Chapter 1

1.1 Brownian Motion

Definition 1.1

A real-valued stochastic process $B = (B_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}; P)$ is called a Brownian motion if it satisfies the following conditions:
a. Almost surely $B_0 = 0$.
b. For all $0 \leq t_1 < \cdots t_n$ the increments $B_{t_n} - B_{t_{n-1}}, \cdots, B_{t_2} - B_{t_1}$ are independent random variables.
c. If $0 \leq s < t$, the increment $B_t - B_s$ is a Gaussian random variable with mean zero and variance $t - s$.
d. With probability one, the map $t \to B_t$ is continuous.
A d-dimensional Brownian motion is defined as an $\mathbb{R}^d$-valued stochastic process $B = (B_t)_{t\geq 0}$, $B_t = (B_t^1, \cdots, B_t^d)$, where $B^1, \cdots, B^d$ are d independent Brownian motions.

Proposition 1.1

Properties (a),(b),(c) are equivalent to that B is a Gaussian process,i.e. for any finite set of indices $t_1, \cdots, t_n$, $(B_{t_1}, \cdots, B_{t_n})$ is a multivariate Gaussian random variable, equivalently, any linear combination of B_{t_i} is normal distributed r.v., with mean zero and covariance function

$$\Gamma(s,t) = \min(s,t)$$

Proof

Suppose (a),(b),(c) holds, then we know $(B_{t_1}, \cdots, B_{t_n})$ is normal for any finite indices and then

$$m(t) = E(B_t) = 0$$
$$\Gamma(s,t) = E(B_s B_t) = E(B^2_{\min(s,t)}) = \min(s,t)$$

Conversly, we know $E(B_0^2) = 0$ and hence $B_0 = 0$ a.s., then we know $E(B_s^2) = s$ and for any $0 < s < t$,

$$E(B_s(B_t - B_s)) = 0$$

and it is easy to check (c), and (b) is deduced by computing the covariance of the increments, notice that two r.v.s are independent iff $\phi_{(X_1, X_2, \cdots, X_n)} = \phi_{X_1}\phi_{X_2}\cdots\phi_{X_n}$ which implies that normal r.v.s are independent iff they have zero covariances.

Theorem 1.1

(Kolmogorov's continuity theorem) Suppose that $X = (X_t)_{t\in[0,T]}$ satisfies

$$E(|X_t - X_s|^\beta) \leq K|t-s|^{1+\alpha}$$

for all $s, t \in [0,T]$ and some constant $\beta, \alpha, K > 0$. Then there exists a version $\tilde{X}$ of X such that if

$$\gamma < \alpha/\beta$$

then

$$|\tilde{X}_t - \tilde{X}_s| \leq G_\gamma |t-s|^\gamma$$

for all $s, t \in [0,T]$ where G_γ is a random variable. The trajectories of $\tilde{X}$ are Holder continuous of order γ for any $\gamma < \alpha/\beta$.

Proposition 1.2

There exists a version of B with Holder-continuous trajectories of order γ for any $\gamma < (k-1)/2k$ on any interval $[0,T]$.

Proof

Since we know $B_t - B_s$ has the normal distribution $\mathcal{N}(0, t-s)$ and then we know

$$E\Big((B_t - B_s)^{2k}\Big) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} x^{2k} \exp^{-\dfrac{x^2}{2(t-s)}} \, dx = (2k-1)!!(t-s)^k = \frac{(2k)!}{2^k k!}(t-s)^k$$

and by the theorem 1.1, the proposition holds.

Proposition 1.3

Brownian motion are basic properties:

a. For any $a > 0$, the process $(a^{-1/2}B_{at})_{t\geq 0}$ is a Brownian motion.

b. For any $h > 0$, the process $(B_{t+h} - B_h)_{t\geq 0}$ is a Brownian motion.

c. The process $(-B_t)_{t\geq 0}$ is a Brownian motion.

d. Almost surely $\lim_{t\to\infty} B_t/t = 0$ and the process $X_t = tB_{1/t}$ for $t > 0$, $X_t = 0$ for $t = 0$ is a Brownian motion.

Proof

a. Consider $0 \leq t_1 < t_2 < \cdots < t_n$ and we may calculate the covariance matrix for

$$a^{-1/2}B_{at_n} - a^{-1/2}B_{at_{n-1}}, \cdots, a^{-1/2}B_{at_2} - a^{1/2}B_{at_1}$$

by

$$\begin{aligned}
&E[(a^{-1/2}B_{at_j} - a^{-1/2}B_{at_{j-1}})(a^{-1/2}B_{at_k} - a^{-1/2}B_{at_{k-1}})] \\
=&a^{-1}(at_j \wedge at_k) - a^{-1}(at_j \wedge at_{k-1}) - a^{-1}(at_{j-1} \wedge at_k) + a^{-1}(at_{j-1} \wedge at_{k-1}) \\
=&\begin{cases} t_j - t_{j-1} - t_{j-1} + t_{j-1} = t_j - t_{j-1} & \text{if } j = k \\ t_j - t_j - t_{j-1} + t_{j-1} = 0 & \text{if } j < k \\ 0 & \text{if } j > k \end{cases}
\end{aligned}$$

and hence $(a^{-1/2}B_{at})_{t\geq 0}$ satisfies the property (b) in definition 1.1, a,d are obvious and c is easy to be checked.

b. Obvious.

c. Obvious.

d. Notice B is Holder continuous. Now we only need to check that

$$E(tB_{1/t}sB_{1/s}) = ts(1/t \wedge 1/s) = (t \wedge s)$$

and the rest is easy to be checked.

Theorem 1.2

(The law of the iterated logarithm)

$$\limsup_{t\to s^+} \frac{|B_t - B_s|}{\sqrt{2|t-s| \ln\ln|t-s|}} = 1, \quad a.s.$$

Proposition 1.4

Fix a time interval $[0, t]$ and consider the following subvision π of this interval:

$$0 = t_0 < t_1 < \cdots < t_n = t$$

The norm of the subdivision π is defined as $|\pi| = \max_{0\leq j\leq n-1}(t_{j+1} - t_j)$. Then

$$\lim_{|\pi|\to 0} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 = t$$

in $L^2(\Omega)$.

Proof

Consider let $\xi_j = (B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)$ and we know ξ_j are independent with mean 0 and hence

$$\begin{aligned} E\Big(\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 - t\Big)^2 = \sum_{j=0}^{n-1} E\xi_j^2 &= \sum_{j=0}^{n-1}(3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2) \\ &= 2\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2 \le 2t|\pi| \to 0 \end{aligned}$$

Proposition 1.5

The total variation of Brownian morion on an interval $[0, t]$ defined by

$$V = \sup_{\pi} \sum_{i=1}^{n-1}[B_{t_{j+1} - B_{t_j}}]$$

where π is any partition of $[0, t]$, is infinite with probability 1.

Proof

Here we know

$$\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 \le V \sup_j |B_{t_{j+1}} - B_{t_j}|$$

and hence if $V < \infty$, then

$$\lim_{|\pi| \to 0} \sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 = 0$$

which means $P(V < \infty) = 0$.

Definition 1.2

(Wiener integral) Let $\mathcal{E}_0$ be the set pf step functions in $\mathbb{R}_+$,i.e.

$$\phi(t) = \sum_{j=0}^{n-1} a_j \chi_{t_j, t_{j+1}}(t)$$

where $n \ge 1$ is an integer,$a_i \in \mathbb{R}$ and $0 = t_0 < \cdot < t_n$. And we may define Wiener integral of a step function by

$$\int_0^\infty \phi dB_t = \sum_{i=0}^{n-1} a_i(B_{t_{i+1}} - B_{t_i})$$

Proposition 1.6

The Wiener integral is a linear isometry from $\epsilon_0 \subset L^2(\mathbb{R}^+)$ to $L^2(\Omega)$.

Proof Notice

$$E[(\int_0^\infty \phi dB_t)^2] = \sum_{i=0}^{\infty} a_i^2(t_{i+1} - t_i) = ||\phi||_2$$

Definition 1.3

We have already know Wiener integral is a linear isometry from a dense subspace from $L^2(\mathbb{R}_+)$ to $L^2(\Omega)$, and hence we may call the extension of the linear isometry to be the Wiener integral and for any $\phi \in L^2(\mathbb{R}_+)$, denote

$$\int_0^\infty \phi dB_t$$

to be its image of the isometry.

Definition 1.4
Let D be a Borel subset of $\mathbb{R}^m$, a white noise on D is a centered Gaussian family of random variables
$$\{W_A, A \subset \mathcal{B}(\mathbb{R}^m), A \subset D, m(A) < \infty\}$$
such that
$$E(W_A W_B) = m(A \cap B)$$
♣

Proposition 1.7
$\chi_A \to W_A$ *is a linear isometry from* $L^2(D) \to L^2(\Omega)$.
♠

Definition 1.5
Similarly, we may define the integral r.s.t. W of $\phi \in L^2(D)$ denoted by
$$\phi \mapsto \int_D \phi W(dx)$$
by extending the linear isometry.
♣

Definition 1.6
Consider a Browian motion B defined on a probability space $(\Omega, \mathcal{F}, P)$. For any time $t \geq 0$, define $\mathcal{F}_t$ the σ-algebra by $B_s, 0 \leq s \leq t$ and the null events in $\mathcal{F}$., we call $\mathcal{F}_t$ the natural filtration of Browiabn motion on the probability space $(\Omega, \mathcal{F}, P)$.
♣

Lemma 1.1
Suppose X and Y
♡

Theorem 1.3
For any measurable and bounded (or nonnegative) function $f : \mathbb{R} \to \mathbb{R}$, $s \geq 0$ and $t \geq 0$, we have
$$E(f(B_{s+t})|\mathcal{F}_s) = (P_t f)(B_s)$$
where
$$(P_t f)(x) = \int_{\mathbb{R}} f(y) p_t(x - y) dy$$
where
$$p_t = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$
♡

Check Durrett Theorem 7.2.1.

Proposition 1.8
The familty of operators P_t satifies the semigroup property $P_t \circ P_s = P_{t+s}$ and $P_0 = Id$.
♠

Proof
$$\begin{aligned}
P_t \circ P_s(f)(x) &= \int_{\mathbb{R}} P_s f(y) p_t(x - y) dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) p_s(y - z) p_t(x - y) dz dy \\
&= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{st}} e^{-\left(\frac{(y-z)^2}{2s} + \frac{(x-y)^2}{2t}\right)} dy dz \\
&= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{st}} e^{-\left(\frac{(\sqrt{s+t}y - (2tz+2sx)/\sqrt{s+t})^2 - (tz+sx)^2/(s+t) + tz^2 + sx^2}{2st}\right)} dy dz
\end{aligned}$$
and the rest is easy to be checked.

Theorem 1.4

The processes $B_t, (B_t^2 - t)$ *and* $e^{aB_t - a^2t/2}, a \in \mathbb{R}$ *are* $\mathcal{F}_t$ *martingales.*

Definition 1.7

The Brownian hitting time is defined by

$$\tau_a = \int \{t \geq 0, B_t = a\}$$

Proposition 1.9

Fix $a > 0$. *Then, for all* $\alpha > 0$

$$E(e^{-\alpha \tau_a}) = e^{-\sqrt{2\alpha} a}$$

Theorem 1.5