Chapter 1 Preliminaries

Lemma 1.1

Denote A, B, C, D are $N \times N$ matrices and we will have

$$1_{\det A \neq 0} \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det \left(\begin{bmatrix} A & 0 \\ C & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} 1 & A^{-1}B \\ 0 & 1 \end{bmatrix} \right) = \det A \det(D - CA^{-1}B)$$

♡

Lemma 1.2

♡

Chapter 2 Real Wigner Matrix

2.1 Wigner theorem

Definition 2.1

$Z_{i,j}, i<j, Y_i$ are two independent families of i.i.d., zero mean and real-valued random variables with $EZ_{1,2}^2 = 1$ and

$$r_k := \max(E|Z_{1,2}|^k, E|Y_1|^k) < \infty$$

and we call

$$X_N(j,i) = X_N(i,j) = Z_{i,j}/\sqrt{N}(i<j) + Y_i/\sqrt{N}(i=j)$$

a Wigner matrix, and if $Z_{i,j}, Y_i$ are Gaussian, we call it Gaussian Wigner matrix.
Let λ_i^N be the eigenvalues of X_N with $\lambda_1^N \leq \lambda_2^N \leq \cdots \leq \lambda_N^N$ and define the empirical distribution to be

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^N}$$

and the standard semicirble distribution as $\sigma(x)dx$ with

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\chi_{|x|\leq 2}$$

♣

Theorem 2.1

For a Wigner matrix, the empirical measure L_N converges weakly in probability to the standard semicircle distribution, i.e. for any $f \in C_b(\mathbb{R}), \epsilon > 0$

$$\lim_{N\to\infty} P(|\langle L_N, f| - |\langle \sigma, f\rangle| > \epsilon) = 0$$

♡

Theorem 2.2

Define the moments $m_k := \langle, \sigma, x^k\rangle$ and we will have

$$m_{2k} = C_k, m_{2k+1} = 0$$

where C_k is the Catalan numbers

$$C_k = C_{2k}^k/(k+1)$$

♡

Definition 2.2

Define the ditribution $\overline{L}_N = EL_N$ by $\langle \overline{L}_N, f\rangle = E\langle L_N, f\rangle$ for $f \in C_b$ and $m_k^N := \langle \overline{L}_N, x^k\rangle$.

♣

Lemma 2.1

a. For $k \in \mathbb{N}$, we have $\lim_{N\to\infty} m_k^N = m_k$.
b. For $k \in \mathbb{N}$ and $\epsilon > 0$, we have

$$\lim_{N\to\infty} P(|\langle L_N, x^k| - \langle \overline{L}_N, x^k\rangle| > \epsilon) = 0$$

♡

Lemma 2.2

(Hoffman-Wielandt) Let A, B be $N \times N$ symmetric matrices, with eigenvalues $\lambda_1^A \leq \lambda_2^A \leq \cdots \leq \lambda_N^A$ and $\lambda_1^B \leq \lambda_2^B \leq \cdots \leq \lambda_N^B$, then

$$\sum_{i=1}^{N} |\lambda_i^A - \lambda_i^B|^2 \leq tr(A-B)^2$$

♡

Theorem 2.3

(Maximal eigenvalue) Consider a Wigner matrix X_N satisfying $r_k \leq k^{Ck}$ for some constant C and all $k \in \mathbb{N}$, we will have λ_N^N converges to 2 in probability. ♡

Theorem 2.4

(CLT for linear statistics of eigenvalues of Wigner matrices)Denote $W_{N,k} := N(\langle L_N, x^k\rangle - \langle \overline{L}_N, x^k\rangle)$ then we will have

$$\lim_{N\to\infty} P\Big(\frac{W_{N,k}}{\sigma_k} \leq x\Big) = \phi(x)$$

where ϕ is the Gaussian distribution and

$$\sigma_k^2 = \lim_{N\to\infty} EW_{N,k}^2$$

♡

2.2 Complex Wigner matrices

Definition 2.3

For two independent families of i.i.d. complex-valued random variables $Z_{i,j}, i < j, Y_i$ such that $EZ_{1,2}^2 = 0, E|Z_{1,2}|^2 = 1$ and

$$r_k := \max(E|Z_{1,2}|^k, E|Y_1|^k) < \infty$$

and $N \times N$ matrix X_N with

$$X_N(j,i)^* = X_N(i,j) = Z_{i,j}/\sqrt{N}(i<j) + Y_i/\sqrt{N}(i=j)$$

is a Hermitian Wigner matrix and define the Gaussian Hermitian Wigner matrix similarly. Since the eigenvalues are real, we may use the old denotation. ♣

Theorem 2.5

For a Hermimtian Wigner matrix, the empirical measure L_N converges weakly in probability to the standard semicircle distribution, i.e. for any $f \in C_b(\mathbb{R}), \epsilon > 0$

$$\lim_{N\to\infty} P(|\langle L_N, f| - |\langle \sigma, f\rangle| > \epsilon) = 0$$

♡

Lemma 2.3

a. For $k \in \mathbb{N}$, we have $\lim_{N\to\infty} m_k^N = m_k$.
b. For $k \in \mathbb{N}$ and $\epsilon > 0$, we have

$$\lim_{N\to\infty} P(|\langle L_N, x^k| - \langle \overline{L}_N, x^k\rangle| > \epsilon) = 0$$

♡

Definition 2.4

Let $\xi_{i,j}, \eta_{i,j}$ to be an i,i,d, family of real mean 0 and variance 1 Gaussian random variables. We define $P_i^{(1)}$ to be the laws of the random matrices $(Z_{i,j}), Z_{i,i} = \sqrt{2}\xi_{i,i}, Z_{i,j} = Z_{j,i} = \xi_{i,j}, i < j$ and $P_i^{(2)}$ is that of $(U_{i,j}), U_{i,i} = \xi_{i,i}, U_{i,j} = \overline{U_{j,i}} = \dfrac{\xi_{i,j} + i\eta_{i,j}}{\sqrt{2}}, i < j$. A random matrix $X \in \mathcal{H}_N^{(\beta)}$ with law $P_N^{(\beta)}$ is said to belong to Gaussian orthogonal ensemble (GOE) or Gaussian unitrary ensemble (GUE). ♣

We know for $X(N)$ in GOR or GUE, we will have $X_N := X(N)/\sqrt{N}$ tends to the semicircle law.

Chapter 3

3.1 The method of Laplace

The method is aiming to deal with an asymptotic integral like

$$\int f(t)^s g(t)dt$$

when $s \to \infty$, with the condition for $f : \mathbb{R} \to \mathbb{R}^+$ and constant a and positive constants s_0, K, L, M and $\mathcal{G}(a, \epsilon_0, s_0, f, K, L, M)$ to be all measurable functions g such that

a. $|g(a)| \leq K$

b. $\sup_{0<|x-a|\leq\epsilon_0} \left|\dfrac{g(x) - g(a)}{x - a}\right| \leq L$

c. $\int f(x)^{s_0}|g(x)|dx \leq M$ then

Theorem 3.1

(Laplace) Let $f : \mathbb{R} \to \mathbb{R}^+$ be a function such that for some $a \in \mathbb{R}$ and some positive constants ϵ_0, c and

a. $f(x) \leq f(x')$ if $a - \epsilon_0 \leq x \leq x' \leq a$ or $a \leq x' \leq x \leq a + \epsilon_0$.

b. For all $\epsilon < \epsilon_0$, $\sup_{|x-a|>\epsilon} f(x) \leq f(a) - c\epsilon^2$.

c. $f(x)$ has two continuous derivatives in $(a - 2\epsilon_0, a + 2\epsilon_0)$.

d. $f''(a) < 0$.

Then for any $g \in \mathcal{G}(a, \epsilon_0, s_0, f, K, L, M)$ we have

$$\lim_{s\to\infty} \sqrt{s} f(a)^{-s} \int f(x)^s g(x)dx = \sqrt{-\frac{2\pi f(a)}{f''(a)}} g(a)$$

and for fixed $a, \epsilon, s_0, K, L, M$ for $g \in \mathcal{G}(a, \epsilon_0, s_0, f, K, L, M)$ the convergence is uniform.

♡