NOTES FOR DIRECTED POLYMER BY F. COMETS

Based on the Lecture Notes by C.S.Z.

Author
Wells Guan

Contents

1 Thermodynamics and Phase Transition

1.1 Useful Conclusions

Definiton 1.1.1. $f : \mathbb{R}^k \to \mathbb{R}^k$ is increasgin if $f(x) < f(y)$ iff $x_i < y_i$.

Definiton 1.1.2. (Positively Associated)

A family $X = (X_i)_{i=1}^k$ of real r.v.s on the same probability space are **postive associated** if for any $f, g : \mathbb{R}^k \to \mathbb{R}$ bounded, increasing

$$Ef(X)g(X) \geq Ef(X)Eg(X)$$

Proposition 1.1.1. (FKG-Harris Inequality)

A family of independent, real random variables is positively associated.

1.2 Markov Property and the Partition Function

Definiton 1.2.1. (Partition Function)

For $n, m \geq 1, x \in \mathbb{Z}^d$, the r.v. on $(\Omega = \mathbb{R}^{\mathbb{N}\times\mathbb{Z}^d}, \mathbb{P})$

$$Z_m^\beta \circ \theta_{n,x}(\omega) = Z_m(\theta_{n,x}\omega, \beta) = E_x \exp\left(\sum_{t=1}^{m} \beta\omega(t+n, S_t)\right) \quad \text{(finite and definitely positive)}$$

is the partition function of the polymer of length m starting at x at time n.

Proposition 1.2.1. $Z_m \circ \theta_{n,x} \stackrel{d}{=} Z_m$.

Proposition 1.2.2. Let $\mathcal{F}_n = \sigma\{S_t, t \leq n\}$ and we will have

$$Z_m \circ \theta_{n,x}(\omega) = E(\exp\beta(H_{n+m}(S) - H_n(S))|\mathcal{F}_n)$$

on the event $\{S_n = x\}$, i.e.

$$\begin{aligned} Z_m \circ \theta_{n,x}(\omega)\chi_{S_n=x} &= E(\exp\left(\beta(H_{n+m}(S) - H_n(S))\right)\chi_{S_n=x}|\mathcal{F}_n) \\ &= E(\exp\left(\beta(H_{n+m}(S) - H_n(S))\right)|\mathcal{F}_n)\chi_{S_n=x} \end{aligned}$$

Proposition 1.2.3. We will have

$$Z_{n+m} = E(\exp\beta H_n(S) Z_m \circ \theta_{n,S_n})$$

which is referred to the Markov property, and we will have

$$Z_{n+m} = Z_n \times E_n^{\beta,\omega}(Z_m \circ \theta_{n,S_n})$$

where $E_n^{\beta,\omega}$ referes the expectation under the polymer measure.

1.3 Markov Chain under the Polymer Measure

Proposition 1.3.1. For all $\beta \in \mathcal{D} := \{\beta, p \text{ differentiable at } \beta\}$ and almost every environment ω, we have

$$\lim_{n\to\infty} E_{P_n^{\beta,\omega}}(H_n(S)/n) = \lim_{n\to\infty} \mathbb{E}(E_{P_n^{\beta,\omega}}(H_n(S)/n)) = p'(\beta)$$

Moreover, for all $\beta \in \mathbb{R}$ we have

$$p'(\beta-) \le \liminf_{n\to\infty} E_{P_n^{\beta,\omega}}(H_n(S)/n) \le \limsup_{n\to\infty} E_{P_n^{\beta,\omega}}(S_n(S)/n) \le p'(\beta+)$$

Proof. Notice that we have already have p_n, p are convex and hence we know $p'(\beta-), p'(\beta+)$ always exists. Let take a look of $p'_n(\beta)$ again:

$$p'_n(\beta) = \frac{1}{nZ_n}\frac{\partial}{\partial\beta}\int \exp(\beta H_n(x))P(dx) = \frac{1}{n}E_{P_n^{\beta,\omega}}(H_n(S))$$

since $\int f(x)P(dx)$ is a finite summation of $f(x)$. Now consider

$$(\mathbb{E}p_n)'(\beta) = \frac{\partial}{\partial\beta}\mathbb{E}p_n = \mathbb{E}p'_n(\beta)$$

since $\sum_x (2d)^{-n}\max\{1, \exp(TH_n(x))\}$ is L^1 and we may apply the DCT for any $\beta \in [0, T)$. Notice $Ep_n \to p$ a.s. for all β, then we know

$$p'(\beta-) = \inf_{\epsilon>0}\frac{p(\beta) - p(\beta-\epsilon)}{\epsilon} = \inf_{\epsilon>0}\lim_{n\to\infty}\frac{\mathbb{E}p_n(\beta) - \mathbb{E}p_n(\beta-\epsilon)}{\epsilon} \le \liminf_{n\to\infty}\mathbb{E}p'_n(\beta)$$

and we can obtain the second inequality similarly. Now, for somewhere $p'(\beta)$ exists, we mya know $\lim_{n\to\infty}\mathbb{E}p'_n(\beta)$ exists and then we may replace $\mathbb{E}p_n$ above with p_n in the view of $\mathbb{P}$-a.s. which means for almost every ω. □

Theorem 1.3.2. The functions $\beta \mapsto \lambda(\beta) - \mathbb{E}p_n$ and $\beta \mapsto \lambda(\beta) - p(\beta)$ are non-decreasing on $\mathbb{R}^+$ and non-increasing on $\mathbb{R}^-$.

Proof. We will conmpute

$$\begin{aligned}\frac{\partial}{\partial\beta}\mathbb{E}\ln Z_n &= \mathbb{E}EZ_n^{-1}H_n(S)\exp(\beta H_n(S)) \\ &= E\mathbb{E}Z_n^{-1}H_n(S)\exp(\beta H_n(S))\end{aligned}$$

by Fubini, and we notice

$$\begin{aligned}Z_n &= \sum_x (2d)^{-n}\exp(\beta\sum_{t=1}^n \omega(t, x_t)) = f(\omega(t,y))_{1\le t\le n, |y|_1\le n} \\ H_n(x) &= \sum_{t=1}^n \omega(t, x_t) = g(\omega(t,y))_{1\le t\le n, |y|_1\le n} \\ \exp(\beta H_n(x)) &= \prod_{t=1}^n \exp(\beta\omega(t, x_t)) = h(\omega(t,y))_{1\le t\le n, |y|_1\le n}\end{aligned}$$

and define

$$f_M = sgn(f)\min\{|f|, M\}, g_M = sgn(g)\min\{|g|, M\}, h_M = sgn(h)\min\{|h|, M\}$$

then for $\beta \geq 0$, we have f, g, h increasing and $\beta \leq 0$ f, h decreasing, and it is easy to check h/f is increasing with $\beta \geq 0$. Now we may use the FKG-Harris and we will have for fixed $x, \beta \geq 0$,

$$\mathbb{E}Z_n^{-1}H_n(x)\exp\left(\beta H_n(x)\right) = \mathbb{E}(h/f)ghh^{-1} =$$

where $f^{-1}gh$ is integrable, so we may use DCT and we will have

$$\mathbb{E}f^{-1}gh = \lim_{M\to\infty} f_M^{-1}g_M h_M \leq \lim_{M\to\infty} \mathbb{E}1/h_M \mathbb{E}h_M/f_M \mathbb{E}g_M h_M = \mathbb{E}1/h\mathbb{E}h/f\mathbb{E}gh$$

since $1/h_M$ decreasing, and we will get an oppositive inequality if $\beta \leq 0$ since h/f decreasing and $g_M h_M$ decreasing. Then

$$\frac{\partial}{\partial\beta}\mathbb{E}\ln Z_n \leq n\lambda'(\beta)E\mathbb{E}Z_n^{-1}\exp\left(\beta H_n(S)\right) = n\lambda'(\beta)$$

and with the opposite inequality when $\beta \leq 0$, so we have

$$\mathbb{E}p_n'(\beta) \leq \lambda'(\beta)$$

on $\mathbb{R}^+$ and the opposite on $\mathbb{R}^-$. The monocity of $\lambda - p$ is induced by limit. $\square$

Theorem 1.3.3. Suppose $d \geq 3$ and the L_2 condition holds, then

$$\lim_{n\to\infty} P_n^{\beta,\omega}$$

2 Martingale Approach and L2 Region

2.1 Checklist

- Proof of $s = \infty$
- compute φ, ψ in 3.3

2.2 Useful Conclusions

Theorem 2.2.1. (Martingale Convergence Theorem)

If X_n is a submartingale with $\sup EX_n^+ < \infty$, then X_n converges to some L^1 limit X a.s. as $n \to \infty$.

Theorem 2.2.2. (Kolmogorov's 0-1 Law)

If $X_1, X_2, \cdots$ are independent and $A \in \mathcal{T} := \cap \mathcal{F}_n'$ then $P(A) = 0$ or 1, where $\mathcal{F}_n' = \sigma(X_n, X_{n+1}, \cdots)$.

Proof. We prove by some steps, first we show that for $A \in \sigma(X_1, \cdots, X_k)$ and $B \in \sigma(X_{k+1}, X_{k+2}, \cdots)$, we have A, B independent. If $B \in \sigma(X_{k+1}, \cdots, X_{k+j})$, we have A, B independent. And we know $\cup_j \sigma(X_{k+1}, \cdots, X_{k+j})$ is a π-system. And we only need to check $\{B, P(A)P(B) = P(A)P(B)\}$ is a λ-system. Ω is obviously in it firstly and for $B \subset B'$, we have

$$P(A(B' - B)) = P(AB') - P(AB) = P(A)(P(B') - P(B)) = P(A)P(B' - B)$$

and if B_n increases to B, we have

$$P(AB) = \lim_{n\to\infty} P(AB_n) = \lim_{n\to\infty} P(A)P(B_n) = P(A)P(B)$$

and we are done.

So we know $A \in \sigma(X_1, \cdots, X_k)$ is independent with B if $B \in \mathcal{T}$ the tail algebra, and similarly we may check that for any $A \in \sigma(X_1, \cdots, X_n)$ is independent with B and hence B is independent with itself, which means $P(B) = 0$ or 1.

□

Proposition 2.2.3. Every convex function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. For any x, let $y < x$ and $z > x$, we know that for any $t \in (y, z)$

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(x) - f(t)}{x - t} \geq \frac{f(x) - f(z)}{x - z}$$

and hence

$$|f(x) - f(t)| \leq \max\left\{\left|\frac{f(x) - f(z)}{x - z}\right|, \left|\frac{f(x) - f(y)}{x - y}\right|\right\} |x - t|$$

and we are done. □

Definiton 2.2.1. (Uniformly Integrable)

A collection of r.v.s $X_i, i \in I$ is **uniformly integrable** if

$$\lim_{M\to\infty}\left(\sup_{i\in I} E(|X_i|; |X_i| > M)\right) = 0$$

Theorem 2.2.4. Let $\phi \geq 0$ be some function with $\phi(x)/x \to \infty$ as $x \to \infty$. If $E\phi(|X_i|) \leq C$ for all $i \in I$, then X_i is uniformly integrable.

Proposition 2.2.5. Suppose that $E|X_n| < \infty$ for all n. If $X_n \to X$ in probability then the following are equivalent:

- $\{X_n\}_{n\geq 0}$ is uniformly integrable.
- $X_n \to X$ in L^1.
- $E|X_n| \to E|X| < \infty$.

Theorem 2.2.6. For a submartingale, the following are equivalent:

- It is uniformly integrable.
- It converges a.s. and in L^1
- It converges in L^1.

Theorem 2.2.7. (L^p convergence theorem)
If X_n is a martingale with $\sup E|X_n|^p$ fintie and $p > 1$, then $X_n \to X$ a.s. and in L_p, where X is given by the martingale convergence theorem.

Theorem 2.2.8. (Skorokhod Representation Theorem)
For some distribution functions F_n, if F_n converges to some F_∞, then there are random variables $Y_n, 1 \leq n \leq \infty$ such that $Y_n \to Y_\infty$ a.s.

Proof. Let $\Omega = (0,1)$ and P to be the Lebegues measure, and let $Y_n(y) = \sup\{x, F_n(x) < y\}$ and we know Y_n is nondecreasing, then if $Y_n(y) \leq a$, notice $F_n(Y_n(y)) \geq y$, then we have $y \leq F_n(a)$, which means

$$P(Y_n \leq a) \leq F_n(a)$$

and if $P(Y_n \leq a) < F_n(a)$, then if $Y_n(z) \leq a$, we know $z < F_n(a)$, however $Y_n(F_n(a)) = a$ which means $F_n(a) < F_n(a)$ and hence a contradiction, so Y_n has the distribution function of F_n.

Let $a_x = Y_\infty(x) = \sup\{y, F(y) < x\}$ and $b_x = \inf\{y, F(y) > x\}$, then we know $a_x \leq b_x$ and define $\Omega_0 = \{x, (a_x, b_x) \neq \emptyset\}$, then Ω_0 is at most countable and we prove for any $x \in \Omega - \Omega_0$, we have $Y_n(x) \to Y_\infty(x)$. Firstly, for these x consider $y < Y_\infty(x)$ for some y such that F is continuous at y, then $F_n(y) \to F(y)$ and notice there is some $y' > y$ such that $F(y') < x$ and hence $F(y) < x$, so there is some N such that for any $m \geq N$ we have $F_m(y) < x$ and hence $y \leq Y_m(x)$ and then

$$y \leq \liminf_{n\to\infty} Y_n(x)$$

for any $y < Y_\infty(x)$, which means

$$\liminf_{n\to\infty} Y_n(x) \geq Y_\infty(x)$$

Similarly for any z such that $z > Y_\infty(x)$ and using the assumption that $a_x = b_x$. $\square$

Theorem 2.2.9. X_n converges to X_∞ weakly if and only if

$$Eg(X_n) \to Eg(X_\infty)$$

for any bounded and continuous function g.

Proof. To see the sufficiency, we know we may find $X_n \overset{d}{=} Y_n$ for $1 \leq \infty$ and $Y_n \to Y_\infty$ a.s. in some probability space, then we have $Eg(X_n) = Eg(Y_n)$ and by the DCT, we know $Eg(Y_n) \to Eg(Y_\infty)$ for any bounded and continuous function g.

For the necessity, we know

$$P(X_n \leq a) = E(\chi_{(-\infty,a]}(X_n))$$

and we consider some slope continuous approaching δ_ϵ for $\chi_{(-\infty,a]}$, now we have

$$P(X_n \leq a) \leq E\delta_\epsilon(X_n) \leq P(X_n \leq a+\epsilon)$$

and if there is $q > 0$ such that $|F_n(a) - F_\infty(a)| > q$ infinitely often, then notice

$$|F_n(a) - F_\infty(a)| \leq |F_n(a) - F_n(a+\epsilon)| + |F_\infty(a) - F_\infty(a+\epsilon)| + |E\delta_\epsilon(X_n) - E\delta_\epsilon(X_\infty)|$$

and then there will be a contradiction and we are done. $\square$

Theorem 2.2.10. (L^p maximum inequality)

If X_n is a submartingale then for $1 < p < \infty$, we have

$$E(\bar{X}_n^p) \leq \left(\frac{p}{p-1}\right) E(X_n^p)^p$$

where $\bar{X}_n = \max_{0 \leq m \leq n} X_m^+$.

2.3 Phase Transition of Weak Disorder and Strong Disorder Phase

Definiton 2.3.1. (Normalized Partition Function)

$$W_n = Z_n(\omega, \beta)\exp\left(-n\lambda(\beta)\right)$$

where $\lambda(\beta) = \mathbb{E}\exp\left(\beta\omega(n,x)\right)$ which is not related to n, x.

Theorem 2.3.1. The limit

$$W_\infty = \lim_{n\to\infty} W_n$$

exists $\mathbb{P}$-a.s. and either the limit W_∞ is a.s. positive or it is a.s. zero.

Remark. We will show W_n is a martingale and use martingale convergence theorem and Kolmogorov's 0-1 law for W_∞.

Proof. Firstly, notice for a fixed path x,

$$\xi_n = \exp\left(\beta H_n(x) - n\lambda(\beta)\right)$$

is a positive martingale w.r.t. the filtration $G_n = \sigma\{\omega(j,x), j \le n\}$. And then we know

$$W_n = E\exp(\beta H_n(S) - n\lambda(\beta)) = \sum_{n\text{-length paths } x} (2d)^{-1}\xi_n(x)$$

is a positive martingale w.r.t. Also, consider

$$\mathbb{E}\xi_n = 1$$

and hence we may know $\mathbb{E}W_n = 1$ and hence we may apply the martingale convergence theorem and get

$$W_\infty = \lim_{n\to\infty} W_n$$

exists and nonnegative $\mathbb{P}$-a.s. and $\mathbb{E}W_\infty < \infty$. Now assueme $\mathcal{F}'_n = \sigma\{\omega(j,x), j \ge n, |x|_1 \le j\}$ and let $\mathcal{F}' = \cap \mathcal{F}'_n$. Then for any $A \in F'$, we have $\mathbb{P}(A) \in \{0,1\}$ since we may apply the proof of Kolmogorov's 0-1 Law by replacing $\sigma(X_k, \cdots, X_{k+j})$ by some family of σ-algebras $\mathcal{F}_i$ and consider $\sigma(\mathcal{F}_k, \cdots, \mathcal{F}_{k+j})$. Now we only need to check$\{W_\infty = 0\} \in \mathcal{F}$, which coms form

$$\begin{aligned} W_\infty &= \lim_{m\to\infty} W_{n+m} \\ &= E(\xi_n(S) \times \lim_{m\to\infty} W_m \circ \theta_{x,S_n}) \\ &= \sum_x P(dx)\exp(\beta H_n(x) - n\lambda(\beta))W_\infty \circ \theta_{n,x_n} \\ &= W_n \sum_x P_n^{\beta,\omega}(S_n = x)W_\infty \circ \theta_{n,x} \end{aligned}$$

and hence

$$\{W_\infty = 0\} = \cap_{P(S_n=x)>0}\{W_\infty \circ \theta_{m,x} = 0\} \in \mathcal{F}_n$$

for any n and we are done. □

Definiton 2.3.2. (Phase Transition)

The polymer is the **weak disorder** phase when $\mathbb{P}(W_\infty > 0) = 1$ and the **strong disorder** phase when $\mathbb{P}(W_\infty = 0) = 1$.

Proposition 2.3.2. If $W_\infty > 0$, then $p(\beta) = \lambda(\beta)$, since

$$p(\beta) = \lambda(\beta) + \lim_{n\to\infty} n^{-1}\ln W_n.$$

Furthermore, we have

$$\lim_{n\to\infty} n^{-1}E_n^{\beta,\omega}H_n = \lambda'(\beta)$$

with $n \to \infty$.

Proof. Notice that we have

$$p(\beta) = \lim_{n\to\infty} p_n(\beta) = \lim_{n\to\infty} n^{-1}\ln W_n + \lambda(\beta) \quad \mathbb{P}\text{-a.s.}$$

so if $W_\infty > 0$, then we will have $p(\beta) = \lambda(\beta)$ but there is no other arguments if $W_\infty = 0$.

Since we have already know p, λ convex, so continuous, and hence $p(\beta) = \lambda(\beta)$ for all $\beta, \mathbb{P}$-a.s. for by choosing β rational and make the union. So we only need to find $p'(\beta)$.

Notice

$$p_n'(\beta) = \frac{\partial}{\partial\beta}\frac{1}{n}\ln(\sum_x (2d)^{-n}\exp(\beta H_n(x))) = \frac{1}{nZ_n}EH_n\exp(\beta H_n(x)) = \frac{1}{n}E_n^{\beta,\omega}H_n$$

and we know for the region where p is differential, which is the whole set since $p = \lambda$. $\square$

Proposition 2.3.3. There exists $\bar{\beta}_c(\mathbb{P}, d) \in [0, \infty]$ such that

$$\begin{cases} W_\infty > 0, \mathbb{P}\text{-a.s.} & \text{if } \beta \in [0, \bar{\beta}_c) \\ W_\infty = 0, \mathbb{P}\text{-a.s.} & \text{if } \beta > \bar{\beta}_c \end{cases}$$

Proof. We know W_n^δ is uniformly integrable since let $\phi = |x|^{1/\delta}$ and we know $E\phi(|W_n|^\delta) = 1$ for all $\delta \in (0,1)$ and hence we have

$$\lim_{n\to\infty}\mathbb{E}W_n^\delta = \mathbb{E}W_\infty^\delta$$

which is either 0 or strictly positive. Now consider

$$\begin{aligned}\frac{\partial}{\partial\beta}\mathbb{E}W_n^\delta &= \mathbb{E}(\delta W_n^{\delta-1}\frac{\partial}{\partial\beta}\sum_x (2d)^{-n}\exp\left(\beta H_n(x) - n\lambda(\beta)\right)) \\ &= \mathbb{E}(\delta W_n^{\delta-1}E((H_n - n\lambda'(\beta))\xi_n))\end{aligned}$$

where since

$$W_n^\delta E(H_n - n\lambda')\xi_n \leq W_n^\delta + Z_n^\delta \sum (2d)^{-1}H_n(x)\xi_n\exp(-n(\delta-1)\lambda(\beta))$$

and notice

$$H_i(x)H_j(x)\exp(\beta(H_i + H_j)(x))$$

are all integrable and the derivative is correct. Then

$$\begin{aligned}\frac{\partial}{\partial\beta}\mathbb{E}W_n^\delta &= \mathbb{E}(\delta W_n^{\delta-1}E((H_n - n\lambda'(\beta))\xi_n)) \\ &= \delta E\mathbb{E}(\xi_n W_n^{\delta-1}(H_n - n\lambda')) \\ &\leq \delta E(\mathbb{E}W_n^{\delta-1}\mathbb{E}(\xi_n(H_n - n\lambda'))) \quad \text{(by FKG)} \\ &= 0\end{aligned}$$

since

$$\mathbb{E}(\exp(\beta\omega)(\omega - \lambda')) = \mathbb{E}(\omega\exp(\beta\omega)) - \mathbb{E}(\omega\exp(\beta\omega)) = 0$$

$\square$

which means $\mathbb{E}W_n^\delta$ is decreasing and hence $\mathbb{E}W_\infty^\delta$ non-increasing

2.4 L^2 Region

Proposition 2.4.1. The return probability

$$\pi_d := P(S_n = 0 \text{ for some } n \geq 1) \text{ is } \begin{cases} 1 & \text{if } d \leq 2 \\ < 1 & \text{if } d \geq 3 \end{cases}$$

and $\pi_{d+1} < \pi_d$ for all $d \geq 3$.

Theorem 2.4.2. Suppose that $d \geq 3$ and the L^2 condition:

$$\lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_d)$$

holds, then $W_\infty > 0$ $\mathbb{P}$-a.s.

Remark. We will show W_n is a L^2 martingale under the condition, and implies that W_∞ have a positive expectation.

Proof. We will use the L^2 martingale to compute $\mathbb{E}W_\infty^2$ and see if $W_\infty = 0$ $\mathbb{P}$-a.s. Since

$$W_n = \exp\left(-n\lambda(\beta)\right) E(\exp\left(\beta H_n(S)\right))$$

we may know that consider an independent copy of S and the product $(\Omega^2, \mathcal{F}^{\otimes 2})$ and then

$$E_{P^{\otimes 2}} \exp\left(\beta\left[H_n(S) + H_n(S')\right] - 2n\lambda(\beta)\right)$$

and then by Fubini

$$\begin{aligned} \mathbb{P}W_n^2 &= E_{P^{\otimes 2}} \mathbb{E} \prod_{t=1}^{n} \exp\left(\beta(\omega(t, S_t) + \omega(t, S_t') - 2\lambda(\beta))\right) \\ &= E_{P^{\otimes 2}} \mathbb{E} \prod_{t=1}^{n} \left(\exp \lambda(2\beta) \chi_{(S_t = S_t')} + \chi_{(S_t \neq S_t')}\right) \\ &= E_{P^{\otimes 2}} \exp\left(\lambda_2(\beta) N_n\right) \end{aligned}$$

where N_n denotes the intersections of S, S' up to time n. Notice N_n increases to N_∞ and and hence $\mathbb{E}W_n^2$ will increase to $E_{P^{\otimes 2}} \exp\left(\lambda_2(\beta) N_\infty\right)$. Consider a simple symmetric random walk $\tilde{S}$ with increment $\tilde{s}_{2k+1} = s_{k+1}, \tilde{s}_{2k+2} = -s_{k+1}'$ and then we know that

$$\{\tilde{S} \text{ return}\} = \{S - S' \text{ return}\}$$

and hence N_∞ must have the geometrically distributed with $p = \pi_d$. Then

$$E_{P^{\otimes 2}} \exp\left(\lambda_2(\beta) N_\infty\right) = \sum_{k=0}^{\infty} (1 - \pi_d) \pi_d^k \exp\left(k\lambda_2(\beta)\right)$$

which is

$$E_{P^{\otimes 2}} \exp\left(\lambda_2(\beta) N_\infty\right) = \begin{cases} \dfrac{1 - \pi_d}{1 - \pi_d \exp\left(\lambda_2(\beta)\right)} & \text{if } \lambda_2(\beta) < -\ln \pi_d \\ \infty & \text{if } \lambda_2(\beta) \geq -\ln \pi_d \end{cases}$$

So we have $\sup_n \mathbb{E}W_n^2$ is finite if and only if $\lambda_2(\beta) < -\ln \pi_d$, then we will have the convergence in L^2 and hence

$$\mathbb{E}W_\infty^2 = \frac{1-\pi_d}{1-\pi_d \exp(\lambda_2(\beta))} > 0,$$

which means $W_\infty > 0$ $\mathbb{P}$-a.s. $\square$

Definiton 2.4.1. (L_2 Region)

The set of β's defined by the L_2 condition is called the L_2 region. For $d \geq 3$, there will be a non-empty interval $(0, \beta_{L_2})$ is in the L_2 region.

Proof. Notice

$$\lambda_2'(\beta) = 2[\lambda'(2\beta) - \lambda'(\beta)]$$

which is nonnegative, and hence increasing on the postive axis and nonpositive, and hence decreassing on the negative axis. Notice

$$1/\pi_d > 1$$

iff $d \geq 3$, and $\lambda_2(\beta) = 0$ at $\beta = 0$, so we may know for $d \geq 3$ we have a nonnegative

$$\beta_{L_2} = \sup\{\beta \geq 0, \lambda_2(\beta) \leq \ln(1/\pi_d)\} > 0$$

and we know $p = \lambda$ when $\beta \leq \beta_{L_2}$. $\square$

Corollary 2.4.3. Let $s = ess\sup_{\mathbb{P}} \omega(t,x)$. We have

$$\lim_{\beta\to\infty} \lambda_2(\beta) = -\ln \mathbb{P}(\omega(t,x) = s)$$

where $s = \infty$ makes the sense that $\mathbb{P}(\omega(t,x) = \infty) = 0$.

Proof. Let q be a measure defined by

$$q(A) = \mathbb{P}(\omega \in A)$$

for borel set A, and then for any t we have

$$e^{\beta(t-h)} q([t-h,t]) \leq \mathbb{E}(e^{\beta,\omega} \omega \in [t-h,t]) \leq e^{\lambda(\beta)}$$

and

$$\beta(t) + \ln q([t,t+h]) \leq \lambda(\beta)$$

On the other hand, we have if $\epsilon > 0$, then for some r,

$$e^{\lambda(\beta)} - \epsilon \leq \mathbb{E}(e^{\beta\omega}; \omega \in [r-h,r]) + \mathbb{E}(e^{\beta\omega}; \omega \leq r-h)$$

and hence

$$\lambda(\beta) - o(\epsilon) \leq \ln(e^{\beta r} q([r-h,r]) + e^{\beta(r-h)}) \leq \beta r + \ln(q([r-h,r]) + e^{-\beta h})$$

Now we have for any $\epsilon > 0$, there exists r such that

$$\lambda_2(\beta) \leq \ln(q([r-h, r]) + e^{-2\beta h}) + o(\epsilon) - 2\ln(q[r, r+h'])$$
$$\lambda_2(\beta) \geq -2\ln(q([r-h, r]) + e^{-\beta h}) - 2o(\epsilon) + \ln(q[r, r+h']).$$

So if s is finite, we can let $r = s$ and $h \to 0$. If not, define $\omega_n = \min\{\omega, n\}$ then we may know $\lambda_2^{(n)}(\beta) \to -\ln(q[n, \infty))$ and since for any β we have $\lambda_2^{(n)}(\beta) \to \lambda_2(\beta)$ by DCT, so we may know by λ_2 is increasing that λ_2 is infinite. $\square$

Theorem 2.4.4. Under the assumptions that $d \geq 3$ and the L_2 condition holds, we have

$$\lim_{n\to\infty} E_{P_n^{\beta,\omega}} \frac{|S_n|^2}{n} = 1$$

for $\mathbb{P}$-a.s. and for all $f \in C(\mathbb{R}^d)$ with at most polynomial growth at infinity

$$\lim_{n\to\infty} E_{P_n^{\beta,\omega}} f(S_n/\sqrt{n}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x/\sqrt{d}) \exp(-|x|^2/2) dx$$

for $\mathbb{P}$-a.s. and in particular, with Z a d-dimensional gaussian vector $Z \sim \mathcal{N}_d(0, d^{-1}I_d)$, we have

$$P_n^{\beta,\omega}(S_n/\sqrt{n} \in A) \to P(Z \in A)$$

for any borel set A in $\mathbb{P}$-a.s.

Remark. We introduce a family of martingales $(M_n)_{n\geq 1}$ on $(\Omega, \mathcal{G}, \mathbb{P})$ of the form

$$M_n = E\varphi(n, S_n) \exp(\beta H_n(S) - n\lambda(\beta))$$

for a path x and $\varphi : \mathbb{N} \times \mathbb{Z}^d \to \mathbb{R}$ is a function for which we assume

- there are constants $C_i, p \in \mathbb{N}, i = 0, 1, 2$ such that

  $$|\varphi(n, x)| \leq C_0 + C_1|x|^p + C_2 n^{p/2}$$

  for all $(n, x) \in \mathbb{N} \times \mathbb{Z}^d$
- $\Phi_n := \varphi(n, S_n)$ is a martingale on $(\Omega_{traj}, \mathcal{F}, P)$ w.r.t the filtration

  $$\mathcal{F}_n = \sigma(S_j; j \leq n)$$

Now consider

$$\begin{aligned}
\mathbb{E}(M_{n+1}|\mathcal{G}_n) &= \mathbb{E}(E\varphi(n+1, S_{n+1}) \exp(\beta H_{n+1}(S) - (n+1)\lambda(\beta))|\mathcal{G}_n) \\
&= E\varphi(n+1, S_{n+1}) \exp(\beta H_n(S) - n\lambda(\beta)) \\
&= EE(\varphi(n+1, S_{n+1}) \exp(\beta H_n(S) - n\lambda(\beta))|\mathcal{F}_n) \\
&= M_n
\end{aligned}$$

by Φ_n is a martingale.

Also we will have a proposition

Proposition 2.4.5. Suppose that $d \geq 3$ and L_2 condition holds, and we have the martingales above with the two properties hold, then there exists $\kappa \in [0, p/2)$ such that

$$\max_{0 \leq j \leq n} |M_j| = O(n^\kappa)$$

with $n \to \infty, \mathbb{P}$-a.s. In addition, $p < \frac{1}{2}d - 1$, then

$$\lim_{n \to \infty} M_n \text{ exists } \mathbb{P}\text{-a.s. and in } L^2(\mathbb{P})$$

if the second property above does not hold, we will have the sequence M_n have a larger bound

$$M_n = O(n^{p/2})$$

for $n \to \infty, \mathbb{P}$-a.s.

Proof. Let $\varphi(n, x) = |x|^2 - n$ and then we may know $p = 2$ and then by the proposition above, mytrgstheree exist $0 \leq \kappa < 1$ such that

$$\max_{0 \leq j \leq n} |M_n| = O(n^\kappa) = o(n)$$

and notice

$$M_n = E(|S_n|^2 - n) \exp(\beta H_n(S) - n\lambda(\beta)) = E_{P^{\beta,\omega}}(|S_n|^2 - n) W_n$$

and hence

$$E_{P^{\beta,\omega}} |S_n|^2 - n = M_n / W_n = o(n)$$

for $\mathbb{P}$-a.s. and hence we have proved the first conclusion.

For the further conclusion, we consider the multi-index a and prove for $f(x) = x^a$ with using the induction on $|a|_1$. Denote

$$\varphi(n, x) = \left(\frac{\partial}{\partial \theta} \right)^a \exp\left(\theta \cdot x - n\rho(\theta)\right) |_{\theta=0}$$
$$\psi(n, x) = \left(\frac{\partial}{\partial \theta} \right)^a \exp\left(\theta \cdot x - n \frac{|\theta|^2}{2d} \right) |_{\theta=0}$$

where $\rho(\theta) = \ln \left(\frac{1}{d} \sum_{j=1}^d \cosh(\theta_j) \right)$, we have $\varphi(n, x) = x^a + \varphi_0(n, x)$ and $\psi(n, x) = x^a + \psi_0(n, x)$ where

$$\varphi_0(n, x) = \sum_{j \geq 1, |b|_1 + 2j \leq |a|_1} A_a(b, j) x^b n^j, \quad \psi_0(n, x) = \sum_{j \geq 1, |b|_1 + 2j = |a|_1} A_a(b, j) x^b n^j$$

for some $A_a(b, j) \in \mathbb{R}$ and hence

$$\begin{aligned} (x/\sqrt{n})^a &= \varphi(n, x) n^{-|a|_1/2} - \varphi_0(n, x) n^{-|a|_1/2} + \psi_0(n, x) n^{-|a|_1/2} - \psi_0(n, x) n^{-|a|_1/2} \\ &= \varphi(n, x) n^{-|a|_1/2} - \psi_0(1, x/\sqrt{n}) + (\psi_0(n, x) - \varphi_0(n, x)) n^{-|a|_1/2} \end{aligned}$$

since where we have

$$\begin{aligned}
\psi_0(n,x)n^{-|a|_1/2} &= \sum_{j\geq 1, |b|_1+2j=|a|_1} A_a(b,j)x^b n^j n^{(-|b|_1/2-j)} \\
&= \sum_{j\geq 1, |b|_1+2j=|a|_1} A_a(b,j)(x/\sqrt{n})^b \\
&= \psi_0(1, x/\sqrt{n}).
\end{aligned}$$

To sum up, we have

$$\begin{aligned}
E_{P_n^{\beta,\omega}}(S_n/\sqrt{n})^a =& E_{P_n^{\beta,\omega}}\varphi(n,S_n)n^{-|a|_1/2} - E_{P_n^{\beta,\omega}}(\psi_0(1,S_n/\sqrt{n})) \\
&+ E_{P_n^{\beta,\omega}}(\psi_0(n,S_n)-\phi_0(n,S_n))n^{-|a|_1/2} \\
=& \frac{1}{W_n}E\varphi(n,S_n)\xi_n n^{-|a|_1/2} - \frac{1}{W_n}E(\psi_0(1,S_n/\sqrt{n})\xi_n) \\
&+ \frac{1}{W_n}E(\psi_0(n,S_n)-\phi_0(n,S_n))\xi_n n^{-|a|_1/2}
\end{aligned}$$

where $\xi_n = \exp(\beta H_n(S) - n\lambda(\beta))$ and the first term and the third term will vanish for $n \to \infty$, since we may check that φ, ψ satifies the first condition in the above proposition with $p = |a|_1$ and then we use the last conclusion in the proposition 2.4.5 and we will see that the third term vanishes. By induction hypothesis, we will know that the second term converges to

$$(2\pi)^{-d/2}\int (x/\sqrt{d})^a e^{-|x|^2/2}dx$$

□

Now we will go through the proof of the proposition 2.4.5.

Proof. Firstly, we assume that we have

$$\mathbb{E}M_n^2 = O(b_n), \quad b_n = \sum_{j=1}^n j^p - d/2$$

and setting $M_n* = \max_{0\leq j\leq n}|M_j|$, and it is sufficent to show that for any $\delta?0$, we have

$$M_n^* = O(n^\delta\sqrt{b_n})$$

for $n \to \infty, \mathbb{P}$-a.s., for $k > 1/\delta$, we have

$$\begin{aligned}
\mathbb{P}(M_{n^k}^* > n^{k\delta}\sqrt{b_{n^k}}) &\leq \mathbb{P}(M_{n^k}^* > n\sqrt{b_n^k}) \\
&\leq \mathbb{E}(M_{n^k}^*)^2/n^2 b_{n^k} \\
&\leq 4\mathbb{E}M_{n^k}^2/(n^2 b_{n^k}) \leq Cn^{-2}
\end{aligned}$$

and hence we know by the BC lemma that

$$M_{n^k}^* \leq n^{k\delta}\sqrt{b_{n^k}} \text{ for large enough } n$$

is almost sure □

3 Semimartingable Approach

3.1 Useful Conclusions

Theorem 3.1.1. (Helly's selection Theorem)

For every sequence F_n of distribution functions, there is a subsequence F_{n_k} and a right continuous nondecresing function F so that F_n converges to F vaguely, i.e. $\lim\limits_{k\to\infty} F_{n_k}(y) = F(y)$ at all continuity points y of F.

Proof. Consider q_i to be all the rational numbers and then we know there has to be a subsequence of F_n such that $F_n(q_1)$ converge to some value, denoted with $F(q)$ and by recursive constructing we will have a function F such that there is a subsequence $F_{n_k}(q_i) \to F(q_i)$ for all the rational numbers. It is easy to check for any $q_i < q_j$, since $F_{n_k}(q_i) \leq F_{n_k}(q_j)$, we know

$$F(q_i) = \lim_{k\to\infty} F_{n_k}(q_i) \leq \lim_{k\to\infty} F_{n_k}(q_j) \leq F(q_j)$$

and hence we may contruct F bye choose

$$F(x) = \inf\{F(q), q \in \mathbb{Q}, q > x\}$$

which is easy to be checked nondecreasing and right continuous.

For any point y such that F is continuous at y, then notice for any $\epsilon > 0$, we have q_1, q_2 rational numbers such that $q_1 < y < q_2$ and

$$F(y) - \epsilon < F(q_1) \leq F(x) \leq F(q_2) < F(y) + \epsilon$$

and let n_k large enough we may have

$$F(y) - \epsilon < F_{n_k}(q_1) \leq F_{n_k}(y) \leq F_{n_k}(q_2) < F(y) + \epsilon$$

and we are done. ☐

Theorem 3.1.2. Every subsequential limit is the distribution function of a probability measure if and only if the sequence F_n is **tight**, i.e. for all $\epsilon > 0$ there is an M_ϵ so that

$$\limsup_{n\to\infty}(1 - F_n(M_\epsilon) + F_n(-M_\epsilon)) \leq \epsilon$$

Proof. To see the sufficiency, assume F_n is tight and $F_{n_k} \overset{v}{\Rightarrow} F$ for some F, let $r < -M_\epsilon, s > M_\epsilon$ be continuity point of F and then we know

$$1 - F(s) + F(r) \leq \limsup_{k\to\infty} 1 - (F_n(M_\epsilon) - F_n(-M_\epsilon)) \leq \epsilon$$

whcih means $\limsup_{x\to\infty} F(x) - F(-x) = 1$ and hence F is a distribution function.

To see the necessity, we may see if F_n not tight, there is an $\epsilon > 0$ and a subsequence $n_k \to \infty$ such that

$$1 - F_{n_k}(k) + F_{n_k}(-k) \geq \epsilon$$

for all k, assume $F_{n_{k_j}}$ converges to F a distribution function weekly, and let $r < 0 < s$

continuity points of F, then

$$1 - F(s) + F(r) = \lim_{j\to\infty} 1 - F_{n_{k_j}}(s) + F_{n_{k_j}}(r) \ge \liminf_{j\to\infty} 1 - F_{n_{k_j}}(k_j) + F_{n_{k_j}}(k_j) \ge \epsilon$$

and let $-r, s \to \infty$ will induce a contradiction. $\square$

Theorem 3.1.3. Consider a sequence of random variables $X_n, 0 \le n \le \infty$, if for any n integer $EX_k^n \to EX_\infty^n$, then X_n converges weakly in X_∞.

Proof. We know $EX_k^2 \to EX^\infty = T$ finite, we have

$$1 - P(-M \le X_k \le M) \le EX_k^2/M^2 \to T/M^2$$

and hence let $M \ge \sqrt{T/\epsilon}$

$$\limsup_{k\to\infty}(1 - P(-M \le X_k \le M)) \le \lim_{k\to\infty} EX_k^2/M^2 \le \epsilon$$

which means F_k is tight where F_k is the distribution function of X_k. Then for any bounded g, we may consider $\delta > 0$ and let M such that

$$\limsup_{k\to\infty}(1 - P(-M \le X_k \le M)) \le \delta/|g|_{L^\infty}$$

and we may find polynomials p_n converges to g uniformly on $[-M, M]$, where we know

$$\lim_{k\to\infty} Ep_n(X_k; |X_k| \le M) = Ep_n(X_\infty; |X_\infty \le M|)$$

and hence

$$\lim_{k\to\infty} Eg(X_k; |X_k| \le M) = Eg(X_\infty; |X_\infty| \le M)$$

since $p_n \to g$ uniformly and then

$$|\liminf_{k\to\infty} Eg(X_k) - Eg(X_\infty)| < 2\delta$$

for any $\delta > 0$ and hence $\liminf_{k\to\infty} Eg(X_k) = E(g)(X_\infty)$ for any bounded and continuous function g, so as for $\limsup$ and we are done. $\square$

Theorem 3.1.4. (Doob's Decomposition)

Any $\mathcal{G}_n$-adapted process $X = \{X_n\}_{n\ge 0} \subset L^1(\mathbb{P})$ can be decomposed in a unique way as

$$X_n = M_n(X) + A_n(X), \quad n \ge 1$$

where $M(X)$ is an $\mathcal{G}_n$-martingale and $A(X)$ is predictable, i.e. $A_n(X)$ is $\mathcal{G}_{n-1}$ measurable with $A_0 = 0$.

Proof. We know if this decomposition exists, then

$$\triangle A_n = \mathbb{E}(\triangle X_n|\mathcal{G}_{n-1})$$

and

$$\triangle M_n = \triangle X_n - \mathbb{E}(\triangle X_n | \mathcal{G}_{n-1})$$

and then

$$A_n = \sum_{i=1}^{n} \mathbb{E}(\triangle X_i | \mathcal{G}_{i-1}), \quad M_n = X_n - \sum_{i=1}^{n} \mathbb{E}(\triangle X_i | \mathcal{G}_{i-1})$$

□

Proposition 3.1.5. If N is a square integrable martingable, then the compensator $A(N^2)$ is denoted by $\langle N \rangle_n$ and is given by

$$\triangle \langle N \rangle_n = E(N_n^2 - N_{n-1}^2 | \mathcal{G}_{n-1}) = E((\triangle N_n)^2 | \mathcal{G}_{n-1})$$

Theorem 3.1.6. $\lim X_n$ exists and finite on $\{A_\infty < \infty\}$.

Theorem 3.1.7. Let $f \geq 1$ be increasing with $\int_0^\infty f(t)^{-2} dt < \infty$. Then $X_n / f(A_n) \to 0$ a.s. on $\{A_\infty = \infty\}$.

3.2 Semimartingable Decomposition

Definiton 3.2.1.

We care about the Doob's decomposition of $X_n = -\ln W_n = M_n + A_n$. Then $-\ln W_n$ is a submartingable and A_n is increasing about n.

Proof. We know $-\ln$ is convex and then

$$\mathbb{E}(-\ln W_n | \mathcal{G}_{n-1}) = \mathbb{E}(\sup\{a W_n + b\} | \mathcal{G}_{n-1}) \geq -\ln(W_{n-1})$$

and hence a submartingable, then

$$\mathbb{E}(M_n + A_n | \mathcal{G}_{n-1}) = M_{n-1} + A_n \geq M_{n-1} + A_{n-1}$$

and hence A_n increasing. □

Definiton 3.2.2.

We introduce

$$U_n = E_{n-1}^{\beta,\omega} \exp(\beta \omega(n, S_n) - \lambda(\beta)) - 1$$

and we will have

$$U_n + 1 = W_n / W_{n-1}$$

and then

$$W_n = \prod_{t=1}^{n} (1 + U_t)$$

and hence

$$\triangle A_n = -\mathbb{E}(\ln(1 + U_n) | \mathcal{G}_{n-1})$$
$$\triangle M_n = -\ln(1 + U_n) + \mathbb{E}(\ln(1 + U_n) | \mathcal{G}_{n-1})$$

Proof. We have

$$\begin{aligned}W_n =&E\exp(\beta H_n(S)-n\lambda(\beta))\\ =&E\exp(\beta H_{n-1}(S)-(n-1)\lambda(\beta))\exp(\beta\omega(n,S_n)-\lambda(\beta))\\ =&E_{n-1}^{\beta,\omega}\exp(\beta\omega(n,S_n)-\lambda(\beta))W_{n-1}\end{aligned}$$

□

Definiton 3.2.3.

Define

$$I_n=\sum_{x\in\mathbb{Z}^d}P_{n-1}^{\beta,\omega}(S_n=x)^2$$

and then consider $\tilde{S}$ an independent copy of S, where S and $\tilde{S}$ are called **replica** and then

$$I_n=(P_{n-1}^{\beta,\omega})^{\otimes 2}(S_n=\tilde{S}_n)$$

Theorem 3.2.1. Let $\beta\neq 0$. Then

$$\{W_\infty=0\}=\left\{\sum_{n\geq 1}I_n=\infty\right\},\quad \mathbb{P}\text{-a.s.}$$

Moreover, if $\mathbb{P}(W_\infty=0)=1$, there exists $c_1,c_2\in(0,\infty)$ depending on $\beta,\mathbb{P}$ such that for $\mathbb{P}$-a.s.

$$c_1\sum_{k=1}^{n}I_k\leq-\ln W_n\leq c_2\sum_{k=1}^{n}I_k\quad\text{for large enough } n$$

and also

$$\lim_{n\to\infty}\frac{-\ln W_n}{A_n}=1\text{ a.s.}$$

Lemma 3.2.2. Let $e_i, 1\leq i\leq m$ be positive, nonconstant i.i.d. random variables on a probability space such that

$$\mathbb{P}(e_1)=1,\quad \mathbb{P}(e_1^3+\ln^2 e_1)<\infty$$

For $\{\alpha_i\}_{i=1}^m$ nonnegative such that $\sum\limits_{i=1}^m\alpha_i=1$, define a centered random variable $U>-1$ by $U=\sum\limits_{i=1}^m\alpha_ie_i-1$. Then, there exists a constant $c\in(0,\infty)$ independent of m and of $\{\alpha_i\}_{i=1}^m$ such that

$$\begin{aligned}\frac{1}{c}\sum_{i=1}^m\alpha_i^2\leq&\mathbb{E}\left(\frac{U^2}{2+U}\right)\\ \frac{1}{c}\sum_{i=1}^m\alpha_i^2\leq&-\mathbb{E}\left(\ln(1+U)\right)\leq c\sum_{i=1}^m\alpha_i^2\\ &\mathbb{E}\left(\ln^2(1+U)\right)\leq c\sum_{i=1}^m\alpha_i^2\end{aligned}$$

Proof. Notice

$$\mathbb{E}(U^2)=\mathbb{E}((\sum_{i=1}^m(a_ie_i))^2-1)=var(e_1)\sum_{i=1}^m\alpha_i^2$$

and

$$\begin{aligned}\mathbb{E}(U^3) =&\mathbb{E}(\sum_{i=1}^m a_i e_i)^2(\sum_{i=1}^m a_i e_i - 1)\\ =&\mathbb{E}(\sum_{i=1}^m a_i e_i)^3 - var(e_1)\sum_{i=1}^m \alpha_i^2\\ \le&(\mathbb{E}(e_1^3)+4)\sum_{i=1}^m \alpha_i^2\end{aligned}$$

and then

$$\begin{aligned}c_1\sum_{i=1}^m \alpha^2 =&\mathbb{E}\left(\frac{U}{\sqrt{2+U}}U\sqrt{2+U}\right)\\ \le&\mathbb{E}\left(\frac{U^2}{2+U}\right)^{1/2}\mathbb{E}(2U^2+U^3)^{1/2}\\ \le&c_3\left(\sum_{i=1}^m \alpha_i^2\right)^{1/2}\mathbb{E}\left(\frac{U^2}{2+U}\right)^{1/2}.\end{aligned}$$

Define $\phi(u) = u - \ln(1+u)$ and then

$$\mathbb{E}\ln(1+U) = -\mathbb{E}(\phi(U))$$

for all $u > -1$. Notice

$$\left(\phi(u) - \frac{1}{4}\frac{u^2}{2+u}\right)' = \frac{3}{4} - \left(\frac{1}{(u+1)(u+2)^2} + \frac{1}{u+2}\right)$$

and we know $\phi(u) \ge \frac{1}{4}\frac{u^2}{2+u}$ which implies the LHS of the second inequality. For the RHS, notice

$$\begin{aligned}\mathbb{E}(\phi(U)) &= \mathbb{E}(\phi(U); 1+U \ge \epsilon) + \mathbb{E}(\phi(U); 1+U < \epsilon)\\ &= \mathbb{E}(\phi(U); 1+U \ge \epsilon) - \mathbb{E}(\ln(1+U); 1+U < \epsilon) + \mathbb{E}(U; 1+U < \epsilon)\\ &\le \mathbb{E}(\phi(U); 1+U \ge \epsilon) - \mathbb{E}(\ln(1+U); 1+U < \epsilon)\end{aligned}$$

for $\epsilon \in (0,1)$, notice $\phi(u) \le \frac{1}{2}(u/\epsilon)^2$ for $1+u \ge \epsilon$ and then

$$\mathbb{E}(\phi(U); 1+U \ge \epsilon) \le \frac{1}{2}\epsilon^{-2}\mathbb{E}U^2 = \frac{1}{2}\epsilon^{-2}c_1\sum_{i=1}^m \alpha_i^2$$

Let $\gamma = -\mathbb{E}\ln(e_1) \ge 0$(which is by the Chebyshev's inequality) and choose ϵ such that $\ln(1/\epsilon) - \gamma \ge 1$. Define

$$V = \sum_{i=1}^m \alpha_i(\ln e_i + \gamma)$$

and by Chebyshev, we have

$$V - \gamma \le \ln(1+U) \le \ln\epsilon$$

and hence

$$-\mathbb{E}(\ln(1+U); 1+U \leq \epsilon) \leq \mathbb{E}(-V; -V \geq 1) + \gamma\mathbb{P}(-V \geq 1) \leq (1+\gamma)\mathbb{E}(V^2)$$

where

$$\mathbb{E}V^2 = \mathbb{E}(\ln e_1 + \gamma)^2 \sum_{i=1}^{m} \alpha_i^2$$

similarly, we have

$$\mathbb{E}(\ln^2(1+U); 1+U \leq \epsilon) \leq (2+2\gamma^2)\mathbb{E}(V^2)$$

and it is easy to check $|\ln(1+U)| \leq \dfrac{-\ln\epsilon}{\epsilon}|u|$ if $\epsilon|leq 1+u$ and hence

$$\mathbb{E}(\ln^2(1+U); \epsilon \leq 1+U) \leq \epsilon^{-2}\ln^2\epsilon^{-1}\mathbb{E}(U^2)$$

and we are done. □

Proof. Use the Lemma 3.3.2. and consider $\alpha_x^n = P_{n-1}^{\beta,\omega}(S_n = x)$ and $e_x^n = \exp(\beta\omega(n,x) - \lambda(\beta))$ which is independent with $\mathcal{G}_{n-1}$ and α_x^n is measurable in $\mathcal{G}_{n-1}$ and hence we may use $\mathbb{P}(|\mathcal{G}_{n-1})$ in the lemma. Notice

$$U_n = \sum a_x^n e_x^n - 1$$

$$\frac{1}{c} I_n \leq \triangle A_n = -\mathbb{E}(\ln(1+U_n)|\mathcal{G}_{n-1}) \leq c I_n$$

$$\mathbb{E}(\ln^2(1+U_n)|\mathcal{G}_{n-1}) \leq c I_n$$

Then if $\sum_{n\geq 1} I_n < \infty$, then we know $\sum \ln^2(1+U_n)$ is integrable and hence M_n^2 is integrable by

$$\mathbb{E}(\triangle M_n)^2 \leq \mathbb{E}\ln(1+U_n)^2$$

by the projection property of conditional expectation. Then we will have

$$\triangle\langle M\rangle_n \leq c I_n$$

and hence we have $A_\infty < \infty$ and $\langle M\rangle_\infty < \infty$, which means $\lim_{n\to\infty} M_n$ exists and finite, which implies $\lim \ln W_n$ exists and finite, so $W_\infty > 0$.

By the approximation above, we have

$$\{\sum I_n = \infty\} = \{A_\infty = \infty\}$$

then if $\langle M\rangle_\infty < \infty$, then we know M_∞ exists and finite. If $\langle M\rangle_\infty = \infty$, then we may know $M_n/\langle M\rangle_n \to 0$ a.s. and it is easy to check that for both cases we have

$$-\frac{\ln W_n}{A_n} \to 1$$

for $\mathbb{P}$-a.s. and we are done. □

Corollary 3.2.3. We have $\mathbb{P}$-a.s.

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}(\ln E_{t-1}^{\beta,\omega} \exp(\beta\omega(t, S_t))|\mathcal{G}_{t-1})$$

Proof. We have

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n}(\ln(W_n) + n\lambda(\beta)) = \lim_{n\to\infty} \frac{1}{n}(-A_n + n\lambda(\beta))$$

and we are done. □

3.3 Size-Biasing Bounds

Definiton 3.3.1. Define

$$\beta_{sb} = \sup\{\beta \geq 0 : E^{\otimes 2}(\exp(\lambda_2(\beta)N_\infty(S, \tilde{S}))|\tilde{S}) < \infty \text{ for } \tilde{S}\text{-a.s.}\}$$

where the event $\{E^{\otimes 2}(\exp(\lambda_2(\beta)N_\infty(S, \tilde{S}))|\tilde{S}) < \infty\}$ belongs to the tail σ-field of $\tilde{S}$, and therefore it has probability 0 or 1.

We also have

$$\beta_{sb} \geq \beta_{L^2}$$

Proposition 3.3.1. Consider $P, \tilde{P}, \mathbb{P}, \tilde{\mathbb{P}}$ to be independent, where

$$\hat{P}(\hat{\omega}(i, x) \in \cdot) = \mathbb{E}(e(i, x); \omega(i, x) \in \cdot)$$

and let $\hat{\omega}$ be an i.i.d. environment and $\hat{e}(i, x) = \exp(\beta\hat{\omega}(i, x) - \lambda(\beta))$ and similarly define $e(i, x)$. Now we define

$$\hat{e}_{\tilde{S}}(i, x) = \begin{cases} \hat{e}(i, x) & \text{if } \tilde{S}_i = x, \\ e(i, x) & \text{if } \tilde{S}_i \neq x \end{cases}$$

and

$$\hat{W}_n^{e,\hat{e},\tilde{S}} = E \prod_{i=1}^{n} \hat{e}_{\tilde{S}}(i, S_i)$$

Then for $f : [0, \infty) \to \mathbb{R}$ bounded measurable,

$$\mathbb{E}W_n f(W_n) = \mathbb{E}\hat{\mathbb{E}}\tilde{E} f(\hat{W}_n^{e,\hat{e},\tilde{S}})$$

Proof. We have

$$\begin{aligned}
\mathbb{E}W_n f(W_n) &= \mathbb{E}\left(\tilde{E}\left(\prod_{i=1}^{n} e(i,\tilde{S}_i)\right) f\left(E\prod_{i=1}^{n} e(i,S_i)\right)\right) \\
&= \tilde{E}\left(\mathbb{E}\left(\prod_{i=1}^{n} e(i,\tilde{S}_i) f\left(E\prod_{i=1}^{n} e(i,S_i)\right)\right)\right) \\
&= \tilde{E}\left(\mathbb{E}\left(\prod_{i=1}^{n} e(i,\tilde{S}_i) f\left(\sum_x P(x)\prod_{i=1}^{n} e(i,x_i)\right)\right)\right) \\
&= \hat{\mathbb{E}} f\left(\sum_x P(x)\prod_{i=1}^{n} \hat{e}_{\tilde{S}}(i,x_i)\right)
\end{aligned}$$

□

Theorem 3.3.2. $W_\infty > 0$ when $\beta < \beta_{sb}$ and hence

$$\beta_{sb} \le \bar{\beta}_c \le \beta_c$$

Proof. Notice

$$\mathbb{E}(W_n f(W_n)) = \mathbb{E}\hat{\mathbb{E}}\tilde{P}(f(\hat{W}_n^{e,\hat{e},\tilde{S}}))$$

and we will know if $\hat{W}_n^{e,\hat{e},\tilde{S}}$ is tight, then for any bounded continuous f, the expectation above will converges and assume $f(x)$ to be some modified $sgn(x)\chi_{(|x|\ge M)}$ and then we know $\mathbb{E}(|W_n|;|W_n| \ge M)$ will converges to some bounded expectaion only related to M, and let M approch infinity we will have W_n uniformly integrable and hence W_∞ positive. And then □

3.4 Localization v.s. Delocalization

Definiton 3.4.1. (The probability of the favourite endpoint)

$$J_n = \max_{x\in\mathbb{Z}} P_{n-1}^{\beta,\omega}\{S_n = x\}$$

and we have

$$J_n^2 \le I_n \le J_n$$

Definiton 3.4.2. We call the polymer is **localized** if

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} J_t > 0, \mathbb{P}\text{-a.s.}$$

and **delocalized** if

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} J_t = 0, \mathbb{P}\text{-a.s.}$$

Theorem 3.4.1. let $\beta \ne 0$. The polymer is localized iff $p < \lambda$ and delocalized iff $p = \lambda$.

Proof. We have

$$\left(\frac{1}{n}\sum_{t=1}^{n} I_t\right)^2 \leq \left(\frac{1}{n}\sum_{t=1}^{n} J_t\right)^2 \leq \frac{1}{n}\sum_{t=1}^{n} J_t^2 \leq \frac{1}{n}\sum_{t=1}^{n} I_t \leq \frac{1}{n}\sum_{t=1}^{n} J_t$$

which implies that

$$c_1\left(\frac{\ln W_n}{n}\right)^2 \leq \left(\frac{1}{n}\sum_{t=1}^{n} J_t\right)^2 \leq \frac{1}{n}\sum_{t=1}^{n} J_t^2 \leq c_2\frac{\ln W_n}{n} \leq c_3\frac{1}{n}\sum_{t=1}^{n} J_t$$

and we are done. □

4 The Localized Phase

4.1 Checklist

- Prove the integration by parts for Gaussian.

4.2 Useful Conclusions

Lemma 4.2.1. (Integration by Part)

If X is centered normal, and f is smooth with

$$\lim_{|x|\to\infty} f(x)\exp(-x^2/(2EX^2)) = 0,$$

then

$$E(Xf(X)) = E(X^2)E(f'(X))$$

Corollary 4.2.2. (Integration by Part for Gaussian Vectors)

If $(X, X-1, \cdots, X_n)$ is a centered, gaussian vector , and F is smooth with

$$\lim_{||x||\to\infty} F(x)\exp(-ax^2) = 0$$

for all $a > 0$, then

$$E(XF(X_1, \cdots, X_n)) = \sum_{i=1}^{n} E(XX_i)E(F_{x_i}(X_1, \cdots, X_n))$$

Theorem 4.2.3. (Chernoff's bound)

For a r.v., assume all the required moment exist, then

$$P(X \geq a) \leq \inf_{t>0} E(e^{tX})e^{-ta}$$

Definiton 4.2.1. (Legendre-Fenchel Transform)

Consider a function $f : \mathbb{R} \to \mathbb{R}$, we define

$$f^*(k) = \sup_{x\in\mathbb{R}}(kx - f(x))$$

Definiton 4.2.2. (Supporting Line)

We call $f : \mathbb{R} \to \mathbb{R}$ has a supporting line at x if there exists $\alpha \in \mathbb{R}$

Proposition 4.2.4.

4.3 Path Localization

Definiton 4.3.1. In this chapter, we consider Gaussian environment

$$\omega(t, x) \sim \mathcal{N}(0, 1)$$

and for $y : \mathbb{N} \to \mathbb{Z}^d$ and S a path, we define

$$N_n(S,y) = \sum_{t=1}^{n} \chi_{\{S_t = y_t\}}$$

and

$$\mathcal{F} = \{\beta > 0, p \text{ is differentiable at } \beta, p'(\beta) < \lambda(\beta)\}$$

Theorem 4.3.1. Assume that the environment is Gaussian. There exists $y^{(n)} : [0,n] \to \mathbb{Z}^d$ such that

$$\liminf_{n\to\infty} \mathbb{E}E_n^{\beta,\omega}\left(\frac{N_n(S,y^{(n)})}{n}\right) \geq 1 - \frac{p'}{\lambda'}(\beta) > 0$$

for all $\beta \in \mathcal{F}$. Moreover,

$$\lim_{\beta\to\infty} \liminf_{n\to\infty} \mathbb{E}E_n^{\beta,\omega}\left(\frac{N_n(S,y^{(n)})}{n}\right) = 1$$

Proof. We have

$$\begin{aligned}
\frac{d}{d\beta}\mathbb{E}p_n(\omega,\beta) &= \frac{1}{n}\frac{d}{d\beta}\mathbb{E}\ln\left(\sum_x P(S^{(n)} = x)\exp(\beta H_n(x))\right) \\
&= \frac{1}{n}\mathbb{E}\frac{1}{Z_n}\sum_x P(S^{(n)} = x)\exp(\beta H_n(x))H_n(x) \\
&= \frac{1}{n}\sum_{t\leq n,x}\mathbb{E}\left(P_n^{\beta,\omega}(S_t = x)\omega(t,x)\right)
\end{aligned}$$

which has a uniform L_1 bound.

Let

$$\begin{aligned}
F(\omega(t,x))_{t\leq n,x} &= P_n^{\beta,\omega}(S_t = x) \\
&= \sum_{S_t = x} P(S)\exp(\beta\sum_{i=1}^{n}\omega(i,S_i))
\end{aligned}$$

and

$$\lim_{x\to\infty}\exp(\beta\sum_{i=1}^{n}x_{i,S_i} - a|x|^2) \leq \lim_{x\to\infty}\exp(|x|(\beta n - a|x|)) = 0$$

which means we may use the integration by parts. Notice

$$\begin{aligned}
P_n^{\beta,\omega}(S_t = x) &= \frac{1}{Z_n}\sum_{x,x_t = x} P(S^{(n)} = x)\exp(\beta H_n(x)) \\
&= \frac{Z_n P_n^{\beta,\omega}(S_t = x)}{Z_n P_n^{\beta,\omega}(S_t = x) + (Z_n - Z_n P_n^{\beta,\omega}(S_t = x))}
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{dP_n^{\beta,\omega}(S_t = x)}{d\beta} &= \frac{1}{Z_n^2}\left(\beta Z_n P_n^{\beta,\omega}(S_t = x)Z_n - \beta Z_n^2 P_n^{\beta,\omega}(S_t = x)^2\right) \\
&= \beta(P_n^{\beta,\omega}(S_t = x) - P_n^{\beta,\omega}(S_t = x)^2)
\end{aligned}$$

so we have

$$\begin{aligned}\frac{d}{d\beta}\mathbb{E}p_n(\omega,\beta) &= \frac{\beta}{n}\sum_{t\le n,x}\mathbb{E}(P_n^{\beta,\omega}(S_t=x)-P_n^{\beta,\omega}(S_t=x)^2)\\ &= \beta\frac{1}{n}\mathbb{E}(\sum_{t\le n}P_n^{\beta,\omega\otimes 2}(S,\tilde{S}\text{ do not coincide at time }t))\\ &= \beta(1-\frac{1}{n}\mathbb{E}\sum_{t\le n}P_n^{\beta,\omega\otimes 2}(S,\tilde{S}\text{ coincide at time }t))\end{aligned}$$

and consider how many contributions to $P_n^{\beta,\omega\otimes 2}(S,\tilde{S}$ coincide at $t_1,\cdots,t_k)$ is k times and hence

$$\beta\left(1-\mathbb{E}E_n^{\beta,\omega\otimes 2}\left(\frac{N_n(S,\tilde{S})}{n}\right)\right)$$

Since $\lambda(\beta)=\beta^2/2$ and we have

$$\lim_{n\to\infty}\mathbb{E}E_n^{\beta,\omega\otimes 2}\left(\frac{N_n(S,\tilde{S})}{n}\right)=1-p'(\beta)/\beta=1-\frac{p'}{\lambda'}(\beta)$$

For fixed n,β,ω, we may define

$$y^{(n)}(t)=arg\max_x P_n^{\beta,\omega}(S_t=x)$$

and then

$$P_n^{\beta,\omega\otimes 2}(S_t=\tilde{S}_t)\le P_n^{\beta,\omega}(S_t=y_t^{(n)})$$

and then

$$\mathbb{E}E_n^{\beta,\omega\otimes 2}\left(\frac{N_n(S,\tilde{S})}{n}\right)\le\mathbb{E}P_n^{\beta,\omega}\left(\frac{N_n(S,y^{(n)})}{n}\right)$$

Recall that p is convex and p is almost linear by

$$p(\beta)\le\beta\inf_{b\in(0,\beta]}\frac{\lambda(b)+\ln(2c)}{b}-\ln(2d)$$

and hence $p'(\beta)$ should be bounded when β is large and hence

$$1\ge\lim_{\beta\to\infty}\liminf_{n\to\infty}\mathbb{E}E_n^{\beta,\omega}\left(\frac{N_n(S,y^{(n)})}{n}\right)\ge\lim_{\beta\to\infty}1-\frac{C}{\beta}$$

for some constant C. □

Theorem 4.3.2. Assuem $d=1$ or $d=2$. Then for all $\beta\neq 0, p(\beta)<\lambda(\beta)$ and therefore $W_\infty=0$.

Proof. Notice for all $z\in\mathbb{Z}^d$, we have

$$P_{t-1}^{\beta,\omega\otimes 2}(S_t=\tilde{S}_t+z)=\sum_x P_{t-1}^{\beta,\omega}(S_t=x)P_{t-1}^{\beta,\omega}(S_t=x+z)\le P_{t-1}^{\beta,\omega\otimes 2}(S_t=\tilde{S}_t)=I_t$$

Notice when $d = 1$,

$$1 = \sum_{z, 2|z, |z| \leq 2t} P_{t-1}^{\beta,\omega \otimes 2}(S_t = \tilde{S}_t + z) \leq 2t + 1 I_t$$

and hence $\sum I_t$ diverge, by

$$-\ln W_n \sim \sum I_t$$

and we know $W_\infty = 0$.

For $d = 2$, if $W_\infty > 0$ a.s., then let

$$A_n = \{|S_n^{(1)}| \leq K\sqrt{n \ln n}, |S_n^{(2)}| \leq K\sqrt{n \ln n}\}$$

and

$$X_n = E(\exp(\beta H_{n-1} - (n-1)\lambda(\beta)); A_n^c).$$

Then

$$\mathbb{P}\left(X_n \geq \exp(-K^2 n \ln n/4)\right) \leq e^{K^2 n \ln n/4}\mathbb{E}(X_n) = e^{K^2 n \ln n/4} P(A_n^c),$$

and by the Chernov's bound:

$$\begin{aligned}
P(\pm S_n^{(1)} > K\sqrt{n \ln n}) &\leq \inf_{t>0} E(\exp(tS_n^{(1)}))e^{-tK(\sqrt{n \ln n})} \\
&= \exp\left[\inf_{t \geq 0}\left(\ln E\exp(tS_n^{(1)}) - tK(\sqrt{n \ln n})\right)\right] \\
&= \exp\left(-\gamma^*(K\sqrt{n \ln n})\right)
\end{aligned}$$

and γ^* is the LF transform of

$$\gamma(u) := \ln E\exp(uS_n^{(1)}) = \ln\frac{1+\cosh u}{2} \leq u^2/2$$

and hence

$$\gamma^*(v) \geq v^2/2$$

and then

$$P(A_n^c) \leq \exp(-K^2 n \ln n)$$

and

$$\mathbb{P}(X_n \geq \exp(-K^2 n \ln n/4)) \leq e^{-3K^2 n \ln n/4}.$$

By BC-lemma, we have $X_n \to 0$ $\mathbb{P}$-a.s., which implies

$$Y_n := P_{n-1}^{\beta,\omega}(A_n^c) = X_n/W_n \to 0/W_\infty = 0$$

for $\mathbb{P}$-a.s. Denote $\mathcal{C}(n,K) = [-K\sqrt{n\ln n}, K\sqrt{n\ln n}]^2$ and

$$\begin{aligned}(1-Y_n)^2 &= \sum_{x,y\in\mathcal{C}(n,K)} {P_{n-1}^{\beta,\omega}}^{\otimes 2}(S_n = x, \tilde{S}_n = y) \\ &\le \sum_{z\in\mathcal{C}(n,2K)} {P_{n-1}^{\beta,\omega}}^{\otimes 2}(S_n = \tilde{S}_n + z) \\ &\le (4K\sqrt{n\ln n})^2 I_n\end{aligned}$$

and hence $\sum I_n$ diverge and we are done. □

Theorem 4.3.3. Assume $\omega(t,x)$ has mean 0 and variance 1. For $d=1$, as $\beta \to 0$ we have

$$\lambda(\beta) - p(\beta) = O(\beta^4)$$

and $d = 2, \beta \to 0$, we have

$$\lambda(\beta) - p(\beta) = \exp(-\pi\beta^{-2}(1+o(1)))$$

5 KPZ Equation and Universality

5.1 Checklist

- Construction of Gaussian field

5.2 Useful Conclusions

5.3 KPZ Equation

Definiton 5.3.1. (KPZ Equation)

$$\frac{\partial h}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,x) + \frac{1}{2}\left(\frac{\partial h}{\partial x}(t,x)\right)^2 + \beta W(t,x)$$

where W is a space-time gaussian white noise ($t \geq 0, x \in \mathbb{R}$). The noise is a distribution-valued Gaussian field on $\mathbb{R}^+ \times \mathbb{R}$ with mean 0 and covariance

$$\mathbb{E}[W(t,x)W(s,y)] = \delta(t-s)\delta(x-y)$$

which means the random variables

$$\left\{\int W(t,x)(f)dtdx | f \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})\right\}$$

is a Gaussian family with mean 0 and

$$\mathbb{E}\left(\int_{\mathbb{R}^+ \times \mathbb{R}} W(f) \int_{\mathbb{R}^+ \times \mathbb{R}} W(g)\right) = \int_{\mathbb{R}^+ \times \mathbb{R}} fg$$

6 Some Paper Conclusion by Stefan Junk 1

6.1 Main Theorem

Theorem 6.1.1. Let $(M_n, \mathcal{F}_n)$ be a non-negative martingable with $M_0 = 1$. Assume that for every $k, l \in \mathbb{N}$ and $f : \mathbb{R}^+ \to \mathbb{R}$ convex,

$$E\left(f\left(\frac{M_{k+l}}{M_k}\right)|\mathcal{F}_k\right) \leq Ef(M_l)$$

Denote $M_n^* = \sup_{k\leq n} M_k$ and $M_\infty = \lim_{n\to\infty} M_n$. Then we have

1. If $P(M_\infty > 0) > 0$, then $E[M_\infty^*] < \infty$.
2. If $P(M_\infty > 0) > 0$ and exists $K > 1$ such that

   $$P(M_{n+1} \leq KM_n) = 1$$

   for all $n \in \mathbb{N}$, then there exists $p > 1$ such that

   $$\sup ||M_n||_p < \infty$$

   Moreover, the set of p's satisfying the above inequality is open.
3. If $P(M_\infty = 0) = 1$ and if $P(M_{n+1} \leq KM_n) = 1$ for all $n \in \mathbb{N}$, we have

   $$P(M_\infty^* > t) > \frac{1}{4K^2 t}$$

4. If $P(M_\infty > 0) = 1$ and if there exists $K > 1$ such that

   $$P(M_{n+1} \geq M_n/K) = 1 \text{ for all } n \in \mathbb{N}$$

   then there exists $p > 0$ such that

   $$\sup_n EM_n^{-p} < \infty$$

   and similarly the set of p's satisfying the above inequality is open.

Remark. According to the Martingable Convergence Theorem, since $EM_n = EM_0 = 1$ and then we know $\sup EM_n$ is always bounded, and hence M_∞ always exists.

Proof. (First step) If we may find $\epsilon, \eta > 0$ such that for any n integer and $t > 1$, we have

$$P(M_n^* > t) \leq P(M_n > t\epsilon)/\eta$$

then we know

$$\begin{aligned} E(M_n^*) &= \int_0^\infty P(M_N^* > t)dt \\ &\le 1 + \int_1^\infty P(M_n^* > t)dt \\ &\le \frac{1}{\eta}\int_1^\infty P(M_n > t\epsilon)dt + 1 \\ &\le \frac{1}{\epsilon\eta} + 1 \end{aligned}$$

Since the LHS converges to EM_∞^* by MCT, then we have $E[M_\infty^*] < \infty$.

We consider

$$f_{\delta,\epsilon} := \delta(x/\epsilon - 1) \wedge 1$$

for $\delta, \epsilon > 0$ and then $f_{\delta,\epsilon}$ concave and

$$\chi_{(\epsilon,\infty)}(x) \ge f_{\delta,\epsilon}(x) \ge \chi_{[(1/\delta+1)\epsilon,\infty)}(x) - \delta\chi_{[0,\epsilon]}(x)$$

(which is actually doing some floor to $f_{\delta,\epsilon}$ by intervals) Let $\tau(t) := \inf\{n \in \mathbb{N} : M_n > t\}$ and then $M_{\tau(t)} > 0$ on $\{\tau < \infty\}$. So

$$\begin{aligned} P(M_n > t\epsilon) &\ge P(\tau \le n, M_n/M_\tau > \epsilon) \\ &= \sum_{k=1}^n E\left(\chi_{(\tau(t)=k)} E(\chi_{(M_n/M_k > \epsilon)} | \mathcal{F}_k)\right) \\ &\ge \sum_{k=1}^n E\left(\chi_{\tau(t)=k} E\left(f_{\delta,\epsilon}\left(\frac{M_n}{M_k}\right) | \mathcal{F}_k\right)\right) \\ &\ge \sum_{k=1}^n E\left(\chi_{\tau(t)=k} E\left(f_{\delta,\epsilon}(M_{n-k})\right)\right) \\ &\ge P(\tau \le n) \inf_{k<n} E(f_{\delta,\epsilon} M_k) \end{aligned}$$

For any $\delta > 0$, we have

$$\begin{aligned} \inf_{k\in\mathbb{N}} E(f_{\delta,\epsilon}(M_k)) &\ge E(\inf_k f_{\delta,\epsilon}(M_k)) \\ &= E(f_{\delta,\epsilon}(\inf_k M_k)) \\ &\ge P(\inf_k M_k \ge (\delta^{-1}+1)\epsilon) - \delta P(\inf_k M_k \le \epsilon) \end{aligned}$$

where the RHS converges to $P(M_\infty > 0) - \delta(M_\infty = 0)$ when $\epsilon = 0$, since we always have

$$E(M_{n+1}\chi_{M_n=0} | \mathcal{F}_\backslash) = (M_n)\chi_{M_n=0} = 0$$

and hence $M_{n+1} = 0$ on $\{M_n = 0\}$ by M nonnegative and hence

$$\{M_\infty > 0\} = \{\inf_k M_k > 0\}$$

If $P(M_\infty > 0) > 0$, we may find $\delta, \epsilon > 0$ such that $\inf_k E(f_{\delta,\epsilon}(M_k)) =: \eta > 0$ and then we are done since$\{\tau \le n\} = \{M_n^* > t\}$.

(Second step) We know $M_n \leq tK\dfrac{M_n}{M_\tau} \Leftrightarrow \{M_\tau \leq tK\}$ on $\{\tau \leq n\}$, then for any $\epsilon > 0$, we have

$$\begin{aligned}
E(M_n^{1+\epsilon}) &\leq t^{1+\epsilon} + E(\chi_{(\tau\leq n)}M_n^{1+\epsilon}) \\
&\leq t^{1+\epsilon} + (Kt)^{1+\epsilon}\sum_{k=1}^{n} E\left(\chi_{\tau(t)=k}E\left(\left(\frac{M_n}{M_k}\right)^{1+\epsilon}|\mathcal{F}_k\right)\right) \\
&\leq t^{1+\epsilon} + (Kt)^{1+\epsilon}\sum_{k=1}^{n} E\left(\chi_{\tau(t)=k}E(M_{n-k})^{1+\epsilon}\right) \\
&\leq t^{1+\epsilon} + (Kt)^{1+\epsilon}P(\tau \leq n)E(M_n)^{1+\epsilon}
\end{aligned}$$

by Chebyshev's inequality at the final step. Since

$$E(M_\infty^\infty) = 1 + \int_1^\infty P(M_\infty^* > t)dt < \infty$$

we may find t such that

$$P(\tau \leq n) \leq P(M_\infty^* > t) \leq \frac{1}{4K^2t}$$

and let ϵ such that $t^{\epsilon \leq 2}$, then we have

$$E(M_n^{1+\epsilon}) \leq 2t^{1+\epsilon}$$

where $1+\epsilon$ is a required p in question 2. If $\sup_n ||M_n||_p < \infty$ for some $p > 1$. We infer the Doob's maximal inequality and we know $||M_\infty^*||$ finite by MCT, so there exists $t > 1$ such that

$$P(M_\infty^* > t) \leq \frac{1}{4K^{p+1}t^p}$$

and for $q \in [p, p+1]$, we have

$$E(M_n^q) \leq t^q + (Kt)^q P(\tau \leq n)E(M_n^q) \leq t^q + \frac{t^{q-p}}{4}E(M_n^q)$$

and choose $q \in (p, p+1)$ such that $t^{q-p} \leq 2$ and then we have $\sup_n ||M_n||_q < \infty$. $\square$