NOTES FOR RENORMALIZATION FLOW

Based on the paper by A.Dunlap and Cole

Author
Wells Guan

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1 Setup

1.1 Semilinear SHE

We consider the semilinear stochastic heat equation

$$du_t^\rho = \frac{1}{2}\Delta u_t^\rho dt + \gamma_\rho \sigma(u_t^\rho) dW_t^\rho, \quad t > 0, x \in \mathbb{R}^2$$

Here σ is a Lipschitz nonlinearity and $dW_t^\rho(x)$ is a Gaussian noise that is white in time and correlated in space at scale $\rho^{1/2} << 1$. We are interested in the pointwise behavior of $u_t^\rho(x)$ as $\rho \to 0$, which calls for an attenuation factor $\gamma_\rho \sim |\ln \rho|^{-1/2}$ due to critical scaling in two dimensions. In fact, we devote most of our attention to a variation on (**??**) in which we first multiply σ and then smooth the noise:

$$dv_t^\rho = \frac{1}{2}\Delta v_t^\rho dt + \gamma_\rho \mathcal{G}_\rho[\sigma(v_t^\rho) dW_t]$$

Definiton 1.1.1.

(Space-time White Noise)

Let $dW = (dW_t(x))_{t\in\mathbb{R}, x\in\mathbb{R}^2}$ be a standard $\mathbb{R}^m$-valued space-time white noise generating a temporal filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$. Writing $dW = (dW^1, \cdots, dW^m)$ in components, then

$$\mathbb{E}[dW_t^i(x) dW_{t'}^{i'}(x')] = \delta_{i,i'}\delta(t-t')\delta(x-x')$$

Proposition 1.1.1. Construct a space-time white noise.

Definiton 1.1.2.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a proability space and fix a target dimension $m \in \mathbb{N}$. The solution $v^\rho : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^m$ is a random vector-valued function parametrized by the correlastion parameter $\rho > 0$. We suppress the dependence of v^p on $\omega \in \Omega$.

Since v is vector-valued, our nonlinearity $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is matrix-valued. Let $\mathcal{H}_+^m$ denote the set of nonnegative-definite symmetric real $m \times m$ matrices, equipped with the metric induced by the Frobenius norm

$$|A|_F^2 := \mathrm{tr}(AA^T) = \mathrm{tr}(A^2)$$

Let σ belong to the space $\mathrm{Lip}(\mathbb{R}^m, \mathcal{H}_+^m)$.

Proposition 1.1.2. Show that $|\cdot|_F$ is a norm on $\mathcal{H}_+^m$.

Definiton 1.1.3. Given $\tau \geq 0$, we define the heat operator

$$\mathcal{G}_\tau v = G_\tau * v$$

where $G_\tau = (2\pi\tau)^{-1}\exp(-\frac{|x|^2}{2\tau})$ denotes the standard heat kernel. Define the spatially-smoothed noise $dW_t^\rho = G_\rho * dW_t$.

Proposition 1.1.3. We have

$$\mathbb{E}[dW_t^{\rho,i}(x) dW_{t'}^{\rho,i'}(x') = \delta_{i,i'}\delta(t-t')G_{2\rho}(x-x')$$

Proof. $\square$

Definiton 1.1.4. Define

$$L(\tau) = \ln(1+\tau) \quad \text{for } \tau \geq 0$$

and set

$$\gamma_\rho = \sqrt{\frac{4\pi}{L(1/\rho)}}$$

Definiton 1.1.5.

(Mild Solution 1)

A mild solution for (**??**) is a predictable random field v^ρ such that for all $s < t$, we have

$$v_t^\rho(x) = \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \mathcal{G}_{t+\rho-r}[\sigma(v_r^\rho) dW_r](x)$$

which means

$$\begin{aligned} v_t^\rho x &= \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \int G_{t+\rho-r}(y)\sigma(v_r^\rho)(x-y) dW_r(x-y) dy \\ &= \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \int G_{t+\rho-r}(y)\sigma(v_r^\rho)(x-y) dW_r(x-y) dy \\ &= \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int \left(\int_s^t G_{t+\rho-r}(y)\sigma(v_r^\rho)(x-y) dW_r(x-y) \right) dy \end{aligned}$$

which can be interpreted as an Ito integral. We only look for the solution v_t^ρ in the spaces $\mathscr{X}_t^l$ of $\mathbb{R}^m$-valued random fields z on $\mathbb{R}^2$ that are $\mathcal{F}_t$-measurable and

$$||z||_l := \sup_{x \in \mathbb{R}^2} (\mathbb{E}|z(x)|^l)^{1/l} < \infty$$

Proposition 1.1.4. For any $l \geq 2$, there is a family of random operators $(\mathcal{V}_{s,t}^{\sigma,\rho})_{s<t}$ such that if $v_s \in \mathscr{X}_s^l$, then $v_t^\rho = \mathcal{V}_{s,t}^{\sigma,\rho} v_s$ is a mild solution of (**??**) for $t \geq s$. We oftten write $\mathcal{V}_t^{\sigma,\rho} := \mathcal{V}_{0,t}^{\sigma,\rho}$.

Shown by some standard fixed-point arguments.

Definiton 1.1.6. (Forward-backward SDE)

The system of SDE:

$$\begin{aligned} d\Gamma_{a,Q}^\sigma(q) &= J_\sigma(Q-q, \Gamma_{a,Q}^\sigma(q)) dB(q), & a \in \mathbb{R}^m, 0 < q < Q \\ \Gamma_{a,Q}^\sigma(0) &= a, & a \in \mathbb{R}^m, Q \geq 0 \\ J_\sigma(q,b) &= [\mathbb{E}\sigma^2(\Gamma_{a,Q}^\sigma(q))]^{1/2}, & q \geq 0, b \in \mathbb{R}^m \end{aligned}$$

for B a standard $\mathbb{R}^m$-valued Brownian motion and $A^{1/2}$ is the unique positive-definite matrix square root of $A \in \mathcal{H}_+^m$.

Definiton 1.1.7. Given $\sigma \in \mathrm{Lip}(\mathbb{R}^m, \mathcal{H}_+^m)$, let $\overline{Q}_{\mathrm{FBSDE}}(\sigma) \in [0, \infty]$ be the supremum of all $Q \geq 0$ such that there is a continuous function $J_\sigma : [0, Q] \times \mathbb{R}^m \to \mathcal{H}_+^m$ satisfyting the FBSDE and

$$\sup_{q \in [0, \overline{Q}]} \mathrm{Lip}(J_\sigma(q, \cdot)) < \infty$$

and define for $M, \beta \in (0, \infty)$

$$\Sigma(M, \beta) := \{\sigma \in \mathrm{Lip}(\mathbb{R}^m, \mathcal{H}_+^m) : |\sigma(u)|_F^2 \le M + \beta^2 |u|^2 \text{ for all } u \in \mathbb{R}^m\}$$

Definiton 1.1.8.

(L^2-subcritical)

A nonlinearity σ is L^2-subcritical if $\sigma \in \Sigma(M, \beta)$ for some $M \in (0, \infty)$ and $\beta \in (0, 1)$ and $\overline{Q}_{\mathrm{FBSDE}}(\sigma) > 1$.

1.2 Main Result

Theorem 1.2.1. Let σ be L^2-subcritical. Fix $Q \in (0, 1)$ and define $\tilde{\sigma} := (1 - Q)^{1/2} J_\sigma(Q, \cdot)$ and $\widetilde{\rho} := \rho^{1-Q}$. Then there is a new white noise $d\widetilde{W}$ such that $\mathcal{G}_{\widetilde{\rho}v}$ is an approximate mild solution of (**??**) with $(\widetilde{\rho}, \widetilde{\sigma}, d\widetilde{W})$ in place of (ρ, σ, dW).

Definiton 1.2.1. Denote $\mathcal{W}_2$ the Wasserstein-2 metric: for any two probability distributions μ, ν

$$\mathcal{W}_2(\mu, \nu) = \inf_{\pi} \left(\int |x - y|^2 \pi(dx, dy) \right)$$

where π owns marginal distributions of μ and ν.

Denote $\langle a \rangle := (|a|^2 + 1)^{1/2}$ the Japanese bracket.

Theorem 1.2.2. For each L^2-subcritical σ and $\overline{T} \in [1, \infty)$, there is a constant $C(\sigma, \overline{T}) \in (e, \infty)$ such that for all $v_0 \in L^\infty(\mathbb{R}^2, \mathbb{R}^m), t \in [\overline{T}^{-1}, \overline{T}]$, and $\rho \in (0, C^1)$, the solution v^ρ of (**??**) satisfies

$$\mathcal{W}_2(v_t^{\rho(x)}, \Gamma_{a,1}^\sigma(1)) \le C \langle ||v_0||_{L^\infty} \rangle \sqrt{\frac{\ln |\ln \rho|}{|\ln \rho|}}$$

where $(\Gamma_{a,1}^\sigma(q))_{q \in [0,1]}$ solves the FBSDE with $Q = 1$ and $a = \mathcal{G}_t v_0(x)$.

1.3 Decoupling Flow

Definiton 1.3.1 (Parabolic Equation)**.**

The following parabolic equation

$$\begin{aligned} \partial_q H(q, b) &= \tfrac{1}{2} [H(q, b) : \nabla_b^2] H(q, b) \\ H(0, b) &= \sigma^2(b) \end{aligned}$$

here for A, B matrices, we denote

$$A : B = \mathrm{tr}[AB]$$

and there is an explicit formula

$$([H(q, b) : \nabla_b^2] H(q, b))_{ij} = \mathrm{tr}[H(q, b) \nabla_b^2] H_{ij}(q, b) = \sum_{k,l=1}^{m} H_{kl}(q, b) \frac{\partial^2 H_{ij}}{\partial b_k \partial b_l}(q, b)$$

Proof. We know

$$\mathrm{tr}[H(q,b)\nabla_b^2] = \sum_{k=1}^{n}\sum_{l=1}^{n} H_{kl}(q,b)\frac{\partial^2}{\partial b_k \partial b_l}$$

$\square$

2 Well-posedness of the FBSDE

2.1 Main Goal

The main result is

Theorem 2.1.1. Let $\sigma \in \mathrm{Lip}(\mathbb{R}^m, \mathcal{H}_+^m)$. For any $Q \in [0, \overline{Q}_{\mathrm{FBSDE}}(\sigma))$, we have

$$\overline{Q}_{\mathrm{FBSDE}}(\sigma) \geq Q + \mathrm{Lip}(J_\sigma(Q, \cdot))^{-2}$$

Definiton 2.1.1. $\mathcal{X}$ is the Banach space of $\mathcal{H}_+^m$-valued continuous functions on $\mathbb{R}^m$ with the norm (and this norm is finite)

$$||\sigma||_{\mathcal{X}} := \sup_{x \in \mathbb{R}^m} \frac{|\sigma(x)|_F}{\langle x \rangle}$$

Proposition 2.1.2. Prove $\mathcal{X}$ is a Banach space.

Proposition 2.1.3. $\mathrm{Lip}(\mathbb{R}^m; \mathcal{H}_+^m) \subset \mathcal{X}$.

2.2 SDE Solution Theory

Definiton 2.2.1. Given a $\mathbb{R}^m$-valued Brownian motion $(B(q))_{q \geq 0}$ adapted to a filtration $\{\mathscr{G}_q\}_{q \geq 0}$. For an adapted process Y on $[0, Q]$, a function $g : [0, Q] \times \mathbb{R}^m \to \mathcal{H}_+^m$ and a constant $a \in \mathbb{R}^m$, we define a new adapted process $\mathcal{R}_{a,Q}^g Y$ on $[0, Q]$ by

$$\mathcal{R}_{a,Q}^g Y(q) := a + \int_0^q g(Q - p, Y(p)) dB(q)$$

whenever this stochastic integral is defined. For $Q > 0$, define

$$\mathcal{A}_Q := \left\{ J : [0, Q] \times \mathbb{R}^m \to \mathcal{H}_+^m \text{ continuous} : \sup_{q \in [0,Q]} \mathrm{Lip}(J(q, \cdot)) < \infty \right\}$$

Proposition 2.2.1. Fix $L < \infty$ and $Q \in (0, \infty)$ and suppose that $g \in \mathcal{A}_Q$ satisfies

$$\sup_{q \in [0,Q]} \mathrm{Lip}(g(q, \cdot)) \leq L$$

Then, for any $a \in \mathbb{R}^m$, there is a unique strong solution $\Theta_{a,Q}^g$ to the SDE

$$\begin{aligned} d\Theta_{a,Q}^g(q) &= g(Q - q, \Theta_{a,Q}^g(q)) dB(q), \quad q \in [0, Q] \\ \Theta_{a,Q}^g(0) &= a \end{aligned}$$

The solution $\Theta_{a,Q}^g$ satisfies the moment bound

$$\sup_{q \in [0,Q]} \mathbb{E}|\Theta_{a,Q}^g(q)|^l < \infty \quad \text{for all } l \in [1, \infty)$$

Moreover, there exists a constant $C = C(L, Q)$ such that for any $Q' \in [0, Q]$ and any adapted process Γ on $[0,Q']$, we have

$$\sup_{q\in[0,Q']}\mathbb{E}|\Gamma(q)-\Theta^g_{a,Q}(q)|^2 \le C\sup_{[0,Q']}\mathbb{E}|\Gamma(q)-\mathcal{R}^g_{a,Q}\Gamma(q)|^2$$

and

$$\mathbb{E}|\Theta^g_{a,Q}(q)-\Theta^g_{\widetilde{a},Q}(q)|^2 \le C|a-\widetilde{a}|^2$$

Proof. Define $\mathcal{K}_{L,Q'}$ on adapted processes on $[0,Q']$ by

$$\sup_{q\in[0,Q']} e^{-L^2 q}\left(\mathbb{E}|\Gamma(q)|^2\right)^{1/2},$$

which makes $\mathcal{R}^g_{a,Q}$ a contraction. □

2.3 Local Solution

Definiton 2.3.1. Let $Q>0$ and let $\sigma\in\mathrm{Lip}(\mathbb{R}^m;\mathcal{H}^m_+)$. We say that $J\in\mathcal{A}_Q$ is a *root decoupling function* for FBSDE on $[0,Q]$ if, for all $q\in[0,Q]$ and all $a\in\mathbb{R}^m$, we have

$$J(q,a)=[\mathbb{E}\sigma(\Theta^J_{a,q}(q))]^{1/2}$$

In this case, we also call J^2 the decoupling function. In the equation above, $\Theta^J_{a,q}$ is as in (**??**).

Definiton 2.3.2. For $\lambda\in(0,\infty)$, we define the set of functions

$$\Lambda(\lambda):=\{\sigma\in\mathrm{Lip}(\mathbb{R}^m;\mathcal{H}^m_+):\mathrm{Lip}(\sigma)\le\lambda\}$$

Proposition 2.3.1. Suppose that $\lambda\in(0,\infty),\sigma\in\Lambda(\lambda),Q_0>0$ and that J is a root decoupling function for FBSDE on $[0,Q_0]$. Then for all $Q\in[0,Q_0\wedge\lambda^{-2})$, we have

$$\mathrm{Lip}(J(Q,\cdot))\le(\lambda^{-2}-Q)^{-1/2}.$$

Proof. □

Lemma 2.3.2. Suppose that $c\in(0,\infty),\overline{Q}<c^{-2}$ and $f:[0,\overline{Q}]\to[c^2,\infty)$ satisfies

$$f(Q)\le c^2\exp\left\{\int_0^Q f(q)dq\right\}$$

for all $Q\in[0,\overline{Q}]$. Then

$$f(Q)\le(c^{-2}-Q)^{-1}\quad\text{for all } Q\in[0,\overline{Q}]$$

Proof. Define $g(Q)=\int_0^Q f(q)dq$ and we have

$$g'(q)=f(q),$$

then

$$(1-e^{-g(q)})'=g'(q)e^{-g(q)}=f(q)e^{-g(q)}$$

and hence

$$1 - e^{-g(Q)} = \int_0^Q f(q)e^{-g(q)}dq \le c^2 Q$$

Therefore, we have

$$f(Q) \le c^2 e^{g(Q)} \le \frac{c^2}{1 - c^2 Q}$$

Here is another proof by (**??**), consider h defined on $[0, \overline{Q}]$ non-negative defined by

$$h(t) = \ln(f(t)/c^2),$$

then we have

$$h(t) \le \int_0^t c^2 \exp(h(s))ds.$$

Define

$$G(x) = -e^{-x} + C$$

and we have

$$h(q) \le -\ln(1 - c^2 q)$$

for any $q \in [0, \overline{Q}]$, which means

$$f(Q) \le c^2/(1 - c^2 Q) = (c^{-2} - Q)^{-1}.$$

□

Definiton 2.3.3. Define

$$\mathcal{Q}_\sigma g(Q, a) = [\mathbb{E}\sigma^2(\Theta^g_{a,Q}(Q))]^{1/2},$$

where $(\Theta^g_{a,Q}(q))_{q\in[0,Q]}$ from (**??**). We note that a fixed point of Q_σ is a root decoupling function for (**??**). We also note that

$$|\mathcal{Q}_\sigma g(Q, a)|_F^2 = |\mathbb{E}\sigma^2(\Theta^g_{a,Q}(Q))]^{1/2}|_F^2 = \mathbb{E}\mathrm{tr}\sigma^2(\Theta^g_{a,Q}(Q)) = \mathbb{E}|\sigma(\Theta^g_{a,Q}(Q))|_F^2$$

Define the set of functions

$$\Lambda(M, \lambda) := \{\sigma \in \mathrm{Lip}(\mathbb{R}^m; \mathcal{H}^m_+) : \mathrm{Lip}(\sigma) \le \lambda \text{ and } |\sigma(u)|_F^2 \le M + \lambda^2|u|^2 \text{ for all } u \in \mathbb{R}^m\}$$

For $Q_0 < \lambda^{-2}$, define the set of functions $\mathcal{Z}_{Q_0,M,\lambda}$ by

$$\mathcal{Z}_{Q_0,M,\lambda} = \big\{g : [0, Q_0] \times \mathbb{R}^m \to \mathcal{H}^m_+ \text{ continuous,} \\ g(q, \cdot) \in \Lambda((1 - \lambda^2 q)^{-2}M, (\lambda^{-2} - q)^{-1/2}) \text{ for all } q \in [0, Q_0]\big\}$$

We will construct the root decoupling function J_σ as a fixed point of Q_σ in a certain $\mathcal{Z}_{Q_0,M,\lambda}$.

Proposition 2.3.3. Fix $\lambda, M \in (0, \infty)$ and $Q_0 \in [0, \lambda^{-2})$. For any $\sigma \in \Lambda(M, \lambda)$, there is a unique root decoupling function $J_\sigma \in \mathcal{A}_{Q_0}$. In particular, we have

$$\overline{Q}_{\mathrm{FBSDE}}(\sigma) \ge \mathrm{Lip}(\sigma)^{-2}.$$

Moreover, there is a $C = C(Q_0, M, \lambda) < \infty$ such that for any $g \in \mathcal{Z}_{Q_0,M,\lambda}$, we have

$$\sup_{q\in[0,Q_0]} ||(g - J_\sigma)(q,\cdot)||_{\mathcal{X}} \le C \sup_{q\in[0,Q_0]} ||(g - \mathcal{Q}_\sigma g)(q,\cdot)||_{\mathcal{X}}$$

and indeed

$$\lim_{n\to\infty} \sup_{q\in[0,Q_0]} ||(\mathcal{Q}_\sigma^n g - J_\sigma)(q,\cdot)||_{\mathcal{X}} = 0$$

where Q_σ^n denotes the n-fold iterated application of Q_σ.

Proof. We know any decoupling function $J \in \mathcal{A}_{Q_0}$ has to be in $\mathcal{Z}_{Q_0,M,\lambda}$ for some $M', \lambda \in (0,\infty)$ $\square$

2.4 Extension of the Solution

Lemma 2.4.1. Let $\sigma \in \mathrm{Lip}(\mathbb{R}^m; \mathcal{H}_+^m)$, whenever $0 \le Q_1 \le Q_2 < \overline{Q}_{\mathrm{FBSDE}}(\sigma)$, we have for any $b \in \mathbb{R}^m$,

$$J_\sigma^2(Q_2, b) = \mathbb{E}[J_\sigma^2(Q_1, \Gamma_{b,Q_2}^\sigma(Q_2 - Q_1))]$$

Proof. Remain. $\square$

Lemma 2.4.2. Let $\sigma \in \mathrm{Lip}(\mathbb{R}^m; \mathcal{H}_+^m)$, whenever $0 \le Q_1 \le Q_2 < \overline{Q}_{\mathrm{FBSDE}}(\sigma)$ and if

$$Q_2 - Q_1 < \overline{Q}_{\mathrm{FBSDE}}(J_\sigma(Q_1,\cdot)),$$

we have

$$J_\sigma(Q_2, b) = J_{J_\sigma(Q_1,\cdot)}(Q_2 - Q_1, b) \quad \text{for all } b \in \mathbb{R}^m.$$

Proof. We have by (**??**) that,for any $Q \in [Q_1, \overline{Q}_{\mathrm{FBSDE}}(\sigma))$ and any $b \in \mathbb{R}^m$,

$$J_\sigma^2(Q, b) = \mathbb{E}[J_\sigma^2(Q_1, \Gamma_{b,Q}^\sigma(Q - Q_1))]$$

By assumption, there is a unique solution to the following FBSDE problem for $Q \in [0, Q_2 - Q_1]$:

$$\begin{aligned}
d\Gamma_{b,Q}^{J_\sigma(Q_1,\cdot)}(q) &= J_{J_\sigma(Q_1,\cdot)}(Q - q, \Gamma_{b,Q}^{J_\sigma(Q_1,\cdot)}(q))dB(q), \quad q \in (0, Q)\\
\Gamma_{b,Q}^{J_\sigma(Q_1,\cdot)}(0) &= b\\
J_{J_\sigma(Q_1,\cdot)}(Q, b) &= (\mathbb{E}[J_\sigma^2(Q_1, \Gamma_{b,Q}^{J_\sigma(Q_1,\cdot)}(Q))])^{1/2}
\end{aligned}$$

Also, if given $Q \in [0, \overline{Q}_{\mathrm{FBSDE}}(\sigma) - Q_1)$, (**??**) and (**??**) with Q replaced by $Q_1 + Q$ will yield

$$\begin{aligned}
d\Gamma_{b,Q+Q_1}^\sigma(q) &= J_\sigma(Q + Q_1 - q, \Gamma_{b,Q+Q_1}^\sigma(q))dB(q), \quad q \in (0, Q)\\
\Gamma_{b,Q+Q_1}^\sigma(0) &= b\\
J_\sigma(Q + Q_1, b) &= (\mathbb{E}[\sigma^2(\Gamma_{b,Q+Q_1}^\sigma(Q))])^{1/2} = (\mathbb{E}[J_\sigma^2(Q_1, \Gamma_{b,Q+Q_1}^\sigma(Q))])^{1/2}
\end{aligned}$$

which means ($\Gamma_{b,Q+Q_1}^\sigma(q)$ and $J_\sigma(Q + Q_1, b)$) will solve the previous FBSDE system, and by the uniqueness of FBSDE problem, we have

$$J_{J_\sigma(Q_1,\cdot)}(Q, b) = J_\sigma(Q + Q_1, b) \quad \text{for all } b \in \mathbb{R}^m \text{ and } Q \in [0, Q_2 - Q_1]$$

$\square$

Proposition 2.4.3. For any $Q' \in [0, \overline{Q}_{\mathrm{FBSDE}}(\sigma))$, we have

$$\overline{Q}_{\mathrm{FBSDE}}(\sigma) \geq Q' + \overline{Q}_{\mathrm{FBSDE}}(J_\sigma(Q', \cdot))$$

Proof. For $Q' < \overline{Q}_{\mathrm{FBSDE}}\sigma$, there is a unique root decoupling function $J_\sigma \in \mathcal{A}_{Q'}$ for (**??**) on $[0, Q']$. Let $P \in [0, \overline{Q}_{\mathrm{FBSDE}}(J_\sigma(Q', \cdot)))$, there is a unique RDF $J_{J_\sigma(Q', \cdot)} \in \mathcal{A}_P$ in (**??**) with σ replaced by $J_\sigma(Q', \cdot)$.

We wish to extend the function J_σ to the time interval by outting $\square$

3 Parabolic Equation for the decoupling function

3.1 Main Theorem

Remark. The space

$$L^1(\mathcal{K}; W^{2,\infty}(U; \mathcal{H}_+^m))$$

means some functions which are L^1 with fixed space point and time varying and $W^{2,\infty}(U : \mathcal{H}_+^m)$ with fixed time point and space varying, with the norm

$$||f|| = \int ||f(t, \cdot)||_{W^{2,\infty}} dt.$$

For two normed spaces, their intersection will be equipped with a norm obtained by adding both the norms together.

Definiton 3.1.1. An *almost classical solution* to (**??**) on a time interval $[0, Q_0)$ is a continuous function $H : [0, Q_0) \times \mathbb{R}^m \to \mathcal{H}_+^m$ such that the following conditions hold:

1. For every compact $\mathcal{K} \subset (0, Q_0)$ and bounded open $U \subset \mathbb{R}^m$,

   $$H_{\mathcal{K}\times U} \in L^1(\mathcal{K}; W^{2,\infty}(U; \mathcal{H}_+^m)) \cap C^1(\mathcal{K} \times U; \mathcal{H}_+^m).$$

   where $W^{2,\infty}(U; \mathcal{H}_+^m)$ is the Sobolev space of functions on U taking values in $\mathcal{H}_+^m$ with weak second derivative mearuable ans essentially bounded.

2. We have $H(0, b) = \sigma^2(b)$ for all $b \in \mathbb{R}^m$;.

3. We have

   $$\partial_q H(q, b) = \tfrac{1}{2}[H(q, b) : \nabla_b^2]H(q, b) \quad \text{for almost all } (q, b) \in (0, Q_0) \times \mathbb{R}^m.$$

Proposition 3.1.1. Fix $Q_0 \in (0, \overline{Q}_{\mathrm{FBSDE}}(\sigma))$. Suppose H is an almost classical solution of (**??**) on $[0, Q_0]$ such that $\sup_{q\in[0,Q_0]} \mathrm{Lip}(\sqrt{H(q, \cdot)}) < \infty$. Also assume that for each compact $\mathcal{K} \subset (0, Q_0)$, there exists a constant $C(\mathcal{K}) \subset (0, Q_0)$, there exists a constant $C(\mathcal{K})$ such that for all $R > 0, U_R = \{b : |b| < R\}$ the open call in $\mathbb{R}^m$ of radius R,

$$||H|_{\mathcal{K}\times U_R}||_{L^1(\mathcal{K};W^{2,\infty}(U_R;\mathcal{H}_+^m))\cap C^1(\mathcal{K}\times U_R;\mathcal{H}_+^m)} \le CR^C$$

then $\sqrt{H} = J_\sigma$ on $[0, Q_0] \times \mathbb{R}^m$.

4 Open Problem

One possible question is that if σ is bounded, then in term of the solution u_ϵ of the stochastic pde

$$\partial_t u_\epsilon = \frac{1}{2}\Delta u_\epsilon + \sigma(u) dW^\epsilon,$$

what will u_ϵ converges to as $\epsilon \to 0$.

A Fundamentals

A.1 Wiener Integral

Let T be a set and $X := \{X(t)\}_{t\in T}$ a T-indexed stochastic process. We recall that X is a Gaussian random field (process when $T \subset \mathbb{R}$) if $(X_{t_1}, \cdots, X_{t_m})$ is a Gaussian random vector for all $t_1, \cdots, t_m \in T$.

Definiton A.1.1. Let $\mathscr{L}(\mathbb{R}^m)$ denote the collection of all Borel-measurable subsets of $\mathbb{R}^m$ that have finite Lebesgue measure. White noise on $\mathbb{R}^m$ is a mean-zero set-indexed Gaussian random field $\xi(A)_{A\in\mathscr{L}(\mathbb{R}^m)}$ with covariance function

$$E[\xi(A_1)\xi(A_2)] := |A_1 \cap A_2| \quad \text{for all } A_1, A_2 \in \mathscr{L}(\mathbb{R}^m),$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^m$ for every m.

A.2 Useful Approximations

Proposition A.2.1 (Bihari-LaSalle Inequality)**.**

Let u and f be non-negative continuous functions defined on the half-infinite ray $[0,\infty)$ and let w be a continuous non-decreasing function defined on $[0,\infty)$ and $w(u) > 0$ on $(0,\infty)$. For u we have

$$u(t) \leq \alpha + \int_0^t f(s)w(u(s))ds, \quad t \in [0,\infty),$$

where α is a non-negative constant, then

$$u(t) \leq G^{-1}\left(G(\alpha) + \int_0^t f(s)ds\right), \quad t \in [0,T]$$

where the function G is defined by

$$G(x) = \int_{x_0}^x \frac{dy}{w(y)}, \quad x \geq 0, x_0 \geq 0,$$

and G^{-1} is the inverse function of G and T is chosen so that

$$G(\alpha) + \int_0^t f(s)ds \in \mathrm{Dom}(G^{-1}), \quad \forall t \in [0,T]$$

Proof. Notice $G(x) = \int_{x_0}^x \frac{dy}{w(y)}$ is increasing and

$$\mathrm{Dom}(G^{-1}) = \left[G(0) = -\int_0^{x_0} \frac{dy}{w(y)}, G(0) + \int_0^\infty \frac{dy}{w(y)}\right]$$

then since $u \geq 0$, then we have

$$G(u(t)) \leq G\left(\alpha + \int_0^t f(s)\omega(u(s))ds\right) := H(t)$$

and we have

$$H'(t) = \frac{f(t)\omega(u(t))}{\omega(\alpha + \int_0^t f(s)\omega(u(s))ds)} \leq f(t)$$

so we have

$$H(t) \leq G(\alpha) + \int_0^t f(s)ds.$$

Therefore, we may have the requested inequality for all $0 \leq t \leq T$ where

$$T := \sup_t \left\{ \int_0^t f(s)ds \leq G(0) - G(\alpha) + \int_0^\infty \frac{dy}{\omega(y)} \right\}$$

□

A.3 Matrix Analysis

Theorem A.3.1 (Spectral Theorem for Real Symmetric Matrices)**.**
Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix, that is, $A^T = A$. Then:

1. All eigenvalues of A are real.

2. Eigenspaces corresponding to distinct eigenvalues are orthogonal.

3. The space $\mathbb{R}^n$ decomposes as an orthogonal direct sum of eigenspaces:

$$\mathbb{R}^n = \bigoplus_{\lambda\in\sigma(A)} \ker(A - \lambda I).$$

4. Equivalently, there exists an orthogonal matrix Q and a real diagonal matrix Λ such that

$$A = Q\Lambda Q^T.$$