The Dynamics of a Blood Cell Model

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Outline

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Introduction

What is a Dynamical System?

- A dynamical system is a process that evolves over time.
- The future behavior depends on the current state and the model parameters.

Examples

- Exploring the complex, always changing behavior of market prices in the stock market.
- The evolution of atmospheric variables such as temperature, pressure, wind, and humidity over time and space.

Goal

- Understanding how the state of a system changes over time.

Problem Statement

Objective Identify a mathematical model to describe the dynamics of the life cycle of blood cells over time.

- One such model is the **Mackey-Glass Model** given by

$$f(x) = \beta \cdot \frac{x_\tau}{1 + x_\tau^n} - \gamma x$$

  - x_τ is the blood cell concentration at time $(t - \tau)$,
  - τ is the delay due to maturation.
  - β is the maximum production rate.
  - n determines sensitivity or steepness of the feedback inhibition.
  - γ represents the natural destruction or removal.

Problem Statement

The Modified Mackey-Glass Model (delay-free):

$$f(x) = \frac{b\theta^m x}{\theta^m + x^m} - cx$$

- **# of Blood Cells:** x
- **Production:** $\frac{b\theta^m x}{\theta^m + x^m}$ — feedback-regulated, non-linear production.
- **Destruction:** $cx, 0 < c < 1$ — linear degradation of blood cells.
- **No delay term:** assumes instantaneous response in production feedback.

Goal: Explore the number of blood cells in a human body over time

Methodology

- Progression of time is modeled using an iterative process.
- **Iterative process** is a numerical method that involves repeatedly applying the model over and over again using the output as next input.
- Using iterative process we explore the long-term behaviour of the blood cell model.
- Steady state, periodicity and instability of the blood cell model will be explored.

Methodology: Analysis of Fixed Points

A system at a fixed point stays at a steady state

- Fixed points are solutions of $f(x) = x$
- For our model, solve

$$\frac{b\theta^m x}{\theta^m + x^m} - cx = x$$

- The Fixed Points are:

$$x^* = \theta\left(\frac{b}{1+c} - 1\right)^{\frac{1}{m}}$$

Methodology: Classification of Fixed Points

- **Attractive Fixed Point** A point x_0 is called an attractive fixed point if the system starts near x_0 and generates a sequence of successive points through iteration, these points eventually converge to x_0.
- Mathematically, x_0 is an attractive fixed point if $|f'(x_0)| < 1$.
- If a system starts near an attractive fixed point, then this indicates stability (steady state) of the system.

Methodology: Classification of Fixed Points

- **Repelling Fixed Point** A point x_0 is called a repelling fixed point if the system starts near x_0 generates a sequence of successive points through iteration that eventually diverges away from x_0.
- Mathematically, x_0 is an attractive fixed point if $|f'(x_0)| > 1$.
- If a system starts near a repelling fixed point, then this indicates sensitivity to initial conditions.

Results: Attractive Fixed Points

Theorem Let

$$f(x) = \frac{b\theta^m x}{\theta^m + x^m} - cx, \text{ with } 0 < c < 1.$$

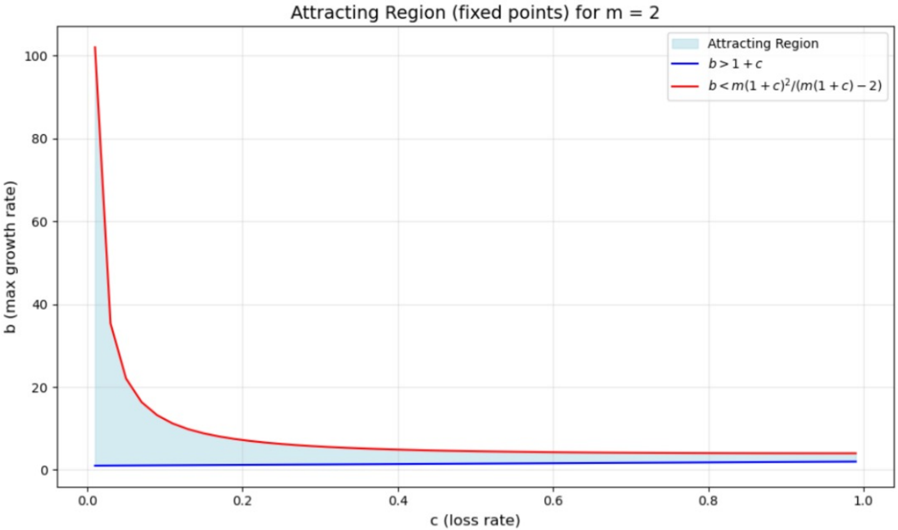
If

$$m > \frac{2}{c+1} \quad \text{and} \;\; 1 + c < b < \frac{m(c+1)^2}{-2 + mc + m}, \text{ then,}$$

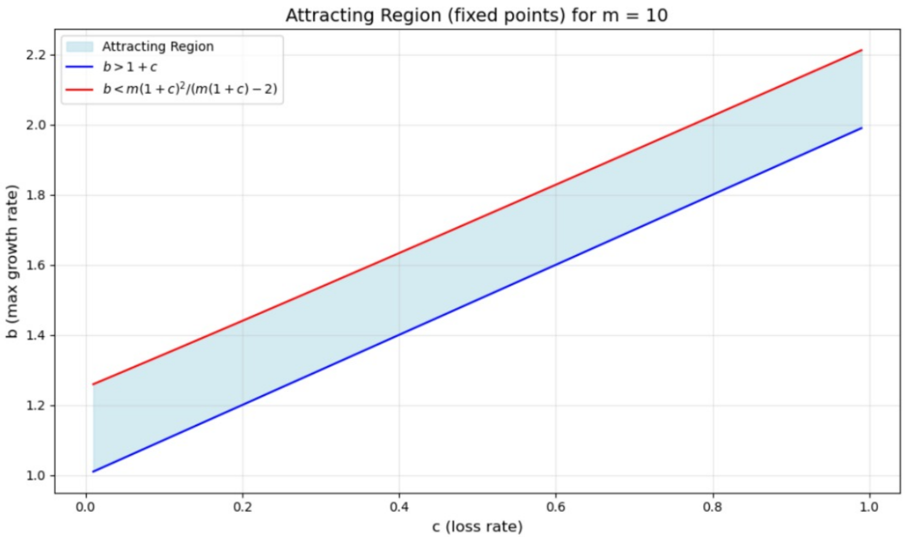
$$x_0 = \theta \left(\frac{b}{1+c} - 1 \right)^{\frac{1}{m}}$$

is an attractive fixed point.

Results: Attractive Fixed Points: EXAMPLE



Results: Attractive Fixed Points: EXAMPLE



Analysis of 2-Cycles

A pair of two points x_1 and x_2 is called a 2- cycle if

$$\mathrm{x}_1 \;\rightarrow\; f(x_1) = x_2 \;\rightarrow\; f(x_2) = x_1$$

To Find a **2 -cycle**, solve,

$$f^2(x) = f(f(x)) = x$$

A 2-cycle with points x_1 and x_2 is called an attractive 2-cycle if

$$|f'(x_1) \cdot f'(x_2)| < 1.$$

Note: A system that starts at a 2-Cycle oscillates between x_1 and x_2 indefinitely, creating a periodic cycle.

Results: Classification of 2-Cycles

Special Case

$$m = 1,\ \theta = 1$$

The model:

$$f(x) = \frac{bx}{1+x} - cx$$

Results: Classification of 2-Cycles

Let

$$\Delta = b^2c^2 - 2b^2c + b^2 - 2bc^3 + 2bc^2 - 2bc + 2b + c^4 - 2c^2 + 1$$

Then the 2-cycle points are:

$$x_1 = 0$$

$$x_2 = \frac{b - c - 1}{c + 1}$$

$$x_3 = \frac{bc - b - c^2 + 2c - \sqrt{\Delta} - 1}{2c^2 - 2c}$$

$$x_4 = \frac{bc - b - c^2 + 2c + \sqrt{\Delta} - 1}{2c^2 - 2c}$$

x_1 and x_2 are trivial cycle points since they are fixed points.

x_3 and x_4 are the real 2-cycle points

Examples of Attractive 2-Cycle Points

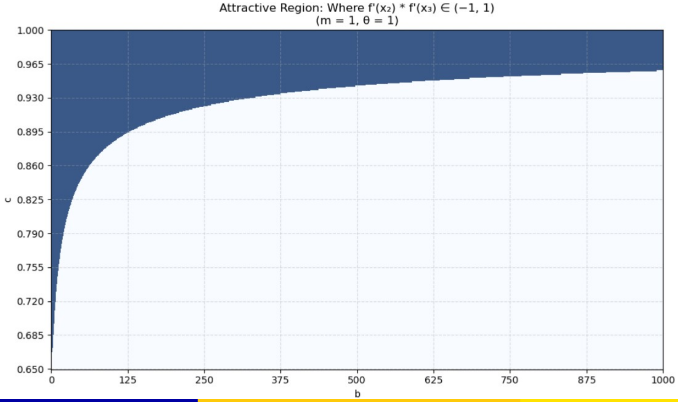
We display the region of of attractive 2-cycles for specific models given by

$$f(x) = \frac{\theta^m bx}{\theta^m + x^m} - cx$$

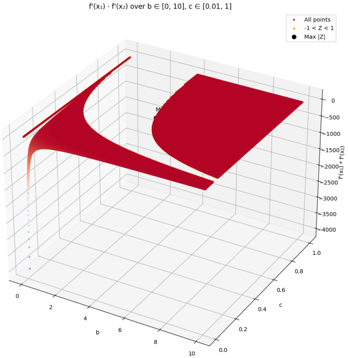
for

$$\theta = 1, m = 1, 2, 3, ...$$

$$f(x) = \frac{bx}{1+x} - cx, m = 1, \theta = 1, c \in (0,1), b > 0$$



$$f(x) = \frac{bx}{1+x^2} - cx, m = 2, \theta = 1, c \in (0,1), b > 0$$



Future Work:

- Find sufficient and necessary conditions for attractive and repelling fixed points for the general model.
- Find and characterize other periodic points
- Explore the possibility of the system undergoing into chaotic states

Conclusion: Biological Interpretation:

In the context of blood cell production, this 2-cycles could represent oscillations in cell production. For instance:

- x_1 might represent a lower or baseline production level.
- x_2 might represent a higher production level, possibly driven by feedback or external factors.
- The system thus oscillates between these two levels, maintaining a dynamic, cyclic rather than a steady state.
- This type of behaviour is critical for understanding how biological systems maintain homeostasis through feedback mechanisms.

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