Задание 4

$$A = \begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{pmatrix} 4-\lambda & -5 & 7 \\ 1 & -4-\lambda & 9 \\ -4 & 0 & 5-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (-4) M_{31} + 0 \cdot M_{32} + (5-\lambda) \cdot M_{33}$$

$$M_{31} = \begin{vmatrix} -5 & 7 \\ -4-\lambda & 9 \end{vmatrix} = (-5) \cdot 9 - 7(-4-\lambda) = -45 + 28 + 7\lambda = -17 + 7\lambda$$

$$M_{33} = \begin{vmatrix} 4-\lambda & -5 \\ 1 & -4-\lambda \end{vmatrix} = (4-\lambda)(-4-\lambda) - (-5) \cdot 1 = -16 - 4\lambda + 4\lambda + \lambda^2 + 5 = \lambda^2 - 11$$

$$\det(A - \lambda I) = (-4)(-17 + 7\lambda) + (5-\lambda)(\lambda^2 - 11) = 68 - 28\lambda + (5\lambda^2 - 55 - \lambda^3 + 11\lambda) = -\lambda^3 + 5\lambda^2 + (-28\lambda + 11\lambda) + (68 - 55) = -\lambda^3 + 5\lambda^2 - 17\lambda + 13$$

$$\lambda^3 - 5\lambda^2 + 17\lambda - 13 = 0$$

$1 - 5 + 17 - 13 = 0 \Rightarrow \lambda = 1$ корень, значит

$$(\lambda - 1)(\lambda^2 - 4\lambda + 13) = 0$$

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\lambda_{2,3} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$\lambda_1 = 1 \; ; \; \lambda_2 = 2 + 3i \; ; \; \lambda_3 = 2 - 3i$$

Найдем собственные векторы:

$\lambda_1 = 1$

$$\begin{cases} 3x - 5y + 7z = 0 & \Rightarrow y = 2x \\ x - 5y + 9z = 0 & \\ -4x + 4z = 0 & \Rightarrow z = x \end{cases}$$

Пусть $x = 1$, тогда

$$\vec{U}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda_2 = 2+3i$
$$\begin{cases} (2-3i)x - 5y + 7z = 0 \\ x + (-6-3i)y + 9z = 0 \\ -4x + (3-3i)z = 0 \end{cases}$$
$\Rightarrow$ (*1) $\Rightarrow$ $y = \frac{5-3i}{4}z$

$\Rightarrow x = \frac{3-3i}{4}z$

(*1)

$$(2-3i)\frac{3-3i}{4}z - 5y + 7z = 0$$

$$(2-3i)(3-3i) = 6-6i-9i+9i^2 = 6-15i-9 = -3-15i$$

$$\left(7 - \frac{3+15i}{4}\right)z - 5y = 0$$

$$\frac{25-15i}{4}z - 5y = 0$$

$$5y = \frac{25-15i}{4}z$$

$$y = \frac{5-3i}{4}z$$

Пусть $z=4$, тогда

$$\vec{U_2} = \begin{pmatrix} 3-3i \\ 5-3i \\ 4 \end{pmatrix}$$

$\lambda_3 = 2-3i$ $\quad \vec{U_3}$ является комплексно сопряженным к вектору $\vec{U_2}$

$$\vec{U_3} = \begin{pmatrix} 3+3i \\ 5+3i \\ 4 \end{pmatrix}$$

Ответ: Собственные числа: $\lambda_1 = 1$; $\lambda_2 = 2+3i$; $\lambda_3 = 2-3i$

Собственные векторы: $\vec{U_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; $\vec{U_2} = \begin{pmatrix} 3-3i \\ 5-3i \\ 4 \end{pmatrix}$; $\vec{U_3} = \begin{pmatrix} 3+3i \\ 5+3i \\ 4 \end{pmatrix}$