Ans. week4

Ex1 :

(1) $n=1$

(a).

$$[\partial_x, x] f = \partial_x(xf) - x\,\partial_x f = f$$

(b)

$$\|f\|_{L^2}^2 = \langle f, f\rangle$$

$$= \langle [\partial_x, x] f, f\rangle$$

$$= \langle \partial_x(xf), f\rangle - \langle x\partial_x f, f\rangle$$

$$\overset{\text{I.B.P.}}{\leq} |\langle xf, \partial_x f\rangle| \times 2$$

$$\overset{\text{C.-S. Ineq.}}{\leq} 2\,\|xf\|_{L^2}\cdot\|\partial_x f\|_{L^2}.$$

(2). the same argument in 1-D. ~~with~~ special attention to the ~~vector~~ inner product in the vector-valued case.

(3).

$$\|f\|_{L^2(\Omega)}^2 = \|f\|_{L^2(\mathbb{R}^n)}^2 \leq \cancel{C}\,\|\nabla_x G\|_{L^2(\mathbb{R}^n)}\cdot\underline{\|xf\|_{L^2(\mathbb{R}^n)}}$$

$$\leq C_\varepsilon\|\nabla_x G\|_{L^2(\mathbb{R}^n)}^2 + \underline{\varepsilon}\,(\text{diam} + \text{dist.}) \times \|f\|_{L^2}^2$$

$\Downarrow$ ε small.

$$\|f\|_{L^2(\Omega)} \leq C(\text{dist. diam}) \|\nabla_x G\|_{L^2}.$$

(4) obviously.

(5).

$$\boxed{\nabla_x \frac{1}{r} = -\frac{\nabla_x r}{r^2} = -\frac{x}{r^3}}$$

$$\left[\frac{1}{r}\nabla_x\left(\frac{1}{r}\cdot\right), x\cdot\right] f$$

$$= \frac{1}{r}\nabla_x\cdot\left(\frac{1}{r}xf\right) - x\cdot\frac{1}{r}\nabla_x\left(\frac{1}{r}f\right)$$

$$= \frac{1}{r}\nabla_x\left(\frac{1}{r}\right)\cdot xf + \frac{1}{r}\cdot\frac{1}{r}\nabla_x\cdot(xf)$$

$$- x\cdot\frac{1}{r}\left[\nabla_x\left(\frac{1}{r}\right)\right]f - x\cdot\frac{1}{r^2}\nabla_x f$$

$$= \frac{1}{r}\left(-\frac{x}{r^3}\right)\cdot xf + \frac{nf + x\cdot\nabla_x f}{r^2}$$

$$- x\cdot\frac{1}{r}\cdot\left(-\frac{x}{r^3}\right)f - \frac{1}{r^2}x\cdot\nabla_x f$$

$$= \frac{n}{r^2}f.$$

$$\Downarrow \cdot f$$

$$n\|r^{-1}f\|_{L^2}^2 = \left\langle \frac{n}{r^2}f, f \right\rangle$$

$$= \left\langle \left[\frac{1}{r}\nabla_x\left(\frac{1}{r}\cdot\right), x\cdot\right] f, f \right\rangle$$

$$= \left\langle \frac{1}{r}\nabla_x\cdot\left(\frac{1}{r}xf\right), f \right\rangle$$

$$- \left\langle \frac{1}{r}x\cdot\nabla_x\left(\frac{1}{r}f\right), f \right\rangle$$

$$= -2\left\langle \frac{1}{r}\nabla_x\left(\frac{1}{r}f\right), xf \right\rangle$$

$$\nabla_x\left(\frac{1}{r}f\right) = \frac{1}{|x|}\nabla_x f + \left(\nabla_x\frac{1}{r}\right)f$$

$$= \frac{1}{|x|}\nabla_x f - \frac{x}{|x|^3}f$$

$$\Rightarrow n\|r^{-1}f\|_{L^2}^2 = -2\left\langle \frac{1}{r}\left(\frac{\nabla_x f}{r} - \frac{x}{|x|^3}f\right), xf \right\rangle$$

$$= -2\left\langle \frac{\nabla_x f}{r^2}, xf \right\rangle$$

$$+ 2\left\langle \frac{1}{r}\cdot\frac{|x|^2}{|x|^3}f, f \right\rangle$$

$$\Rightarrow (n-2)\|r^{-1}f\|_{L^2}^2 = -2\left\langle \nabla_x f \cdot \frac{x}{r^2}f \right\rangle.$$

Ex2

$$\int \left(\int |f(x,y)|^p \, d\mu(x) \right)^{\frac{q}{p}} d\nu(y)$$

$$= \int \left(\int |f(x,y)|^p \, d\mu(x) \right)^{\frac{q}{p}-1} \cdot \left(\int |f(x,y)|^p \, d\mu(x) \right) d\nu(y)$$

$$\overset{\text{Fubini}}{=} \int \int |f(x,y)|^p \cdot \left(\int \quad \right)^{\frac{q}{p}-1} d\nu(y) \, d$$

$$\overset{\text{Hölder}}{\leq} \int \left(\int |f(x,y)|^{p \cdot \frac{q}{p}} d\nu(y) \right)^{\frac{p}{q}} \cdot \left(\int \left(\int \cdots \right)^{\left(\frac{q}{p}-1\right) \cdot \frac{1}{1-p/q}} d\nu(y) \right)^{1-\frac{p}{q}} d\mu$$

$$\frac{q}{p}$$

$$= \int \left(\int |f(x,y)|^q \, d\nu(y) \right)^{\frac{p}{q}} d\mu(x) \cdot \left(\int \left(\int |f(x,y)|^p \, d\mu(x) \right)^{\frac{q}{p}} \right.$$

$$\Downarrow$$

$$\left(\int \left(\int |f(x,y)|^p \, d\mu(x) \right)^{\frac{q}{p}} d\nu(y) \right)^{\frac{1}{q}}$$

$$\leq \left(\int \left(\int |f(x,y)|^q \, d\nu(y) \right)^{p/q} d\mu(x) \right)^{\frac{1}{p}}.$$

(1): for $r \geq 1$,

$$\|f*g\|_{L^r} = \Big\| \|f(y)g(x-y)\|_{L^1_y} \Big\|_{L^r_x}$$

$$\overset{\cancel{\text{Young}}\ \text{Minkowski}}{\leq} \Big\| \|f(y)g(x-y)\|_{L^r_x} \Big\|_{L^1_y}$$

$$\overset{\text{Invariance of Leb. meas. under translation}}{=} \|g\|_{L^r} \|f\|_{L^1}$$

Generally:

$$\|f*g\|_{L^r} \leq \Big\| \|f(y)g(x-y)\|_{L^1_y} \Big\|_{L^r_x}$$

$$1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$$

$$= \Big\| \Big\| |f(y)|^{p/q'} \left(|f(y)|^{\frac{p}{r}} |g(x-y)|^{\frac{q}{r}}\right) \cdot |g(x-y)|^{\frac{q}{p'}} \Big\|_{L^1_y} \Big\|_{L^r_x}$$

$$\leq \Big\| \|f\|_{L^p}^{p/q'} \left(\int |f(y)|^p |g(x-y)|^q \, dy\right)^{\frac{1}{r}} \cdot \left(\int |g(x-y)|^q dy\right)^{\frac{1}{p'}} \Big\|_{L^r_x}$$

$$\overset{\text{Invar of Leb. under reflection}}{\leq} \|f\|_{L^p}^{p/q'} \cdot \|g\|_{L^q}^{p/q'} \cdot \Big\| \left(\int |f(y)|^p \cdot |g(x-y)|^q dy\right)^{\frac{1}{r}} \Big\|_{L^r_x}$$

$$\overset{\text{Fubini}}{=} \|f\|_{L^p}^{p/q'} \|g\|_{L^q}^{p/q'} \cdot \|f\|_{L^p}^{\frac{p}{r}} \|g\|_{L^q}^{q/r}$$

$$= \|f\|_{L^p} \|g\|_{L^q}.$$

(2) the standard interpolation.

(3). $\int_{\mathbb{R}^n} |x|^{-s \cdot \frac{n}{s}}$

$$= C \cdot \int_0^\infty r^{-\alpha} \cdot r^{n-1}\, dr = \infty.$$

the proof of HL Ineq. can be found in the Book by Lieb-Loss.

Ex 3.
Ex 4.
Ex 6

follow the notes.

Ex5 is standard, one can found their proof in the references cited by Javier.