Chapter 1

m.s. for measure space
mrb. for measurable
t.v.s. for a topological vector space

1.1 Test functions and Distributions

Theorem 1.1

Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X. Associate to each $p \in \mathcal{P}$ and to each positive n the set

$$V(p,n) = \{x : p(x) < n^{-1}\}$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p,n)$. Then $\mathscr{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that
a. every $p \in \mathcal{P}$ is continuous
b. a set $E \subset X$ is bounded, i.e. for any neighbourhood V of 0, there exists s real positive such that $E \subset rV$ for any $|r| \geq s$, if and only if every $p \in \mathcal{P}$ is bounded on E.

♡

Proof

Obviously, $\mathscr{B}$ is a convex balanced local base for τ. Let $A \subset X$ be open iff A is a union of translates of members of $\mathscr{B}$, which defines a topology τ on X. By the way, it is easy to check that $p(0) = 0$ for all $p \in \mathcal{P}$, and if $x_n \to y$, then we know $x_n - y \to 0$, which means for any integer m, there exists N such that $x_n - y \in V(p,m)$ for any $n \geq N$ and hence $p(x_n - y) \to 0$, which means p is continuous under τ.

Then we consider if $x + y \in U$ for some $x, y \in X, U$ open, then we know $U - (x+y)$ is an open neighbourhood of 0 and hence there exists a union of finite elements of $V(p,n)$ denoted as V such that $0 \in V + q$ and hence there exists p_i, n_i such that $V' = \bigcup_{i=1}^m V(p_i, n_i) \subset V + q$, then we know let $T = \bigcup_{i=1}^m V(p_i, 2n_i)$ and $T + T \subset V' \subset V + Q$. Now we know $(T+T) + (x+y) \subset U$ and hence $(T+x) + (T+y) \subset U$, which means addition is continuous under τ.

Now consider if αx for some $\alpha \in \mathbb{K}, x \in X$ such that $\alpha x \in U$ for some U open, then if $\alpha = 0$, then we may find δ and a neighbourhood V of x such that $diam(B(\alpha,\delta)V)$ is small sufficiently and hence $B(\alpha,\delta)V \subset U$. Now consider if $\alpha \neq 0$, then we know we may find $V = \bigcup_{i=1}^m V(p_i, n_i)$ and $V + y \subset U$ for some $y \in X$ with $\alpha x \in V + y$, then we know $\alpha(x - y') \in V$ where $y' = \alpha^{-1} y$. Then we may find V' an open neighbourhood of $x - y'$ and B centered at α such that $BV' \subset V$ and hence multiplication is continuous under τ.

To sum up, (X,τ) is a locally convex space. (b) is obviously then.

Theorem 1.2

For the conditions provided in theorem 1.1., if we know $\mathcal{P}$ is a countable separating family of seminorms on X, we claim that

$$d(x,y) = \max_i \frac{c_i p_i(x-y)}{1 + p_i(x-y)}$$

where $c_i \to 0$ positive, is a metric on X metrize τ.

♡

Proof It is easy to check that d is a metric on X and consider $B_r = B_d(0,r)$, then we know

$$B_r = \bigcap_{i=1}^{\infty} \{x, \frac{c_i p_i(x)}{1 + p_i(x)} < r\} = \bigcap_{i=1}^{\infty} \{x, (c_i - r)p_i(x) < r\} = \bigcap_{c_i > r} V(p_i, \frac{c_i - r}{r})$$

which is an open set in τ and for any $V(p,n)$, we may find r small enoguh such that $B_r \subset V(p,n)$, which means for any open set U, it can be a union of open balls of d and hence they induce the same topology.

Definition 1.1
(Frechet space) A local convex t.v.s. with the topology induced by a translation-invariant complete metric. ♣

Definition 1.2
If K is a compact set in an open set Ω, then $\mathscr{D}_K$ denotes the space of all $f \in C^\infty(\Omega)$ whose support lies in K. ♣

Proposition 1.1
There exists a topology in $C^\infty(\Omega)$ makes $C^\infty(\Omega)$ into a Frechet space with the Heine-Borel property, i.e. any bounded closed set in $C^\infty(\Omega)$ is compact, such that $\mathscr{D}_K$ is a closed subspace of C^∞ whenever $K \subset \Omega$. ♠

Proof

We choose compact sets K_i such that K_i lies in the interior of K_{i+1} at first with $\Omega = \bigcup K_i$. Define seminorms p_n by

$$p_n = \max\{|\partial^\alpha(x)| : x \in K_n, |\alpha| \leq n\}$$

Then by thoerem 1.1 and 1.2. we know it defines a metrizable locally convex topology on $C^\infty(\Omega)$ and for each $x \in \Omega$, the functional $f \mapsto f(x)$ is continuous in this topology. Since

$$\mathscr{D}_K = \bigcap_{x \in K^c} \mathscr{N}(f \mapsto f(x))$$

and hence $\mathscr{D}_K$ is closed under this toplogy in $C^\infty(\Omega)$.

It is easy to check that

$$V_n = \{f \in C^\infty(\Omega), p_n(f) < n^{-1}\}$$

then if f_i is Cauchy in $C^\infty(\Omega)$ and then we know $f_i - f_j \in V_n$ for fixed n if i, j large sufficiently. Then it is easy to see that $\partial^\alpha f_i$ converges to some function g_α uniformly since it is Cauchy uniformly. And hence it is easy to check that $g_0 \in C^\infty(\Omega)$, then we know C^∞ is a Frechet space and hence $\mathscr{D}_K$ is because it is a closed subspace.

We skip the proof of Heine-Borel property of $C^\infty(\Omega)$.

Definition 1.3
Consider a nonempty open set $\Omega \subset \mathbb{R}^n$, then define

$$\mathscr{D}(\Omega) = \bigcup_{K \subset \Omega \text{ compact}} \mathscr{D}_K$$

as the $test\ function\ space$ $\mathscr{D}(\Omega)$. The norms

$$||\phi||_N = \max\{|\partial^\alpha \phi(x)|, x \in \Omega, |\alpha| \leq N\}$$

is defined. ♣

Proposition 1.2
The restrictions of these norms to any D_K where $K \subset \Omega$ compact induce the same topology on $\mathscr{D}_K$ as do the seminorms p_N in proposition 1.1. ♠

Proof

For each K. we know there exists N such that K is in the interior of K_N for N large enough and we have

$$||\phi||_N \leq p_N(\phi)$$

if $\phi \in \mathscr{D}_K$ and then the problem goes since notice $||\phi||_n \leq ||\phi||_{n+1}, p_n \leq p_{n+1}$. Then we may know

$$V_N = \{\phi \in \mathscr{D}_K, ||\phi||_N < N^{-1}\}$$

will become a local base.

Definition 1.4

Let Ω be a nonempty open set in $\mathbb{R}^n$.
a. For every compact $K \subset \Omega, \tau_K$ denotes the Frechet space topology of $\mathscr{D}_K$ as described above.
b. β is the collection of all convex balanced sets $W \subset \mathscr{D}(\Omega)$ such that $\mathscr{D}_K \cap W \in \tau_J$ for every compact $K \subset \Omega$.
c. τ is the collection of all unions of sets of the form $\phi + W$ with $\phi \in \mathscr{D}(\Omega)$ and $W \in \beta$.

♣

We can see that

$$\{\phi \in \mathscr{D}(\Omega), |\phi(x_m)| < c_m, m \geq 1\}$$

for a sequence x_m without limit point in Ω and c_m a sequence of positive numbers belongs to β.

Theorem 1.3

a. τ is a topology in $\mathscr{D}(\Omega)$ and β is a local base for τ.
b. τ makes $\mathscr{D}(\Omega)$ into a locally convew topological vector space.

♡

Proof We claim that for any $V_1, V_2 \in \tau, \phi \in V_1 \cap V_2$

$$\phi + W \subset V_1 \cap V_2$$

for some $W \in \beta$. We know there exist $\phi_1, \phi_2, \in \mathscr{D}(\Omega)$ and $W_1, W_2 \in \beta$ such that

$$\phi \in \phi_i + W_i \in V_i$$

and we may choose K so that $\phi_1, \phi_2 \in \mathscr{D}_k$ and then since $\mathscr{D}_K \cap W_i$ is open, we have

$$\phi - \phi_i \in (1 - \epsilon_i) W_i$$

for some $\epsilon_i > 0$ and hence

$$\phi - \phi_i + \epsilon_i W_i \subset W_i$$

by the convexity of W_i, then

$$\phi + \epsilon_i W_i \subset \phi_i + W_i \subset V_i$$

and let $W = (\epsilon_1 W_1) \cap (\epsilon_2 W_2)$ and we are done. Then we know any intersection of two open sets in τ is open in τ and if let $\phi = 0, V_1 = V_2 = V$, then we know there is always some $W \in \beta$ such that $W \subset V$ for any open set V and hence β is a local base.