Chapter 1

Fundamental Concepts

Definition 1.1

If $U \subset \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$ is a continuous function, then f is called C^1 on U if $\partial f/\partial x, \partial f/\partial y$ exist and are continous on U. ♣

Definition 1.2

We define for $f = u + iv : U \to \mathbb{C}$ a C_1 function

$$\frac{\partial}{\partial z} f := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f$$
$$\frac{\partial}{\partial \bar{z}} f := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f$$

which is easy to be checked linear and the chain rules. ♣

where we may check let $z = x + iy, \bar{z} = x - iy$, we have

$$\frac{\partial}{\partial z} z = 1, \quad \frac{\partial}{\partial z} \bar{z} = 0$$
$$\frac{\partial}{\partial \bar{z}} z = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{z} = 1$$

Proposition 1.1

(The Leibniz Rules) We have for any $F, G \in C^1$

$$\frac{\partial}{\partial z}(F \cdot G) = \frac{\partial F}{\partial z} \cdot G + F \cdot \frac{\partial G}{\partial z}$$
$$\frac{\partial}{\partial \bar{z}}(F \cdot G) = \frac{\partial F}{\partial \bar{z}} \cdot G + F \cdot \frac{\partial G}{\partial \bar{z}}$$

♠

Proposition 1.2

We have for $l \leq j, m \leq k$ nonnegative integers and then

$$(\frac{\partial^l}{\partial z^l})(\frac{\partial^m}{\partial \bar{z}^m})(z^j \bar{z}^k) = \frac{j!}{l!}\frac{k!}{m!} z^{j-l}\bar{z}^{k-m}$$

♠

Proposition 1.3

If $p(z, \bar{z}) = \sum a_{lm} z^l \bar{z}^m$ is a polynomaial, then p contains no term with $m > 0$ iff $\frac{\partial p}{\partial \bar{z}} \equiv 0$. ♠

Corollary 1.1

If $p(z, \bar{z}) = qz, \bar{z}$ are polynomials, then they have same coefficients. ♡

Definition 1.3

A C_1 function $f : U \mapsto \mathbb{C}$ is said to be holomorphic if

$$\frac{\partial f}{\partial \bar{z}} = 0$$

at every point of U. ♣

Definition 1.4

A C^1 function $f = u(x, y) + iv(x, y) : U \to \mathbb{C}$ is holomorphic if

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

at every point of U, which is called the $Cauchy$-$Riemann$ equations. ♣

Proposition 1.4

If $f : U \to \mathbb{C}$ is C^1 and if f satisfies the C-R equations, then

$$\frac{\partial}{\partial z} f = \frac{\partial}{\partial x} f = -i \frac{\partial}{\partial y} f$$

on U. ♠

Proof

We have

$$\begin{aligned} \frac{\partial}{\partial x} f &= \frac{\partial}{\partial x} u + i \frac{\partial}{\partial x} v = (\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) u = 2 \frac{\partial}{\partial z} u \\ \frac{\partial}{\partial x} f &= \frac{\partial}{\partial x} u + i \frac{\partial}{\partial x} v = i(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) v = 2 \frac{\partial}{\partial z} iv \\ -i \frac{\partial}{\partial y} f &= -i \frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v = (\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) u = 2 \frac{\partial}{\partial z} u \\ -i \frac{\partial}{\partial y} f &= -i \frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v = i(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) v = 2 \frac{\partial}{\partial z} iv \end{aligned}$$

on U.

Definition 1.5

If $U \subset \mathbb{C}$ is open and $u \in C^2(U)$, then u is called $harmonic$ if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where we also denote it as

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

where the operator is called the $Laplace$ $operator$. ♣

Here we have

$$4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} u = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = \Delta u$$

Proposition 1.5

The real and imaginary parts of a holomorphic C^2 function are harmonic. ♠

Proof

Assume $f = u + iv$ and then according to C-R equations, we have

$$\frac{\partial^2}{\partial x^2} u = \frac{\partial^2}{\partial x \partial y} v = \frac{\partial^2}{\partial y \partial x} v = -\frac{\partial^2}{\partial y^2} u$$

and

$$\frac{\partial^2}{\partial x^2} v = -\frac{\partial^2}{\partial x \partial y} u = -\frac{\partial^2}{\partial y \partial x} u = -\frac{\partial^2}{\partial y^2} v$$

Lemma 1.1

It $u(x, y)$ is a real-valued polynomial with $\Delta u = 0$, then there exists a (holomorphic) $Q(z)$ such that $ReQ = u$. ♡

Proof

Consider $u(x,y) = u(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}) = P(z,\bar{z}) = \sum a_{lm} z^l \bar{z}^m$, we know $\Delta u = 0$ and hence

$$P(z,\bar{z}) = a_0 0 + \sum^m a_k z^k + \sum^n b_k \bar{z}^k$$

P is real-valued and we know

$$a_0 0 + \sum^m a_k z^k + \sum^n b_k \bar{z}^k = \bar{a_0} 0 + \sum^m \bar{a_k} \bar{z}^k + \sum^n \bar{b_k} z^k$$

and hence $a_0 0 \in \mathbb{R}, a_k = \bar{b_k}$ and hence

$$u(z) = c + \sum^n a_k z^k + \sum^n \bar{a_k} \bar{z}^k = Re(c + 2\sum^n a_k z^k) = Re(Q)$$

where Q is obviously holomorphic.

Theorem 1.1

If f, g are C^1 functions on the rectangle

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : |x-a| < \delta, |y-b| < \epsilon\}$$

and if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \text{ on } \mathcal{R}$$

then there is a function $h \in C^(\mathcal{R})$ such that

$$\frac{\partial}{\partial x} h = f, \frac{\partial}{\partial y} h = g$$

on $\mathcal{R}$. If f, g are real-valuedd, the nwe may take h to be real-valued also.

♡

Proof

For $(x,y) \in \mathcal{R}$, define

$$h(x,y) = \int_a^x f(t,b)dt + \int_b^y g(x,s)ds$$

and we know

$$\frac{\partial}{\partial y} h(x,y) = g(x,y)$$

and

$$\frac{\partial}{\partial x} h(x,y) = f(x,b) + \frac{\partial}{\partial x} \int_b^y g(x,s)ds = f(x,b) + \int_b^y \frac{\partial}{\partial y} f(x,s) = f(x,b) + f(x,y) - f(x,b) = f(x,y)$$

and hence $h \in C^2(\mathcal{R})$ and real-valued if f, g are.

Corollary 1.2

If $\mathcal{R}$ is an open rectangle (or open disc) and if u is a real-valued harmonic function on $\mathbb{R}$, then there is a holomorphic function F on $\mathbb{R}$ such that $ReF = u$.

♡

Proof

We know

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$

and hence there exists v real-valued such taht

$$\frac{\partial}{\partial x} v = -\frac{\partial}{\partial y} u, \frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u$$

and hence $F = u + iv$ is a holomorphic function with $Re(F) = u$.

Theorem 1.2

If $U \subset \mathbb{C}$ is either an open rectangle or an open disc and if F is holomorphic on U, then there is a holomorphic function H on U such that $\partial H/\partial z = F$ on U.

Proof

Consider $H = h_1 + ih_2$ and we have $F = u(z) + iv(z)$, then we let $f = u, g = -v$ and we will have

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial x} g$$

and hence we have a real C^2 function h_1 such that

$$\frac{\partial}{\partial x} h_1 = u, \frac{\partial}{\partial y} h_1 = -v$$

and $h_2 \in C^2$ with

$$\frac{\partial}{\partial x} h_2 = v, \frac{\partial}{\partial y} h_2 = u$$

Then

$$\frac{\partial}{\partial z} H = \frac{1}{2}(\frac{\partial}{\partial x} h_1 + \frac{\partial}{\partial y} h_2) + \frac{i}{2}(\frac{\partial}{\partial x} h_2 - \frac{\partial}{\partial y} h_1) = u + iv = F$$

Definition 1.6

A function $\phi : [a, b] \to \mathbb{R}$ is called $continuously\ differentiable$ and we write $\phi \in C^1([]a, b)$ if

(a) ϕ is continous on $[a, b]$

(b) ϕ' exists on (a, b)

(c) ϕ' has a continuous extension to $[a, b]$, i.e.

$$\lim_{t \to a^+} \phi'(t) \text{ and } \lim_{t \to b^-} \phi'(t)$$

both exists. Then $\phi(b) - \phi(a) = \int_a^b \phi'(t)dt$.

Proof

Here notice that ϕ is absolutely continuous on $[a, b]$ respect to m, then we know $\phi(b - \epsilon) - \phi(a + \epsilon) = \int_{a+\epsilon}^{b-\epsilon} \phi'(t)dt$ for any $epsilon > 0$, and hence

$$\phi(b) - \phi(a) = \int_a^b \phi'(t)dt$$

Definition 1.7

A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be continuous on $[a, b]$ if both γ_1 and γ_2 are, $\gamma = \gamma_1 + i\gamma_2$. The curve is C_1 on $[a, b]$ if γ_1, γ_2 are C_1 on $[a, b]$ and then we may denote

$$\frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt}$$

Definition 1.8

Let $\varphi : [a, b] \to \mathbb{C}$ be continuous on $[a, b]$. Write $\varphi(t) = \varphi_1(t) + i\varphi_2(t)$. Then we define

$$\int_a^b \varphi(t)dt = \int_a^b \varphi_1(t)dt + i \int_a^b \varphi_2(t)dt$$

Proposition 1.6

Let $U \subset \mathbb{C}$ be open and let $\gamma : [a, b] \to U$ be a C_1 curve. If $f : U \to \mathbb{R}$ and $f \in C^1(U)$, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \left(\frac{\partial}{\partial x} f(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial}{\partial y} f(\gamma(t))\frac{d\gamma_2}{dt}\right)dt$$

This is due to the chain rule.

Proposition 1.7

Repalce f above as complex-valued and holomorphic, then we have

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial}{\partial z} f(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)dt$$

Proof

Notice

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &= \int_a^b \left(\frac{\partial}{\partial x}u(\gamma(t))\frac{d\gamma_1}{dt}(t) + \frac{\partial}{\partial y}u(\gamma(t))\frac{d\gamma_2}{dt}(t)\right) + i\left(\frac{\partial}{\partial x}v(\gamma(t))\frac{d\gamma_1}{dt}(t) + \frac{\partial}{\partial y}v(\gamma(t))\frac{d\gamma_2}{dt}(t)\right)dt \\ &= \frac{\partial}{\partial x} f(\gamma(t))\frac{d\gamma}{dt}(t) = \int_a^b \frac{\partial}{\partial z} f(\gamma(t))\frac{d\gamma}{dt}(t)dt \end{aligned}$$

Definition 1.9

If $U \subset \mathbb{C}$ open and $F : U \to \mathbb{C}$ is continuous on U and $\gamma : [a,b] \to U$ is a C_1 curve, then we define the complex line integral

$$\int_\gamma F(z)dz = \int_a^b F(\gamma(t))\frac{d\gamma}{dt}dt$$

Proposition 1.8

Let $U \subset \mathbb{C}$ be open and let $\gamma : [a,b] \to U$ be a C^1 curve. If f is a holomorphic function on U, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_\gamma \frac{\partial}{\partial z} f(z)dz$$

Proposition 1.9

If $\phi : [a,b] \to \mathbb{C}$ is continuous, then

$$|\int_a^b \phi(t)dt| \le \int_a^b |\phi(t)|dt$$

Proposition 1.10

Let $U \subset \mathbb{C}$ be open and $f \in C^0(U)$. If $\gamma : [a,b] \to U$ is a C^1 curve, then

$$|\int_\gamma f(z)dz| \le (\sup_{t\in[a,b]} |f(\gamma(t))|) \cdot l(\gamma)$$

where

$$l(\gamma) = \int_a^b |\frac{d\gamma}{dt}(t)|dt$$

Proposition 1.11

Let $U \subset \mathbb{C}$ be an open set and $F : U \to \mathbb{C}$ a continuous function. Let $\gamma : [a,b] \to U$ be a C^1 curve. Suppose that $\theta : [c,d] \to [a,b]$ is a one-to-one, onto, increasing C^1 function with a C^1 inverse. Let $\tilde{\gamma} = \gamma \circ \phi$. Then

$$\int_{\tilde{\gamma}} f dz = \int_\gamma f dz$$

Proof

We have

$$\int_{\tilde{\gamma}} f dz = \int_c^d f(\gamma(\phi(t)))\frac{d\gamma(\phi(t))}{dt}dt = \int_a^b f(\gamma(s))\frac{\gamma(s)}{ds}\phi'(\phi^{-1}(s))(\phi^{-1})'(s)ds = \int_\gamma f dz$$

since $\phi'(\phi^{-1}(s))(\phi^{-1})' = 1$.

Definition 1.10
Let f be a function on the open set U in $\mathbb{C}$ and consider if
$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$
exists then we say that f has a complex derivative at z_0. We denote the complex derivative by $f'(z_0)$. ♣

Theorem 1.3
Let $U\subset\mathbb{C}$ be an open set and let f be holomorphic on U. Then f' exists at each point of U and
$$f'(z)=\frac{\partial}{\partial z}f$$
for all $z\in U$. ♡

Proof

Consider
$$\gamma(t)=(1-t)z_0+tz$$
and then we know
$$f(z)-f(z_0)=f(\gamma(1))-f(\gamma_0)=\int_\gamma\frac{\partial}{\partial z}fdz=(z-z_0)\int_0^1\frac{\partial}{\partial z}f(\gamma(t))dt=\frac{\partial}{\partial z}f(z_0)+\int_0^1(\frac{\partial}{\partial z}f(\gamma(t))-\frac{\partial}{\partial z}f(z_0))dt$$
and hence
$$|\frac{f(z)-f(z_0)}{z-z_0}-\frac{\partial}{\partial z}f(z_0)|\le\int|\frac{\partial}{\partial z}f(\gamma(t))-\frac{\partial}{\partial z}f(z_0)|dt\to 0$$
when $z\to z_0$.

Theorem 1.4
If $f\in C^1(U)$ and f has a complex derivative at each point of U, then f is holomorphic on U. In particular, if a continuous, complex-valued function f on U has a complex derivative at each point and if f' is continuous on U, then f is holomorphic on U. ♡

Proof

It is easy to check
$$\lim_{h\to 0,h\in\mathbb{R}}\frac{f(z_0+h)-f(z_0)}{h}=\frac{\partial}{\partial x}u(x_0,y_0)+i\frac{\partial}{\partial x}v(x_0,y_0)$$
and
$$\lim_{h\to 0,h\in\mathbb{R}}\frac{f(z_0+ih)-f(z_0)}{h}=-i\frac{\partial}{\partial y}u(x_0,y_0)+\frac{\partial}{\partial y}v(x_0,y_0)$$
and hence f satisfies the C-R equations so holomorphic.

Notice the continuity of f' may implies that $f\in C^1(U)$ and hence the problem goes.

Theorem 1.5
Let f be holomorphic in a neighborhood of $P\in\mathbb{C}$. Let ω_1,ω_2 be complex numbers of unit modulus. Consider the directional derivatives
$$D_{\omega_1}f(P)=\lim_{t\to 0}\frac{f(P+t\omega_1)-f(P)}{t}$$
and
$$D_{\omega_2}f(P)=\lim_{t\to 0}\frac{f(P+t\omega_2)-f(P)}{t}$$
then
a. $|D_{\omega_1}f(P)|=|D_{\omega_2}f(P)|$
b. If $f'(P)\neq 0$, then the directed angle from ω_1 to ω_2 equals the directed angle from $D_{\omega_1}f(P)$ to $D_{\omega_2}f(P)$. ♡

Proof

Notice that

$$D_{\omega_j} = f'(P)\omega_j, j = 1, 2$$

and then the conclusions go.

Lemma 1.2

Let $(\alpha, \beta) \subset \mathbb{R}$ be an open interval and let $H : (\alpha, \beta) \to \mathbb{R}, F : (\alpha, \beta) \to \mathbb{R}$ be continuous functions. Let $p \in (\alpha, \beta)$ and suppose that dH/dx exists and equals $F(x)$ for all $x \in (\alpha, \beta) - \{p\}$. Then $(dH/dx)(p)$ exists and $(dH/dx)(x) = F(x)$ for all $x \in (\alpha, \beta)$. ♡

Proof

Assume $[a, b] \subset (\alpha, \beta)$ and then $K(x) = H(a) + \int_a^x F(t)dt$ on $[a, b]$, so we know $K - H$ is continuous on $[a, b]$ and constant on $[a, p) \cup (p, b]$, which means $K = H$ on $[a, b]$.

Theorem 1.6

Let $U \subset \mathbb{C}$ be either an open rectangle or an open disc and let $P \in U$. Let f and g be continuous, real-valued functions on U which are continuously differentiable on $U - \{P\}$. Suppose further that

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial x} g \text{ on } U - \{P\}$$

Then there exists a C^1 function $h : U \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x} = f, \frac{\partial}{\partial y} = g$$

at every point of U. ♡

Proof

Consider a closed rectangle containing p inside in U and define $h(x, y) = \int_a^x f(t, b)dt + \int_b^y g(x, s)ds$ and we know that $\frac{\partial}{\partial y} h = g(x, y)$ and $\frac{\partial}{\partial x} h = f(x, y)$ for any $x \neq P_x$, then for a fixed y, we know $dh(x, y)/dx = f(x, y)$ exists for all points in U except for (p_x, y) and hence $dh(x, y)/dx = f(x, y)$ at (p_x, y). Then we know $\frac{\partial}{\partial x} h = f, \frac{\partial}{\partial y} h = g$ on U.

Theorem 1.7

Let $U \subset \mathbb{C}$ be either an open rectangle or an open disc. Let $P \in U$ be fixed. Suppose that F is continuous on U and holomorphic on $U - \{P\}$. Then there is a holomorphic H on U such that U such that $\frac{\partial}{\partial z} H = F$. ♡

Proof

Consider $F = u + iv$, then we have

$$\frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u \text{ and } \frac{\partial}{\partial y} u = \frac{\partial}{\partial x} (-v)$$

on $U - \{P\}$, then we know there exists h_1, h_2 on U such that $\frac{\partial}{\partial x} h_1 = u$, $\frac{\partial}{\partial y} h_1 = (-v)$, $\frac{\partial}{\partial x} h_2 = v$, $\frac{\partial}{\partial y} h_2 = u$ and let $H = h_1 + ih_2$, we have

$$\frac{\partial}{\partial z} H = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(h_1 + ih_2) = (u + u) + i(v + v) = F$$

Definition 1.11

The boundary $\partial D(P, r)$ of the disc $D(P, r)$ can be parametrized as a simple closed curve $\gamma : [0, 1] \to \mathbb{C}$ by setting

$$\gamma(t) = P + re^{2\pi it}$$

we call it counterclockwise orientation. ♣

Lemma 1.3

Let γ be the boundary of a disc $D(z_0, r)$ in the complex plane, equipped with counterclockwise orientation. Let z be a point inside the circle $\partial D(z_0, \gamma)$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\xi - z}d\xi = 1$$

♡

Proof

Consider $I(z) = \int_\gamma \frac{1}{\xi - z}d\xi = \int_0^1 \frac{1}{(z_0 + e^{2\pi it}) - z}(2\pi i)e^{2\pi it}dt$ and since

$$\frac{\partial}{\partial x}\frac{1}{\xi - z} = \frac{1}{(\xi - z)^2}, \quad \frac{\partial}{\partial y}\frac{1}{\xi - z} = i\frac{1}{(\xi - z)^2}$$

and hence we have

$$\frac{\partial}{\partial \bar{z}}I(z) = \int_\gamma \frac{\partial}{\partial \bar{z}}(\frac{1}{\xi - z})d\xi = 0 \quad \frac{\partial}{\partial z}I(z) = \int_\gamma \frac{\partial}{\partial z}(\frac{1}{\xi - z})d\xi = \int_\gamma \frac{1}{(\xi - z)^2}d\xi$$

where $\frac{1}{(\xi - z)^2}$ is the complex derivative of the holomorphic function $\frac{-1}{\xi - z}$ and hence

$$\frac{\partial}{\partial z}I(z) = \int_\gamma \frac{1}{(\xi - z)^2}d\xi = 0$$

Therefore, $I(z)$ is holomorphic on $D(z_0, r)$ and $\frac{\partial}{\partial z}I = 0$ which means I is constant on $D(z_0, r)$ and notice

$$I(z_0) = 2\pi i$$

and hence the equation holds.

Theorem 1.8

(The Cauchy integral fomula) Suppose that U is an open set in $\mathbb{C}$ and that f is a holomorphic function on U. Let $z_0 \in U$ and let $r > 0$ be such that $\overline{D}(z_0, r) \subset U$. Let $\gamma : [0, 1] \to \mathbb{C}$ be the C^1 curve $\gamma(t) = z_0 + r\cos(2\pi t) + ir\sin(2\pi t)$. Then for each $z \in D(z_0, r)$,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi$$

♡

Proof By theorem 1.7, there is H such that

$$\frac{\partial}{\partial z}H = \frac{f(\xi) - f(z)}{\xi - z}$$

if $\xi \neq z$ and $\frac{\partial}{\partial z}H(z) = f'(z)$ holomorphic on $D(z_0, r + \epsilon)$ and hence

$$\int_\gamma \frac{f(\xi) - f(z)}{\xi - z}d\xi = 0$$

and the equation holds by the lemma 1.3.

Theorem 1.9

(The Cauchy integral theorem) If f is a holomorphic function on an open disc U in the complex plane, and if $\gamma : [a, b] \to U$ is a C^1 curve in U with $\gamma(a) = \gamma(b)$, then

$$\int_\gamma f(z)dz = 0$$

♡

Proof Only need to pick G such that $\frac{\partial}{\partial z}G = f$ on U is fine.

Definition 1.12

A piecewise C^1 curve $\gamma : [a, b] \to \mathbb{C}, a < b, a, b \in \mathbb{R}$ is a continuous function such that there exists a finite set of numbers $a_1 \leq a_2 \leq \cdots \leq a_k$ satisfying $a_1 = a$ and $a_k = b$ and with the property that for every $1 \leq j \leq k - 1$,

$\gamma|_{[a_j,a_{j+1}]}$ *is a* C^1 *curve. As before,* γ *is a piecewise* C^1 *curve in an open set* U *if* $\gamma_{[a,b]} \subset U$. ♣

Definition 1.13

If $U \subset \mathbb{C}$ *is open and* $\gamma : [a,b] \to U$ *is a piecewise* C^1 *curve in* U *and if* $f : U \to \mathbb{C}$ *is a continuous, complex-valued function on* U*, then*

$$\int_\gamma f(z)dz = \sum_{j=1}^{k} \int_{\gamma|_{[a_j,a_{j+1}]}} f(z)dz$$

and the definition is well-defined. ♣

Proof

We need to show for any $\{a_j\}_1^k, \{b_i\}_1^m$, the RHS determined by the chosen sequence is the same. Assume $a_{j_t} = b_{i_t}, 1 \le t \le q$, with $\{a_j\}_{j_t+1}^{j_{t+1}-1} \cap \{b_i\}_{i_t+1}^{j_{i+1}-1} = \emptyset$, then we know $\gamma|_{a_{j_t},a_{j_{t+1}}}$ is a C_1 curve and hence the integral over the curve is the same.

Lemma 1.4

Let $\gamma : [a,b] \to U$ *open in* $\mathbb{C}$ *to be a piece wise* C^1 *curve. Let* $\phi : [c,d] \to [a.b]$ *be a piecewise* C^1 *strictly monotone increasing function with* $\phi(c) = a, \phi(d) = b$*. Let* $f : U \to \mathbb{C}$ *be a continuous function on* U*. Then the function* $\gamma \circ \phi : [c,d] \to U$ *is a piecewise* C^1 *curve and*

$$\int_\gamma f(z)dz = \int_{\gamma\circ\phi} f(z)dz$$

♡

Proof Use the proposition 1.11.

Lemma 1.5

If $f : U \to \mathbb{C}$ *is a holomorphic function and if* $\gamma : [a,b] \to U$ *is a piecewise* C^1 *curve, then*

$$f(\gamma(b)) - f(\gamma(a)) = \int_\gamma f' dz$$

♡

Proof Use the proposition 1.7.

Proposition 1.12

If $f : \mathbb{C} - \{0\} \to \mathbb{C}$ *is a holomorphic function, and if* γ_r *describes the circle of radius* r *around* 0*, traversed once around counter-clockwise, then, for any two positive numbers* $r_1 < r_2$*,*

$$\int_{\gamma_{r_1}} f(z)dz = \int_{\gamma_{r_2}} f(z)dz$$

♠

Proposition 1.13

Let $0 < r < R < \infty$ *and define the annulus* $\mathcal{A} = \{z \in \mathbb{C} : r < |z| < R\}$*. Let* $f; \mathcal{A} \to \mathbb{C}$ *be a holomorphic function. If* $r < r_1 < r_2 < R$ *and if for each* j *the curve* γ_{r_j} *describes the circle pf radius* r_j *around* 0*, traversed once counter clockwise, then we have*

$$\int_{\gamma_{r_1}} f dz = \int_{\gamma_{r_2}} f dz$$

♠

Applications of the Cauchy integral

Theorem 1.10

Let $U \subset \mathbb{C}$ be an open set and let f be holomorphic on U. Then $f \in C^\infty(U)$. Moreover, if $\overline{D}(P,r) \subset U$ and $z \in D(P,r)$, then

$$\Big(\frac{\partial}{\partial z}\Big)^k f(z) = \frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+1}}d\xi$$

for any integer k.

Proof

Use the induction to f, assume

$$(\frac{\partial}{\partial z})^k f(z) = \frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+1}}d\xi$$

and $(\frac{\partial}{\partial z})^k f(z)$ is holomorphic, then we gonna prove that

$$(\frac{\partial}{\partial z})^{k+1} f(z) = \frac{(k+1)!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+2}}d\xi$$

and $(\frac{\partial}{\partial z})^{k+1} f(z)$ is holomorphic. Consider

$$\begin{aligned}\Big|\frac{f(\xi)}{(\xi-\omega)^{k+1}}-\frac{f(\xi)}{(\xi-z)^{k+1}}\Big| &\le \sup_{\xi\in\partial D(P,r)}|f(\xi)|\epsilon^{-2k-2}\Big|\sum_{i=1}^{k+1}C_{k+1}^i(2r)^{k+1-i}(\omega-z)^i\Big| \\ &\le |\omega-z|(k+1)\Big(\sup_{\xi\in\partial D(P,r)}|f(\xi)|\epsilon^{-2k-2}\Big|\sum_{i=0}^{k}C_k^i(2r)^{k-i}(\omega-z)^i\Big|\Big) \\ &\le |\omega-z|(k+1)\Big(\sup_{\xi\in\partial D(P,r)}|f(\xi)|\epsilon^{-2k-2}(2r+1)^k\Big)\end{aligned}$$

for all $|\omega - z|$ small enough and hence

$$\frac{f(\xi)}{(\xi-\omega)^{k+1}} \to \frac{f(\xi)}{(\xi-z)^{k+1}}$$

uniformly when $\omega \to z$, so may know

$$\lim_{\omega\to z}\frac{(\frac{\partial}{\partial z})^{k+1}f(\omega)-(\frac{\partial}{\partial z})^{k+1}f(z)}{\omega-z} = \lim_{\omega\to z}\frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{\frac{f(\xi)}{(\xi-\omega)^{k+1}}-\frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega-z}d\xi$$

and we know that

$$\lim_{\omega\to z}\frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{\frac{f(\xi)}{(\xi-z)^{k+1}}-\frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega-z}d\xi = \frac{k!}{2\pi i}\int_{|\xi-P|=r}\lim_{\omega\to z}\frac{\frac{f(\xi)}{(\xi-\omega)^{k+1}}-\frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega-z}d\xi$$

by the DCT and hence

$$\lim_{\omega\to z}\frac{(\frac{\partial}{\partial z})^{k+1}f(\omega)-(\frac{\partial}{\partial z})^{k+1}f(z)}{\omega-z} = \frac{(k+1)!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+2}}d\xi$$

which means $(\frac{\partial}{\partial z})^k f(z)$ is holomorphic and the equality holds. Then we use the induction, and the conclusion goes.

Corollary 1.3

If $f : U \to \mathbb{C}$ is holomorphic, then $f' : U \to \mathbb{C}$ is holomorphic.

Theorem 1.11

If ϕ is a continuous function on $\{\xi : |\xi - P| = r\}$, then the function f given by

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - P| = r} \frac{\phi(\xi)}{\xi - z} d\xi$$

is defined and holomorphic on $D(P, r)$.

♡

Theorem 1.12

(Morera) Suppose that $f : U \to \mathbb{C}$ is a continuous function on a connected open subset U of $\mathbb{C}$. Assume that for every closed, piecewise C^1 curve $\gamma : [0, 1] \to U$, $\gamma(0) = \gamma(1)$, it holds that

$$\int_\gamma f(\xi) d\xi = 0$$

Then f is holomorphic on U.

♡

Proof Consider $x \in U$ and define $F(y) = \int_\phi f dz$ for any $y \in U$ where ϕ is a picewise C^1 curve from x to y, where we know the integral is well-defined since any integral of f on a closed, piece wise C^1 curve is 0. Then for any $y \in U$, consider a segment from $y + h$ where $|h|$ is small enough and we know

$$\lim_{|h| \to 0} \frac{F(y + h) - F(y)}{h} = \lim_{|h| \to 0} \frac{1}{h} \int_0^h f(y + z) dz = f(y)$$

which means F is holomorphic on U and $F' = f$ on U, and hence f is holomorphic on U.

Definition 1.14

let $P \in \mathbb{C}$ be fixed. A complex power series centered at P is an expression of the form

$$\sum a_k (z - P)^k$$

where a_k is complex valued.

♣

Lemma 1.6

(Abel) If $\sum a_k (z - P)^k$ converges at some z, then the series converges at each $\omega \in D(P, r)$, where $r = |z - P|$.

♡

Proof

Since $\sum a_k (z - P)^k$ converges, we know $a_k (z - P)^k \to 0$ and hence bounded, then we know

$$|a_k| \leq M r^{-k}$$

for some $M > 0$ and then for any $\omega \in D(P, r)$, assume $|\omega - P| = \delta < r$, then we know

$$|a_k (\omega - P)^k| \leq |a_k| \delta^k \leq M (\delta / r)^{-k}$$

and hence

$$\sum |a_k (\omega - P)^k| \leq M \sum (\delta / r)^{-k} < \infty$$

which means $\sum a_k (\omega - P)^k$ converges.

Definition 1.15

Let $\sum a_k (z - P)^k$ be a power series. Then

$$r = \sup\{|\omega - P| : \sum a_k (\omega - P)^k \textit{ converges}\}$$

is called the $radius\ of\ convergence$ of the power series.

♣

Lemma 1.7

If $\sum a_k (z - P)^k$ is a power series with radius of convergence r, then the series converges for each $\omega \in D(P, r)$ and diverges for each ω such that $|\omega - P| > r$.

♡

Lemma 1.8

(The root test) The radius of convergence of the power series $\sum a_k(z-P)^k$ *is*

$$\frac{1}{\limsup |a_k|^{1/k}}$$

if $\limsup |a_k|^{1/k} > 0$ *or*

$$\infty$$

if $\limsup |a_k|^{1/k} = 0$.

Proof

Assume $\alpha = \limsup |a_k|^{1/k}$, if $|\omega - P| > 1/\alpha$, then denote $|\omega - P| = c/\alpha, c > 1$ and we know

$$|a_k(z-P)^k| = (c|a_k|^{1/k}/\alpha)^k$$

and we know there are infinitly many a_k such that $|a_k|^{1/k}/\alpha > 1/c$ and hence the series diverge.

For $|\omega - P| < 1/\alpha$, we denote $|\omega - P| = d/\alpha, d < 1-\epsilon$ for some $\epsilon > 0$ and we have

$$|a_k(\omega-P)^k| \le (|a_k|^{1/k}d/\alpha)^k \le (1-\epsilon)^k$$

when k is sufficiently large and hence the series is absolutely convergent and the condition for $\alpha = 0$ is similar.

Definition 1.16

Let $\sum f_k(z)$ *be a series of functions on a set* E. *The series is said to be uniformly Cauchy if for any* $\epsilon > 0$, *these is an integer* N *such that*

$$|\sum_{k=m}^{n} f_k(z)| < \epsilon$$

on E *for any* $n \ge m \ge N$.

Proposition 1.14

Let $\sum a_k(z-P)^k$ *be a power series with radius of convergence* r. *Then, for any number* R *with* $0 \le R < r$ *,the series* $\sum |a_k(z-P)|^k$ *converges uniformly on* $\overline{D}(P,R)$ *and hence* $\sum a_k(z-P)^k$ *converges uniformly and absolutely on* $\overline{D}(P,R)$.

Proof We know

$$\lim_{k\to\infty} |a_k r^k| \to 0$$

and hence there exists $M > 0$ such that

$$|a_k| \le \frac{M}{r^k}$$

then we know

$$\sum_{k=0}^{n} |a_k(z-P)^k| \le \sum_{k=0}^{n} M(r/R)^k$$

on $\overline{D}(P,R)$ and hence the series converges uniformly.

Lemma 1.9

If a power series

$$\sum_{j=0}^{\infty} a_j(z-P)^j$$

has radius of convergence $r > 0$, *then the series defines a* C^∞ *function* $f(z)$ *on* $D(P,r)$. *The function* f *is* *holomorphic on $D(P,r)$. The series obtained by termwise differentiation k times of the original power series,*

$$\sum_{j=k}^{\infty} \frac{j!}{k!} a_j (z-P)^{j-k}$$

converges on $D(P,r)$ and its sum is $(\partial/\partial z)^k f(z)$ for each $z \in D(P,r)$. ♡

Proof

For any $z \in D(P,r)$, we know the series is abosolutely convergent at z, and hence

$$D_h f(z) = \lim_{d \to 0} \sum_{j=0}^{\infty} a_j \frac{(z+dh-P)^j - (z-P)^j}{d} = \sum_{j=0}^{\infty} a_j j (z-P)^{j-1}$$

since

$$\sum_{j=0}^{\infty} j|a_j r'^{j-1}| \leq C + \sum_{j=m}^{\infty} |a_j (r'+\epsilon)^{j-1}/(r'+\epsilon)^j|$$

for some $\epsilon > 0$ and integer m big sufficiently, and hence we may exchange the summation and the limit. Then we know f is holomorphic and hence in C^∞ and we may use the induction to $\frac{\partial}{\partial z}^k f$.

Proposition 1.15

If both series $\sum_{j=0}^{\infty} a_j(z-P)^j$ and $\sum_{j=0}^{\infty} b_j(z-P)^j$ converge on a disc $D(P,r), r > 0$ and if $\sum_{j=0}^{\infty} a_j(z-P)^j = \sum_{j=0}^{\infty} b_j(z-P)^j$ on $D(P,r)$, then $a_j = b_j$ for every j. ♠

Proof

Use the lemma 1.9. directly.

Theorem 1.13

Let $U \subset \mathbb{C}$ be an open set and let f be holomorphic on U. Let $P \in U$ and suppose that $D(P,r) \subset U$. Then the comnplex power series

$$\sum_{k=0}^{\infty} \frac{(\frac{\partial}{\partial z})^k f(P)}{k!} (z-P)^k$$

has radius of convergence at least r. It converges to $f(z)$ on $D(P,r)$. ♡

Proof

For $z \in D(P,r)$, we know

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - P| = r'} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{|\xi-P|=r'} \frac{f(\xi)}{\xi - P} \sum_{n \geq 0} ((z-P)(\xi-P)^{-1})^n d\xi$$

for $r' > |z-P|$ and $D(z,r') \subset D(P,r)$ and then we know

$$f(z) = \lim_{N \to \infty} \frac{1}{2\pi i} \int_{|\xi-P|=r'} \frac{f(\xi)}{\xi - P} \sum_{n=0}^{N} ((z-P)(\xi-P)^{-1})^n d\xi$$

since the series converges uniformly. Then

$$f(z) = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{2\pi i} \int_{|\xi-P|=r'} \frac{f(\xi)}{(\xi-P)^{n+1}} (z-P)^n = \sum_{k=0}^{\infty} \frac{(\frac{\partial}{\partial z})^k f(P)}{k!} (z-P)^k$$

Theorem 1.14

(The Cauchy estimates) Let $f : U \to \mathbb{C}$ be a holomorphic function on an open set $U, P \in U$ and assume that the

closed disc $\overline{D}(P,r), r>0$ is contained in U. Set $M=\sup_{z\in\overline{D}(P,r)}|f(z)|$, then for $k\geq 1$ we have

$$\left|\frac{\partial^k f}{\partial z^k}(P)\right|\leq\frac{Mk!}{r^k}$$

♡

Proof

We know

$$\left|\frac{\partial^k f}{\partial z^k}(P)\right|=\left|\frac{k!}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-z)^{k+1}}d\xi\right|\leq\frac{Mk!}{r^k}$$

Lemma 1.10

Suppose that f is a holomorphic function on a connected open set $U\subset\mathbb{C}$. If $\partial f/\partial z=0$ on U, then f is constant on U. ♡

Proof Notice $\dfrac{\partial}{\partial x}f=\dfrac{\partial}{\partial y}f=0$ on U.

Definition 1.17

A function f is said to be entire if it is defined and holomorphic on all of $\mathbb{C}$, that is, $f:\mathbb{C}\to\mathbb{C}$ is holomorphic. ♣

Theorem 1.15

(Liouville's theorem) A bounded entire function is constant. ♡

Proof For any $P\in\mathbb{C}$, we may know

$$\left|\frac{\partial}{\partial z}f(P)\right|\leq M/r$$

for any $r>0$ and hence $\dfrac{\partial}{\partial z}f=0$ on $\mathbb{C}$ and hence it is a constant on $\mathbb{C}$.

Theorem 1.16

If $f:\mathbb{C}\to\mathbb{C}$ is an entire function and if for some real number C and some postive integer k it holds that

$$|f(z)|\leq C|z|^k$$

for all z with $|z|>1$, then f is a polynomial in z of degree at most k. ♡

Proof We know

$$\left|\left(\frac{\partial}{\partial z}\right)^{k+l}f(0)\right|\leq C(k+l)!/r^l$$

for any $r\geq 0$ and hence $\left(\dfrac{\partial}{\partial z}\right)^{k+l}f(0)=0$.

Theorem 1.17

Let $p(z)$ be a nonconstant polynomial. Then p has a root. ♡

Proof

If not, we know $g(z)=1/p(z)$ is holomorphic on $\mathbb{C}$ and bounded since $|p(z)|\to\infty, |z|\to\infty$, so by the Liouville's theorem, we know $p(z)$ is constant and hence a contradiction.

Corollary 1.4

If $p(z)$ is a holomorphic polynomial of $\deg k$, then there are k complex numbers $\alpha_1,\cdots,\alpha_k$ and a constant C such *that*

$$p(z) = C\prod_{i=1}^{k}(z-\alpha_i)$$

Theorem 1.18

Le $f_j : U \to \mathbb{C}, j \geq 1$ be a sequence of holomorphic functions on an open set U in $\mathbb{C}$. Suppose that there is a function $f : U \to \mathbb{C}$ such that, for each compact subset E of U, the sequence $f_j|_E$ converges uniformly to $f|_E$. Then f is holomorphic on U.

Proof

Firstly, it is easy to check f is continuous on U.

For $z \in U$, we may consider $D_z = \overline{D}(z,r) \subset U$ is a compact set, and we know $f_j \to f$ uniformly on D_z, then

$$f(z+d) = \lim_{n\to\infty} f_n(z+d) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{|\xi-z|=r}\frac{f_n(\xi)}{\xi-(z+d)}d\xi = \frac{1}{2\pi i}\int_{|\xi-z|=r}\frac{f(\xi)}{\xi-(z+d)}d\xi$$

then we know

$$f'(z) = \lim_{|d|\to 0}\frac{1}{2\pi i}\int_{|\xi-z|=r} f(\xi)\Big(\Big|\frac{1}{\xi-(z+d)} - \frac{1}{\xi-z}\Big|/d\Big)d\xi = \frac{1}{2\pi i}\int_{|\xi-z|=r}\frac{f(\xi)}{(\xi-z)^2}d\xi$$

and hence f is holomorphic on U.

Corollary 1.5

If f_j, f, U are as in the theorem above, then for any integer $k \geq 0$, we have

$$\Big(\frac{\partial}{\partial z}\Big)^k f_j(z) \to \Big(\frac{\partial}{\partial z}\Big)^k f(z)$$

uniformly on compact sets.

Proof We have

$$\Big|\Big(\frac{\partial}{\partial z}\Big)^k f_j(z) - \Big(\frac{\partial}{\partial z}\Big)^k f(z)\Big| \leq \frac{k!}{r^k}\sup_{\overline{D}(z,r)}|f(z)-f_j(z)|$$

if $\overline{D}(z,r) \subset U$ and the rest is easy to be checked.

Theorem 1.19

Let $U \subset \mathbb{C}$ be a connected open set and let $f : U \to \mathbb{C}$ be holomorphic. Let $Z = \{z \in U, f(z) = 0\}$. If there are a $z_0 \in Z$ and $\{z_j\}_{j=1}^{\infty} \in Z - \{z_0\}$ such that $z_j \to z_0$, then $f = 0$ on U.

Proof

Consider

$$E = \{z, \Big(\frac{\partial}{\partial z}\Big)^k f(z) = 0 \text{ for any interger } k \geq 0\}$$

and we claim $z_0 \in E$, if not there exists n_0 such that

$$\Big(\frac{\partial}{\partial z}\Big)_0^n f(z_0) \neq 0$$

and hence

$$g(z) = \sum_{i=n_0}^{\infty}\Big(\frac{\partial}{\partial z}\Big)^i f(z)\frac{(z-z_0)^{i-n_0}}{i!}$$

is not 0 at z_0 but $g(z_j) = 0$ for any z_j, and hence $g(z_0) = 0$ by the continuity, which is a contradiction and hence $z_0 \in E$.

Now it is easy to check E is closed about U and also E is open since for any $z \in E$, we we know

$$f(z+d) = \sum \partial^j f(z)/j!d^j$$

for any d in some open call centered at z and hence the ball is in E. Notice U is connected and we know $E = U$ and theorem is proved.

Corollary 1.6

Let $U \subset \mathbb{C}$ be a connected open set and $D(P,r) \subset U$. If f is holomorphic on U and $f|_{D(P,r)} = 0$, then $f = 0$ on U.

Corollary 1.7

Let $U \subset \mathbb{C}$ be a connected open set and $D(P,r) \subset U$. Let f, g be holomorphic on U. If $\{z, f(z) = g(z)\}$ has an accumulation in U, then $f = g$ on U.

Corollary 1.8

Let $U \subset \mathbb{C}$ be a connected open set and $D(P,r) \subset U$. Let f, g be holomorphic on U. If $fg = 0$ on U, then either $f = 0$ on U or $g = 0$ on U.

Proof Choose a point z, $f(z) \neq 0$ is fine.

Corollary 1.9

Let $U \subset \mathbb{C}$ be connected and open and let f be holomorphic on U. If there is a $P \in U$ such that

$$\left(\frac{\partial}{\partial z}\right)^j f(P) = 0$$

for every j, then $f = 0$ on U.

Corollary 1.10

If f and g are entire holomorphic functions and if $f = g$ for all $x \in \mathbb{R} \subset \mathbb{C}$, then $f = g$.

Definition 1.18

Let $U \subset \mathbb{C}$ be an open set and $P \in U$. Suppose that $f : U - \{P\} \to \mathbb{C}$ is holomorphic. In this situation we say that f has an isolated singular point ar P

Definition 1.19

If $\lim_{z \to P} |f(z)| = +\infty$, then we call f has a pole at P. If P is not a pole or a removable singularity, we call f has an essential singularity at P.

Theorem 1.20

(The Riemann removable singularities theorem) Let $f : D(P,r) - \{P\} \to \mathbb{C}$ be holomorphic and bounded. Then
a. $\lim_{z \to P} f(z)$ exists
b. the function $\hat{f} : D(P,r) \to \mathbb{C}$ defined by

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \neq P \\ \lim_{\xi \to P} f(\xi) & \text{if } z = P \end{cases}$$

Proof

Consider

$$g(z) = \begin{cases} (z-P)^2 f(z) & \text{if } z \in D(P,r) - \{P\} \\ 0 & \text{if } z = P \end{cases}$$

we claim that $g \in C^1(D(P,r))$. Since we know

$$\frac{\partial g}{\partial \bar{z}} = 0$$

on $D(P,r) - \{P\}$ and if $g \in C^1(D(P,r))$, then g is holomorphic. Notice

$$g'(z) = 2(z-P)f(z) + (z-P)^2 f'(z)$$

on $D(P,r) \to \mathbb{C}$ and

$$\frac{\partial g}{\partial x}(P) = \lim_{h\to 0} hf(P+h) = 0$$

and similarly $\partial g/\partial y(P) = 0$ and it suffices to show

$$\lim_{z\to P} g'(z) = \lim_{z\to P} 2(z-P)f(z) + (z-P)^2 f'(z)$$

equals to 0, which can be implied by the Cauchy estimation. Now we know g is C^1 and hence homorphic on $D(P,r)$. Then let

$$H(z) = \sum_{n=2}^{\infty} \left(\frac{\partial}{\partial z}\right)^n g(P)/n!(z-P)^2$$

which has radius convergence at least r and holomorphic on $D(P,r)$, which equals to $f(z)$ on $D(P,r) - \{P\}$ and satisfies the requirements.

Theorem 1.21

(Casorati-Weierstrass) If $f : D(P,r_0) - \{P\}$ is holomorphic and P is an essential singularity of f, then $f(D(P,r) - \{P\})$ is dense in $\mathbb{C}$ for any $0 < r < r_0$. ♡

Proof

It suffices to show $r = r_0$, then there is $\lambda \in \mathbb{C}$ and an $\epsilon > 0$ such that

$$|f(z) - \lambda| > \epsilon$$

for all $z \in D(P,r_0) - \{P\}$. Consider the function $g : D(P,r_0) - \{P\} \to \mathbb{C}$ defined by

$$g(z) = \frac{1}{f(z) - \lambda}$$

then P is a removable singularity for g with

$$f(z) = \lambda + \frac{1}{\hat{g}(z)}$$

on $D(P,r) - \{P\}$ where $\hat{g} \neq 0$, so if $\hat{g}(P) = 0$, then it is easy to check that P is a pole of f, which is a contradiction, so $\lim_{z\to P} f(z)$ exists and finite, which means P is a removable singularity of f and hence contradictory.

Definition 1.20

A Laurent series on $D(P,r)$ is a expression of the form

$$\sum_{-\infty}^{\infty} a_j (z-P)^j$$

and when we say a series with double infinities converges, we mean $\sum_{n\geq 0} \alpha_n$ and $\sum_{n\leq 0} \alpha_n$ converge both. ♣

Lemma 1.11

If $\sum_{-\infty}^{\infty} a_j(z-P)^j$ converges at $z_1 \neq P$ and $z_2 \neq P$ with $|z_1 - P| < |z_2 - P|$, then the series converges for all z such that $|z_1 - P| < |z - P| < |z_2 - P|$. ♡

Proof

Assume $S_n(z) = \sum_{j=0}^{n} a_j(z-P)^j$ and $W_n(z) = \sum_{j=-1}^{-n} a_j(z-P)^j$ and we know $S_n(z_2), W_n(z_1)$ converges and hence there exists M such that

$$|a_{-j}||z_1 - P|^{-j}, |a_j||z_2 - P|^j < M$$

then for any z in the annulus, we know

$$\sum_{j=0}^{n} |a_j||z-P|^j \leq M \sum_{j=0}^{n} \left(\frac{|z-P|}{|z_2 - P|}\right)^j$$

and

$$\sum_{j=1}^{n}|a_{-j}||z-P|^{-j}\leq M\sum_{j=1}^{n}\left(\frac{|z-P|}{|z_1-P|}\right)^{-j}$$

which means $S_n(z), W_n(z)$ are both absolutely convergent and the conclusion holds.

Proposition 1.16

Let $0\leq r_1<r_2\leq\infty$. If the Laurent series $\sum_{-\infty}^{\infty}a_j(z-P)^j$ converges on $D(P,r_2)-\overline{D}(P,r_1)$ to a function f, then for any r satisfying $r_1<r<r_2$, and each $j\in\mathbb{Z}$,

$$a_j=\frac{1}{2\pi i}\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-P)^{j+1}}d\xi$$

Proof

We know since the series converges uniformly on the circle $|z-P|=r$, then

$$\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-P)^{j+1}}d\xi=\int_{|\xi-P|=r}\sum_{-\infty}^{\infty}a_k(\xi-P)^{k-j-1}d\xi=\sum_{-\infty}^{\infty}a_k\int_{|\xi-P|=r}(\xi-P)^{k-j-1}d\xi$$

and then we know

$$\int_{|\xi-P|=r}(\xi-P)^{k-j-1}d\xi=\begin{cases}0 & \text{if } k-j-1\neq-1\\ 2\pi i & \text{if } k-j-1=-1\end{cases}$$

and hence

$$\int_{|\xi-P|=r}\frac{f(\xi)}{(\xi-P)^{j+1}}d\xi=2\pi i a_j$$

Theorem 1.22

(The Cauchy integral formula for an annulus) Suppose that $0\leq r_1<r_2\leq+\infty$ and that $f:D(P,r_2)-\overline{D}(P,r_1)\to\mathbb{C}$ is holomorphic. Then for each s_1,s_2 such that $r_1<s_1<s_2<r_2$ and each $z\in D(P,s_2)-\overline{D}(P,r_1)$, it holds that

$$f(z)=\frac{1}{2\pi i}\int_{|\xi-P|=s_2}\frac{f(\xi)}{\xi-z}d\xi-\frac{1}{2\pi i}\int_{|\xi-P|=s_1}\frac{f(\xi)}{\xi-z}d\xi$$

Theorem 1.23

(The existence of Laurent expansions) If $0\leq r_1<r_2\leq\infty$ and $f:D(P,r_2)-\overline{D}(P,r_1)\to\mathbb{C}$ is holomorphic, then there exist complex numbers a_j such that

$$\sum_{-\infty}^{\infty}a_j(z-P)^j$$

converges on $D(P,r_2)-\overline{D}(P,r_1)$ to f. If $r_1<s_1<s_2<r_2$, then the series converges absolutely and uniformly on $D(P,s_2)-\overline{D}(P,s_1)$.

Proof

Notice

$$\int_{|\xi-P|=s_2}\frac{f(\xi)}{\xi-z}d\xi=\int_{|\xi-P|=s_2}\frac{f(\xi)}{\xi-P}\frac{1}{1-\dfrac{z-P}{\xi-P}}=\frac{1}{2\pi i}\int_{|\xi-P|=s_2}\sum_{n\geq0}\left(\frac{f(\xi)(z-P)^n}{(\xi-P)^{n+1}}\right)d\xi$$

and notice the series converge uniformly on $|\xi-P|=s_2$, so we know

$$\int_{|\xi-P|=s_2}\frac{f(\xi)}{\xi-z}d\xi=\sum_{n\geq0}\left(\int_{|\xi-P|=s_2}\frac{f(\xi)}{(\xi-P)^{n+1}}d\xi\right)(z-P)^n$$

and similarly, we may know

$$\int_{|\xi-P|=s_1} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n\geq 1} \Big(\int_{|\xi-P|=s_1} \frac{f(\xi)}{(\xi-P)^{-n+1}} d\xi \Big) (z-P)^{-n}$$

and the rest is by the theorem 1.22.

Proposition 1.17

If $f : D(P,r) - \{P\} \to \mathbb{C}$ is holomorphic, then f has a unique Laurent series expansion

$$f(z) = \sum_{-\infty}^{\infty} a_j (z-P)^j$$

which converges absolutely for $z \in D(P,r) - \{P\}$. The convergence is uniform on compact subsets of $D(P,r) - \{P\}$. ♠

Proposition 1.18

There are three possibilities for the Laurent series of a holomorphic function f,

a. $a_j = 0$ for all $j < 0$;

b. for some $k > 0, a_j = 0$ for all $-\infty < j < -k$;

c. neither (a) or (b). ♠

Proof

(a) implies P is removable is obviously, conversely, consider the series expansion of the holomorphic expansion $\hat{f}$.

(b) implies P is a pole can be seen by

$$|f(z)| \geq (z-P)^{-k} \Big(|a_{-k}| - \sum_{j=-k+1} +\infty |a_j| (z-P)^{j+k} \Big)$$

and hence $f(z) \to \infty, z \to P$.

For the other direction, we may consider there exists $D(P,r)$ such that $f(z)$ is nonzero there and let $g(z) = 1/f(z)$ which is holomorphic on $D(P,r)$ and P is a removable singularity of g, hence we may find $\hat{g}$ is holomorphic on $D(P,r)$ and hence

$$H(z) = (z-P)^m Q(z)$$

for some integer Q and some function Q nonzero at P, which measn $Q(z)$ is nonzero on $D(P,r)$ and we may find $1/Q(z)$ holomorphic on $D(P,r)$, then we will find a series of f.

Definition 1.21

If a function f has a Laurent expansion

$$f(z) = \sum_{j=-k}^{\infty} a_j (z-P)^j$$

for some $k > 0$ and $a_{-k} \neq 0$, then we say that f has a pole of order k at P. ♣

Proposition 1.19

Let f be holomorphic on $D(P,r) - \{P\}$ and suppose that f has a pole of order k at P. Then the Laurent serues coefficients a_j of f expanded about the point P, for $j = -k, -k+1, -k+2, \cdots$ are given by the formula

$$a_j = \frac{1}{(k+j)!} \Big(\frac{\partial}{\partial z} \Big)^{k+j} ((z-P)^k f)|_{z=P}$$

♠

Definition 1.22

An open set $U \subset \mathbb{C}$ is holomorphically simplt connected if U is connected and if, for each holomorphic function $f : U \to \mathbb{C}$, there is a holomorphic function $F : U \to \mathbb{C}$ such that $F' = f$. ♣

Lemma 1.12

A connected open set U is holomorphically simply connected if and only if for each holomorphic function $f : U \to \mathbb{C}$ and each piecewise C^1 closed curve γ in U,

$$\int_\gamma f = 0$$

Definition 1.23

If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1 curve and if $P \notin \tilde{\gamma} = \gamma([a, b])$, then the index of γ with respect to P, witten $Ind_\gamma(P)$ is defined to be the number

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\xi - P}d\xi$$

Lemma 1.13

If $\gamma : [a, b] \to \mathbb{C} - \{P\}$ is a piecewise C^1 closed curve and if P is a point not on that curvem then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\xi - P} = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t) - P}dt$$

is an integer.

Proof

Consider

$$g(t) = (\gamma(t) - P)\exp\Big(-\int_a^t \gamma'(s)/[\gamma(s) - P]ds\Big)$$

then g is continuous and we also have

$$g'(t) = \gamma'(t)\exp\Big(-\int_a^t \frac{\gamma'(s)}{\gamma(s) - P}ds\Big) + (\gamma(t) - P)\frac{-\gamma'(t)}{\gamma(t) - P}\exp\Big(-\int_a^t \frac{\gamma'(s)}{\gamma(s) - P}ds\Big) = 0$$

and it is easy to check $g(a) = g(b)$ and hence

$$-\int_a^b \frac{\gamma'(s)}{\gamma(s) - P}ds$$

must be and integer multiple of $2\pi i$.

Definition 1.24

The residue of f at P, written as $Res_f(P)$ is define by the coefficient of $(z - P)^{-1}$ in the Laurent expansion of f about P.

Theorem 1.24

(The residue theorem) Suppose that $U \subset \mathbb{C}$ is an h.s.c. open set in C and that $P_1, \cdots, P_n$ are distinct points of U. Suppose that $f : U - \{P_1, \cdots, P_n\} \to \mathbb{C}$ is a holomorphic function and γ is a closed, piecewise C^1 curve in $U - \{P_1, \cdots, P_n\}$. Then

$$\int_\gamma f = \sum_{j=1}^n Res_f(P_j)\Big(\int_\gamma \frac{1}{\xi - P_j}d\xi\Big)$$

Proof

Let s_j be the negative part of the Laurent series of f at P_j and we know $f - s_j$ is holomorphic on some neighbourhood of P_j and hence we may know

$$\int_\gamma (f - \sum s_j) = 0$$

and hence

$$\int_\gamma f = \int_\gamma s_j$$

where

$$\int_\gamma s_j(\xi)d\xi = \sum_{k=1}^{\infty} a_{-k}^{(j)} \int_\gamma (\xi - P_j)^{-k} d\xi = 2\pi i a_{-1}^{(j)} Ind_\gamma(P_j)$$

Proposition 1.20

Let f be a function with a pole of order k at P. Then

$$Res_f(P) = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial z}\right)^{k-1}((z-P)^k f(z))|_{z=P}$$

♠

Definition 1.25

A set S in $\mathbb{C}$ is discrete iff for each $z \in S$ there is a positve number r such that $S \cap D(z,r) = \{z\}$

♣

Definition 1.26

A meromorphic function f on an open set U with singular set S is a function $f : U - \{S\} \to \mathbb{C}$ such that

a. the set S is closed in U and is discrete.

b. the function F is holomorphic on $U - \{S\}$.

c. for each $z \in S$ and $r > 0$ such that $D(z.r) \subset U$ and $S \cap D(z,r) = \{z\}$, the function $f|_{D(z,r)-\{z\}}$ has a pole at z.

♣

Lemma 1.14

If U is a connected open set in $\mathbb{C}$ and if $f : U \to \mathbb{C}$ is a holomorphic function with $f \neq 0$, then the function

$$F : U - \{z : f(z) = - \to C\}$$

define by $F(z) = 1/f(z), z \in U - \{z, f(z) = 0\}$ is a meromorphic function on U with singular set equal to $\{z \in U, f(z) = 0\}$.

♡

Proof It is easy to check $S = \{z, f(z) = 0\}$ is closed and discrete by theorem 1.19 and F is obviously holomorphicon $U - S$. The rest is easy to check.(connected open set is for the theorem 1.19)

Definition 1.27

Suppose that $f : U \to \mathbb{C}$ is a holomorphic function on an open set $U \subset \mathbb{C}$ and that for some $R > 0, U \supset \{z : |z| > R\}$. Define $G : \{z : 0 > |z| < 1/R \to \mathbb{C}\}$ by $G(z) = f(1/z)$, then we say that

a. f has a removable singularity at ∞ if G has a removable singularity at 0.

b. f has a pole at ∞ if G has a pole at 0.

c. f has an essential singularity at ∞ if G has an essential singularity at 0.

♣

Theorem 1.25

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Then $\lim_{|z\|\to+\ infty}$ iff f is a non constant polynomial.

♡

Proof

Consider the series expansion of f which is exactly the Laurent expansion of G and we are done.

Definition 1.28

Suppose that f is a meromorphic function defined on an open set $U \subset \mathbb{C}$ such that for some $R > 0$, we have $U \supset \{z, |z| > R\}$. We say that f is a meromorphic at ∞ at ∞ if the function $G(z) = f(1/z)$ is meromorphic in the usual sense on $\{z, |z| < 1/R\}$.

♣

Theorem 1.26

A meromorphic function f on $\mathbb{C}$ which is also meromorphic at ∞ must be a rational funtion, i.e. a quotient of polynomials in z. Conversely, every rational function is meromorphic on $\mathbb{C}$ and at ∞. ♡

Proof

We know there has to be $R > 0$ such that all finite poles of f is in $\overline{D}(0, R)$, denoted as $P_1, P_2, \cdots, P_k$ and we may know

$$F(z) = (z - P_1)^{n_1} \cdots (z - P_k)^{n_k} f(z)$$

has removable singularities on $\mathbb{C}$ and then it suffices to show that F is rational. If ∞ is a removable singularity or pole of F, then the problem goes. If not, we know $F(1/z)$ has an essential singularity at 0 and then we may find $G(1/z)$ has infinitly many negative items, which is a contradiction.

Definition 1.29

Let $f : U \to \mathbb{C}$ be holomorphic and has zeros but not identically zero, then we know f has the series expansion and call the first nonzero term determined by the least positive integer n as the order of z_0 as a zero of f. ♣

Lemma 1.15

If f is holomorphic on a neighborhood of a disc $\overline{D}(z_0, r)$ and has a zero of order n at z_0 and no other zeros in the closed disc, then

$$\frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f'(\xi)}{f(\xi)} d\xi = n$$

♡

Proof We know

$$f(z) = (z - z_0)^n \sum_{j=n}^{\infty} \Big[\sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0)(z - z_0)^{j-n} \Big]$$

where we define

$$H(z) = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0)(z - z_0)^{j-n}$$

which is holomorphic on an open disc containing $\overline{D}(z_0, r)$ and nonzero on the closed disc, so we may know that H'/H is holomorphic on some neighbourhood of the closed disc and since

$$\frac{f'(\xi)}{f(\xi)} = \frac{H'(\xi)}{H(\xi)} + \frac{n}{\xi - z_0}$$

we may know that the integral equals to n.

Proposition 1.21

Suppose that $f : U \to \mathbb{C}$ is a holomorphic on an open set $U \subset \mathbb{C}$ and that $\overline{D}(P, r) \subset U$. Suppose that f is nonvanishing on $\partial D(P, r)$ and that $z_1, z_2, \cdots, z_k$ are the zeros of f in the interior of the disc. Let n_l be the order of the zero of f at z_l, then

$$\frac{1}{2\pi i} \int_{|\xi - P| = r} \frac{f'(\xi)}{f(\xi)} d\xi = \sum_{l=1}^{k} n_l$$

♠

Proof Let

$$H(z) = \frac{f(z)}{(z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_k)^{n_k}}$$

and the rest is easy to be checked.

Lemma 1.16

If $f: U - \{Q\} \to \mathbb{C}$ is a nowhere zero holomorphic function on $U - \{Q\}$ wutg a pole of order n at Q and if $\overline{D}(Q, r) \subset U$, then

$$\frac{1}{2\pi i}\int_{\partial D(Q,r)} \frac{f'(\xi)}{f(\xi)} d\xi = -n$$

Proof We know $H(z) = (z - Q)^n f(z)$ has a removable singularity at Q where

$$\frac{H'(\xi)}{H(\xi)} = \frac{n}{\xi - Q} + \frac{f'(\xi)}{f(\xi)}$$

and the rest is easy to be checked.

Theorem 1.27

Suppose that f is a meromorphic function on an open set $U \subset \mathbb{C}$, that $\overline{D}(P, r) \subset U$ and that f has neither poles nor zeros on $\partial D(P, r)$. Then

$$\frac{1}{2\pi i}\frac{f'(\xi)}{f(\xi)} d\xi = \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k$$

where $n_1, n_2, \cdots, n_p$ are the multiplicities of the zeros $z_1, z_2, \cdots, z_p$ of f in $D(P, r)$ and $m_1, m_2, \cdots, m_p$ are the orders of the poles $w_1, w_2, \cdots, w_q$ of f in $D(P, r)$.

Proof Multiplying $(z - P_k)^{m_k}$ for the poles and dividing $(z - z_i)^{n_i}$ for the zeros.

Theorem 1.28

(The open mapping theorem) If $f: U \to \mathbb{C}$ is a nonconstant holomorphic function on a connected open set U, then $f(U)$ is an open set in $\mathbb{C}$.

Proof It suffices to show for any $Q \in f(U)$, there exists $\epsilon > 0$ such that $D(Q, \epsilon) \subset f(U)$. Assume that $f(P) = Q$ let let $g(z) = f(z) - Q$ and we know there exists an $r > 0$ such that g can not be zero on $\overline{D}(P, r) - \{P\}$ by considering the series expansion and we know

$$\frac{1}{2\pi i}\frac{f'(\xi)}{f(\xi) - Q} d\xi = n$$

where n is the order of P as a zero of g, so we know there exists $\epsilon > 0$ such that $|g(\xi)| > \epsilon$ on $\partial D(P, r)$ by the compactness and we claim that $D(Q, \epsilon)$ is in $f(U)$. Define

$$N(z) = \frac{1}{2\pi i}\int_{\partial D(P,r)} \frac{f'(\xi)}{f(\xi) - z} d\xi$$

for $z \in D(Q, \epsilon)$ and it is well-defined since

$$|f(\xi) - z| \geq |g(\xi)| - |z - Q| > \epsilon - |z - Q| > 0$$

and then it is easy to check N is continuous on $D(Q, \epsilon)$, but it is interger-valued and hence it has to be n on $D(Q, \epsilon)$, which means there exists zeros for $f(\xi) - z$ inside the $D(P, r)$ and hence $D(Q, \epsilon) \subset f(D(P, r)) \subset f(U)$.

Lemma 1.17

Let $f: U \to \mathbb{C}$ be a noncanstant holomorphic function on a connected open set $U \subset \mathbb{C}$. Then the multiple points of f in U are isolated.

Proof Since f is noncanstant, the holomorphic function f' is not identically zero, and then we know the zeros of f' is isolated by theorem.1.19. And any multiple point of f is a zero of f' and hence the lemma holds.

Theorem 1.29

Suppose that $f: U \to \mathbb{C}$ be a nonconstant holomorphic function on a connected open set U such that $P \in U$ and $f(P) = Q$ with order k. Then there are numbers $\delta, \epsilon > 0$ such that each $q \in D(Q, \epsilon) - \{Q\}$ has exactly k distinct *preimages in $D(P, \delta)$ and each preimage is a simple point of f.* ♡

Proof There exist δ_1 such that $D(P, \delta_1) - \{P\}$ is a simple point of f. THen let $\delta, \epsilon > 0$ such that $Q \in f(D(P, \delta) - \{P\})$ and $D(Q, \epsilon) \subset f(D(P, \delta))$ without meeting $f\partial(D(P, \delta))$ since $f(D(P, \delta))$ is open, then for any $q \in D(Q, \epsilon) - \{Q\}$, we know

$$\frac{1}{2\pi i} \int_{\partial D(P,\delta)} \frac{f'(\xi)}{f(\xi) - q} d\xi = k$$

since the integral is continuous as a function of q and the problem goes.

Theorem 1.30

(Rouche's theorem) Suppose that $f, g : U \to \mathbb{C}$ are holomorphic functions on an open set $U \subset \mathbb{C}$. Suppose also that $\overline{D}(P, r) \subset \mathbb{C}$ and that, for each $\xi \in \partial D(P, r)$,

$$f(\xi) - g(\xi) < |f(\xi)| + |g(\xi)|$$

Then

$$\frac{1}{2\pi i} \int_{\partial D(P,r)} \frac{f'(\xi)}{f(\xi)} d\xi = \frac{1}{2\pi i} \int_{\partial D(P,r)} \frac{g'(\xi)}{g(\xi)}$$

♡