Computer Graphics (COMP3271)
Written Assignment 1 Solution

1. (10 marks)

Given a unit quaternion $q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and a 3D vector $\mathbf{v} \in \mathbb{R}^3$, find the rotated vector $\mathbf{v}'$ opereted by q.

(a) $q = \frac{\sqrt{2}}{2} + 0\mathbf{i} + 0\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$, $\mathbf{v} = (1, 2, 1)^T$

(b) $q = -\frac{1}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$, $\mathbf{v} = (1, 0, 0)^T$

Solution

(a) The operation equals to rotating $\mathbf{v}$ through 90 degrees about $(0, 0, 1)^T$, and $\mathbf{v}' = (-2, 1, 1)^T$.

(b) The operation equals to rotating $\mathbf{v}$ through 240 degrees about $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T$, and $\mathbf{v}' = (0, 0, 1)^T$.

2. (10 marks)

Derive the rotation transformation of θ degrees about the point $(a, b)^T$ in the 2D plane.

(a) Write the transformation in the form of $X' = MX + B$.

(b) Write the transformation in the form of $X' = NX$ in homogeneous coordinates.

Solution

The idea is:

1. translate $(a, b)^T$ to the origin;

2. rotate θ degrees about the origin;

3. translate the point back.

(a)

$$X' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a - a * \cos\theta + b * \sin\theta \\ b - a * \sin\theta - b * \cos\theta \end{bmatrix}$$

(1)

(b)

$$X' = \begin{bmatrix} \cos\theta & -\sin\theta & a - a * \cos\theta + b * \sin\theta \\ \sin\theta & \cos\theta & b - a * \sin\theta - b * \cos\theta \\ 0 & 0 & 1 \end{bmatrix} X$$

(2)

3. (20 marks)

Derive the 3×3 homogeneous transformation matrix for the reflection about the line $y = cx + d$ in the $2D$ plane.

Solution

The idea is:

1. translate the line along y-axis by $-d$ to the origin;

2. rotate the line to x-axis about the origin;

3. reflect across x-axis;

4. rotate the line back;

5. translate the line back.

Suppose $\tan\theta = c$, the transformation matrix M is,

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin^2\theta + \cos^2\theta & 2\sin\theta\cos\theta & -2d\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & -d\left(\sin^2\theta - \cos^2\theta\right) + d \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta & -d\sin 2\theta \\ \sin 2\theta & -\cos 2\theta & d(1 + \cos 2\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

Because $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-c^2}{1+c^2}$, $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2c}{1+c^2}$, we get the result as

$$M = \begin{bmatrix} \frac{1-c^2}{1+c^2} & \frac{2c}{1+c^2} & -\frac{2cd}{1+c^2} \\ \frac{2c}{1+c^2} & -\frac{1-c^2}{1+c^2} & d\frac{1-c^2}{1+c^2} + d \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

4. (10 marks)

Given two triangles ΔABC and $\Delta A'B'C'$ in 2D plane with $A = (0,0)^T$, $B = (2,0)^T$, $C = (0,1)^T$, $A' = (0,1)^T$, $B' = (0,2)^T$, and $C' = (-1,-2)^T$. Derive the affine transformation matrix for the transformation T that maps ΔABC to $\Delta A'B'C'$ such that $T(A) = A'$, $T(B) = B'$, $T(C) = C'$.

solution

We represent ΔABC in the matrix form.

$$M = \begin{bmatrix} A & B & C \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{5}$$

The matrix form of $\Delta A'B'C'$ is

$$M' = \begin{bmatrix} A' & B' & C' \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix} \tag{6}$$

The transformation can be represented as

$$M' = TM \tag{7}$$

So we can compute the transformation matrix as follows.

$$\begin{aligned} T &= M'M^{-1} \\ &= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{2} & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{8}$$

5. (10 marks)

Give a sequence of 4 x 4 matrices that transforms the unit square in the left figure to the parallelogram in the right. Find a sequence of OpenGL function calls that implements these transformations.
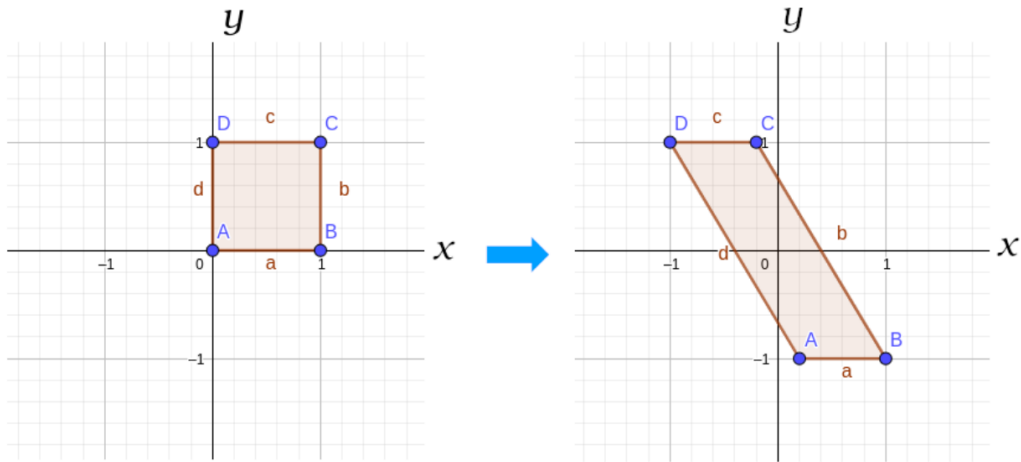
solution

The transformation matrix is,

$$
\begin{bmatrix}
1 & -0.6 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\times
\begin{bmatrix}
1 & 0 & 0 & -0.4 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\times
\begin{bmatrix}
0.8 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\tag{9}
$$

$$\qquad\quad Shearing \qquad\qquad\quad Translation \qquad\qquad Scaling$$

The sequence of OpenGL function calls is:

GLfloat M[16] = {1, 0, 0, 0, -0.6, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1};
glMultMatrixf(M);
glTranslatef(-0.4, -1.0, 0.0);
glScalef(0.8, 2.0, 1.0);

6. **(10 marks)** Show that any sequence of rotations and translations can be replaced by a single rotation about the origin followed by a translation.

solution

We can show by simply multiplying 4×4 matrices that the concatenation of two rotations yields a rotation and than the concatenation of two translations yields a translation. By looking at the product of a rotation and a translation, we find left three columns of RT are the left three columns of R and the right column of RT is the right column of the translation matrix. If we now consider RTR' where R' is a 4×4 rotation matrix, the left three columns are exactly the same as the left three columns of RR' and the right column still has 1 as its bottom element. Thus, the form is the same as RT with an altered rotation (which is the concatenation of the two rotations) and an altered

translation. Inductively, we can see that any further concatenations with rotations and translations do no alter this form.

7. **(10 marks)** Consider the line in $\mathbb{R}^3$ given by

$$L(t) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let Q_1 and Q_2 be two points on this line that are at distance 1 apart from each other. We now transform the line by applying transformation T given by

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What is the distance between $T(Q_1)$ and $T(Q_2)$?

Solution 1

Suppose

$$Q_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Because the distance of Q_1Q_2 is 1, we have $\|a - b\| = \frac{\sqrt{2}}{2}$.

$$T(Q_1) = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ a+1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a+2 \\ a+2 \\ 5 \\ 1 \end{pmatrix}$$

$$T(Q_2) = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ b+1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2b+2 \\ b+2 \\ 5 \\ 1 \end{pmatrix}$$

$$\tag{11}$$

Distance of $T(Q_1)T(Q_2)$ is $\sqrt{(2a - 2b)^2 + (a - b)^2} = \sqrt{\frac{5}{2}}$.

Solution 2

The unit vector of the direction of Q_1Q_2 is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T$ and the length of Q_1Q_2 is 1.

Apply the transformation matrix T to the unit vector:

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

So we have the length of the transformed unit vector as $\sqrt{(\sqrt{2})^2 + (\frac{1}{\sqrt{2}})^2} = \sqrt{\frac{5}{2}}$.
Since the original length of $Q_1 Q_2$ is 1, we have the length of the transformed $Q_1 Q_2$ as $\sqrt{\frac{5}{2}} \times 1 = \sqrt{\frac{5}{2}}$.

8. **(20 marks)**

Given an affine transformation $X' = MX + B$, where M is a 2 by 2 matrix and B is a 2D vector, find the equation of the image E of the circle $x^2 + y^2 - 1 = 0$ under this transformation. Show that E is an ellipse.

Solution

We denote the unit circle as $X^T X = 1$, where $X = (x, y)^T$, and define an arbitrary 2D affine transformation $X' = MX + B$, where M is a nonsingular 2×2 matrix, and B is a translation vector of 2×1. So we have $X = M^{-1}(X' - B)$. Substitute it into $X^T X = 1$, and we have

$$\begin{aligned} &\left[M^{-1}(X' - B) \right]^T \left[M^{-1}(X' - B) \right] = 1 \\ \Leftrightarrow &(X' - B)^T \left(M^{-1} \right)^T M^{-1} (X' - B) = 1 \end{aligned} \tag{13}$$

Now we show that $(M^{-1})^T M^{-1}$ is a symmetric positive definite matrix. This is because $((M^{-1})^T M^{-1})^T = (M^{-1})^T M^{-1}$ and for any nonzero vector Y,

$$Y^T (M^{-1})^T M^{-1} Y = (M^{-1}Y)^T M^{-1} Y = \| M^{-1} Y \|^2 > 0, \tag{14}$$

since M is nonsingular and $M^{-1}Y \neq 0$.
And one definition of ellipse is

$$\left\{ X | (X - X_c)^T A (X - X_c) = 1 \right\} \tag{15}$$

with $X, X_c \in R^n$, X_c the center of ellipse, and $A_{n \times n}$ a symmetric positive definite matrix. Therefore X' is on an ellipse and B is the center.