Chapter 1

Fundamental Concepts

Definition 1.1

If $U \subset \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$ is a continuous function, then f is called C^1 on U if $\partial f/\partial x, \partial f/\partial y$ exist and are continous on U. ♣

Definition 1.2

We define for $f = u + iv : U \to \mathbb{C}$ a C_1 function

$$\frac{\partial}{\partial z} f := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f$$
$$\frac{\partial}{\partial \bar{z}} f := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f$$

which is easy to be checked linear and the chain rules. ♣

where we may check let $z = x + iy, \bar{z} = x - iy$, we have

$$\frac{\partial}{\partial z} z = 1, \quad \frac{\partial}{\partial z}\bar{z} = 0$$
$$\frac{\partial}{\partial \bar{z}} z = 0, \quad \frac{\partial}{\partial \bar{z}}\bar{z} = 1$$

Proposition 1.1

(The Leibniz Rules) We have for any $F, G \in C^1$

$$\frac{\partial}{\partial z}(F \cdot G) = \frac{\partial F}{\partial z} \cdot G + F \cdot \frac{\partial G}{\partial z}$$
$$\frac{\partial}{\partial \bar{z}}(F \cdot G) = \frac{\partial F}{\partial \bar{z}} \cdot G + F \cdot \frac{\partial G}{\partial \bar{z}}$$

♠

Proposition 1.2

We have for $l \leq j, m \leq k$ nonnegative integers and then

$$(\frac{\partial^l}{\partial z^l})(\frac{\partial^m}{\partial \bar{z}^m})(z^j \bar{z}^k) = \frac{j!}{l!}\frac{k!}{m!} z^{j-l}\bar{z}^{k-m}$$

♠

Proposition 1.3

If $p(z, \bar{z}) = \sum a_{lm} z^l \bar{z}^m$ is a polynomaial, then p contains no term with $m > 0$ iff $\frac{\partial p}{\partial \bar{z}} \equiv 0$. ♠

Corollary 1.1

If $p(z, \bar{z}) = qz, \bar{z}$ are polynomials, then they have same coefficients. ♡

Definition 1.3

A C_1 function $f : U \mapsto \mathbb{C}$ is said to be holomorphic if

$$\frac{\partial f}{\partial \bar{z}} = 0$$

at every point of U. ♣

Definition 1.4

A C^1 function $f = u(x,y) + iv(x,y) : U \to \mathbb{C}$ is holomorphic if

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

at every point of U, which is called the $Cauchy$-$Riemann$ equations. ♣

Proposition 1.4

If $f : U \to \mathbb{C}$ is C^1 and if f satisfies the C-R equations, then

$$\frac{\partial}{\partial z} f = \frac{\partial}{\partial x} f = -i\frac{\partial}{\partial y} f$$

on U. ♠

Proof

We have

$$\begin{aligned} \frac{\partial}{\partial x} f &= \frac{\partial}{\partial x} u + i\frac{\partial}{\partial x} v = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})u = 2\frac{\partial}{\partial z} u \\ \frac{\partial}{\partial x} f &= \frac{\partial}{\partial x} u + i\frac{\partial}{\partial x} v = i(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})v = 2\frac{\partial}{\partial z} iv \\ -i\frac{\partial}{\partial y} f &= -i\frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})u = 2\frac{\partial}{\partial z} u \\ -i\frac{\partial}{\partial y} f &= -i\frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v = i(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})v = 2\frac{\partial}{\partial z} iv \end{aligned}$$

on U.

Definition 1.5

If $U \subset \mathbb{C}$ is open and $u \in C^2(U)$, then u is called harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where we also denote it as

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

where the operator is called the $Laplace$ $operator$. ♣

Here we have

$$4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} u = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \Delta u$$

Proposition 1.5

The real and imaginary parts of a holomorphic C^2 function are harmonic. ♠

Proof

Assume $f = u + iv$ and then according to C-R equations, we have

$$\frac{\partial^2}{\partial x^2} u = \frac{\partial^2}{\partial x \partial y} v = \frac{\partial^2}{\partial y \partial x} v = -\frac{\partial^2}{\partial y^2} u$$

and

$$\frac{\partial^2}{\partial x^2} v = -\frac{\partial^2}{\partial x \partial y} u = -\frac{\partial^2}{\partial y \partial x} u = -\frac{\partial^2}{\partial y^2} v$$

Lemma 1.1

It $u(x,y)$ is a real-valued polynomial with $\Delta u = 0$, then there exists a (holomorphic) $Q(z)$ such that $ReQ = u$. ♡

Proof

Consider $u(x,y) = u(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}) = P(z,\bar{z}) = \sum a_{lm} z^l \bar{z}^m$, we know $\Delta u = 0$ and hence

$$P(z,\bar{z}) = a_0 0 + \sum^m a_k z^k + \sum^n b_k \bar{z}^k$$

P is real-valued and we know

$$a_0 0 + \sum^m a_k z^k + \sum^n b_k \bar{z}^k = \bar{a_0} 0 + \sum^m \bar{a_k} \bar{z}^k + \sum^n \bar{b_k} z^k$$

and hence $a_0 0 \in \mathbb{R}$, $a_k = \bar{b_k}$ and hence

$$u(z) = c + \sum^n a_k z^k + \sum^n \bar{a_k} \bar{z}^k = Re(c + 2\sum^n a_k z^k) = Re(Q)$$

where Q is obviously holomorphic.

Theorem 1.1

If f, g are C^1 functions on the rectangle

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : |x-a| < \delta, |y-b| < \epsilon\}$$

and if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \text{ on } \mathcal{R}$$

then there is a function $h \in C^(\mathcal{R})$ such that

$$\frac{\partial}{\partial x} h = f, \frac{\partial}{\partial y} h = g$$

on $\mathcal{R}$. If f, g are real-valuedd, the nwe may take h to be real-valued also.

Proof

For $(x,y) \in \mathcal{R}$, define

$$h(x,y) = \int_a^x f(t,b)dt + \int_b^y g(x,s)ds$$

and we know

$$\frac{\partial}{\partial y} h(x,y) = g(x,y)$$

and

$$\frac{\partial}{\partial x} h(x,y) = f(x,b) + \frac{\partial}{\partial x} \int_b^y g(x,s)ds = f(x,b) + \int_b^y \frac{\partial}{\partial y} f(x,s) = f(x,b) + f(x,y) - f(x,b) = f(x,y)$$

and hence $h \in C^2(\mathcal{R})$ and real-valued if f, g are.

Corollary 1.2

If $\mathcal{R}$ is an open rectangle (or open disc) and if u is a real-valued harmonic function on $\mathbb{R}$, then there is a holomorphic function F on $\mathbb{R}$ such that $ReF = u$.

Proof

We know

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$

and hence there exists v real-valued such taht

$$\frac{\partial}{\partial x} v = -\frac{\partial}{\partial y} u, \frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u$$

and hence $F = u + iv$ is a holomorphic function with $Re(F) = u$.

Theorem 1.2

If $U \subset \mathbb{C}$ is either an open rectangle or an open disc and if F is holomorphic on U, then there is a holomorphic function H on U such that $\partial H/\partial z = F$ on U.

Proof

Consider $H = h_1 + ih_2$ and we have $F = u(z) + iv(z)$, then we let $f = u, g = -v$ and we will have

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial x} g$$

and hence we have a real C^2 function h_1 such that

$$\frac{\partial}{\partial x} h_1 = u, \frac{\partial}{\partial y} h_1 = -v$$

and $h_2 \in C^2$ with

$$\frac{\partial}{\partial x} h_2 = v, \frac{\partial}{\partial y} h_2 = u$$

Then

$$\frac{\partial}{\partial z} H = \frac{1}{2}(\frac{\partial}{\partial x} h_1 + \frac{\partial}{\partial y} h_2) + \frac{i}{2}(\frac{\partial}{\partial x} h_2 - \frac{\partial}{\partial y} h_1) = u + iv = F$$

Definition 1.6

A function $\phi : [a, b] \to \mathbb{R}$ is called $continuously\ differentiable$ and we write $\phi \in C^1([]a, b)$ if

(a) ϕ is continous on $[a, b]$

(b) ϕ' exists on (a, b)

(c) ϕ' has a continuous extension to $[a, b]$, i.e.

$$\lim_{t \to a^+} \phi'(t) \text{ and } \lim_{t \to b^-} \phi'(t)$$

both exists. Then $\phi(b) - \phi(a) = \int_a^b \phi'(t)dt$.

Proof

Here notice that ϕ is absolutely continuous on $[a, b]$ respect to m, then we know $\phi(b - \epsilon) - \phi(a + \epsilon) = \int_{a+\epsilon}^{b-\epsilon} \phi'(t)dt$ for any $epsilon > 0$, and hence

$$\phi(b) - \phi(a) = \int_a^b \phi'(t)dt$$

Definition 1.7

A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be continuous on $[a, b]$ if both γ_1 and γ_2 are, $\gamma = \gamma_1 + i\gamma_2$. The curve is C_1 on $[a, b]$ if γ_1, γ_2 are C_1 on $[a, b]$ and then we may denote

$$\frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt}$$

Definition 1.8

Let $\varphi : [a, b] \to \mathbb{C}$ be continuous on $[a, b]$. Write $\varphi(t) = \varphi_1(t) + i\varphi_2(t)$. Then we define

$$\int_a^b \varphi(t)dt = \int_a^b \varphi_1(t)dt + i\int_a^b \varphi_2(t)dt$$

Proposition 1.6

Let $U \subset \mathbb{C}$ be open and let $\gamma : [a, b] \to U$ be a C_1 curve. If $f : U \to \mathbb{R}$ and $f \in C^1(U)$, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \left(\frac{\partial}{\partial x} f(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial}{\partial y} f(\gamma(t))\frac{d\gamma_2}{dt}\right)dt$$

This is due to the chain rule.

Proposition 1.7

Repalce f above as complex-valued and holomorphic, then we have

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial}{\partial z} f(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)dt$$

Proof

Notice

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &= \int_a^b \left(\frac{\partial}{\partial x} u(\gamma(t))\frac{d\gamma_1}{dt}(t) + \frac{\partial}{\partial y} u(\gamma(t))\frac{d\gamma_2}{dt}(t)\right) + i\left(\frac{\partial}{\partial x} v(\gamma(t))\frac{d\gamma_1}{dt}(t) + \frac{\partial}{\partial y} v(\gamma(t))\frac{d\gamma_2}{dt}(t)\right)dt \\ &= \frac{\partial}{\partial x} f(\gamma(t))\frac{d\gamma}{dt}(t) = \int_a^b \frac{\partial}{\partial z} f(\gamma(t))\frac{d\gamma}{dt}(t)dt \end{aligned}$$

Definition 1.9

If $U \subset \mathbb{C}$ open and $F : U \to \mathbb{C}$ is continuous on U and $\gamma : [a,b] \to U$ is a C_1 curve, then we define the complex line integral

$$\int_\gamma F(z)dz = \int_a^b F(\gamma(t))\frac{d\gamma}{dt}dt$$

Proposition 1.8

Let $U \subset \mathbb{C}$ be open and let $\gamma : [a,b] \to U$ be a C^1 curve. If f is a holomorphic function on U, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_\gamma \frac{\partial}{\partial z} f(z)dz$$

Proposition 1.9

If $\phi : [a,b] \to \mathbb{C}$ is continuous, then

$$|\int_a^b \phi(t)dt| \leq \int_a^b |\phi(t)|dt$$

Proposition 1.10

Let $U \subset \mathbb{C}$ be open and $f \in C^0(U)$. If $\gamma : [a,b] \to U$ is a C^1 curve, then

$$|\int_\gamma f(z)dz| \leq (\sup_{t\in[a,b]} |f(\gamma(t))|) \cdot l(\gamma)$$

where

$$l(\gamma) = \int_a^b |\frac{d\gamma}{dt}(t)|dt$$

Proposition 1.11

Let $U \subset \mathbb{C}$ be an open set and $F : U \to \mathbb{C}$ a continuous function. Let $\gamma : [a,b] \to U$ be a C^1 curve. Suppose that $\theta : [c,d] \to [a,b]$ is a one-to-one, onto, increasing C^1 function with a C^1 inverse. Let $\tilde{\gamma} = \gamma \circ \phi$. Then

$$\int_{\tilde{\gamma}} f dz = \int_\gamma f dz$$

Proof

We have

$$\int_{\tilde{\gamma}} f dz = \int_c^d f(\gamma(\phi(t)))\frac{d\gamma(\phi(t))}{dt}dt = \int_a^b f(\gamma(s))\frac{\gamma(s)}{ds}\phi'(\phi^{-1}(s))(\phi^{-1})'(s)ds = \int_\gamma f dz$$

since $\phi'(\phi^{-1}(s))(\phi^{-1})' = 1$.

Definition 1.10

Let f be a function on the open set U in $\mathbb{C}$ and consider if

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists then we say that f has a complex derivative at z_0. We denote the complex derivative by $f'(z_0)$.

Theorem 1.3

Let $U\subset\mathbb{C}$ be an open set and let f be holomorphic on U. Then f' exists at each point of U and

$$f'(z)=\frac{\partial}{\partial z}f$$

for all $z\in U$.

Proof

Consider

$$\gamma(t)=(1-t)z_0+tz$$

and then we know

$$f(z)-f(z_0)=f(\gamma(1))-f(\gamma_0)=\int_\gamma\frac{\partial}{\partial z}fdz=(z-z_0)\int_0^1\frac{\partial}{\partial z}f(\gamma(t))dt=\frac{\partial}{\partial z}f(z_0)+\int_0^1(\frac{\partial}{\partial z}f(\gamma(t))-\frac{\partial}{\partial z}f(z_0))dt$$

and hence

$$|\frac{f(z)-f(z_0)}{z-z_0}-\frac{\partial}{\partial z}f(z_0)|\le\int|\frac{\partial}{\partial z}f(\gamma(t))-\frac{\partial}{\partial z}f(z_0)|dt\to 0$$

when $z\to z_0$.

Theorem 1.4

If $f\in C^1(U)$ and f has a complex derivative at each point of U, then f is holomorphic on U. In particular, if a continuous, complex-valued function f on U has a complex derivative at each point and if f' is continuous on U, then f is holomorphic on U.

Proof

It is easy to check

$$\lim_{h\to 0,h\in\mathbb{R}}\frac{f(z_0+h)-f(z_0)}{h}=\frac{\partial}{\partial x}u(x_0,y_0)+i\frac{\partial}{\partial x}v(x_0,y_0)$$

and

$$\lim_{h\to 0,h\in\mathbb{R}}\frac{f(z_0+h)-f(z_0)}{h}=-i\frac{\partial}{\partial y}u(x_0,y_0)+\frac{\partial}{\partial y}v(x_0,y_0)$$

and hence f satisfies the C-R equations so holomorphic.

Notice the continuity of f' may implies that $f\in C^1(U)$ and hence the problem goes.

Theorem 1.5

Let f be holomorphic in a neighborhood of $P\in\mathbb{C}$. Let ω_1,ω_2 be complex numbers of unit modulus. Consider the directional derivatives

$$D_{\omega_1}f(P)=\lim_{t\to 0}\frac{f(P+t\omega_1)-f(P)}{t}$$

and

$$D_{\omega_2}f(P)=\lim_{t\to 0}\frac{f(P+t\omega_2)-f(P)}{t}$$

then

a. $|D_{\omega_1}f(P)|=|D_{\omega_2}f(P)|$

b. If $f'(P)\neq 0$, then the directed angle from ω_1 to ω_2 equals the directed angle from $D_{\omega_1}f(P)$ to $D_{\omega_2}f(P)$.

Proof

Notice that

$$D_{\omega_j} = f'(P)\omega_j, j = 1, 2$$

and then the conclusions go.

Lemma 1.2

Let $(\alpha, \beta) \subset \mathbb{R}$ be an open interval and let $H : (\alpha, \beta) \to \mathbb{R}, F : (\alpha, \beta) \to \mathbb{R}$ be continuous functions. Let $p \in (\alpha, \beta)$ and suppose that dH/dx exists and equals $F(x)$ for all $x \in (\alpha, \beta) - \{p\}$. Then $(dH/dx)(p)$ exists and $(dH/dx)(x) = F(x)$ for all $x \in (\alpha, \beta)$. ♡

Proof

Assume $[a, b] \subset (\alpha, \beta)$ and then $K(x) = H(a) + \int_a^x F(t)dt$ on $[a, b]$, so we know $K - H$ is continuous on $[a, b]$ and constant on $[a, p) \cup (p, b]$, which means $K = H$ on $[a, b]$.

Theorem 1.6

Let $U \subset \mathbb{C}$ be either an open rectangle or an open disc and let $P \in U$. Let f and g be continuous, real-valued functions on U which are continuously differentiable on $U - \{P\}$. Suppose further that

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial x} g \text{ on } U - \{P\}$$

Then there exists a C^1 function $h : U \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x} = f, \frac{\partial}{\partial y} = g$$

at every point of U. ♡

Proof

Consider a closed rectangle containing p inside in U and define $h(x, y) = \int_a^x f(t, b)dt + \int_b^y g(x, s)ds$ and we know that $\frac{\partial}{\partial y} h = g(x, y)$ and $\frac{\partial}{\partial x} h = f(x, y)$ for any $x \neq P_x$, then for a fixed y, we know $dh(x, y)/dx = f(x, y)$ exists for all points in U except for (p_x, y) and hence $dh(x, y)/dx = f(x, y)$ at (p_x, y). Then we know $\frac{\partial}{\partial x} h = f, \frac{\partial}{\partial y} h = g$ on U.

Theorem 1.7

Let $U \subset \mathbb{C}$ be either an open rectangle or an open disc. Let $P \in U$ be fixed. Suppose that F is continuous on U and holomorphic on $U - \{P\}$. Then there is a holomorphic H on U such that U such that $\frac{\partial}{\partial z} H = F$. ♡

Proof

Consider $F = u + iv$, then we have

$$\frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u \text{ and } \frac{\partial}{\partial y} u = \frac{\partial}{\partial x}(-v)$$

on $U - \{P\}$, then we know there exists h_1, h_2 on U such that $\frac{\partial}{\partial x} h_1 = u$, $\frac{\partial}{\partial y} h_1 = (-v)$, $\frac{\partial}{\partial x} h_2 = v$, $\frac{\partial}{\partial y} h_2 = u$ and let $H = h_1 + ih_2$, we have

$$\frac{\partial}{\partial z} H = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(h_1 + ih_2) = (u + u) + i(v + v) = F$$

Definition 1.11

The boundary $\partial D(P, r)$ of the disc $D(P, r)$ can be parametrized as a simple closed curve $\gamma : [0, 1] \to \mathbb{C}$ by setting

$$\gamma(t) = P + re^{2\pi it}$$

we call it counterclockwise orientation. ♣

Lemma 1.3

Let γ be the boundary of a disc $D(z_0, r)$ in the complex plane, equipped with counterclockwise orientation. Let z be a point inside the circle $\partial D(z_0, \gamma)$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\xi - z}d\xi = 1$$

♡

Proof

Consider $I(z) = \int_\gamma \frac{1}{\xi - z}d\xi = \int_0^1 \frac{1}{(z_0 + e^{2\pi i t}) - z}(2\pi i)e^{2\pi i t}dt$ and since

$$\frac{\partial}{\partial x}\frac{1}{\xi - z} = \frac{1}{(\xi - z)^2}, \quad \frac{\partial}{\partial y}\frac{1}{\xi - z} = i\frac{1}{(\xi - z)^2}$$

and hence we have

$$\frac{\partial}{\partial \bar{z}}I(z) = \int_\gamma \frac{\partial}{\partial \bar{z}}(\frac{1}{\xi - z})d\xi = 0 \quad \frac{\partial}{\partial z}I(z) = \int_\gamma \frac{\partial}{\partial z}(\frac{1}{\xi - z})d\xi = \int_\gamma \frac{1}{(\xi - z)^2}d\xi$$

where $\frac{1}{(\xi - z)^2}$ is the complex derivative of the holomorphic function $\frac{-1}{\xi - z}$ and hence

$$\frac{\partial}{\partial z}I(z) = \int_\gamma \frac{1}{(\xi - z)^2}d\xi = 0$$

Therefore, $I(z)$ is holomorphic on $D(z_0, r)$ and $\frac{\partial}{\partial z}I = 0$ which means I is constant on $D(z_0, r)$ and notice

$$I(z_0) = 2\pi i$$

and hence the equation holds.

Theorem 1.8

(The Cauchy integral fomula) Suppose that U is an open set in $\mathbb{C}$ and that f is a holomorphic function on U. Let $z_0 \in U$ and let $r > 0$ be such that $\overline{D}(z_0, r) \subset U$. Let $\gamma : [0, 1] \to \mathbb{C}$ be the C^1 curve $\gamma(t) = z_0 + r\cos(2\pi t) + ir\sin(2\pi t)$. Then for each $z \in D(z_0, r)$,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi$$

♡

Proof By theorem 1.7, there is H such that

$$\frac{\partial}{\partial z}H = \frac{f(\xi) - f(z)}{\xi - z}$$

if $\xi \neq z$ and $\frac{\partial}{\partial z}H(z) = f'(z)$ holomorphic on $D(z_0, r + \epsilon)$ and hence

$$\int_\gamma \frac{f(\xi) - f(z)}{\xi - z}d\xi = 0$$

and the equation holds by the lemma 1.3.

Theorem 1.9

(The Cauchy integral theorem) If f is a holomorphic function on an open disc U in the complex plane, and if $\gamma : [a, b] \to U$ is a C^1 curve in U with $\gamma(a) = \gamma(b)$, then

$$\int_\gamma f(z)dz = 0$$

♡

Proof Only need to pick G such that $\frac{\partial}{\partial z}G = f$ on U is fine.

Definition 1.12

A piecewise C^1 curve $\gamma : [a, b] \to \mathbb{C}, a < b, a, b \in \mathbb{R}$ is a continuous function such that there exists a finite set of numbers $a_1 \leq a_2 \leq \cdots \leq a_k$ satisfying $a_1 = a$ and $a_k = b$ and with the property that for every $1 \leq j \leq k - 1$,

$\gamma|_{[a_j,a_{j+1}]}$ *is a* C^1 *curve. As before,* γ *is a piecewise* C^1 *curve in an open set* U *if* $\gamma_{[a,b]} \subset U$. ♣

Definition 1.13

If $U \subset \mathbb{C}$ *is open and* $\gamma : [a,b] \to U$ *is a piecewise* C^1 *curve in* U *and if* $f : U \to \mathbb{C}$ *is a continuous, complex-valued function on* U*, then*

$$\int_\gamma f(z)dz = \sum_{j=1}^{k} \int_{\gamma|_{[a_j,a_{j+1}]}} f(z)dz$$

and the definition is well-defined. ♣

Proof

We need to show for any $\{a_j\}_1^k, \{b_i\}_1^m$, the RHS determined by the chosen sequence is the same. Assume $a_{j_t} = b_{i_t}, 1 \le t \le q$, with $\{a_j\}_{j_t+1}^{j_{t+1}-1} \cap \{b_i\}_{i_t+1}^{j_{i+1}-1} = \emptyset$, then we know $\gamma|_{a_{j_t},a_{j_{t+1}}}$ is a C_1 curve and hence the integral over the curve is the same.

Lemma 1.4

Let $\gamma : [a,b] \to U$ *open in* $\mathbb{C}$ *to be a piece wise* C^1 *curve. Let* $\phi : [c,d] \to [a.b]$ *be a piecewise* C^1 *strictly monotone increasing function with* $\phi(c) = a, \phi(d) = b$*. Let* $f : U \to \mathbb{C}$ *be a continuous function on* U*. Then the function* $\gamma \circ \phi : [c,d] \to U$ *is a piecewise* C^1 *curve and*

$$\int_\gamma f(z)dz = \int_{\gamma\circ\phi} f(z)dz$$

♡

Proof Use the proposition 1.11.

Lemma 1.5

If $f : U \to \mathbb{C}$ *is a holomorphic function and if* $\gamma : [a,b] \to U$ *is a piecewise* C^1 *curve, then*

$$f(\gamma(b)) - f(\gamma(a)) = \int_\gamma f' dz$$

♡

Proof Use the proposition 1.7.

Proposition 1.12

If $f : \mathbb{C} - \{0\} \to \mathbb{C}$ *is a holomorphic function, and if* γ_r *describes the circle of radius* r *around* 0*, traversed once around counter-clockwise, then, for any two positive numbers* $r_1 < r_2$*,*

$$\int_{\gamma_{r_1}} f(z)dz = \int_{\gamma_{r_2}} f(z)dz$$

♠

Proposition 1.13

Let $0 < r < R < \infty$ *and define the annulus* $\mathcal{A} = \{z \in \mathbb{C} : r < |z| < R\}$*. Let* $f; \mathcal{A} \to \mathbb{C}$ *be a holomorphic function. If* $r < r_1 < r_2 < R$ *and if for each* j *the curve* γ_{r_j} *describes the circle pf radius* r_j *around* 0*, traversed once counter clockwise, then we have*

$$\int_{\gamma_{r_1}} f dz = \int_{\gamma_{r_2}} f dz$$

♠

Applications of the Cauchy integral

Theorem 1.10

Let $U \subset \mathbb{C}$ be an open set and let f be holomorphic on U. Then $f \in C^\infty(U)$. Moreover, if $\overline{D}(P,r) \subset U$ and $z \in D(P,r)$, then

$$\Big(\frac{\partial}{\partial z}\Big)^k f(z) = \frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+1}}d\xi$$

for any integer k.

Proof

Use the induction to f, assume

$$(\frac{\partial}{\partial z})^k f(z) = \frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+1}}d\xi$$

and $(\frac{\partial}{\partial z})^k f(z)$ is holomorphic, then we gonna prove that

$$(\frac{\partial}{\partial z})^{k+1} f(z) = \frac{(k+1)!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+2}}d\xi$$

and $(\frac{\partial}{\partial z})^{k+1} f(z)$ is holomorphic. Consider

$$\begin{aligned}\Big|\frac{f(\xi)}{(\xi-\omega)^{k+1}} - \frac{f(\xi)}{(\xi-z)^{k+1}}\Big| &\leq \sup_{\xi\in\partial D(P,r)}|f(\xi)|\epsilon^{-2k-2}\Big|\sum_{i=1}^{k+1}C_{k+1}^i(2r)^{k+1-i}(\omega-z)^i\Big| \\ &\leq |\omega-z|(k+1)\Big(\sup_{\xi\in\partial D(P,r)}|f(\xi)|\epsilon^{-2k-2}\Big|\sum_{i=0}^{k}C_k^i(2r)^{k-i}(\omega-z)^i\Big|\Big) \\ &\leq |\omega-z|(k+1)\Big(\sup_{\xi\in\partial D(P,r)}|f(\xi)|\epsilon^{-2k-2}(2r+1)^k\Big)\end{aligned}$$

for all $|\omega - z|$ small enough and hence

$$\frac{f(\xi)}{(\xi-\omega)^{k+1}} \to \frac{f(\xi)}{(\xi-z)^{k+1}}$$

uniformly when $\omega \to z$, so may know

$$\lim_{\omega\to z}\frac{(\frac{\partial}{\partial z})^{k+1}f(\omega) - (\frac{\partial}{\partial z})^{k+1}f(z)}{\omega - z} = \lim_{\omega\to z}\frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{\frac{f(\xi)}{(\xi-\omega)^{k+1}} - \frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega-z}d\xi$$

and we know that

$$\lim_{\omega\to z}\frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{\frac{f(\xi)}{(\xi-z)^{k+1}} - \frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega-z}d\xi = \frac{k!}{2\pi i}\int_{|\xi-P|=r}\lim_{\omega\to z}\frac{\frac{f(\xi)}{(\xi-\omega)^{k+1}} - \frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega-z}d\xi$$

by the DCT and hence

$$\lim_{\omega\to z}\frac{(\frac{\partial}{\partial z})^{k+1}f(\omega) - (\frac{\partial}{\partial z})^{k+1}f(z)}{\omega - z} = \frac{(k+1)!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+2}}d\xi$$

which means $(\frac{\partial}{\partial z})^k f(z)$ is holomorphic and the equality holds. Then we use the induction, and the conclusion goes.

Corollary 1.3

If $f : U \to \mathbb{C}$ is holomorphic, then $f' : U \to \mathbb{C}$ is holomorphic.

Theorem 1.11

If ϕ is a continuous function on $\{\xi : |\xi - P| = r\}$, then the function f given by

$$f(z) = \frac{1}{2\pi i}\int_{|\xi - P| = r} \frac{\phi(\xi)}{\xi - z} d\xi$$

is defined and holomorphic on $D(P, r)$.

♡

Theorem 1.12

(Morera) Suppose that $f : U \to \mathbb{C}$ is a continuous function on a connected open subset U of $\mathbb{C}$. Assume that for every closed, piecewise C^1 curve $\gamma : [0, 1] \to U$, $\gamma(0) = \gamma(1)$, it holds that

$$\int_\gamma f(\xi) d\xi = 0$$

Then f is holomorphic on U.

♡

Proof Consider $x \in U$ and define $F(y) = \int_\phi f dz$ for any $y \in U$ where ϕ is a picewise C^1 curve from x to y, where we know the integral is well-defined since any integral of f on a closed, piece wise C^1 curve is 0. Then for any $y \in U$, consider a segment from $y + h$ where $|h|$ is small enough and we know

$$\lim_{|h|\to 0} \frac{F(y + h) - F(y)}{h} = \lim_{|h|\to 0} \frac{1}{h}\int_0^h f(y + z)dz = f(y)$$

which means F is holomorphic on U and $F' = f$ on U, and hence f is holomorphic on U.

Definition 1.14

let $P \in \mathbb{C}$ be fixed. A complex power series centered at P is an expression of the form

$$\sum a_k(z - P)^k$$

where a_k is complex valued.

♣

Lemma 1.6

(Abel) If $\sum a_k(z - P)^k$ converges at some z, then the series converges at each $\omega \in D(P, r)$, where $r = |z - P|$.

♡

Proof

Since $\sum a_k(z - P)^k$ converges, we know $a_k(z - P)^k \to 0$ and hence bounded, then we know

$$|a_k| \le Mr^{-k}$$

for some $M > 0$ and then for any $\omega \in D(P, r)$, assume $|\omega - P| = \delta < r$, then we know

$$|a_k(\omega - P)^k| \le |a_k|\delta^k \le M(\delta/r)^{-k}$$

and hence

$$\sum |a_k(\omega - P)^k| \le M \sum (\delta/r)^{-k} < \infty$$

which means $\sum a_k(\omega - P)^k$ converges.

Definition 1.15

Let $\sum a_k(z - P)^k$ be a power series. Then

$$r = \sup\{|\omega - P| : \sum a_k(\omega - P)^k \text{ converges}\}$$

is called the radius of convergence of the power series.

♣

Lemma 1.7

If $\sum a_k(z - P)^k$ is a power series with radius of convergence r, then the series converges for each $\omega \in D(P, r)$ and diverges for each ω such that $|\omega - P| > r$.

♡

Lemma 1.8

(The root test) The radius of convergence of the power series $\sum a_k(z-P)^k$ *is*

$$\frac{1}{\limsup |a_k|^{1/k}}$$

if $\limsup |a_k|^{1/k} > 0$ *or*

$$\infty$$

if $\limsup |a_k|^{1/k} = 0$.

Proof

Assume $\alpha = \limsup |a_k|^{1/k}$, if $|\omega - P| > 1/\alpha$, then denote $|\omega - P| = c/\alpha, c > 1$ and we know

$$|a_k(z-P)^k| = (c|a_k|^{1/k}/\alpha)^k$$

and we know there are infinitly many a_k such that $|a_k|^{1/k}/\alpha > 1/c$ and hence the series diverge.

For $|\omega - P| < 1/\alpha$, we denote $|\omega - P| = d/\alpha, d < 1 - \epsilon$ for some $\epsilon > 0$ and we have

$$|a_k(\omega - P)^k| \leq (|a_k|^{1/k} d/\alpha)^k \leq (1-\epsilon)^k$$

when k is sufficiently large and hence the series is absolutely convergent and the condition for $\alpha = 0$ is similar.

Definition 1.16

Let $\sum f_k(z)$ *be a series of functions on a set* E. *The series is said to be uniformly Cauchy if for any* $\epsilon > 0$, *these is an integer* N *such that*

$$|\sum_{k=m}^{n} f_k(z)| < \epsilon$$

on E *for any* $n \geq m \geq N$.

Proposition 1.14

Let $\sum a_k(z-P)^k$ *be a power series with radius of convergence* r. *Then, for any number* R *with* $0 \leq R < r$ *,the series* $\sum |a_k(z-P)|^k$ *converges uniformly on* $\overline{D}(P,R)$ *and hence* $\sum a_k(z-P)^k$ *converges uniformly and absolutely on* $\overline{D}(P,R)$.

Proof