



Three-body problem — From Newton to supercomputer plus machine learning

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ABSTRACT

The famous three-body problem can be traced back to Newton in 1687, but quite few families of periodic orbits were found in 300 years thereafter. In this paper, we propose an effective approach and roadmap to numerically gain planar periodic orbits of three-body systems with arbitrary masses by means of machine learning based on an artificial neural network (ANN) model. Given any a known periodic orbit as a starting point, this approach can provide more and more periodic orbits (of the same family name) with variable masses, while the mass domain having periodic orbits becomes larger and larger, and the ANN model becomes wiser and wiser. Finally we have an ANN model trained by means of all obtained periodic orbits of the same family, which provides a convenient way to give accurate enough predictions of periodic orbits with arbitrary masses for physicists and astronomers. It suggests that the high-performance computer and artificial intelligence (including machine learning) should be the key to gain periodic orbits of the famous three-body problem.

1. Introduction

How are the trajectories of three point masses m_1, m_2 and m_3 that are attracted each other by Newton's gravitational law? This so-called "three-body problem" can be traced back to Newton (1687) in 1687. According to Newton's second law and gravitational law, the related governing equations about N -body problem read

$$m_k \frac{d^2 \mathbf{r}_k}{dt^2} = \sum_{j=1, j \neq k}^N \frac{G m_k m_j (\mathbf{r}_j - \mathbf{r}_k)}{\left| \mathbf{r}_j - \mathbf{r}_k \right|^3}, \quad 1 \leq k \leq N, \quad (1)$$

where \mathbf{r}_k and m_k are the position vector and mass of the k th-body, t denotes the time, respectively, with a given initial position $\mathbf{r}_{0,k}$ and velocity $\mathbf{v}_{0,k}$, i.e.

$$\mathbf{r}_k(0) = \mathbf{r}_{0,k}, \quad \dot{\mathbf{r}}_k(0) = \mathbf{v}_{0,k}, \quad 1 \leq k \leq N. \quad (2)$$

Here the dot denotes the derivative with respect to t . Note that \mathbf{r}_k, m_k and t are dimensionless using a characteristic length L , a characteristic mass M and the Newton's gravitational constant G . If the trajectory of each body at $t = T$ exactly returns its initial status, say,

$$\mathbf{r}_k(T) = \mathbf{r}_k(0) = \mathbf{r}_{0,k}, \quad \dot{\mathbf{r}}_k(T) = \dot{\mathbf{r}}_k(0) = \mathbf{v}_{0,k}, \quad 1 \leq k \leq N, \quad (3)$$

one has a periodic solution of the N -body problem. The famous three-body problem (Poincaré, 1890; Broucke, 1975; Broucke and Boggs, 1975; Hadjidemetriou, 1975; Hénon, 1976; Moore, 1993; Montgomery, 1998; Li and Liao, 2017; Li et al., 2018; Sun, 2018; Kwiecinski et al., 2018; Stone and Leigh, 2019; Li and Liao, 2019; Lin et al., 2019; Tanikawa et al., 2019; Abouelmagd et al., 2020; Li et al., 2021) corresponds to $N = 3$.

1.1. Times of Newton, Euler, Lagrange and Poincaré

For the two-body problem, corresponding to $N = 2$, Newton gave a closed-form periodic solution. However, for a three-body problem ($N = 3$), it becomes extremely difficult to find periodic orbits: no periodic orbits had been found until Euler reported one in 1740 and Lagrange published one in 1772. However, according to Montgomery's topological method (Montgomery, 1998) of classifying periodic orbits of three-body systems, they belong to the *same* family, namely the "Euler-Lagrange family". Thereafter, *no* new periodic orbits were reported in about two centuries.

Why is it so difficult to find periodic orbits of three-body systems? The mysterious reasons were revealed by Poincaré (1890) in 1890,

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who proved that, unlike the two-body problem that is integrable and thus its solutions is completely understood, the three-body problem is *not* integrable. This well explains why *only* the Euler–Lagrange family (in closed form) were found in more than two hundred years, since closed-form orbits do *not* exist at all in general. It implies that one generally *had* to use numerical algorithms to solve three-body problem, but unfortunately the electronic computer was even *not* invented in the times of Poincaré (1890). In addition, Poincaré (1890) also found that trajectories of three-body system are generally rather sensitive to initial conditions (i.e. the butterfly-effect), which leaded to the foundation of a new field of modern science, i.e. chaotic dynamics. Nowadays, it is well-known that, due to the famous butterfly-effect, i.e. a hurricane in North America might be created by a flapping of wings of a distant butterfly in South America several weeks earlier, it is very difficult to gain reliable trajectories of chaotic systems, especially in a long interval of time. All of these explain why periodic orbits of three-body problem are so difficult to obtain and why it becomes one of the oldest open question in science.

In most cases, trajectories of three-body system are chaotic, i.e. non-periodic, as discovered by Poincaré (1890) in 1890 and confirmed again by Lorenz (1963) in 1963 and Stone and Leigh (2019) in 2019. However, in some special cases, there indeed exist periodic orbits, i.e. the three bodies exactly return to their initial positions and initial velocities after a period T . The periodic orbits of the three-body problems are very important, since they are “the only opening through which we can try to penetrate in a place which, up to now, was supposed to be inaccessible”, as pointed out by Poincaré (1890). However, the question is: how can we find them *effectively*?

1.2. Times of supercomputer

The excellent work of Poincaré (1890) made a historical turning point of searching for periodic orbits of three-body system. The non-existence of the uniform first integral of triple system reveals the impossibility of finding closed-form analytic solutions of three-body problems in general cases: it clearly indicates that we mostly *had to* (i.e. must) use numerical algorithms to solve this problem. This was indeed a revolutionary contribution of Poincaré (1890) with great foresightedness at that times when there was even *no* electronic computers at all: about a half century later, Turing (1936, 1950) published his epoch-making papers that became the foundation of modern computer and artificial intelligence. Thanks to Von Neumann (1958), who proposed the so-called Von Neumann - Machine for modern computer, also to Jack S. Kilby, who won the Nobel prize in Physics 2000 for his taking part in the invention of the integrated circuit, and to many scientists, mathematicians, engineers and so on whose names we have no space to mention here, the performance of supercomputer becomes better and better, together with more and more advanced numerical algorithms. This provides us a strong support of hardware and software for discovering new periodic orbits of three-body problem now.

With the ceaseless progress of electronic computers, more and more researchers followed the way, which Poincaré (1890) suggested, to numerically search for periodic orbits of three-body problem. According to Montgomery’s topological method (Montgomery, 1998) of classifying periodic orbits of three-body systems, only three families of periodic orbits were found in 300 years after Newton, i.e. (1) the so-called “Euler–Lagrange family” found by Euler in 1740 and Lagrange in 1772; (2) the so-called “BHH family” *numerically* found by Broucke (1975) and Broucke and Boggs (1975) in 1975, Hadjidemetriou (1975) in 1975 and Hénon (1976) in 1976, respectively; (3) the so-called “figure-eight family” for three equal masses *numerically* found by Moore (1993) in 1993, until 2013 when Šuvakov and Dmitrašinović (2013) *numerically* gained 11 new families of periodic orbits of triple system with three equal masses. All of these suggest that finding periodic orbits of three-body problem by means of numerical methods should be a correct way.

Note that, among the three families of periodic orbits mentioned above, the BHH family and figure-8 family were found by numerical methods respectively in 1975s and 1993, when the performance of computer might be not good enough. However, in 2010s we had supercomputer with peak performance about 1,000 petaflops, i.e. several billion billion fundamental calculations per second, together with lots of powerful numerical algorithms. What prevents us from *effectively* finding thousands of new families of periodic orbits of three-body problem though we have so powerful supercomputer?

The key of finding periodic orbits is to gain reliable computer-generated trajectories of three-body system under arbitrary initial conditions in a *long enough* interval of time. However, as discovered by Poincaré (1890) and rediscovered by Lorenz (1963), computer-generated trajectories of chaotic systems are sensitive to initial conditions, i.e. the famous butterfly-effect. In other words, a tiny difference on initial conditions might lead to a huge deviation of computer-generated simulation after a long time. In addition, Lorenz (1989, 2006) further found that computer-generated trajectories of chaotic systems are also sensitive to algorithms: different numerical algorithms might give distinctly different computer-generated trajectories of chaotic systems after a long time. This is indeed a great obstacle! This kind of sensitive dependence on numerical algorithm (SDNA) for a chaotic system had been also observed and reported by many researchers (Li et al., 2000, 2001; Teixeira et al., 2007), which however unavoidably leaded to some intense arguments (Yao and Hughes, 2008): some researchers even suggested that “all chaotic responses should be simply numerical noises” and might “have nothing to do with differential equations”. Besides, it is currently reported by Chandramoorthy and Wang (2021) that “shadowing solutions can be almost surely nonphysical”, which “invalidates the argument that small perturbations in a chaotic system can only have a small impact on its statistical behavior”. Thus, by means of traditional numerical algorithms (mostly in double precision), it is rather difficult to gain reliable/convergent computer-generated trajectories of three-body system under arbitrary initial conditions in a long *enough* interval of time.

To overcome this obstacle, Liao (2009) proposed the so-called “clean numerical simulation” (CNS) for chaotic systems. Unlike other traditional numerical algorithms, which mostly use double precision, the CNS can greatly reduce *not only* truncation error *but also* the round-off error to keep the total “numerical noises” in such a required tiny level that a reliable (or convergent) computer-generated simulation can be obtained in a long enough interval of time. In the frame of the CNS, the truncation error is decreased by means of numerical algorithms at high enough order in time and space, and the round-off error is reduced by using multiple-precision with many enough digits for all parameters and variables. The CNS has been successfully applied to many chaotic systems, such as Lorenz equation, chaotic three-body systems, some spatio-temporal chaotic systems and so on (Liao, 2014; Liao and Wang, 2014; Liao and Li, 2015; Hu and Liao, 2020; Liao and Qin, 2022). As reported by Hu and Liao (2020), the use of double precision might lead to huge deviations of computer-generated simulations of spatio-temporal chaos *even in statistics*, not only quantitatively but also qualitatively, particularly in a long interval of time. This indicates that we must be very careful to numerically simulate chaotic systems. Fortunately, the CNS can provide us a guaranteed tool to gain reliable/convergent trajectories of chaotic systems (such as three-body systems with arbitrary initial conditions) during a long enough time.

Using the CNS as an integrator of the governing equations and combining the grid search method and the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011), Li and Liao (2017) in 2017 successfully found 695 families of periodic planar collisionless orbits of three-body systems with *three* equal masses and zero angular momentum, including the figure-eight family, the 11 families found by Šuvakov and Dmitrašinović in 2013, and besides more than 600 new families that have never been reported. Similarly, Li et al.

(2018) further found 1349 new families of periodic planar collisionless orbits of the three-body system with only *two* equal masses. In 2020, starting from a known periodic orbit with three equal masses and using the CNS to integrate the governing equations, Li et al. (2021) successfully obtained 135 445 new periodic orbits with arbitrarily *unequal* masses by means of combining the numerical continuation method (Allgower and Georg, 2003) and the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011), including 13 315 stable ones. Therefore, in only four years, using high-performance computer and our new strategy based on the CNS, we successfully increased the family number of the known periodic orbits of three-body systems by nearly *four* orders of magnitude! This strongly indicates that our numerical strategy is correct, powerful and rather *effective* for finding new periodic orbits of three-body systems. It should be emphasized that this great progress is mainly due to the use of the new numerical strategy based on the CNS, since the performance of supercomputer is good enough for three-body systems even at the beginning of the 21st century.

Triple stars systems are key objects in astronomy. All observed periodic triple stars belong to the BHH family (Janković and Dmitrašinović, 2016). So it is an open problem whether there exist other families of periodic triple stars in our universe. In this paper, we present a new method to obtain periodic orbits of the three-body problem with different masses. This will enrich our understanding about the triple system.

2. An approach based on machine learning

However, it is time-consuming to use the numerical continuation method (Allgower and Georg, 2003) to find the 135 445 periodic orbits (with *unequal* masses) reported by Li et al. (2021).

The numerical continuation method was applied to obtain solutions of differential equations $\dot{\mathbf{u}} = F(\mathbf{u}, \lambda)$, where λ is a physical parameter, also called “natural parameter”. Assume \mathbf{u}_0 is a solution of the system at a natural parameter $\lambda = \lambda_0$. Using this solution \mathbf{u}_0 at $\lambda = \lambda_0$ as an initial guess, a new solution \mathbf{u}' can be obtained at a new natural parameter $\lambda = \lambda_0 + \Delta\lambda$ through the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002) and the clean numerical simulation (CNS) (Liao, 2009; Liao and Wang, 2014; Hu and Liao, 2020), if the increment $\Delta\lambda$ is small enough to make sure iterations converge. Besides, these periodic orbits are in essence *discrete*, say, only for some *specific* values of m_1 and m_2 in an irregular domain (in case of $m_3 = 1$ since we use the mass of the 3rd body as a characteristic mass M , without loss of generality). Can we gain a periodic orbit more *efficiently* for *arbitrary* values of masses m_1 and m_2 ? Thanks to the times of machine learning, the answer is positive and rather attractive, as described below. For example, let us use a known BHH orbit as a starting point to illustrate this. Fifty-seven satellite periodic orbits of the BHH family of three-body systems with three equal masses were found (Janković and Dmitrašinović, 2016) in 2016, and this number was extended to ninety-nine (Janković et al., 2020) in 2020. The initial configuration of the BHH satellite periodic orbits with zero angular momentum is described by

$$\mathbf{r}_1(0) = (x_1, 0), \mathbf{r}_2(0) = (1, 0), \mathbf{r}_3(0) = (0, 0),$$

and

$$\dot{\mathbf{r}}_1(0) = (0, v_1), \dot{\mathbf{r}}_2(0) = (0, v_2),$$

$$\dot{\mathbf{r}}_3(0) = (0, -(m_1 v_1 + m_2 v_2)/m_3),$$

where \mathbf{r}_i , $\dot{\mathbf{r}}_i$ and m_i is the position vector, velocity vector and mass of the i th body, respectively. Thus, for given m_1 , m_2 and m_3 (we assume $m_3 = 1$ thereafter), we should determine four unknown physical variables x_1, v_1, v_2 and the period T . Note that these orbits are periodic in a rotating frame of reference, say, the frame of coordinates rotates an angle θ_T in the corresponding period T .

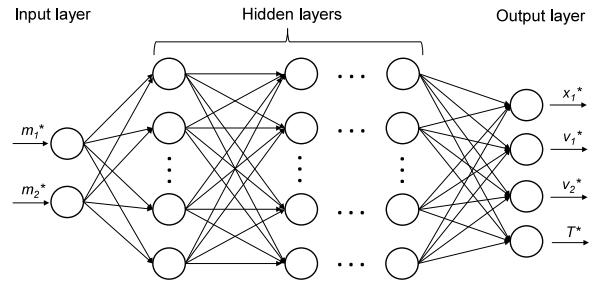


Fig. 1. The artificial neural network (ANN) model, where the input (m_1^*, m_2^*) and output $(x_1^*, v_1^*, v_2^*, T^*)$ are the normalized data of (m_1, m_2) and (x_1, v_1, v_2, T) , respectively.

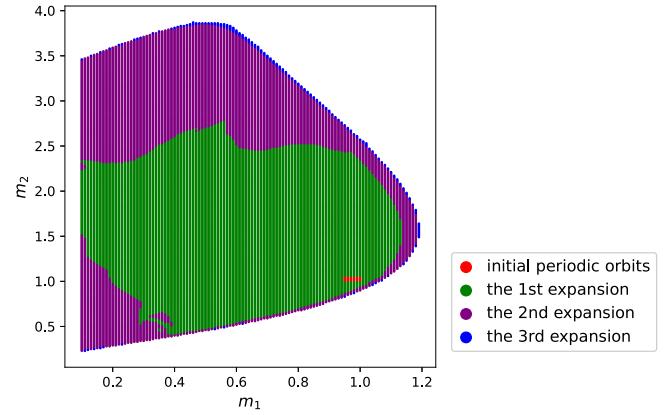


Fig. 2. The periodic orbits found by expansion on the mass region. Red dot (S_1): initial periodic orbits; green dot (S_2): the 1st extrapolation/expansion; purple dot (S_3): the 2nd extrapolation/expansion; blue dot (S_4): the 3rd extrapolation/expansion.

Without loss of generality, let us consider such a known BHH periodic orbit with the initial condition $x_1 = -1.325626981682458, v_1 = -0.8933877752879044, v_2 = -0.2885702941263346$, the period $T = 9.199307755830397$ and the rotation angel $\theta_T = 0.383160887655628$ of the coordinate frame, where $m_1 = m_2 = m_3 = 1$. Using this periodic orbit as a starting point, we first follow Li et al. (2021) to obtain only 36 periodic orbits for different masses in a small domain $m_1 \in [0.95, 1.00], m_2 \in [1.00, 1.05]$ (marked by S_1 , with the mass increment $\Delta m_1 = \Delta m_2 = 0.01$) by means of the numerical continuation method (Allgower and Georg, 2003) and the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011). Firstly, using the periodic orbit with equal masses, the periodic orbits with different masses m_1 and $m_2 = m_3 = 1$ can be obtained by means of the continuation method and the Newton–Raphson method. Secondly, using these periodic orbits with different masses m_1 and $m_2 = m_3 = 1$ as starting points, we obtain periodic orbits with different m_2 by means of the continuation method and the Newton–Raphson method. Finally, we can obtain periodic orbits with the mass domain $m_1 \in [0.95, 1.00], m_2 \in [1.00, 1.05]$ and $m_3 = 1$. For details, please refer to Li et al. (2021). The return distance (deviation) of these periodic orbits is defined by

$$\delta_T = \sqrt{\sum_{i=1}^3 (|\mathbf{r}_i(T) - \mathbf{r}_i(0)|^2 + |\dot{\mathbf{r}}_i(T) - \dot{\mathbf{r}}_i(0)|^2)}$$

in the rotating frame of reference, where T is the period. Note that all of the 36 periodic orbits satisfy the criteria $\delta_T < 10^{-10}$. Besides, they all have the same family name, which is defined by the so-called “free group element” according to Montgomery’s topological method (Montgomery, 1998).

Artificial neural network (Hassoun et al., 1995; Gevrey et al., 2003; Andrews et al., 1995; Abiodun et al., 2018) is a machine learning technique evolved from the idea of simulating the human brain, which

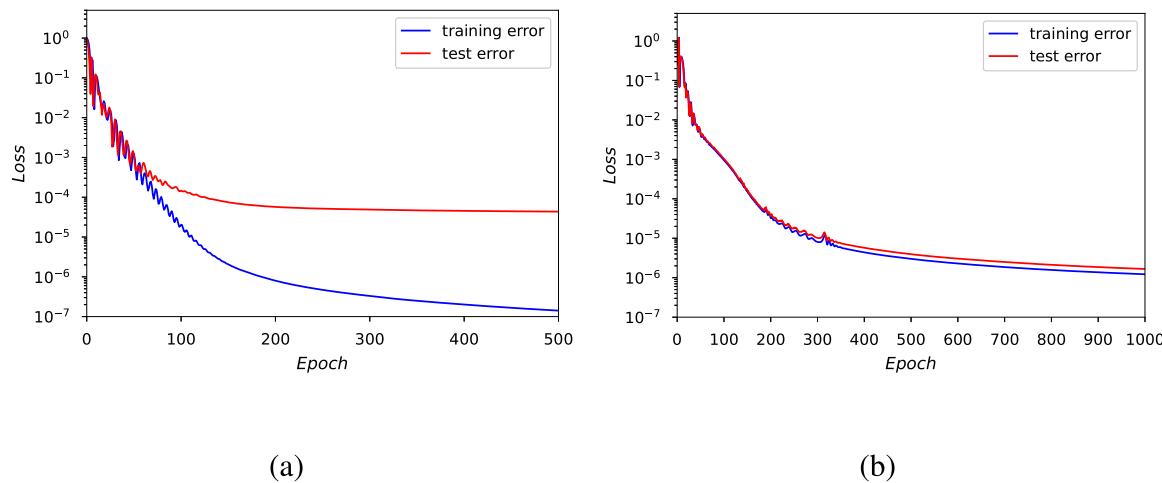


Fig. 3. (a) Loss function of the neural network for the first training with 36 periodic orbits. (b) Loss function of the neural network for the second training with 17 457 periodic orbits. Blue line: training error. Red line: test error.

can be used to perform statistical modeling. Compared with traditional regression approaches, the main advantages of ANN are as follows. First, the ANN does not require information about the complex nature of the underlying process to be explicitly expressed in mathematical form (Sudheer et al., 2002). Besides, the ANN has capability of modeling more complex nonlinear relationships (Livingstone, 2008). Therefore, generally speaking, the ANN is applicable across a wider range of problems than the traditional regression approaches. Especially, the ANN can easily deal with classification problems with complicated boundary (Bala and Kumar, 2017). Thus, we use the ANN in the following parts to model the relationship between the parameters of three-body problem and to classify the types of orbits and their stability.

In the next step, we use the ANN model to gain a relation between the input vector (m_1, m_2) and the output vector (x_1, v_1, v_2, T) of the periodic orbits. The ANN model we used here consists of multiple fully connected layers, say, one input layer, six hidden layers and one output layer. The number of neurons for the input layer, the hidden layers and the output layer is 2, 1024 and 4, respectively, as shown in Fig. 1. We use the optimization algorithm AMSGrad (Reddi et al., 2019) as an optimizer to minimize the mean square error for training the ANN model. At beginning, we use the results of the 36 known periodic orbits gained by means of the numerical continuation method (Allgower and Georg, 2003) and the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011) as the training set to train the ANN model. The trained ANN model provides us a kind of relationship (expressed by \mathcal{F}_1) between (m_1, m_2) and (x_1, v_1, v_2, T) , which can be further used to predict the initial conditions x_1, v_1, v_2 and the period T of candidates (i.e. possible periodic orbits) for various masses (m_1, m_2) outside of the original small domain $S_1 = \{(m_1, m_2) : m_1 \in [0.95, 1.00], m_2 \in [1.00, 1.05]\}$. Using the neural network’s predictions as the initial guesses for the Newton–Raphson method, we first obtain the periodic orbits with various m_2 when $m_1 = 1$, and with various m_1 when $m_2 = 1$ in order to give a reference of the mass region of possible candidates. Then, we set the candidates of periodic orbits within the mass region $(m_1, m_2) \in [0.1, 1.2], [0.4, 2.8]$ with $\Delta m = 0.01$ and exclude the mass region where the periodic orbits have been found. Finally, we find the 17 421 new periodic orbits (66.1%) within this region of 26 364 candidates with $\delta_T < 10^{-10}$ for various (m_1, m_2) outside of S_1 , which are marked in green in Fig. 2 and expressed by S_2 . In the same way, we further use the results of all (i.e. $36 + 17\,421 = 17\,457$) known periodic orbits as a training set to train our ANN model so as to gain a better relationship \mathcal{F}_2 between (m_1, m_2) and (x_1, v_1, v_2, T) for extrapolation/expansion outside of the mass domain. We set the candidates of periodic orbits

within the mass domain $(m_1, m_2) \in ([0.1, 1.2], [0.2, 3.9])$ with $\Delta m = 0.01$ and exclude the mass region where the periodic orbits have been found. Finally, we find the new 11 473 periodic orbits marked in purple in Fig. 2 and expressed by S_3 about 49.4% of all the 23 243 candidates. Similarly, we use the results of *all* (i.e. $36 + 17,421 + 11,473 = 28,930$) known periodic orbits as the training set to further train our ANN model so as to give a relationship F_3 between (m_1, m_2) and (x_1, v_1, v_2, T) . The chosen mass region (m_1, m_2) of possible candidates is just beyond the boundary of previous obtained periodic orbits, where the horizontal or vertical distance from the (m_1, m_2) to the boundary is less than $7\Delta m (\Delta m = 0.01, m_1 > 0.1, m_2 > 0.1)$. Then, we find the 220 new periodic orbits as marked in blue in Fig. 2 and expressed by S_4 , only about 7.1% out of all the 3104 candidates. This indicates that we come to the boundary of the mass domain (m_1, m_2) where the periodic orbits exit. In other words, there are no periodic orbits beyond that boundary. Thus, we find the nearly largest mass domain $S^* = S_1 \cup S_2 \cup S_3 \cup S_4$ for the existence of periodic orbits with some similar properties.

Note that, for each training, we have examined that there is no overfitting phenomenon before we use the neural network for further prediction. As we randomly divide the whole dataset into the training set (90%) and test set (10%), and find that there is no distinct difference between the errors of these two sets. Then, we train the ANN with all the data including the training and test set to make the best use of the known data for further prediction. For example, for the first training, we randomly divide the dataset with 36 examples into training set (90%) and test set (10%). The loss function of training is shown in Fig. 3(a). The loss is defined as the mean squared error of the standardized data. The mean relative errors in the training set and test set are about 8.3×10^{-7} and 5.1×10^{-5} , respectively. We find that as the training epoch increases, the error in the test set first decreases and then converges. It is not the phenomenon of overfitting where the loss function of test set first decreases and then increases. As the number of examples is limited, the test error is small while not as small as the training error. Then, we combine the test set into the training set to make use of all the data and use it for further extrapolation prediction. Note that, unlike the traditional use of the ANN, we apply the ANN model here to make *extrapolation* predictions of periodic orbits. As for the prediction error of *extrapolation* for (x_1, v_1, v_2, T) , we calculate the mean relative error by comparing the ANN's predictions with the "exact" results which are obtained by modifying the ANN's predictions via the Newton–Raphson method. The mean relative error of the ANN's extrapolation is about 0.05 for the first expansion of 17 421 new periodic orbits. For the second training with 17 457 (=17 421+36) periodic orbits, we randomly divide the whole dataset into the training set (90%) and test set (10%). The loss function is shown in Fig. 3(b). The

Table 1

Mean relative errors of different models for the first training with 36 data.

Model	Training set (32 data)	Test set (4 data)
Linear model	3.09×10^{-5}	3.40×10^{-5}
Two-dimensional interpolation	1.95×10^{-7}	2.51×10^{-7}
ANN	8.32×10^{-7}	5.11×10^{-5}

mean relative errors in the training set and test set are about 2.2×10^{-5} and 2.3×10^{-5} , respectively. Thus, as the number of data is sufficient, the difference between the errors of training set and test set is very close. Then, we train all the 17 457 examples for further prediction. The same approach is repeated in the similar way until no periodic orbits can be found by the ANN's extrapolation prediction. The mean relative error of the ANN's extrapolation prediction for the second and third expansion are about 0.009 and 0.00007, respectively. As shown in Fig. 2, starting from the 36 periodic orbits in the original small domain $(m_1, m_2) \in S_1$ (marked in red), we totally gain the 29 150 periodic orbits in the mass domain $(m_1, m_2) \in S^* = S_1 \cup S_2 \cup S_3 \cup S_4$ by the three times extrapolations/expansions. Finally, training our ANN model by using all (i.e. 29 150) these known results as a training set, we obtain a relationship \mathcal{F}^* between $(m_1, m_2) \in S^*$ and (x_1, v_1, v_2, T) , which can give the ANN's predictions of periodic orbits for arbitrary values of $(m_1, m_2) \in S^*$ in the accuracy level of 10^{-4} for the return distance (deviation) δ_T . All of these ANN's predictions of periodic orbits are accurate enough for normal use, although their accuracy can be further modified to arbitrary level by means of the Newton–Raphson method via the CNS and multiple-precision (MP), as mentioned below. Note that the mass domain (m_1, m_2) with the periodic orbits becomes larger and larger, i.e. from S_1 to $S_1 \cup S_2$ to $S_1 \cup S_2 \cup S_3$ to $S^* = S_1 \cup S_2 \cup S_3 \cup S_4$, and the relationship between (m_1, m_2) and (x_1, v_1, v_2, T) becomes better and better, from \mathcal{F}_1 to \mathcal{F}_2 to \mathcal{F}_3 to \mathcal{F}^* , implying that our ANN model becomes wiser and wiser!

Obviously, the smaller the return distance (deviation) δ_T is, the more accurate the periodic orbit given by the numerical strategy. Note that $\delta_T = 0$ exactly corresponds to a closed-form periodic orbit. But unfortunately, except the Euler–Lagrange family, nearly all known periodic orbits of three-body systems were gained by numerical methods, and this fact is consistent with Poincaré's famous proof of the non-existence of the uniform first integral of three-body systems (Poincaré, 1890). It is found that, since the CNS and the MP (multiple precision) are used in our numerical strategy, the return distance (deviation) δ_T of these periodic orbits can be reduced to *any* given value, say, the initial conditions (x_1, v_1, v_2) and the period T of these relatively periodic orbits can be at *arbitrary* level of precision, i.e. as accurate as one would like, as long as the performance of supercomputer is good enough.

In the present work, we use the ANN model as it is capable of modeling the complex nonlinear relationship, compared with the traditional approaches such as the linear model or two-dimensional regression. In the first training with 32 data, as the data is limited and the relationship is simple, the traditional approaches indeed have smaller mean relative errors in the test set, as shown in Table 1. However, in the second training with 17 457 data when the data is sufficient and the relationship between the outputs and the inputs becomes more complex, the ANN model has better performance with smaller error in the test set than the linear model and the two-dimensional regression, as shown in Table 2. It reflects that one of the advantages of ANN is its ability to model more complex relationship. Thus, the ANN can be applicable across a wider range of problems.

The relationship \mathcal{F}^* between $(m_1, m_2) \in S^*$ and (x_1, v_1, v_2, T) given by the ANN model is fundamentally different from the set of the original 29 150 periodic orbits in the following aspects.

(A) For *arbitrary* values of $(m_1, m_2) \in S^*$, \mathcal{F}^* can always give a good enough prediction of the initial condition (x_1, v_1, v_2) and the period T of the corresponding periodic orbit, which can be used as a starting point to gain a more accurate periodic orbit with a tiny return distance (deviation) $\delta_T < 10^{-10}$.

Table 2

Mean relative errors of different models for the second training with 17 457 data.

Model	Training set (15 711 data)	Test set (1746 data)
Linear model	1.54×10^{-2}	1.54×10^{-2}
Two-dimensional interpolation	2.49×10^{-3}	2.49×10^{-3}
ANN	2.18×10^{-5}	2.26×10^{-5}

Table 3

Initial conditions and periods T predicted by our trained ANN model for the six examples of BHH satellites with various masses m_1 and m_2 in case of $r_1(0) = (x_1, 0)$, $r_2(0) = (1, 0)$, $r_3(0) = (0, 0)$, $\dot{r}_1(0) = (0, v_1)$, $\dot{r}_2(0) = (0, v_2)$, $\dot{r}_3(0) = (0, -(m_1 v_1 + m_2 v_2)/m_3)$, where m_i , x_i and v_i are the mass, initial position and velocity of the i th body, respectively, with the same rotation angle $\theta = 0.38316088765562$ of the reference frame for relatively periodic orbits.

Case	m_1	m_2	x_1	v_1	v_2	T
(a)	1.0124	0.9968	-1.32962	-0.88963	-0.28501	9.2111
(b)	0.5312	2.2837	-0.97138	-1.37584	-0.34528	7.0421
(c)	0.8056	2.0394	-1.05795	-1.24044	-0.26723	7.4389
(d)	0.3916	0.8341	-1.21503	-1.00328	-0.53749	9.2807
(e)	0.1472	3.4219	-0.80027	-1.74811	-0.42176	5.8793
(f)	0.8413	1.4155	-1.17777	-1.05903	-0.29934	8.3444

(B) For each prediction of a periodic orbit given by our ANN model for arbitrary values of $(m_1, m_2) \in S^*$, we can always modify it by means of the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011), until a very accurate periodic orbit with $\delta_T < 10^{-60}$ is gained. For example, for the randomly chosen masses $m_1 = 0.550073$ and $m_2 = 1.738802$, the initial condition and period of the periodic orbit predicted by \mathcal{F}^* is $x_1 = -1.0509$, $v_1 = -1.2291$, $v_2 = -0.3751$ and $T = 7.7189$. The return distance δ_T of this predicted periodic orbit is about 6×10^{-4} , which can be reduced to 10^{-60} by means of the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011) and the CNS, with the initial condition and period in accuracy of hundred significant digits:

$$\begin{aligned} x_1 &= -1.0509175496041811604923698392786632425650672636140394304186656973, \\ v_1 &= -1.2291270518667041014360821708612407200218851479344331903064815068, \\ v_2 &= -0.3751072767005198189484967986510621349079165896727957038682248759, \\ T &= +7.7189475555888051714951891045174205372467192448577195610635970572. \end{aligned}$$

Note that the prediction given by our ANN model are in the accuracy of four significant digits. Similarly, we *randomly* check one thousand cases of $(m_1, m_2) \in S^*$, and found that, in every case, we indeed can always gain a periodic orbit with a quite tiny return distance (deviation) $\delta_T < 10^{-60}$. All of these suggest that, statistically speaking, for *arbitrary* mass values $(m_1, m_2) \in S^*$, every prediction given by our ANN model could lead to a periodic orbit that could be in *arbitrary* accuracy as long as the performance of computer is good enough.

(C) It is found that the mean absolute errors of predictions for randomly selected one thousand cases of $(m_1, m_2) \in S^*$ given by our trained ANN model \mathcal{F}^* are in the level of 10^{-5} for the initial conditions (x_1, v_1, v_2) and 10^{-4} for the period T . Although we indeed can further gain a periodic orbit in accuracy of one hundred significant digits by means of the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011) and the CNS, the periodic orbits predicted by our trained ANN model are good enough from practical viewpoint, since it is *unnecessary* to have so accurate trajectory in practice. For example, there also exists error in measurement for the actual astronomical observation. In practice, it is rather convenient to gain a periodic orbit for *arbitrary* masses $(m_1, m_2) \in S^*$ by means of the initial conditions (x_1, v_1, v_2) and period T predicted by the trained ANN model, for example, as shown in Fig. 4 for randomly chosen six different masses of m_1 and m_2 . Compared with using

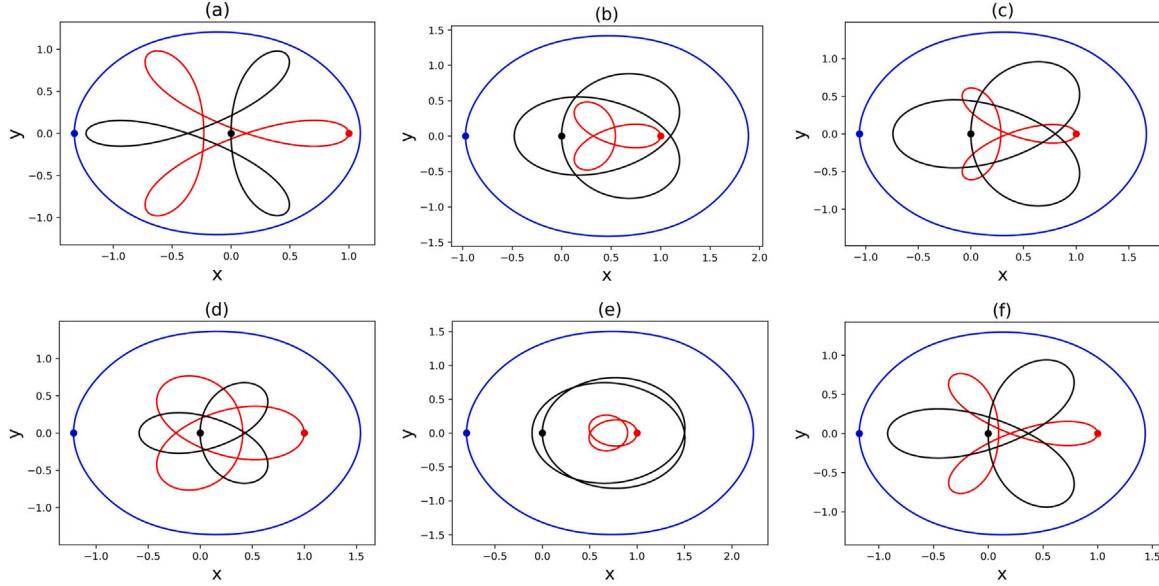


Fig. 4. The relatively periodic BHH satellites orbits of the three-body system with various masses m_1 and m_2 in a rotating frame of reference. The corresponding physical parameters are given by ANN in Table 3. Blue line: body-1; red line: body-2; black line: body-3.

the numerical continuation method to find a periodic orbit with arbitrary mass $(m_1, m_2) \in S^*$, using the neural network can directly provide a good enough prediction of periodic orbit in practice. While for the numerical continuation method, we need to modify the initial conditions by the Newton–Raphson method to give a good enough solution. Even if we aim to obtain the periodic orbits with high precision, the neural network can give more accurate initial solution for the Newton–Raphson method. Thus, our ANN model provides us great convenience in practice.

(D) The ANN model can ceaselessly learn and thus be further modified when some new periodic orbits are gained, say, the trained ANN model could become wiser and wiser.

In summary, we illustrate that the trained ANN model can provide us accurate enough periodic orbits for *arbitrary* values of (m_1, m_2) in an irregular domain S^* .

How accurate is a periodic orbit with return distance (deviation) $d_T < 10^{-60}$? Even if we use the diameter of universe $d_u = 930$ light year $= 8.8 \times 10^{18}$ meter as the characteristic length L , we have the corresponding inaccuracy of the dimensional initial position $|\Delta x_1|d_u < 10^{-41}$ meter that is even smaller than the Planck length $l_p \approx 1.62 \times 10^{-35}$ meter. Note that Planck length is a lower bound to physical proper length in any space–time: it is impossible to measure length scales smaller than Planck length, according to Padmanabhan (1985). Besides, it should be emphasized that all of these periodic orbits are *stable*, say, a tiny disturbance does not increase exponentially. So, from physical view-point, a stable periodic orbit with return distance (deviation) $d_T < 10^{-60}$ gained by numerical method is *equivalent* to $d_T = 0$ that corresponds to an *exact* solution of periodic orbit in a closed-form.

Therefore, the high-performance computer and the machine learning play very important role in finding periodic orbits of three-body systems with arbitrary masses. It should be emphasized here that it is Turing (1936, 1950) who laid the foundations of modern computer and artificial intelligence (including machine learning).

The above approach has general meaning. To show this point, let us further consider another BHH satellite periodic orbit with three equal masses $m_1 = m_2 = m_3 = 1$, $x_1 = -1.609965115714630$, $v_1 = -0.6656909425824538$, $v_2 = -0.1529561125709906$, the period $T = 6.879203007710456$ and the rotation angel $\theta = 0.0105056462558377$ of the reference frame (for relatively periodic orbits). At first, using this known periodic orbit as a starting point, we can obtain 36 periodic

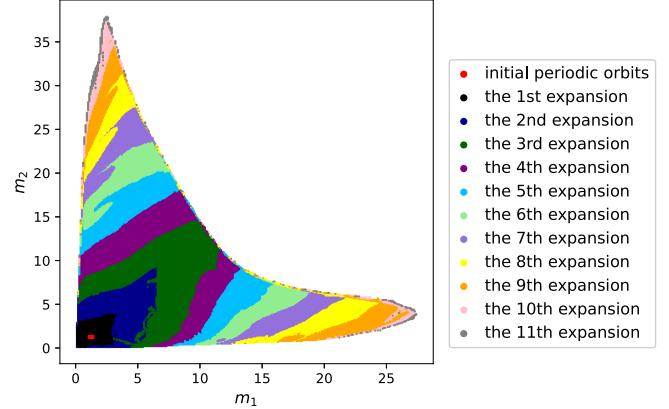


Fig. 5. The relatively periodic orbits with the same rotation angle $\theta = 0.0105056462558377$ of reference frame, found in each extrapolation/expansion on the various mass regions. Red dot: initial periodic orbits; black dot: 1st expansion; dark blue dot: 2nd expansion; dark green dot: the 3rd expansion; dark purple dot: the 4th expansion; light blue dot: the 5th expansion; light green dot: the 6th expansion; light purple dot: the 7th expansion; yellow dot: the 8th expansion; orange dot: the 9th expansion; pink dot: the 10th expansion; gray dot: the 11th expansion.

orbits with various masses in a small mass domain $m_1 \in [1.0, 1.5]$ and $m_2 \in [1.0, 1.5]$ (with mass increment $\Delta m_1 = \Delta m_2 = 0.1$) by combining the numerical continuation method (Allgower and Georg, 2003) and the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011).

Similarly, we train a ANN model (with the same structure as mentioned above) by means of the initial conditions and periods of these 36 known periodic orbits and then use the trained ANN model to predict the initial conditions and periods of some candidates of possible periodic orbits *outside* the previous mass domain, while the initial conditions and periods of each candidate are modified by the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011) so as to confirm whether or not the candidate is a periodic one, say, its return distance can be reduced to the tiny level of 10^{-10} . The same process repeats 11 times, while the number of the known relatively periodic orbits becomes more and more, the mass domain becomes larger and larger, and the ANN model becomes wiser and

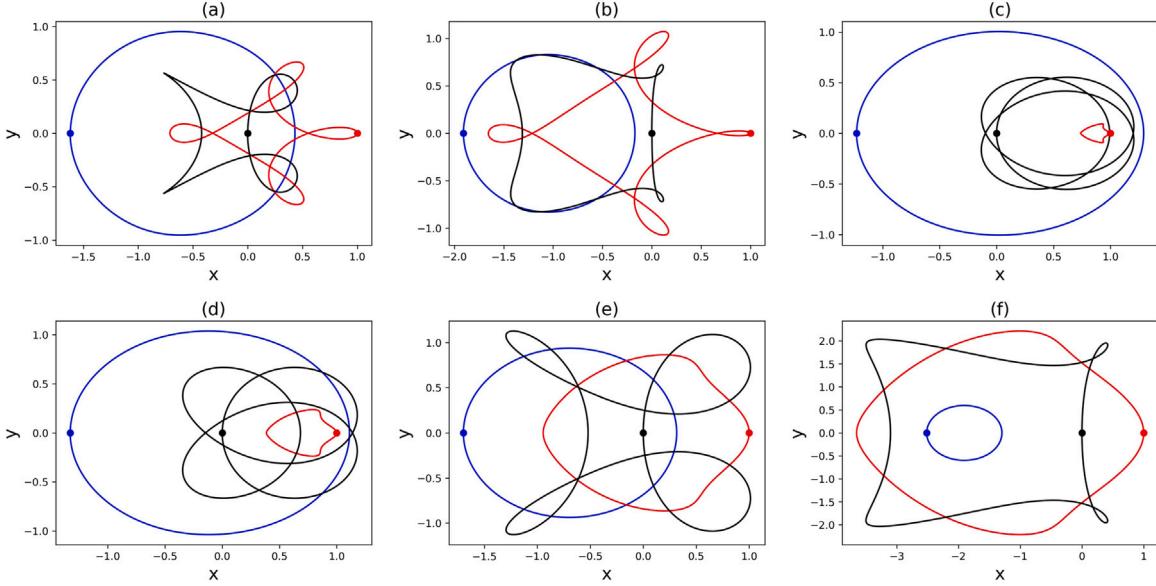


Fig. 6. The relatively periodic BHH satellites orbits of the three-body system with various masses m_1 and m_2 in a rotating frame of reference. The corresponding physical parameters are given by ANN in Table 4. Blue line: body-1; red line: body-2; black line: body-3.

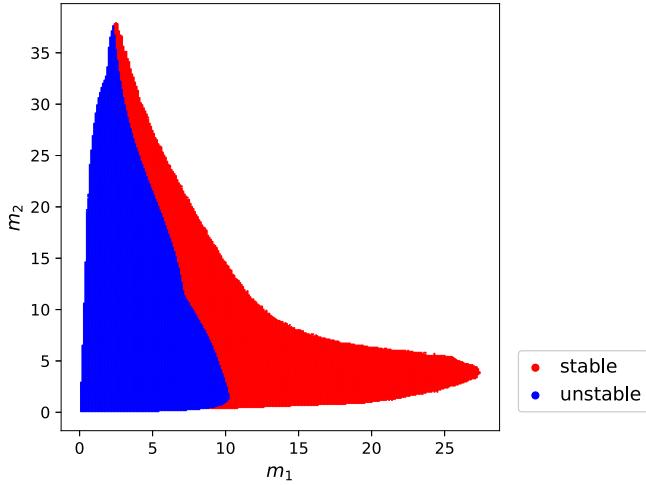


Fig. 7. The linear stability of the relatively periodic orbits in the second case with the rotation angle $\theta = 0.0105056462558377$. Red domain: stable; blue domain: unstable.

Table 4

Initial conditions and periods T of six examples of BHH satellites with various masses m_1 and m_2 , predicted by our trained ANN model in case of $r_1(0) = (x_1, 0)$, $r_2(0) = (1, 0)$, $r_3(0) = (0, 0)$, $r_1(0) = (0, v_1)$, $r_2(0) = (0, v_2)$, $r_3(0) = (0, -(m_1 v_1 + m_2 v_2)/m_3)$, where m_i , x_i and v_i are the mass, initial position and velocity of the i th body, respectively, with the same rotation angle $\theta = 0.0105056462558377$ of the reference frame for relatively periodic orbits.

No.	m_1	m_2	x_1	v_1	v_2	T
(a)	1.0283	0.9879	-1.62064	-0.65955	-0.14784	6.9193
(b)	1.5142	0.4968	-1.90809	-0.50283	-0.08614	8.2924
(c)	2.9216	31.9067	-1.22892	-2.37239	0.09481	1.6822
(d)	4.4143	18.6575	-1.32880	-1.86103	0.25507	2.3306
(e)	10.3501	10.4522	-1.69797	-1.22777	0.86183	3.5607
(f)	18.7011	4.2388	-2.51585	-0.53262	1.49754	5.9612

wiser! Finally, we totally obtain 35 895 relatively periodic orbits in a mass domain \bar{S}^* whose area is about 1000 times larger than the initial one $m_1 \in [1.0, 1.5]$ and $m_2 \in [1.0, 1.5]$, as shown in Fig. 5.

Similarly, we further train our ANN model by using the known 35 895 relatively periodic orbits. To show the validation and accuracy

of this ANN's model, we randomly chose 1000 cases of *arbitrary* mass $(m_1, m_2) \in \bar{S}^*$, and it is found that the initial conditions and periods predicted by our trained ANN model can provide us good enough periodic orbits with a return distance d_T in a tiny level of 10^{-3} (about 71%), 10^{-4} (about 2%), and 10^{-2} (about 27%), which can be further decreased by means of the clean numerical simulation (CNS) and the Newton–Raphson method with multiple-precision (MP) until a very small return distance (such as $d_T < 10^{-60}$) is satisfied. Thus, generally speaking, our trained ANN model can always give periodic orbits for an arbitrary mass $(m_1, m_2) \in S_2^*$, which is accurate enough from practical viewpoint. For example, randomly choosing six cases with various masses m_1 and m_2 , where $(m_1, m_2) \in \bar{S}^*$, our ANN model can quickly predict their initial conditions and periods, as listed in Table 4, and give the corresponding trajectories in a satisfied level of accuracy, as shown in Fig. 6. It is found that the differences between the periodic orbits predicted by the ANN's model with $10^{-4} < d_T < 10^{-2}$ and the “exact” ones further modified by the Newton–Raphson method and the multiple-precision (MP) with $d_T < 10^{-60}$ are negligible for practical uses. Thus, our approach based on the ANN model has indeed general meanings.

3. Classification for orbits based on machine learning

Stability is an important property of periodic orbits, because only stable triple systems can be observed. We employ a theorem given by Kapela and Simó (2007) to determine the linear stability of periodic orbits of three-body problem through the monodromy matrix (Simó, 2002). It is found that all relatively periodic orbits for the first case with the rotation angle $\theta = 0.3831608876556280$ are linearly stable. For the second case with the rotation angle $\theta = 0.0105056462558377$, there are 16 739 linearly stable periodic orbits among the 35 895 computer-generated periodic orbits, as shown in Fig. 7. Note that, the ANN can be applicable to deal with complicated classification problems (Bala and Kumar, 2017). We use the ANN to classify different types of orbits for different masses which have complex shapes of the boundary. We can give an ANN classifier model to classify the periodicity and stability of the orbits for *arbitrary* masses m_1 and m_2 , especially for the masses nearly the boundary of each type. The orbits are classified into three categories: stable periodic orbits, unstable periodic orbits and non-periodic orbits, expressed by $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$, respectively. The ANN classifier model consists of eight fully connected layers with

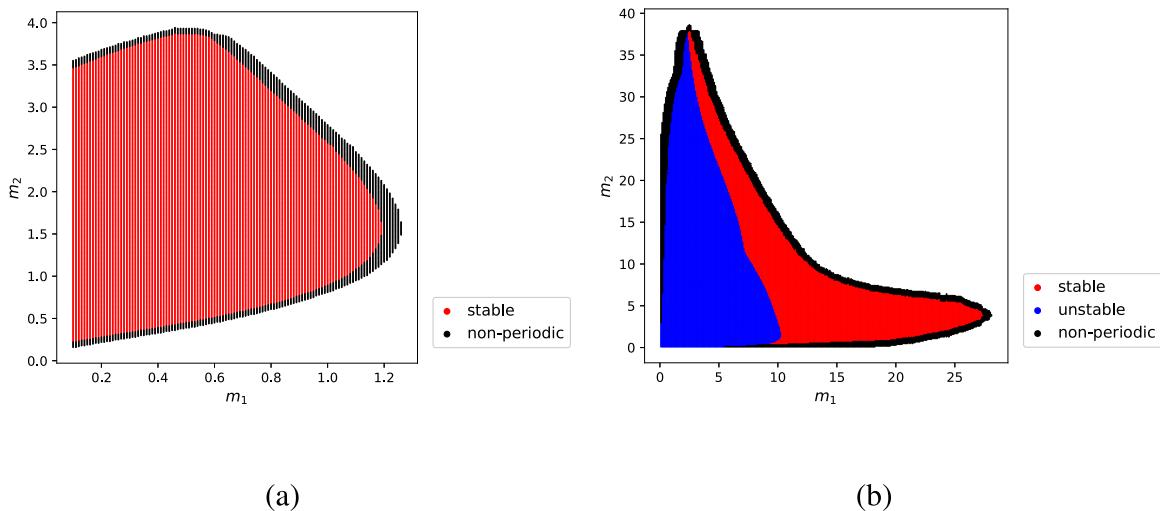


Fig. 8. The linear stability and periodicity of the orbits for the first case (a) and the second case (b). Red dot: linear stable; blue dot: unstable; black dot: non-periodic.

Table 5
Confusion matrix for the ANN predictions in test set of the first case.

Type	Stable periodic orbits	Non-periodic orbits
Stable periodic orbits	1475	2
Non-periodic orbits	0	137

Table 6
Confusion matrix for the ANN predictions in test set of the second case.

Type	Stable periodic orbits	Unstable periodic orbits	Non-periodic orbits
Stable periodic orbits	846	1	5
Unstable periodic orbits	1	943	2
Non-periodic orbits	1	0	291

one input layer, six hidden layers and one output layer. The numbers of neurons for input layer, hidden layers and output layer are 2, 256 and 3, respectively. Different from the above-mentioned ANN regression model, the activation function of the ANN classifier model is softmax function. The loss function is cross entropy. As shown in Fig. 8, for the first case, the mass domain of 3128 non-periodic orbits are outside the boundary of periodic orbits. The numbers of stable periodic orbits, unstable periodic orbits and non-periodic orbits are 29150, 0 and 3128, respectively. For the second case, the mass domain of 5900 non-periodic orbits are outside the boundary of periodic orbits. The numbers of stable periodic orbits, unstable periodic orbits and non-periodic orbits are 16739, 19156 and 5900, respectively. For each case, the whole dataset is randomly divided into three sets, the training set (90%), validation set (5%) and test set (5%).

The early stopping (Prechelt, 1998) is used when training, and we save the ANN model with the maximum accuracy in the validation set. The test set is used to verify the generalization capability of the ANN model. For the first case, the accuracies of the training set, validation set and test set are about 99.98%, 99.94% and 99.94%, respectively. And the macro F1-scores (Dimitrov et al., 2017) of the training set, validation set and test set are about 0.9995, 0.9983 and 0.9980, respectively. For the second case, the accuracies of the training set, validation set and test set are about 99.62%, 99.57% and 99.52%, respectively. And the macro F1-scores of the training set, validation set and test set are about 0.9941, 0.9935 and 0.9932, respectively. As the accuracies and F1-scores of the training set and test set are close, overfitting does not exist in these two cases. The confusion matrixes for the ANN predictions in the test sets of the first case and the second case are shown in Tables 5 and 6, respectively, which illustrate the good performances of the ANN. Therefore, we can use the ANN classifier

to predict the periodicity and stability of orbits for any given masses in each case.

4. A roadmap of searching for periodic orbits of three-body problem

The successful examples mentioned above suggest us a general road map for finding new periodic orbits of three-body system (in case of $m_3 = 1$) with the same “free group element” (word) given by Montgomery’s topological method (Montgomery, 1998):

- (1) For a three-body system with three or two equal masses, first find candidates of the initial conditions for possible periodic orbits by means of the grid search method, and then modify these candidates by means of the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011), until a satisfied periodic orbit with a tiny enough return distance (deviation) is obtained;
- (2) Given a known periodic orbit, use it as a starting point to gain a few of new periodic orbits with various masses in a small domain of mass (m_1, m_2) by combining the numerical continuation method (Allgower and Georg, 2003) and the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011). The initial conditions and periods of all these known periodic orbits form a training set for a ANN model.
- (3) For a given training set in a mass domain of (m_1, m_2) , train the ANN model to predict initial conditions and periods so that some new periodic orbits *outside* of the previous mass domain of (m_1, m_2) could be found by modifying these predictions via the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011). Then, combining the results of these new periodic orbits with the previous training set, we further have a new training set with more elements, which could further provide us some new periodic orbits in a even larger mass domain of (m_1, m_2) in a similar way, *outside* of the previous one. The same process can repeat again and again, so that more and more periodic orbits are found in a larger and larger mass domain of (m_1, m_2) , and the ANN model becomes wiser and wiser, until quit few or no new periodic orbits can be found in a larger domain of (m_1, m_2) . Finally, we have a trained ANN model \mathcal{F}^* of all periodic orbits in the final mass domain $(m_1, m_2) \in S^*$.
- (4) Randomly choose hundreds or thousands of *arbitrary* masses $(m_1, m_2) \in S^*$. For each case, check whether or not the trained ANN model \mathcal{F}^* could give an accurate enough prediction, and in addition whether or not the corresponding return distance

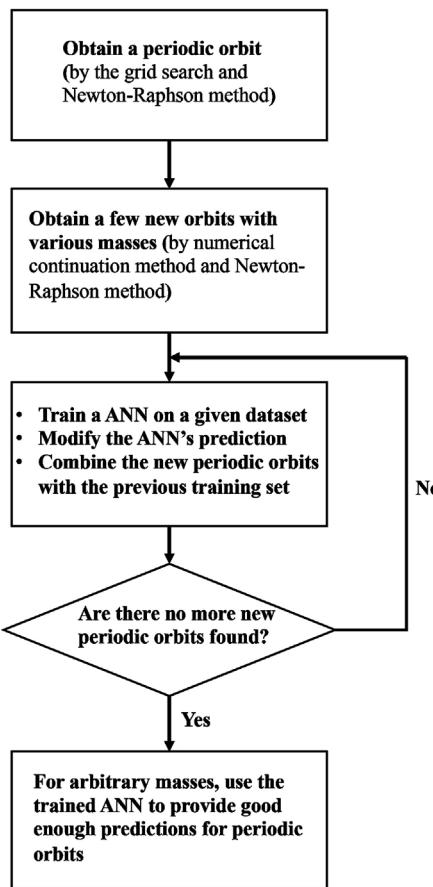


Fig. 9. A roadmap for searching the periodic orbits of three-body problem.

(deviation) could be indeed reduced to 10^{-60} . If yes, then the ANN model \mathcal{F}^* can provide a good enough prediction of periodic orbits in the domain $(m_1, m_2) \in S^*$, from practical viewpoint.

The pipeline for searching the periodic orbits of three-body problem is shown in Fig. 9. Note that it is better to use the CNS as integrator so as to guarantee the reliability and convergence of computer-generated trajectories of chaotic three-body system under arbitrary initial conditions in a required long interval of time. Besides, the periodicity and stability of orbits for *arbitrary* masses m_1 and m_2 can be well predicted by an ANN classifier model.

5. Concluding remarks and discussions

The famous three-body problem can be traced back to Newton in 1687, but quite few families of periodic orbits were found in 300 years thereafter. As proved by Poincaré, the first integral does not exist for three-body systems, which implies that numerical approach had to be used in general.

Artificial neural network (Hassoun et al., 1995; Gevrey et al., 2003; Andrews et al., 1995; Abiodun et al., 2018) is a machine learning technique evolved from simulating the human brain, which has been widely proved to be rather powerful. Compared with the traditional regression approaches, the ANN has many advantages. First, the ANN does not require information about the complex nature of the underlying process to be explicitly expressed in mathematical form (Sudheer et al., 2002). Besides, generally speaking, the ANN can handle more complex nonlinear relationships than the traditional regression approaches (Livingstone, 2008), so that the ANN is applicable across a wider range of problems. Especially, the ANN can be easily applied to deal with classification problems with complicated boundary (Bala-

and Kumar, 2017). Therefore, we use the ANN here to model the relationship between the parameters of three-body problem and to classify the types of orbits and their stability.

In this paper, we propose an effective approach and roadmap to numerically gain planar periodic orbits of three-body systems with arbitrary masses by means of machine learning based on an artificial neural network (ANN) model. Given any a known periodic orbit as a starting point, this approach can provide more and more periodic orbits (of the same family name) with variable masses, while the mass domain having periodic orbits becomes larger and larger, and the ANN model becomes wiser and wiser. Finally we have an ANN model trained by means of all obtained periodic orbits of the same family, which provides a convenient way to give accurate enough periodic orbits with *arbitrary* masses for physicists and astronomers. In addition, the periodicity and stability of orbits for *arbitrary* masses can be well predicted by an ANN classifier model.

It must be emphasized that high performance computer and artificial intelligence (including machine learning) play important roles in solving periodic orbits of triple systems. Today, nothing can prevent us from obtaining massive periodic solutions of three-body problem. This is due to the great contributions of some great mathematicians, scientists and engineers in more than three hundred years! Especially, it is Poincaré (1890) who made a historical turning point by proving the non-existence of the uniform first integral of triple system, which implies that we had to use numerical approach in general. It is Turing (1936, 1950) who published his epoch-making papers that became the foundation of modern computer and artificial intelligence. It is Von Neumann (1958) who proposed the so-called “Von Neumann - Machine” for modern computer. It is Jack S. Kilby, the winner of Nobel prize for physics in 2000, who took part in the invention of the integrated circuit. And so on. The famous three-body problem might

be an excellent example to illustrate the importance of inventing new tools for human being to better understand and explore the nature.

The approach and roadmap mentioned in this paper has general meanings. Note that thousands of families of periodic orbits of triple systems with three or two equal masses have currently been found (for example, please visit the website <https://github.com/sjtu-liao/three-body>). Using any of them as a starting point, we can similarly gain a trained ANN model to give accurate enough predictions of periodic orbits of the same family of triple system in a corresponding mass domain. All of these might form a massive data base for periodic orbits of triple systems, which should be helpful to enrich and deepen our understandings about the famous three-body systems. Besides, hopefully, some physical laws, such as generalized Kepler's law (Li and Liao, 2017) for periodic orbits of triple systems with *arbitrary* masses, could be found in future by analyzing these massive data by means of machine learning.

Note also that nearly all known periodic orbits of triple systems are planar, i.e. two-dimensional. In theory, the same ideas mentioned in this paper can be used to search for periodic orbits of triple systems in three dimensions. Although, according to our results reported here and in the previous papers (Li et al., 2018, 2021), given a pair (m_1, m_2) , there is only one possible set (x_1, v_1, v_2, T) that would lead to a periodic orbit of the three-body problem, it should be interesting to investigate whether there exist bifurcations of periodic orbits in future.

CRediT authorship contribution statement

Shijun Liao: Conceptualization, Methodology, Writing – original draft, Supervision, Funding acquisition. **Xiaoming Li:** Data curation, Formal analysis, Validation, Writing – original draft. **Yu Yang:** Data curation, Formal analysis, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statements

The data and the code related to this paper can be seen in the website (<https://github.com/sjtu-liao/three-body>), including the trained ANN models of the two cases, all the periodic orbits found (also including those reported in Li and Liao (2017), Li et al. (2018, 2021)), the code to train the ANN and use the ANN to predict the periodic orbits, and the code to train the ANN for classifying the orbits.

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Appendix. Direct search for periodic orbits with arbitrary masses

The phase space of the planar three-body problem has 12 dimensions. The grid searching method suffers the curse of dimensionality in such high dimensions. In order to reduce the dimensionality of the searching space, we can fix some parameters of initial conditions. Since the collinear instant of three bodies is common in the three-body system, it is reasonable to consider three bodies are in a line at the start of periodic orbits. The three bodies are in a line for the BHH family of periodic orbits: $\mathbf{r}_1(0) = (x_1, 0)$, $\mathbf{r}_2(0) = (x_2, 0)$, $\mathbf{r}_3(0) = (x_3, 0)$, and their

initial velocities are orthogonal to the line: $\dot{\mathbf{r}}_1(0) = (0, v_1)$, $\dot{\mathbf{r}}_2(0) = (0, v_2)$, $\dot{\mathbf{r}}_3(0) = (0, v_3)$.

According to the scaling law of the three-body system (Li et al., 2021), we can fix $x_2 = 1$ and $x_3 = 0$. Without loss of generality, we assume momentum of the system is equal to zero, thus $v_3 = -(m_1 v_1 + m_2 v_2)/m_3$.

The approximated initial conditions and periods of periodic orbits can be obtained through grid searching method, then these approximated periodic orbits will be corrected by means of the Newton–Raphson method (Farantos, 1995; Lara and Pelaez, 2002; Abad et al., 2011) and the clean numerical simulation (CNS) (Liao, 2009; Liao and Wang, 2014; Hu and Liao, 2020). The differential equations of the planar three-body problem can be described as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}); \quad \mathbf{x}(0) = \mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^{12}, \quad (4)$$

and $\mathbf{x} = \mathbf{x}(t, \mathbf{y})$, $t \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{12}$ is the solution of these equations, where \mathbf{y} is the initial condition.

For the initial condition \mathbf{y} and period T , a relative periodic orbit with a rotation angle θ means that

$$\mathbf{x}(T, \mathbf{y}) - \mathbf{P}(\theta)\mathbf{y} = 0, \quad (5)$$

where $\mathbf{P}(\theta)$ is the rotation matrix.

Now we correct the approximated initial conditions and periods of periodic orbits. Let us assume the approximated initial conditions and periods are (\mathbf{y}_i, T_i) at i th step. To improve accuracy of the approximated initial conditions and periods, the corrections of initial conditions and periods $(\Delta\mathbf{y}_i, \Delta T_i)$ should be satisfied the following equations:

$$\mathbf{x}(T_i + \Delta T_i, \mathbf{y}_i + \Delta\mathbf{y}_i) - \mathbf{P}(\theta)(\mathbf{y}_i + \Delta\mathbf{y}_i) = 0. \quad (6)$$

Using the first order Taylor approximation of the above equations, we have

$$\mathbf{x}(T_i, \mathbf{y}_i) - \mathbf{P}(\theta)\mathbf{y}_i + \left(\frac{\partial \mathbf{x}}{\partial \mathbf{y}} - \mathbf{P}(\theta) \right) \Delta\mathbf{y}_i + \frac{\partial \mathbf{x}}{\partial t} \Delta T_i = 0, \quad (7)$$

where $\mathbf{P}(\theta)$ is the rotation matrix, $\mathbf{M} = \partial\mathbf{x}/\partial\mathbf{y}$ is the solution of variational equations. $\partial\mathbf{x}/\partial t$ is the derivative of the solution at $t = T_i$, i.e., $\mathbf{f}(\mathbf{y}_{T_i})$, where $\mathbf{y}_{T_i} = \mathbf{x}(T_i, \mathbf{y}_i)$.

Finally, we obtain the following linear system:

$$\begin{pmatrix} \mathbf{M} - \mathbf{P}(\theta) & \mathbf{f}(\mathbf{y}_{T_i}) \end{pmatrix} \begin{pmatrix} \Delta\mathbf{y}_i \\ \Delta T_i \end{pmatrix} = (\mathbf{P}(\theta)\mathbf{y}_i - \mathbf{y}_{T_i}). \quad (8)$$

Then the correction of initial conditions and periods of periodic orbits can be computed through solving these linear equations. As we fix some parameters of the initial conditions, only variables x_1, v_1, v_2 and T will be modified. Then we have linear equations $\mathbf{A}\mathbf{z} = \mathbf{b}$, where \mathbf{A} is a 12×4 matrix, $\mathbf{z} = (\Delta x_1 \Delta v_1 \Delta v_2 \Delta T)^T$ is a 4-vector and \mathbf{b} is a 12-vector. Since the matrix \mathbf{A} is not a square matrix, we solve this system by means of the least-norm method with SVD (Trefethen and Bau III, 1997).

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