NOTES FOR C*-ALGEBRA

Based on the John Conway

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Contents

1 C*-Algebras

1.1 Basic Concepts

Definiton 1.1.1. (Involution)

For a Banach algebra $\mathcal{A}$(which is not required to have an identity), the involution is a map $\mathcal{A} \to \mathcal{A}$ denoted by $a \mapsto a^*$ such that for any $a, b \in \mathcal{A}$

- $(a^*)^* = a$
- $(ab)^* = b^* a^*$
- $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$ for any $\alpha \in \mathbb{C}$

Definiton 1.1.2. (C*-algebra)

A **C*-algebra** is a Banach algebra $\mathcal{A}$ such that

$$||a^* a|| = ||a||^2$$

for any $a \in \mathcal{A}$.

Proposition 1.1.1. Suppose $\mathcal{A}$ is a C*-algebra, then the involution keeps norm, i.e. for any $a \in \mathcal{A}$ we have $||a^*|| = ||a||$.

Proof.

Notice

$$||a||^2 = ||aa^*|| \leq ||a|| ||a^*||$$

which implies $||a|| \leq ||a^*||$ and since $(a^*)^* = a$ and we are done.

Proposition 1.1.2. Suppose $\mathcal{A}$ is a C*-algebra, $a \in \mathcal{A}$, then

$$||a|| = \sup\{||ax||, x \in \mathcal{A}, ||x|| \leq 1\}$$

Proof.

1.2 The Positive Elements in a C*-Algebra

Definiton 1.2.1. If $\mathcal{A}$ is a C*-algebra, then a is positive if $a \in \mathrm{Re}\mathcal{A}$(the hermitian elements of $\mathcal{A}$) and $\sigma(a) \subset [0, \infty)$. If a is positive, this is denoted by $a \geq 0$. Let $\mathcal{A}_+$ be the set of all positive elements of $\mathcal{A}$.

Proposition 1.2.1. If $a \in \int$

1.3 Ideals and Quotients of C*-Algebras

Proposition 1.3.1. If I is a closed left or right ideal in the C*-algebra $\mathcal{A}$, $a \in I$ with $a = a^*$ and if $f \in C(\sigma(a))$ with $f(0) = 0$, then $f(a) \in I$.

Corollary 1.3.2. If I is a closed left or right ideal,$a \in I$ with $a = a^*$ then $a_+, a_-, |a|$ and $|a|^{1/2} \in I$.

Theorem 1.3.3. If I is a closed ideal in the C*-algebra $\mathcal{A}$, then $a^* \in I$ if $a \in I$.

Proposition 1.3.4. If $\mathcal{A}$ is a C*-algebra and I is an ideal of $\mathcal{A}$, then for every a in I there is a sequence $\{e_n\}$ of positive elements in I such that

- $e_1 \leq e_2 \leq \cdots$ and $||e_n|| \leq 1$
- $||ae_n - a|| \to 0$ as $n \to \infty$

Lemma 1.3.5. If I is an ideal in a C*-algebra $\mathcal{A}$ and $a \in \mathcal{A}$, then $||a+I|| := \inf\{||a-x||, x \in I\} = \inf\{||a - ax|| : x \in I, x \geq 0, ||x|| \leq 1\}$.

Proof.

We know

$$||a + I|| \leq \inf\{||a - ax||, x \geq 0, ||x|| \leq 1\}$$

and let $e_n \in I, e_n \leq 1$ such that $||y - ye_n|| \to 0$ for some $y \in I$ then since we know $0 \leq 1 - e_n \leq 1$, so $||(a + y)(1 - e_n)|| \leq ||a + y||$ and hence

$$||a + y|| \geq \liminf ||a - ae_n|| \geq \inf\{||a - ax||, x \geq 0, ||x|| \leq 1, x \in I\}$$

Theorem 1.3.6. If $\mathcal{A}$ is a C*-algebra and I is a closed ideal of $\mathcal{A}$, then for each $a + I$ in $\mathcal{A}/I$ define $(a + I)^* = a^* + I$. Then A/I with its quotient norm is a C*-algebra.

Proof.

By the lemma 1.3.5. we know

$$||a + I||^2 = \inf\{||(1 - x)a^*a(1 - x)||, x \geq 0, ||x|| \leq 1, x \in I\} \leq ||a^*a + I||$$

and since $||a^* + I|| = ||a + I||$ by proposition 1.3.4. and we are done.