Bonus 04 - MATH 722

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Problem

Prove if f is a polynomial on $\mathbb{C}$, then the zeros of f' are contained in the closed convex hull of the zeros of f.

Sol.

We know f can be expressed by

$$f = C\prod_{i=1}^{m}(z - z_i)^{n_i}$$

for some $z_i, 1 \le i \le m$ complex and $n_i, 1 \le i \le n$ integers. If $m = 1$, then the problem is trivial, we assume $m \ge 2$ and then we may know

$$f' = C\prod_{i=1}^{m}(z - z_i)^{n_i - 1}\Big[\sum_{i=1}^{m} n_i \prod_{1\le j\le m, j\ne i}(z - z_j)\Big]$$

and it suffices to show that the zeros of

$$\sum_{i=1}^{m} n_i \prod_{1\le j\le m, j\ne i}(z - z_j)$$

are contained in the convex hull of the zeros of $\{z_1, \cdots, z_m\}$, now we consider if ξ is a zero of the polynomial above and not in the convex hull of $\{z_1, \cdots, z_m\}$, then we may find a straight line on the complex plane separting ξ and $z_1, z_2, \cdots, z_m$ and hence if we denote $\theta_i = Arg(z_i - \xi), 1 \le i \le m$, without loss of the generality, we may assume $\theta_1 \le \theta_2 \le \cdots \le \theta_n$ and

$$\theta_n - \theta_1 < \pi$$

Then we let

$$Arg\Big(\frac{\prod_{i=1}^{m}(z_i - \xi)}{(z_j - \xi)}\Big) = \phi_j$$

for $1 \le j \le m$ and we have

$$\phi_j \in [\sum_{i=1}^{m}\theta_i - \theta_n, \sum_{i=1}^{m}\theta_i - \theta_1] \in (a - \pi/2, a + \pi/2)$$

for some $a \in \mathbb{R}$ and hence

$$\sum_{i=1}^{m} n_i \prod_{1\le j\le m, j\ne i}(\xi - z_j) \ne 0$$

which can be implied by considering $\Lambda(z)$ be the image of z by $\mathbb{C} \to \mathbb{R}^2$ and then we know

$$\Lambda\Big[\sum_{i=1}^{m} n_i \prod_{1\le j\le m, j\ne i}(\xi - z_j)\Big] \cdot \Lambda(e^{i\alpha}) < 0$$

since $n_i \geq 1$, which means

$$\sum_{i=1}^{m} n_i \prod_{1 \leq j \leq m, j \neq i} (\xi - z_j) \neq 0$$

Therefore, ξ cannot be a zero of

$$\sum_{i=1}^{m} n_i \prod_{1 \leq j \leq m, j \neq i} (z - z_j)$$

and we are done.