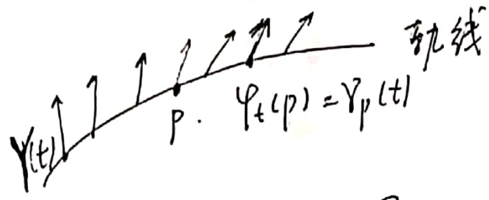
·M: 光滑流形, X: 向量场, φ_t: X生成(局部)单参数变换群.



$$\varphi_t: M\to M,\quad p\mapsto \varphi_t(p)$$

$$\varphi_t^{-1}: M\to M,\quad \varphi_t(p)\to p$$

$$(\varphi_t^{-1})_*: T_{\varphi_t(p)}M\to T_pM$$

$Y(t)\triangleq Y|_{\gamma_p(t)}$ 也是M光滑向量场, $Y(t)\in T_{\varphi_t(p)}M$.

当t变化, $(\varphi_t^{-1})_*Y(t)=(\varphi_t^{-1})_*Y|_{\varphi_t(p)}\subset T_pM$中一条曲线

$$\frac{d}{dt}\Big|_{t=0}(\varphi_t^{-1})_*Y|_{\varphi_t(p)}=?\quad\in T_pM.$$

Def. (李导数) M: m-dim 光滑流形, X,Y: 光滑向量场, φ_t: X生成(局部)单参数变换群, 称

$$\mathcal{L}_XY\triangleq\frac{d}{dt}\Big|_{t=0}(\varphi_t^{-1})_*Y=\lim_{t\to0}\frac{(\varphi_t^{-1})_*Y|_{\varphi_t(p)}-Y|_p}{t}$$ 为<u>Y关于X的李导数</u>.

Prop. $\mathcal{L}_XY=[X,Y]$.

Pf: Step1. $\forall p\in M$, $f\in C^\infty$, $F(t)\triangleq f(\varphi_t^{-1}(p))=f(\varphi_{-t}(p))$, 则

$$F(t)-F(0)=\int_0^1\frac{dF(st)}{ds}ds=t\int_0^1F'(u)|_{u=st}ds$$

记 $g_t(p)=\int_0^1F'(u)|_{u=st}ds$, 则 $g_0(p)=\int_0^1\frac{df(\varphi_u^{-1}(p))}{du}\Big|_{u=0}ds=-X_pf\Rightarrow g_0=-Xf$

且 $F(t)=F(0)+tg_t(p)=f(p)+tg_t(p)$

Step2. $\forall q\in M$,
$$\begin{aligned}(\mathcal{L}_XY)_qf&=\lim_{t\to0}\frac{(\varphi_t^{-1})_*Y|_{\varphi_t(q)}-Y|_q}{t}f\\&=\lim_{t\to0}\frac{(\varphi_t^{-1})_*Y|_{\varphi_t(q)}f-Y_qf}{t}\\&=\lim_{t\to0}\frac{Y|_{\varphi_t(q)}(f\circ\varphi_t^{-1})-Y_qf}{t}\\&\overset{f\circ\varphi_t^{-1}}{=}\lim_{t\to0}\frac{Y|_{\varphi_t(q)}(f+tg_t)-Y_qf}{t}\\&=\lim_{t\to0}\frac{Y_{\varphi_t(q)}f-Y_qf}{t}+\lim_{t\to0}Y_{\varphi_t(q)}g_t\triangleq\mathrm{I}+\mathrm{II}\end{aligned}$$

其中, $\mathrm{I}=\lim_{t\to0}\frac{Yf|_{\varphi_t(q)}-Yf|_q}{t}=X_q(Yf)$, $\mathrm{II}=Y_q(-Xf)=-Y_q(Xf)$

$\therefore(\mathcal{L}_XY)_qf=X_q(Yf)-Y_q(Xf)=[X,Y]|_qf$. #

Def. (李导数 for Ω^k). M: 光滑流形, X: 光滑向量场, φ_t: X生成(局部)单参数变换群, $\omega\in\Omega^k(M)$. 称 $\mathcal{L}_X\omega\triangleq\frac{d}{dt}\big|_{t=0}\varphi_t^*\omega$ 为ω的<u>李导数</u>.

Ex. 类似于向量场, 微分形式, 可定义一般(r,s)型张量场李导数

Prop. $\forall Y_1,\cdots,Y_k\in\Gamma(TM)$，有 $(\mathcal{L}_X\omega)(Y_1,\cdots,Y_k)=X(\omega(Y_1,\cdots,Y_k))-\sum_{l=1}^{k}\omega(Y_1,\cdots,\mathcal{L}_XY_l,\cdots,Y_k)$.

Pf: 仅证$k=1$情形.

$$\forall p\in M,\ (\mathcal{L}_X\omega)_p(Y_p)=\left(\frac{d}{dt}\Big|_{t=0}\varphi_t^*\omega|_{\varphi_t(p)}\right)(Y_p)=\lim_{t\to0}\frac{\varphi_t^*\omega|_{\varphi_t(p)}-\omega_p}{t}(Y_p)$$

其中
$$\varphi_t^*\omega|_{\varphi_t(p)}(Y_p)-\omega_p(Y_p)=\varphi_t^*\omega|_{\varphi_t(p)}(Y_p)-\omega|_{\varphi_t(p)}(Y|_{\varphi_t(p)})+\omega|_{\varphi_t(p)}(Y|_{\varphi_t(p)})-\omega_p(Y_p)$$
$$=-\varphi_t^*\omega|_{\varphi_t(p)}\left(-Y_p+(\varphi_t^{-1})_*Y|_{\varphi_t(p)}\right)+\omega(Y)|_{\varphi_t(p)}-\omega(Y)|_p.$$

$$\therefore(\mathcal{L}_X\omega)_p(Y_p)=-\omega_p((\mathcal{L}_XY)_p)+X_p(\omega(Y))\Rightarrow(\mathcal{L}_X\omega)(Y)=X(\omega(Y))-\omega(\mathcal{L}_XY).\ \#$$

Rmk. 也可写为 $X(\omega(Y))=(\mathcal{L}_X\omega)(Y)+\omega(\mathcal{L}_XY)$. (Leibniz法则)

Thm. G:李群，$X,Y\in T_eG$，$ad(X)(Y)=[X,Y]$.

Recall: $\forall g\in G$，$\alpha_g: G\to G,\ h\mapsto ghg^{-1}=L_g\circ R_{g^{-1}}(h)$；$Ad: G\to GL(\mathfrak{g},\mathbb{R}),\ g\mapsto(\alpha_g)_*$；$ad=(Ad)_*: T_eG\to gl(\mathfrak{g},\mathbb{R})$

Pf: Step1. 设$X\in T_eG$决定的左不变向量场为$\tilde{X}$，φ_t: $\tilde{X}$生成的单参数变换群，则$a_t=\varphi_t(e)$是$\tilde{X}$生成的单参数子群

$$\forall g\in G,\ \varphi_t(g)=\varphi_t L_g(e)\overset{(L_g)_*\tilde{X}=\tilde{X}}{=}L_g\varphi_t(e)=L_g(a_t)=g\cdot a_t=R_{a_t}(g)\Rightarrow\varphi_t=R_{a_t}$$

即，$\tilde{X}$生成单参数变换群为$\tilde{X}$生成的单参数子群的右移动.（之前HW的解答）

Step2.
$$ad(X)(Y)=\frac{d}{dt}\Big|_{t=0}Ad(a_t)(Y)=\lim_{t\to0}\frac{Ad(a_t)(Y)-Ad(e)(Y)}{t}=\lim_{t\to0}\frac{(L_{a_t})_*(R_{a_t^{-1}})_*(Y)-Y}{t}$$
$$=\lim_{t\to0}\frac{(R_{a_t^{-1}})_*(\tilde{Y}|_{\varphi_t(e)})-\tilde{Y}|_e}{t}\overset{\text{step1}}{=}\lim_{t\to0}\frac{(\varphi_t^{-1})_*(\tilde{Y}|_{\varphi_t(e)})-\tilde{Y}|_e}{t}=(\mathcal{L}_{\tilde{X}}(\tilde{Y}))_e=[\tilde{X},\tilde{Y}]|_e=[X,Y].\ \#$$

Rmk. 约定 $\forall f\in\Omega^0(M)$，$\mathcal{L}_X(f)\triangleq X(f)$

Def (interior product) 设$X\in\Gamma(TM)$，定义 $C_X:\Omega^k(M)\longrightarrow\Omega^{k-1}(M)$
$$\omega\mapsto C_X\omega:\ C_X\omega(Y_1,\cdots,Y_{k-1})\triangleq\omega(X,Y_1,\cdots,Y_{k-1}),\ \forall Y_l\in\Gamma(TM)$$

Ex. $\omega\in\Omega^k(M)$，$\theta\in\Omega^l(M)$，则 $C_X(\omega\wedge\theta)=C_X\omega\wedge\theta+(-1)^k\omega\wedge C_X\theta$.

Prop. (H. Cartan) ① $d\circ\mathcal{L}_X=\mathcal{L}_X\circ d$ ② $\mathcal{L}_X\mathcal{L}_Y-\mathcal{L}_Y\mathcal{L}_X=\mathcal{L}_{[X,Y]}$ ③ $\mathcal{L}_X=d\circ C_X+C_X\circ d$ ④ $\mathcal{L}_X\circ C_Y-C_Y\circ\mathcal{L}_X=C_{[X,Y]}$

Pf of ③: $k=1$情形. $\forall\omega\in\Omega^1(M)$，$Y\in\Gamma(TM)$

LHS: $(\mathcal{L}_X\omega)(Y)=X(\omega(Y))-\omega(\mathcal{L}_XY)=X(\omega(Y))-\omega([X,Y])$

RHS: $(dC_X\omega)(Y)=(d\omega(X))(Y)=Y(\omega(X))$

$(C_Xd\omega)(Y)=d\omega(X,Y)=X(\omega(Y))-Y(\omega(X))-\omega([X,Y])$ #.

· $\varphi:\mathbb{R}\times M\to M$，$(t,p)\mapsto\varphi_t(p)=\varphi(t,p)$ 有 ① $\varphi_0(p)=p$，② $\varphi_{t+s}=\varphi_t\circ\varphi_s$ ∴ 故$\mathbb{R}$可看成1-维李群作用在M.

Def(李氏变换群) M: m-dim光滑流形，G: r维李群. 若$\exists$光滑$\theta: G\times M\to M$，$(g,x)\mapsto g\cdot x\triangleq\theta(g,x)$，有 ① $\forall x\in M$，$e\cdot x=x$ ② $\forall g_1,g_2\in G$，$x\in M$，$g_1\cdot(g_2\cdot x)=(g_1\cdot g_2)x$

则称G为左作用在M上的李氏变换群.