EXERCISE 1

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1. Suppose $X_1, \dots, X_n \sim F$, $F_n(x)$ is the EDF. For a fixed x, solve the asymptotic distribution of $\sqrt{F_n(x)}$.

Solve. The *Central Limit Theorem* (CLT) gives that

$$\sqrt{n}\left(F_n(x) - F(x)\right) \to N\left(0, F(x)(1-F(x)\right).$$

Applying *Delta Method* to $g(x) = \sqrt{x}$, we have

$$\sqrt{n}\left(\sqrt{F_n(x)} - \sqrt{F(x)}\right) \to N\big(0, F(x)(1-F(x)\left(g'(F(x))\right)^2\big) = N\left(0, \frac{1-F(x)}{4}\right).$$

Thus,

$$\sqrt{F_n(x)} \to N\left(\sqrt{F(x)}, \frac{1-F(x)}{4n}\right).$$

□

2. Suppose $x \neq y \in \mathbb{R}$, $F_n(x)$ is the EDF. Solve $Cov\left(F_n(x), F_n(y)\right)$.

Solve. Without loss of generality (WLOG), suppose $x < y$. We have that $\mathbb{E}F_n(x) = F(x)$, $\mathbb{E}F_n(y) = F(y)$. We only need to calculate $\mathbb{E}F_n(x)F_n(y)$. Notice that

$$P(F_n(x) = \frac{i}{n}, F_n(y) = \frac{j}{n}) = \frac{n!}{i!(j-i)!(n-j)!}F(x)^i\left(F(y)-F(x)\right)^{j-i}\left(1-F(y)\right)^{n-j},\ i \leq j,$$

therefore we have

$$\begin{aligned}
&\mathbb{E}F_n(x)F_n(y)\\
=&\sum_{i\leq j}\frac{ij}{n^2}\frac{n!}{i!(j-i)!(n-j)!}F(x)^i\left(F(y)-F(x)\right)^{j-i}\left(1-F(y)\right)^{n-j}\\
=&\sum_{j=1}^{n}\frac{j}{n(n-j)!}\left(1-F(y)\right)^{n-j}\sum_{i=1}^{j}\frac{(n-1)!}{(i-1)!(j-i)!}F(x)^i\left(F(y)-F(x)\right)^{j-i}\\
=&\sum_{j=1}^{n}\frac{j(n-1)!}{n(n-j)!(j-1)!}\left(1-F(y)\right)^{n-j}F(x)F(y)^{j-1}\sum_{i=0}^{j-1}\frac{(j-1)!}{i!(j-i-1)!}\left(\frac{F(x)}{F(y)}\right)^i\left(1-\frac{F(x)}{F(y)}\right)^{j-i-1}\\
=&F(x)\sum_{j=0}^{n-1}\frac{j+1}{n}\frac{(n-1)!}{(n-j-1)!j!}F(y)^j\left(1-F(y)\right)^{n-j-1}
\end{aligned}$$

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$$=F(x)\frac{(n-1)F(y)+1}{n}.$$

Now we conclude that

$$\begin{aligned}Cov\left(F_n(x),F_n(y)\right)&=\mathbb{E}F_n(x)F_n(y)-\mathbb{E}F_n(x)\mathbb{E}F_n(y)\\&=\frac{(n-1)F(x)F(y)+F(x)}{n}-F(x)F(y)\\&=\frac{F(x)\left(1-F(y)\right)}{n},\end{aligned}$$

if $x<y$. Otherwise, change the order of x and y. □

Remark 1. *Another way is to write $F_n(x)=\sum_{i=1}^n \mathbf{1}_{X_i\le x}$, $F_n(y)=\sum_{j=1}^n \mathbf{1}_{X_j\le y}$. Then*

$$\begin{aligned}Cov\big(F_n(x),F_n(y)\big)=&\sum_{i,j}Cov\big(\mathbf{1}_{X_i\le x},\mathbf{1}_{X_j\le y}\big)\\=&nCov\big(\mathbf{1}_{X_1\le x},\mathbf{1}_{X_1\le y}\big)+n(n-1)Cov\big(\mathbf{1}_{X_1\le x},\mathbf{1}_{X_2\le y}\big),\end{aligned}$$

where the two covariances at the right-hand side is easy to compute.

3. Suppose the order statistics $X_{(1)}\le\ldots\le X_{(n)}$ are from the law F. Prove that for any $0<\beta<1$, we have $P\big(F(X_{(n)})-F(X_{(1)})>\beta\big)=1-n\beta^{n-1}+(n-1)\beta^n$.

Proof. Notice that $F(X)\sim U[0,1]$, and $F(X_{(1)})\le\ldots\le F(X_{(1)})$ are the ordered statistics from the law $U[0,1]$. The proof comes from conditioning on $F(X_{(n)})$. For any $x\in(0,1)$,

$$P(F(X_{n)})<x)=P(F(X_1),\ldots,F(X_n<x)=x^n,$$

thus the pdf of $F(X_{(n)})$ is $f_{F(X_{(n)})}(x)=nx^{n-1}$. Conditioned on $F(X_{(n)})$, we have

$$\begin{aligned}P\big(F(X_{(n)})-F(X_{(1)})>\beta|F(X_{(n)})\big)=&P\big(F(X_{(1)})<F(X_{(n)})-\beta|F(X_{(n)})\big)\\=&1-(\frac{\beta}{F(X_{(n)})})^{n-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}P\big(F(X_{(n)})-F(X_{(1)})>\beta\big)=&\int_\beta^1\left[1-(\frac{\beta}{x})^{n-1}\right]nx^{n-1}dx\\=&1-\beta^n-n\beta^{n-1}(1-\beta)\\=&1-n\beta^{n-1}+(n-1)\beta^n.\end{aligned}$$

□

4. Suppose $X_1,\ldots,X_n$ are simple samples from the distribution $U(0,1)$. Prove that the medium $\hat{\xi}_{n,1/2}$ has asymptotic distribution $N(\frac{1}{2},\frac{1}{4n})$.

Proof. According to the large sample property (page 25, Lec1.pdf), we have

$$\sqrt{n}(\hat{\xi}_{n,1/2}-\xi_{1/2})\to N\big(0,\frac{1}{4f^2(\xi_{1/2})}\big).$$

Plugging in $\xi_{1/2}=\frac{1}{2}$ and that f is continuous at $\frac{1}{2}$ with $f(\frac{1}{2})=1$ completes the proof. □