Chapter 1

dyn.sys. for dynamical system.
mrb. for measurable.
topo. for topological.
m.p. for measure-preserving.

Definition 1.1

A measurable space $(X, \mathcal{M})$ is a set with a σ-algebra $\mathcal{M}$.
A measurable/topological dynamical system is a mrb./topo. space X and a mrb./continuous function f.
A system is measure-preserving if there is a measure μ on X s.t. for any set U, $\mu(f^{-1}(U)) = \mu(U)$, then the data (X, f, μ) is a m.p.s and in particular a p.m.p.s if $\mu(X) = 1$. ♣

Definition 1.2

If G is a topological group, then G is a topo. space and a group as well, where group multiplication and inversion is continuous.
A measure μ on G is translation-invariant if $\mu(gA) = \mu(A)$ for any mrb. subset A and $g \in G$. ♣

Proposition 1.1

L measure is the only translation invariant measure on $\mathbb{R}^n$ and $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. ♠

Proof

It suffices to show that if μ is translation invariant on $\mathbb{R}^n$, then it is a L measure. Assume $\mu([0,1]^n)$ is $a < \infty$ and then we may know all single point set has 0 measure, and then we may know $\mu([0, \frac{1}{k}]^n) = k^{-n}a$ and we may know for any rectangle R with rational vertices, $\mu(R) = am(R)$ and hence $\mu = am$ for all rectangles and then we know $\mu = am$.

It is similar when replacing $\mathbb{R}^n$ by $\mathbb{T}^n$.

Example 1.1 Here are some examples of m.p.s.s.

a. Circle rotations, i.e. $f(x) = x + \alpha (mod\, 1), \alpha \in \mathbb{R}$.
b.Translations on tori, i.e. $f(x_1, \cdots, x_n) = (x_1 + \alpha_1, \cdots, x_n + \alpha_n)(mod\, 1), \alpha_i \in \mathbb{R}, 1 \le i \le n$.
c.Translations on $\mathbb{R}^n$, i.e. $f(x) = x + v, v \in \mathbb{R}^n$.
d. Circle doubling, e.g. $f(x) = 2x(mod\, 1)$.
e. Toral Automorphisms, $A \in GL_{n\times n}(\mathbb{Z})$.
f. Linear maps of $\mathbb{R}^n$ with determinant 1.

Proof

We may skip the proof for a,b,c and consider a set $\{A, m(f^{-1}(A)) = m(A), A \subset \mathbb{R}^n\}$ which is apparently a σ-algebra and hence all the Borel sets since all rectangles are in it. We left the proof of e below.

To prove e. we need a lemma.

Definition 1.3

For $f : X \to Y$ mrb.m the $push forward$ measure of a measure μ on X is defined by $f_*\mu(U) = \mu(f^{-1}(U))$. ♣

Lemma 1.1

A measure ν on Y is $f_*\mu$ iff for any $g \in L^1(Y, f_*\mu)$

$$\int_X g \circ f d\mu = \int_Y g d\nu$$
♡

Proof

To show the sufficiency, consider

$$\int_Y \chi_U d(f_*\mu) = \mu(f^{-1}(U)) = \int_X \chi_U \circ f d\mu$$

for any mrb. set U on Y, and hence the equation holds for any simple function, and hence for all $g \in L^1(Y, f_*\mu)$ by DCT.

To show the necessity, considering any characteristic function is fine.

Corollary 1.1

If $A \in M_{n\times n}(\mathbb{Z})$ has nonzero determinant and $f : \mathbb{T}^n \to \mathbb{T}^n$ is the map it induces on the torus, then f preserves L measure. ♡

Proof

It suffices to show $m = f_*m$. Notice

$$f_*m(U+v) = m(f^{-1}(U+v)) = f_*m(U)$$

and hence $f_*m = am$ for some $a \in \mathbb{R}$. Then it is easy to check $f_*m = m$ by consider $\mu(\mathbb{T}^n) = \mu(f^{-1}(\mathbb{T}^n))$.

Definition 1.4

For a mrb. dyn.sys (X, f) and a mrb. set U, a point p in U recurs to U if it returns to U i.o. and for a topo. dyn.sys (X, f), call p recurrent if p recurs to any open set containing it. ♣

Theorem 1.1

(Measurable Poincare Recurrence) If (X, f, μ) is a p.m.ps. and U is a mrb. set, then p recurs to U a.s. on U. ♡

Proof

Consider $B = \{\text{the set of points in} U \text{ never come back}\}$, then we know $B = \bigcap_{n=0}^{\infty} E_n$ where $E_n = f^{-n}(U^c)$ and hence B is mrb., and it is easy to check $f^m(B) \cap f^n(B) = \emptyset, n \neq m$ and hence $P(B) = 0$, which means $P(\bigcup_{n\geq 0} f^{-n}B) = 0$.

Theorem 1.2

(Topological Poincare Recurrence)A point is a.s. recurrent in a second countable topological p.m.p.s. ♡

Proof

By Poincare Recurrence, we may find a countable open cover of X, then the conclusion goes.

Definition 1.5

We say a m.p.s. (X, T, μ) is ergodic if the only T-invariant measurable sets, i.e. a mrb. set A is T- invariant means $T^{-1}(A) = A$ are null or conull, which is equalivalent to the almost T-invariant mrb. sets are null or conull. ♣

Proof

The necessity is trivial, to see the sufficiency, consider U is an almost T-invariant set, then we assume $A' = \bigcap_{N\geq 0} \bigcup_{n\geq N} T^{-n}(A)$, we know

$$T^{-1}(A') = T^{-1}(A') = \bigcap_{N\geq 0} \bigcup_{n\geq N} T^{-(n+1)}(A) = A'$$

and hence A' is null or conull. Then it is easy to check $\mu(A \triangle T^{-k}(A)) = 0$ for any integer k and hence $A \triangle A'$ is null, which means A is null or conull.

Lemma 1.2

A m.p.s. (X, T, μ) is ergodic iff for any two positive measure sets A and B there is some n so that $T^{-n}(A) \cap B$ has positive measure. ♡

Proof

To see the sufficiency, if there are two positive measure sets A, B such that $T^{-n}(A) \cap B = 0$ for any interger n, then we know $\mu(A), \mu(B) \in (0,1)$. And we know $\bigcup_{n\geq 0} T^{-n}(A)$ is almost T-invariant and hence is null or conull, which is a contradiction.

To see the necessity, we consider a positive measure T-invariant mrb. set A, then we know $\bigcup_{n\geq 0} T^{-n}(A) \cap A^c$ is the emptyset and hence A^c is a null set.

Lemma 1.3

Let $T : \mathbb{T}^n \to \mathbb{T}^n$ be given by $T(x) = x + v$ where $v \in \mathbb{T}^n$. If $\{mv\}_{m\geq 0}$ is dense in $\mathbb{T}^n$, then T is ergodic with respect to Lebesgue measure. ♡

Proof

By the LRN theorem, there exists $\epsilon > 0$ such that $m(B(a,\epsilon) \cap A)/m(A), m(B(b,\epsilon) \cap B) > 0.9$ for some $a \in A, b \in B$. Then we know there exists $m \geq 0$ such that $m(T^m(B(a,\epsilon)) \cap B(b,\epsilon))/B(a,\epsilon) > 0.99$ and hence $m(T^m(A \cap B(a,\epsilon))) > 0.89m(T^m(B(a,\epsilon)) \cap B(a,\epsilon)), m(B \cap B(b,\epsilon)) > 0.89m(T^m(B(a,\epsilon)), B(b,\epsilon))$ and hence $m(T^m(A) \cap B) \geq 0.5m(B(a,\epsilon)) > 0$. Then by Lemma.6, the conclusion goes.

Remark $\{mv\}_{m\geq 0}$ is dense in $\mathbb{T}^n$ iff the smallest closed subgroup of $\mathbb{T}^n$ containing v is $\mathbb{T}^n$ itself.

Proof

(?)Only need to show $\overline{\{mv\}_{m\geq 0}}$ is a subgroup.

Definition 1.6

A function is T-invariant if $f \circ T = f$. ♣

Lemma 1.4

A m.p.s. (X, T, μ) is ergodic , then any a.e. bounded mrb. T-invariant function is constant.
If any $T-invariant$ simple function has to be constant a.e., then (X, T, μ) is ergodic. ♡

Proof

(Change to finite a.e.?)We know $T(A) \subset A, T(A^c) \subset A^c$ and hence $\chi_A \circ T = \chi_A$ and hence χ_A is 0 or 1 a.e. and the necessity goes.

To see the sufficiency, for any f mrb. and T-invariant then we know $\{f \leq c\}$ is T-invariant and hence null or conull for any $c \in \mathbb{R}$. Notice $\bigcup_{q\in\mathbb{Q}}\{f \leq q\}$ is X and $\bigcup_{q\in\mathbb{Q}}\{f \leq q\}$ is null and then e may find $\sup\{q \in \mathbb{Q}, \{f \leq q\} \text{ null}\} = \inf\{\{f \leq q\} \text{ conull}\} = a$ and hence $\{f = a\}$ is conull.

Lemma 1.5

Let T be the action induced by $A \in GL_{n\times n}(\mathbb{Z}) : \mathbb{T}^n \to \mathbb{T}^n$. Then T is ergodic iff A does not have a root of unity as an eigenvalue. ♡

Proof

Skip temporarily.

The Birkhoff Ergodic Theorem

Definition 1.7

Given a p.m.p.s. (X, T, μ) and $f : X \to \mathbb{R}$ a function in L^1, set $S_0(f) := 0$,

$$S_n(f) := \sum k = 0^{n-1} f(T^k) \quad \text{and} \quad Av_n(f) := \frac{S_n(f)}{n}$$

♣

Theorem 1.3

(The Maximal Ergodic Theorem) For $\alpha \in \mathbb{R}$, let E_α be the points in X so that $Av_n(f) > \alpha$ for some n. Then $\alpha\mu(E_\alpha) \leq \int_{E_\alpha} f$.

Proof

Assume $\alpha = 0$, then let $M_n(f) = \max_{0 \leq k \leq n}(S_n(f))$ and $P_n = \{x, M_n(f)(x) > 0\}$, and notice

$$M_n(f) \circ T \geq S_k(f) \circ T + f = S_{k+1}(f)$$

for $0 \leq k \leq n$, so notice $M_n \geq 0$ and $M_n = 0$ on $X - P_n$, we have

$$\int_{P_n} f \geq \int_{P_n} M_n(f) d\mu - \int_{P_n} \circ d\mu \geq \int_X M_n(f) d\mu - \int_X M_n(f) \circ T d\mu = 0$$

and since $E_0 = \bigcup_{n \geq 0} P_n$, so $\int_{E_0} f d\mu = \lim \int f\chi_{P_n} d\mu \geq 0$ by DCT.

Then we may replace f by $f - \alpha$ to obtain the required general conclusion.

Theorem 1.4

(The Birkhoff Ergodic Theorem) If $f^*(x) = \limsup_n Av_n(f)$ and $f_*(x) = \liminf_n Av_n(f)$, then $f_* = f^*$, these functions are T-invariant and $\int f^* = \int f$. In particular, if (X, μ, T) is ergodic, $Av_n(f)$ converges pointwise a.e. to $\int f$.

Proof

Notice

$$\frac{n}{n+1} Av_n(f)(T(x)) + \frac{1}{n+1} f(x) = Av_{n+1}(f)(x)$$

and hence $f^* \circ T = f^*, f_* \circ T = f_*$.

Then for rational p, q, let $E(p,q) = \{x, f_*(x) \leq p| < q \leq f^*(x)\}$ we know

$$q\mu(E(p,q)) \leq \int_{E(p,q)} f \leq p\mu(E(p,q))$$

and hence $E(p,q) = 0$. So $f^* = f_*$ a.s. and hence $Av_n(f)$ converges to f^* a.s.

We know $\int f^* = \int f$ when f is bounded since

$$\int f^* d\mu = \lim \int_X Av_n(f) d\mu = \int_X f\mu$$

and for f unbounded, we may find $g_n \to f$ uniformly and we may find $||g_n - f||_1 < \frac{\epsilon}{3}$, we have already know $Av_k(g_n)$ converges a.s. and hence in L^1 and then we have

$$||Av_k(f) - Av_m(f)||_1 < \epsilon$$

for k, m big enough and hence $Av_k(f)$ is Cauchy in L^1 and hence it is convergent to f^* in measure.

Then we may use the Lemma 1.4, we know f^* is constant and hence $Av_n(f)$ converges to $f^* = \int f^* = \int f$ a.s.

The Riesz Representation Theorem

Lemma 1.6

Supposre that X is a compact metric space. If K is a closed subset and μ is a finite measure, then

$$\mu(K) = \inf\{\int_X f d\mu . \chi_K \leq f \in C(X)\}$$

Proof

It is easy to show that if $f \in C(X)$,then f is bounded on X a.e. and hence $\mu(K) \leq f d\mu$ for any $f \geq \chi_K$ continuous.

Then we may use the Urysohn's Lemma to complete the proof.

Lemma 1.7

If X is a compact metric space, then $\mathcal{M}(X)$ injects into $C(X)^*$.

Proof

If $||f-g||_u < \epsilon$, then we know

$$|\mu(f)-\mu(g)| \leq \epsilon\mu(X)$$

which means μ is continuous as a map from $C(X)$ to $\mathbb{R}$, the linerity of μ is obviously and the injective is secured by lemma 1.6., which means if $\mu = \nu$ as a bounded linear map of $C(X)$, then $\mu = \nu$ on all campact sets and hence the problem goes.

Lemma 1.8

If X is a compact metric space, then there is a continuous surjection from the Cantor set to X.

Proof

Consider we may find a 2^{q_1} cover of X, then we can find a 2^{q_2} cover of each balls of the first cover and repeat, then we may consider there will be a natural continuous map from $\{0,1\}^{\mathbb{N}}$ to $\prod_{i\geq 0}\{1,2,\cdots,2^{q_i}\}$ which determine a singleton and any point in X can be represented like this.

Definition 1.8

We call a functional $\mu : C(X) \to \mathbb{R}$ is positive if $\mu(f) \geq 0$ for any $f : X \to (0,\infty)$. This forms a cone, i.e. a subset of v.s. closed under addition and positive scalar multiplication.

Lemma 1.9

Suppose that X is the Cantor set. Then the cone of positive linear functionals in $C(X)^*$ can be identified with $\mathcal{M}(X)$.

Proof

Let $\phi \in C(X)^*$ be a positive linear functional. Then consider $\mathcal{B}$ is the finite union of subsets with the first n positions are the same for some integer n, we can check any subsets of $\{B\}$ is open and closed at the same time and hence $\{\chi_B\}_{B\in\mathcal{B}}$ are continuous and also $\mathcal{B}$ is an algebra, which is easy to check that ϕ is σ-additive on $\mathcal{B}$ and hence it determine a measure μ on X. Then ϕ and μ agreee on a dense set of $C(X)$ and hence they are the same.

Lemma 1.10

A nonzero linear functional $\mu \in C(X)^*$ is positive iff $\mu(\chi_X) = ||\mu||$.

Proof

Firstly, notice $\mu(fg)$ defines a nonnegative semidefinite bilinear form For any $f \in C(X)$,

$$|\mu(f)|^2 = |\mu(f\cdot\chi_X)|^2 \leq \mu(f^2)\mu(\chi_X) \leq \mu(||f||^2\chi_X)\mu(\chi_X) = ||f||^2\mu(\chi_X)^2$$

and hence $||\mu|| \leq \mu(\chi_X)$. And the equality holds when $f = \chi_X$.

For any $f : X \to [a,1], a > 0$, we have

$$\mu(f) - \frac{1+m}{2} = |\mu(f) - \mu(\frac{1+m}{2}\chi_X)\mu(\chi_X)| \leq ||f - \frac{1+m}{2}||\mu(\chi_X) \leq \frac{1-m}{2}\mu(\chi_X)$$

and hence $\mu(f) \in [m,1]\mu(\chi_X)$, which means μ is positive.

Theorem 1.5

(Riesz Representation Theorem) If X is any compact metric space, the nthe cone of postive linear functionals in $C(X)^*$ can be identified with $\mathcal{M}(X)$.

Proof

Let C be the Cantor set and $p : C \to X$ a continuous surjection. $C(X) \to C(C)$ given by $p^*(f) = f \circ p$ is a linear isometry. Let $\phi \in C(X)^*$ be a postive linear functional and $p^*\phi(f \circ p) = \phi(f) \leq ||\phi|| ||f||$ and hence is can be extended to a functional $\varphi : C(C) \to \mathbb{R}$ with the same norm. By the lemma 1.6. and 1.7., the extension is positive and can be given by integration against a measure μ on C. Then

$$\phi(f) = \varphi(p \circ f) = \int_C f(p) d\mu = \int_X f dp_* \mu$$

Theorem 1.6

(Banach-Alaoglu) The set of contracting functionals in $C(X)^*$ is compact in the weak topology.* ♡

Corollary 1.2

$\mathcal{M}^1(X)$ is compact and convex in the weak topology.* ♡

Proposition 1.2

For $g : X \to X$, $g^* \mathcal{M}^1(X) \to \mathcal{M}^1(X)$ is cotinuous. ♠

Proof

For any $\mu_n \to \mu$ in the weak* topology, we know

$$g^* \mu_n(f) = \mu_n(f \circ g) \to \mu(f \circ g) = g^* \mu(f)$$

for any $f \in C(x)$.

Definition 1.9

A topological semigroup is a group G together with a topology so that multiplication $m : G \times G \to G$ given by $m(g, h) = gh$ is continuous.

A semigroup G is amenable if every continuous action of G on a compact metric space X admits a G-invariant measure. ♣

Lemma 1.11

(Markov-Kakutani) If G is abelian then it is amenable. ♡

Proof

Let G act continuously on a compact metric space X. Set $\mathcal{M} = \mathcal{M}^1(X)$ and then let $A_{n,g}(\mu) = \frac{1}{n} \sum\limits_{i=1}^{n} (g^i)_* \mu$. Let S be the set of finite composition of $\{A_n, g\}$. This ia an abelian semigroup since G is abelian. Notice $g(\mathcal{M})$ is a closed set for each $g \in S$, then for finite elements in S, the intersection of their images are nonempty and hence $\bigcap_{s \in S} s(\mathcal{M})$ is nonempty, consider $\mu \in \bigcap_{s \in S} s(\mathcal{M})$, then for any $n \in \mathbb{N}, g \in G$, we have

$$||\mu - g_* \mu|| = \tfrac{1}{n} || \sum_{i=1}^{n} (g^i)_* \mu_n - \sum_{i=2}^{n+1} (g^i)_8 \mu_n || \leq \frac{1}{n}$$

and hence μ is a G-invariant measure.

Corollary 1.3

(Krylov-Bogolyubov) If X is a compact metric space and $T : X \to X$ is continuous then there is an T-invariant measure. ♡

Definition 1.10

(Haar measure) A left-invariant Borel measure on a topological group. ♣

Corollary 1.4

Compact groups are amenable. ♡

Proof

Let G act on a compact space X (where we assume the action is continuous, i.e. $G \times X \to X$ is continuous) and μ the Haar measure on X, then let $\phi(g) = gx$ where $x \in X$ and then for any $A \subset X$ and let $\nu = \phi_* \mu$

$$\nu(g^{-1}A) = \mu(\phi^{-1}g^{-1}A) = \mu(g^{-1}\phi^{-1}A) = \mu(\phi^{-1}A) = \nu(A)$$

Definition 1.11

The compact – open topology on the space $C(X,X)$ of continuous self-maps of X is that of uniform convergence, i.e. the one metrized by

$$d_{C(X,X)}(f,g) := \sup_{x \in X} d_X(f(x), g(x))$$

♣

Lemma 1.12

If G acts continuously on X, the the homomorphism $G \to C(X,X)$ is continuous.

♡

Proof

Fix $\epsilon > 0, h \in G$, for each x there exists an U_x, W_x open in G, X such that $W_x H_x \subset B(hx, \epsilon/2)$, we may find a finte collection of x_i such that $X = \bigcup_i U_{x_i}$ and then if $g \in \bigcap_i W_{x_i}$, we will know for any $x \in X$ we have

$$|g(x) - h(x)| \le |g(x) - h(x_i)| + |h(x_i) - h(x)| < \epsilon$$

Definition 1.12

For a topological semigroup G acting continuously on a metric space X, each element $g \in G$ defines a linear contraction $g^* : C(X) \to C(X)$ where $g^*(f) = f \circ g$. This is homomorphism from the opposite semigroup G^{op} to $B(C(X))$.

♣

Proof

We know $g_* h_*(f) = f \circ h \circ g = (hg)_*(f)$.

Lemma 1.13

The homomorphism from G^{op} to $B(C(X))$ that sends g to g^* is continuous. Moreover, G acts continuously on $\mathcal{M}^1(X)$.

♡

Lemma 1.14

Let G be a compact group. If G acts continuously and transitively on a Hausdorff space X with point stabilizer H, then X is homeomorphic to G/H.

In factm the conclusion hols when G is a locally compact Hausdorff group that is σ-compact and X is a locally compact Hausdorff space.

♡

Proof

If H stabilize $x \in X$, then $\phi : G/H \to X, \phi(g) = gx$, then ϕ is a continuous surjection and injective. Then notice a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, which can be shown by thinking a closed set has to be mapped to a closed set.

Definition 1.13

$Gr_d(\mathbb{R}^n)$ is the Grassmannian of d-dimensional subspaces in $\mathbb{R}^n$. If V_n is a sequence of subspaces, then we say that $V_n \to V$ if a basis of V_n converges to a basis of V.

♣

Corollary 1.5

Since $O(n)$ is compact and acts transitively on the Grassmannian, $Gr_d(\mathbb{R}^n)$ is compact for all d and n. In particular, it is homeomorphic to $O(n)/O(d) \times O(n-d)$.

♡

Lemma 1.15

Suppose that (g_m) is a sequence of matrices in $SL(n, \mathbb{R})$ with unbounded entries. Suppose that for $\mu, \nu \in \mathcal{M}^1(\mathbb{P}(\mathbb{R}^n))$, $(g_m)_*\mu$ weak converges to ν. Then there are proper subspaces $\mathcal{R}$ and V of $\mathbb{R}^n$ so that ν is supported on $\mathbb{P}(\mathcal{R}) \cup \mathbb{P}(V)$.*

♡

Proof Assume $g_m/||g_m||_\infty$ converges to a matrix g elementwise, then we know $\det g = 0$, then let $N, \mathcal{R}$ be the null and the range of g respectively, then $g_m \cdot N$ will converge for some subsequence to a subspace V. If l is a line in $\mathbb{R}^n$, then $g_m l$ converges to a line in V or $\mathcal{R}$, and hences l is in N or not in N. Let $\mu_1(A) = \mu(A \cap \mathcal{P}(N))$ and $\mu_2(A) = \mu(A - \mathcal{P}(N))$. Then we may know $(g_m)_*\mu_i$ weak*ly converge to ν_i and ν_1, ν_2 are supported on $\mathcal{P}, \mathcal{R}$.

Definition 1.14

For a seq $\{a_n\}$ real and we call

$$A_n = \frac{1}{n}\sum_{i=1}^{n} a_i$$

as its $Cesaro\ averages$.
We call the seq Cesaro converges to L iff

$$\lim_{n\to\infty} A_n = L$$

and Cesaro absolutely converges to L iff

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |a_i - L| = 0$$

♣

Lemma 1.16

Suppose that (X, T, μ) is a p.m.p.s. THe system is ergodic if and only if for any $f, g \in L^2$, $\int f(T^n x) g(x) d\mu$ Cesaro-converges to $\int_X \int_X g$.

♡

Proof