# Bonus 02 - MATH 722

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### Problem

Let $f$ be holomorphic on a neighbourhood of $\overline{D}(0,1)$. Assume that the restriction of $f$ to $\overline{D}(0,1)$ is one-to-one and $f'$ is nowhere zero on $\overline{D}(0,1)$. Prove that in fact $f$ is one-to-one on a neigbourhood of $\overline{D}(0,1)$.

### Sol.

Let $D_n = D(0, 1+n^{-1})$ and if the conclusion is not correct, then we can always find $x_n, y_n \in D(0, 1+n^{-1})$ such that $f(x_n) = f(y_n)$, and since $x_n, y_n$ is in $\overline{D}(0,2)$. Then there exists $x, y$ such that $x_n \to x, y_n \to y$ with $|x|, |y| \leq 1$, and $f(x) - f(y) = \lim_{n\to\ infty} f(x_n) - f(y_n) = 0$, if $x \neq y$ there will be a contradiction and hence $x = y$, which is also impossible since we know $f'(x) \neq 0$ and we may assume there exists $\delta > 0$ such that $Ref'(y) > \epsilon > 0$ for some $\epsilon$ whenever $|y - x| < \delta$, then for any $p, q \in D(x, \delta)$, we have

$$Re(f(p) - f(q)) = Re(\int_{\overline{qp}} f'(\xi)d\xi) = \int_{\overline{qp}} Ref'(\xi)d\xi \geq \epsilon|p - q|$$

which means $f$ should be one-to-one around $x$, the proof is the same when $Im(f'(x)) \neq 0$. Then we know $x_n, y_n \to x$ with $f(x_n) = f(y_n)$, which is a contradiction. Therefore, $f$ should be one-to-one on $D_n$ for some integer $n$.