NOTES FOR STOCHASTIC ANALYSIS

Based on the notes provided by Timo Seppalainen

Author
Wells Guan

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1 Stochastic Process

1.1 Path Spaces

Definiton 1.1.1. (Coordinate Variables and Shift maps) On the path space D, the **coordinate variables** are defined by $X_t(\omega) = \omega(t)$ for $\omega \in D$ and it can generate the natrual filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$.

The **shift maps** $\theta_s : D \to D$ are defined by $(\theta_s w)(t) = \omega(s+t)$, for an event $A \in \mathcal{B}_D$, the inverse image

$$\theta_s^{-1}(A) = \theta_s \omega \in A$$

Definiton 1.1.2. (Markov Process)

An $\mathbb{R}^d$-valued Markov process is a collection $\{P^x, x \in \mathbb{R}^d\}$ of probability measures on D such that

- $P^x(X_0 = x) = 1$.
- For each $A \in \mathcal{B}_D$, $x \mapsto P^x(A)$ is measurable on $\mathbb{R}^d$.
- $P^x[\theta_t^{-1} A | \mathcal{F}_t](\omega) = P^{X_t(\omega)}(A)$ for P^x-almost every ω, any x, A.

1.2 Brownian Motion

Definiton 1.2.1. (Brownian Motion)

For a probability space $(\Omega, \mathcal{F}, P)$, let $\mathcal{F}_t$ be a filtration and $B = \{B_t, 0 \leq t < \infty\}$ an adapted real-valued stochastic process. Then B is a one-dimensional **Brownian motion** w.r.t. $\{\mathcal{F}_t\}$ if

- $t \mapsto B_t(\omega)$ is continuous for a.s. ω.
- For $0 \leq s < t$, $B_t - B_s$ is independent of $\mathcal{F}_s$ and has normal distribution with mean zero and variance $t - s$.
- If $B_0 = 0$ a.s., then call B a standard BM.

Proposition 1.2.1. The second property is equivalent with

$$E[Zh(B_t - B_s)] = E(Z) \cdot \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} h(x) \exp\left\{-\frac{x^2}{2(t-s)}\right\} dx$$

for some any bounded r.v. $Z \in \mathcal{F}_s$ and bounded borel function h.

Proof.

We know for any $h = \chi_B, Z = \chi_D$, B is Borel and $D \in \mathcal{F}_s$, the equality holds. Then we may use the DCT to obtain the conclusion.

Proposition 1.2.2. The Brownian motion has stationary, independent increments.

Definiton 1.2.2. (Multi-dimensional BM)

A d-dimensional standard Brownian motion is an $\mathbb{R}^d$-valued process $B_t = (B_t^1, \cdots, B_t^d)$ with the property that each component B_t^i is a one-dimensional standard Brownian motion and coordinates $B_1, B_2, \cdots, B_d$ are independent.

Theorem 1.2.3. There exists a Borel probability measure P^0 on the path space $C = C_{\mathbb{R}}[0,\infty)$, which is metrized by

$$r(\eta,\xi) = \sum_{k=1}^{\infty} 2^{-k}(1 \wedge \sup_{0\le t\le k} |\eta - \xi|)$$

such that B the coordinate process on $(C, \mathcal{B}_C, P^0)$ is a standard one-dimensional Brownian motion w.r.t. $\{\mathcal{F}_t^B\}$.

Proposition 1.2.4. Suppose $B = \{B_t\}$ is a Brownian motion w.r.t. $\{\mathcal{F}_t\}$ on $(\Omega, \mathcal{F}, P)$. Then B_t and $B_t^2 - t$ are martingales w.r.t. $\mathcal{F}_t$.

Proof.

We know for $0 \le s < t$,

$$E(B_t|\mathcal{F}_s) = E(B_t - B_s + B_s|\mathcal{F}_s) = B_s$$

and

$$E(B_t^2 - t|\mathcal{F}_s) = E((B_t - B_s)^2 - B_s^2 + 2B_tB_s - t|\mathcal{F}_s) = B_s^2 - s$$

Proposition 1.2.5. Suppose $B = \{B_t\}$ is a Brownian motion w.r.t. $\{\mathcal{F}_t\}$ on $(\Omega, \mathcal{F}, P)$.

- We can assume that $\mathcal{F}_t$ contains every set A for which there exists $N \in \mathcal{F}$ such that $A \subset N$ and $P(N) = 0$. Furthermore, $B = \{B_t\}$ is also a Brownian motion w.r.t. $\{\mathcal{F}_{t+}\}$.

- Fix $s \in \mathbb{R}_+$ and define $Y_t = B_{s+t} - B_s$, then Y is independent of $\mathcal{F}_{s+}$ and it is a standard Brownian motion w.r.t. $\mathcal{G} = \{\mathcal{G}_t = \mathcal{F}_{(s+t)+}\}$.

Proof.

It remains to check that $B_t - B_s$ is independence of $\overline{F}_s$. For any $G \in \overline{F}_s$, there exists A such that $P(A \triangle G) = 0$ and hence $P(GB) = P(AB) = P(A)P(B) = P(G)P(B)$ for any $B \in \sigma(B_t - B_s)$ and we are done.

For any $Z \in \mathcal{F}_{s+}$ bounded and h Borel bounded, we have for any $0 \le s < s' < t$,

$$E[Zh(B_t - B_s')] = E(Z) \cdot \frac{1}{\sqrt{2\pi(t-s')}} \int h(x) \exp\left\{-\frac{x^2}{2(t-s')}\right\} dx$$

and use DCT to let $s' \to s$ and we are done.

The rest part is easy to be checked.

Lemma 1.2.6. Suppose X is a right-continuous process adapted to a filtration $\{\mathcal{F}_t\}$ and for all $s < t$ the increment $X_t - X_s$ is independent of $\mathcal{F}_s$, then $X_t - X_s$ is independent of $\overline{\mathcal{F}}_{s+}$.

Definiton 1.2.3. (Path Space)

In the following part, we will consider the path space C and the coordinate process $B_t(\omega) = \omega(t)$ and the filtration $\mathcal{F}_t^B$ generated by B. For any x there exists a probability measure P^x such that B is a Brownian motion started at x. Expectation under P^x is denoted by E^x.

Proof.

We know $\omega \mapsto x + \omega$ is a homeomorphism on C and hence we may define $P^x(A) = P^0(-x + A)$ and then $P^x(B_0 = x) = P^0(-x + \{B_0 = x\}) = P^0(\{B_0 = 0\}) = 1$. The rest is similar to be checked.

Definiton 1.2.4. (Shift map)

The shift maps $\{\theta_s : 0 \leq s < \infty\}$ defined by $(\theta_x \omega)(t) = \omega(s + t)$, the shift acts on B is defined by $\theta_s B = \{B_{s+t}, t \geq 0\}$.

Proposition 1.2.7. Let H be a bounded $\mathcal{B}_C$-measurable function on C.

- $E^x[H]$ is a Borel measurable function of x.
- For each $x \in \mathbb{R}$

$$E^x[H \circ \theta_s | \mathcal{F}^B_{s+}](\omega) = E^{B_s(\omega)}[H] \quad \text{for } P^x - \text{ almost every } \omega$$

In particular, $\{P^x\}$ on C is a Markov process w.r.t. $\mathcal{F}^B_t$.

2 Martingales

2.1 Basic Conclusions

Proposition 2.1.1.

- If M is a martingale and ϕ is a convex function such that $\phi(M_t)$ is integrable for all $t \geq 0$, then $\phi(M_t)$ is a submartingale.
- If M is a submartingable and ϕ a decreassing convex function such that $\phi(M_t)$ is integrable for all $t \geq 0$, then $\phi(M_t)$ is a submartingale.

Proof.

We only need to consider $S_\phi = \{l(x) = ax + b, l(x) \leq \phi(x) \text{ for any } x\}$ and $\phi(x) = \sup_{l \in S_\phi} l(x)$, then

$$E(\phi(M_t)|\mathcal{F}_s) = E(\sup l(M_t)|\mathcal{F}_s) \geq \sup l(E(M_t|\mathcal{F}_s)) = \phi(M_s)$$

and for M is submartingale, we have

$$E(\phi(M_t)|\mathcal{F}_s) = E(\sup l(M_t)|\mathcal{F}_s) \geq \sup l(E(M_t|\mathcal{F}_s)) = \phi(E(M_t|\mathcal{F}_s)) \geq \phi(M_s)$$

Definiton 2.1.1. (Uniformly Integrable)

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of random variables, call them **uniformly integrable** if

$$\lim_{M \to \infty} \sup_{\alpha \in A} E[|X_\alpha|; |X_\alpha| \geq M] = 0$$

Lemma 2.1.2. Let X be an integrable random variable, then $\{E(X|\mathcal{A})\}_{\mathcal{A} \subset \mathcal{F} \text{ sub-}\sigma\text{-algebra}}$ is uniformly integrable.

Proof.

Recall that if X is integrable, we may know that $E(|X|;|X| \geq M) \to 0$ with $M \to \infty$, and $P(|X| \geq M) \to 0$, so for any $\epsilon > 0$, we may choose M such that $E(|X|;|X| \geq M) < \epsilon/2$ and then $\delta = \epsilon/2M$ will satisfy that for any $A, P(A) < \delta$, $E(|X|;A) < \epsilon$.

With the fact above, we know that since

$$|E(X|\mathcal{A})| \leq E(|X||\mathcal{A})$$

then

$$P(|E(X|\mathcal{A})| \geq M) \leq M^{-1}E(|E(X|\mathcal{A})|) \leq M^{-1}E|X|$$

and for any $\epsilon > 0$, let M such that $M^{-1}E|X| < \delta$, then

$$E(|E(X|\mathcal{A})|;|E(X|\mathcal{A})| \geq M) \leq E(|X|;|E(X|\mathcal{A})| \geq M) < \epsilon$$

and we are done.

Lemma 2.1.3. Suppose $X_n \to X$ in L^1, for any sub-σ-algebra $\mathcal{A} \subset \mathcal{F}$, there exists $\{n_j\}$ such that $E[X_{n_j}|\mathcal{A}] \to E[X|\mathcal{A}]$ a.s.

Proof.

It suffices to show that $E(X_n|\mathcal{A}) \to E(X|\mathcal{A})$ in L^1, which is because

$$E(|E(X_n|\mathcal{A}) - E(X|\mathcal{A})|) = E(|E(X_n - X|\mathcal{A})|) \leq E|X_n - X|$$

and we are done.

Proposition 2.1.4. Suppose M is a right-continuous submartingale w.r.t. a filtration $\{\mathcal{F}_t\}$, then M is a submartingale also w.r.t. $\{\mathcal{F}_{t+}\}$.

Proof.

To show this conclusion, we shall consider $M \vee c$ is a submartingale and then

$$E[M_t \vee c|\mathcal{F}_{s+}] \geq E[M_{s+n^{-1}} \vee c|\mathcal{F}_{s+}]$$

for some n, since $M_{s+n^{-1}} \vee c \to M_s \vee c$, then by lemma 2.1.2, we will have this is also a convergence in L^1 and hence we will find a subsequence such that $E(M_{s+n_j^{-1}} \vee c|\mathcal{F}_{s+}) \to E(M_s \vee c|\mathcal{F}_{s+})$ and we have

$$E[M_t \vee c|\mathcal{F}_{s+}] \geq M_s$$

and let $c \to \infty$, we are done.

Proposition 2.1.5. Suppose the filtration $\{\mathcal{F}_t\}$ satisfies the usual events, in other words $\mathcal{F}$ is complete and $\mathcal{F}_t = \mathcal{F}_{t+}$. Let M be a submartingale, such that $t \to EM_t$ is right-continuous. Then there exists a cadlag modification of M that is an $\mathcal{F}_t$-submartingale.

2.2 Optimal Stopping

Lemma 2.2.1. Let M be a submaringale. Let σ, τ two stopping times whose values lie in an ordered countable set $\{s_1 < s_2 < s_3 < \cdots\} \cup \{\infty\}$ where s_j increases to ∞. Then for any $T < \infty$,

$$E[M_{\tau\wedge T}\mathcal{F}_\sigma] \geq M_{\sigma\wedge\tau\wedge T}$$

Proof.