Week 10

Ex1 Strongly singular integrals

In this exercise, we try to weaken the Hörmander condition in C-3. theorem at the sacrafice that we should strengthen the conditions on the Fourier transform.

(1) let $\phi \in C^\infty(\mathbb{R}^n)$ be a function that vanishes near the origin and is identically equal one outside a compact set. for $0<a<1$, $b>0$, define

$$T_{a,b} f = K_{a,b} * f$$

where

$$\widehat{K_{a,b}}(\xi) = \phi(\xi) \frac{e^{i|\xi|^a}}{|\xi|^b}.$$

Show that

(a)
$$|\widehat{K_{a,b}}(\xi)| \leq \frac{A}{(1+|\xi|)^b}$$
for some constant $A>0$;

(b)
$$|K_{a,b}(x)| \approx \frac{B}{|x|^{n+\delta}}, \quad \delta = \frac{\frac{na}{2}-b}{1-a}$$
for x close to the origin.

(c) show that the kernel $K_{a,b}(x)$ does not satisfy the gradient condition

(2) by admitting

thm A the operator $T_{a,b}$ is bounded on L^p if

$$(*) \qquad \left|\frac{1}{2}-\frac{1}{p}\right| < \frac{(b/n)\cdot(n/2+\delta)}{b+\delta}, \quad \delta = \frac{\frac{na}{2}-b}{1-a}$$

and is unbounded if the reverse inequality holds

show that the Hörmander condition fails to hold as well.

(3) Admitting

thm B: let K be a tempered distribution on $\mathbb{R}^n$ with compact support which coincides with a locally integrable function away from the origin. Suppose there exists $\theta \in (0,1)$ s.t.

$$|\hat{K}(\xi)| \leq \frac{A}{(1+|\xi|)^{n\theta/2}}, \quad \xi \in \mathbb{R}^n$$

$$\int_{|x|>2|y|^{1-\theta}} |K(x-y)-K(x)|\,dx \leq B, \quad |y| \leq 1$$

then the operator $Tf = K*f$ is bounded on L^p, $1<p<\infty$, and is weak(1,1).

show that the operator $T_{a,b}$ is bounded on L^p for p satisfying

$$\left|\frac{1}{2}-\frac{1}{p}\right| \leq \frac{\frac{b}{n}\left(\frac{n}{2}+\delta\right)}{b+\delta}, \quad \delta = \frac{\frac{na}{2}-b}{1-a}$$