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# NOTES FOR ABSTRACT ALGEBRA

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# Contents

# 1 Rings and Ideals

## 1.1 Rings

### Definiton 1.1.1. (Ring)

A ring  $R$  is an abelian group with an associative multiplication distributive over the addition. (We always assume a ring has a multiplicative identity and commutative if not marked)

A unit is an element  $u$  with a reciprocal  $1/u$  such that  $u \cdot 1/u = 1$ , which is also denoted  $u^{-1}$  and called a numtiplicative inverse and the units form a multiplicative group, denoted  $R^\times$ .

### Definiton 1.1.2. (Homomorphism)

A ring homomorphism is a ring map  $\phi : R \rightarrow R'$  which preserving sums, products and 1. If  $R' = R$  we call  $\phi$  an endomorphism and if it is also bijective we call it an automorphism.

### Definiton 1.1.3. (Subring)

A subset  $R'' \subset R$  is a buting if  $R''$  is a ring and the inclusion  $R'' \hookrightarrow R$  is a ring map. We call  $R$  a extension of  $R''$  and the inclusion an extension.

### Definiton 1.1.4. (Algebra)

An  $R$ -algebra is a ring  $R'$  that comes equipped with a ring homomorphism  $\phi : R \rightarrow R'$  called the structure map. An  $R$ -algebta homormorphism  $R' \rightarrow R''$  is a ring homomorphism between  $R$ -algebtas compatible with structure maps.

### Definiton 1.1.5. (Group action)

A group  $G$  is said to act on  $R$  if there is a homomorphism given from  $G$  into the group of automorphisms of  $R$ . The ring of invariants  $R^G$  is the subring defined by

$$R^G := \{x \in R \mid gx = x \text{ for all } g \in G\}$$

### Definiton 1.1.6. (Boolean)

A ring  $B$  is called Boolean if  $f^2 = f$  for all  $f \in B$ , then  $2f = 0$  since

$$2f = (f + f)^2 = 4f$$

### Definiton 1.1.7. (Polynomial rings)

Let  $R$  be a ring,  $P := R[X_1, \dots, X_n]$  the polynomial ring in  $n$  variables.  $P$  has the Universal Mapping Property (UMP), i.e. given a ring homomorphism  $\phi : R \rightarrow R'$  and given an element  $x_i$  of  $R'$  for each  $i$ , there is a unique ring map  $\pi : P \rightarrow R'$  with  $\pi|_R = \phi$  and  $\pi(X_i) = x_i$ .

Similarly, let  $X := \{X_\lambda\}_{\lambda \in \Lambda}$  be any set of variables. Set  $P' := R[X]$  the elements of  $P'$  are the polynomials in any finitely many of  $X$ .

### Definiton 1.1.8. (Ideals)

Let  $R$  be a ring. An ideal  $I$  is a subset containing 0 of  $R$  such that  $xa \in I$  for any  $x \in R, a \in I$  and closed under addition.

For a subset  $S \subset R$ ,  $\langle S \rangle$  means the smallest ideal containing  $S$ .

Given a single element  $a$ , we say that the ideal  $\langle a \rangle$  is principal. For a number of ideals  $I_\lambda$ , the sum  $\sum I_\lambda$  mean the set of all finite linear combinations  $\sum x_\lambda a_\lambda$  for  $x_\lambda \in R, a_\lambda \in I_\lambda$ . If

$\Lambda$  is finite, then the product  $\prod I_\lambda$  means the ideal generated by all products  $\prod a_\lambda, a_\lambda \in I_\lambda$ .

For two ideals  $I$  and  $J$ , the transporter of  $J$  into  $I$  mean the set

$$(I : J) := \{x \in R | xJ \subset I\}$$

If  $I \subset J$  a subring such that  $I \neq J$ , then we call  $I$  proper.

For a ring homomorphism  $\phi : R \rightarrow R'$ ,  $I \subset R$  a subring, denote by  $IR'$  or  $I^e$  the ideal of  $R'$  generated by  $\phi(I)$  can we call it the extension of  $I$ .

Given an ideal  $J$  of  $R'$  and its preimage  $\phi^{-1}(J)$  is an ideal of  $R$  and we call it the contraction of  $J$  denoted with  $J^c$ .

**Definiton 1.1.9.** (Residue Rings)

Let  $I$  be an ideal of  $R$  and the cosets of  $I$

$$R/I := \{x + I | x \in R\}$$

have a ring structure and it will be called the residue ring or quotient ring or factor ring of  $R$  modulo  $I$  and the quotient map:

$$\kappa : R \rightarrow R/I, \quad \kappa(x) = x + I$$

and  $\kappa x$  is called the residue of  $x$ .

**Proposition 1.1.1.**

For  $I \subset R$  a subring and a ring homomorphism from  $R$  to  $R'$ , then  $\ker(\phi) \supset I$  implies that is a ring homomorphism  $\psi : R/I \rightarrow R'$  with  $\psi\kappa = \phi$ .

$\psi$  is surjective iff  $\phi$  is surjective.  $\psi$  is injective iff  $I = \ker(\phi)$ .

**Corollary 1.1.2.**  $R/\ker(\phi) \cong Im(\phi)$

**Proposition 1.1.3.**

$R/I$  is universal among  $R$ -algebras  $R'$  such that  $IR' = 0$ , i.e. for  $\phi : R \rightarrow R'$  such that  $\phi(I) = 0$ , there is a unique ring homomorphism  $\psi : R/I \rightarrow R'$  such that  $\psi\kappa = \phi$ .

**Definiton 1.1.10.** The UMP serves to determine  $R/I$  up to unique isomorphism, i.e. if  $R'$  equipped with  $\phi : R \rightarrow R'$  has the UMP too, then  $R'$  is isomorphic to  $R/I$ .

*Proof.*

If  $R'$  has the UMP among the  $R$ -algebras  $R''$  such that  $IR'' = 0$ , then  $\phi(I) = 0$  and hence there is a unique  $\psi : R/I \rightarrow R'$  such that  $\psi\kappa = \phi$  and since  $\kappa I = 0$ , we know there exists unique  $\psi'$  such that  $\psi'\phi = \kappa$  and then  $(\psi'\psi)\kappa = \kappa$  and hence  $\psi'\psi = 1$  and we are done by the uniqueness.

**Proposition 1.1.4.** Let  $R$  be a ring,  $P := R[X]$  the polynomial ring in one variable,  $a \in R$  and  $\pi : P \rightarrow R$  the  $R$ -algebra map define by  $\pi(X) := a$ , then

- $\ker \pi = \{F(X) \in P | F(a) = 0\} = \langle X - a \rangle$
- $P/\langle X - a \rangle \cong R$

**Definiton 1.1.11.** (Order of a polynomial)

Let  $R$  be a ring,  $P$  the polynomial ring in variables  $X_\lambda$  for  $\lambda \in \Lambda$  and  $(x_\lambda) \in R^\Lambda$  a vector. Let  $\phi_{(x_\lambda)} : P \rightarrow P$  denote the  $R$ -algebra homomorphism defined by  $\phi_{(x_\lambda)} X_\mu := X_\mu + x_\mu$ .

The order of  $F$  at the vector  $(x_\lambda)$  is defined as the smallest degree of monomials  $M$  in  $(\phi_{(x_\lambda)} F)$ .

We know  $\text{ord}_{(x_\lambda)} F = 0$  iff  $F(x_\lambda) \neq 0$ .

**Definiton 1.1.12.** Let  $R$  be a ring,  $I$  an ideal and  $\kappa$  the quotient map. Given an ideal  $J \supset I$  then the cosets

$$J/I := \{b + I | b \in J\} = \kappa(J)$$

and then  $J/I$  is an ideal of  $R/I$  and also  $J/I = J(R/I)$ .

**Proposition 1.1.5.** Given  $J \supset I$  and we know

$$\phi : R \rightarrow R/I \rightarrow (R/I)/(J/I)$$

then we have the commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & R/J \\ \downarrow & & \downarrow \cong \\ R/I & \longrightarrow & (R/I)/(J/I) \end{array}$$

*Proof.*

Since  $\phi(J) = 0$ , so there exists unique  $\psi : R/J \rightarrow (R/I)/(J/I)$  such that  $\psi\kappa_J = \phi$  and since  $\kappa_J(I) = 0$  and there exists  $p$  such that  $p\kappa_I = \kappa_J$  and consider  $p(J/I) = 0$  and there exists  $h$  such that  $h\kappa_{(J/I)} = p$  and it is easy to check  $h\psi = 1$  by uniqueness and we are done.

**Definiton 1.1.13.** Let  $R$  be a ring. Let  $e \in R$  be an idempotent, i.e.  $e^2 = e$  then  $Re$  is a ring with  $e$  as multiplication unit, but  $Re$  is not a subring unless  $e = 1$ .

Let  $e' := 1 - e$ , then  $e'$  is idempotent and  $ee' = 0$  and we call them complementary idempotents.

Denote  $\text{Idem}(R)$  the set of all idempotents, which is close under a ring homomorphism.

**Proposition 1.1.6.** If  $e_1, e_2 \in R$  such that  $e_1 + e_2 = 1$  and  $e_1 e_2 = 0$ , then they are complementary idempotents.

**Definiton 1.1.14.** Let  $R : R' \times R''$  be a product of two rings with componentwise operations.

**Proposition 1.1.7.** Let  $R$  be a ring and  $e', e''$  complementary idempotents. Set  $R' := Re'$  and  $R'' = Re''$ . Define  $\phi : R \rightarrow R' \times R''$  by  $\phi(x) = (xe', xe'')$  and then  $\phi$  is a ring isomorphism.  $R' = R/Re''$  and  $R'' = R/Re'$ .

*Proof.*

Check  $\phi$  is surjective and injective.

There is a natrual isomorphism between  $I = \{(0, xe'')\} \subset R' \times R''$  and  $R''$ , and consider the diagram

$$\begin{array}{ccc} R & \longleftarrow & R' \times R'' \\ \downarrow & & \downarrow \\ R/R'' & & R' \times R''/I \end{array}$$

and use the UMP.

## 1.2 Prime Ideals

**Definiton 1.2.1.** (Zerodivisors)

Let  $R$  be a ring. An element  $x$  is called a zerodivisor if there is a nonzero  $y$  such that  $xy = 0$ ; otherwise,  $x$  is called a nonzerodivisor. Denote the set of zerodivisors by  $\text{z.div}(R)$  and the nonzerodivisors by  $S_0$ .

**Definiton 1.2.2.** (Multiplicative subsets, prime ideals)

Let  $R$  be a ring. A subset  $S$  is called multiplicative if  $1 \in S$  and  $x, y \in S$  implies  $xy \in S$ .

An ideal  $P$  is called prime if its complement  $R - P$  is multiplicative, or equivalently, if  $1 \notin P$  and  $xy \in P$  implies  $x \in P$  or  $y \in P$ .

**Definiton 1.2.3.** (Fields, domains)

A ring is called a field if  $1 \neq 0$  and if every nonzero element is a unit.

A ring is called an integral domain, or a domain if  $\langle 0 \rangle$  or equivalently, if  $R$  is nonzero and has no nonzero zerodivisors.

Every domain  $R$  is a subring of its fraction field  $\text{Frac}(R) := \{x/y, x, y \in R \text{ and } y \neq 0\}$ .

**Proposition 1.2.1.** Any subring  $R$  of a field  $K$  is a domain, and for a domain  $R$ ,  $\text{Frac}(R)$  has the UMP: the inclusion of  $R$  into any field  $L$  extends uniquely to an inclusion of  $\text{Frac}(R)$  into  $L$ .

*Proof.*

For any subring  $R$  of a field,  $a, b \in R$ , if  $ab = 0$ , and  $a$  nonzero, then  $b = 0$  and we are done.

If  $\phi : R \hookrightarrow L$ , then  $\phi(x/y) = \phi(x)\phi(y)^{-1}$  is well-defined and obviously a ring homomorphism and we are done.

**Definiton 1.2.4.** (Polynomials over a domain)

Let  $R$  be a domain,  $X$  a set of variable.  $P := R[X]$  and then  $P$  is a domain, and  $\text{Frac}(P)$  is called the rational functions.

**Definiton 1.2.5.** (Unique factorization)

Let  $R$  be a domain,  $p$  a nonzero nonunit. We call  $p$  prime if  $p|xy$  implies  $p|x$  or  $p|y$ , which is equivalent with  $\langle p \rangle$  is prime.

For  $x, y \in R$ , we call  $d \in R$  their gcd if  $d|x$  and  $d|y$  and if  $c|x, c|y$  then  $c|d$ .

$p$  is irreducible if  $p = yz$  implies  $y$  or  $z$  is a unit. We call  $R$  is a UFG if every nonzero nonunit factors into a product of irreducibles and the factorization is unique to order and units.

**Proposition 1.2.2.** If every nonzero nonunit factors have a factorization of a product of irreducible elements, then the factorization is unique up to order and units iff every irreducible element is prime.

*Proof.*

**Lemma 1.2.3.** Let  $\phi : R \rightarrow R'$  be a ring homomorphism, and  $T \subset R'$  a subset. If  $T$  is multiplicative, then  $\phi^{-1}T$  is multiplicative; the converse holds if  $\phi$  is surjective.

*Proof.*

**Proposition 1.2.4.** Let  $\phi : R \rightarrow R'$  be a ring map, and  $J \subset R'$  an ideal. Set  $I := \phi^{-1}J$ . If  $J$  is prime, then  $I$  is prime; the converse holds if  $\phi$  is surjective.

**Corollary 1.2.5.** Let  $R$  be a ring,  $I$  an ideal. Then  $I$  is prime iff  $R/I$  is a domain.

*Proof.*

Consider

$$\kappa : R \rightarrow R/I$$

the quotient map and  $I$  prime implies  $\langle 0 \rangle$  is prime in  $R/I$  and hence  $R/I$  is a domain.

**Definiton 1.2.6.** (Maximal ideal)

Let  $R$  be a ring. An ideal  $I$  is said to be maximal if  $I$  is proper and there is no proper ideal  $J$  such that  $I \subset J, I \neq J$ .

**Proposition 1.2.6.** A ring  $R$  is a field iff  $\langle 0 \rangle$  is a maximal ideal.

**Corollary 1.2.7.** Let  $R$  be a ring,  $I$  an ideal. Then  $I$  is maximal iff  $R/I$  is a field.

*Proof.*

Only need to check  $\langle 0 \rangle$  is maximal in  $R/I$ .

**Corollary 1.2.8.** In a ring, every maximal ideal is prime.

**Definiton 1.2.7.** (Coprime)

Let  $R$  be a ring, and  $x, y \in R$ . We say  $x$  and  $y$  are coprime if their ideals  $\langle x \rangle$  and  $\langle y \rangle$  are comaximal.

$x$  and  $y$  are coprime if and only if there are  $a, b \in R$  such that  $ax + by = 1$ .

**Definiton 1.2.8.** A domain  $R$  is called a Principal Ideal Domain if every ideal is principal. A PID is a UFD.

**Theorem 1.2.9.** Let  $R$  be a PID. Let  $P := R[X]$  be the polynomial ring in one variable  $X$ , and  $I$  a nonzero prime ideal of  $P$ . Then  $P = \langle F \rangle$  with  $F$  prime, or  $P$  is maximal. Assume  $P$  is maximal. Then either  $P = \langle F \rangle$  with  $F$  prime, or  $P = \langle p, G \rangle$  with  $p \in R$  prime,  $pR = P \cap R$  and  $G \in P$  prime with image  $G' \in (R/pR)[X]$  prime.

**Theorem 1.2.10.** Every proper ideal  $I$  is contained in some maximal ideal.

**Corollary 1.2.11.** Let  $R$  be a ring,  $x \in R$ . Then  $x$  is a unit iff  $x$  belongs to no maximal ideal.

### 1.3 Radicals

**Definiton 1.3.1.** (Radical)

Let  $R$  be a ring. Its radical  $\text{rad}(R)$  is defined to be the intersection of all its maximal ideals.

**Proposition 1.3.1.** Let  $R$  be a ring,  $I$  an ideal,  $x \in R$  and  $u \in R^\times$ . Then  $x \in \text{rad}(R)$  iff  $u - xy \in R^\times$  for all  $y \in R$ . In particular, the sum of an element of  $\text{rad}(R)$  and a unit is a unit, and  $I \subset \text{rad}(R)$  if  $1 - I \subset R^\times$ .

*Proof.*

For a maximal ideal  $J$ , if  $u - xy \in J$ , then  $u \in J$  which is a contradiction and hence  $u - xy$  is a unit. Conversely, if there exists  $J$  maximal such that  $x \in J$ , then  $\langle x \rangle + J = R$  and hence there exists  $m \in J$  such that  $u - xy = m$  for some unit  $u$ , which is a contradiction.

**Corollary 1.3.2.** Let  $R$  be a ring,  $I$  an ideal,  $\kappa : R \rightarrow R/I$  the quotient map. Assume  $I \subset \text{rad}(R)$ , then  $\kappa$  is injective on  $\text{Idem}(R)$ .

*Proof.*

For  $e, e' \in \text{Idem}(R)$  and  $x = e - e'$ , if  $\kappa(x) = 0$ , then  $x^3 = x$  and hence  $x(1 - x^2) = 0$ , so  $1 - x^2$  is a unit and hence  $x$  is 0 and we are done.

**Definiton 1.3.2.** (Local ring)

A ring is called local if it has exactly one maximal ideal, and semilocal if it has at least one and at most finitely many.

By the residue field of a local ring  $A$ , we mean the field  $A/M$  where  $M$  is the maximal ideal of  $A$ .

**Lemma 1.3.3.** Let  $A$  be a ring,  $N$  the set of nonunits. Then  $A$  is local iff  $N$  is an ideal, if so, then  $N$  is the maximal ideal.

*Proof.*

Only need to check the sufficiency, if  $A$  is local, then we know  $M$  is contained in  $N$ , and if there is  $y \in M - N$ , then  $\langle y \rangle$  is a proper ideal and hence  $y \in N$ , which is a contradiction and hence  $M = N$  and we are done.

**Proposition 1.3.4.** Let  $R$  be a ring,  $S$  a multiplicative subset, and  $I$  an ideal with  $I \cap S = \emptyset$ . Set  $\mathcal{S} := \{J, J \supset I, J \cap S = \emptyset\}$ , then  $\mathcal{S}$  has a maximal element  $P$  and every such  $P$  is prime.

*Proof.*

By Zorn's lemma, there is a maximal element  $P$  in  $\mathcal{S}$ , for  $x, y \in R - P$ , there exists  $p, q \in P, a, b \in R$  such that  $p + ax \in S, q + by \in S$  and hence  $pq + pby + qax + abxy \in S$ , and hence  $xy \notin P$  and we are done.

**Definiton 1.3.3.** (Saturated multiplicative subsets)

Let  $R$  be a ring, and  $S$  a multiplicative subset. We say  $S$  is saturated if for  $x, y \in R, xy \in S$ , then  $x, y \in S$ .

**Lemma 1.3.5.** Let  $R$  be a ring,  $I$  a subset of  $R$  that is stable under addition and multiplication, and  $P_1, \dots, P_n$  ideals such that  $P_3, \dots, P_n$  are prime. If  $I$  is not contained in  $P_j$  for all  $j$ , then there is an  $x \in I$  such that  $x \in P_j$  for  $j$  or equivalently, if  $I \subset \bigcup_{i=1}^n P_i$ , then  $I \subset P_i$  for some  $i$ .

*Proof.*

If  $n = 1$  then we are done. We may use the induction, assume that  $n \geq 2$ , then by induction, for each  $i$ , there is  $x_i \in I$  such that  $x_i$  is not in  $P_j, i \neq j$  and  $x_i \in P_i$ , so then  $x_1 + x_2 \notin P_2$  if  $n = 2$ . For other  $n$ , we will know  $(x_1 \cdots x_{n-1}) \notin P_j$  for all  $j$ .

**Definiton 1.3.4.** Let  $R$  be a ring,  $S$  a subset, its radical  $\sqrt{S}$  is the set

$$\sqrt{S} := \{x \in R | x^n \in S \text{ for some } n\}$$

If  $I$  is an ideal and  $I = \sqrt{I}$ , then call  $I$  to be radical.

We call  $\sqrt{0}$  is the nilradical and denoted as  $\text{nil}(R)$ . We call  $x \in R$  nilpotent if  $x \in \text{nil}(0)$ , we call an ideal  $I$  nilpotent if  $a^n = 0$  for some  $n \geq 1$ .

**Theorem 1.3.6.** Let  $R$  be a ring,  $I$  an ideal, then

$$\sqrt{I} = \bigcap_{P \supset I, P \text{ prime}} P$$



*Proof.*

For  $x \notin \sqrt{I}$ , let  $S$  contains all the expotents of  $x$  and  $S$  is multiplicative, then  $I \cap S = \emptyset$  and then there is an  $P$  prime containing  $I$  with not containing  $x$  and hence  $\sqrt{a}$  contains the union.

Converse direction is easy.

**Proposition 1.3.7.** Let  $R$  be a ring,  $I$  an ideal. Then  $\sqrt{I}$  is an ideal.

**Definiton 1.3.5.** (Minimal primes)

Let  $R$  be a ring,  $I$  an ideal and  $P$  prime. We call  $P$  a minimal prime of  $I$  if  $P$  is minimal in the set of primes containing  $I$ , we all  $P$  a minimal prime of  $R$  if  $P$  is a minimal prime of  $\langle 0 \rangle$ .

**Proposition 1.3.8.** A ring  $R$  is reduced, i.e. 0 is the only nilpotent, and has only one minial prime iff  $R$  is a domain.

*Proof.*

Converse direction is obvious. If 0 is the only nilpotent elements,  $Q$  is a minimal prime ideal, then  $Q = 0$  since 0 is the intersection of all the minimal primes, and we are done.

## 1.4 Modules

**Definiton 1.4.1.** (Modules)

Let  $R$  be a ring. An  $R$ -module  $M$  is an abelian group with a scalar multiplication  $R \times M \rightarrow M$  which is

- $x(m + n) = xm + xn$  and  $(x + y)m = xm + ym$
- $x(y m) = (xy)m$
- $1m = m$

A submodule  $N$  of  $M$  closed under scalar multiplication.

Given  $m \in M$ , its annihilator

$$\text{Ann}(m) := \{x \in R | xm = 0\}$$

and the annilhilator of  $M$  is

$$\text{Ann}(M) := \{x \in R | xm = 0 \text{ for all } m \in M\}$$

We call the intersection of all maximal ideals containing  $\text{Ann}(M)$  the radical of  $M$ , denoted as  $\text{rad}(M)$ .

**Proposition 1.4.1.** There is a bijection between the maximal ideals containing  $\text{Ann}(M)$  and the maximal ideals of  $R/\text{Ann}(M)$ , and hence

$$\text{rad}(R/\text{Ann}(M)) = \text{rad}(M)/\text{Ann}(M)$$

**Proposition 1.4.2.** Given a submodule  $N$  of  $M$ , and then  $\text{Ann}(M) \subset \text{Ann}(N)$  and we also have  $\text{Ann}(M) \subset \text{Ann}(M/N)$ .

**Definiton 1.4.2.** (Semilocal)

We call  $M$  semilocal if there are only finitely many maximal ideals containing  $\text{Ann}(M)$ . If  $R$  is semilocal, so is  $M$  and we will know  $M$  is semilocal iff  $R/\text{Ann}(M)$  is a semilocal ring.

**Definiton 1.4.3.** (Polynomials)

The sets of polynomials

$$M[X] := \left\{ \sum_{i=0}^n m_i M_i, M_i \text{ monomials} \right\}$$

and then  $M[X]$  is an  $R[X]$  – module.

**Definiton 1.4.4.** (Homomorphisms)

Let  $R$  be aring,  $M$  and  $N$  modules. A  $R$ -linear map is a map  $\alpha : M \rightarrow N$  such that

$$\alpha(xm + yn) = x\alpha m + y\alpha n$$

Let  $\iota : \ker \alpha \rightarrow M$  be the inclusion and then  $\ker \alpha$  has the UMP:  $\alpha \iota = 0$  and for a homomorphism  $\beta : K \rightarrow M$  with  $\alpha \beta = 0$ , there is a unique homomorphism  $\gamma : K \rightarrow \ker \alpha$  with  $\iota \gamma = \beta$  as shown below

$$\begin{array}{ccccc} \ker \alpha & \xrightarrow{\iota} & M & \xrightarrow{\alpha} & N \\ & \swarrow \gamma & \uparrow \beta & \searrow 0 & \\ & & K & & \end{array}$$

**Definiton 1.4.5.** (Endomorphism)

An endomorphism of  $M$  a self-homomorphism denoted as  $\text{End}_R(M) \subset \text{End}_{\mathbb{Z}}(M)$ .

For  $x \in R$ , let  $\mu_x$  the self map of multiplication by  $x$  and then  $x \mapsto \mu_x$  denoted as

$$\mu_R : R \rightarrow \text{End}_R(M)$$

and note that  $\ker \mu_R = \text{Ann}(M)$ . We call  $M$  faithful if  $\mu_R$  is injective.

**Definiton 1.4.6.** For two rings  $R$  and  $R'$ , suppose  $R'$  is an  $R$ -algebra and  $M'$  an  $R'$ -module, then  $M'$  is also an  $R$ -module by  $xm := \phi(x)m$ .

A subalgebra  $R''$  of  $R'$  is a subring such that the structure map owning image in  $R''$ . The subalgebra generated by  $x_\lambda \in R'$  for  $\lambda \in \Lambda$  is the smallest  $R$ -subalgebra containing  $x_\lambda$  and we denote it by  $R[\{x_\lambda\}]$  and we call  $x_\lambda$  the generators.

We say  $R'$  is a finitely generated  $R$ -algebra if there exists  $x_i, 1 \leq i \leq n$  such that  $R' = R[x_1, \dots, x_n]$ .

**Definiton 1.4.7.** (Residue modules)

Let  $R$  be a ring,  $M$  a module and  $M' \subset M$  a submodule. Then

$$M/M' := \{m + M' | m \in M\}$$

which is the residue module or  $M$  modulo  $M'$ , form the quotient map

$$\kappa : M \rightarrow M/M', \quad m \mapsto m + M'$$

**Definiton 1.4.8.** (Cyclic Modules)

Let  $R$  be a ring. A module  $M$  is said to be cyclic if there exists  $m \in M$  such that  $m = Rm$ , then  $\alpha : x \mapsto xm$  induces an isomorphism  $R/\text{Ann}(m) \cong M$ .

**Definiton 1.4.9.** (Noether Isomorphisms)

Let  $R$  be a ring,  $N$  a module, and  $L$  and  $M$  submodules.

Assume  $L \subset M$ , and

$$\alpha : N \rightarrow N/L \rightarrow (N/L)/(M/L)$$

and we may know  $\ker \alpha = M$ . then  $\alpha$  factors through the isomorphism  $\beta$  in  $N \rightarrow N/M \rightarrow (N/L)/(M/L)$  since  $\alpha$  is surjective and  $\ker \alpha = M$ , so

$$\begin{array}{ccc} N & \longrightarrow & N/M \\ \downarrow & & \downarrow \beta \\ N/L & \longrightarrow & (N/L)/(M/L) \end{array}$$

Assume  $L$  not in  $M$  and

$$L + M := \{l + m, l \in L, m \in M\}$$

and it will be a submodule, then similarly

$$\begin{array}{ccc} L & \longrightarrow & L/(L \cap M) \\ \downarrow & & \downarrow \beta \\ L + M & \longrightarrow & (L + M)/M \end{array}$$

**Definiton 1.4.10.** (Cokernels, coimages)

Let  $R$  be a ring,  $\alpha : M \rightarrow N$  linear. Associated to  $\alpha$  there are its cokernel and its coimage

$$\text{Coker}(\alpha) := N/\text{Im}(\alpha) \quad \text{Coim}(\alpha) := M/\ker \alpha$$

**Definiton 1.4.11.** (Generators, free modules)

Let  $R$  be a ring,  $M$  a module. Given some submodules  $N_\lambda$ , by the sum  $\sum N_\lambda$ , we mean the set of all finite linear combinations  $\sum x_\lambda m_\lambda, m_\lambda \in N_\lambda$ .

Elements  $m_\lambda$  are said to be free of linearly independent if the linear combination equals to zero implies zero coefficients. If  $m_\lambda$  are said to be form a (free) basis of  $M$ , then they are free and generate  $M$  and we say  $M$  is free on  $m_\lambda$ .

We say  $M$  is finitely generated if it has a finite set of generators and  $M$  is free if it has a free basis.

**Theorem 1.4.3.** Let  $R$  be a PID,  $E$  a free module with  $e_\lambda$  a basis, and  $F$  a submodule, then  $F$  is free and has a basis indexed by a subset of  $\lambda$ .

**Definiton 1.4.12.** Let  $R$  be a ring,  $\Lambda$  a set,  $M_\lambda$  a module for  $\lambda \in \Lambda$ . The direct product of  $M_\lambda$  is the set of any vectors

$$\prod M_\lambda := \{(m_{m_\lambda})\}$$

which is a module under componentwise addition and scalar multiplication.

The direct sum of  $M_\lambda$  is the subset of restricted vectors:

$$\bigoplus M_\lambda := \{(m_\lambda), m_\lambda \text{ nonzero for only finite elements}\}$$

**Proposition 1.4.4.**  $\prod M_\lambda$  has the UMP, for  $R$ -homomorphism  $\alpha_\kappa : L \rightarrow M_\kappa$ , there is a unique  $R$ -homomorphism  $L \rightarrow \prod M_\lambda$  such that  $\pi_\kappa \alpha = \alpha_\kappa$ , in other words,  $\pi_\lambda$  induce a bijection of

$$\text{Hom}(L, \prod M_\lambda) \cong \prod \text{Hom}(L, M_\lambda)$$

Similarly, the direct sum comes equipped with injections

$$\iota_\kappa \rightarrow \bigoplus M_\lambda$$

and it has the UMP: given  $\beta_\kappa : M_\kappa \rightarrow N$ , there is a unique  $R$ -homomorphism  $\beta : \bigoplus M_\lambda \rightarrow N$  such that  $\beta \iota_\kappa = \beta_\kappa$  and  $\iota_\kappa$  induce the bijection:

$$\text{Hom}(\bigoplus, N) \rightarrow \bigoplus \text{Hom}(M_{\lambda,N})$$

## 1.5 Exact Sequences

**Definiton 1.5.1.** (Exact)

A sequence of module homomorphisms

$$\cdots \rightarrow M_{k-1} \xrightarrow{\alpha_{k-1}} M_k \xrightarrow{\alpha_k} M_{k+1} \rightarrow \cdots$$

is said to be exact at  $M_k$  if  $\ker \alpha_k = \text{Im}(\alpha_k)$ . The sequence is said to be exact if it is exact at every  $M_k$ , except an initial source or final target.

**Definiton 1.5.2.** (Short exact sequences)

A sequence  $0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$  is exact if and only if  $\alpha$  is injective and  $N \cong \text{Coker} \alpha$  or dually if and only if  $\beta$  is surjective and  $L = \ker \beta$ . Then the sequence is called short exact and we often regard  $L$  as a submodule of  $M$  and  $N$  the quotient  $M/L$ .

*Proof.*

**Proposition 1.5.1.** For  $\lambda \in \Lambda$ , let  $M'_\lambda \rightarrow M_\lambda \rightarrow M''_\lambda$  be sequence of module homomorphisms. If every sequence is exact, then so are the two induced sequences

$$\bigoplus M'_\lambda \rightarrow \bigoplus M_\lambda \rightarrow \bigoplus M''_\lambda, \quad \prod M'_\lambda \rightarrow \prod M_\lambda \rightarrow \prod M''_\lambda$$

Conversely, if either induced sequence is exact then so is every original one.

*Proof.*

**Proposition 1.5.2.** Let  $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$  be a short exact sequence, and  $N \subset M$  a submodule. Set  $N' := \alpha^{-1}(N)$  and  $N'' := \beta(N)$ . Then the induced sequence  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  is short exact.

**Definiton 1.5.3.** (Retraction, section, splits)

A linear map  $\rho : M \rightarrow M'$  is a retraction of another  $\alpha : M' \rightarrow M$  if  $\rho \alpha = 1_{M'}$ , then  $\alpha$  is injective and  $\rho$  is surjective.

Dually, we call  $\sigma : M'' \rightarrow M$  a section of another  $\beta : M \rightarrow M''$  if  $\beta\sigma = 1_{M''}$ , then  $\beta$  is surjective and  $\sigma$  is injective.

We call a 3-term exact sequence  $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$  splits if there is an isomorphism  $\phi : M \cong M' \oplus M''$  with  $\phi\alpha = \iota_{M'}$  and  $\beta = \pi_{M''}\phi$ .

**Proposition 1.5.3.** Let  $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$  be a 3-term exact sequence. Then the following conditions are equivalent

- The sequence splits
- There exists a retraction  $\rho : M \rightarrow M'$  of  $\alpha$  and  $\beta$  is surjective.
- There exists a section  $\sigma : M'' \rightarrow M$  of  $\beta$  and  $\alpha$  is injective

*Proof.*

Assume the sequence is splits, then we have the commuting diagram

$$\begin{array}{ccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\ & \searrow \iota_{M'} & \downarrow \phi(\cong) & \nearrow \pi_{M''} & \\ & & M' \oplus M'' & & \end{array}$$

then let  $\rho = \pi_{M'}\phi$ , then  $\rho\alpha = \pi_{M'}\phi\phi^{-1}\iota_{M'} = 1_{M'}$ . Let  $\sigma = \phi^{-1}\iota_{M''}$  and then  $\beta\sigma = \pi_{M''}\phi\phi^{-1}\iota_{M''} = 1_{M''}$  and then  $\beta$  is surjective and  $\alpha$  is injective.

Now assume there is such a retraction  $\rho$  and  $\beta$  is surjective, then define  $\sigma = 1_M - \alpha\rho$  and  $\phi : M \rightarrow M' \oplus M''$  by  $m \mapsto (\rho(m), \beta\sigma(m))$ , if  $\phi(m) = 0$ , then  $\rho(m) = 0$  and  $\sigma(m) = m$ , which means  $\beta(m) = 0$ . There exists  $a \in M'$  such that  $m = \alpha(a)$  and hence  $a = 0$  which means  $m = 0$ , so  $\ker \phi = 0$ . For  $(a, b) \in M' \oplus M''$ , assume  $\beta(m) = b$ , then  $\phi(\alpha(a) + \sigma(m)) = (a + \rho(m - \alpha\rho(m)), \beta(\alpha(a) + \beta\sigma(m))) = (a, b)$  and hence  $\phi$  is surjective. And  $\phi\alpha(a) = (a, \beta\sigma\alpha(a)) = (a, 0)$  and  $\pi_{M''}\phi(m) = \beta(\sigma(m)) = \beta(m)$  and we are done.

**Lemma 1.5.4.** Consider this commutative diagram with exact rows:

$$\begin{array}{ccccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\ & \downarrow \gamma' & \downarrow \gamma & & \downarrow \gamma'' & & \\ 0 & \longrightarrow & N' & \xrightarrow{\alpha'} & N & \xrightarrow{\beta'} & N'' \end{array}$$

It yields the following exact sequence:

$$\ker \gamma' \xrightarrow{\varphi} \ker \gamma \xrightarrow{\psi} \ker \gamma'' \xrightarrow{\partial} \operatorname{coker} \gamma' \xrightarrow{\varphi'} \operatorname{coker} \gamma \xrightarrow{\psi'} \operatorname{coker} \gamma''$$

Moreover, if  $\alpha$  is injective, then so is  $\varphi$ ; dually, if  $\beta'$  is surjective, then so is  $\psi'$ .

*Proof.*

Notice  $\alpha'\gamma' = \gamma\alpha$ ,  $\beta'\gamma = \gamma''\beta$  and let  $\varphi = \alpha|_{\ker \gamma'}$ ,  $\psi = \beta|_{\ker \gamma}$  and we know  $\varphi(\ker \gamma') \subset \ker \gamma$ ,  $\psi(\ker \gamma) \subset \ker \gamma''$ . Obviously,  $\operatorname{Im}(\varphi) \subset \ker \psi$  and for any  $b \in \ker \psi$ , it is in  $\ker \gamma \cap \operatorname{Im} \alpha$ , since  $\alpha'$  is injective and hence its preimage has to be contained in  $\ker \gamma'$  and hence it is in  $\operatorname{Im}(\varphi)$ .

$\alpha', \beta'$  will induce natural  $\varphi', \psi'$  on  $\text{coker}\gamma', \text{coker}\gamma$  by defining  $n' + \text{Im}\gamma' \mapsto \alpha'(n') + \text{Im}\gamma, n + \text{Im}\gamma \mapsto \beta'(n) + \text{Im}\gamma''$ , which is well-defined since  $\alpha'(\text{Im}\gamma') \subset \text{Im}\gamma, \beta'(\text{Im}\gamma) \subset \text{Im}\gamma''$  and the exactness is similarly checked.

Define  $\partial$  by the following, if  $\gamma''m'' = 0$ , consider  $m$  is one of preimage of  $m''$  and let  $a$  to be the preimage of  $\gamma(m)$ , then let  $\partial m'' = a + \text{Im}\gamma'$ . It is well-defined since if  $\beta m = \beta n = m''$ , then  $m - n \in \ker \beta$ , which means the preimages of  $\gamma m, \gamma n$  are in the same coset. For  $m \in \ker \gamma$ ,  $\partial(\psi(m)) = \alpha'^{-1}\gamma(m) + \text{Im}\gamma' = 0$  and if  $\partial(m'') = 0$ , then assume  $\beta m = m''$  and we know  $\alpha^{-1}\gamma(m) \in \text{Im}\gamma'$  and hence there exists  $x \in M'$  such that  $\gamma\alpha x = \gamma m$  and we know  $\beta(m - \alpha(x)) = m''$  and  $\gamma(m - \alpha x) = 0$ , which means  $\ker \partial = \text{Im}\psi$ . If  $a = \alpha'^{-1}(\gamma(m))$  with  $m'' = \beta m \in \ker \gamma''$ , then  $\varphi'(a + \text{Im}(\gamma')) = \alpha'a + \text{Im}\gamma = 0$  and if  $\varphi'(a + \text{Im}(\gamma')) = 0$ , then there exists  $m$  such that  $\alpha'(a) = \gamma m$  and then  $\partial(\beta(m)) = a + \text{Im}\gamma'$  and we are done.

**Theorem 1.5.5.** (Left exactness of Hom)

- Let  $M' \rightarrow M \rightarrow M'' \rightarrow 0$  be a sequence of linear maps. Then it is exact iff for all modules  $N$ , the following induced sequence is exact

$$0 \rightarrow \text{hom}(M'', N) \rightarrow \text{hom}(M, N) \rightarrow \text{hom}(M', N)$$

- Let  $0 \rightarrow N' \rightarrow N \rightarrow N''$  be as sequence of linear maps. Then it is exact iff for all modules  $M$ , the following induced sequence is exact.

$$0 \rightarrow \text{hom}(M, N') \rightarrow \text{hom}(M, N) \rightarrow \text{hom}(M, N'')$$

*Proof.*

Assume  $M' \xrightarrow{\phi} M \xrightarrow{\psi} M''$  and then the induced map will be  $\tilde{\psi} : f \mapsto f \circ \psi$  and  $\tilde{\phi} : g \mapsto g \circ \phi$ . If  $\psi$  is surjective, then  $\tilde{\psi}$  will be an injective since  $f \circ \psi = 0$  implies  $f = 0$ , and if  $g \circ \phi = 0$ , then  $\ker \psi = \text{Im}\phi \subset \ker g$  and hence there will be  $g' : M'' \cong M/\ker \psi \rightarrow N$  such that  $g'\psi = g$  by the UMP and we are done. We know for  $g : M \rightarrow N, g \circ \phi = 0$ , equivalently  $\text{Im}\phi \subset \ker g$  iff there exists unique  $g' : M'' \rightarrow N$  such that  $g' \circ \psi = g$ , which means  $M'' \cong \text{coker}\phi$  and the diagram

$$\begin{array}{ccccccc} M' & \xrightarrow{\phi} & M & \xrightarrow{\psi} & M'' & \longrightarrow & 0 \\ & & & \searrow \kappa & \updownarrow & \nearrow & \\ & & & & \text{coker}\phi & & \end{array}$$

commutes and we are done.

Similarly assume that  $N' \xrightarrow{\phi} N \xrightarrow{\psi} N''$ , then  $\tilde{\phi} : f \mapsto \phi \circ f$  and  $\tilde{\psi} : g \mapsto \psi \circ g$ , which means  $\ker \psi = N' \hookrightarrow N$ . It is easy to check  $\ker \tilde{\phi} = 0$  and  $\text{Im}\tilde{\phi} \subset \ker \tilde{\psi}$ . For  $g \in \ker \tilde{\psi}$ , since  $\text{Im}g \subset \ker \psi = \text{Im}\phi$ , then let  $g' = g|_N$  will satisfy that  $\phi \circ g' = g$ . For the converse direction, we know for any  $g : M \rightarrow N$ ,  $\text{Im}g \subset \ker \psi$  iff there exists a unique  $g' : M \rightarrow N'$  such that

$\phi \circ g' = g$ , then we may, which is

$$\begin{array}{ccccccc} 0 & \longrightarrow & N' & \xrightarrow{\phi} & N & \xrightarrow{\psi} & N'' \\ & & \searrow & & \swarrow & & \\ & & \ker \psi & & & & \end{array}$$

**Definiton 1.5.4.** (Presentation)

A (free) presentation of a module  $M$  is an exact sequence

$$G \rightarrow F \rightarrow M \rightarrow 0$$

with  $G$  and  $F$  free. If  $G$  and  $F$  are free of finite rank, then the presentation is called finite. If  $M$  has a finite presentation, then call  $M$  finitely presented.

**Proposition 1.5.6.** Let  $R$  be a ring,  $M$  a module,  $m_\lambda$  generators. Then there is an exact sequence  $0 \rightarrow K \rightarrow R^{\oplus \Lambda} \xrightarrow{\alpha} M \rightarrow 0$  with  $\alpha e_\lambda = m_\lambda$  where  $e_\lambda$  the standard basis and there is a presentation.

*Remark.*

Choose  $K = \ker \alpha$  and  $k_\sigma, \sigma \in \Sigma$  to be generators of  $K$ , then

$$R^{\oplus \Sigma} \rightarrow R^{\oplus \Lambda} \rightarrow M \rightarrow 0$$

is a presentation.

**Definiton 1.5.5.** (Projective Module)

A module  $P$  is called projective if given any surjective linear map  $\beta : M \rightarrow N$ , every linear map  $\alpha : P \rightarrow N$  lifts to one  $\gamma : P \rightarrow M$ , i.e.  $\alpha = \beta\gamma$ .

**Theorem 1.5.7.** The following conditions on an  $R$ -module  $P$  are equivalent

- The module  $P$  is projective
- Every short exact sequence  $0 \rightarrow K \rightarrow M \rightarrow P \rightarrow 0$  splits
- There is a module  $K$  such that  $K \oplus P$  is free
- Every exact sequence  $N' \rightarrow N \rightarrow N''$  induces an exact sequence

$$\text{hom}(P, N') \rightarrow \text{hom}(P, N) \rightarrow \text{hom}(P, N'')$$

- Every surjective homomorphism  $\beta : M \rightarrow N$  induces a surjection

$$\text{hom}(P, \beta) : \text{hom}(P, M) \rightarrow \text{hom}(P, N)$$

*Proof.*

By considering the  $P \cong M / \ker \phi$  it will induce a section of  $\psi : M \rightarrow P$  and obviously  $\phi : K \rightarrow M$  is injective and we are done for (1) implies (2). Use proposition 1.5.6. and we will know there exists  $K$  such that  $K \oplus P \cong R^{\oplus \Lambda}$  which is free, which is for (2) implies (3).

Assume (3), then there exists  $\Lambda$  such that  $K \oplus P \cong R^{\oplus \Lambda}$ . Also notice that we will have

$$\prod N'_\lambda \rightarrow \prod N_\lambda \rightarrow \prod N''_\lambda$$

is exact, which implies that

$$\text{hom}(R^{\oplus \Lambda}, N') \rightarrow \text{hom}(R^{\oplus \Lambda}, N) \rightarrow \text{hom}(R^{\oplus \Lambda}, N'')$$

is exact since  $\text{hom}(R^{\oplus \Lambda}, N) \cong \prod N_\lambda$  and hence

$$\text{hom}(K \oplus P, N') \rightarrow \text{hom}(K \oplus P, N) \rightarrow \text{hom}(K \oplus P, N'')$$

which implies

$$\text{hom}(K, N') \oplus \text{hom}(P, N') \rightarrow \text{hom}(K, N) \oplus \text{hom}(P, N) \rightarrow \text{hom}(K, N'') \oplus \text{hom}(P, N'')$$

by isomorphism and hence the conclusion goes.

Assume (4), we know  $M \rightarrow N \rightarrow 0$  is exact and we are done.

Assume (5), which is exactly the definition of projective module.

**Lemma 1.5.8.** (Schanuel)

Any two short exact sequences

$$0 \rightarrow L \xrightarrow{i} P \xrightarrow{\alpha} M \rightarrow 0, \quad 0 \rightarrow L' \xrightarrow{i'} P' \xrightarrow{\alpha'} M \rightarrow 0$$

with  $P$  and  $P'$  projective are essentially isomorphic; i.e. there is the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L \oplus P' & \xrightarrow{i \oplus 1_{P'}} & P \oplus P' & \xrightarrow{\alpha \oplus 0} & M \longrightarrow 0 \\ & & \downarrow \cong \beta & & \downarrow \cong \gamma & & \downarrow = \\ 0 & \longrightarrow & P \oplus L' & \xrightarrow{1_P \oplus i'} & P \oplus P' & \xrightarrow{0 \oplus \alpha'} & M \longrightarrow 0 \end{array}$$

*Proof.*

Firstly, it is easy to check the two exact sequences are exact. Then consider

$$0 \rightarrow K := \ker(\alpha \oplus \alpha') \rightarrow P \oplus P' \rightarrow M \rightarrow 0$$

which is exact, there exists  $\pi : P' \rightarrow P$  such that  $\alpha\pi = \alpha'$ , so we may define  $\phi : P \oplus P' \rightarrow P \oplus P'$  by  $\begin{pmatrix} 1_P & \pi \\ 0 & 1_{P'} \end{pmatrix}$  which means  $(p, p') \mapsto (p + \pi p', p')$  and then  $\alpha p + \alpha' p' = (\alpha \oplus$

$0)\phi(p, p') = (\alpha \oplus \alpha')(p, p')$  where the inverse of  $\phi$  will be  $\begin{pmatrix} 1_P & -\pi \\ 0 & 1_{P'} \end{pmatrix}$  and hence  $\phi$  is an automorphism.

Notice  $L$  is  $\ker \alpha$ , and for  $(p, p') \in L \oplus P'$ , denoted  $\psi : L \oplus P' \rightarrow K$  the induced map by  $\phi^{-1}$  and then  $\psi(p, p') = (p - \pi p', p')$  which is in  $\ker(\alpha \oplus \alpha')$  and it has inverse obviously, and hence  $L \oplus P' \cong K$ , and use the similar construction to  $P \oplus L'$  and we are done.

**Proposition 1.5.9.** Let  $R$  be a ring, and  $0 \rightarrow M \rightarrow N \rightarrow M' \rightarrow 0$  an exact sequence. Prove  $M, M'$  are finitely generated implies  $N$  is finitely generated.



**Proposition 1.5.10.** Let  $R$  be a ring, and  $0 \rightarrow L \rightarrow R^n \rightarrow M \rightarrow 0$  an exact sequence. Prove  $M$  is finitely generated iff  $L$  is finitely presented.

**Proposition 1.5.11.** Let  $0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$  be a short exact sequence with  $L$  finitely generated and  $M$  finitely presented. Then  $N$  is finitely presented.

*Proof.*

There exists  $G \rightarrow F \rightarrow M \rightarrow 0$  exact with  $G, F$  free of finite rank. Let  $\mu : R^m \rightarrow M$  any surjection and  $\nu := \beta\mu$ , let  $K = \ker \nu$  and  $\lambda = \mu|_K$ , then the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & R^m & \xrightarrow{\nu} & N \longrightarrow 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow 1_N \\ 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N \longrightarrow 0 \end{array}$$

commutes and the snake lemma ensure that  $\ker \lambda \cong \ker \mu$ , however  $\ker \mu$  is finitely generated and hence  $\ker \lambda$  is finitely generated, and snake lemma ensured that  $\text{coker} \lambda = 0$  and hence  $0 \rightarrow \ker \lambda \rightarrow K \rightarrow L \rightarrow 0$  is exact and hence  $K$  is finitely generated and hence  $N$  is finitely presented.

**Proposition 1.5.12.** Let  $0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$  be a short exact sequence with  $L, N$  finitely presented. Then  $M$  is finitely presented.

*Proof.*

Let  $\lambda : R^l \rightarrow L, \nu : R^n \rightarrow N$  any two surjections and define  $\gamma := \alpha\lambda$  and since  $R^n$  is projective, then define  $\delta : R^n \rightarrow M$  by lifting  $\nu$  and  $\mu : R^l \oplus R^n \rightarrow M$  by  $\gamma + \delta$  and the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^l & \longrightarrow & R^l \oplus R^n & \xrightarrow{\nu} & R^n \longrightarrow 0 \\ & & \downarrow \lambda & & \downarrow \mu & & \downarrow \nu \\ 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N \longrightarrow 0 \end{array}$$

commutes, and the snake lemma yields that

$$0 \rightarrow \ker \lambda \rightarrow \ker \mu \rightarrow \ker \nu \rightarrow 0$$

exact and  $\text{coker} \mu = 0$  and  $\ker \lambda, \ker \mu$  are finitely generated and hence  $\ker \mu$  is finitely generated and hence  $M$  is finitely presented.

## 1.6 Direct Limits

**Definiton 1.6.1.** (Categories)

A category  $\mathcal{C}$  is a collection of elements, called objects. Each pair of objects  $A, B$  is equipped with a set  $\text{hom}_{\mathcal{C}}(A, B)$  called maps or morphisms. For objects  $A, B, C$ , there is a composition law

$$\text{hom}_{\mathcal{C}}(A, B) \times \text{hom}_{\mathcal{C}}(B, C) \rightarrow \text{hom}_{\mathcal{C}}(A, C), \quad (a, \beta) \rightarrow \beta a$$

and there is a distinguished map  $1_B \in \text{hom}_{\mathcal{C}}(B, B)$  such that

$$\gamma(\beta\alpha) = (\gamma\beta)\alpha \text{ for any } \gamma : C \rightarrow D, \quad \text{and } 1_B\alpha = \alpha, \beta 1_B = \beta$$

and we say  $\alpha$  is an isomorphism with inverse  $\beta : B \rightarrow A$  such that  $\alpha\beta = 1_B$  and  $\beta\alpha = 1_A$ .

**Definiton 1.6.2.** (Functors)

A map of categories is known as a functor. Namely, given categories  $\mathcal{C}$  and  $\mathcal{C}'$ , a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  is a rule that assigns to each object  $A$  of  $\mathcal{C}$  and  $F(A)$  of  $\mathcal{C}'$  and to each map  $\alpha$  such that  $F(\alpha) : F(A) \rightarrow F(B)$

$$F(\beta\alpha) = F(\beta)F(\alpha), \quad F(1_A) = 1_{F(A)}$$

A map of functors is known as a natural transformation. Given two functors  $F, F' : \mathcal{C} \rightarrow \mathcal{C}'$ , a natural transformation  $\theta : F \rightarrow F'$  is a collection of maps  $\theta(A) : F(A) \rightarrow F'(A)$  such that  $\theta(B)F(\alpha) = F'(\alpha)\theta(A)$  for any  $\alpha$  and  $1_{F(A)}$  trivially form a natural transformation  $1_F$ . We call  $F$  and  $F'$  isomorphic if there are natural transformation  $\theta : F \rightarrow F'$  and  $\theta' : F' \rightarrow F$  such that  $\theta'\theta = 1_F$  and  $\theta\theta' = 1_{F'}$ .

A contravariant functor  $G$  from  $\mathcal{C}$  to  $\mathcal{C}'$  is a rule similar to  $F$  but  $G(\alpha) : G(B) \rightarrow G(A)$  with analogous properties with functors.

**Definiton 1.6.3.** (Adjoint)

Let  $F : \mathcal{C} \rightarrow \mathcal{C}'$  and  $F' : \mathcal{C}' \rightarrow \mathcal{C}$  be functors. We call  $(F, F')$  an adjoint pair,  $F$  the left adjoint of  $F'$  and  $F'$  the right-adjoint of  $F$  if for any  $A \in \mathcal{C}$  and  $A' \in \mathcal{C}'$ , there is given a natural bijection

$$\text{hom}_{\mathcal{C}'}(F(A), A') \cong \text{hom}_{\mathcal{C}}(A, F'(A'))$$

here natural means that maps  $B \rightarrow A$  and  $A' \rightarrow B'$  induce a commutative diagram:

$$\begin{array}{ccc} \text{hom}_{\mathcal{C}'}(F(A), A') & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}}(A, F'(A')) \\ \downarrow & & \downarrow \\ \text{hom}_{\mathcal{C}'}(F(B), B') & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}}(B, F'(B')) \end{array}$$

**Proposition 1.6.1.** Naturality serves to determine an adjoint up to canonical isomorphism. Namely, let  $F$  and  $G$  be two left adjoints of  $F'$  and then  $F$  and  $G$  are isomorphic.

*Proof.*

Define  $\theta(A) : G(A) \rightarrow F(A)$  by the image of  $1_{F(A)}$  under the isomorphism

$$\text{hom}(F(A), F(A)) \cong \text{hom}(A, F'F(A)) \cong \text{hom}(G(A), F(A))$$

for  $\alpha : A \rightarrow B$  it will induce the commutative diagram

$$\begin{array}{ccccc} \text{hom}_{\mathcal{C}'}(F(A), F(A)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}}(A, F'F(A)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}'}(G(A), F(A)) \\ \downarrow & & \downarrow & & \downarrow \\ \text{hom}_{\mathcal{C}'}(F(A), F(B)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}}(A, F'F(B)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}'}(G(A), F(B)) \\ \uparrow & & \uparrow & & \uparrow \\ \text{hom}_{\mathcal{C}'}(F(B), F(B)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}}(B, F'F(B)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}'}(G(B), F(B)) \end{array}$$

where we may know  $\theta(B)G(\alpha) = F(\alpha)\theta_A$  and hence  $\theta$  is a natural transformation, similarly, define  $\theta' : F \rightarrow G$  and we will have

$$\begin{array}{ccccc} \text{hom}_{\mathcal{C}'}(F(A), F(A)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}}(A, F'F(A)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}'}(G(A), F(A)) \\ \downarrow & & \downarrow & & \downarrow \\ \text{hom}_{\mathcal{C}'}(F(A), G(A)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}}(A, F'G(A)) & \xrightarrow{\cong} & \text{hom}_{\mathcal{C}'}(G(A), G(A)) \end{array}$$

which is induced by  $\theta'(A)$  and then  $\theta'(A)\theta(A) = 1_G(A)$  and we are done.

**Definition 1.6.4.** (Direct limits)

Let  $\Lambda, \mathcal{C}$  categories and  $\Lambda$  is small, i.e. its objects form a set. Given a functor  $\lambda \mapsto M_\lambda$  from  $\Lambda$  to  $\mathcal{C}$ , its direct limit denoted with  $\varinjlim M_\lambda$  is defined to be the object of  $\mathcal{C}$  universal among objects  $P$  equipped with maps  $\beta_\mu : M_\mu \rightarrow P$  what are compatible with the transition map  $\alpha_\mu^\kappa : M_\kappa \rightarrow M_\mu$ , i.e. there is a unique map  $\beta$  such that all the diagrams

$$\begin{array}{ccccc} M_\kappa & \xrightarrow{\alpha_\mu^\kappa} & M_\mu & \xrightarrow{\alpha_\mu} & \varinjlim M_\lambda \\ \downarrow \beta_\kappa & & \downarrow \beta_\mu & & \downarrow \beta \\ P & \xrightarrow{1_P} & P & \xrightarrow{1_P} & P \end{array}$$

where  $\lambda \mapsto M_\lambda$  is often called a direct system. We know the limit is determined up to unique isomorphism.

We say  $\mathcal{C}$  has direct limits indexed by  $\Lambda$  if for every functor  $\lambda \mapsto M_\lambda$ , the direct limit exists. We say that  $\mathcal{C}$  has direct limits if it has direct limits indexed by every small category.

Given a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$ , note that a functor  $\lambda \mapsto M_\lambda$  from  $\Lambda$  to  $\mathcal{C}$  yields a functor from  $\Lambda$  to  $\mathcal{C}'$ . Furthermore, whenever the corresponding two direct limits exist, the maps  $F(\alpha_\mu) : F(M_\mu) \rightarrow F(\varinjlim M_\lambda)$  induce a canonical map

$$\phi_F : \varinjlim F(M_\lambda) \rightarrow F(\varinjlim M_\lambda)$$

If  $\phi_F$  is always an isomorphism, we say  $F$  preserves direct limits.

**Proposition 1.6.2.** Assume  $\mathcal{C}$  has direct limits indexed by  $\Lambda$ . Then, given a natural transformation from  $\lambda \mapsto M_\lambda$  to  $\lambda \mapsto N_\lambda$ , universality yields unique commutative diagrams

$$\begin{array}{ccc} M_\mu & \longrightarrow & \varinjlim M_\lambda \\ \downarrow & & \downarrow \\ N_\mu & \longrightarrow & \varinjlim N_\lambda \end{array}$$

*Proof.*

We know

$$\theta(\mu) : M_\mu \rightarrow N_\mu, \theta(\mu)\alpha_\mu^\lambda = \beta_\mu^\lambda \theta(\lambda)$$

and hence consider

$$\begin{array}{ccccc}
M_\lambda & \longrightarrow & M_\mu & \longrightarrow & \varinjlim M_\lambda \\
\downarrow & & \downarrow & & \downarrow \alpha \\
N_\lambda & \longrightarrow & N_\mu & \longrightarrow & \varinjlim N_\lambda \\
\downarrow & & \downarrow & & \\
P & \xrightarrow{=} & P & \xrightarrow{=} & P
\end{array}$$

**Definiton 1.6.5.** (Functor category)

The functor category  $\mathcal{C}^\Lambda$ , i.e. a category with objects to be the functors from  $\Lambda$  to  $\mathcal{C}$  and the maps are the natural transformation, then  $\varinjlim$  yields a functor from  $\mathcal{C}^\Lambda$  to  $\mathcal{C}$ .

The direct limit functor is the left adjoint of the diagonal function  $\Delta : \mathcal{C} \rightarrow \mathcal{C}^\Lambda$  which send  $M$  to the constant functor  $\Delta M$  which has the same value  $M$  at every  $\lambda$  and  $1_M$  at every map of  $\Lambda$ ; for  $\gamma : M \rightarrow N$  it carries  $\gamma$  to  $\Delta\gamma : \Delta M \rightarrow \Delta N$  which has the same value  $\gamma$  at every  $\lambda$ .

*Proof.*

By proposition 1.6.2. we assume  $\lambda \mapsto M_\lambda, \lambda \mapsto N_\lambda$  and  $\theta$  a natural transformation, then

$$\varinjlim(\theta) : \varinjlim M_\lambda \rightarrow \varinjlim N_\lambda$$

which is uniquely determined.

Notice

$$\varinjlim : \mathcal{C}^\Lambda \rightarrow \mathcal{C}, \quad \Delta : \mathcal{C} \rightarrow \mathcal{C}^\Lambda$$

and we want to check

$$\text{hom}(\varinjlim(\lambda \mapsto M_\lambda), N) \cong \text{hom}(\lambda \mapsto M_\lambda, \Delta N)$$

assume  $\gamma : \varinjlim(\lambda \mapsto M_\lambda) \rightarrow N$  and then we would like  $\gamma \mapsto \Delta\gamma$  is an isomorphism, which is obviously an injection and assume  $\delta : \lambda \mapsto M_\lambda \rightarrow \Delta N$  where we know  $\delta(\lambda) : M_\lambda \rightarrow N$  which satisfies some commutative diagram and hence there exists a unique  $\gamma : \varinjlim(\lambda \mapsto M_\lambda) \rightarrow N$ .

**Definiton 1.6.6.** (Coproduct)

Let  $\mathcal{C}$  be a category,  $\Lambda$  a set and  $M_\lambda$  an object for each  $\lambda \in \Lambda$ . The coproduct  $\coprod_{\lambda \in \Lambda} M_\lambda$  is defined as the object of  $\mathcal{C}$  universal among objects  $P$  equipped with a map  $\beta_\mu : M_\mu \rightarrow P$  and the maps  $\iota_\lambda : M_\lambda \rightarrow \coprod M_\lambda$  is call the inclusions.

If  $\Lambda$  is empty then  $B$  is an object with a unique map  $\beta$  to other  $P$  and such  $B$  is called an initial object.

**Definiton 1.6.7.** (Coequalizers)

Let  $\alpha, \alpha' : M \rightarrow N$  their coequalizer is the object universal among  $P$  with  $\eta : N \rightarrow P$  such that  $\eta\alpha = \eta\alpha'$ .

**Lemma 1.6.3.** A category has direct limits iff it has coproducts and coequalizers. If a category has direct limits, then a functor preserves them iff it preserves coproduct and coequalizers.

*Proof.*

Let  $\Lambda \mapsto M_\lambda$  where  $\text{hom}(\mu, \nu)$  is empty for any  $\mu \neq \nu$  and then the corresponding direct limit is the coproduct. For  $M, N \in \mathcal{C}$  and two morphisms, then the inclusion of them two is a small category and the direct limit will be the coequalizer. If  $F$  preserves direct limits, since we have shown that coproduct and coequalizer is special direct limits and we are done.

Conversely, if  $\mathcal{C}$  has coproducts and coequalizers. Assume  $\Lambda$  a small category and  $\lambda \mapsto M_\lambda$  a functor, let  $\Sigma$  all transition maps and for each  $\sigma = \alpha_\mu^\lambda \in \Sigma$ , set  $M_\Sigma := M_\lambda$  and let  $M := \coprod M_\sigma$  and  $N = \coprod M_\lambda$ , for each  $\sigma$ , there are two maps  $M_\sigma \rightarrow N$  which is  $\iota_\lambda$  and the composition  $\iota_\mu \alpha_\mu^\lambda$ , then let  $C$  be the coequalizer of corresponding maps  $\alpha, \alpha' : M \rightarrow N$  and  $\eta : N \rightarrow C$  the insertion. So if  $\beta_\lambda : M_\lambda \rightarrow P$  compatible with the transition maps, then there is a unique  $\beta : N \rightarrow P$  such that  $\beta \iota_\lambda = \beta_\lambda$  and hence  $\beta \alpha = \beta \alpha'$  and we are done.

If  $F$  preserves coproduct and coequalizers, then  $F$  preserves the construction and we are done.

**Theorem 1.6.4.** The categories  $R$ -module and sets have direct limits.

**Theorem 1.6.5.** Every left adjoint  $F : \mathcal{C} \rightarrow \mathcal{C}'$  preserves direct limits.

**Proposition 1.6.6.** Let  $\mathcal{C}$  be a category,  $\Lambda$  and  $\Sigma$  small categories. Assume  $\mathcal{C}$  has direct limits indexed by  $\Sigma$ . Then the functor category  $\mathcal{C}^\Lambda$  does too.

**Theorem 1.6.7.** Let  $\mathcal{C}$  be a category with direct limits indexed by small categories  $\Sigma$  and  $\Lambda$ . Let  $\sigma \mapsto (\lambda \mapsto M_{\sigma\lambda})$  be a functor from  $\Sigma$  to  $\mathcal{C}^\Lambda$ . Then

$$\varinjlim_{\sigma} \varinjlim_{\lambda} M_{\sigma\lambda} = \varinjlim_{\lambda} \varinjlim_{\sigma} M_{\sigma\lambda}$$

**Corollary 1.6.8.** Let  $\Lambda$  be a small category,  $R$  a ring, and  $\mathcal{C}$  is sets or  $R$ -modules. Then functor  $\varinjlim : \mathcal{C}^\Lambda \rightarrow \mathcal{C}$  preserves coproducts and coequalizers.

## 1.7 Tensor Products

**Definiton 1.7.1.** (Bilinear maps)

Let  $R$  be a ring and  $M, N, P$  modules. We call a map  $\alpha : M \times N \rightarrow P$  bilinear if it is linear in each variable. Denote the set of all these maps by  $\text{Bil}_R(M, N; P)$ , it is clearly an  $R$ -module with sum and scalar multiplication performed valuewise.

**Definiton 1.7.2.** (Tensor product)

Let  $R$  be a ring and  $M, N$  modules. Their tensor product denoted  $M \otimes_R N$  is constructed as the quotient of the free module  $R^{\oplus(M \times N)}$  modulo the submodule generated by the following elements, where  $(m, n)$  stands for the standard basis element  $e_{(m, n)}$ :

$$(m + m', n) - (m, n) - (m', n), (m, n + n') - (m, n) - (m, n'), (xm, n), (m, xn) - x(m, n)$$

and the above construction yields a canonical bilinear map

$$\beta : M \times N \rightarrow M \otimes N$$

and set  $m \otimes n := \beta(m, n)$

**Theorem 1.7.1.** (UMP of tensor product)

Let  $R$  be a ring,  $M, N$  modules. Then  $\beta : M \times N \rightarrow M \otimes N$  is the universal bilinear

map with source  $M \times N$ ; in fact,  $\beta$  induces a module isomorphism

$$\theta : \text{hom}_R(M \otimes_R N, P) \cong \text{Bil}_R(M, N; P)$$

**Corollary 1.7.2.** (Bifunctoriality)

Let  $R$  be a ring,  $\alpha : M \rightarrow M'$  and  $\alpha' : N \rightarrow N'$  module homomorphisms. Then there is a canonical commutative diagram:

$$\begin{array}{ccc} M \times N & \xrightarrow{\alpha \times \alpha'} & M' \times N' \\ \downarrow \beta & & \downarrow \beta' \\ M \otimes N & \xrightarrow{\alpha \otimes \alpha'} & M' \otimes N' \end{array}$$

*Proof.*

Notice

$$(\alpha \otimes \alpha')(m \otimes n) = \alpha m \otimes \alpha' n$$

**Proposition 1.7.3.** Let  $R$  be a ring,  $M$  and  $N$  modules,

- Then the switch map  $(m, n) \mapsto (n, m)$  induces an isomorphism

$$M \otimes_R N = N \otimes_R M$$

- The multiplication on  $M$  induces an isomorphism

$$R \otimes_R M = M$$

*Proof.*

The switch map induces an isomorphism between  $M \otimes_R N = N \otimes_R M$ .

Define  $\beta : R \times M \rightarrow M$  by  $\beta(x, m) := xm$ , then  $\beta$  is bilinear and we have for any  $\alpha : R \times M \rightarrow P$ , define  $\gamma : M \rightarrow P$  by  $\gamma(m) = \alpha(1, m)$  and then  $\alpha = \gamma\beta$ , where  $\gamma$  is unique since  $\beta$  surjective and hence  $M \cong R \otimes M$  since

$$\begin{array}{ccc} R \times M & \xrightarrow{\beta'} & P \\ \downarrow & \searrow \beta'' \quad \nearrow \beta & \uparrow \gamma \\ R \otimes M & & M \end{array}$$

let  $P$  be  $M$  and  $R \otimes M$  and we are done.

**Definiton 1.7.3.** Let  $R$  and  $R'$  be rings. An abelian group  $N$  is an  $(R, R')$ -bimodule if it is both an  $R$ -module and an  $R'$ -module if  $x(x'n) = x'(xn)$  for all  $x \in R, x' \in R'$  and  $n \in N$ .