Chapter 1

m.s. for measure space
mrb. for measurable

1.1 L^p spaces

Definition 1.1

For a fixed m.s. $(X, \mathcal{M}, \mu)$*, if* f *is a measurable function on* X *and* $0 < p < \infty$*, we define*

$$||f||_p = \left[\int |f|^p d\mu\right]^{1/p}$$

and

$$L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C}, f \text{ mrb and } ||f||_p < \infty\}$$

♣

Lemma 1.1

(Yooung's inequality) If $a, b \geq 0$ *and* $0 < \lambda < 1$*, then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$$

with equality iff $a = b$*.*

♡

Proof

If $b = 0$, the inequality goes. Then assume $b > 0$, and it suffices to show that

$$\frac{a}{b}^\lambda \leq \lambda\frac{a}{b} + (1-\lambda)$$

and consider the function $f(x) = x^\lambda - \lambda x - (1-\lambda)$, we have $f'(x) = \lambda x^{1-\lambda} - \lambda$ which is less than zero if $x > 1$ and greater than zero if $x < 1$, so we know $f(x) \leq f(1) = 0$ and the inequality holds.

Theorem 1.1

(Holder Inequality) Suppose $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$*. If* f *and* g *are measurable functions on* X*, then*

$$||fg||_1 \leq ||f||_p ||g||_1$$

In particular, if $f \in L^p, g \in L^q$*, then* $fg \in L^1$ *and in this case equality holds iff* $\alpha|f|^p = \beta|g|^q$ *a.e. for some constants* α, β*.*

♡

Proof

Consider we should show that

$$\int |fg| d\mu \leq \int |f|^p d\mu \int |g|^q d\mu$$

and if $||f||_p = 0$ or $||g||_q = 0$, then the LHS equals to 0. Now we consider let replace f, g with $f/||f||_p, g/||g||_q$ and it is suffices to show

$$\int |fg| d\mu \leq 1$$

and notice we have

$$\int |fg| d\mu \leq \int \frac{1}{p}|f|^p + \frac{1}{q}|g|^q d\mu = 1$$

and the equality holds iff $|fg| = p^{-1}|f|^p + q^{-1}|g|^q$ a.e. which means $|f|^p = |g|^q$ a.e. for the replaced f, g.

Theorem 1.2

(Minkowski's Inequality) If $1 \leq p < \infty$ *and* $f, g \in L^p$*, then*

$$||f + g||_p \leq ||f||_p + ||g||_p$$

♡

Proof

Consider

$$\int |f+g|^p d\mu \leq \int |f+g|^{p-1}(|f|+|g|) \leq |||f+g|^{p-1}||_q(||f||_p+||g||_p) = ||f+g||_p^{(p-1)/p}$$

and the inequality holds.

Theorem 1.3

For $1 \leq p < \infty$, L^p is a Banach space. ♡

Proof

It suffices to show that L^p is complete, which can be induced from any absolutely convergence series $S = \sum f_i$ converges. Let $S_n = \sum_{i=1}^{n} f_i$ and it is easy to check that S_n is Cauchy in L^p, then let $G = \sum |f_i|$ and we have $|G|_p = \lim |G_n|_p < \infty$ by the MCT where $G_n = \sum_{i=1}^{n} |f_i|$ and hence $G \in L^p$ which means S converges a.e. and consider

$$\lim ||S - S_n||_p = || \lim S - S_n||_p = 0$$

by the DCT.

Proposition 1.1

For $1 \leq p < \infty$, the set of simple functions $f = \sum_{1}^{n} a_j \chi_{E_j}$, where $\mu(E_j) < \infty$ for all j is dense in L^p. ♠

Proof

For $f \in L^p$, we may find $|f_j| \uparrow |f|$ and f_j converges to f pointwise, then we assume $f_j = \sum_1^n a_j \chi_{E_j}$ and then we have

$$\sum_{1}^{n} a_j^p \mu(E_j) = \int |f_j|^p d\mu \leq \int |f|^p d\mu < \infty$$

and hence f_j is just in the required set, and by the DCT we know $||f - f_j||_p \to 0$.

Definition 1.2

$$||f||_\infty = \int \{a \geq 0 : \mu(\{x : |f(x)| > \alpha\}) = 0\}$$

with the convention that $\inf \emptyset = \infty$ *and then it is called the essential supremum of* $|f|$. *And define*

$$L^\infty = \{f : X \to \mathbb{C}, f \text{ mrb and } ||f||_\infty < \infty\}$$

♣

Theorem 1.4

a. If f and g are measurable functions on X, then $||fg||_1 \leq ||f||_1||g||_\infty$, if $f \in L^1$ and $g \in L^\infty$, $||fg||_1 = ||f||_1||g||_\infty$ iff $|g(x)| = ||g||_\infty$ a.e. on the set where $f(x) \neq 0$.
b. $|| \cdot ||_\infty$ is a norm on L^∞.
c. $||f_n - f||_\infty \to 0$ iff $f_n \to f$ uniformly a.e.
d. L^∞ is a Banach space.
e. The simple functions are dense in L^∞. ♡

Proof a. Let $E = \{|g| \leq |g|_\infty\}$ and then E is conull, so

$$\int |fg| d\mu = \int_E |fg| d\mu \leq ||g||_\infty \int_E |f| d\mu = \int |f| d\mu ||g||_\infty$$

where the equality can be reached when $g(x) = ||g||_\infty$ a.e. on E.

b. It suffices to show the triangle inequality where notice $|f| \leq ||f||_\infty, g \leq ||g||_\infty$ a.e. and hence $|f+g| \leq ||f||_\infty + ||g||_\infty$ a.e.

c. Let $E_n = \{|f_n - f| \leq ||f_n - f||_\infty\}$ and then let $E = \bigcap E_n$ conull and hence $f_n \to f$ on E uniformly.

d. If suffices to show that an absolutely convergent series $\sum f_i$ converges in L^∞ where we may know $f_i \leq ||f_i||_\infty$ a.e. on X for any integer i and hence the we will know $\sum |f_i| \leq \sum ||f_i||_\infty$ a.e. and hence $\sum f_i$ converges a.e. and we have $|\sum f_i - \sum_1^n f_i| \leq \sum_{n+1}^\infty ||f_i||_\infty \to 0$ a.e.

e. Let $f_j \to f$ be the simple functions converges to f uniformly where f is bounded and hence $f_j \to f$ uniformly a.e. and hence $||f_j - f||_\infty \to 0$.

Proposition 1.2

If $0 < p < q < r \leq \infty$, then $L^q \subset L^p + L^r$; that is, each $f \in L^q$ is the sum of a function in L^p and a function in L^r.

Proof

Considering $|f| > 1$ and $|f| \leq 1$ separately will be fine.

Proposition 1.3

If $0 < p < q < r \leq \infty$, then $L^p \cap L^r \subset L^q$ and $||f||_q \leq ||f||_p^\lambda ||f||_r^{1-\lambda}$ where $q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1}$.

Proof

Here we know

$$\int |f|^q d\mu = \int |f|^{\lambda q} |f|^{(1-\lambda)q} d\mu \leq |||f|^{\lambda q}||_{p/\lambda q} |||f|^{(1-\lambda)q}||_{r/(1-\lambda)q} = ||f||_p^{\lambda q} ||f||_r^{(1-\lambda)q}$$

by the Holder's inequality and the inequality holds.

Proposition 1.4

If A is any set and $0 < p < q \leq \infty$, then $l^p(A) \subset l^q(A)$ and $||f||_q \leq ||f||_p$.

Proof If $q = \infty$, then $||f||_\infty = \sup |f(\alpha)| \leq ||f||_p$. If $q < \infty$, then consider

$$||f||_q \leq ||f||_p^\lambda ||f||_\infty^{1-\lambda} \leq ||f||_p$$

Proposition 1.5

If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $||f||_p \leq ||f||_q \mu(X)^{(p^{-1}-q^{-1})}$.

Proof

Consider if $q = \infty$, then

$$\int |f|^p d\mu \leq \int |f|_\infty^p d\mu = ||f||_\infty^p \mu(X)$$

and if $q < \infty$, then

$$\int |f|^p d\mu = \int (|f|^q)^{p/q} (1)^{(q-p)/q} \leq |f^p|_{q/p} |1|_{q/(q-p)} = ||f||_q^p \mu(X)^{(1-p/q)}$$

by the Holder's inequality.

Proposition 1.6

Suppose that p and q are conjugate exponents and $1 \leq q < \infty$. If $g \in L^q$, then

$$||g||_q = ||\phi_g|| = \sup\{|\int fg|, ||f||_p = 1\}$$

If μ is semifinite, this result holds also for $q = \infty$, where define

$$\phi_g(f) = \int fg$$

Proof

It suffices to show that $||\phi_g|| \geq ||g||_q$. Let

$$f = \frac{|g|^{q-1}\overline{sgn(g)}}{||g||_q^{q-1}}$$

and we have

$$||f||_p = \frac{\int |g|^{(q-1)p}}{||g||_p^{q-1}} = 1$$

and $|\phi_g(f)| = \int fg = \dfrac{\int |g|^q}{||g||_q^{q-1}} = ||g||_q$.

If $q = \infty$, we know there exists $B \subset \{|g| > ||g||_\infty - \epsilon\}$ for any $\epsilon > 0$ such that $\mu(B) < \infty$, then let

$$f = \mu(B)^{-1}\chi_B\overline{sgn(g)}$$

and we have $||f||_1 = 1$ and

$$|\phi_g(f)| = \mu(B)^{-1}\int_B |g| \geq ||g||_\infty - \epsilon$$

and hence $||\phi_g|| = ||g||_\infty$.

Theorem 1.5

Let p and q be conjugate exponents. Suppose that g is a measurable function on X such that $fg \in L^1$ for all f in Σ which is the space of all simple functions with a finite measure support, and the quantity

$$M_q(g) = \sup\{|\int fg|, f \in \Sigma \text{ and } ||f||_p = 1\}$$

is finite. Also, suppose either that $S_g = \{x, g(x) \neq 0\}$ is σ-finite or that μ is semifinite. Then $g \in L^q$ and $M_q(g) = ||g||_q$.

Proof

Notice for any f bounded with a finite measure support and $||f||_p = 1$, we know $|f| \leq ||f||_\infty\chi_E$ where E is a finite support of f and consider f_n is simple function converge to f with $|f_n| \leq |f|$ and then we know

$$|\int fg| = \lim |\int f_n g| \leq M_q(g)$$

by the DCT.

Suppose $q < \infty$ and S_g is $\sigma - finite$, then we may find E_n increasing to S_g with $\mu(E_n) < \infty$, we may find $\phi_n \to g$ and let $g_n = \phi_n\chi_{E_n}$. Then $g_n \to g$ pointwise and let

$$f_n = \frac{g_n^{q-1}\overline{sgn(g)}}{||g_n||_q^{q-1}}$$

then we have

$$||f_n||_p = \frac{\int |g_n|^q}{||g_n||_q^q} = 1$$

and

$$|\int f_n g| = \int \frac{|g_n|^{q-1}|g|}{||g_n||_q^{q-1}} \geq ||g_n||_q$$

which means $M_q(g) \geq ||g_n||_q$ for any integer n and hence $M_q(g) \geq ||g||_q$ by the MCT, which means $g \in L^q$.

If μ is semifinite, then let $E = \{|g| > \epsilon\}$ and then we know there is $A \subset E$ with $\mu(A) < \infty$ if $\mu(E) > 0$, and we have

$$M_q(g) \geq |\int \mu(A)^{-p^{-1}}\chi_A\overline{sgn(g)}g| \geq \epsilon\mu(A)^{1-p^{-1}}$$

where $\mu(A)$ can be arbitrarily large if $\mu(E) = \infty$ and which is a contradiction. Therefore, μ is semifinite will imply that S_g is σ-finite.

If $q = \infty$, then let $A = \{|g| \geq M_\infty(g) + \epsilon\}$, if $\mu(A)$ is positive, then we let $f = \mu(A)^{-1}\chi_A sgn(g)$ and we know

$$|\int fg| \geq M_\infty(g) + \epsilon$$

which is a contradiction and hence $||g||_\infty \le M_\infty(g)$.

Theorem 1.6

Let p and g be conjugate exponents. If $1 < p < \infty$, for each $\phi \in (L^p)^*$ there exists $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$ and hence L^q is isometrically isomorphic to $(L^p)^*$. The same conclusion holds for $p = 1$ if μ is σ-finite.

♡

Proof

Firstly assume μ is finite, the all simple functions are in L^p, and then consider for disjoint sets E_j and $E = \bigcup_j E_j$, we have

$$||\chi_E - \sum_{i=1}^{n} \chi_{E_j}||_p = \mu(\bigcup_{n+1}^{\infty}) \to 0$$

then let $\nu(E) = \phi(\chi_E)$ and

$$\nu(E) = \phi(E) = \lim \phi(\sum_{i=1}^{n} \chi_{E_i}) = \lim \sum_{i=1}^{n} \nu(E_j)$$

and hence ν is a complex measure. Also if $\mu(E) = 0$, then $\nu(E) = \phi(\chi_E) = 0$, so there is an g measurable such that $\phi(\chi_E) = \nu(E) = \int_E g$ and notice

$$|\int fg| \le ||\phi||||f||_p$$

for any simple function in L^p and hence $g \in L^q$ by theorem 1.5 and then we know $fg \in L^1$ for any $f \in L^p$ and hence $\phi(f) = \int fg$ for any $f \in L^p$.

If μ is σ-finite, let E_n increasing X, $\mu(E_n) > 0$ and then we know there is $g_n \in L^q(E_n)$ on E_n such that $\phi(f) = \int fg_n$ for any $f \in L^p(E_n)$ and $g_n = g_m$ on E_n a.e., then we define $g = g_n$ on E_n and we know $||g||_q = \lim ||g_n||_q \le ||\phi||$ by the MCT, now we know

$$\int fg = \lim \int f\chi_{E_n} g = \lim \int fg_n = \lim \phi(f\chi_{E_n}) = \phi(f)$$

For general μ, for a σ-finite subset E, there is $g_E \in L^q(E)$ and $\phi(f) = \int fg_E$ for any $f \in L^p(E)$ and $||g_E||_q \le ||\phi||$, so we may find E_n such that $||g_{E_n}||_q \to \sup ||g_E||_q$ and let $F = \bigcup_E$ which is σ-finite, then we know $||g_F||_q \ge ||g_{E_n}||_q$ for any integer n and hence $||g_F||_q = M$. Then for any A σ-finite, we will know

$$\int |g_F|^q + \int |g_{A-F}|^q = \int |g_{A\cup F}|^q \le M = \int |g_F|^q$$

and hence $g_{A-F} = 0$ a.e. and hence $g_{A\cup F} = g_F$ a.e. for all A σ-finite subset. If $g \in L^p$, we know S_f is σ-finite and hence $\phi(f) = \int fg_{S_g \cup F} = \int fg_F$ for any $f \in L^p$.

Corollary 1.1

If $1 < p < \infty$, L^p is reflexive.

♡

Theorem 1.7

(Chebyshev's Inequality) If $f \in L^p (0 < p < \infty)$, then for any $\alpha > 0$,

$$\mu(\{x : |f| > \alpha\}) \le \left[\frac{||f||_p}{\alpha}\right]^p$$

♡

Theorem 1.8

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let K be an $(\mathcal{M}\otimes\mathcal{N})$-measurable function on $X \times Y$. Suppose that there exists $C > 0$ such that $\int |K(x, y)d\mu(x)| \le C$ for a.e. $y \in Y$ and $\int |K(x, y)d\nu(y)| \le C$ for a.e. $x \in X$ and that $1 \le p \le \infty$. If $f \in L^p(\nu)$, then the integral

$$Tf(x) = \int K(x, y)f(y)d\nu(y)$$

converges absolutely for a.e. $x \in X$, the function Tf thus defines is in $L^p(\mu)$ and $||Tf||_p \leq C||f||_p$. ♡

Proof Consider

$$\int |K(x,y)f(y)|d\nu(y) \leq ||K(x,\cdot)^{q^{-1}}||_q||K(x,y)^{p^{-1}}|f(y)|||_p \leq C^{q^{-1}}\left[\int |K(x,y)||f(y)|^p d\nu(y)\right]^{p^{-1}}$$

for a.e. $x \in X$, then we know

$$\begin{aligned}\int |Tf(x)|^p d\mu(x) &= \int |\int K(x,y)f(y)d\nu(y)|^p d\mu(x)\\ &\leq \int C^{p/q}\int |K(x,y)||f(y)|^p d\nu(y)d\mu(x)\\ &= C^{p/q}\int\int |K(x,y)|d\mu(x)|f(y)|^p d\nu(y)\\ &\leq C^{p/q+1}||f||_p^p < \infty\end{aligned}$$

and hence $Tf \in L^p(\mu)$ and $||Tf||_p \leq C||f||_p$.

Theorem 1.9

Suppose that $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ are σ-finite measure spaces, and let f be an $(\mathcal{M}\otimes\mathcal{N})$-measurable function on $X \times Y$.
a. If $f \geq 0$ and $1 \leq p < \infty$, then

$$\left[\int\left(\int f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int\left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y)$$

b. If $1 \leq p \leq \infty$, $f(\cdot,y) \in L^p(\mu)$ for a.e. y, and the function $y \mapsto ||f(\cdot,y)||_p$ is in $L^1(\nu)$, then $f(x,\cdot) \in L^1(\nu)$ for a.e. x, the function $x \mapsto \int f(x,y)d\nu(y)$ is in $L^p(\mu)$ and

$$||\int f(\cdot,y)d\nu(y)||_p \leq \int ||f(\cdot,y)||_p d\nu(y)$$

♡

Proof

a. Let $g \in L^q(\mu)$ and we have

$$\int\int f(x,y)d\nu(y)|g(x)|d\mu(x) \leq ||g||_q\int\left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y)$$

and hence $||\int f(x,y)d\nu(y)||_p \leq \int\left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y)$ by theorem 1.5.

b. This conclusion is obvious and by (a) if $p < \infty$ and it goes when $q = \infty$.

Theorem 1.10

Let K be a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that $K(\lambda x,\lambda y) = \lambda^{-1}K(x,y)$ for all $\lambda > 0$ and $\int_0^\infty |K(x,1)|x^{-1/p}dx \leq C < \infty$ for some $p \in [1,\infty]$, and let q be the conjuate exponent to p. For $f \in L^p$ and $g \in L^q$, let

$$Tf(y) = \int_0^\infty K(x,y)f(x)dx, \quad Sg(x) = \int_0^\infty K(x,y)g(y)dy$$

Then Tf and Sg are defined a.e. and $||Tf||_p \leq C||f||_p$ and $||Sg||_q \leq C||g||_q$. ♡

Proof Consider

$$\begin{aligned}\left(\int |Tf(y)|^p dy\right)^{1/p} = \left(\int |\int K(x,y)f(x)dx|^p dy\right)^{1/p} &\leq \left(\int\left(\int |K(x,y)f(x)|dx\right)^p dy\right)^{1/p}\\ &= \left(\int\left(\int |K(z,1)f(yz)|dz\right)^p dy\right)^{1/p}\\ &\leq \int ||f(\cdot z)||_p|K(z,1)|dz\\ &\leq C||f||_p\end{aligned}$$

by the Minkowski's inequaltiy for integral and $||f(yz)||_p = z^{-1/p}||f||_p$ and the other conclusion is the same since

$$\begin{aligned}\int_0^\infty |K(1,y)|y^{-1/q}dy &= \int_0^\infty |K(y^{-1},1)|y^{1-1/q}dy \\ &= -\int_0^\infty |K(u,1)|u^{1/q+1}(-u^{-2})du = \int_0^\infty |K(u,1)|u^{-1/p}du \le C\end{aligned}$$

Corollary 1.2

Let

$$Tf(y) = y^{-1}\int_0^y f(x)dx, \quad Sg(x) = \int_x^\infty y^{-1}g(y)dy$$

Then for $1 < p \le \infty$ and $1 \le q < \infty$,

$$||Tf||_p \le \frac{p}{p-1}||f||_p, \quad ||Sg||_q \le q||g||_q$$

♡

Proof

Let $K(x,y) = y^{-1}\chi_{(x<y)}$ and we know

$$\int |K(x,y)|x^{-1/p}dx = y^{-1}qx^{1/q}|_0^y = q = \frac{p}{p-1}$$

Definition 1.3

If f is a measurable function on $(X, \mathcal{M}, \mu)$, its $distribution\ function$ $\lambda_f : (0,\infty) \to [0,\infty]$ by

$$\lambda_f(\alpha) = \mu(|f| > \alpha)$$

♣

Proposition 1.7

a. λ_f is decreasing and right continuous.
b. If $|f| \le |g|$, then $\lambda_f \le \lambda_g$.
c. If $|f_n|$ increases to $|f|$, then λ_{f_n} increases to λ_f.
d. If $f = g + h$, then $\lambda_f(\alpha) \le \lambda_g(\frac{1}{2}\alpha) + \lambda_h(\frac{1}{2}\alpha)$.

♠

Proof

a. Trivial.
b. $\lambda_g(\alpha) = \mu(|g| > \alpha) \ge \mu(|f| > \alpha) = \lambda_f(\alpha)$.
c. $\{|f| > \alpha\} = \bigcup\{|f_n| > \alpha\}$.
d. $\{|f+g| > \alpha\} \subset \{|f| > \frac{1}{2}\alpha\}$ and $\{|g| > \frac{1}{2}\alpha\}$.

Proposition 1.8

If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and ϕ is a nonnegative Borel measurable function on $(0,\infty)$, then

$$\int_X \phi \circ |f| d\mu = -\int_0^\infty d\lambda_f(\alpha)$$

where $d\lambda_f = d\nu$, which is the negative Borel measure defined by λ_f.

♠

Proposition 1.9

Consider for a h-interval $(a,b]$, we have

$$\int_X \chi_{(a,b]}(|f|)d\mu = \mu(b \le |f| > a) = -\nu((a,b]) = -\int_0^\infty \chi_{(a,b]}d\lambda_f$$

and hence the equality holds for all Borel set E. The rest can be obtained by the MCT.

♠

Proposition 1.10

If $0 < p < \infty$, *then*

$$\int |f|^p d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha$$

Proof

If $\lambda_f(\alpha) = \infty$ for some α, then we know the both sides are infinity. Then we assume $\lambda_f < \infty$ and if f is simple, then λ_f should be bounded and vanish when $\alpha \to \infty$ and the integration by parts will show it immediately.

For general case, let $\{g_n\}$ be simple functions increase to $|f|^p$ and the MCT will guarantee the equality.

Definition 1.4

If f is a measurable function on X and $0 < p < \infty$, *we define*

$$[f]_p = (\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha))^{1/p}$$

and the weak L^p space is all f such that $[f]_p < \infty$.

We have

$$L^p \subset \textit{weak } L^p, \quad [f]_p \leq ||f||_p$$

Proposition 1.11

If f is a measurable function and $A > 0$, *let* $E(A) = \{x, |f| > A\}$ *and set*

$$h_A = f\chi_{X-E(A)} + A(sgn(f))\chi_{E(A)} \quad g_A = f - h_A = (sgn(f))(|f| - A)\chi_{E(A)}$$

then

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A), \quad \lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \textit{if } \alpha < A \\ 0 & \textit{if } \alpha \geq A \end{cases}$$

Proof

Here we have

$$\lambda_{g_A}(\alpha) = \mu(\{|g_A| > \alpha\}) \leq \mu(\{|f| > \alpha + A\})$$

and by the way

$$\lambda_f(\alpha + A) = \mu(\{|f| - A > \alpha\}) \leq \mu(\{|g_A| > \alpha\})$$

Then we know

$$\lambda_{h_A}(\alpha) = \mu(\{|f||\chi_{X-E(A)}| > \alpha\}) + \mu(\{A|\chi_{E(A)}| > \alpha)\}) = \chi_{\alpha<A}(\lambda_f(\alpha) - \lambda_f(A) + \lambda_f(A)) = \chi_{\alpha<A}\lambda_f(\alpha)$$

Lemma 1.2

Let ϕ be a counded continuous function on the strip $0 \leq Rez \leq 1$ *that is holomorphic on the interior of the strip. If* $|\phi(z)| \leq M_0$ *for* $Rez = 0$ *and* $|\phi(z)| \leq M_1$ *for* $Rez = 1$, *then* $|\phi(z)| \leq M_0^{1-t} M_1^t$ *for* $Rez = t, 0 < t < 1$.

Proof

Let $\phi_n(z) = \phi(z) M_0^{z-1} M_1^{-z} e^{n^{-1}z(z-1)}$ and we know $|\phi_n(0)|, |\phi_n(1)| \leq 1$ when $Rez = 0, 1$ and notice $|\phi_n| \to 0$ when $|Imz| \to \infty$ since let $z = x + iy$ and

$$|\phi_n(z)| = |\phi(z)||M_0^{x-1}||M_1^{-x}|e^{n^{-1}(x(x-1)-y^2)}| \to 0, y \to \infty$$

and then we know $\phi_n(z) \leq 1$ on the strip by the maximal modulus principle, then we have

$$|\phi(z)| M_0^{t-1} M_1^{-t} = \lim_{n \to \infty} |\phi_n(z)| \leq 1$$

Theorem 1.11

(The Riesz-Thorin Interpolation Theorem)

Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are mesure spaces and $p_0, p_1, q_0, q_1 \in [1, \infty]$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. For $0 < t < 1$, define

$$p_t^{-1} = (1-t)p_0^{-1} + tp_1^{-1}, \quad q_t^{-1} = (1-t)q_0^{-1} + tq_1^{-1}$$

If T is a linear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ into $L^{q_0}(\nu) + L^{q_1}(\nu)$ such that $||Tf||_{q_0} \le M_0||f||_{p_0}$ for $f \in L^{p_0}(\mu)$ and $||Tf||_{q_1} \le M_1||f||_{p_1}$ for $f \in L^{p_1}(\mu)$, then $||Tf||_{q_t} \le M_0^{1-t}M_1^t||f||_{p_t}$ for $f \in L^{p_t}(\mu), 0 < t < 1$. ♡

Proof

We know

$$||Tf||_{q_t} = \sup\{|\int (Tf)g|, g \in \Sigma_X, ||g||_{\tilde{q_t}} = 1\}$$

where $\tilde{q_t}$ is the conjugate of q_t and then we only need to show that

$$|\int (Tf)g| \le M_0^{1-t}M_1^t$$

for any $g \in \Sigma_X$ and $||f||_{p_t} = 1$. We assume $f = \sum a_j \chi_{E_j}$ and $g = \sum b_k \chi_{F_k}$. Define

$$\alpha(z) = (1-t)p_0^{-1} + tp_1^{-1}, \quad \beta(z)(1-t)q_0^{-1} + tq_1^{-1}$$

and let

$$f_z = \sum |a_j|^{\alpha(z)/\alpha(t)} e^{i\theta_j} \chi_{E_j}$$
$$g_z = \sum |b_k|^{(1-\beta(z))/(1-\beta(t))} e^{i\varphi_k} \chi_{F_k}$$

where $\theta_j = Arg(a_j), \varphi_k = Arg(b_k)$ and

$$\phi(z) = \int (Tf_z)g_z$$

here we assume $\alpha(t) =\neq 0, \beta(t) \neq 1$ and hence $(p_0, p_1) \neq (\infty, \infty), (q_0, q_1) \neq (1, 1)$. Then we know

$$\phi(z) = \sum |a_j|^{\alpha(z)/\alpha(t)} |b_k|^{(1-\beta(z))/(1-\beta(t))} e^{i(\varphi_k + \theta_j)} \int (T\chi_{E_j})\chi_{F_k}$$

which is an entire function and we have

$$\begin{aligned} |\phi(ir)| &\le ||Tf_{ir}||_{q_0}||g_{ir}||_{\tilde{q_0}} \le M_0||f_{ir}||_{p_0}||g_{ir}||_{\tilde{q_0}} \\ &= M_0|\int |f|^{p_0 Re\alpha(ir)/\alpha(t)}|^{1/p_0}|\int |g|^{\tilde{q_0} Re(1-\beta(ir))/(1-\beta(t))}|^{1/\tilde{q_0}} \\ &= M_0 \end{aligned}$$

and

$$\begin{aligned} |\phi(1+ir)| &\le ||Tf_{1+ir}||_{q_1}||g_{ir}||_{\tilde{q_1}} \le M_1||f_{1+ir}||_{p_1}||g_{ir}||_{\tilde{q_0}} \\ &= M_1|\int |f|^{p_1 Re\alpha(1+ir)/\alpha(t)}|^{1/p_1}|\int |g|^{\tilde{q_1} Re(1-\beta(1+ir))/(1-\beta(t))}|^{1/\tilde{q_1}} \\ &= M_1 \end{aligned}$$

Therefore, we will know $|\int (Tf)g| = |\phi(t)| \le M_0^{1-t}M_1^t$ by the lemma 1.2. When $p_0 = p_1 = \infty$, the inequality is trivial and when $q_0 = q_1 = 1$, let $g_z = g$ and the proof is fine.

Now we only need to prove that $Tf = \lim Tf_n$ for any $f \in L^{p_t}$ where $f_n \in \Sigma_X$ and $f_n \to f$ pointwise with $|f_n| \le |f|$. Consider $g = f\chi_{|f|<1}$ and $h = f\chi_{|f|>1}$, then we know $g \in L^{p_0}$ and $h \in L^{p_1}$, then we know $||Tg_n - Tg||_{q_0} \le M_0||g_n - g||_{p_0} \to 0$ and $||Th_n - Th||_{q_1} \le M_1||h_n - h||_{p_1} \to 0$ by the DCT and hence there exists subsequence n_k such that $Tg_{n_k} \to Tg, Th_{n_k} \to Th$ pointwise and hence $Tf_{n_k} \to Tf$ pointwise, and

$$||Tf||_{q_t} \le \liminf ||Tf_n||_{n_k} \le \liminf M_0^{1-t}M_1^t||f_{n_k}||_{p_t} = M_0^{1-t}M_1^t||f||_{p_t}$$

and the problem goes.

Definition 1.5

For $T : X \to Y$ where X, Y are normed vector spaces and T is called sublinear if

$$|T(f+g)| \le |Tf| + |Tg| \quad |T(cf)|c|Tf|$$

for any $f, g \in X, c > 0$.
Then we call a sublinear map T is strong type (p,q) if $L^p(\mu) \subset X$ and T maps $L^p(\mu)$ into $L^q(\nu)$, then there exists $C > 0$ such that $||Tf||_q \le C||f||_p$ for all $f \in L^p(\mu)$ for any $1 \le p, q \le \infty$.
T is weak type (p,q) if $L^p(\mu) \subset X$ and T maps $L^p(\mu)$ into weak $L^q(\nu)$ and there exists $C > 0$ such that $[Tf]_q \le C||f||_p$ for all $f \in L^p(\mu)$ for any $1 \le p \le \infty$ and $1 \le q < \infty$. ♣

Theorem 1.12

(The Marcinkiewicz Interpolation Theorem)
Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are mesure spaces and $p_0, p_1, q_0, q_1 \in [1, \infty]$ such that $p_0 \le q_0, p_1 \le q_1$ and $q_0 \ne q_1$ and

$$p^{-1} = (1-t)p_0^{-1} + tp_1^{-1}, \quad q^{-1} = (1-t)q_0^{-1} + tq_1^{-1}$$

where $0 < t < 1$. If T is a sublinear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ to the space of measurable functions on Y that is weak types (p_0, q_0) and (p_1, q_1), then T is strong type (p,q). More precisely, if $[Tf]_{q_j} \le C_j||f||_{p_j}$ for $j = 0, 1$, then $||Tf||_q \le B_p||f||_p$ where B_p depends only on p_j, q_j, C_j in addition to p; and for $j = 0, 1$, $B_p|p - p_j|$ remains bounded as $p \to p_j$ if $p_j < \infty$. ♡

Proof

Assume $p_0 = p_1, q_0 < q_1$, then we know $q < \infty$ and

$$C_0||f||_{p_0} \ge [Tf]_{q_0}, \quad C_1||f||_{p_0} \ge [Tf]_{q_1}$$

and we know if $q_1 < \infty$ then for any f with $||f||_{p_0} = ||f||_{p_1} = 1$

$$\begin{aligned} \int |Tf|^q = q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha &\le q\Big[\int_0^1 \alpha^{q-1}\Big(\frac{C_0||f||_{p_0}}{\alpha}\Big)^{q_0} + \int_1^\infty \alpha^{q-1}\Big(\frac{C_1||f||_{p_1}}{\alpha}\Big)^{q_1}\Big]d\alpha \\ &= qC_0^{q_0}\int_0^1 \alpha^{q-q_0-1}d\alpha + qC_1^{q_1}\int_1^\infty \alpha^{q-q_1-1}d\alpha \\ &= \frac{q}{q-q_0}C_0^{q_0} + \frac{q}{q_1-q}C_1^{q_1} = B_p^q \end{aligned}$$

If $q_1 = \infty$, then assume $||f||_{p_0} = 1$, we have

$$\int |Tf|^q = q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha \le q\int_0^{C_1||f||_{p_0}} \alpha^{q-1}\Big(\frac{C_0||f||_{p_0}}{\alpha}\Big)^{q_0}d\alpha = \frac{q}{q-q_0}C_0^{q_0}C_1^{q-q_0}$$

and hence

$$||Tf||_q = ||||f||_{p_0}T(f/||f_{p_0}||)||_q \le B_p||f||_{p_0}$$

where

$$B_p = \Big(\Big(\frac{q}{q-q_0}C_0^{q_0}C_1^{q-q_0}\Big)^{1/q}\chi_{q_1=\infty} + \Big(\frac{q}{q-q_0}C_0^{q_0} + \frac{q}{q_1-q}C_1^{q_1}\Big)^{1/q}\chi_{q_1<\infty}\Big)$$

when $p_0 = p_1, q_0 < q_1$ and we know B_p is a constant respect to p and obviously we have $B_p|p - p_j|$ is bounded when $p \to p_j$. Then we assume $p_0 < p_1$, then we have for any $f \in L^p(\mu)$

$$\int |g_A|^{p_0} = p_0\int_0^\infty \alpha^{p_0-1}\lambda_{g_A}(\alpha)d\alpha \le p_0\int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha)d\alpha$$

$$\int |h_A|^{p_1} = p_1\int_0^\infty \alpha^{p_1-1}\lambda_{h_A}(\alpha)d\alpha \le p_1\int_0^A \alpha^{p_1-1}\lambda_f(\alpha)d\alpha$$

Let $A = A(\alpha)$ and

$$\int |Tf|^q = q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha \le 2^q q\int_0^\infty \alpha^{q-1}(\lambda_{g_A}(\alpha) + \lambda_{h_A}(\alpha))d\alpha$$

and notice

$$\lambda_{g_A}(\alpha) \leq \Big(\frac{C_0||g_A||_{p_0}}{\alpha}\Big)^{q_0}, \quad \lambda_{h_A}(\alpha) \leq \Big(\frac{C_1||h_A||_{p_1}}{\alpha}\Big)^{q_1}$$

where we may see $g_A \in L^{p_0}, h_A \in L^{p_1}$ by consider $f' = f/A$, then $g_1' = g_A/A, h_1' = h_A/A$ and we have

$$\int |h_1'|^{p_1} \leq \int |f'|^p, \quad \int |g_1'|^{p_0} \leq \int (|g_1'|+1)^{p_0} \leq \int |f'|^p$$

and hence $h_1' \in L^{p_1}, g_1' \in L^{p_0}$, which means the inequalities above holds for f and then we have

$$\begin{aligned}\int |Tf|^q &\leq 2^q q \int_0^\infty \alpha^{q-1}\Big[\Big(\frac{C_0||g_A||_{p_0}}{\alpha}\Big)^{q_0} + \Big(\frac{C_1||h_A||_{p_1}}{\alpha}\Big)^{q_1}\Big]d\alpha \\ &= 2^q q\Big[C_0^{q_0} p_0^{q_0/p_0}\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0} d\alpha \\ &\quad + C_1^{q_1} p_1^{q_1/p_1}\int_0^\infty \alpha^{q-q_1-1}\Big(\int_0^{A(\alpha)} \beta^{p_1-1}\lambda_f(\beta)d\beta\Big)^{q_1/p_1} d\alpha\Big]\end{aligned}$$

where we have

$$\begin{aligned}\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0} d\alpha &\leq \Big[\int_0^\infty \Big(\int_{A(\alpha)\leq\beta} [\alpha^{p_0(q-q_0-1)/q_0}\beta^{p_0-1}\lambda_f(\beta)]^{q_0/p_0} d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0} \\ &= \Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_{A(\alpha)\leq\beta} \alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0}\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \alpha^{q-q_1-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_1-1}\lambda_f(\beta)d\beta\Big)^{q_1/p_1} d\alpha &\leq \Big[\int_0^\infty \Big(\int_{A(\alpha)>\beta} [\alpha^{p_1(q-q_1-1)/q_1}\beta^{p_1-1}\lambda_f(\beta)]^{q_0/p_0} d\alpha\Big)^{p_1/q_1} d\beta\Big]^{q_1/p_1} \\ &= \Big[\int_0^\infty \beta^{p_1-1}\lambda_f(\beta)\Big(\int_{A(\alpha)>\beta} \alpha^{q-q_1-1}d\alpha\Big)^{p_1/q_1} d\beta\Big]^{q_1/p_1}\end{aligned}$$

then we may consider if $q_0 < q_1$ then let $A(\alpha) = \alpha^r$ and we have

$$\begin{aligned}\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0} d\alpha &\leq \Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_0^{\beta^{1/r}} \alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0} \\ &= \frac{1}{q-q_0}\Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\beta^{p_0(q-q_0)/rq_0}\beta\Big]^{q_0/p_0}\end{aligned}$$

and let

$$r = \frac{p_0}{q_0}\frac{q-q_0}{p-p_0} = \frac{q_0^{-1}-q^{-1}}{q^{-1}}\frac{p^{-1}}{p_0^{-1}-p^{-1}} = \frac{q_0^{-1}-q_1^{-1}}{p_0^{-1}-p_1^{-1}}\frac{p^{-1}}{q^{-1}} = \frac{q_1^{-1}-q^{-1}}{p_1^{-1}-p^{-1}}\frac{p^{-1}}{q^{-1}} = \frac{p_1}{q_1}\frac{q-q_1}{p-p_1}$$

and we know if $||f||_p = 1$ then

$$\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0} d\alpha \leq \frac{1}{q-q_0}\Big(\frac{||f||_p^p}{p}\Big)^{q_0/p_0} = |q-q_0|^{-1}p^{-q_0/p_0}$$

and similarly

$$\begin{aligned}\int_0^\infty \alpha^{q-q_1-1}\Big(\int_0^{A(\alpha)} \beta^{p_1-1}\lambda_f(\beta)d\beta\Big)^{q_1/p_1} d\alpha &\leq \Big[\int_0^\infty \beta^{p_1-1}\lambda_f(\beta)\Big(\int_{\beta^{1/r}}^\infty \alpha^{q-q_1-1}d\alpha\Big)^{p_1/q_1} d\beta\Big]^{q_1/p_1} \\ &= \frac{1}{q_1-q}\Big[\int_0^\infty \beta^{p_1-1}\lambda_f(\beta)\beta^{p_1(q-q_1)/rq_1}\beta\Big]^{q_1/p_1}\end{aligned}$$

and then

$$\int_0^\infty \alpha^{q-q_1-1}\Big(\int_0^{A(\alpha)} \beta^{p_1-1}\lambda_f(\beta)d\beta\Big)^{q_1/p_1} d\alpha \leq \frac{1}{q_1-q}\Big(\frac{||f||_p^p}{p}\Big)^{q_1/p_1} = |q-q_1|^{-1}p^{-q_1/p_1}$$

Therefore, we have

$$\int |Tf|^q \leq 2^q q\Big[C_0^{q_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1} + C_1^{q_1}(p_1/p)^{q_1/p_1}|q-q_1|^{-1}\Big]$$

when $q_0 < q_1$ and if $q_0 > q_1$, let $A(\alpha) = \alpha^r$ and notice $r < 0$ so we have

$$\begin{aligned}\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0} d\alpha &\le \Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_{\beta^{1/r}}^\infty \alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0}\\ &= \frac{1}{q_0-q}\Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\beta^{p_0(q-q_0)/rq_0}\beta\Big]^{q_0/p_0}\end{aligned}$$

and the rest calculation are similar, we can still get

$$\int |Tf|^q \le 2^q q\Big[C_0^{q_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1} + C_1^{q_1}(p_1/p)^{q_1/p_1}|q-q_1|^{-1}\Big] = B_t$$

and to show $B_p|p-p_j|$ is bounded when $p \to p_j, j = 0, 1$, it suffices to show that $|(p-p_j)/(q-q_j)|$ is bounded when $p \to p_j$ and which is easy to check by r.

For the rest conditions, we assume $p_1 = q_1 = \infty$ at first, we know

$$||Th_A||_\infty \le C_1||h_A||_\infty$$

and let $A(\alpha) = \alpha/C_1$ then $\lambda_{Th_A}(\alpha) = 0$ and then

$$\begin{aligned}\int |Tf|^q &\le 2^q q C_0^{q_0} p_0^{q_0/p_0}\Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_0^{C_1\beta} \alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0}\\ &= 2^q q C_0^{q_0} C_1^{q-q_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1}\end{aligned}$$

when $||f||_p = 1$, and hence

$$B_p = 2\Big[C_0^{q_0}C_1^{q-q_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1}\Big]^{1/q}$$

at this considition, which is bounded when $p \to p_j, j = 0, 1$.

Then assume $q_0 < q_1 = \infty$, we have

$$||Th_A||_\infty \le C_1||h_A||_{p_1} \le C_1\Big(p_1\int_0^A \alpha^{p_1-1}\lambda_f(\alpha)d\alpha\Big)^{1/p_1} \le C_1 p_1^{1/p_1}A^{(p_1-p)/p_1}(||f||_p^p/p)^{1/p_1}$$

and let $A(\alpha) = [\alpha/[C_1(p_1||f||_p^p/p)^{1/p_1}]]^{\frac{p_1}{p_1-p}}$ and we get $||Th_{A(\alpha)}||_\infty \le \alpha$ and

$$\begin{aligned}\int |Tf|^q &\le 2^q q C_0^{q_0} p_0^{q_0/p_0}\Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_0^{d\beta^{(p_1-p)/p_1}} \alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0}\\ &= 2^q q C_0^{q_0} d^{q-q_0} p_0^{q_0/p_0}|q-q_0|^{-1}\Big[\int_0^\infty \beta^{p_0-1+p_0(q-q_0)(p_1-p)/p_1q_0}\lambda_f(\beta)d\beta\Big]^{q_0/p_0}\\ &= 2^q q C_0^{q_0}\Big(C_1(p_1||f||_p^p/p)^{1/p_1}\Big)^{q-q_0} p_0^{q_0/p_0}|q-q_0|^{-1}\Big(\frac{||f||_p^p}{p}\Big)^{q_0/p_0}\end{aligned}$$

For $q_1 < q_0 = \infty$, we have

$$||Tg_A||_\infty \le C_0||g_A||_{p_0} \le C_0\Big(p_0\int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha)d\alpha\Big)^{1/p_0} \le C_0 p_0^{1/p_0}A^{(p_0-p)/p_0}(||f||_p^p/p)^{1/p_0}$$

and let $A(\alpha) = [\alpha/[C_0(p_0||f||_p^p/p)^{1/p_0}]]^{\frac{p_0}{p_0-p}}$ and we get $||T_{g_{A(\alpha)}}||_\infty \le \alpha$ and then the rest are the same.

Fourier analysis

Definition 1.6

For this chapter we work on $\mathbb{R}^n$. $C^k(U)$ is the space of all functions on U with continuous partial derivatives of order $\leq k$ an $C^\infty(U) = \bigcap_{i=1}^\infty C^k(U)$. For any $E \subset \mathbb{R}^n$, $C_c^\infty(E)$ is the space of all C^∞ functions on $\mathbb{R}^n$ with compact support contained in E. If we miss U, E, it means $U = \mathbb{R}^n$ or $E = \mathbb{R}^n$.
For $x, y \in \mathbb{R}^n$, we define

$$x \cdot y = \sum_{i=1}^n x_i y_i, \quad |x| = \sqrt{x \cdot x}$$

A $multi-index$ is an ordered n-tuple of nonnegative integers α with

$$|\alpha| = \sum_{i=1}^n \alpha_j, \quad \alpha! = \prod_{i=1}^n \alpha_j!, \quad \partial^\alpha = \left(\frac{\partial}{\partial x_1}^{\alpha^1}\right) \cdots \left(\frac{\partial}{\partial x_n}^{\alpha^n}\right)$$

and for $x \in \mathbb{R}^n$, we define

$$x^\alpha = \prod_{i=1}^n x_j^{\alpha_j}$$

♣

Definition 1.7

(Schwarz space) $\mathcal{S}$ is consisted of functions f in C^∞ such that for any nonnegative integer N and multi-index α, define

$$||f||_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

then

$$\mathcal{S} = \{f \in C^\infty : ||f||_{(N,\alpha)} < \infty \text{ for all } N, \alpha\}$$

♣

Proposition 1.12

If $f \in \mathcal{S}$, then $\partial^\alpha f \in L^p$ for all α and all $p \in [1, \infty]$.

♠

Proof We know

$$|\partial^\alpha f(x)| \leq C_N (1 + |x|)^{-N}$$

for all N and $(1 + x)^{-N} \in L^p$ for all $N > n/p$.

Proposition 1.13

$\mathcal{S}$ is a Frechet space, i.e. a complete Hausdorff topological vector space whose topology is defined by a countable family of seminorms, with the topology defined by the norms $|| \cdot ||_{N,\alpha}$.

♠

Proof

We only need to show the completness of $\mathcal{S}$, which means for $\{f_j\}_1^\infty$ Cauchy in $\mathcal{S}$, i.e. $||f_m - f_n||_{(N,\alpha)} \to 0, n, m \to \infty$, there exists $g \in \mathcal{S}$ and $||f_n - g||_{N,\alpha} \to 0, n \to \infty$.

Notice

$$\sup_{x \in \mathbb{R}^n} |\partial^\alpha f_n - \partial^\alpha f_m| = ||f_n - f_m||_{(0,\alpha)} \to 0, n, m \to \infty$$

and hence $\partial^\alpha f_n$ converges uniformly to some g^α for any multi-index α. Now we consider

$$\partial^\alpha f_k(x + te_j) - \partial^\alpha f_k(x) = \int_0^t \partial^{\alpha + e_j} f_k(x + se_j) ds$$

then by DCT we know

$$g^\alpha(x + te_j) - g^\alpha(x) = \int_0^t g^{\alpha + e_j}(x + se_j) ds$$

and henc $\partial^{e_j} g^\alpha = g^{\alpha+e_j}$ which means $g^\alpha = \partial^\alpha g^0$ by the induction. Then notice

$$||f_k - g||_{(N,\alpha)} = \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f_k(x) - \partial^\alpha g(x)|$$

and we know for $\epsilon > 0$, there exists an integer N such that

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f_k(x) - \partial^\alpha f_j(x)| < \epsilon/2$$

then we know

$$\begin{aligned}(1+x)^N|\partial^\alpha f_k(x) - \partial^\alpha g(x)| &\le (1+x)^N|\partial^\alpha f_k(x) - \partial^\alpha f_(k+m)(x)| + (1+x)^N|\partial^\alpha f_{k+m}(x) - \partial^\alpha g(x)| \\ &< \epsilon/2 + (1+x)^N|\partial^\alpha f_{k+m}(x) - \partial^\alpha g(x)|\end{aligned}$$

for any integer m and hence

$$(1+x)^N|\partial^\alpha f_k(x) - \partial^\alpha g(x)| \le \epsilon$$

for any $k \ge N$, which means $||f_k - g||_{(N,\alpha)} \to 0, k \to \infty$ and hence $g \in \mathcal{S}$.

Proposition 1.14

(The product rule)For $|\alpha| = N, f, g \in C^N$, we have

$$\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$$

Proof We use the induction to N, if we have the formula for any $|\alpha| = N-1$, we will know

$$\begin{aligned}\partial^{\alpha+e_j}(fg) = \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\partial^{e_j}[(\partial^\beta f)(\partial^\gamma g)] &= \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}[(\partial^{\beta+e_j} f)(\partial^\gamma g) + (\partial^\beta f)(\partial^{\gamma+e_j} g)] \\ &= \sum_{\beta+\gamma=\alpha+e_j}\left(\frac{\alpha!}{(\beta-e_j)!\gamma!} + \frac{\alpha!}{\beta!(\gamma-e_j)!}\right)(\partial^\beta f)(\partial^\gamma g) \\ &= \sum_{\beta+\gamma=\alpha+e_j}\frac{(\alpha+e_j)!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)\end{aligned}$$

Corollary 1.3

We may know

$$\begin{aligned}\partial^\alpha(x^\beta f) &= x^\beta\partial^\alpha f + \sum c_{\gamma\delta}x^\delta\partial^\gamma f \\ x^\beta\partial^\alpha f &= \partial^\alpha(x^\beta f) + \sum c'_{\gamma\delta}\partial^\gamma(x^\delta f)\end{aligned}$$

for some constants $c_{\gamma\delta}, c'_{\gamma\delta} = 0$ unless $|\gamma| < |\alpha|$ and $|\delta| < |\beta|$.

Proof We know

$$\partial^\alpha(x^\beta f) = \sum_{\gamma+\delta=\alpha}\frac{\alpha!}{\gamma!\delta!}(\partial^\delta x^\beta)(\partial^\gamma f)$$

and the first conclusion goes, and hence the second equality goes by elimination.

Proposition 1.15

If $f \in C^\infty$, then $f \in \mathcal{S}$ iff $x^\beta\partial^\alpha f$ is bounded for all multi-indices α, β iff $\partial^\alpha(x^\beta f)$ is bounded for all multi-indices α, β.

Proof

For the first equivalence, notice

$$|x^\beta\partial^\alpha f| \le (1+|x|)^{|\beta|}|\partial^\alpha f|$$

is bounded. And notice $\sum_{j=1}^n |x_j|^N$ is strictly postive when $|x| = 1$ for any integer N, then we know it has a minimum when

$|x| = 1$ and denote it as δ, we know $\sum_{j=1}^{n} |x_j|^N \geq \delta_N |x|^N$, then we know

$$(1 + |x|)^N \leq 2^N(1 + |x|^N) \leq 2^N(1 + \delta^{-1} \sum_{j=1}^{n} |x_j|^N) \leq 2^N + 2^N\delta^{-1} \sum_{|\beta| \leq N} |x^\beta|$$

and hence $f \in \mathcal{S}$.

The second equivalence can be deduced by the corollary 1.3.

Definition 1.8

If f is a function on $\mathbb{R}^n$ and $y \in \mathbb{R}^n$, we call

$$\tau_y f(x) = f(x - y)$$

and we know $||\tau_y f||_p = ||f||_p$ for $1 \leq p \leq \infty$ and $||\tau_y f||_u = ||f||_u$. A function f is called uniformly continuous if $||\tau_y f - f||_u \to 0, y \to 0$.

♣

Lemma 1.3

If $f \in C_c(\mathbb{R}^n)$, then f is uniformly continuous.

♡

Proposition 1.16

If $1 \leq p < \infty$, translation is continuous in the L^p norm, i.e. if $f \in L^p$ and $z \in \mathbb{R}^n$, then

$$\lim_{y \to 0} ||\tau_{y+z} f - \tau_z f||_p = 0$$

♠

Proof

Notice C_c is dense in L^p is fine.

Definition 1.9

Le f and g be measurable functions on $\mathbb{R}^n$. The convolution of f and g is the function $f * g$ defined by

$$f * g(x) = \int f(x - y)g(y)dy$$

We may prove that $f * g$ is measurable.

♣

Proposition 1.17

Assumming that all integrals in question exist, we have

a. $f * g = g * f$

b. $(f * g) * h = f * (g * h)$

c. For $z \in \mathbb{R}^n, \tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$

d. If A is the closure of $\{x + y, x \in supp(f), y \in supp(g)\}$, then $supp(f * g) \subset A$.

♠

Proof

a. Trivial.

b. We know

$$\begin{aligned}(f * g) * h(x) &= \int \int f(z)g(x - y - z)dzh(y)dy \\ &= \int f(z)(g * h)(x - z)dz = f * (g * h)(x)\end{aligned}$$

c.

$$\tau_z(f * g) = f * g(x + z) = \int f(x + z - y)g(y)dy = \int \tau_z f(x - y)g(y)dy = \tau_z f * g(x)$$

and hence $\tau_z(f * g) = \tau_z(g * f) = \tau_z g * f = f * \tau_z g$.

d. Trivial.

Theorem 1.13

(Young's inequality) If $f \in L^1$ and $g \in L^p (1 \le p \le \infty)$, then $f * g(x)$ exists for almost every x, $f * g \in L^p$ and $||f * g||_p \le ||f||_1 ||g||_p$.

Proof Notice

$$||f * g||_p = (\int |\int f(y)g(x-y)dy|^p dx)^{1/p} \le \int (\int |f(y)g(x-y)|^p dx)^{1/p} dy = \int |f(y)| ||g||_p dy = ||f||_1 ||g||_p$$

by the Minkowski's inequality for integrals.

Proposition 1.18

If p and q are conjugate exponents, $f \in L^p$ and $g \in L^q$, then $f * g(x)$ exists for every x, $f * g$ is bounded and uniformly continuous and $||f * g||_u \le ||f||_p ||g||_q$. If $1 < p < \infty$, then $f * g \in C_0(\mathbb{R}^n)$, i.e. f vaninshed at infinity, i.e. $\{|f| \ge \epsilon\}$ is compact for any $\epsilon > 0$.

Proof We know

$$|f * g(x)| = |\int f(y)g(x-y)dy| \le |f(\cdot)g(x-\cdot)|_1 \le ||f||_p ||g(x-\cdot)||_q = ||f||_p ||g||_q$$

by the Holder's inequality and hence $||f * g||_u \le ||f||_p ||g||_q$. Then for any $y \in \mathbb{R}^n$,

$$||\tau_y f * g - f * g||_u = ||(\tau_y f - f) * g||_u \le ||\tau_y f - f||_p ||g||_q \to 0, y \to 0$$

if $1 \le p < \infty$. If $p = \infty$, exchange f, g.

Consider $f_n, g_n \in C_c$ and $f_n \to f$ in L^p, $g_n \to g$ in L^q, then we know

$$||f_n * g_n - f * g||_u \le ||f_n * g_n - f_n * g||_u + ||f_n * g - f * g||_u \le ||f_n||_p ||g_n - g||_q + ||f_n - f||_p ||g||_q \to 0$$

and notice $f_n * g_n \in C_c$ and hence $f * g \in C_0$.

Proposition 1.19

Suppose $1 \le p, q, r \le \infty$ and $p^{-1} + q^{-1} = r^{-1} + 1$, then

a. (Young's inequality, general form) If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and $||f * g||_r \le ||f||_p ||g||_q$.

b. Suppose also that $p > 1, q > 1$ and $r < \infty$. If $f \in L^p$ and $g \in weak\ L^q$,then $f * g \in L^r$ and $||f * g||_r \le C_{pq} ||f||_p [g]_q$ where C_{pq} is independent of f, g.

c. Suppose that $p = 1$ and $r = q > 1$. If $f \in L^1$ and $g \in weak\ L^q m$ then $f * g \in weak\ L^q$ and $[f * g]_q \le C_q ||f||_1$ where C_q is independent of f and g.

Proof

a. Notice we have the inequality holds when $p = 1, r = q$ and $r = \infty$, then for fixed q and g, then we may use the Riesz-Thorin Interpolation Theorem.

We will skip the proof for b.c.

Proposition 1.20

If $f \in L^1, g \in C^k$ and $\partial^\alpha g$ is bounded for $|\alpha| \le k$, then $f * g \in C^k$ and $\partial^\alpha (f * g) = f * (\partial^\alpha g)$ for $|\alpha| \le k$.

Proof

If $\alpha^\alpha (f * g) = f * (\partial^\alpha g)$, then

$$\partial^{\alpha + e_j}(f * g) = \partial^{e_j} f * (\partial^\alpha g) = \partial^{e_j} \int f(y) \partial^\alpha g(x-y)dy = \int f(y) \partial^{e_j} \partial^\alpha g(x-y)dy$$

if $\partial^{\alpha + e_j} g$ is bounded and hence the conclusion holds by the induction.

Proposition 1.21

If $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$.

Proof We know $f * g \in C^\infty$ by proposition 1.20. and then notice

$$1+|x| \leq (1+|x-y|)(1+|y|)$$

so we have

$$\begin{aligned}(1+|x|)^N|\partial^\alpha(f*g)(x)| &\leq \int (1+|x-y|)^N|\partial^\alpha f(x-y)|(1+|y|)^N|g(y)|dy \\ &\leq ||f||_{(N,\alpha)}||g||_{(N+n+1,0)}\int (1+|y|)^{-n-1}dy < \infty\end{aligned}$$

Theorem 1.14

Suppose $\phi \in L^1$ and $\int \phi(x)dx = a$, define $\phi_t(x) = t^{-n}\phi(t^{-1}x)$ then
a. If $f \in L^p, 1 \leq p < \infty$, then $f * \phi_t \to af, t \to 0$ in L^p.
b. If f is bounded and uniformly continuous then $f * \phi_t \to af$ uniformly as $t \to 0$.
c. If $f \in L^\infty$ and f is continuous on an open set U, then $f * \phi_t \to af$ uniformly on compact subsets of U as $t \to 0$. ♡

Proof

a. Notice that

$$f*\phi_t - af = \int f(x-y)\phi_t(y)dy - \int f(x)\phi(y)dy = \int (f-\tau_{tz}f)(x)\phi(z)dz$$

and then by the Minkowski's inequality for integrals, we know

$$||f*\phi_t - af||_p \leq \int ||f-\tau_{tz}f||_p|\phi(z)|dz$$

and by DCT we know

$$||f*\phi_t - af||_p \to 0, t \to 0$$

b. Notice

$$||f*\phi_t - af||_u \leq \int ||f-\tau_{tz}f||_u|\phi(z)|dz \leq \int_E ||f-\tau_{tz}f||_u|\phi(z)|dz + \int_{E^c} 2||f||_u|\phi(z)|dz$$

for any measurable set and choose E as a property compact set is fine.

c. Still refer the equality and then we know for a compact subset E of U, and $\epsilon > 0$, we may choose a compact set K such that

$$\int_{K^c} 2||f||_\infty|\phi(z)|dz < \epsilon/2$$

then choose d small enough such that $dK + E$ is in a compact subset E' of U and notice f is bounded and uniformly continuous on E', so we know

$$||(f*\phi_t - af)|_E||_u = ||f|_E * \phi_t - af|_E||_u$$

and the rest is similar.

Theorem 1.15

Suppose $|\phi(x)| \leq C(1+|x|)^{-n-\epsilon}$ for some $C, \epsilon > 0$ and $\int |\phi(x)|dx = a$. If $f \in L^p$, then $f * \phi_t(x) \to af(x)$ as $t \to 0$ for every x in the Lebesgue set of f, inparticular, for almost every x and for every x at which f is continuous. ♡

Proof

Firstly, let us recall that if $f \in L^p$, then $L \in L^1_{loc}$, since

$$\int_K |f| \leq \int_{K\cap\{|f|\geq 1\}} |f|^p + m(K) < ||f||_p^p + m(K) < \infty$$

for any compact set K.

We are going to show

$$\int |f(x-y) - f(x)||\phi_t(y)|dy \to 0, t \to 0$$

if $r^{-n}\int_{|y|<r}|f(x-y)-f(x)|dy \to 0, r \to 0$.

We know for any $\delta > 0$, there exists $\eta > 0$ such

$$\int_{|y|<r} |f(x-y) - f(x)| dy < \delta r^n$$

for any $r < \eta$. We have

$$\int_{|y|\geq\eta} |f(x-y) - f(x)||\phi_t(y)| dy \leq ||f||_p ||\chi_{|y|\geq\eta}\phi_t(y)||_p' + |f(x)|||\chi_{|y|\geq\eta}\phi_t(y)||_1$$

Now we consider $||\chi_{|y|\geq\eta}\phi_t(y)||_q$, if $q = \infty$, we know

$$||\chi_{|y|\geq\eta}\phi_t y||_\infty \leq \sup_{|y|\geq\eta} Ct^{-n}(1+|y/t|)^{-n-\epsilon} \leq Ct^{-n}(1+|\eta/t|)^{-n-\epsilon} \leq C|\eta|^{-n-\epsilon}t^\epsilon \to 0$$

if $t \to \infty$. For $q < \infty$, we know

$$||\chi_{|y|\geq\eta}\phi_t(y)||_q^q = \int_{|y|\geq\eta} t^{-nq}|\phi(t^{-1}y)|^q dy \leq Ct^{\epsilon q}\int_{r\geq\eta/t} r^{n-1-(n+\epsilon)q} dr \leq C_1 t^{\epsilon q}$$

for some constant C_1 by the proposition 2.51. on Folland.

Now we consider

$$\int_{|y|<\eta} |f(x-y) - f(x)||\phi_t(y)| dy$$

for fixed t, we consider

$$\begin{aligned}
\int_{|y|<\eta} |f(x-y) - f(x)||\phi_t(y)| dy &= \sum_{i=1}^{K} \int_{2^{-i}\eta\leq|y|<2^{1-i}\eta} |f(x-y) - f(x)||\phi_t(y)| dy \\
&+ \int_{|y|\leq 2^{-K}\eta} |f(x-y) - f(x)||\phi_t(y)| dy \\
&\leq \sum_{i=1}^{K} \int_{2^{-i}\eta\leq|y|<2^{1-i}\eta} |f(x-y) - f(x)|(Ct^{-n}|2^{-i}\eta/t|^{-n-\epsilon}) dy \\
&+ \int_{|y|\leq 2^{-K}\eta} |f(x-y) - f(x)|(Ct^{-n}) dy \\
&\leq C\delta t^\epsilon \sum_{i=1}^{K} [2^{i(n+\epsilon)}/\eta^{n+\epsilon}](2^{1-i}\eta)^n + C\delta t^{-n}(2^{-K}\eta)^n \\
&\leq 2^n C\delta (t/\eta)^\epsilon \frac{2^{K\epsilon}-1}{2^\epsilon - 1} + C\delta(2^{-K}\eta/t)^n
\end{aligned}$$

so let $2^{K-1} < \eta/t \leq 2^K$, then we know

$$\int_{|y|<\eta} |f(x-y) - f(x)||\phi_t(y)| dy \leq C\delta + 2^n C\delta (t/\eta)^\epsilon \frac{(2(\eta/t))^\epsilon - 1}{2^\epsilon - 1} \leq 2^n C\delta(1 + \frac{2^\epsilon}{2^\epsilon - 1})$$

for any $\delta > 0$, and the conclusion holds.

Definition 1.10

If $a = 1$, then call $\{\phi_t\}_{t>0}$ an approximate identity.

Proposition 1.22

C_c^∞ (and hence also $\mathcal{S}$) is dense in $L^p (1 \leq p < \infty)$ and in C_0.

Proof

If there exists $\phi \in C_c^\infty$, we will know that $g * \phi_t \to g$ in L^p and $g * \phi_t \in C_c^\infty$ if $g \in C_c, g \in L^1 \cap L^p$. Notice for any $f \in L^p$,$\epsilon > 0$, we can find $g \in C_c, g \in L^1 \cap L^p$ such that $||f - g|| < \epsilon$, and hence we know C_c^∞ is dense in L^p. Also if $f \in C_0$, then for any $\epsilon > 0$, we may find $g \in C_c$ such that $||f - g||_u \leq \epsilon$, since g is bounded and uniformly continuous, we know $||g * \phi_t - g||_u \to 0$ if $t \to 0$. So now we only need to check that C_c^∞ is nonempty, which can be given by

$$\phi(x) = e^{-\frac{1}{1-|x|^2}}\chi_{1-|x|^2>0}$$

since $e^{-1/t}\chi_{(0,\infty)}(t)$ is smooth and $1 - |x|^2$ is obviously smooth.

Theorem 1.16

(The C^∞ Urysohn lemma) If $K \subset \mathbb{R}^n$ is compact and U is an open set containing K, there exists $f \in C_c^\infty$ such that $0 \leq f \leq 1$ and $f = 1$ on K and supp$(f) \subset U$.

Proof

Consider $\delta = d(K, U)$ and we may find $\phi \in C_c^\infty$ such that supp$(\phi) \subset D(0, d/3)$. Let $V = \bigcup_{x \in K} D(x, d/3)$, then $\chi_V * \phi$ is the function we would like.

Theorem 1.17

If ϕ is a measurable function on $\mathbb{R}^n$(resp. $\mathbb{T}^n$), such that $\phi(x+y) = \phi(x)\phi(y)$ and $|\phi| = 1$, there exists $\xi \in \mathbb{R}^n$(resp. $\mathbb{T}^n$) such that $\phi(x) = e^{2\pi\xi \cdot x}$.

Proof

We first prove the conclusion for $\mathbb{R}$, let $a \in \mathbb{R}$ such that $\int_0^a \phi(t)dt \neq 0$ and let $A = a^{-1}$, then we know

$$\phi(x) = A \int_0^a \phi(x)\phi(t)dt = A \int_x^{x+a} \phi(t)dt$$

and hence $\phi(x)$ is continuous, and then $\phi(x) \in C^1$ with

$$\phi'(x) = A[\phi(a) - 1]\phi(x) = B\phi(x)$$

Therefore,

$$[e^{-Bx}\phi(x)]' = -B\phi(x) + \phi'(x) = 0$$

and hence $\phi(x) = ce^{Bx}$ for some constant C. Notice $\phi(0) = 1$, so we know $\phi(x) = e^{Bx}$ and hence B is a pure imaginary number, so we may let $B = 2\pi i\xi$ for some contant ξ.

If ϕ is defined on T, we may expand it into a period function on $\mathbb{R}$ with the same property and hence $\xi \in \mathbb{Z}$.

For ϕ defined on $\mathbb{R}^n$, we may consider $\phi^j(t) = \phi(te_j)$ and then we know $\phi^j(t) = e^{2\pi i\xi_j}$ for some constant ξ_j, then for $x \in \mathbb{R}^n$, we know $\phi(x) = \prod_{i=1}^n \phi(x_i e_i) = \prod_{i=1}^n \phi^i(x_i) = e^{2\pi i\xi \cdot x}$ for some $\xi \in \mathbb{R}$ and the conclusion is similar for $\mathbb{T}^n$.

Before entering the next theorem, we recall a lemma we did not prove it formally before.

Lemma 1.4

If $\{u_\alpha\}$ is an orthonormal set in a Hilbert space $\mathcal{H}$, if the finite linear combination of $\{u_\alpha\}$ is dense in $\mathcal{H}$, then it is a orthonormal basis.

Proof Assume $\langle x, u_\alpha \rangle = 0$ for any α, then if $x \neq 0$, then we may find $x_n \in$ span$\{u_\alpha\}$ converges to x in $\mathcal{H}$ and hence

$$||x||^2 = \lim \langle x_n, x \rangle = 0$$

and the conclusion holds.

Theorem 1.18

Let $E_\mathcal{K}(x) = e^{2\pi i\mathcal{K} \cdot x}$, then $\{E_\mathcal{K}, \mathcal{K} \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$.

Proof

It is easy to verify that $\{E_\kappa\}_{\kappa \in \mathbb{Z}^n}$ is an orthonormal basis since

$$\langle E_{\kappa_1}, E_{\kappa_2} \rangle = \int E_{\kappa_1 - \kappa_2} = \delta\kappa_1 - \kappa_2$$

Now we consider $A =$ span$\{E_\kappa\}_{\kappa \in \mathbb{Z}^n}$, then we know A is separating points and hence we know A is dense in $C(\mathbb{T}^n)$ by the Stone-Weierstrass' Theorem and hence it is dense in $L^2(\mathbb{T}^n)$. The rest is by the lemma 1.4.

Definition 1.11

If $f \in L^2(\mathbb{T}^n)$, we define its Fourier transform $\hat{f}$ a function on $\mathbb{Z}^n$ by

$$\hat{f}(\kappa) = \langle f, E_\kappa \rangle = \int_{\mathbb{T}^n} f(x) e^{-2\pi i \kappa \cdot x} dx$$

and we call the series

$$\sum_{\kappa \in \mathbb{Z}^n} \hat{f}(\kappa) E_\kappa$$

the Fourier series of f.

♣

Theorem 1.19

(The Hausdorff-Young Inequality) Suppose that $1 \leq p \leq 2$ and q is the conjugate exponent to p. If $f \in L^p(\mathbb{T}^n)$, then $\hat{f} \in l^q(\mathbb{Z}^n)$ and $||\hat{f}||_q \leq ||f||_p$.

♡

Proof

Use the Riesz-Thorin Interpolation Theorem directly.

Definition 1.12

For $f \in L^1$, define the Fourier Transform of f by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx$$

with $||\hat{f}||_u \leq ||f||_1$ and continuous by the DCT and we know

$$\mathcal{F} : L^1 \to BC(\mathbb{R}^n)$$

♣

Theorem 1.20

Suppose $f, g \in L^1$.
a. $(\hat{\tau_y f})(\xi) = e^{-2\pi i \xi \cdot y} \hat{f}(\xi)$ and $\tau_\eta(\hat{f}) = \hat{h}$ where $h = e^{2\pi i \eta \cdot x} f(x)$.
b.If T is an invertible linear transformation of $\mathbb{R}^n$ and $S = (T^*)^{-1}$ is its inverse transpose, then $(f \hat{\circ} T) = |\det T|^{-1} \hat{f} \circ S$. In particular, if T is a rotation, then $(f \hat{\circ} T) = \hat{f} \circ T$ and if $Tx = t^{-1}x (t > 0)$, then $(f \hat{\circ} T)(\xi) = t^n \hat{f}(t\xi)$, so that $(\hat{f_t})(\xi) = \hat{f}(t\xi)$.
c. $(f \hat{*} g) = \hat{f}\hat{g}$.
d. If $x^\alpha f \in L^1$ for $|\alpha| \leq k$, then $\hat{f} \in C^k$ and $\partial^\alpha \hat{f} = [(-2\pi \hat{i} x)^\alpha f]$.
e. If $f \in C^k, \partial^\alpha f \in L^1$ for $|\alpha| \leq k$, and $\partial^\alpha f \in C_0$ for $|\alpha| \leq k-1$, then $(\partial^{\hat{\alpha}} f)(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$.
f.(The Riemann-Lebesgue Lemma)$\mathcal{F}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$.

♡

Proof

a. We know

$$(\hat{\tau_y f})(\xi) = \int f(x-y) e^{-2\pi i \xi \cdot x} dx = e^{-2\pi i \xi \cdot y} \hat{f}(\xi)$$

and

$$\tau_\eta \hat{f}(\xi) = \hat{f}(\xi + \eta) = \int f(x) e^{-2\pi i (\xi - \eta) \cdot x} dx = \int h(x) e^{-2\pi i \xi \cdot x} dx = \hat{h}(\xi)$$

b. We know

$$(f \hat{\circ} T)(\xi) = \int f(Tx) e^{-2\pi i \xi \cdot x} dx = \int f(Tx) e^{-2\pi i \xi^* T^{-1} T x} dx = |\det T|^{-1} \int f(x) e^{-2\pi (S\xi) \cdot x} dx = |\det T^{-1}| \hat{f} \circ S$$

and the rest is easy to check.

c. We know

$$(f \hat{*} g)(\xi) = \int \int f(x-y) g(y) dy e^{-2\pi i \xi \cdot x} dx = \int (\int f(x-y) e^{-2\pi i \xi \cdot (x-y)} dx) g(y) e^{-2\pi i \xi \cdot y} dy = \hat{f}(\xi) \hat{g}(\xi)$$

by the Fubini's theorem.

d. We assume $\partial^\alpha = [(-2\hat{\pi i x})^\alpha f]$ and then

$$\begin{aligned}\partial^{\alpha+e_j}\hat{f}(\xi) &= \partial^{e_j}[(-2\pi i\hat{x})^\alpha f(\xi)] \\ &= \partial^{e_j}\int(-2\pi ix)^\alpha f(x)e^{-2\pi i\xi\cdot x}dx \\ &= \lim_{t\to 0}\frac{\int(-2\pi ix)^\alpha f(x)(e^{-2\pi i(\xi+te_j)\cdot x}-e^{-2\pi i\xi\cdot x})dx}{t}\end{aligned}$$

since

$$|(-2\pi ix)^\alpha f(x)(e^{-2\pi i(\xi+te_j)\cdot x}-e^{-2\pi i\xi\cdot x})| \le |(-2\pi ix)^\alpha f(x)||2\pi ix^{e_j}| = C|x^{\alpha+e_j}f(x)| \in L^1$$

so we know

$$\partial^{\alpha+e_j}\hat{f}(\xi) = \int(-2\pi ix)^{\alpha+e_j}f(x)e^{-2\pi i\xi\cdot x}dx$$

if $|\alpha+e_j|\le k$ and by the induction, we are done.

e. We know if the equality if true for α, then

$$\begin{aligned}(\partial^{\alpha\hat{+}e_j}f)(\xi)-(2\pi i\xi)^{\alpha+e_j}\hat{f}(\xi) &= \int\partial^{\alpha+e_j}f(x)e^{-2\pi i\xi\cdot x}dx-(2\pi i\xi)^{\alpha+e_j}\int f(x)e^{-2\pi i\xi\cdot x}dx \\ &= \int\partial^{e_j}\partial^\alpha f(x)e^{-2\pi i\xi\cdot x}dx-(2\pi i\xi)^{e_j}\int\partial^\alpha f(x)e^{-2\pi i\xi\cdot x}dx \\ &= \int\partial^{e_j}[\partial^\alpha f(x)e^{-2\pi i\xi\cdot x}]dx \\ &= \int\int_{-\infty}^{\infty}\partial^{e_j}[\partial^\alpha f(x)e^{-2\pi i\xi\cdot x}]dx_jdx' = 0\end{aligned}$$

by the Fubini's theorem if $\partial^\alpha f\in C_0$. And the conclusion holds by the induction.

f. If $f\in C^1\cap C_c$, then we know $\partial^\alpha f\in L^1$ for any $|\alpha|\le 1$ and then we know

$$2\pi i|\xi|^\alpha\hat{f}(\xi) = (\hat{\partial^\alpha}f)(\xi)$$

is bounded and continuous, and hence

$$|\xi|\hat{f}(\xi) = \sqrt{\sum_1^n[|\xi|^\alpha\hat{f}(\xi)]^2}$$

is bounded and continuous, and hence $\hat{f}\in C_0$. Now notice $C^1\cap C_c$ is dense in L^1 and hence $\hat{f}_n\to\hat{f}$ uniformly if $f_n\to f$ in L^1 and hence $\mathcal{F}(C^1\cap C_c)$ is dense in $\mathcal{F}(L^1)$ under the uniform norm, notice C_0 is closed under the uniform norm and the conclusion holds.

Corollary 1.4

$\mathcal{F}$ maps the Schwartz space $\mathcal{S}$ continuously to itself.

Proof

Notice we have $x^\alpha\partial^\beta f\in L^1\cap C_0$ and $f\in C^\infty$ then $\hat{f}\in C^\infty$ and

$$(x^\alpha\hat{\partial}^\beta f)(\xi) = (-2\pi i)^{-|\alpha|}\partial^\alpha(\partial^{\hat{\beta}}f) = (-1)^{|\alpha|}(2\pi i)^{|\alpha|-|\beta|}\partial^\alpha(\xi^\beta\hat{f})$$

which means $\partial^\alpha(\xi^\beta\hat{f})$ is bounded for any α,β and hence $\hat{f}\in\mathcal{S}$.

By the way, notic $\int(1+|x|)^{-n-1}dx<|infty$ and we have

$$||(x^\alpha\hat{\partial}^\beta f)||_u \le ||(x^\alpha\partial^\beta f)||_1 \le C||(1+|x|)^{n+1}x^\alpha\partial^\beta f||_u$$

so we know

$$||\hat{f}||_{(N,\beta)} = ||(1+|\xi|)^N\partial^\beta\hat{f}||_u$$

is less than a linear combination of $\partial^\beta(\xi^\gamma\hat{f})$ with $|\gamma|\le N$ and hence

$$||\hat{f}||_{(N,\beta)} \le \sum_{\gamma\le|\beta|}C_\gamma||f||_{(N+n+1,\gamma)} < \infty$$

and hence $\mathcal{F}$ is continuous on $\mathcal{S}$.

Proposition 1.23

If $f(x) = e^{-\pi a|x|^2}$ where $a > 0$, then $\hat{f}(\xi) = a^{-n/2}e^{-\pi|\xi|^2/a}$.

Proof

If $n = 1$,then we know

$$f' = -2\pi a x f$$

and hence

$$(\hat{f})' = (-2\hat{\pi i x f}) = \frac{i}{a}\hat{f}' = -\frac{2\pi}{a}(\cdot)\hat{f}$$

since $xf = cf'$ is in L^1 and $f \in C^\infty$, $f' \in L^1$ and $f' \in C_0$, so we know

$$(e^{\pi\xi^2/a}\hat{f}(\xi))' = 0$$

and since $\hat{f}(0) = a^{-1/2}$, we have

$$\hat{f} = a^{-1/2}e^{-\pi\xi^2/a}$$

For general n, use the Fubini's theorem:

$$\int e^{-\pi a|x|^2}e^{-2\pi i\xi\cdot x} = \prod \int e^{-\pi a x_j^2}e^{-2\pi i \xi_j x_j} = a^{-n/2}\prod e^{-\pi\xi_j^2/a} = a^{-n/2}e^{-\pi|\xi|^2/a}$$

Definition 1.13

If $f \in L^1$, we define

$$f^\vee = \hat{f}(-x) = \int f(\xi)e^{2\pi i\xi\cdot x}d\xi$$

Lemma 1.5

If $f, g \in L^1$ then $\int \hat{f}g = \int f\hat{g}$.

Proof

We know

$$\int \hat{f}g = \int f(x)e^{-2\pi i\xi\cdot x}g(\xi)dxd\xi = \int f\hat{g}$$

by the Fubini's theorem.

Theorem 1.21

(The Fourier Inversion Theorem) If $f \in L^1$ and $\hat{f} \in L^1$, the n f agrees almost everywhere with a continuous function f_0 and $(\hat{f})^\vee = (\hat{f^\vee}) = f_0$.

Proof

Let $\phi_{x,t}(\xi) = e^{2\pi i\xi\cdot x - \pi t^2|\xi|^2}$ and then we know

$$(\hat{\phi_{x,t}})(y) = \int e^{2\pi i\xi\cdot x - \pi t^2|\xi|^2 - 2\pi i\xi\cdot y}d\xi = \int e^{-\pi t^2|\xi|^2}e^{-2\pi i(y-x)}d\xi = t^{-n}e^{-\pi|x-y|^2/t^2} = g_t(x-y)$$

where $g = e^{-\pi|x|^2}$ by proposition 1.23.

Then we know

$$f * g_t(x) = \int f(y)g_t(x-y) = \int f(\hat{\phi_{x,t}})(y) = \int \hat{f}\phi_{(x,t)}$$

so we know $\int \hat{f}\phi_{(x,y)} \to f, t \to 0$ in L^1, however

$$\lim_{t\to 0}\int \hat{f}\phi_{(x,t)} = \lim_{t\to 0}\int \hat{f}(\xi)e^{2\pi i\xi\cdot x - \pi t^2|\xi|^2}d\xi = \int \lim_{t\to 0}\hat{f}(\xi)e^{2\pi i\xi\cdot x - \pi t^2|\xi|^2}d\xi = (\hat{f})^\vee$$

by the DCT and hence $f = (\hat{f})^\vee$ a.e. where we know $(\hat{f})^\vee$ is a continuous function. Then notice

$$(\hat{f^\vee})(x) = \int f^\vee(\xi)e^{-2\pi i\xi\cdot x}d\xi = \int \hat{f}(-\xi)e^{2\pi i(-\xi)\cdot x}d\xi = (\hat{f})^\vee(x)$$

and the problem goes.

Corollary 1.5

If $f \in L^1$ and $\hat{f} = 0$, then $f = 0$ a.e.

Corollary 1.6

$\mathcal{F}$ is an isomorphism of $\mathcal{S}$ onto itself.

Theorem 1.22

Let $\mathcal{X} = \{f \in L^1, \hat{f} \in L^1\}$, then we can extend $\mathcal{F}$ from $\mathcal{X}$ to $L^1 + L^2$.

Proof

Notice $\hat{f} \in L^1$ implies that $f \in L^\infty$ and hence $f \in L^2$, so we know $\mathcal{X} \in L^1 \cap L^2$. Then since $\mathcal{S} \subset \mathcal{X}$ and dense in both L^1 and L^2, so we may extend $\hat{f}$ by L^∞ on L^1 and by L^2 on L^2, however the Fourier transform on L^1 has been defined.

For L^2 case, we may consider $f, g \in \mathcal{X}$ and $h = \bar{\hat{g}}$ which implies $\mathcal{F}$ keeps the L^2 inner product on $\mathcal{X}$ since

$$\int f\bar{g} = \int f\hat{h} = \int \hat{f}h = \int \hat{f}\bar{\hat{g}}$$

so, if $f_n, g_n \in \mathcal{X}$ converges to f, g, then we know

$$\langle f, g\rangle = \lim_{n\to\infty} \langle f_n, g_n\rangle = \lim_{n\to\infty} \langle \hat{f}_n, \hat{g_n}\rangle = \langle \hat{f}, \hat{g}\rangle$$

which means $\mathcal{F}$ is even a unitary isomorphism on L^2.

Now we only need to check that the expansion from $\mathcal{X}$ agree on $L^1 \cap L^2$. For $f \in L^1 \cap L^2$, we may consider $g(x) = e^{-\pi|x|^2}$ and we know $f \cdot g_t \in L^1$ and $(f \hat{*} g_t) = \hat{f}\hat{g}_t = e^{-\pi t^2|\xi|^2}\hat{f}$, and hence $(f * g_t) \in \mathcal{X}$, then we know $(f * g_t) \to f$ in both L^1 and L^2, so $f \hat{*} g_t \to \hat{f}$ in both L^∞ and L^2 and hence the extension agrees, so we know $||\hat{f}||_2 = \lim ||f \hat{*} g_t||_2 = \lim ||f * g_t||_2 = ||f||_2 < \infty$.

Theorem 1.23

Suppose that $1 \le p \le 2$ and q is the conjugate exponent to p. If $f \in L^p(\mathbb{R}^n)$, then $\hat{f} \in L^q$ and $||\hat{f}||_q \le ||f||_p$.

Theorem 1.24

If $f \in L^1$, the series $\sum_{k\in\mathbb{Z}^n} \tau_k f$ converges pointwise a.e. and in $L^1(\mathbb{T}^n)$ to a function Pf such that $||Pf||_1 \le ||f||_1$. Moreover, for $\kappa \in \mathbb{Z}^n$, $(\hat{Pf})(\kappa)$ equals $\hat{f}(\kappa)$.

Proof

Let $Q = [-1/2, 1/2)^n$ and we know

$$\int_Q \sum_{k\in\mathbb{Z}^n} |f(x-k)|dx = \sum_{k\in\mathbb{Z}^n} \int_{Q+k} |f(x)|dx = \int |f|$$

by the MCT and hence $\sum \tau_k f$ converges a.e. and in $L^1(\mathbb{T}^n)$ to a function $Pf \in L^1(\mathbb{T}^n)$ with $||Pf||_1 \le ||f||_1$. And

$$(\hat{Pf})(\kappa) = \int_Q \sum_{k\in\mathbb{Z}^n} f(x-k)e^{2\pi i\kappa\cdot x}dx = \sum_{k\in\mathbb{Z}^n} \int_{Q+k} f(x)e^{-2\pi i\kappa\cdot(x+k)}dx = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\kappa\cdot x}dx = \hat{f}(\kappa)$$

by the DCT.

Theorem 1.25

(The Poisson Summation Formula) Suppose $f \in C(\mathbb{R}^n)$ such that $|f| \le C(1+|x|)^{-n-\epsilon}$ and $|\hat{f}(\xi)| \le C(1+|\xi|)^{-n-\epsilon}$ for some $C, \epsilon > 0$, then

$$\sum_{k\in\mathbb{Z}^n} f(x+k) = \sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa)e^{2\pi i\kappa\cdot x}$$

where both series converge absolutely and uniformly on $\mathbb{T}^n$.

Proof

The absolute and uniformly convergence of the series follows that $\sum(1+|k|)^{-n-\epsilon} < \infty$, so $Pf = \sum_k \tau_k f$ is in $C(\mathbb{T}^n)$ and hence in $L^2(\mathbb{T}^n)$. Then by the theorem 1.24, we know $\sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa)e^{2\pi i\kappa\cdot x}$ converges to Pf in $L^2(\mathbb{T}^n)$ and since the right series is also continuous and hence they are the same pointwise.

Theorem 1.26

Suppose that f is periodic and absolutely continuous on $\mathbb{R}$ and that $f' \in L^p(\mathbb{T})$ for some $p > 1$, then $\hat{f} \in l^1(\mathbb{Z})$.

Proof

For $p > 1$, we know $C_p = \sum_1^\infty k^{-p} < \infty$ and since $L^p(\mathbb{T}) \subset L^2(\mathbb{T})$, then we assume $p \leq 2$, then we know

$$\widehat{(f')}(k) = \int_{\mathbb{T}} f'(x)e^{-2\pi ikx}dx = f(x)e^{-2\pi ikx}|_0^1 - \int f(x)(-2\pi ik)e^{-2\pi ikx} = 2\pi ik\hat{f}(k)$$

by the Integration by parts and then

$$\sum_{k\neq 0} |\hat{f}(k)| \leq [\sum_{k\neq 0}(2\pi|k|)^{-p}]^{1/p}[\sum_{k\neq 0}(2\pi|k\hat{f}(k)|^q)]^{1/q} \leq C||\widehat{(f')}||_q \leq C||f'||_p$$

by the Hausdorff-Young inequality and hence $||\hat{f}||_1 < \infty$.

Lemma 1.6

If $f, g \in L^2$,then $(\hat{f}\hat{g})^\vee = f * g$.

Proof We know

$$||\hat{f}\hat{g}||_1 \leq ||\hat{f}||_2||\hat{g}||_2 = ||f||_2||g||_2 < \infty$$

and hence $(\hat{f}\hat{g})^\vee$ exists and

$$f * g(x) = \int f(y)g(x-y)dy = \int \hat{f}(\xi)\widehat{g(x-\cdot)}(\xi)d\xi = (\hat{f}\hat{g})^\vee(x)$$

Theorem 1.27

Suppose that $\Phi \in L^1 \cap C_0, \Phi(0) = 1$ and $\phi = \Phi^\vee \in L^1$. For $f \in L^1 + L^2$, for $t > 0$ set

$$f^t(x) = \int \hat{f}(\xi)\Phi(t\xi)e^{2\pi\xi\cdot x}d\xi$$

a. If $f \in L^p, 1 \leq p < \infty$, then $f^t \in L^p$ and $||f^t - f||_p \to 0, t \to 0$.
b. If f is bounded and uniformly continuous, then so is f^t and $f^t \to f$ uniformly as $t \to 0$.
c. Suppose that $|\phi(x)| \leq C(1+|x|)^{-n-\epsilon}$ for some $C, \epsilon > 0$. Then $f^t(x) \to f(x)$ for every x in the Lebesgue set of f.

Proof

Let $f = f_1 + f_2, f_1 \in L^1, f_2 \in L^2$ and we know $\Phi \in L^1 \cap L^2$, so

$$\int \hat{f}_1(\xi)\Phi(t\xi)e^{2\pi i\xi\cdot x}d\xi = \int \hat{f}_1(\xi)\widehat{(\phi_t)}(\xi)e^{2\pi i\xi\cdot x}d\xi$$

since

$$\int \phi_t(\xi)e^{-2\pi i\xi\cdot x}d\xi = t^{-n}\int \phi(t^{-1}\xi)e^{-2\pi i\xi\cdot x}d\xi = \hat{\phi}(tx) = \Phi(tx)$$

a.e. so

$$\int \hat{f}_1(\xi)\Phi(t\xi)e^{2\pi i\xi\cdot x}d\xi = f_1 * \phi_t$$

since $\hat{f}_1\phi \in L^1$ and $f * \phi_t \in L61$. Then

$$\int \hat{f}_2(\xi)\Phi(t\xi)e^{2\pi i\xi\cdot x}d\xi = f_2 * \phi_t(\xi)$$

by the lemma 1.6. and we know $f^t = f * \phi_t$. Then by the theorem 1.14. we have (a),(b) and (c) is according to theorem 1.15.

Theorem 1.28

Suppsoe that $\Phi \in C$ satisfies $|\Phi(\xi)| \leq C(1+|\xi|)^{-n-\epsilon}$, $|\Phi^{\vee}(x)| \leq C(1+|x|)^{-n-\epsilon}$ and $\Phi(0) = 1$. Given $f \in L^1(\mathbb{T}^n)$ for $t > 0$, set

$$f^t(x) = \sum_{\kappa \in \mathbb{Z}^n} \hat{f}(\kappa)\Phi(t\kappa)e^{2\pi i \kappa \cdot x}$$

a. If $f \in L^p(\mathbb{T}^n), 1 \leq p < \infty$, then $||f^t - f||_p \to 0$ as $t \to 0$ and if $f \in C(\mathbb{T}^n)$, then $f^t \to f$ uniformly as $t \to 0$.
b. $f^t(x) \to f(x)$ for every x in the Lebesgue set of f.

♡