NOTES FOR DIFFERENTIAL MANIFOLD

Based on the notes provided by Alex Waldron on MATH 761 2024 FALL

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1 Immersions, submersions, submanifolds

1.1 The Inverse Function Theorem

Theorem 1.1.1. (Inverse Function Theorem)

Let $F : M \to N$ be a smooth map and $p \in M$. Suppose $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism. Then there exist neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

1.2 Locall Diffeomorphisms and Covering Maps

Definiton 1.2.1. (Local Diffeomorphism)

A smooth map $F : M \to N$ is called a **local diffeomorphism** if dF_x is an isomorphism for all $x \in M$.

Definiton 1.2.2. A map $\pi : X \to Y$ between two topological spaces is called a covering map is every point $y \in Y$ there is a neighborhood V that is evenly covered.

Proposition 1.2.1. Let $F : M \to N$ be a local diffeomorphism which is **proper** (i.e. the inverse image of compact subset is compact) and assume N is connected. Then F is a covering map.

Proof.

The fiber is consisted of discrete points by local diffeomorphism, then we know it has to be finite, and then we may find the covering by choose finite intersection of neighborhoods of b. The connectness of N secure F to be surjective.

Proposition 1.2.2. Suppose $\pi : X \to N$ is a covering map, with N a smmoth manifold. There exists a unique smooth structure on X such that π is a local diffeomorphism.

1.3 Immersions and Embeddings

Definiton 1.3.1. A smooth map $F : M \to N$ is an **immersion** if dF_x is injective for all $x \in M$.

Proposition 1.3.1. Suppose dF_p is injective. There exists a coordinate system on N near $F(p)$ such that F takes the form

$$F(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)$$

2 Vector Fields

2.1 Integral Curves and Flows

Definiton 2.1.1. Let $X \in \mathfrak{X}(M)$ and $J \subset \mathbb{R}$ an open interval. A path $\gamma : J \to M$ is said to be an integral curve of X if

$$\gamma'(t) = X_{\gamma(t)}$$

for all $t \in J$.