdots,X_k)|\mathcal{T})$ a.s.

下证 2.，即 $\lim A_n(\varphi) \in \mathcal{T}$.

$\forall m$，$n>m$ 时，记 $I_{n,k}$ 为 $\{i_1,\cdots,i_k\} \in [1,n]$ 互不同，$I'_{n,k}$ 为 $\{i_1,\cdots,i_k\} \in [m,n]$ 互不同

则 $|I_{n,k}| - |I'_{n,k}| = \frac{n!}{(n-k)!} - \frac{(n-m)!}{(n-m-k)!}$

$\frac{(n-k)!}{n!} \left| \sum\limits_{I_{n,k}} - \sum\limits_{I'_{n,k}} \varphi(X_{i_1},\cdots,X_{i_k}) \right| \le \frac{(n-k)!}{n!} \left(\frac{n!}{(n-k)!} - \frac{(n-m)!}{(n-m-k)!} \right) \sup|\varphi| \to 0 \quad (n\to\infty)$

$\because \lim A_n(\varphi) = \lim \frac{(n-k)!}{n!} \sum\limits_{I'_{n,k}} \varphi(X_{i_1},\cdots,X_{i_k}) \in \sigma(X_{m+1}, X_{m+2}, \cdots)$

$\therefore \lim A_n(\varphi) \in \bigcap\limits_m \sigma(X_{m+1},\cdots) \in \mathcal{T}$ #

Pf of 鞅停时：(1) $\Rightarrow$ (2)，只证鞅. $\forall A \in \mathcal{F}_S$，$M \triangleq S 1_A + T 1_{A^c}$，则 $\{M \le n\} = A \cap \{S\le n\} + A^c \cap \{T \le n\} \in \mathcal{F}_n$

$\therefore M$ 为停时，$M \le T \overset{(1)}{\Rightarrow} EX_M = EX_T$

$\Rightarrow EX_S 1_A = EX_T 1_A$，$\forall A \in \mathcal{F}_S \Rightarrow E(X_T|\mathcal{F}_S) = X_S$ a.s.

(2) $\Rightarrow$ (3) $\{S\le T\}$ 上，$E(X_T|\mathcal{F}_S) = E(X_T|\mathcal{F}_{S\wedge T}) \overset{(2)}{=} X_{S\wedge T}$, a.s.

$\{S>T\}$ 上，$E(X_T|\mathcal{F}_S) = E(X_{T\wedge S}|\mathcal{F}_S) = X_{S\wedge T}$, a.s.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1)，trivial. #

E.g. 2 $\forall \{\mathcal{F}_n\}$ 流，$A_n \in \mathcal{F}_n$，则 $\{A_n, \text{i.o.}\} = \{\sum P(A_n|\mathcal{F}_{n-1}) = \infty\}$ a.s.

Pf: $\{A_n, \text{i.o.}\} = \{\sum 1_{A_n} = \infty\}$

$M_n = \sum\limits_{k=1}^{n} (1_{A_k} - P(A_k|\mathcal{F}_{k-1}))$ 为 mar，且 $|\Delta M| \le 1$

由上一 e.g. 记号，在 D 上，$\{\sum 1_{A_n} = \infty\}$ 且 $\{\sum P(A_n|\mathcal{F}_{n-1}) = \infty\}$ ……成立

在 C 上，若 $\sum 1_{A_n} = \infty$，则 $\sum P(A_n|\mathcal{F}_{n-1}) = \infty$，反之亦然（即同时成立）

由上一 e.g. $P(C\cup D) = 1$ 知 $\{A_n, \text{i.o.}\} = \{\sum P(A_n|\mathcal{F}_{n-1}) = \infty\}$ a.s. #

· 可交换序列

Def. $(X_1,\cdots,X_N)$ 称为 exchangable，若 $(X_{\pi(1)},\cdots,X_{\pi(N)}) \overset{d}{=} (X_1,\cdots,X_N)$，$\forall \{1,\cdots,N\}$ 置换 π

无穷 $(X_1, X_2, \cdots)$ 可交换，若 $\forall N \in \mathbb{N}$，$(X_1,\cdots,X_N)$ 可交换

$\mathcal{E}_n = \{A: \exists\, \mathbb{R}^\infty$ 可测 B，s.t. $A = \{(X_1, X_2, \cdots) \in B\}$ 且 $\forall \{1,\cdots,n\}$ 置换 π，有 $A = \{(X_{\pi(1)},\cdots,X_{\pi(n)}, X_{n+1},\cdots) \in B\}.\}$，则 $\mathcal{E} = \bigcap\limits_n \mathcal{E}_n$ 称为可交换 σ 域

Thm (de Finelli) 设 $X_1, X_2, \cdots$ 可交换，则在 $\mathcal{E}$ 的条件下，$X_1, X_2, \cdots$ 条件独立同分布.

Rmk. $X_1, X_2, \cdots$ 关于 $\mathcal{E}$ 条件 i.i.d.，还关于尾事件 $\mathcal{T}$ 条件 i.i.d.

Thm (Hewitt-Savage 0-1律)，$X_1, X_2, \cdots$ i.i.d.，$A \in \mathcal{E}$. 则 $P(A) = 0$ 或 1.

Pf: $\forall \varphi$ 可测有界，$E(\varphi(X_1,\cdots,X_k)|\mathcal{E}) = E(\varphi(X_1,\cdots,X_k)|\mathcal{T})$ a.s.

(i.i.d. 尾 r.v. 为 const) $= E\varphi(X_1,\cdots,X_k)$

$\therefore X_1,\cdots,X_k \perp \mathcal{E} \Rightarrow \mathcal{E} \perp \sigma\{X_1, X_2, \cdots\} \Rightarrow \mathcal{E}$ 与 $\mathcal{E}$ 独立 $\Rightarrow \forall A\in\mathcal{E}$，$P(A) = P(A\cap A) = P(A)^2$

$\Rightarrow P(A) = 0$ 或 1. #

Thm (鞅停时) $(X_n, \mathcal{F}_n)$ 为鞅（下鞅），则 (1) $\forall S \le T$，$EX_S \underset{(\le)}{=} EX_T$, a.s.

$\Leftrightarrow$ (2) $\forall S \le T$，$E(X_T|\mathcal{F}_S) \underset{(\ge)}{=} X_S$, a.s.

$\Leftrightarrow$ (3) $\forall S, T$，$E(X_T|\mathcal{F}_S) \underset{(\ge)}{=} X_{S\wedge T}$, a.s.

· $EX_T = EX_0$，$\forall$ 停时 T，则 X_n 为鞅

C.Eg. (停时定理不成立) SRW，$S_n = \sum\limits_{i=1}^{n} X_i$. $S \equiv 1$，$T = \inf\{n: S_n = 1\}$，则 $\overline{\lim} S_n = \infty$，$\underline{\lim} S_n = -\infty$ (a.s.)

$T < \infty$, a.s.

但 $ES_T \ne ES_S$.

Thm (有界停时) $(X_n, \mathcal{F}_n)$ 为鞅（下鞅），停时 S, T，T 有界，则 $E(X_T|\mathcal{F}_S) \underset{(\ge)}{=} X_{S\wedge T}$, a.s. 即证 $\forall S\le T$，$EX_S \underset{(\le)}{=} EX_T$, a.s.

Pf: $(X_{T\wedge n})$ 为 mar $\Rightarrow EX_{T\wedge n} = EX_0$，设 $S \le T \le M \in \mathbb{N}$，则 $EX_T = EX_0$，$EX_S = EX_0$ $\therefore EX_T = EX_S$ #

(X_n) 为 sub，$K_k = 1_{\{S<k\le T\}} \in \mathcal{F}_{k-1}$，则 $(K\cdot X)_n$ 为 sub，$(K\cdot X)_n = X_{T\wedge n} - X_{S\wedge n} \Rightarrow E(X_{T\wedge n}) \ge E(X_{S\wedge n})$

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Pf of Lem: $E|X_{T\wedge n}|1_{\{|X_{T\wedge n}|>k\}} \le E|X_T|1_{\{|X_T|>k,\, T\le n\}} + E|X_n|1_{\{|X_n|>k,\, T>n\}}$

$\le E|X_T|1_{\{|X_T|>k\}} + E|X_n|1_{\{|X_n|>k\}}$

由(X_n)一致可积，只须 $E|X_T|<\infty \Rightarrow \sup_n E|X_{T\wedge n}|1_{\{|X_{T\wedge n}|>k\}} \le E|X_T|1_{\{|X_T|>k\}} + \sup_n E|X_n|1_{\{|X_n|>k\}} \to 0\ (k\to\infty)$

下证 $E|X_T|<\infty$

$\because \sup_n EX_{T\wedge n}^+ \le \sup_n EX_n^+ \le \sup_n E|X_n| \overset{\text{一致可积}}{<\infty}$

$\therefore$ 由鞅收敛 $X_{T\wedge n} \xrightarrow{a.s.} X_T \in L^1$ #

Pf of 五条件 $\Rightarrow E(X_T|\mathcal{F}_S)=(\ge)X_{S\wedge T}$, a.s.: 只须证 $(X_{T\wedge n_k})$一致可积（由上一定理）

① trivial

② $\underline{\lim}\, E|X_n|1_{\{T>n\}}=0 \Rightarrow \overset{\exists 子列\{n_k\}}{E|X_{n_k}|1_{\{T>n_k\}}} \to 0$.

$E|X_{T\wedge n_k}|1_{\{|X_{T\wedge n_k}|>M\}} = E|X_T|1_{\{|X_T|>M,\, T\le n_k\}} + E|X_{n_k}|1_{\{|X_{n_k}|>M,\, T>n_k\}}$

$\le E|X_T|1_{\{|X_T|>M\}} + E|X_{n_k}|1_{\{T>n_k\}} \to 0$

(即 $\lim_{M\to\infty}\overline{\lim}_{n_k\to\infty} E|X_{T\wedge n_k}|1_{\{|X_{T\wedge n_k}|>M\}}=0$.) $\Rightarrow (X_{T\wedge n_k})$一致可积.

$\Leftrightarrow \lim_{M\to\infty}\sup_k E|X_{T\wedge n_k}|1_{\{|X_{T\wedge n_k}|>M\}}=0$.

③ $E|X_{T\wedge n}|1_{\{|X_{T\wedge n}|>M\}} \le E|X_T|1_{\{|X_T|>M\}} + E|X_n|1_{\{|X_n|>M,\, T>n\}}$

$\overset{Y_n=X_n1_{\{T>n\}}}{=} E|X_T| + E|Y_n|1_{\{|Y_n|>M\}} \Rightarrow (X_{T\wedge n})$一致可积

④$\Rightarrow$⑤ $\Rightarrow$ ~~④~~, ~~⑥~~ ③: $|X_T| \le |X_0| + \sum_{n=1}^{T}|X_n-X_{n-1}|$

令 $Y_n=X_n-X_{n-1}$，则 $E\sum_{n=1}^{T}|Y_n| = E\sum_{n=1}^{\infty}|Y_n|1_{\{T\ge n\}}$

$=\sum_{n=1}^{\infty}E|Y_n|1_{\{T\ge n\}}$

$=\sum_{n=1}^{\infty}E\big(E(|Y_n|1_{\{T\ge n\}}|\mathcal{F}_{n-1})\big)$

$=\sum_{n=1}^{\infty}E\big(1_{\{T\ge n\}}E(|Y_n||\mathcal{F}_{n-1})\big)$

$=E\Big(\sum_{n=1}^{T}E(|Y_n||\mathcal{F}_{n-1})\Big)<\infty$

$\therefore E|X_T|<\infty$. $|X_n|1_{\{T>n\}} \le |X_0| + \sum_{n=1}^{T}|X_n-X_{n-1}| \in L^1$ $\therefore |X_n|1_{\{T>n\}}$一致可积，由③得 #

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Thm X: 非负上鞅. $S\le T$, $E(X_T|\mathcal{F}_S)\le X_S$, a.s. （若$T=\infty$, $X_T=X_\infty \triangleq \lim X_n$）

Pf: $E(X_{T\wedge n}|\mathcal{F}_S)\le X_{S\wedge n}$ a.s.（有界停时）

而 $E(X_T|\mathcal{F}_S) \overset{\text{Fatou}}{\le} \underline{\lim}_{n\to\infty} E(X_{T\wedge n}|\mathcal{F}_S) \le \underline{\lim}_{n\to\infty} X_{S\wedge n} = X_S$, a.s. #

Thm. $(X_{T\wedge n})$一致可积鞅$_{(下)}$，则 $E(X_T|\mathcal{F}_S) \underset{(\ge)}{=} X_{S\wedge T}$ a.s. Rmk. 只要子列$(X_{T\wedge n_k})$一致可积.

Lem (X_n)一致可积下鞅，则 $\forall T$, $X_{T\wedge n}$一致可积

Rmk. (X_n)一致可积鞅, $X_n \xrightarrow{L^1,a.s.} X_\infty$, $X_n=E(X_\infty|\mathcal{F}_n)$, $X_{T\wedge n}=E(X_\infty|\mathcal{F}_{T\wedge n})$.

Pf: 由有界停时, $E(X_{T\wedge n}|\mathcal{F}_S)=(\ge)X_{S\wedge T\wedge n}$, a.s.

$\because X_{S\wedge T\wedge n} \xrightarrow{L^1 \& a.s.} X_{S\wedge T}$（由一致可积）

$E(X_{T\wedge n}|\mathcal{F}_S) \xrightarrow{L^1} E(X_T|\mathcal{F}_S)$（由 $X_{T\wedge n}\xrightarrow{a.s.}X_T$ 且一致可积）

$\therefore E(X_T|\mathcal{F}_S)=(\ge)X_{S\wedge T}$, a.s.（取子列有$L^1$&a.s.收敛）.

Thm $(X_n,\mathcal{F}_n)$鞅$_{(下)}$, S,T停时，若下述之一成立：

∨① $X_n^{(+)}$一致可积, $\forall T$

② $E|X_T|<\infty$, 且 $\underline{\lim}\, E|X_n|1_{\{T>n\}}=0$

③ $E|X_T|<\infty$, 且 $X_n1_{\{T>n\}}$一致可积

∨④ $\overset{ET<\infty}{\cancel{E|X_T|<\infty}}$ 且 $E[|X_{n+1}-X_n||\mathcal{F}_n]\le B$, a.s.

⑤ T有限，且 $E\Big[\sum_{n=1}^{T}E(|X_n-X_{n-1}||\mathcal{F}_{n-1})\Big]<\infty$

则 $E(X_T|\mathcal{F}_S)=(\ge)X_{S\wedge T}$, a.s.

• $X_i \overset{i.i.d.}{\sim}$ $EX_1=0$, $EX_1^2=1$, $ET<\infty$，则 $ES_T^2=ET$ （S_n^2-n 为 mar）

△总结：停时定理 $E(X_T|\mathcal{F}_S)=(\ge)X_{S\wedge T}$ 成立条件

① T有界

或② (X_n)一致可积.

Eg. $Y\in L^1$, S,T停时，则 $E(E(Y|\mathcal{F}_T)|\mathcal{F}_S)=E(E(Y|\mathcal{F}_S)|\mathcal{F}_T)=E(Y|\mathcal{F}_{S\wedge T})$

Pf: $X_n=E(Y|\mathcal{F}_n)$ mar，则 $LHS=E(X_T|\mathcal{F}_S)=X_{S\wedge T}$ #.

Eg. $X\ge 0$ super, $\forall n\ge 0$, 有 $P(X_n>0, \inf_{0\le k\le n}X_k=0)=0$.

Pf: $\sigma\triangleq\inf\{n: X_n=0\}$, $\forall a>0$, $\tau\triangleq\inf\{n>\sigma: X_n\ge a\}$, 下证 $\tau=\infty$, a.s.

$0\ge EX_{\tau\wedge n}-EX_{\sigma\wedge n} = E(\underset{\ge a}{X_\tau}-\underset{=0}{X_\sigma}; \tau\le n) + E(\underset{\ge 0}{X_n}-X_\sigma; \sigma<n<\tau)$

$\ge a\cdot P(\tau\le n) + 0$

$\therefore P(\tau\le n)=0 \Rightarrow \tau=\infty$, a.s. #

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Pf of Thm: 记 $\tau = \inf\{k \geq 0, X_k \geq \lambda\}$，则

$$LHS = \lambda P(\tau \leq n) = \lambda E I(\tau \leq n) \leq E X_{\tau\wedge n} I(\tau \leq n) \overset{停时}{\leq} E X_n I(\tau \leq n) \leq E X_n^+$$

($\because E(X_n|\mathcal{F}_\tau) \geq X_{\tau\wedge n}$ 且 $\{\tau \leq n\} \in \mathcal{F}_\tau$)

记 $\sigma = \inf\{k \geq 0, X_k \leq -\lambda\}$，则

$$\lambda P(\max_{0\leq k\leq n}(-X_k) \geq \lambda) = \lambda P(\sigma \leq n) = \lambda E I(\sigma \leq n) \leq E(-X_{\sigma\wedge n}) I(\sigma \leq n)$$
$$= -EX_{\sigma\wedge n} + E X_{\sigma\wedge n} I(\sigma > n)$$
$$\leq -EX_0 + E X_n I(\sigma > n) \leq EX_n^+ - EX_0$$ #

Pf of L^p-maximal: 记 $X^* = \max X_k^+$

① $$E(X^*)^p = p\int_0^\infty x^{p-1} P(X^* \geq x)\,dx$$
$$= p\int_0^\infty x^{p-2} E(X_n^+; X^* \geq x)\,dx$$
$$= p E\left[X_n^+ \int_0^\infty x^{p-2} 1(X^* \geq x)\,dx\right]$$
$$= \frac{p}{p-1} EX_n^+ (X^*)^{p-1}$$
$$\overset{Hölder}{\leq} \frac{p}{p-1}(E(X_n^+)^p)^{\frac{1}{p}}(E(X^*)^p)^{\frac{p-1}{p}}$$

② $$EX^* = \int_0^\infty P(X^* \geq x)\,dx$$
$$\leq \int_1^\infty \frac{1}{x} E(X_n^+; X^* \geq x)\,dx + 1$$
$$\leq 1 + EX_n^+ \log^+ X^*$$
(由 $a\log^+ b \leq a\log^+ a + \frac{1}{e} b$, $\forall a, b > 0$)
$$\leq 1 + EX_n^+ \log^+ X_n^+ + \frac{1}{e} EX^*.$$ #

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Eg. (Wald 等式) X_n i.i.d. $S_n = \sum_{k=1}^n X_k$. T 停时, $ET < \infty$，则 $ES_T = EX_1 \cdot ET$.
若 $EX_1 = 0$, $EX_1^2 = 1$，则 $ES_T^2 = ET$.

Pf: $\{S_n - nEX_1\}$ 为 mar. 由 $ET < \infty$, $E(|X_n - EX_1| \,|\, \mathcal{F}_{n-1}) \leq E|X_1 - EX_1| < \infty$
由停时定理, $E(S_T - TEX_1) = 0$

$\{S_n^2 - n\}$ 为 mar. $E(S_{T\wedge n}^2) = E(T\wedge n) \to ET$

$$\because E(S_{T\wedge(n+1)}^2) = E(S_T^2 \cdot 1_{\{T\leq n\}}) + E(S_{n+1}^2 \cdot 1_{\{T\geq n+1\}})$$
$$= ES_T^2 1_{\{T\leq n\}} + ES_n^2 1_{\{T\geq n+1\}} + EX_{n+1}^2 1_{\{T\geq n+1\}}$$
$$= ES_{T\wedge n}^2 + P(T \geq n+1)$$

$$\therefore E(S_{T\wedge m} - S_{T\wedge n})^2 = ES_{T\wedge m}^2 - ES_{T\wedge n}^2 = \sum_{k=n+1}^m P(T \geq k) \to 0 \ (m, n \to \infty)$$
(由于 $ET = \sum_{k=1}^\infty P(T \geq k) < \infty$.)

由 Fatou, $E(S_T^2 - ES_{T\wedge n}^2) = E(S_T - S_{T\wedge n})^2$
$$\leq \varliminf_{m\to\infty} E(S_{T\wedge m} - S_{T\wedge n})^2 \leq \sum_{k=n+1}^\infty P(T \geq k) \to 0. \ (n \to \infty)$$ #

Thm. (X_k): sub，则 $\forall \lambda > 0$, $n \geq 0$，有
$$\lambda P(\max_{0\leq k\leq n} X_k \geq \lambda) \leq E(X_n; \max_{0\leq k\leq n} X_k \geq \lambda) \leq EX_n^+$$
$$\lambda P(\max_{0\leq k\leq n} |X_k| \geq \lambda) \leq 2EX_n^+ - EX_0 \leq 3\max_{0\leq k\leq n} E|X_k|$$

Rmk. 若 (X_k) mar: $|X_k|$ sub $\Rightarrow \lambda P(\max |X_k| \geq \lambda) \leq E|X_n|$

Thm (L^p 最大值不等式) (X_n): sub ① $p > 1$，有 $E(\max_{0\leq k\leq n} X_k^+)^p \leq (\frac{p}{p-1})^p E(X_n^+)^p$

② $p = 1$, $\log^+ x = \log x \vee 0$，有 $E(\max X_k^+) \leq \frac{e}{e-1}(1 + EX_n^+ \log^+ X_n^+)$