Bonus 01 - MATH 722

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Problem

Prove that there is a constant C, indeoendent of n such that if $\{z_j\}$ are complex numbers and if

$$\sum_{j=1}^{n} |z_j| \geq 1$$

then there is a subcollection $\{z_{j_1}, \cdots, z_{j_k}\} \subset \{z_1, \cdots, z_n\}$ such that

$$|\sum_{m=1}^{k} z_{j_m}| \geq C$$

Can you find the best constant C?

Sol.

Here we consider $v_\theta = e^{i\theta}$ and for any $z_j, 1 \leq j \leq n$ such that

$$\sum_{j=1}^{n} |z_j| \geq 1$$

assume $Arg z_j = \phi_j, 1 \leq j \leq n$ and denote $f(\theta) = \sum_{i=1}^{n} |z_j|(\cos(\theta-\phi_j)\vee 0)$ which is the sum of projections of z_j on v_θ for $1 \leq j \leq n$, then notice

$$\int_0^{2\pi} f(\theta)d\theta = \sum_{i=1}^{n} |z_j| \int_0^{2\pi} (\cos(\theta - \phi_j) \vee 0)d\theta = \int_0^{2\pi} (\cos(\theta) \vee 0)d\theta = 2$$

and hence there have to be $\theta \in [0, 2\pi)$ such that $f(\theta) \geq \frac{1}{\pi}$, since for any $n \geq 0$, there has to be θ_n such that $f(\theta) > \frac{1}{\pi} - n^{-1}$ and notice f is continuous on $[0, 2\pi]$. Then we consider all $\{z_{j_m}\}$ such that $cos(\theta - \phi_{j_m}) \geq 0$ and then we know

$$|\sum_{m=1}^{k} z_{j_m}| \geq f(\theta) \geq \frac{1}{\pi}$$

Then we will show that $C = \frac{1}{\pi}$ is the best bound.

Consider $z_j = (4n)^{-1} e^{i\frac{j\pi}{2n}} 1 \leq j \leq 4n$ and for any subcollection $\{z_{j_m}\}$ let $v = (\sum_{m=1}^{k} z_{j_m})/|(\sum_{m=1}^{k} z_{j_m})|$ and then we know that $f(Arg_v) \geq |\sum_{m=1}^{k} z_{j_m}|$. And then for any $\theta \in [0, 2\pi)$, we know

$$f(\theta) = (4n)^{-1} \sum_{j=1}^{4n} (\cos(\theta - \frac{j\pi}{2n}) \vee 0) = (4n)^{-1} (\sum_{j=0}^{n-1} \Theta_1 + \frac{j\pi}{2n}) + (\sum_{j=0}^{n-1} \Theta_2 + \frac{j\pi}{2n}) \leq (2n)^{-1} \sum_{j=0}^{n-1} \cos()\frac{j\pi}{2n})$$

where $\Theta_1 = \theta(mod\ \frac{\pi}{2n})$ and $\Theta_2 = \frac{\pi}{2n} - \Theta_1$, then notice that

$$\lim_{n\to\infty}(2n)^{-1}\sum_{j=0}^{n-1}\cos(\frac{j\pi}{2n}) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\cos t dt = \frac{1}{\pi}$$

since $\cos(x)$ is Riemann-integrable on $[0, \frac{\pi}{2}]$, which means $\limsup|\sum_{m=1}^{k} z_{j_m}| \leq \lim_{n\to\infty}\sup(f(\theta, n)) = \dfrac{1}{\pi}$ and hence $\dfrac{1}{\pi}$ is the best $\boldsymbol{C}$.