NOTES FOR STOCHASTIC CALCULUS

Based on the Paolo Baldi

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Definiton 2.1.1. (Stochastic Process)

A **stochastic process** is an object of the form

$$X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, (X_t)_{t \in T}, P)$$

where

- $(\Omega, \mathcal{F}, P)$ is a probability space
- T is a subset of $\mathbb{R}^+$
- $(\mathcal{F}_t)_{t \in T}$ is a filtration, i.e. an increasing family of sub-σ-algebras of $\mathcal{F}$
- $(X_t)_{t \in T}$ is a family of r.v.'s on $(\Omega, \mathcal{F})$ taking values in a measurable space $(E, \mathcal{E})$ **adapted** to $(\mathcal{F}_t)$.

The **natural filtration** $(\mathcal{G}_t)_t$ is defined as

$$\mathcal{G}_t = \sigma(X_s, s \leq t)$$

and the **augmented natural filtration** $(\overline{\mathcal{G}}_t)$ is defined by

$$\overline{\mathcal{G}}_t = \sigma(\mathcal{G}_t, \mathcal{N})$$

where $\mathcal{N} = \{A; A \in \mathcal{F}, P(A) = 0\}$. Denote $\mathcal{F}_\infty = \sigma(\bigcup_t \mathcal{F}_t)$ for a filtration $(\mathcal{F}_t)_t$

Definiton 2.1.2. (Space of paths)

Ω can be considered as a subset of $E^T := \{\text{all functions } T \to E\}$ by the map

$$\omega \mapsto (t \mapsto X_t(\omega))$$

and hence is called the **space of paths** and E is called the **state space**.

Definiton 2.1.3. (Equivalent and modification)

For two processes $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, (X_t)_{t \in T}, P)$ and $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \in T}, (X'_t)_{t \in T}, P')$, they are **equivalent** if for any $t_1, \cdots, t_m \in T$, $(X_{t_1}, \cdots, X_{t_m})$ and $(X'_{t_1}, \cdots, X'_{t_m})$ have the same law.

X is called a **modification** of X' if $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, P) = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \in T}, P')$ and for every $t \in T$, $X_t = X'_t$, P-a.s., and they are **indistinguishable** if X is a modification of X' and

$$P(X_t = X'_t \text{ for every } t \in T) = 1$$

Example 2.1.1. Here is a counter example that if X is a modification of X', then X and X' are not necessarily indistinguishable. Let $\Omega = [0, 1]$ and $\mathcal{F} = \mathcal{B}([0, 1])$ and P the Lebesgue measure, and

$$X_t(\omega) = 1_{\{\omega\}}(t), \quad X'_t(\omega) = 0$$

Definiton 2.1.4. (Topological state space)

Assume the state space is a topological space endowed with its Borel σ-algebra $\mathcal{B}(E)$ and T an intercal of $\mathbb{R}^+$.

A process is said to be (a.s.) **continuous** if for every (a.e.) ω the map $t \mapsto X_t(\omega)$ is continuous. And the definitions of one side continuity is similar.

X is **measurable** if the map $(t,\omega) \mapsto X_t(\omega)$ is measurable $(T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F}) \to (E, \mathcal{B}(E))$. It is said to be **progressively measurable** if for every $u \in T$ the map $(t,\omega) \to X_t(\omega)$ is measurable $([0,u] \times \Omega, \mathcal{B}([0,u]) \otimes \mathcal{F}_u) \to (E, \mathcal{B}(E))$.

Proposition 2.1.1. Let $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, (X_t)_t, P)$ be a right-continuous process. Then it is progressively measurable.

Proof.

For a fixed $u \in T$, we define $X^{(n)}$ by

$$X_s^{(n)} = X_{(k+1)u/2^n} \text{ for } s \in [ku/2^n, (k+1)u/2^n) \quad \text{ and } X_s^{(n)} = X_u \text{ if } s \geq u$$

and then we know $X_s^{(n)} = X_{s_n}$ for some $s_n > s$ and $|s_n - s| \leq u/2^n$ if $s \leq u$ and then we know $X_s^{(n)} \to X_s$ as $n \to \infty$ for $s \leq u$ since $s_n \downarrow s$. Consider $B \in \mathcal{B}(E)$ and then

$$\begin{aligned}
&\{X^{(n)} \in B\} \cap \{s \leq u\} \\
=& \left(\bigcup_{k=0}^{2^n-1} \{X^{(n)} \in B\} \cap \{s \in [ku/2^n, (k+1)u/2^n)\} \right) \cup \Big(\{X^{(n)} \in B\} \cap \{s = u\} \Big) \\
=& \left(\bigcup_{k=0}^{2^n-1} [ku/2^n, (k+1)u/2^n) \times \{X_{(k+1)u/2^n} \in B\} \right) \cup (\{u\} \times \{X_u \in B\}) \in \mathcal{B}([0,u], \mathcal{F}_u)
\end{aligned}$$

which means $X^(n)$ is progressively measurable and hence X is progressively measurable.

Definiton 2.1.5. (Standard process)

Denote $\mathcal{F}_{t+} = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$ and we say the filtration is **right-contunuous** if $\mathcal{F}_{t+} = \mathcal{F}_t$ for every t.

A process is said to be **standard** if $(\mathcal{F}_t)$ is right-continuous and $\mathcal{F}_t$ contains the negligible events of $\mathcal{F}$ for each t.

2.2 Kolmogorov's continuity theorem

Theorem 2.2.1. (Kolmogorob's continuity theorem)

Let $D \subset \mathbb{R}^m$ an open set and $(X_y)_{y\in D}$ a family of d-dimensional r.v.'s on $(\Omega, \mathcal{F}, P)$ such that there exists $\alpha > 0, \beta > 0, c > 0$ such that

$$E[|X_y - X_z|^\beta] \leq c|y - z|^{m+\alpha}$$

Then there exists a family $(\widetilde{X}_y)_t$ of $\mathbb{R}^d$-valued r.v.'s such that

$$X_y = \widetilde{X}_y$$

a.s. for every $y \in D$ and that the map $y \mapsto \widetilde{X}_y(\omega)$ is Holder continuous with exponent γ for every $\gamma < \alpha/\beta$ on every compaact subset of D for every $\omega \in \Omega$.