Homework0 - Kuijlaars

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Exercise 0.1

Prove that

$$|x-y| \le \sqrt{|x|^2+1}\sqrt{|y|^2+1}$$

holds for every $x, y \in \mathbb{C}$.

Proof. Consider $x = x_1 + ix_2, y = y_1 + iy_2$ and we will have

$$|x-y|^2 = |x|^2 + |y|^2 - 2(x_1y_1 + x_2y_2) \le |x|^2 + |y|^2 + 2|x||y| \le (|x|^2+1)(|y|^2+1)$$

by Cauchy's inequality. □

Exercise 0.2

Let $\Sigma \subset \mathbb{R}$ be a closed interval and V an admissible external field on Σ with equilibruim measure μ_V.

a. Suppose V is convex on Σ. Then prove that the support of μ_V is an interval.

b. Suppose $\Sigma = [0, \infty)$, V is differentiable on $(0, \infty)$ and $x \mapsto xV'(x)$ is increasing on $(0, \infty)$. Prove that the support of μ_V is an interval containing 0.

Proof. a. If not, there will be $c \in [a, b]$ not in the support of μ_V with x_1, x_2 in the support of μ_V, $c = \lambda x_1 + (1-\lambda)x_2$ and we will have

$$\begin{aligned}
2U^\mu(c) + V(c) &\ge l \\
&\ge \lambda(2U^\mu(x_1) + V(x_1)) + (1-\lambda)(2U^\mu(x_2) + V(x_2)) \\
&\ge \lambda 2U^\mu(x_1) + (1-\lambda)2U^\mu(x_2) + V(c)
\end{aligned}$$

and hence

$$\int \log\frac{1}{|c-s|} d\mu_V(s) \ge \int [\lambda \log\frac{1}{|x_1-s|} + (1-\lambda)\log\frac{1}{|x_2-s|}] d\mu_V(s)$$

which is a contradiction since $\log\frac{1}{|x-s|}$ is strictly convex.

b. Consider $G(x) = V(x^2)$ which is also admissible, which correspond with a measure μ_G and we know $G : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex and then we obtain the support of μ_G is interval contaning 0, and then we may obtain $supp(\mu_V) = f(supp(\mu_G))$ where $f(x) = x^2$. □

Exercise 0.3

Let $V : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ to be an admissible external field on $\mathbb{R}$ that is even, i.e. $V(-x) = V(x)$.

a. Show that the equilibrium measure μ_V.

b. Let μ_V^* be the pushforward of μ_V under the

Proof. If $x \notin \mathrm{supp}(\mu)$, we know there exists $\epsilon > 0$ such that $\mu((x_0 - \epsilon, x_0 + \epsilon)) = 0$ and we will know

$$2U^\mu(x_0) + V(x_0)$$

□

Exercise 0.4

Let $\Sigma \subset \mathbb{C}$ be closed and let $V : \Sigma \to \mathbb{R} \cup \{+\infty\}$ be admissible. Let σ be a measure with $d\sigma > 1$ (it could be ∞) and $supp(\sigma) = \Sigma$, with the additional property that that there exist $\mu \in P_\sigma(\Sigma)$ for which $I(\mu)$ and $I_V(\mu)$ are finite.

Proof. We recall the conclusion

a. If $E_V := \inf_{\mu \in P(\Sigma)} I_V(\mu)$, then $\infty < E_V < +\infty$.

b. Any sequence (μ_n) in $P(\Sigma)$ for which $\sup_n I_V(\mu_n) < +\infty$ is tight.

c. If $\mu \in P(\Sigma)$ satisfies $I_V(\mu) = E_V$, then μ has compact support.

d. If $\mu_n \to \mu$ weakly in $P(\Sigma)$, then $I_V(\mu) \leq \liminf_{n\to\infty} I_V(\mu_n)$.

e. If μ and ν have compact support then

$$I_V(\frac{\mu + \nu}{2}) \leq \frac{1}{2}(I_V(\mu) + I_V(\nu))$$

with strictly inequality if $\mu \neq \nu$ and both $I_V(\mu)$ and $I_V(\nu)$ are finite. Then similarly we can choose a sequence in $P^\sigma(\Sigma)$ and pick a limit from a convergence subsequence, then it suffices to show that $P^\sigma(\Sigma)$ is closed under the weak topology.

b. □