Potential theory for random matrices

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Week 0: Introduction to potential theory

0.1 Introduction

A large part of random matrix theory deals with the limiting behavior of eigenvalues as the size of the random matrix tends to infinity. Well known limiting distributions are the semi-circle law given by the probability measure

$$d\mu_{sc} = \frac{1}{2\pi}\sqrt{4-x^2}\,dx \qquad \text{on } [-2,2] \tag{0.1}$$

or the Marchenko-Pastur law

$$d\mu_{MP} = \frac{1}{\pi}\sqrt{\frac{4-x}{x}}dx \qquad \text{on } [0,4]. \tag{0.2}$$

The semi-circle law appears as the limiting empirical measure for the eigenvalues of GOE/GUE ensembles (after suitable scaling), or more general Wigner ensembles, i.e., matrices with i.i.d. entries. The Marchenko-Pastur law appears as the limiting empirical measure for the eigenvalues of LOE/LUE ensembles and for the singular values of many random matrix ensembles.

These limiting distributions are characterized by logarithmic potential theory. For example, the semi-circle law is the unique probability measure on the real line that minimizes the weighted logarithmic energy

$$\iint \log\frac{1}{|x-y|}d\mu(x)d\mu(y) + \int x^2 d\mu(x) \tag{0.3}$$

among all probability measures on $\mathbb{R}$ and a similar characterization exists for the Marchenko-Pastur law.

During the course we study logarithmic potential theory and we develop tools to study such minimization problems. Familiarity with measure theory is necessary. Logarithmic potential theory is naturally studied in the complex plane, and a good understanding of complex analysis will be helpful as well. Several calculations are easier done in the complex plane and the first homework exercise (to be done before the course) is about that.

Exercise 0.1. The semi-circle law is given by (0.1).

(a) Compute its Stieltjes transform $F(z) := \int_{-2}^{2} \frac{d\mu_{sc}(x)}{z-x}$ for $z \in \mathbb{C} \setminus [-2,2]$, by means of contour integration. That is, write the integral as an integral over a contour in the complex x-plane surrounding the interval $[-2,2]$, and then evaluate the integral by a residue calculation. There will be a contribution from infinity and from a pole at z.

(b) Verify that

$$\lim_{\varepsilon \to 0+} \frac{1}{\pi} \operatorname{Im} F(x - i\varepsilon) = \frac{d\mu_{sc}}{dx}, \qquad \text{for } x \in [-2,2]. \tag{0.4}$$

(c) Let $m_k = \int_{-2}^{2} x^k d\mu_{sc}(x)$ be the kth moment of the semi-circle law. Use the Laurent expansion of F around infinity to compute $m_{2k} = \frac{1}{k+1}\binom{2k}{k}$ for $k = 0, 1, 2, \ldots$. The number m_{2k} is the kth Catalan number.

The relation (0.4) is a special case of a general formula on how to recover a measure μ from its Stieltjes transform $F_\mu(z) = \int \frac{1}{z-x} d\mu(x)$. This formula is known as the Stieltjes-Perron inversion formula. For a compactly supported measure μ on $\mathbb{R}$ with a continuous density $d\mu = v dx$ it says that

$$v(x) = \lim_{\varepsilon \to 0+} \frac{1}{\pi} \operatorname{Im} F_\mu(x - i\varepsilon). \tag{0.5}$$

0.2 Harmonic and superharmonic functions

We recall the definitions and some basic facts that you may have seen in a course on PDEs.

Let Ω be an open subset of $\mathbb{C}$. A function $h : \Omega \to \mathbb{R}$ is harmonic in Ω if it is C^2 and satisfies $\Delta h = 0$. Locally a harmonic function is the real part of an analytic function.

A function $h : \Omega \to \mathbb{R}$ is harmonic on Ω if and only if, it is continuous and has the local mean value property, which means that for every $z_0 \in \Omega$ there is $r_0 > 0$ such that for every $r \in (0, r_0)$ one has $D_r(z_0) \subset \Omega$ and

$$h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta}) d\theta.$$

A harmonic function satisfies the maximum and minimum principle that we state here in the following strong form.

Proposition 0.1 (maximum/minimum principle for harmonic functions)**.** *Suppose Ω is a bounded connected open set and h is a harmonic function in Ω with a continuous extension to the closure $\overline{\Omega}$. Then h assumes its maximum and minimum on the boundary of Ω. If h assumes its maximum or minimum value in a point of Ω as well, then h is constant.*

A function $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ is superharmonic on Ω if it is lower semi-continuous and it has the local super mean value property, which means that for every $z_0 \in \Omega$ there is $r_0 > 0$ such that for every $r \in (0, r_0)$ one has $D_r(z_0) \subset \Omega$ and

$$u(z_0) \geq \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) d\theta.$$

A superharmonic function has the minimum principle.

Proposition 0.2 (minimum principle for superharmonic functions)**.** *Suppose u is a superharmonic function on the connected open set Ω and let $m \in \mathbb{R}$. If $\liminf_{z \to \zeta} u(z) \geq m$ holds for every $\zeta \in \partial\Omega$, then $u \geq m$ in Ω. If in addition $u = m$ somewhere in Ω, then u is constant.*

There is an analogous notion of subharmonic functions. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if and only if $-u$ is superharmonic. A subharmonic function satisfies the maximum principle.

Logarithmic potentials are important examples of superharmonic functions.

The logarithmic potential of a compactly supported measure μ is the function $U^\mu : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$U^\mu(z) = \int \log \frac{1}{|z - s|} d\mu(s).$$

For us, a measure is always positive and very often a probability measure.

Exercise 0.2. Let μ be a measure on $\mathbb{C}$ with compact support. Show that U^μ is superharmonic on $\mathbb{C}$, and harmonic on $\mathbb{C} \setminus \operatorname{supp}(\mu)$.

You have to show that U^μ is lower semi-continuous, i.e., if $(z_n)_n$ is a sequence in $\mathbb{C}$ with limit z_0 then

$$U^\mu(z_0) \leq \liminf_{n \to \infty} U^\mu(z_n),$$

and that U^μ satisfies the super mean value property, i.e.,

$$U^\mu(z_0) \geq \frac{1}{2\pi} \int_0^{2\pi} U^\mu \left(z_0 + re^{i\theta} \right) d\theta$$

for $z_0 \in \mathbb{C}$ and every $r > 0$.

Exercise 0.3. In our study of equilibrium measures we will associate with a function $V : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ a probability measure μ on $\mathbb{R}$ and a constant ℓ such that

$$\begin{aligned} 2U^\mu + V &= \ell, \quad \text{on the support of } \mu, \\ 2U^\mu + V &\geq \ell, \quad \text{on } \mathbb{R}. \end{aligned} \tag{0.6}$$

Suppose that μ is a probability measure satisfying (0.6) for some constant ℓ. Let x_0 be a point where V assumes its minimum on $\mathbb{R}$. Prove that $x_0 \in \operatorname{supp}(\mu)$. You may assume in your proof that U^μ is continuous.

The function V is known as an external field, and it is assumed to be continuous on $\mathbb{R}$ and it tends to $+\infty$ at infinity sufficiently fast, namely $\frac{V(x)}{\log(1+x^2)} \to +\infty$ as $x \to \pm\infty$.

It will be good to know that one can recover a measure from its logarithmic potential.

Proposition 0.3. *Let μ be a measure with compact support and U^μ its logarithmic potential. Then one can recover μ from its logarithmic potential U^μ since one has*

$$\Delta U^\mu = -2\pi\mu,$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian.

This means that for compactly supported C^2 functions $f : \mathbb{C} \to \mathbb{R}$ one has

$$\int_{\mathbb{C}} \Delta f \cdot U^\mu dm_2 = -2\pi \int f d\mu,$$

with dm_2 the planar Lebesgue measure on $\mathbb{C}$.

Exercise 0.4 (optional)**.** Prove Proposition 0.3 or look it up in the literature.