Stochastic Processes, MA04243, Spring 2019, Homework 6

due: Sunday afternoon, 28 April

1. Let N be a Poisson random variable with parameter λ. Prove that
$$Ee^{i\theta N} = \exp(-\lambda(1-e^{i\theta})), \quad \theta \in \mathbb{R}.$$

2. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with parameter λ, and $(\mathcal{F}_t)_{t\geq 0}$ be the canonical (or natural, or induced) filtration of N. Define $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ respectively by
$$X_t = (N_t - \lambda t)^2 - \lambda t \quad \text{and} \quad Y_t = \exp(\theta N_t - \lambda t(e^\theta - 1)).$$
Prove that X and Y are martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$.

3. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with parameter λ, and $(\xi, \xi_1, \xi_2, \cdots)$ be an independent sequence of i.i.d. random variables (independent of N) such that the random variable ξ has the distribution μ. Define the compound Poisson process $X = (X_t)_{t\geq 0}$ by $X_t = \xi_1 + \cdots + \xi_{N_t}$. Prove that
$$Ee^{i\theta X_t} = \exp\left(-\lambda t \int (1-e^{i\theta x})\mu(dx)\right), \quad \theta \in \mathbb{R}.$$

4. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with parameter λ, and $(\xi, \xi_1, \xi_2, \cdots)$ be an independent sequence of i.i.d. random variables (independent of N) such that $P(\xi = 1) = p$ and $P(\xi = -1) = q$ with $p+q = 1$. Define $N^p = (N_t^p)_{t\geq 0}$ by letting N_t^p be the total number of appearances of "1" in the sequence $(\xi_1, \cdots, \xi_{N_t})$, and similarly define $N^q = (N_t^q)_{t\geq 0}$ by letting N_t^q be the total number of appearances of "-1" in the sequence $(\xi_1, \cdots, \xi_{N_t})$. Prove that N^p is a Poisson process with parameter λp, N^q is a Poisson process with parameter λq, and N^p and N^q are independent.

5. Let $(S, \mathcal{S}, \mu)$ be a σ-finite measure space, the random measure M on $(S, \mathcal{S})$ (that is, $M = (M(\omega, B))_{\omega\in\Omega, B\in\mathcal{S}})$ be a Poisson measure with intensity μ, and f be a nonnegative measurable function on $(S, \mathcal{S})$. Write $\mu f = \int_S f(x)\mu(dx)$ and $(Mf)(\omega) = \int_S f(x)M(\omega, dx)$. Prove that
$$E(Mf) = \mu f \quad \text{and} \quad E\exp(-Mf) = \exp(-\mu(1-e^{-f})).$$