RMT2024 at U of M – Paquette
Assignment 0

Resolvents. This course will make substantial use of the resolvent $R(z; A)$ for square matrices A (real or complex), which we define as

$$R(z; A) \coloneqq (A - z\,\mathrm{Id})^{-1}, \quad \text{for all} \quad z \notin \mathrm{Spec}(A),$$

with $\mathrm{Spec}(A)$ the set of eigenvalues of A, and which is a continuous (in fact analytic) function of both arguments off of the set $\{(z, A) : z \notin \mathrm{Spec}(A)\}$. A fundamental identity, which will be helpful in what follows is

$$R(z; A) - R(z; B) = R(z; A)(B - A)R(z; B),$$

for all z for which both resolvents are defined. A second key estimate is the operator-norm bound, which holds for symmetric A:

$$\|R(z; A)\|_{\mathrm{op}} \le \frac{1}{d(z, \mathrm{Spec}(A))} \le \frac{1}{|\Im z|}.$$

Stein's Lemma. We will also use Stein's Lemma. Say a function f between Banach spaces $(X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is α-pseudo-Lipschitz with constant L (for $\alpha \ge 0$) if

$$\|f(x_1) - f(x_2)\|_Y \le L(1 + \|x_1\|_X + \|x_2\|_X)^\alpha.$$

Say f is pseudo-Lipschitz if it is α-pseudo-Lipschitz for some non-negative (α, L). Pseudo-Lipschitz functions on $\mathbb{R}^d$ are differentiable almost-everywhere, and their derivative is at most of polynomial-growth.

For univariate standard normals, this follows from integrating by parts against the Gaussian density:

$$\mathbb{E}Zf(Z) = \mathbb{E}f'(Z) \quad \text{where} \quad Z \overset{\mathscr{L}}{=} N(0, 1).$$

where $f : \mathbb{R} \to \mathbb{R}$ is pseudo-Lipschitz. By changing variables, this extends to the multivariate version of Stein's Lemma:

$$\mathbb{E}Zf(Z) = \Sigma\,(\mathbb{E}\nabla f(Z)) \quad \text{where} \quad Z \overset{\mathscr{L}}{=} N(0, \Sigma),$$

again for $f : \mathbb{R}^n \to \mathbb{R}$ pseudo-Lipschitz and $\Sigma \succeq 0$ an $n \times n$ matrix. (Deriving this from the univariate Stein's lemma is a good exercise!)

GOE. An $n \times n$ symmetric matrix G has the n-dimensional GOE (Gaussian orthogonal ensemble) distribution if $\{G_{ij} : i \ge j\}$ are normally distributed, mean 0, and have the normalization $\mathbb{E}G_{ij}^2 = (1 + \delta_{ij})$.

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Exercises.

1. Show the Woodbury formula, for n-dimensional vectors U, V and a square matrix A

$$R(z; A + UV^T) - R(z; A) = -\frac{R(z; A)UV^T R(z; A)}{1 + U^T R(z; A)V},$$

provided $z \notin \mathrm{Spec}(A + UV^T)$ and $z \notin \mathrm{Spec}(A)$.

2. Show the directional derivative of $R(z; A)$ in in its A variable in the direction of B is

$$\lim_{\epsilon \to 0} \epsilon^{-1} \left(R(z; A + \epsilon B) - R(z; A)\right) = -R(z; A)BR(z; A),$$

which therefore gives us an expression for all partial derivatives in A.

3. Suppose that S is a symmetric matrix, G is GOE, and set $A = SGS$. Show that for z with $\Im z > 0$

$$\mathbb{E}\left(R(z; A)A\right) = -\mathbb{E}\left(R(z; A)S^2 R(z; A)S^2 + R(z; A)S^2 \operatorname{tr}(R(z; A)S^2)\right).$$

4. Suppose that X is a vector of iid real random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$. Show that for any $p \in \mathbb{N}$ there is a constant C_p (not depending on the law of X_1) so that for any (complex) matrix A

$$\mathbb{E}|\langle X, AX\rangle - \operatorname{tr}(A)|^{2p} \le C_p(\mathbb{E}X_1^{4p})\|A\|_F^{2p}.$$

Here $\|A\|_F = (\operatorname{tr}(AA^*))^{1/2}$. *At least do this for $p = 2$. If you do the general case, you may try the following approach: first reduce to the case A is real symmetric, then reduce to the case $A_{ii} = 0$ for all i. Induction on p is one way forward.*