國立臺灣大學

2021 數位控制系統

Digital Control System

Project #1

組別：第十三組

Group #13

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Problem A-1

Target

使用 Method #1 求出θ與i_a之間的 k-domain 差分方程式與 Z-domain 轉移函式，並將結果以 Fig. 2 Discrete-time model of position motion system 的 Block Diagram 形式表示。

Solution

Step1:

為了求$\theta(t)$與$i_a(t)$的關係，先將兩者的關係以動態方程式列出，其 O.D.E.如下。

$$J_m \frac{d^2\theta(t)}{dt^2} = K_T i_a(t)$$

Step2:

移項以便於列出微分方程式。

$$\frac{d^2\theta(t)}{dt^2} = \frac{K_T}{J_m} i_a(t)$$

Step3:

將微分方程式等號左右兩邊同取 Laplace 轉換。

$$\mathcal{L}\left\{\frac{d^2\theta(t)}{dt^2}\right\} = \mathcal{L}\left\{\frac{K_T}{J_m} i_a(t)\right\}$$

$$s^2\Theta(s) - s\theta(0) - \dot{\theta}(0) = \frac{K_T}{J_m} I_a(s)$$

Step4:

解 Laplace Function。

$$\Theta(s) = \frac{1}{s}\theta(0) + \frac{1}{s^2}\dot{\theta}(0) + \frac{K_T}{J_m s^2} I_a(s)$$

Step5:

考慮包含 Z.O.H. Input 之數學模型。

$$I_a(s)|_{at\ 0\sim T} = \frac{1}{s} I_a(0)$$

$$\Theta(s)|_{at\ 0\sim T}=\frac{1}{s}\theta(0)+\frac{1}{s^2}\dot{\theta}(0)+\frac{K_T}{J_m s^3}I_a(0)$$

Step6:

取 Laplace 反轉換。

$$\mathcal{L}^{-1}\{\Theta(s)|_{at\ 0\sim T}\}=\mathcal{L}^{-1}\left\{\frac{1}{s}\theta(0)+\frac{1}{s^2}\dot{\theta}(0)+\frac{K_T}{J_m s^3}I_a(0)\right\}$$

$$\mathcal{L}^{-1}\{\Theta(s)|_{at\ 0\sim T}\}=\theta(0)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\dot{\theta}(0)\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+\frac{K_T i_a(0)}{J_m}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$\theta(t)|_{at\ 0\sim T}=\theta(0)+\dot{\theta}(0)t+\frac{K_T i_a(0)}{2J_m}t^2$$

Step7:

找取樣瞬間的響應，滿足：t = kT = T。

$$\theta(T)=\theta(0)+\dot{\theta}(0)T+\frac{K_T i_a(0)}{2J_m}T^2$$

Step8:

建立0T~∞T的 Discrete-Time Model。

0T~1T	:	$\theta(T)=\theta(0)+\dot{\theta}(0)T+\frac{K_T T^2}{2J_m}i_a(0)$
1T~2T	:	$\theta(2T)=\theta(T)+\dot{\theta}(T)T+\frac{K_T T^2}{2J_m}i_a(T)$
2T~3T	:	$\theta(3T)=\theta(2T)+\dot{\theta}(2T)T+\frac{K_T T^2}{2J_m}i_a(2T)$
⋮	⋮	⋮
$(k-1)$T~kT	:	$\theta(kT)=\theta[(k-1)T]+T\dot{\theta}[(k-1)T]+\frac{K_T T^2}{2J_m}i_a[(k-1)T]$

Step9:

Z-Transform，並簡化 1T, 2T, …, kT 至 1, 2, …, k。

$$\mathcal{Z}\{\theta(k)\}=\mathcal{Z}\left\{\theta(k-1)+\dot{\theta}(k-1)T+\frac{K_T T^2}{2J_m}i_a(k-1)\right\}$$

$$\mathcal{Z}\{\theta(k)\}=\mathcal{Z}\{\theta(k-1)\}+T\mathcal{Z}\{\dot{\theta}(k-1)\}+\frac{K_T T^2}{2J_m}\mathcal{Z}\{i_a(k-1)\}$$

$$\Theta(Z) = Z^{-1}\Theta(Z) + TZ^{-1}\dot{\Theta}(Z) + \frac{K_T T^2}{2J_m} Z^{-1} I_a(Z) ---------(1)$$

因此我們還需要再額外求出$\dot{\Theta}(Z)$。

求$\dot{\Theta}(Z)$，根據物理定義，我們知道：$\dot{\theta}(t) = \omega_m(t)$，將 *Step1* 到 *Step9* 再做一次。
Step1: 為了求$\theta(t)$與$i_a(t)$的關係，先將兩者的關係以動態方程式列出，其 O.D.E.如下。 $$J_m \frac{d\omega_m(t)}{dt} = K_T i_a(t)$$
Step2: 移項以便於列出微分方程式。 $$\frac{d\omega_m(t)}{dt} = \frac{K_T}{J_m} i_a(t)$$
Step3: 將微分方程式等號左右兩邊同取 Laplace 轉換。 $$\mathcal{L}\left\{\frac{d\omega_m(t)}{dt}\right\} = \mathcal{L}\left\{\frac{K_T}{J_m} i_a(t)\right\}$$ $$s\mathcal{W}_m(s) - \omega_m(0) = \frac{K_T}{J_m} I_a(s)$$
Step4: 解 Laplace Function。 $$\mathcal{W}_m(s) = \omega_m(0)\frac{1}{s} + \frac{K_T}{J_m s} I_a(s)$$
Step5: 考慮包含 Z.O.H. Input 之數學模型。 $$I_a(s)\|_{at\ 0\sim T} = \frac{1}{s} I_a(0)$$ $$\mathcal{W}_m(s)\|_{at\ 0\sim T} = \frac{1}{s}\omega_m(0) + \frac{K_T}{J_m s^2} I_a(0)$$
Step6: 取 Laplace 反轉換。 $$\mathcal{L}^{-1}\{\mathcal{W}_m(s)\|_{at\ 0\sim T}\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\omega_m(0) + \frac{K_T}{J_m s^2} I_a(0)\right\}$$ $$\mathcal{L}^{-1}\{\mathcal{W}_m(s)\|_{at\ 0\sim T}\} = \omega_m(0)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{K_T i_a(0)}{J_m}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$ $$\omega_m(t)\|_{at\ 0\sim T} = \omega_m(0) + \frac{K_T i_a(0)}{J_m} t$$

Step7 & Step8:

找取樣瞬間的響應，滿足：t = kT = T。

$$\omega_m(T) = \omega_m(0) + \frac{K_T i_a(0)}{J_m} T$$

延伸至：

$$\omega_m(kT) = \omega_m\big((k-1)T\big) + \frac{K_T T}{J_m} i_a\big((k-1)T\big)$$

Step9:

Z-Transform，並簡化 1T, 2T, …, kT 至 1, 2, …, k。

$$\mathcal{Z}\{\omega_m(k)\} = \mathcal{Z}\left\{\omega_m(k-1) + \frac{K_T T}{J_m} i_a(\mathrm{k}-1)\right\}$$

$$\mathcal{W}_m(Z) = Z^{-1}\mathcal{W}_m(Z) + \frac{K_T T}{J_m} Z^{-1} I_a(Z)$$

Step10:

Find **Z-domain Transfer Function**。

$$\frac{\mathcal{W}_m(Z)}{I_a(Z)} = \frac{K_T T}{J_m} \frac{Z^{-1}}{1-Z^{-1}}$$

$$\mathcal{W}_m(Z) = \frac{K_T T}{J_m} \frac{Z^{-1}}{1-Z^{-1}} I_a(Z) = \dot{\Theta}(Z) ---------(2)$$

將(2)代入(1)，並且做 Z 反轉換：

$$\Theta(Z) = Z^{-1}\Theta(Z) + TZ^{-1}\frac{K_T T}{J_m}\frac{Z^{-1}}{1-Z^{-1}} I_a(Z) + \frac{K_T T^2}{2J_m} Z^{-1} I_a(Z)$$

$$(1-2Z^{-1}+Z^{-2})\Theta(Z) = \frac{K_T T^2}{2J_m}(Z^{-1}+Z^{-2}) I_a(Z)$$

$$\mathcal{Z}^{-1} \Rightarrow \boldsymbol{\theta(k) = 2\theta(k-1) - \theta(k-2) + \frac{K_T T^2}{2J_m}[i_a(k-1) + i_a(k-2)]}$$

Step10:

Find **Z-domain Transfer Function**。

$$\frac{\Theta(\mathrm{Z})}{\mathrm{I_a(Z)}} = \frac{\mathrm{K_T T^2}}{\mathrm{2J_m}} \frac{\mathrm{Z^{-1}(1+Z^{-1})}}{\mathrm{(1-2Z^{-1}+Z^{-2})}}$$

Step11:

Revisit the Z-domain equations to get the Z-domain block diagram.

$$\frac{\Theta(Z)}{\mathcal{W}_m(Z)} = \frac{\Theta(\mathrm{Z})}{\mathrm{I_a(Z)}}\frac{\mathrm{I_a(Z)}}{\mathcal{W}_m(Z)} = \frac{\Theta(\mathrm{Z})}{\mathrm{M_{air\_gap}(Z)}}\frac{\mathrm{M_{air\_gap}(Z)}}{\mathcal{W}_\mathrm{m}(\mathrm{Z})} = \frac{T}{2}\frac{1+Z^{-1}}{1-Z^{-1}}$$

***Result:**k-domain difference equations*

1. θ與i_a

✓ Analytical Value:

$$\theta(k) = 2\theta(k-1) - \theta(k-2) + \frac{K_T T^2}{2J_m}[i_a(\mathrm{k}-1) + i_a(\mathrm{k}-2)]$$

✓ Numerical Value:

$$\theta(k) = 2\theta(k-1) - \theta(k-2) + 1.867 \times 10^{-4}[i_a(\mathrm{k}-1) + i_a(\mathrm{k}-2)]$$

2. ω_m與i_a

✓ Analytical Value:

$$\omega_m(k) = \omega_m(k-1) + \frac{K_T T}{J_m} i_a(\mathrm{k}-1)$$

✓ Numerical Value:

$$\omega_m(k) = \omega_m(k-1) + 1.867 i_a(\mathrm{k}-1)$$

***Result:**Z-domain transfer functions*

1. θ與i_a

✓ Analytical Value:

$$\frac{\Theta(\mathrm{Z})}{\mathrm{I_a(Z)}} = \frac{\mathrm{K_T T^2}}{2\mathrm{J_m}} \frac{\mathrm{Z^{-1}(1+Z^{-1})}}{\mathrm{(1-2Z^{-1}+Z^{-2})}}$$

✓ Numerical Value:

$$\frac{\Theta(\mathrm{Z})}{\mathrm{I_a(Z)}} = 1.867 \times 10^{-4} \frac{\mathrm{Z^{-1}(1+Z^{-1})}}{\mathrm{(1-2Z^{-1}+Z^{-2})}}$$

2. ω_m與i_a

✓ Analytical Value:

$$\frac{\mathcal{W}_\mathrm{m}(\mathrm{Z})}{\mathrm{I_a(Z)}} = \frac{\mathrm{K_T T}}{\mathrm{J_m}} \frac{\mathrm{Z^{-1}}}{\mathrm{1-Z^{-1}}}$$

✓ Numerical Value:

$$\frac{\mathcal{W}_m(Z)}{I_a(Z)} = 0.14 \times 13.333\,\frac{Z^{-1}}{1-Z^{-1}} = 1.867\,\frac{Z^{-1}}{1-Z^{-1}}$$

Result:*Z-domain transfer functions in block diagram based on Fig. 2.*

✓ Analytical Value:

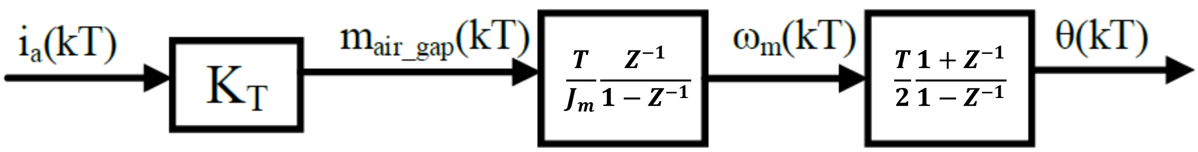


Fig. 2 Discrete-time model of position motion system

✓ Numerical Value:

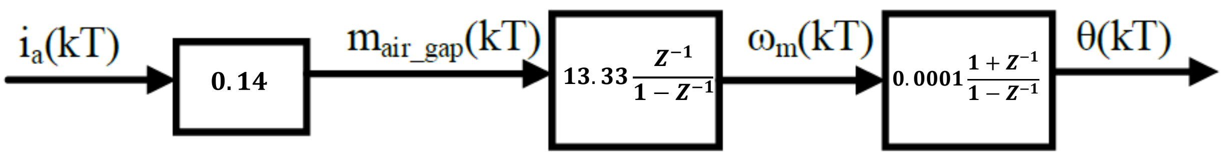


Fig. 2 Discrete-time model of position motion system

Problem A-2

Target

使用 Method #2 求出e_a與i_a之間的 k-domain 差分方程式與 Z-domain 轉移函式，並將結果以 Fig. 3 Discrete-time model of motor electric subsystem 的 Block Diagram 形式表示。

Solution

Step1:

為了求e_a與i_a的關係，先將兩者的關係以動態方程式列出，其 O.D.E.如下。

$$e_a(t) = R_p i_a(t) + L_p \frac{di_a(t)}{dt} + K_e \omega_m(t) = R_p i_a(t) + L_p \frac{di_a(t)}{dt} + e_b(t)$$

$$e_a(t) - e_b(t) = R_p i_a(t) + L_p \frac{di_a(t)}{dt}$$

Step2:

將微分方程式等號左右兩邊同取$Laplace|_{b.s.}$轉換。

$$\mathcal{L}\{e_a(t) - e_b(t)\} = \mathcal{L}\left\{R_p i_a(t) + L_p \frac{di_a(t)}{dt}\right\}$$

$$E_a(s) - E_b(s) = R_p I_a(s) + L_p[sI_a(s) - i_a(0)]$$

$$= R_p I_a(s) + L_p s I_a(s) = (L_p s + R_p) I_a(s)$$

$$\left.\frac{I_a(s)}{E_a(s) - E_b(s)}\right|_{b.s.} = \frac{1}{(L_p s + R_p)}$$

Step3:

由於輸入為數位訊號，因此要先經過一層 Z.O.H.將其轉換為類比訊號，在 s-domain 的實現方式如下：

$$\left.\frac{I_a(s)}{E_a(s) - E_b(s)}\right|_{b.s.} = \frac{1}{(L_p s + R_p)} \Rightarrow \left.\frac{I_a(s)}{E_{aZOH}(s) - E_{bZOH}(s)}\right|_{b.s.} = ZOH(s) \times \frac{1}{(L_p s + R_p)}$$

$$\left.\frac{I_a(s)}{[E_a(s) - E_b(s)]|_{ZOH}}\right|_{b.s.} = \frac{1 - e^{-sT}}{s} \times \frac{1}{(L_p s + R_p)}$$

$$= (1 - e^{-sT}) \times \frac{1}{(L_p s^2 + R_p s)}$$

$$= (1 - e^{-sT}) \times \frac{\frac{R_p}{L_p}}{s\left(s + \frac{R_p}{L_p}\right)} \times \frac{1}{R_p}$$

$$= (1 - e^{-sT}) \frac{\tau}{s(s+\tau)} \frac{1}{R_p} \quad , \tau = \frac{R_p}{L_p}$$

$$= NSD(s)|_{b.s.}$$

Step4:

因為現在已經是考慮 Z.O.H 的$NSD(s)|_{b.s.}$轉移函數了，故可以直接參考 Z-Table 將 s-domain 進行 Z 轉換至 Z-domain。

$$NSD(Z) = \mathcal{Z}\{NSD(s)|_{b.s.}\} = \mathcal{Z}\left\{\frac{I_a(s)}{E_{aZOH}(s) - E_{bZOH}(s)}\Bigg|_{b.s.}\right\}$$

$$= \mathcal{Z}\left\{(1 - e^{-sT}) \frac{\tau}{s(s+\tau)} \frac{1}{R_p}\right\} \quad , \tau = \frac{R_p}{L_p}$$

$$= \mathcal{Z}\{(1 - e^{-sT})\}\, \mathcal{Z}\left\{\frac{\tau}{s(s+\tau)}\right\} \frac{1}{R_p} \quad , \tau = \frac{R_p}{L_p}$$

$$= (1 - Z^{-1}) \frac{Z(1 - e^{-\tau T})}{(Z-1)(Z - e^{-\tau T})} \frac{1}{R_p} \quad , \tau = \frac{R_p}{L_p}$$

$$= \frac{1 - e^{-\tau T}}{Z - e^{-\tau T}} \frac{1}{R_p} \quad , \tau = \frac{R_p}{L_p}$$

$$\Rightarrow \mathbf{NSD(Z)} = \frac{\mathbf{I_a(Z)}}{\mathbf{E_a(Z) - E_b(Z)}} = \frac{\mathbf{1 - e^{-\tau T}}}{\mathbf{Z - e^{-\tau T}}} \frac{\mathbf{1}}{\mathbf{R_p}} \quad , \boldsymbol{\tau} = \frac{\mathbf{R_p}}{\mathbf{L_p}}$$

$$\Rightarrow \mathbf{NSD(Z)} = \frac{\mathbf{I_a(Z)}}{\mathbf{E_a(Z) - E_b(Z)}} = \frac{\mathbf{1}}{\mathbf{R_p}} \frac{\mathbf{1 - e^{-\frac{R_p}{L_p}T}}}{\mathbf{Z - e^{-\frac{R_p}{L_p}T}}}$$

$$R_p(Z - e^{-\tau T}) I_a(Z) = (1 - e^{-\tau T})[E_a(Z) - E_b(Z)] \quad , \tau = \frac{R_p}{L_p}$$

Step5:

對 Z-domain 之結果做 inverse Z-transformation，求得差分方程式。

$$\mathcal{Z}^{-1}\{R_p(Z-e^{-\tau T})I_a(Z)\}=\mathcal{Z}^{-1}\{(1-e^{-\tau T})[E_a(Z)-E_b(Z)]\}\ ,\tau=\frac{R_p}{L_p}$$

$$R_p\{i_a[(k+1)T]-e^{-\tau T}i_a(kT)\}=(1-e^{-\tau T})[e_a(kT)-e_b(kT)]\ ,\tau=\frac{R_p}{L_p}$$

$$\mathbf{i_a[(k+1)T]=e^{-\tau T}i_a(kT)+\frac{1-e^{-\tau T}}{R_p}[e_a(kT)-e_b(kT)]\ ,\tau=\frac{R_p}{L_p}}$$

$$\mathbf{\Rightarrow i_a[(k+1)T]=e^{-\frac{R_p}{L_p}T}i_a(kT)+\frac{1-e^{-\frac{R_p}{L_p}T}}{R_p}[e_a(kT)-e_b(kT)]}$$

Step6:

最終結果用 Block Diagram 形式表現，同時將數值帶入並進行計算。

$$\tau=\frac{R_p}{L_p}=\frac{2.6}{4.3}\frac{\Omega}{\mathrm{mH}}=\frac{2.6}{4.3}\frac{\Omega}{10^{-3}\Omega\cdot sec}=\frac{2600}{4.3}\frac{1}{sec}$$

$$e^{-\tau T}=e^{-\frac{R_p}{L_p}T}\cong 0.8861$$

$$\frac{1-e^{-\frac{R_p}{L_p}T}}{R_p}\cong 0.0438\frac{1}{\Omega}$$

Result:*k-domain difference equations*

- ✓ Analytical Value:

$$i_a[(k+1)T]=e^{-\frac{R_p}{L_p}T}i_a(kT)+\frac{1-e^{-\frac{R_p}{L_p}T}}{R_p}[e_a(kT)-e_b(kT)]$$

$$\xrightarrow{\text{簡化}} i_a(k+1)=e^{-\frac{R_p}{L_p}T}i_a(k)+\frac{1-e^{-\frac{R_p}{L_p}T}}{R_p}[e_a(k)-e_b(k)]$$

- ✓ Numerical Value:

$$i_a(k+1)=0.8861\ i_a(k)+0.0438\ [e_a(k)-e_b(k)]$$

Result:*Z-domain transfer functions*

✓ Analytical Value:

$$NSD(Z) = \frac{I_a(Z)}{E_a(Z) - E_b(Z)} = \frac{1}{R_p} \frac{1 - e^{-\frac{R_p}{L_p}T}}{Z - e^{-\frac{R_p}{L_p}T}}$$

✓ Numerical Value:

$$NSD(Z) = \frac{I_a(Z)}{E_a(Z) - E_b(Z)} = \frac{1}{2.6} \frac{1 - 0.8861}{Z - 0.8861}$$

$$\cong 0.0438 \frac{1}{Z - 0.8861} = 0.0438 \frac{Z^{-1}}{1 - 0.8861Z^{-1}}$$

Result:*Z-domain transfer functions in block diagram based on Fig. 3.*

✓ Analytical Value:

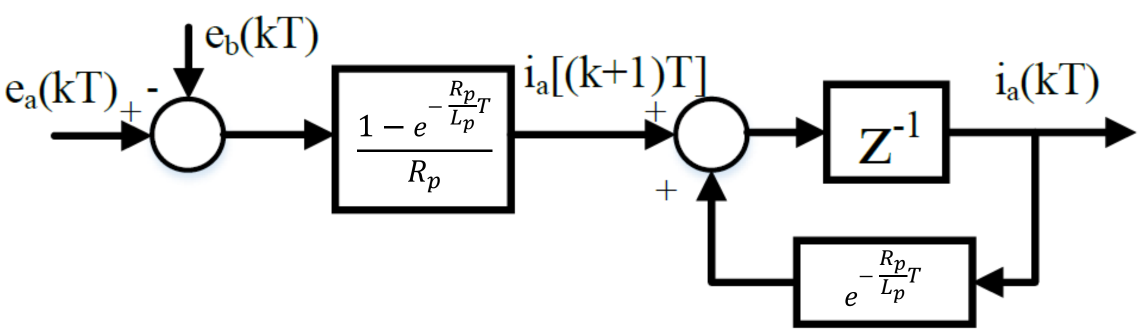


Fig. 3 Discrete-time model of motor electric subsystem

✓ Numerical Value:

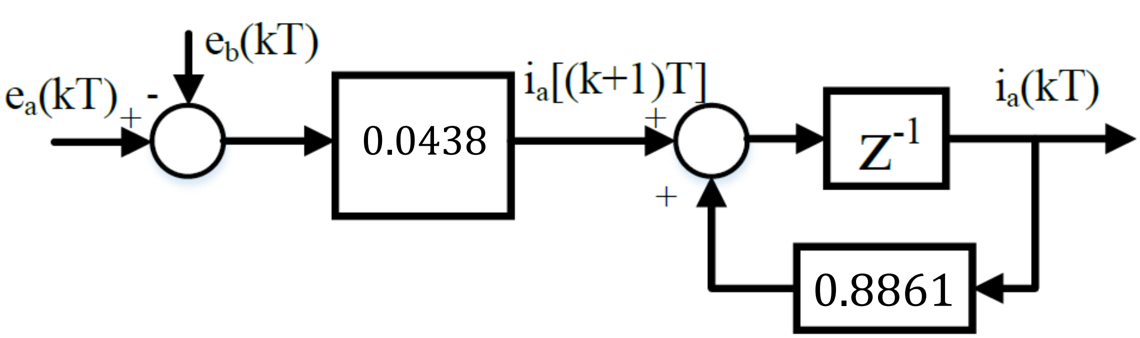


Fig. 3 Discrete-time model of motor electric subsystem

Problem B-1

Target

分析完整的馬達系統，其中輸入為$e_a(t=kT)$，輸出為$\omega_m(t=kT)$。藉由上述關係分析 Equations (4) (5)（如下所示），求解其各個係數。

$$i_a(k) = \frac{\frac{1}{L_p}e_a(k-1) - \frac{K_e}{L_p}\omega_m(k-1)}{\omega_n^2} C_{1T} + \frac{i_a(k-1)}{\omega_n^2} C_{2T} \tag{4}$$

$$\omega_m(k) = \left[\frac{\frac{K_T}{L_p J_p}e_a(k-1)}{\omega_n^2} - \omega_m(k-1)\right] C_{3T} + \frac{\frac{K_T}{J_p}i_a(k-1)}{\omega_n^2} C_{1T} + \omega_m(k-1) \tag{5}$$

已知題目給予三條微分方程式，分別為 Equations (1) (2) (3)，其表達如下所示：

$$e_a(t) = R_p i_a(t) + L_p\frac{di_a(t)}{dt} + K_e\omega_m(t) \quad (1),\quad J_m\frac{d\theta^2(t)}{dt^2} = K_T i_a(t) \quad (2),\ \text{and}\ J_m\frac{d\omega_m(t)}{dt} = K_T i_a(t) \quad (3)$$

不過在開始討論以前，這邊先做一個勘誤部份。

Handout #1 勘誤

6. Find the continuous time step response solution (cross-coupled initial conditions) (反拉式轉換求 $i_a(t)$ 及 $\omega_m(t)$)

$\mathcal{L}^{-1}$ (Ø(s)) → ø(t) terms (the response to a step input given initial conditions)

請以

$$C_{1t} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t)$$

$$C_{2t} = \mathcal{L}^{-1}\left\{\frac{s\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} = -\frac{\omega_n^2}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t-\varphi)$$

$$C_{3t} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\right\} = 1 - \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t+\varphi)$$

化簡式子

$$C_{3t} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)} - \frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}\right\}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left\{\sqrt{1-\zeta^2}\cos\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t\right] + \zeta\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)\right]\right\}$$ $$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right)$$
故圈起來的部份應為 1，而非 ω_n。

因此本大題的所有討論，$C_{3t} = 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \phi\right)$。

Solution

Step1:

藉由觀察 Equation (4)的關係式，發現要將 Equation (1)與 Equation (3)進行合併。

因此首先針對 Equation(1)的 O.D.E.進行拉式轉換，過程如下：

$$\text{O.D.E} : \mathrm{e_a}(t) = R_p i_a(t) + L_p\frac{di_a(t)}{dt} + K_e\omega_m(t)$$

$$\mathcal{L}\{e_a(t)\} = \mathcal{L}\{R_p i_a(t)\} + \mathcal{L}\left\{L_p\frac{di_a(t)}{dt}\right\} + \mathcal{L}\{K_e\omega_m(t)\}$$

$$\Rightarrow \mathrm{E_a}(s) = R_p I_a(s) + L_p\left(sI_a(s) - i_a(0)\right) + K_e\mathcal{W}_m(s)$$

Step2:

接著針對 Equation (3)的 O.D.E.進行拉式轉換，過程如下：

$$\text{O.D.E:}\ \mathrm{J_m}\frac{d\omega_m(t)}{dt} = K_T i_a(t)$$

$$\mathcal{L}\left\{\mathrm{J_m}\frac{d\omega_m(t)}{dt}\right\} = \mathcal{L}\{K_T i_a(t)\}$$

$$\Rightarrow s\mathcal{W}_m(s) - \omega_m(0) = \frac{K_T}{J_m}\mathrm{I_a}(s)$$

$$\Rightarrow \mathcal{W}_m(s) = \frac{1}{s}\omega_m(0) + \frac{K_T}{J_m}\frac{1}{s}I_a(s)$$

Step3:

接著將上述求出的兩條頻域式子進行合併，合併過程如下所示：

計算 Equation (4):

$$\mathrm{E_a}(s) = R_p I_a(s) + L_p\left(sI_a(s) - i_a(0)\right) + K_e\mathcal{W}_m(s) ------(1)$$

$$\mathcal{W}_m(s)=\frac{1}{s}\omega_m(0)+\frac{K_T}{J_m}\frac{1}{s}I_a(s)----------------(2)$$

將 (2)代入 (1)得到：

$$\mathrm{E_a}(s)=R_pI_a(s)+L_p\left(sI_a(s)-i_a(0)\right)+\frac{K_e}{s}\omega_m(0)+\frac{K_TK_e}{J_ms}I_a(s)$$

$$\mathrm{sE_a}(s)=R_psI_a(s)+L_ps^2I_a(s)-L_psi_a(0)+K_e\omega_m(0)+\frac{K_TK_e}{J_m}I_a(s)$$

$$=\left(\mathrm{L_p}\mathrm{s}^2+\mathrm{R_p}\mathrm{s}+\frac{\mathrm{K_eK_T}}{\mathrm{J_m}}\right)\mathrm{I_a}(s)-L_psi_a(0)+K_e\omega_m(0)$$

$$\frac{1}{\mathrm{L_p}}sE_a(s)=\left(s^2+\frac{R_p}{L_p}s+\frac{K_eK_T}{J_mL_p}\right)I_a(s)-si_a(0)+\frac{K_e}{L_p}\omega_m(0)$$

$$\text{Set }\omega_n=\sqrt{\frac{K_eK_T}{J_mL_p}},\xi=\frac{R_p}{2\omega_nL_p}$$

$$\frac{1}{\mathrm{L_p}}\mathrm{sE_a}(s)=(s^2+2\xi\omega_ns+\omega_n^2)I_a(s)-si_a(0)+\frac{K_e}{L_p}\omega_m(0)$$

$$\mathrm{I_a}(s)=\frac{1}{s^2+2\xi\omega_ns+\omega_n^2}\left\{\frac{1}{\mathrm{L_p}}sE_a(s)+si_a(0)-\frac{K_e}{L_p}\omega_m(0)\right\}$$

$$=\frac{\omega_n^2}{s^2+2\xi\omega_ns+\omega_n^2}\left\{\frac{\frac{1}{\mathrm{L_p}}sE_a(s)-\frac{K_e}{L_p}\omega_m(0)}{\omega_n^2}+\frac{sI_a(0)}{\omega_n^2}\right\}$$

Step4:

由於輸入$e_a(t=kT)$為離散訊號，需要使用 Z.O.H.將之改變為連續訊號，這裡使用 Method #1 先針對 0~T 時刻的訊號做 Z.O.H.，亦即將輸入的頻域訊號改變為 $\frac{1}{s}\,e_a(0)$，最後再將之代入上述合併的式子，計算過程如下所示：

Z.O.H. input:

$$E_a(s)|_{at\ 0\sim T}=\frac{1}{s}e_a(0)$$

$$\mathrm{I_a}(s)=\frac{\omega_n^2}{s^2+2\xi\omega_ns+\omega_n^2}\left\{\frac{\frac{1}{\mathrm{L_p}}sE_a(s)|_{at\ 0\sim T}-\frac{K_e}{L_p}\omega_m(0)}{\omega_n^2}+\frac{si_a(0)}{\omega_n^2}\right\}\ for\ 0\sim T$$

$$\mathrm{I_a}(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\left\{\frac{\frac{1}{\mathrm{L_p}}e_a(0)-\frac{K_e}{L_p}\omega_m(0)}{\omega_n^2}+\frac{s i_a(0)}{\omega_n^2}\right\}\ for\ 0\sim T$$

Step5:

將之展開為題目要求的形式，並對其求反拉式轉換，其中使用 Method #1 的特性，同時考慮 0~T 時刻、T~2T 時刻 … [(nT)-1]~nT 時刻的情況，將之歸納成離散的差分方程式形式，最後使用對照法得到C_{1t}與C_{2t}，過程如下所示：

$$\mathcal{L}.\mathrm{T.}^{-1}\Rightarrow\mathcal{L}^{-1}\{\mathrm{I_a}(s)\}=\mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\left(\frac{\frac{1}{L_p}e_a(0)-\frac{K_e}{L_p}\omega_m(0)}{\omega_n}+\frac{i_a(0)}{\omega_n^2}s\right)\right\}$$

$$\mathrm{i_a}(t)=\frac{\frac{1}{L_p}e_a(0)-\frac{K_e}{L_p}\omega_m(0)}{\omega_n^2}C_{1t}+\frac{i_a(0)}{\omega_n^2}C_{2t}\qquad \mathrm{for}\quad 0\sim\mathrm{T}$$

$$\mathrm{i_a}(T)=\frac{\frac{1}{L_p}e_a(0)-\frac{K_e}{L_p}\omega_m(0)}{\omega_n^2}C_{1T}+\frac{i_a(0)}{\omega_n^2}C_{2T}\qquad \mathrm{for}\quad \mathrm{t}=\mathrm{T}$$

$$\boxed{\mathrm{i_a}(k)=\frac{\frac{1}{L_p}e_a(k-1)-\frac{K_e}{L_p}\omega_m(k-1)}{\omega_n^2}C_{1T}+\frac{i_a(k-1)}{\omega_n^2}C_{2T}}$$

→Difference Equation (4)

$$\mathrm{C_{1t}}=\mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right\}$$

$$\mathrm{C_{2t}}=\mathcal{L}^{-1}\left\{\frac{\omega_n^2 s}{s^2+2\xi\omega_n s+\omega_n^2}\right\}$$

Step6:

藉由觀察 Equation (5)的關係式，發現同樣是要將 Equation (1)與 Equation (3) 的兩條頻域式子進行合併，合併過程如下所示:

計算 Equation (5):

$$\mathrm{E_a}(s)=R_pI_a(s)+L_p\big(sI_a(s)-i_a(0)\big)+K_e\mathcal{W}_m(s)------(1)$$

$$\mathcal{W}_m(s)=\frac{1}{s}\omega_m(0)+\frac{K_T}{J_m}\frac{1}{s}I_a(s)---------------(2)$$

From (1)

$$(\mathrm{R_p}+\mathrm{L_p}s)\mathrm{I_a}(s)=E_a(s)-K_e\mathcal{W}_m(s)+L_pi_a(0)$$

$$\mathrm{I_a}(s)=\frac{1}{R_p+L_ps}\left(E_a(s)-K_e\mathcal{W}_m(s)+L_pi_a(0)\right)$$

Step7:

由於輸入$e_a(t=kT)$為離散訊號，需要使用 Z.O.H.將之改變為連續訊號，這裡使用 Method #1 先針對 0~T 時刻的訊號做 Z.O.H.，亦即將輸入的頻域訊號改變為 $\frac{1}{s}\,e_a(0)$，最後再將之代入上述合併的式子，計算過程如下所示：

Z.O.H. input:

$$E_a(s)|_{at\ 0\sim T}=\frac{1}{s}e_a(0)$$

$$\mathrm{I_a}(s)=\frac{1}{R_p+L_ps}\left(E_a(s)-K_e\mathcal{W}_m(s)+L_pi_a(0)\right)$$

$$\Rightarrow \mathrm{I_a}(s)=\frac{1}{R_p+L_ps}\left(\frac{1}{s}e_a(0)-K_e\mathcal{W}_m(s)+L_pi_a(0)\right)\quad for\ 0\sim T----(3)$$

將 (3)代入 (2) 得到：

$$\mathcal{W}_m(s)=\frac{1}{s}\omega_m(0)+\frac{K_T}{J_m}\frac{1}{R_ps+L_ps^2}\left(\frac{1}{\mathrm{s}}\mathrm{e_a}(0)-\mathrm{K_e}\mathcal{W}_m(\mathrm{s})+\mathrm{L_pi_a}(0)\right)$$

$$=\frac{1}{\mathrm{s}}\omega_m(0)+\frac{K_T}{J_m}\frac{1}{\mathrm{L_p}}\frac{1}{s^2+\frac{R_p}{L_p}s}\left(\frac{1}{\mathrm{s}}\mathrm{e_a}(0)-\mathrm{K_e}\mathcal{W}_m(\mathrm{s})+\mathrm{L_pi_a}(0)\right)$$

$$=\frac{1}{\mathrm{s}}\omega_m(0)+\frac{K_T}{J_m}\frac{1}{\mathrm{L_p}}\frac{1}{s^2+2\xi\omega_ns}\left(\frac{1}{\mathrm{s}}\mathrm{e_a}(0)-\mathrm{K_e}\mathcal{W}_m(\mathrm{s})+\mathrm{L_pi_a}(0)\right)$$

$$,\text{where }\omega_n=\sqrt{\frac{K_eK_T}{J_mL_p}}\ ,\xi=\frac{R_p}{2\omega_nL_p}$$

$$\mathcal{W}_m(s)-\frac{1}{s}\omega_m(0)=\frac{1}{s^2+2\xi\omega_ns}\left(\frac{K_T}{J_mL_p}\frac{1}{s}e_a(0)-\ \omega_n^2\mathcal{W}_m(s)+\frac{K_T}{J_m}i_a(0)\right)$$

$$\Rightarrow(\mathrm{s}^2+2\xi\omega_ns)\left(\mathcal{W}_m(\mathrm{s})-\frac{1}{\mathrm{s}}\omega_m(0)\right)+\omega_n^2\mathcal{W}_m(\mathrm{s})=\frac{\mathrm{K_T}}{J_mL_p}\frac{1}{s}e_a(0)+\frac{K_T}{J_m}i_a(0)$$

$$\Rightarrow(\mathrm{s}^2+2\xi\omega_n\mathrm{s}+\omega_n^2)\mathcal{W}_m(\mathrm{s})=\frac{K_T}{J_mL_p}\frac{1}{s}e_a(0)+\frac{K_T}{J_m}i_a(0)+(\mathrm{s}^2+2\xi\omega_n\mathrm{s})\frac{1}{s}\omega_m(0)$$

$$=\frac{K_T}{J_mL_p}\frac{1}{s}e_a(0)+\frac{K_T}{J_m}i_a(0)+(\mathrm{s}^2+2\xi\omega_n\mathrm{s}+\omega_n^2)\frac{1}{s}\omega_m(0)-\frac{\omega_n^2}{s}\omega_m(0)$$

$\Rightarrow \mathcal{W}_m(s)$

$$= \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\left(\frac{1}{\omega_n^2}\frac{K_T}{J_m L_p}\frac{1}{s}e_a(0) + \frac{1}{\omega_n^2}\frac{K_T}{J_m}i_a(0)\right) + \frac{1}{s}\omega_m(0) - \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}\omega_m(0)$$

$$= \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}\left(\frac{1}{\omega_n^2}\frac{K_T}{J_m L_p}e_a(0) - \omega_m(0)\right) + \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\left(\frac{1}{\omega_n^2}\frac{K_T}{J_m}i_a(0)\right) + \frac{1}{s}\omega_m(0)$$

Step8:

將之展開為題目要求的形式，並對其求反拉式轉換，其中使用 Method #1 的特性，同時考慮 0~T 時刻、T~2T 時刻 … [(nT)-1]~nT 時刻的情況，將之歸納成離散的差分方程式形式，最後使用對照法得到C_{1t}與C_{2t}，過程如下所示：

$\mathcal{L}.\mathrm{T}.^{-1} \Rightarrow \mathcal{L}^{-1}\{\mathcal{W}_m(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}\right\}\left(\frac{1}{\omega_n}\frac{K_T}{J_m L_p}e_a(0) - \omega_m(0)\right) + \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\}\left(\frac{1}{\omega_m^2}\frac{\mathrm{K_T}}{\mathrm{J_m}}\mathrm{i_a}(0)\right) + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\omega_m(0)$$

$$\Rightarrow \omega_m(\mathrm{t}) = \left(\frac{1}{\omega_n^2}\frac{K_T}{J_m L_p}e_a(0) - \omega_m(0)\right)\mathrm{C_{3t}} + \left(\frac{1}{\omega_m^2}\frac{\mathrm{K_T}}{\mathrm{J_m}}\mathrm{i_a}(0)\right)C_{1t} + \omega_m(0) \quad for\ 0\sim T$$

$$\omega_m(\mathrm{T}) = \left(\frac{1}{\omega_n^2}\frac{K_T}{J_m L_p}e_a(0) - \omega_m(0)\right)\mathrm{C_{3T}} + \left(\frac{1}{\omega_m^2}\frac{\mathrm{K_T}}{\mathrm{J_m}}\mathrm{i_a}(0)\right)C_{1T} + \omega_m(0) \quad for\ t = T$$

$$\boxed{\omega_m(\mathrm{k}) = \left(\frac{1}{\omega_n^2}\frac{K_T}{J_m L_p}e_a(k-1) - \omega_m(k-1)\right)\mathrm{C_{3T}} + \left(\frac{1}{\omega_m^2}\frac{\mathrm{K_T}}{\mathrm{J_m}}\mathrm{i_a}(\mathrm{k}-1)\right)C_{1T} + \omega_m(k-1)}$$

↓

Difference Equation (5)

$$\mathrm{C_{1t}} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\}$$

$$\mathrm{C_{3t}} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}\right\}$$

Step9:

最後，將係數分別代入，整理出如下方之表格。

$C_{1t} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t)$ $C_{2t} = \mathcal{L}^{-1}\left\{\frac{s\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} = -\frac{\omega_n^2}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t-\varphi)$ $\cancel{C_{3t} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\right\} = 1-\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t+\varphi)}$ $C_{3t} = 1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$	$\omega_n = 551.2495$ $\xi = 0.5484$ $\varphi = 0.9903$ $C_{1T} = 57.1283$ $C_{2T} = 2.6756\times 10^5$ $C_{3T} = 0.0058$

Result: Difference Equation (4)

✓ Analytical Value:

$$\mathrm{i_a}(k) = \frac{\frac{1}{L_p}e_a(k-1)-\frac{K_e}{L_p}\omega_m(k-1)}{\omega_n^2}C_{1T}+\frac{i_a(k-1)}{\omega_n^2}C_{2T}$$

$\Rightarrow \mathrm{i_a}(k) =$

$$\left(7.653\times 10^{-4}e_a(k-1)-1.0714\times 10^{-4}\omega_m(k-1)\right)C_{1T}+\left(3.29\times 10^{-6}\mathrm{i_a}(\mathrm{k}-1)\right)C_{2T}$$

✓ Numerical Value:

$$\Rightarrow \mathrm{i_a}(k) = 0.0437\ e_a(k-1)-0.0061\ \omega_m(k-1)+0.8805\ \mathrm{i_a}(\mathrm{k}-1)$$

Result: Difference Equation (5)

✓ Analytical Value:

$$\omega_\mathrm{m}(\mathrm{k}) = \left(\frac{\mathrm{K_T}}{\omega_\mathrm{n}^2\mathrm{L_p J_m}}\mathrm{e_a}(\mathrm{k}-1)-\omega_\mathrm{m}(\mathrm{k}-1)\right)C_{3T}+\frac{\frac{K_T}{J_m}i_a(k-1)}{\omega_\mathrm{n}^2}C_{1T}+\omega_\mathrm{m}(k-1)$$

$$\Rightarrow \omega_\mathrm{m}(\mathrm{k}) = \left(7.1429\mathrm{e_a}(\mathrm{k}-1)-\omega_\mathrm{m}(\mathrm{k}-1)\right)C_{3T}+0.0307i_a(k-1)C_{1T}+\omega_\mathrm{m}(k-1)$$

✓ Numerical Value:

$$\Rightarrow \omega_\mathrm{m}(\mathrm{k}) = 0.0417\ \mathrm{e_a}(\mathrm{k}-1)-5.8\times 10^{-3}\omega_\mathrm{m}(\mathrm{k}-1)+1.7547\ i_a(k-1)+\omega_\mathrm{m}(k-1)$$

Problem B-2

Target

將 Transfer Function 重新繪製成如 Fig. 4 所示，並將其缺漏的係數填上。

Solution

Step1:

針對 *Problem B-1* 所求出的 Difference Equations (4)進行 Z 轉換，並將其化簡為 Fig. 4 的 Block Diagram，並對照其係數分別求解之，計算過程如下。

Difference Equation (4):

$$\mathrm{i_a}(k) = \frac{\frac{1}{L_p} e_a(k-1) - \frac{K_e}{L_p}\omega_m(k-1)}{\omega_n^2} C_{1T} + \frac{i_a(k-1)}{\omega_n^2} C_{2T} - - - - - - (4)$$

將 (4)進行 Z 轉換：

$$\mathrm{I_a}(Z) = \frac{C_{1T}}{L_p\omega_n^2}\left(Z^{-1}E_a(z) - Z^{-1}K_e\mathcal{W}_m(Z)\right) + \frac{C_{2T}}{\omega_n^2} Z^{-1} I_a(Z)$$

$$\left(1 - \frac{\mathrm{C_{2T}}}{\omega_n^2}\mathrm{Z}^{-1}\right)\mathrm{I_a}(z) = \frac{C_{1T}}{L_p\omega_n^2} Z^{-1}\left(E_a(z) - K_e\mathcal{W}_m(z)\right)$$

$$\frac{\mathrm{I_a}(z)}{E_a(z) - K_e\mathcal{W}_m(z)} = \frac{\frac{C_{1T}}{L_p\omega_n^2} Z^{-1}}{1 - \frac{C_{2T}}{\omega_n^2} Z^{-1}} = \frac{B_1 Z^{-1}}{1 - A_1 Z^{-1}}$$

$$\boldsymbol{A_1} = \frac{\mathbf{C_{2T}}}{\boldsymbol{\omega_n^2}}$$

$$\mathbf{B_1} = \frac{\boldsymbol{C_{1T}}}{\boldsymbol{L_p\omega_n^2}}$$

Step2:

針對 *Problem B-1* 所求出的 Difference Equations (5)進行 Z 轉換，並將其化簡為 Fig. 4 的 Block Diagram，並對照其係數分別求解之，計算過程如下。

Difference Equation (5):

$$\omega_m(k) = \left(\frac{K_T}{\omega_n^2 L_p J_m} e_a(k-1) - \omega_m(k-1)\right) C_{3T} + \frac{\frac{K_T}{J_m} i_a(k-1)}{\omega_n^2} C_{1T} + \omega_m(k-1) - -(5)$$

$$, where\ J_m = J_p = 0.015 \times 10^{-3}\ \mathrm{kg} - m^2$$

將 (5)進行 Z 轉換：

$$\mathcal{W}_m(z) = \left(\frac{\frac{\omega_n^2}{K_e} Z^{-1} E_a(z)}{\omega_n^2} - Z^{-1}\mathcal{W}_m(z)\right) C_{3T} + \frac{\frac{K_T}{J_p} Z^{-1} I_a(z)}{\omega_n^2} C_{1T} + Z^{-1}\mathcal{W}_m(z)$$

$$= \left(\frac{1}{K_e} Z^{-1} E_a(z) - Z^{-1}\mathcal{W}_m(z)\right) C_{3T} + \frac{\frac{K_T}{J_p}}{\omega_n^2} Z^{-1} I_a(z) C_{1T} + Z^{-1}\mathcal{W}_m(z)$$

$$\Rightarrow (1 - Z^{-1})\mathcal{W}_m(z) = \frac{1}{K_e} Z^{-1}\big(E_a(z) - K_e \mathcal{W}_m(z)\big) C_{3T} + \frac{K_T}{J_p} \frac{1}{\omega_n^2} Z^{-1} I_a(z) C_{1T}$$

$$= \frac{1}{K_e} Z^{-1} \frac{1 - \frac{C_{2T}}{\omega_n^2} Z^{-1}}{\frac{C_{1T}}{L_p \omega_n^2} Z^{-1}} I_a(z) C_{3T} + \frac{K_T}{J_p} \frac{1}{\omega_n^2} Z^{-1} I_a(z) C_{1T}$$

$$= \frac{1}{K_e} \frac{1 - \frac{C_{2T}}{\omega_n^2} Z^{-1}}{\frac{C_{1T}}{L_p \omega_n^2}} I_a(z) C_{3T} + \frac{K_T}{J_p} \frac{1}{\omega_n^2} Z^{-1} I_a(z) C_{1T}$$

$$= \frac{1}{K_e} \frac{L_p \omega_n^2}{C_{1T}} \left(C_{3T} + Z^{-1}\left(\frac{C_{1T}^2}{\omega_n^2} - \frac{C_{2T} C_{3T}}{\omega_n^2}\right)\right) I_a(z)$$

$$\Rightarrow (1 - Z^{-1})\mathcal{W}_m(z) = \frac{1}{K_e} \frac{L_p \omega_n^2}{C_{1T}} \left(C_{3T} + Z^{-1}\left(\frac{C_{1T}^2}{\omega_n^2} - \frac{C_{2T} C_{3T}}{\omega_n^2}\right)\right) I_a(z)$$

$$\frac{\mathcal{W}_m(z)}{I_a(z)} = \frac{L_p \omega_n}{K_e} \frac{C_{3T} + \frac{1}{\omega_n^2}(C_{1T}^2 - C_{2T} C_{3T}) Z^{-1}}{\frac{C_{1T}}{\omega_n}(1 - Z^{-1})} = \frac{L_p \omega_n}{K_e} \frac{B_{\omega e1} + B_{\omega e2} Z^{-1}}{B_{ie}(1 - Z^{-1})}$$

$$\mathbf{B_{ie}} = \frac{\boldsymbol{C_{1T}}}{\boldsymbol{\omega_n}}$$

$$\mathbf{B_{\omega e1}} = \boldsymbol{C_{3T}}$$

$$\mathbf{B_{\omega e2}} = \frac{\mathbf{1}}{\boldsymbol{\omega_n^2}}(\boldsymbol{C_{1T}^2} - \boldsymbol{C_{2T}C_{3T}})$$

Step3:

最後，將係數分別代入，整理出如下方之表格。

$C_{1t} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t)$ $C_{2t} = \mathcal{L}^{-1}\left\{\frac{s\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} = -\frac{\omega_n^2}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t-\varphi)$ $\cancel{C_{3t} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\right\} = 1-\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t+\varphi)}$ $C_{3t} = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$ $A_1 = \frac{C_{2T}}{\omega_n^2}$ $B_1 = \frac{C_{1T}}{L_p\omega_n^2}$ $B_{ie} = \frac{C_{1T}}{\omega_n}$ $B_{\omega e1} = C_{3T}$ $B_{\omega e2} = \frac{1}{\omega_n^2}(C_{1T}^2 - C_{2T}C_{3T})$	$\boldsymbol{\omega_n = 551.2495}$ $\boldsymbol{\xi = 0.5484}$ $\boldsymbol{\varphi = 0.9903}$ $\boldsymbol{C_{1T} = 57.1283}$ $\boldsymbol{C_{2T} = 2.6756 \times 10^5}$ $\boldsymbol{C_{3T} = 0.0058}$ $\boldsymbol{A_1 = 0.8805}$ $\boldsymbol{B_1 = 0.0437}$ $\boldsymbol{B_{ie} = 0.1036}$ $\boldsymbol{B_{\omega e1} = 0.0058}$ $\boldsymbol{B_{\omega e2} = 0.0056}$

Results

✓ Analytical Value:

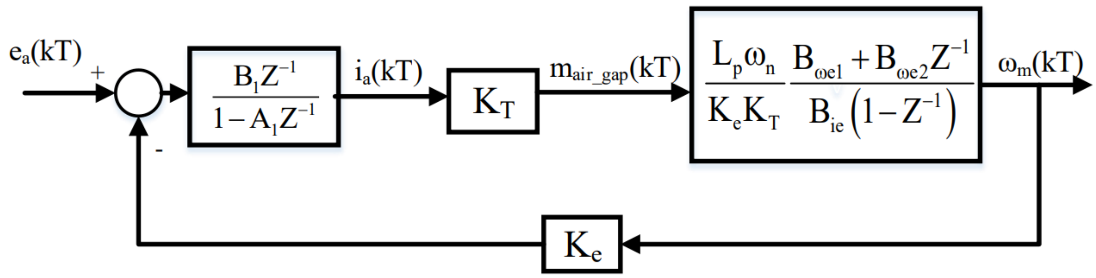


Fig. 4 Discrete-time model of motor drive system

✓ Numerical Value (1):

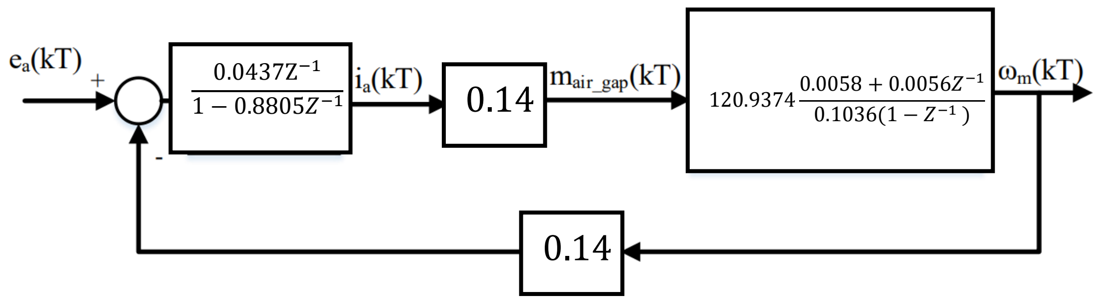


Fig. 4 Discrete-time model of motor drive system

✓ Numerical Value (2):

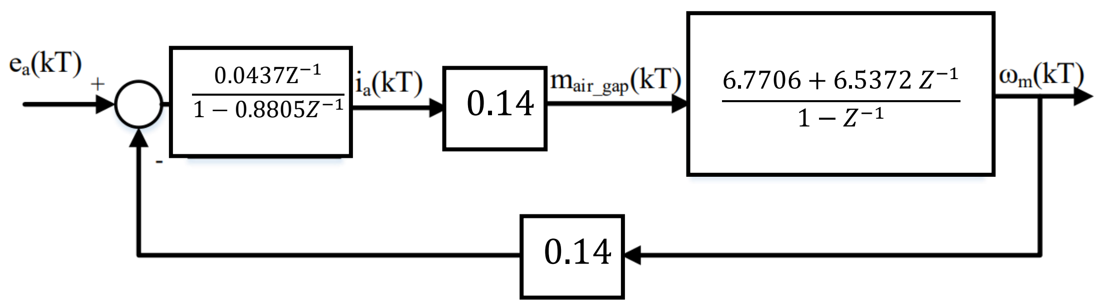


Fig. 4 Discrete-time model of motor drive system

Problem B-3

Results

Problem A 和 *Problem B* 的 Z 轉換後之轉移函數不會一樣，理由我們認為有二：

1. 由自動控制理論可知，常見的馬達必滿足如 *Problem A-1* 和 *Problem A-2* 中所見的方程式，分別為機械方程式與電方程式。由於 *Problem A-2* 中的輸入為端電壓$e_a(t)$、輸出為電流$i_a(t)$，因此是沒有考慮到反電動勢(Back EMF Voltage) $e_b(t) = K_e \cdot \omega_m(t)$，因此 Fig. 2 與 Fig. 3 合併後的數學模型亦不包含反電動勢。而乃探討整個馬達系統的轉移函數，因此必須考慮負回授反電動勢所造成的響應，故兩者轉移函數有所差異。

2. 在 *Problem A* 中，我們考慮到 Z.O.H.(Zero Order Hold) ，因此在 Fig. 3 中新的轉移函數 NSD(s)中，需額外考量到 Z.O.H.的轉移函數：$\mathrm{Z.O.H.}(s) = \frac{1-e^{-sT}}{s}$。至於 *Problem B* 中，我們直接令 Z.O.H. Input $E_a(s)|_{at\ 0\sim T} = \frac{1}{s}e_a(0)$，而電流$i_a(t)$是會受到電阻$R_p$和電感$L_p$影響的，並且在計算過程中，我們也發現$K_e$會影響整個系統。

   簡言之，在 Fig. 3 中的$\mathrm{Z.O.H.}(s) = \frac{1-e^{-sT}}{s}$因為會作用在$[E_a(s) - E_b(s)]$上，故$E_b(s)$之初始值並非真實回授，因此需考量 Z.O.H.之效應，而不能像 *Problem A* 那樣可以忽略之。

總結：由 1. & 2. 可知，若探討一回授系統，同時系統分別由數個獨立的動態方程式/微分方程式(O.D.E.) 所組成，即使做 Laplace 轉換後在 s-domain 中可以各自算完彼此轉移函式後再進行合併，但在 Z 轉換中卻不能這麼做，主要是因為 A/D 在常見的模型中必須考慮到 Z.O.H. 之影響，然而它們卻可能不相同，因此不能任意在 Block Diagram 中進行合併。

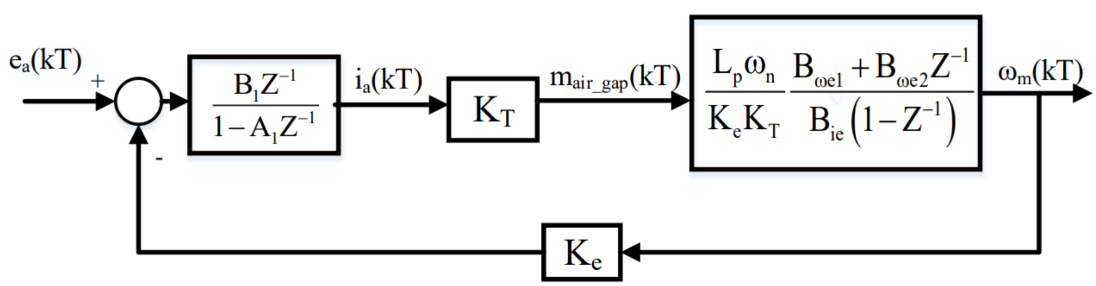


Fig. 4 Discrete-time model of motor drive system

Problem C-1

Target: Fig. 4 與 Fig. 5 兩者方塊圖雖不同，但模擬結果相同。

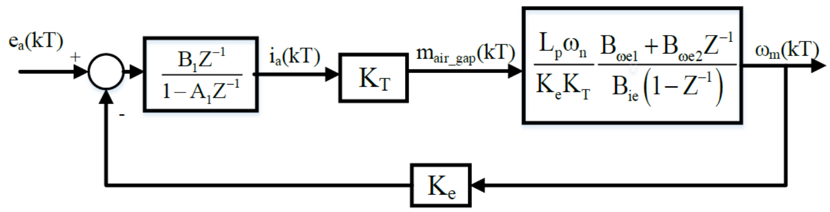


Fig. 4 Discrete-time model of motor drive system

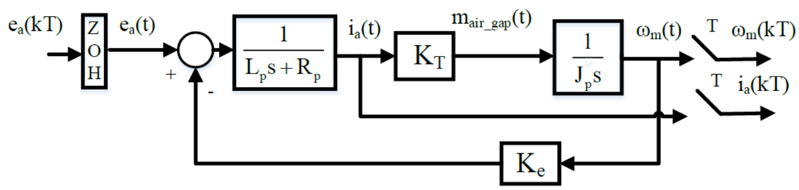
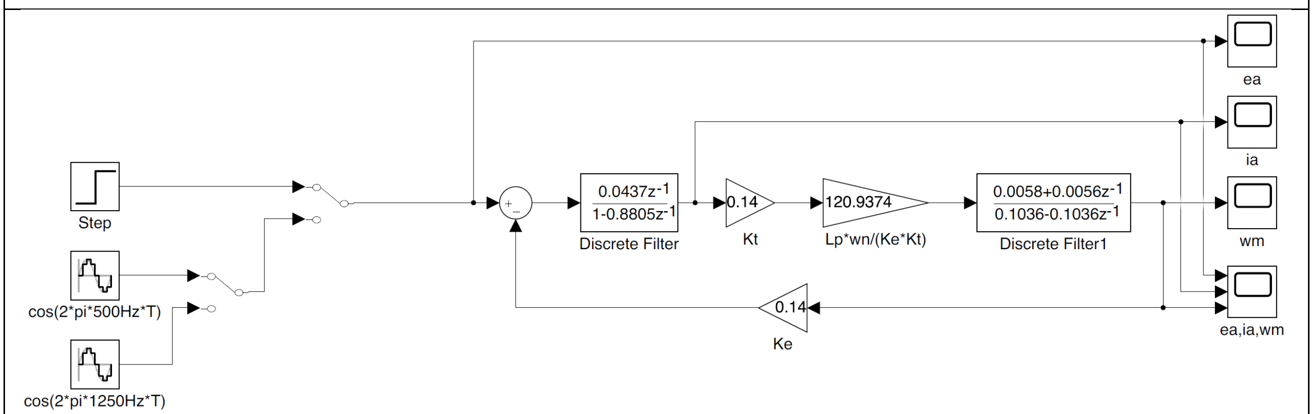


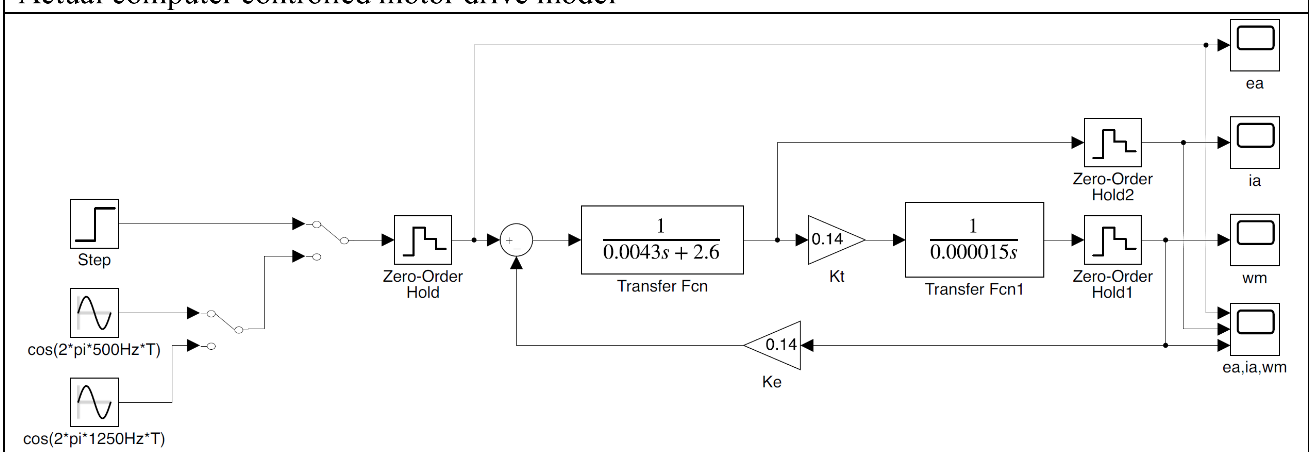
Fig. 5 Actual computer controlled motor drive

Solution:

Discrete-Time model

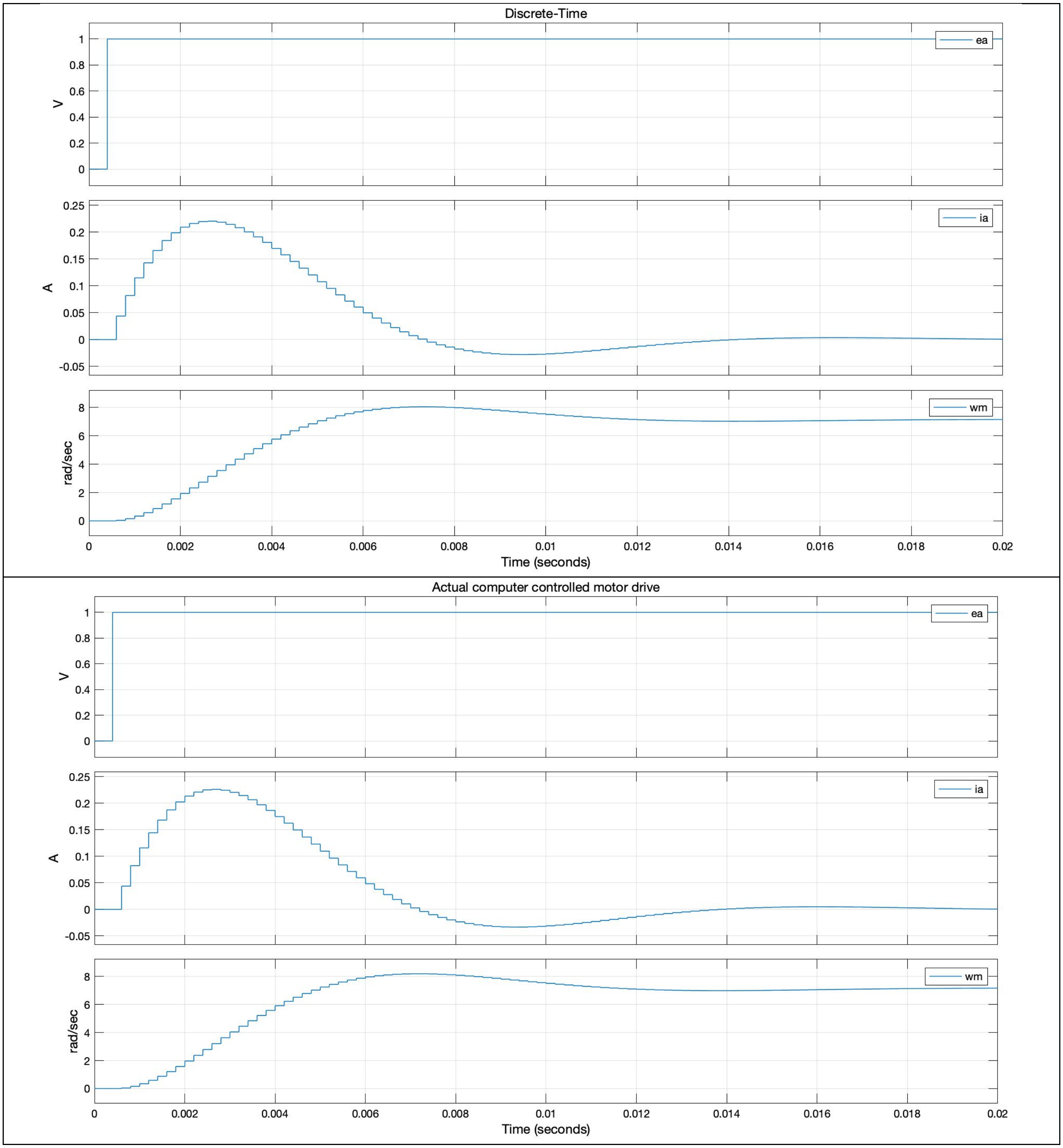


Actual computer controlled motor drive model



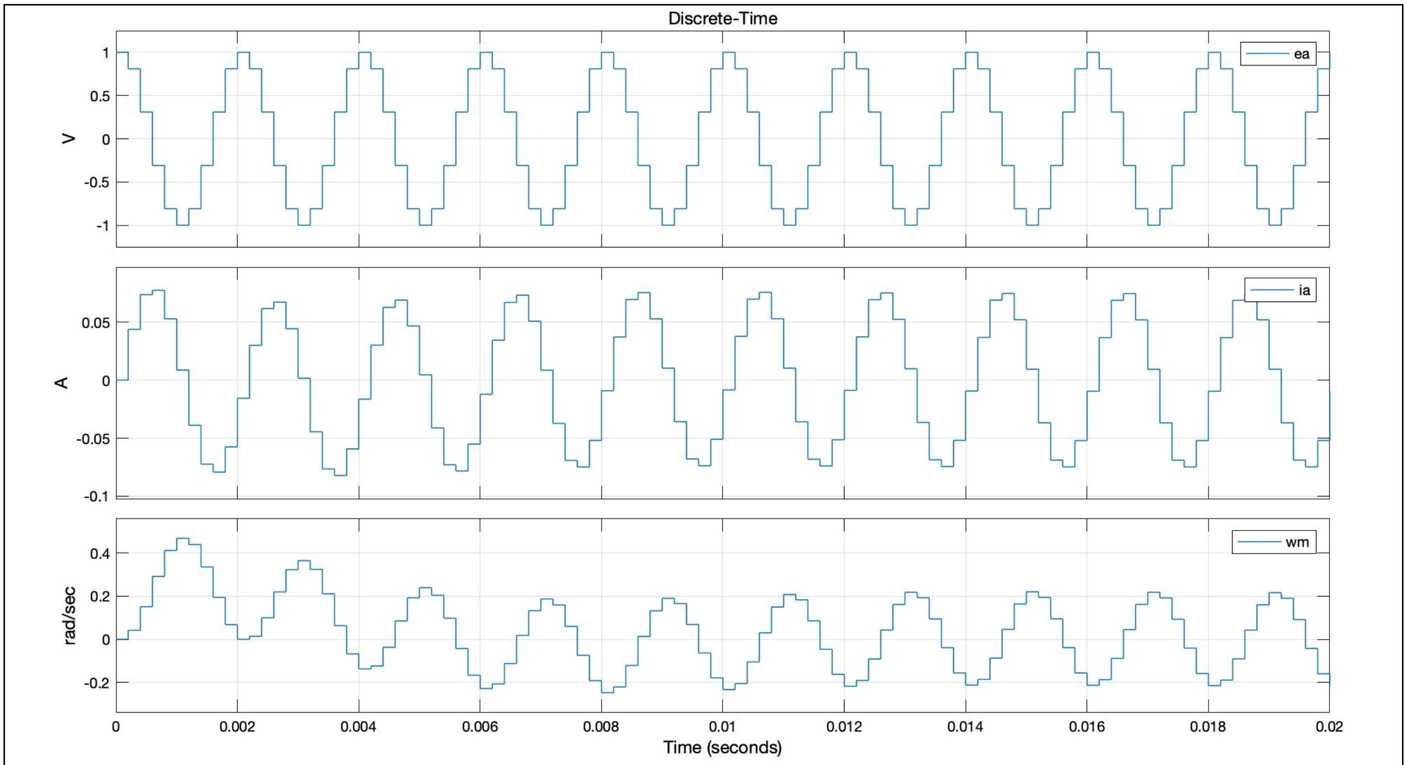
(a)

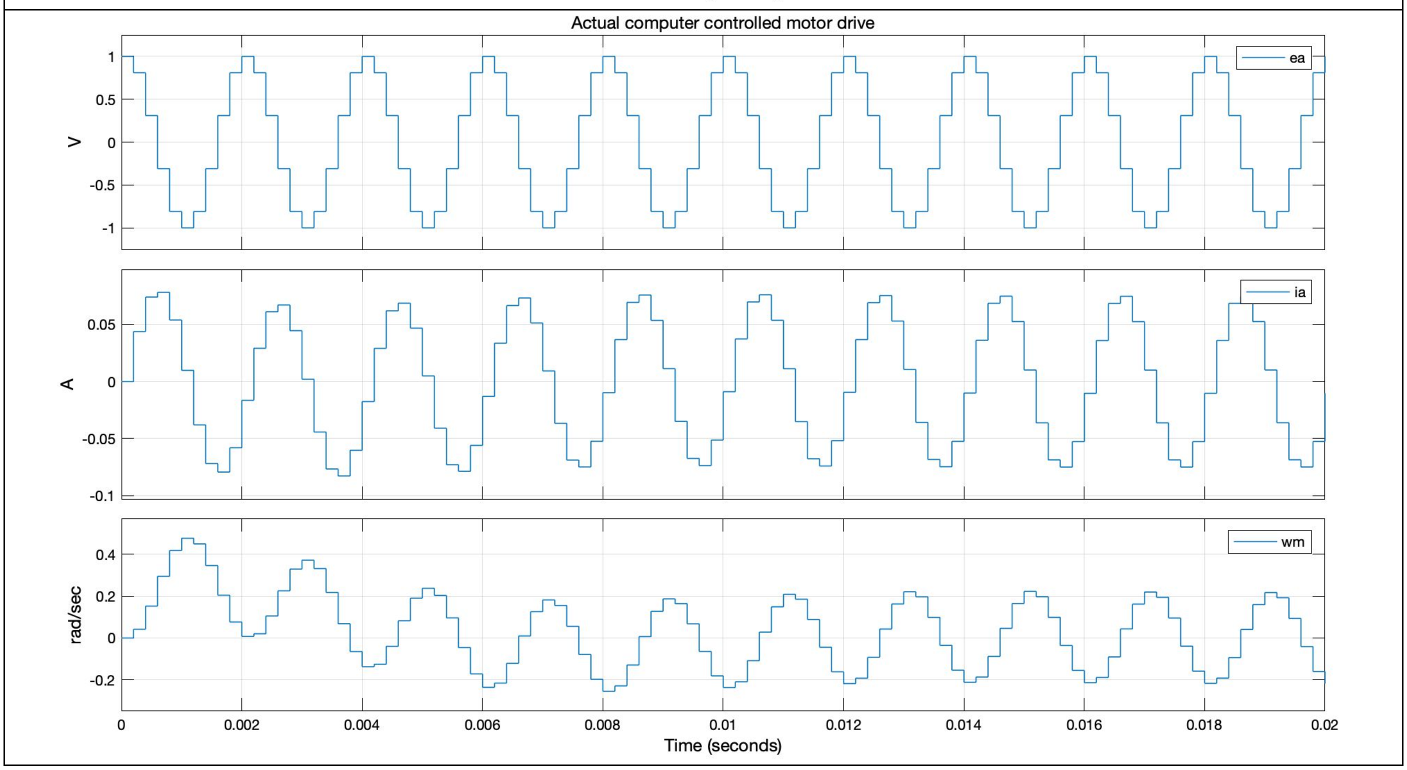
Result: $e_a(kT) = 1 \ at \ kT = 2T$



(b)

Result: $e_a(kT) = \cos(2\pi \times 500Hz \times T)$





Result: $e_a(kT) = \cos(2\pi \times 1.25kHz \times T)$