Chapter 1

Fundamental Concepts

Definition 1.1

A complex algebra is a complex v.s. A with a multiplication with $x(yz) = (xy)z, (x+y)z = xz+yz, x(y+z) = xy + xz$ and $cxy = (cx)y = x(cy)$ for any $x, y, z \in A$ and $c \in \mathbb{C}$.
If A is a Banach space with $||xy|| \leq ||x||||y||$ and there is a unit element e in A i.e. $xe = ex = x, ||e|| = 1$ for any $x \in A$, then we call A is a Banach algebra.

♣

Lemma 1.1

If Y is a complete n.v.s, then so is $L(X, Y)$.

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Lemma 1.2

(The Open Mapping Theorem) Let X, Y be Banach spaces. If $T \in L(X, Y)$ is surjective, then T is open.

♡

Theorem 1.1

Assume that A is a Banach space as well as a complex algebra with unit element $e \neq 0$, in which multiplication is left-continuous and right-continuous. Then there is a norm on A which induces the same topology as the given one and which makes A into a Banach algebra.

♡

Proof

Consider $T : X \to B$ by $x \mapsto M_x$ where $M_x(y) = xy$ which is obviously a bounded linear map and B is the subspace of $L(X, X)$ of all these maps. Then we have

$$||M_x M_y|| \leq ||M_x||||M_y|| \quad ||M_e|| = 1$$

and notice

$$||x|| \leq ||M_x||||e||$$

which means T^{-1} is continuous, and notice if M_{x_n} Cauchy, then

$$M'y = xy$$

and hence $M' = M_x$, which means B will become a Banach space, and hence T is open by the Open Mapping Theorem, which means T is continuous and then a isometry from X to B, where B is a Banach algebra and so does X.

Definition 1.2

Suppose A is a complex algebra and ϕ is a linear functional on A which is not identically 0. If

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in A$, then ϕ is called a complex homomorphism on A.

♣

Proposition 1.1

If ϕ is a complex homomorphism on a complex algebra A with unit e, then $\phi(e) = 1$ and $\phi(x) \neq 0$ for every invertible $x \in A$.

♠

Proof

There is y such that $\phi(y) \neq 0$, then it is easy to check $\phi(e) = 1$ and hence the invertible element can not be mapped to 0.

Theorem 1.2

Suppose A is a Banch algebra, $x \in A$, $||x|| < 1$. Then

a. $e - x$ is invertible,

b. $||(e-x)^{-1} - e - x|| \leq \dfrac{||x||^2}{1-||x||}$.

c. $|\phi(x)| < 1$ for every complex homomorphism ϕ on A.

Proof We only need to prove c, since for any $|\lambda| \geq 1$, we know

$$\phi(e - \lambda^{-1}x) = 1 - \lambda^{-1}\phi(x) \neq 0$$

which means $\phi(x)$ has to be strictly less than 1.

Lemma 1.3

Suppose f is an entire function of one complex variable, $f(0) = 1$, $f'(0) = 0$, and

$$0 < |f(\lambda)| \leq e^{|\lambda|}$$

Then $f(\lambda) = 1$ for all $\lambda \in \mathbb{C}$.

Theorem 1.3

If ϕ is a linear functional on a Banach algebra A, such that $\phi(e) = 1$ and $\phi(x) \neq 0$ for every invertible $x \in A$, then

$$\phi(xy) = \phi(x)\phi(y)$$

and ϕ is continuous.

Proof Here consider N to be the null space of ϕ, then for any $x \in A$, $x = a + \phi(x)e$ where $a \in N$ and then

$$\phi(xy) = \phi(ab) + \phi(x)\phi(y)$$

so it suffices to show $ab \in N$ for any $a \in N, b \in N$, which is equivalent to $a^2 \in N$ for any $a \in N$, since if so, then $\phi(x)^2 = \phi(x)^2$, and we have

$$\phi(xy + yx) = 2\phi(x)\phi(y)$$

which means $ax + xa \in N$ for any $a \in N, x \in A$, then consider

$$(xy - yx)^2 + (xy + yx)^2 = 2x(yxy) + 2(yxy)x$$

and hence if $x \in N$, then $xyxy + yxyx, (xy + yx)^2 \in N$ which means $(xy - yx)^2 \in N$, and then $xy - yx \in N$.

Now we will show that $a^2 \in N$ for any $a \in N$, consider since x invertible is not in N, so $||e - x|| \geq 1$ for any $x \in N$ and hence

$$||\lambda e - x|| \geq |\lambda| = (\phi(\lambda e - x))$$

for any $x \in N, \lambda \in C$ which means $||\phi(x)|| \leq ||x||$ for any $x \in A$ and hence ϕ is continuous.

Then we may assume $a \in N, ||a|| = 1$ and consider

$$f(\lambda) = \sum \frac{\phi(a^n)}{n!}\lambda^n$$

where $f(0) = 1, f'(0) = 0$ entire.

To show f is nonzero, consider

$$E(\lambda) == \sum \frac{\lambda^n}{n!}a^n$$

where $a^0 = e$ and then we know $f(\lambda) = \phi(E(\lambda))$, and notice $E(\lambda + \mu) = E(\lambda)E(\mu)$ since

$$||x_n y_n - xy|| \leq ||x_n||||y_n - y|| + ||y||||x - x_n||$$

and then $E(\lambda)E(-\lambda) = e$ and hence $E(\lambda)$ is invertible, so $\phi(E(\lambda)) \neq 0$. To sum up, $f = 1$ on $\mathbb{C}$ and hence $0 = f'' = 0\phi(a^2)$.

Definition 1.3

Let A be a Banach algebra, let $GL(A)$ be the set of all invertible elements of A, which is a group under the multiplication.
For $x \in A$, the spectrum $\sigma(x)$ of x is the set of all complex numbers λ such that $\lambda e - x$ is not invertible.
The spectral radius of x is the number

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

♣

Theorem 1.4

Suppose A is a Banach algebra, $x \in GL(A), h \in A$ and $||h|| < \frac{1}{2}||x^{-1}||^{-1}$. Then $x + h \in G(A)$ and

$$||(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}|| \leq 2||x^{-1}||^3||h||^2$$

♡

Proof $(x+h) = x(e + x^{-1}h)$ and hence $x + h \in GL(A)$ since $||x^{-1}h|| < \frac{1}{2}$. And

$$||(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}|| \leq ||(e + x^{-1}h)^{-1} - e + hx^{-1}||||x^{-1}|| \leq \frac{||x^{-1}h||^2}{1 - ||x^{-1}h||} \leq 2||x^{-1}||^3||h||^2$$

Corollary 1.1

If A is a Banach algebra, then $G(A)$ is an open subset of A and the mapping $x \to x^{-1}$ is a homeomorphism of $GL(A)$ onto $GL(A)$.

♡

Theorem 1.5

If A is a Banach algebra and $x \in A$, then
a. the spectrum $\sigma(x)$ of x is compact and nonempty
b. the spectral readius $\rho(x)$ of x satisfies

$$\rho(x) = \lim_{n\to\infty} ||x^n||^{1/n} = \inf_{n\geq 1} ||x^n||^{1/n}$$

♡

Proof

a. Consider $\phi : \mathbb{C} \to A$ by $\lambda \mapsto \lambda e - x$ and it is easy to check ϕ is continuous, and then consider $\phi^{-1}\sigma(x)^c$ is an open set and hence $\sigma(x)$ is closed. For $|\lambda| > ||x||$, $e - \lambda^{-1}x \in GL(A)$ and hence $\sigma(x) \subset \overline{B(0, ||x||)}$ and hence $\sigma(x)$ is a bounded closed set in $\mathbb{C}$, so it is compact.

Now denote $U = \sigma(x)^c$ and define $f : U \to A$ by $f(\lambda) = (\lambda e - x)^{-1}$ which is continuous, then we know

$$f(\mu) - f(\lambda) + (\mu - \lambda)f(\lambda)^2 \leq 2||f(\lambda)||^3||\mu - \lambda||^2$$

for μ close to λ and hence

$$\lim_{\mu\to\lambda} \frac{f(\mu) - f(\lambda)}{\mu - \lambda} = -f^2(\lambda)$$

which means f is a holomorphic A-valued function, then notice for $|\lambda| > ||x||$

$$f(\lambda) = \lambda^{-1} \sum \lambda^{-n}x^n$$

which converges uniformly on $\partial D(0, r)$, then we may know

$$\frac{1}{2\pi i} \int_{\partial D(0,r)} \lambda^k f(\lambda)d\lambda = \frac{1}{2\pi i} \sum \int_{\partial D(0,r)} \lambda^{k-n-1}x^n = x^k$$

but if $\sigma(x)$ is empty, then let $k = 0$ we know the integral should be 0 since f is holomorphic on $\mathbb{C}$, but then we will know $e = 0$ which is a contradiction, so $\sigma(x)$ is nonempty.

b. We use the f above, and we know f is holomorphic on $\partial D(0, r)$ for any $r > \rho(x)$, then we consider

$$M(r) = \sup_\theta |f(re^{i\theta})| < \infty$$

and $||x^n|| \leq r^{n+1}M_r$ and hence

$$\limsup_{n\to\infty} ||x^n||^{1/n} \leq r$$

for any $r > \rho(x)$, which means $\rho(x) \leq \limsup ||x^n||^{1/n}$.

Notice

$$(\lambda e^n - x^n)y = (\lambda e - x)(\lambda^{n-1}e + \cdots + x^{n-1})y$$

for any $y \in A$, and hence $\lambda \in \sigma(x)$ will imply $\lambda^n \in \sigma(x^n)$, so we have

$$\lambda \leq ||x^n||^{1/n}$$

for any integer n and hence $\rho(x) = \sup\{|\lambda|, \lambda \in \sigma(x)\} \leq \inf ||x^n||^{1/n}$.