

Chapter 1

1.1 Basics of Stochastic Processes

We will refer X_t to be real or \mathbb{R}^d -valued continuous-time stochastic processes defined on a probability space (Ω, \mathcal{F}, P) . For every fixed $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is called a trajectory or sample path of the process.

For a real-valued stochastic process, let $-\leq t_1 < \dots < t_n$ be fixed. Then we know

$$P_{t_1, \dots, t_n} = P \circ (X_{t_1}, \dots, X_{t_n})^{-1}$$

is a probability distribution on \mathbb{R}^n , which is called the finite-dimensional marginal distribution of the process.

Theorem 1.1

(Kolmogorov's extension theorem) Consider a family of probability measures

$$\{P_{t_1, \dots, t_n}, t_1 < \dots < t_n, n \geq 1, t_i \geq 0\}$$

such that

- P_{t_1, \dots, t_n} is a probability on \mathbb{R}^n .
- For $\{t_{k_1} < \dots < t_{k_m}\} \subset \{t_1 < t_2 < \dots < t_n\}$, $P_{t_{k_1}, \dots, t_{k_m}}$ is required to be a marginal of P_{t_1, \dots, t_n} , then there exists a real-valued stochastic process X_t owning finite-dimensional marginal distributions of $\{P_{t_1, \dots, t_n}\}$.



Definition 1.1

A real-valued process X_t is a second-order process iff $EX_t^2 < \infty, t \geq 0$, define

$$m_X(t) = EX_t, \Gamma_X(s, t) = \text{cov}(X_s, X_t)$$



Definition 1.2

A real-valued process X_t is said to be Gaussian if its finite-dimensional marginal distributions are multidimensional Gaussian laws.



Proposition 1.1

A Gaussian process is determined by m_X and Γ_X , conversely, for any $m : \mathbb{R}_+ \rightarrow \mathbb{R}$ and a symmetric $\Gamma : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ which is nonnegative definite, there always exists a Gaussian process with mean m and covariance function Γ by Kolmogorov's extension theorem.



Definition 1.3

We call two processes X, Y are equivalent if for all $t \geq 0$, $X_t = Y_t$ a.s. And we call them indistinguishable if $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$ and for all ω in some set with probability 1.



Proposition 1.2

Two equivalent processes with right-continuous trajectories are indistinguishable.



Proof Let $X_q = Y_q$ on Ω_q for $q \in \mathbb{Q}$ and let $\Omega' = \bigcap_{q \in \mathbb{Q}} \Omega_q$ and we know Ω' has the probability 1. And it is easy to check that $X_t = Y_t$ on Ω' for all t .

Theorem 1.2


(Kolmogorov's continuity theorem) Suppose that $X = X_t, t \in [0, T]$ satisfies

$$E(|X_t - X_s|^\beta) \leq K|t - s|^{1+\alpha}$$


for all $s, t \in [0, T]$ and for some constants $\beta, \alpha, K > 0$. Then there exists a version \tilde{X} of X such that, if $\gamma < \alpha/\beta$,

then

$$|\tilde{X}_t| - \tilde{X}_s \leq G_\gamma |t - s|^\gamma$$


for all $s, t \in [0, T]$, where G_γ is a random variable. The trajectories of \tilde{X} are Hölder continuous of γ for any $\gamma < \alpha/\beta$. 

Definition 1.4

\mathcal{F}_t is an increasing family of sub- σ -field of \mathcal{F} . A process X_t is \mathcal{F}_t -adapted if X_t is \mathcal{F}_t -measurable for all $t \geq 0$. 


Definition 1.5

An adapted process $X_t, t \geq 0$ is a Markov process w.r.t. a filtration \mathcal{F}_t if for any $s \geq 0, t > 0$ and any measurable and bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$E(f(X_{s+t})|\mathcal{F}_s) = E(f(X_{s+t})|X_s) \text{ a.s.}$$


Proposition 1.3

A \mathcal{F}_t -Markov process X_t is also a \mathcal{F}_t^X -Markov process where

$$\mathcal{F}_t^X = \sigma\{X_u, 0 \leq u \leq t\}$$


Proof Notice

$$E(f(X_{s+t})|\mathcal{F}_s^X) = E(E(f(X_{s+t})|\mathcal{F}_s)|\mathcal{F}_s^X) = E(E(f(X_{s+t})|X_s)|\mathcal{F}_s^X) = E(f(X_{s+t})|X_s)$$

since $\sigma(X_s) \subset \mathcal{F}_s^X \subset \mathcal{F}_t$.


Definition 1.6

Assume a filtration \mathcal{F}_t on (Ω, \mathcal{F}, P) satisfies that for any $P(A) = 0, A \in \mathcal{F}, A \in \mathcal{F}_0$ and it is right-continuous, i.e.

$$\mathcal{F}_t = \bigcap_{n \geq 1} \mathcal{F}_{t+n^{-1}}$$

Then consider a r.v. $T : \Omega \rightarrow [0, \infty]$ is a stopping time w.r.t. to the filtration if

$$\{T \leq t \in \mathcal{F}_t\}$$

for any $t \geq 0$. 

Proposition 1.4

a. T is a stopping time iff $\{T < t\} \in \mathcal{F}_t$ for all $t \geq 0$.

b. $S \vee T$ and $S \wedge T$ are stopping times.

c. Given a stopping time T ,

$$\mathcal{F}_T = \{A, A \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$$

is a σ -algebra.

d. If $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.

e. Let $X_t, t \geq 0$ be a continuous and adapted process. The hitting time of a set $A \subset \mathbb{R}$ is defined by

$$T_A = \inf\{t \geq 0, X_t \in A\}$$

and whether A is open or closed, T_A is a stopping time.

f. Let X_t be an adapted stochastic process with right-continuous paths and let $T < \infty$ be a stopping time. Then the random variable

$$X_T(\omega) = X_{T(\omega)}(\omega)$$

is \mathcal{F}_T -measurable. 

Definition 1.7

An adapted process $M = M_t, t \geq 0$ is called a martingale w.r.t. a filtration $\mathcal{F}_t, t \geq 0$ if

- a. for all $t \geq 0$, $E(|M_t|) < \infty$
- b. for each $s \leq t$, $E(M_t | \mathcal{F}_s) = M_s$

**Proposition 1.5**

- a. For any integrable random variable X , $E(X | \mathcal{F}_t)$ is a martingale.
- b. If M_t is a submartingale then $t \rightarrow E(M_t)$ is nondecreasing.
- c. If M_t is a martingale and φ is a convex function such that $E|\varphi(M_t)| < \infty$ for all $t \geq 0$ then $\varphi(M_t)$ is a submartingale.



Proof Only (c) is needed to be proved. Consider $ax + b \leq \varphi(x)$ and we know

$$E(\varphi(M_t) | \mathcal{F}_s) \geq aE(M_t | \mathcal{F}_s) + b$$

for any such a, b and hence

$$E(\varphi(M_t) | \mathcal{F}_s) \geq \varphi(M_s)$$

Definition 1.8

An adapted process $M_t, t \geq 0$ is called a local martingale if there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that, for any $n \geq 1$ $M_{t \wedge \tau_n}$ is a martingale.

**Theorem 1.3**

Let $M_t, t \geq 0$ be a continuous local martingale such that $M_0 = 0$. Let $\pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$ be a partition of $[0, t]$. Then we have

$$\sum_{j=0}^{n-1} (M_{t_{j+1}} - M_{t_j})^2 \rightarrow \langle M \rangle_t, |\pi| \rightarrow 0$$

in probability, where $\langle M \rangle_t, t \geq 0$ is called the quadratic variation of the local martingale. Moreover, if $M_t, t \geq 0$ is a martingale then the convergence holds in $L^1(\Omega)$.

**Theorem 1.4**

The quadratic variation is the unique continuous and increasing process satisfying $\langle M \rangle_0 = 0$ and

$$M_t^2 - \langle M \rangle_t$$

is a local martingale.



1.2 Brownian Motion

Definition 1.9

A real-valued stochastic process $B = (B_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}; P)$ is called a Brownian motion if it satisfies the following conditions:

- a. Almost surely $B_0 = 0$.
- b. For all $0 \leq t_1 < \dots < t_n$ the increments $B_{t_n} - B_{t_{n-1}}, \dots, B_{t_2} - B_{t_1}$ are independent random variables.
- c. If $0 \leq s < t$, the increment $B_t - B_s$ is a Gaussian random variable with mean zero and variance $t - s$.
- d. With probability one, the map $t \rightarrow B_t$ is continuous.

A d -dimensional Brownian motion is defined as an \mathbb{R}^d -valued stochastic process $B = (B_t)_{t \geq 0}$, $B_t = (B_t^1, \dots, B_t^d)$, where B^1, \dots, B^d are d independent Brownian motions.



Proposition 1.6

Properties (a),(b),(c) are equivalent to that B is a Gaussian process, i.e. for any finite set of indices t_1, \dots, t_n , $(B_{t_1}, \dots, B_{t_n})$ is a multivariate Gaussian random variable, equivalently, any linear combination of B_{t_i} is normal distributed r.v., with mean zero and covariance function

$$\Gamma(s, t) = \min(s, t)$$

Proof

Suppose (a),(b),(c) holds, then we know $(B_{t_1}, \dots, B_{t_n})$ is normal for any finite indices and then

$$m(t) = E(B_t) = 0$$

$$\Gamma(s, t) = E(B_s B_t) = E(B_{\min(s, t)}^2) = \min(s, t)$$

Conversely, we know $E(B_0^2) = 0$ and hence $B_0 = 0$ a.s., then we know $E(B_s^2) = s$ and for any $0 < s < t$,

$$E(B_s(B_t - B_s)) = 0$$

and it is easy to check (c), and (b) is deduced by computing the covariance of the increments, notice that two r.v.s are independent iff $\phi_{(X_1, X_2, \dots, X_n)} = \phi_{X_1} \phi_{X_2} \dots \phi_{X_n}$ which implies that normal r.v.s are independent iff they have zero covariances.

Theorem 1.5

(Kolmogorov's continuity theorem) Suppose that $X = (X_t)_{t \in [0, T]}$ satisfies

$$E(|X_t - X_s|^\beta) \leq K|t - s|^{1+\alpha}$$

for all $s, t \in [0, T]$ and some constant $\beta, \alpha, K > 0$. Then there exists a version \tilde{X} of X such that if

$$\gamma < \alpha/\beta$$

then

$$|\tilde{X}_t - \tilde{X}_s| \leq G_\gamma |t - s|^\gamma$$

for all $s, t \in [0, T]$ where G_γ is a random variable. The trajectories of \tilde{X} are Holder continuous of order γ for any $\gamma < \alpha/\beta$.

Proposition 1.7

There exists a version of B with Holder-continuous trajectories of order γ for any $\gamma < (k-1)/2k$ on any interval $[0, T]$.

Proof

Since we know $B_t - B_s$ has the normal distribution $\mathcal{N}(0, t-s)$ and then we know

$$E((B_t - B_s)^{2k}) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} x^{2k} \exp\left(-\frac{x^2}{2(t-s)}\right) dx = (2k-1)!!(t-s)^k = \frac{(2k)!}{2^k k!} (t-s)^k$$

and by the theorem 1.1, the proposition holds.

Proposition 1.8

Brownian motion are basic properties:

- For any $a > 0$, the process $(a^{-1/2} B_{at})_{t \geq 0}$ is a Brownian motion.
- For any $h > 0$, the process $(B_{t+h} - B_h)_{t \geq 0}$ is a Brownian motion.
- The process $(-B_t)_{t \geq 0}$ is a Brownian motion.
- Almost surely $\lim_{t \rightarrow \infty} B_t/t = 0$ and the process $X_t = tB_{1/t}$ for $t > 0$, $X_t = 0$ for $t = 0$ is a Brownian motion.

Proof

a. Consider $0 \leq t_1 < t_2 < \dots < t_n$ and we may calculate the covariance matrix for

$$a^{-1/2}B_{at_n} - a^{-1/2}B_{at_{n-1}}, \dots, a^{-1/2}B_{at_2} - a^{-1/2}B_{at_1}$$

by

$$\begin{aligned} & E[(a^{-1/2}B_{at_j} - a^{-1/2}B_{at_{j-1}})(a^{-1/2}B_{at_k} - a^{-1/2}B_{at_{k-1}})] \\ &= a^{-1}(at_j \wedge at_k) - a^{-1}(at_j \wedge at_{k-1}) - a^{-1}(at_{j-1} \wedge at_k) + a^{-1}(at_{j-1} \wedge at_{k-1}) \\ &= \begin{cases} t_j - t_{j-1} - t_{j-1} + t_{j-1} = t_j - t_{j-1} & \text{if } j = k \\ t_j - t_j - t_{j-1} + t_{j-1} = 0 & \text{if } j < k \\ 0 & \text{if } j > k \end{cases} \end{aligned}$$

and hence $(a^{-1/2}B_{at})_{t \geq 0}$ satisfies the property (b) in definition 1.1, a,d are obvious and c is easy to be checked.

b. Obvious.

c. Obvious.

d. Notice B is Holder continuous. Now we only need to check that

$$E(tB_{1/t}sB_{1/s}) = ts(1/t \wedge 1/s) = (t \wedge s)$$

and the rest is easy to be checked.

Theorem 1.6

(The law of the iterated logarithm)

$$\limsup_{t \rightarrow s^+} \frac{|B_t - B_s|}{\sqrt{2|t - s| \ln \ln |t - s|}} = 1, \quad a.s.$$



Proposition 1.9

Fix a time interval $[0, t]$ and consider the following subdivision π of this interval:

$$0 = t_0 < t_1 < \dots < t_n = t$$

The norm of the subdivision π is defined as $|\pi| = \max_{0 \leq j \leq n-1} (t_{j+1} - t_j)$. Then

$$\lim_{|\pi| \rightarrow 0} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 = t$$

in $L^2(\Omega)$.



Proof

Consider let $\xi_j = (B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)$ and we know ξ_j are independent with mean 0 and hence

$$\begin{aligned} E\left(\sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 - t\right)^2 &= \sum_{j=0}^{n-1} E\xi_j^2 = \sum_{j=0}^{n-1} (3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2) \\ &= 2 \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 \leq 2t|\pi| \rightarrow 0 \end{aligned}$$

Proposition 1.10

The total variation of Brownian morion on an interval $[0, t]$ defined by

$$V = \sup_{\pi} \sum_{i=1}^{n-1} [B_{t_{i+1}} - B_{t_i}]$$

where π is any partition of $[0, t]$, is infinite with probability 1.



Proof

Here we know

$$\sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 \leq V \sup_j |B_{t_{j+1}} - B_{t_j}|$$

and hence if $V < \infty$, then

$$\lim_{|\pi| \rightarrow 0} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 = 0$$

which means $P(V < \infty) = 0$.

Definition 1.10

(Wiener integral) Let \mathcal{E}_0 be the set of step functions in \mathbb{R}_+ , i.e.

$$\phi(t) = \sum_{j=0}^{n-1} a_j \chi_{t_j, t_{j+1}}(t)$$

where $n \geq 1$ is an integer, $a_i \in \mathbb{R}$ and $0 = t_0 < \dots < t_n$. And we may define Wiener integral of a step function by

$$\int_0^\infty \phi dB_t = \sum_{i=0}^{n-1} a_i (B_{t_{i+1}} - B_{t_i})$$



Proposition 1.11

The Wiener integral is a linear isometry from $\mathcal{E}_0 \subset L^2(\mathbb{R}_+)$ to $L^2(\Omega)$.



Proof Notice

$$E[(\int_0^\infty \phi dB_t)^2] = \sum_{i=0}^{\infty} a_i^2 (t_{i+1} - t_i) = \|\phi\|_2^2$$

Definition 1.11

We have already know Wiener integral is a linear isometry from a dense subspace from $L^2(\mathbb{R}_+)$ to $L^2(\Omega)$, and hence we may call the extension of the linear isometry to be the Wiener integral and for any $\phi \in L^2(\mathbb{R}_+)$, denote

$$\int_0^\infty \phi dB_t$$

to be its image of the isometry.



Definition 1.12

Let D be a Borel subset of \mathbb{R}^m , a white noise on D is a centered Gaussian family of random variables

$$\{W_A, A \subset \mathcal{B}(\mathbb{R}^m), A \subset D, m(A) < \infty\}$$

such that

$$E(W_A W_B) = m(A \cap B)$$



Proposition 1.12

$\chi_A \rightarrow W_A$ is a linear isometry from $L^2(D) \rightarrow L^2(\Omega)$.



Definition 1.13

Similarly, we may define the integral r.s.t. W of $\phi \in L^2(D)$ denoted by

$$\phi \mapsto \int_D \phi W(dx)$$

by extending the linear isometry.



Definition 1.14

Consider a Brownian motion B defined on a probability space (Ω, \mathcal{F}, P) . For any time $t \geq 0$, define \mathcal{F}_t the σ -algebra by $B_s, 0 \leq s \leq t$ and the null events in \mathcal{F} , we call \mathcal{F}_t the natural filtration of Brownian motion on the probability space (Ω, \mathcal{F}, P) .

**Lemma 1.1**

Suppose X and Y

**Theorem 1.7**

For any measurable and bounded (or nonnegative) function $f : \mathbb{R} \rightarrow \mathbb{R}, s \geq 0$ and $t \geq 0$, we have

$$E(f(B_{s+t})|\mathcal{F}_s) = (P_t f)(B_s)$$

where

$$(P_t f)(x) = \int_{\mathbb{R}} f(y) p_t(x - y) dy$$

where

$$p_t = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$



Check Durrett Theorem 7.2.1.

Proposition 1.13

The family of operators P_t satisfies the semigroup property $P_t \circ P_s = P_{t+s}$ and $P_0 = \text{Id}$.

**Proof**

$$\begin{aligned} P_t \circ P_s(f)(x) &= \int_{\mathbb{R}} P_s f(y) p_t(x - y) dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) p_s(y - z) p_t(x - y) dz dy \\ &= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{st}} e^{-\left(\frac{(y-z)^2}{2s} + \frac{(x-y)^2}{2t}\right)} dy dz \\ &= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{st}} e^{-\left(\frac{(\sqrt{s+t}y - (2tz+2sx)/\sqrt{s+t})^2 - (tz+sx)^2/(s+t) + tz^2 + sx^2}{2st}\right)} dy dz \end{aligned}$$

and the rest is easy to be checked.

Theorem 1.8

The processes $B_t, (B_t^2 - t)$ and $e^{aB_t - a^2 t/2}, a \in \mathbb{R}$ are \mathcal{F}_t martingales.



Proof B_t is obviously a \mathcal{F}_t martingale. Notice

$$E[(B_t^2 - t)|\mathcal{F}_s] = E[(B_t - B_s)^2 + B_s^2 + 2B_s(B_t - B_s) - t|\mathcal{F}_s] = t - s + B_s^2 - t = B_s^2 - s$$

and

$$E(e^{aB_t - a^2 t/2}|\mathcal{F}_s) = e^{-a^2 t/2} E(e^{a(B_t - B_s)} e^{aB_s}|\mathcal{F}_s) = e^{aB_s} E(e^{a(B_t - B_s)}|\mathcal{F}_s) = e^{aB_s} E(e^{a(B_t - B_s) - a^2(t-s)/2}) = e^{aB_s - a^2 s/2}$$

Definition 1.15

The Brownian hitting time is defined by

$$\tau_a = \inf\{t \geq 0, B_t = a\}$$



Proposition 1.14

Fix $a > 0$. Then, for all $\alpha > 0$

$$E(e^{-\alpha\tau_a}) = e^{-\sqrt{2\alpha}a}$$



1.3 Stochastic Integrals

Definition 1.16

We say that a stochastic process u_t is progressively measurable if, for any $t \geq 0$ the restriction of u to $\Omega \times [0, t]$ is $\mathcal{F}_t \times B([0, t])$ -measurable.

**Definition 1.17**

Let P be the σ -field of sets $A \subset \Omega \times \mathbb{R}^+$ such that χ_A is progressively measurable. We denote by $L^2(P)$ the Hilbert space $L^2(\Omega \times \mathbb{R}^+, P, P \times m)$ equipped the norm

$$\|u\|^2 = E\left(\int_0^\infty u_s^2 ds\right) = \int_0^\infty Eu_s^2 ds$$

by Fubini's theorem.

**Definition 1.18**

A process $u = u_t$ is called a simple process if it is of the form

$$u_t = \sum_{j=0}^{n-1} \phi_j \chi_{(t_j, t_{j+1}]}(t)$$

where $0 \leq t_0 < t_1 < \dots < t_n$ and the ϕ_j are \mathcal{F}_{t_j} -measurable random variables such that $E\phi_j^2 < \infty$ and denote the space of simple processes as \mathcal{E} .

We define the stochastic integral of a process $u \in \mathcal{E}$ of u_t as

$$I(u) = \int_0^\infty u_t dB_t = \sum_{j=0}^{n-1} \phi_j (B_{t_{j+1}} - B_{t_j})$$



Here it is easy to see that $\phi_j \chi_{(t_j, t_{j+1}]}(t)$ is progressively measurable.

Proposition 1.15

Here are some properties of stochastic integrals.

a. For any $a, b \in \mathbb{R}$ and simple process $u, v \in \mathcal{E}$, we know

$$\int_0^\infty (au_t + bv_t) dB_t = a \int_0^\infty u_t dB_t + b \int_0^\infty v_t dB_t$$

b. For any $u \in \mathcal{E}$, we have

$$E\left(\int_0^\infty u_t dB_t\right) = 0$$

c. For any $u \in \mathcal{E}$, we know

$$E\left(\left(\int_0^\infty u_t dB_t\right)^2\right) = E\left(\int_0^\infty u_t^2 dt\right)$$



Proof (a) is trivial. (b) can be shown by the independence of ϕ_j and $B_{t_{j+1}} - B_{t_j}$. (c) Can be shown by

$$E\left(\left(\int_0^\infty u_t dB_t\right)^2\right) = E\left(\left(\sum_{j=0}^{n-1} \phi_j (B_{t_{j+1}} - B_{t_j})\right)^2\right) = \sum_{j=0}^{n-1} E\phi_j^2 (t_{j+1} - t_j)$$

Proposition 1.16

The space \mathcal{E} of simple processes is dense in $L^2(P)$. 

Proof For $u \in L^2(P)$, we define

$$u_t^{(n)} = n \int_{(t-1/n) \wedge 0}^t u_s ds$$

we know $u_t^{(n)}$ is continuous as $\mathbb{R} \rightarrow L^2(\Omega)$ and hence we also know

$$\lim_{n \rightarrow \infty} E \left(\int_0^\infty |u_t - u_t^{(n)}|^2 dt \right) = 0$$

since $\lim_{n \rightarrow \infty} u_t^{(n)} = u_t$ a.s. by Lebesgue differential, then we may know the limit above holds by the DCT since

$$\int_0^\infty |u_t^{(n)}(\omega)|^2 dt = \int_0^\infty n^2 \left| \int_{t-1/n}^t u_s(\omega) ds \right|^2 dt \leq \|u\|_2^2$$

where $f(s, t) = u_s \chi_{(t-1/n, t]}$ by the Minkowski's inequality of integrals.

For $u \in L^2(P)$ continuous in $L^2(\Omega)$, we define


$$u_t^{(n, N)} = \sum_{j=0}^{(n, N)} u_{t_j} \chi_{(t_j, t_{j+1}]}(t)$$

where $t_j = jN/n$. The continuity in $L^2(\Omega)$ implies that

$$E \left(\int_0^J |u_t - u_t^{n, N}|^2 dt \right) \leq E \left(\int_N^\infty u_t^2 dt \right) + N \sup_{|t-s| \leq N/n, t, s \leq N} E(|u_t - u_s|^2)$$

and we let $N \rightarrow \infty$ and $n \rightarrow \infty$ we may get the conclusion.

Proposition 1.17

The stochastic integral can be extended to a linear isometry. 


Proposition 1.18

Here are some properties, for any $u, v \in L^2(P)$, we know

$$E(I(u)) = 0, \quad E(I(u)I(v)) = E\left(\int_0^\infty u_s v_s ds\right)$$

and for any T we set

$$\int_0^T u_s dB_s = \int_0^\infty u_s \chi_{[0, T]}(s) dB_s$$


which is the indefinite integral of u with respect to B , where requiring $u \in L_T^2(P)$. 

Definition 1.19

Define $L_\infty^2(P)$ the set of progressively processes such that

$$E \left(\int_0^t u_s^2 ds \right) < \infty$$

for each $t > 0$, for any process $u \in L_\infty^2(P)$, we can define the indefinite integral process

$$\int_0^t u_s dB_s$$


Proposition 1.19

Here are some properties of indefinite integral process.

a. For any $a \leq b \leq c$, we have

$$\int_a^b u_s dB_s + \int_b^c u_s dB_s = \int_a^c u_s dB_s$$

b. If $a < b$ and F is a bounded and \mathcal{F}_a -measurable random variable then

$$\int_a^b F u_s dB_s = F \int_a^b u_s dB_s$$

c. Let $u \in L^2 \infty(P)$, then the indefinite stochastic integral

$$M_t = \int_0^t u_s dB_s, t \geq 0$$

is a square integrable martingale w.r.t. the filtration \mathcal{F}_t and admits a continuous version.

Proof (a) is trivial. For (b), we only need to consider $u_t^{(n)}$ simple and converging to u_t in $L^2(P)$, and we have

$$\int_a^b F u_s^{(n)} dB_s = F \sum \phi_{t_j}(B_{t_{j+1}} - B_{t_j}) = F \int_a^b u_s^{(n)} dB_s$$

and we are done since F is bounded.

(c) For a simple process

$$u_t = \sum_{j=0}^{n-1} \phi_j \chi_{(t_j, t_{j+1}]}(t)$$

then we know

$$\begin{aligned} E\left(\int_0^t u_v dB_v \middle| \mathcal{F}_s\right) &= \sum_{j=0}^{n-1} E(\phi_j(B_{t_{j+1} \wedge t} - B_{t_j \wedge t}) \middle| \mathcal{F}_s) \\ &= \sum_{j=0}^{n-1} E(E(\phi_j(B_{t_{j+1} \wedge t} - B_{t_j \wedge t}) \middle| \mathcal{F}_{t_j \wedge s}) \middle| \mathcal{F}_s) \\ &= \int_0^s u_v dB_v \end{aligned}$$

and hence we know M_t is an \mathcal{F}_t -martingale if u is simple. For $T > 0$, let $u^{(n)}$ converges to u in $L_T^2(P)$, then we know $t \in [0, T]$, we have

$$\int_0^t u_s^{(n)} dB_s \rightarrow \int_0^t u_s dB_s$$

in $L^2(\Omega)$ and we know the convergence of the conditional expectations by $E(Z(X - E(X|\mathcal{F}))) = 0$ for any $Z \in L^2(\mathcal{F})$ and hence $\int_0^t u_s dB_s$ is a martingale.

To show that the indefinite integral has a continuous version. Let $u \in L^2$ and fix $T > 0$. Consider a sequence of simple processes $u^{(n)}$ which converges to u in $L_T^2(P)$, by the continuity of the paths of Brownian motion, we know

$$M_t^{(n)} = \int_0^t u_s^{(n)} dB_s$$

has continuous trajectories. Then since $M^{(n)}$ is a martingale, so we know

$$\begin{aligned} P\left(\sup_{0 \leq t \leq T} |M_t^{(n)} - M_t^{(m)}| > \lambda\right) &\leq \frac{1}{\lambda^2} E(|M_t^{(n)} - M_t^{(m)}|^2) \\ &= \frac{1}{\lambda^2} E\left(\int_0^T |u_t^{(n)} - u_t^{(m)}|^2\right) \rightarrow 0 \end{aligned}$$

as $n, m \rightarrow \infty$ and we may choose n_k such that

$$P\left(\sup_{0 \leq t \leq T} |M_t^{(n_{k+1})} - M_t^{(n_k)}| > 2^{-k}\right) \leq 2^{-k}$$

and we may know that for any $\omega \in \Omega$, there is $k_1(\omega)$ such that

$$\sup_{0 \leq t \leq T} |M_t^{(n_{k+1})}(\omega) - M_t^{(n_k)}(\omega)| \leq 2^{-k}$$

has probability of 1 and then we know $M_t^{(n_k)}(\omega)$ is uniformly convergent to a continuous function $J_t(\omega)$ a.s. and then we know $J_t(\omega) = \int_0^t u_s dB_s$ a.s. Then we may choose different t and construct a continuous version of $\int_0^t u_s dB_s$ inductively.

Proposition 1.20

For any $T, \lambda > 0$ and $u \in L^2_\infty(P)$, we know

$$P\left(\sup_{t \in [0, T]} |M_t| > \lambda\right) \leq \frac{1}{\lambda^2} E\left(\int_0^T u_t^2 dt\right)$$

and

$$E\left(\sup_{t \in [0, T]} |M_t|^2\right) \leq 4E\left(\int_0^T u_t^2 dt\right)$$

by Doob's inequality and L^p maximum inequalities.

**Proposition 1.21**

Let $u \in L^2_\infty(P)$. Consider a subdivision of the interval $[0, t]$

$$\pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$$

then as $|\pi| \rightarrow 0$, we have

$$S_\pi^2(u) = \sum_{j=0}^{n-1} \left(\int_{t_j}^{t_{j+1}} u_s dB_s \right)^2 \rightarrow \int_0^t u_s^2 ds$$

in $L^1(\Omega)$.



1.4 Derivative and Divergence Operators

Definition 1.20

For this chapter, we will consider the probability space (Ω, \mathcal{F}, P) such that $\Omega = \mathbb{R}^n$, $\mathcal{F} = \mathcal{B}(\mathbb{R}^n)$ to be the Borel σ -field of \mathbb{R}^n and P to be the standard Gaussian probability with density

$$p(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$$

and we define the derivative operator for differentiable function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right)$$

and the divergence operator for differentiable vector-valued functions $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$\delta(u) = \sum_{i=1}^n \left(u_i x_i - \frac{\partial u_i}{\partial x_i} \right) = \langle u, x \rangle - \text{div} u$$

**Proposition 1.22**

The operator δ is the adjoint of ∇ that is

$$E(\langle u, \nabla F \rangle) = E(F \delta(u))$$

for $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuously differentiable functions satisfying integral by parts.



Proof Since $\partial p / \partial x_i = -x_i p$ and we have

$$\begin{aligned} \int \langle \nabla F, u \rangle p dx &= \sum_{i=1}^n \int \frac{\partial F}{\partial x_i} u_i p dx \\ &= \sum_{i=1}^n \left(- \int F \frac{\partial u_i}{\partial x_i} p dx + \int F u_i x_i p dx \right) \\ &= \int F \delta(u) p dx \end{aligned}$$

Definition 1.21

Consider the Hilbert space $h \in H = L^2(\mathbb{R}^+)$ and the Wiener integral

$$B(h) = \int_0^\infty h(t)dB_t$$

and S the set of smooth and cylindrical random variables of the form

$$F = f(B(h_1), \dots, B(h_n))$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in H$.

**Definition 1.22**

For $F \in S$, we define the Malliavin derivative DF to be the H -valued random variable defined by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(h_1), \dots, B(h_n)) h_i(t)$$

which is well-defined and a linear but unbounded operator from S into $L^2(\Omega; H)$.

Let S_H to be the element $u = (u_t)$ with the form

$$u_t = \sum_{j=1}^n F_j h_j(t)$$

with $F_j \in S$ and $h_j \in H$. And the divergence of an element u in S_H will be given by

$$\delta(u) = \sum_{j=1}^n F_j B(h_j) - \sum_{j=1}^n \langle DF_j, h_j \rangle_H$$

**Proposition 1.23**

For $F \in S$ and $u \in S_H$, then

$$E(F\delta(u)) = E(\langle DF, u \rangle_H)$$



Proof Consider h_i, h_j orthonormal and then we may know that $B(h_i)$ are i.i.d., then for $F = f(B(h_1), \dots, B(h_n))$ and

$$u = \sum_{j=1}^n g_j(B(h_1), \dots, B(h_n)) h_j$$

and

$$\begin{aligned} E(\langle DF, u \rangle_H) &= \int \sum_{j=1}^n E(D_t F g_j(B(h_1), \dots, B(h_n)) h_j(t)) \\ &= \sum_{i,j=1}^n \int E\left(\frac{\partial f}{\partial x_i}(B(h_1), \dots, B(h_n)) g_j(B(h_1), \dots, B(h_n)) h_i(t) h_j(t)\right) \\ &= \sum_{i=1}^n E\left(\frac{\partial f}{\partial x_i}(B(h_1), \dots, B(h_n)) g_j(B(h_1), \dots, B(h_n))\right) \\ &= \sum_{j=1}^n E\left(F g_j(B(h_1), \dots, B(h_n)) B(h_j) - F \frac{\partial g_j}{\partial x_j}(B(h_1), \dots, B(h_n))\right) \\ &= E(F\delta(u)) \end{aligned}$$

since

$$\langle DF_j, h_j \rangle = \frac{\partial g_j}{\partial x_j}(B(h_1), \dots, B(h_n))$$

Proposition 1.24

Suppose that $u, v \in S_H$, $F \in S$ and $h \in H$. Then for a complete orthonormal system in H denoted as e_i , we have

$$E(\delta(u)\delta(v)) = E(\langle u, v \rangle_H) + E\left(\sum_{i,j=1}^{\infty} D_{e_i}\langle u, e_j \rangle_H D_{e_j}\langle v, e_i \rangle_H\right)$$

$$D_h(\delta(u)) = \delta(D_h(u)) + \langle h, u \rangle_H$$

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_H$$



Proof We know

$$\begin{aligned} D_h(\delta(u)) &= D_h\left(\sum_{j=1}^n F_j B(h_j) - \sum_{j=1}^n \langle DF_j, h_j \rangle_H\right) \\ &= \sum_{j=1}^n D_h F_j B(h_j) + \sum_{j=1}^n F_j \langle h, h_j \rangle_H - \sum_{j=1}^n D_h \left\langle \sum_{i=1}^n \frac{\partial F_j}{\partial x_i}(B(h_1), \dots, B(h_n)) h_i, h_j \right\rangle_H \\ &= \sum_{j=1}^n F_j \langle h, h_j \rangle_H + \sum_{j=1}^n \left(D_h F_j B(h_j) - \langle D_h(DF_j), h_j \rangle_H \right) \\ &= \langle u, h \rangle_H + \delta(D_h u) \end{aligned}$$

by Fubini's Theorem.

Then

$$\begin{aligned} E(\delta(u)\delta(v)) &= E\left(\langle v, D(\delta(u)) \rangle_H\right) \\ &= \sum_{n \geq 1} \left(\langle v, e_i \rangle_H D_{e_i}(\delta(u)) \right) \\ &= E(\langle u, v \rangle_H) + \sum_{i=1}^{\infty} E(\langle v, e_i \rangle_H \delta(D_{e_i} u)) \\ &= E(\langle u, v \rangle_H) + \sum_{i=1}^{\infty} E(\langle D\langle v, e_i \rangle_H, D_{e_i} u y u u \rangle_H) \end{aligned}$$