

Chapter 1

Fundamental Concepts

Definition 1.1

If $U \subset \mathbb{R}^2$ is open and $f : U \rightarrow \mathbb{R}$ is a continuous function, then f is called C^1 on U if $\partial f / \partial x, \partial f / \partial y$ exist and are continuous on U .



Definition 1.2

We define for $f = u + iv : U \rightarrow \mathbb{C}$ a C_1 function

$$\begin{aligned}\frac{\partial}{\partial z} f &:= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f \\ \frac{\partial}{\partial \bar{z}} f &:= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f\end{aligned}$$

which is easy to be checked linear and the chain rules.



where we may check let $z = x + iy, \bar{z} = x - iy$, we have

$$\begin{aligned}\frac{\partial}{\partial z} z &= 1, & \frac{\partial}{\partial z} \bar{z} &= 0 \\ \frac{\partial}{\partial \bar{z}} z &= 0, & \frac{\partial}{\partial \bar{z}} \bar{z} &= 1\end{aligned}$$

Proposition 1.1

(The Leibniz Rules) We have for any $F, G \in C^1$

$$\begin{aligned}\frac{\partial}{\partial z} (F \cdot G) &= \frac{\partial F}{\partial z} \cdot G + F \cdot \frac{\partial G}{\partial z} \\ \frac{\partial}{\partial \bar{z}} (F \cdot G) &= \frac{\partial F}{\partial \bar{z}} \cdot G + F \cdot \frac{\partial G}{\partial \bar{z}}\end{aligned}$$



Proposition 1.2

We have for $l \leq j, m \leq k$ nonnegative integers and then

$$\left(\frac{\partial^l}{\partial z^l} \right) \left(\frac{\partial^m}{\partial \bar{z}^m} \right) (z^j \bar{z}^k) = \frac{j!}{l!} \frac{k!}{m!} z^{j-l} \bar{z}^{k-m}$$



Proposition 1.3

If $p(z, \bar{z}) = \sum a_{lm} z^l \bar{z}^m$ is a polynomial, then p contains no term with $m > 0$ iff $\frac{\partial p}{\partial \bar{z}} \equiv 0$.



Corollary 1.1

If $p(z, \bar{z}) = qz, \bar{z}$ are polynomials, then they have same coefficients.



Definition 1.3

A C_1 function $f : U \rightarrow \mathbb{C}$ is said to be holomorphic if

$$\frac{\partial f}{\partial \bar{z}} = 0$$


at every point of U .



Definition 1.4

A C^1 function $f = u(x, y) + iv(x, y) : U \rightarrow \mathbb{C}$ is holomorphic if

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

at every point of U , which is called the Cauchy-Riemann equations. 

Proposition 1.4

If $f : U \rightarrow \mathbb{C}$ is C^1 and if f satisfies the C-R equations, then

$$\frac{\partial}{\partial z} f = \frac{\partial}{\partial x} f = -i \frac{\partial}{\partial y} f$$

on U . 

Proof

We have

$$\begin{aligned} \frac{\partial}{\partial x} f &= \frac{\partial}{\partial x} u + i \frac{\partial}{\partial x} v = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) u = 2 \frac{\partial}{\partial z} u \\ \frac{\partial}{\partial x} f &= \frac{\partial}{\partial x} u + i \frac{\partial}{\partial x} v = i \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) v = 2 \frac{\partial}{\partial z} iv \\ -i \frac{\partial}{\partial y} f &= -i \frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) u = 2 \frac{\partial}{\partial z} u \\ -i \frac{\partial}{\partial y} f &= -i \frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v = i \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) v = 2 \frac{\partial}{\partial z} iv \end{aligned}$$

on U .

Definition 1.5

If $U \subset \mathbb{C}$ is open and $u \in C^2(U)$, then u is called harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where we also denote it as


$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

where the operator is called the Laplace operator. 

Here we have

$$4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} u = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} u = \Delta u$$

Proposition 1.5

The real and imaginary parts of a holomorphic C^2 function are harmonic. 

Proof


Assume $f = u + iv$ and then according to C-R equations, we have

$$\frac{\partial^2}{\partial x^2} u = \frac{\partial^2}{\partial x \partial y} v = \frac{\partial^2}{\partial y \partial x} v = -\frac{\partial^2}{\partial y^2} u$$

and

$$\frac{\partial^2}{\partial x^2} v = -\frac{\partial^2}{\partial x \partial y} u = -\frac{\partial^2}{\partial y \partial x} u = -\frac{\partial^2}{\partial y^2} v$$

Lemma 1.1

If $u(x, y)$ is a real-valued polynomial with $\Delta u = 0$, then there exists a (holomorphic) $Q(z)$ such that $\text{Re} Q = u$. 

Proof

Consider $u(x, y) = u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2}\right) = P(z, \bar{z}) = \sum a_{lm} z^l \bar{z}^m$, we know $\Delta u = 0$ and hence

$$P(z, \bar{z}) = a_0 + \sum_{k=1}^m a_k z^k + \sum_{k=1}^n b_k \bar{z}^k$$

P is real-valued and we know

$$a_0 + \sum_{k=1}^m a_k z^k + \sum_{k=1}^n b_k \bar{z}^k = \bar{a}_0 + \sum_{k=1}^m \bar{a}_k \bar{z}^k + \sum_{k=1}^n \bar{b}_k z^k$$

and hence $a_0 \in \mathbb{R}$, $a_k = \bar{b}_k$ and hence

$$u(z) = c + \sum_{k=1}^n a_k z^k + \sum_{k=1}^n \bar{a}_k \bar{z}^k = \operatorname{Re}(c + 2 \sum_{k=1}^n a_k z^k) = \operatorname{Re}(Q)$$

where Q is obviously holomorphic.

Theorem 1.1

If f, g are C^1 functions on the rectangle

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : |x - a| < \delta, |y - b| < \epsilon\}$$

and if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \text{ on } \mathcal{R}$$

then there is a function $h \in C^1(\mathcal{R})$ such that

$$\frac{\partial}{\partial x} h = f, \frac{\partial}{\partial y} h = g$$

on \mathcal{R} . If f, g are real-valued, then we may take h to be real-valued also.

**Proof**

For $(x, y) \in \mathcal{R}$, define

$$h(x, y) = \int_a^x f(t, b) dt + \int_b^y g(x, s) ds$$

and we know

$$\frac{\partial}{\partial y} h(x, y) = g(x, y)$$

and

$$\frac{\partial}{\partial x} h(x, y) = f(x, b) + \frac{\partial}{\partial x} \int_b^y g(x, s) ds = f(x, b) + \int_b^y \frac{\partial}{\partial x} g(x, s) ds = f(x, b) + \int_b^y \frac{\partial}{\partial y} f(x, s) ds = f(x, b) + f(x, y) - f(x, b) = f(x, y)$$

and hence $h \in C^2(\mathcal{R})$ and real-valued if f, g are.

Corollary 1.2

If \mathcal{R} is an open rectangle (or open disc) and if u is a real-valued harmonic function on \mathbb{R} , then there is a holomorphic function F on \mathbb{R} such that $\operatorname{Re} F = u$.

**Proof**

We know

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$

and hence there exists v real-valued such that

$$\frac{\partial}{\partial x} v = -\frac{\partial}{\partial y} u, \frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u$$

and hence $F = u + iv$ is a holomorphic function with $\operatorname{Re}(F) = u$.

Theorem 1.2

If $U \subset \mathbb{C}$ is either an open rectangle or an open disc and if F is holomorphic on U , then there is a holomorphic function H on U such that $\partial H / \partial z = F$ on U .

**Proof**

Consider $H = h_1 + ih_2$ and we have $F = u(z) + iv(z)$, then we let $f = u, g = -v$ and we will have

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial x} g$$

and hence we have a real C^2 function h_1 such that

$$\frac{\partial}{\partial x} h_1 = u, \frac{\partial}{\partial y} h_1 = -v$$

and $h_2 \in C^2$ with

$$\frac{\partial}{\partial x} h_2 = v, \frac{\partial}{\partial y} h_2 = u$$

Then

$$\frac{\partial}{\partial z} H = \frac{1}{2} \left(\frac{\partial}{\partial x} h_1 + \frac{\partial}{\partial y} h_2 \right) + \frac{i}{2} \left(\frac{\partial}{\partial x} h_2 - \frac{\partial}{\partial y} h_1 \right) = u + iv = F$$

Definition 1.6

A function $\phi : [a, b] \rightarrow \mathbb{R}$ is called continuously differentiable and we write $\phi \in C^1([a, b])$ if

- (a) ϕ is continuous on $[a, b]$
- (b) ϕ' exists on (a, b)
- (c) ϕ' has a continuous extension to $[a, b]$, i.e.

$$\lim_{t \rightarrow a^+} \phi'(t) \text{ and } \lim_{t \rightarrow b^-} \phi'(t)$$

both exists. Then $\phi(b) - \phi(a) = \int_a^b \phi'(t) dt$.

**Proof**

Here notice that ϕ is absolutely continuous on $[a, b]$ respect to m , then we know $\phi(b - \epsilon) - \phi(a + \epsilon) = \int_{a+\epsilon}^{b-\epsilon} \phi'(t) dt$ for any $\epsilon > 0$, and hence

$$\phi(b) - \phi(a) = \int_a^b \phi'(t) dt$$

Definition 1.7

A curve $\gamma : [a, b] \rightarrow \mathbb{C}$ is said to be continuous on $[a, b]$ if both γ_1 and γ_2 are, $\gamma = \gamma_1 + i\gamma_2$. The curve is C_1 on $[a, b]$ if γ_1, γ_2 are C_1 on $[a, b]$ and then we may denote

$$\frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i \frac{d\gamma_2}{dt}$$

**Definition 1.8**

Let $\varphi : [a, b] \rightarrow \mathbb{C}$ be continuous on $[a, b]$. Write $\varphi(t) = \varphi_1(t) + i\varphi_2(t)$. Then we define

$$\int_a^b \varphi(t) dt = \int_a^b \varphi_1(t) dt + i \int_a^b \varphi_2(t) dt$$

**Proposition 1.6**

Let $U \subset \mathbb{C}$ be open and let $\gamma : [a, b] \rightarrow U$ be a C_1 curve. If $f : U \rightarrow \mathbb{R}$ and $f \in C^1(U)$, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \left(\frac{\partial}{\partial x} f(\gamma(t)) \frac{d\gamma_1}{dt} + \frac{\partial}{\partial y} f(\gamma(t)) \frac{d\gamma_2}{dt} \right) dt$$



This is due to the chain rule.

Proposition 1.7

Repalce f above as complex-valued and holomorphic, then we have

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial}{\partial z} f(\gamma(t)) \cdot \frac{d\gamma}{dt}(t) dt$$

Proof

Notice

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &= \int_a^b \left(\frac{\partial}{\partial x} u(\gamma(t)) \frac{d\gamma_1}{dt}(t) + \frac{\partial}{\partial y} u(\gamma(t)) \frac{d\gamma_2}{dt}(t) \right) + i \left(\frac{\partial}{\partial x} v(\gamma(t)) \frac{d\gamma_1}{dt}(t) + \frac{\partial}{\partial y} v(\gamma(t)) \frac{d\gamma_2}{dt}(t) \right) dt \\ &= \frac{\partial}{\partial x} f(\gamma(t)) \frac{d\gamma}{dt}(t) = \int_a^b \frac{\partial}{\partial z} f(\gamma(t)) \frac{d\gamma}{dt}(t) dt \end{aligned}$$

Definition 1.9

If $U \subset \mathbb{C}$ open and $F : U \rightarrow \mathbb{C}$ is continuous on U and $\gamma : [a, b] \rightarrow U$ is a C_1 curve, then we define the complex line integral

$$\int_{\gamma} F(z) dz = \int_a^b F(\gamma(t)) \frac{d\gamma}{dt} dt$$

Proposition 1.8

Let $U \subset \mathbb{C}$ be open and let $\gamma : [a, b] \rightarrow U$ be a C^1 curve. If f is a holomorphic function on U , then

$$f(\gamma(b)) - f(\gamma(a)) = \int_{\gamma} \frac{\partial}{\partial z} f(z) dz$$

Proposition 1.9

If $\phi : [a, b] \rightarrow \mathbb{C}$ is continuous, then

$$\left| \int_a^b \phi(t) dt \right| \leq \int_a^b |\phi(t)| dt$$

Proposition 1.10

Let $U \subset \mathbb{C}$ be open and $f \in C^0(U)$. If $\gamma : [a, b] \rightarrow U$ is a C^1 curve, then

$$\left| \int_{\gamma} f(z) dz \right| \leq \left(\sup_{t \in [a, b]} |f(\gamma(t))| \right) \cdot l(\gamma)$$

where

$$l(\gamma) = \int_a^b \left| \frac{d\gamma}{dt}(t) \right| dt$$

Proposition 1.11

Let $U \subset \mathbb{C}$ be an open set and $F : U \rightarrow \mathbb{C}$ a continuous function. Let $\gamma : [a, b] \rightarrow U$ be a C^1 curve. Suppose that $\theta : [c, d] \rightarrow [a, b]$ is a one-to-one, onto, increasing C^1 function with a C^1 inverse. Let $\tilde{\gamma} = \gamma \circ \phi$. Then

$$\int_{\tilde{\gamma}} f dz = \int_{\gamma} f dz$$

Proof

We have


$$\int_{\tilde{\gamma}} f dz = \int_c^d f(\gamma(\phi(t))) \frac{d\gamma(\phi(t))}{dt} dt = \int_a^b f(\gamma(s)) \frac{\gamma(s)}{ds} \phi'(\phi^{-1}(s)) (\phi^{-1})'(s) ds = \int_{\gamma} f dz$$

since $\phi'(\phi^{-1}(s))(\phi^{-1})' = 1$.

Definition 1.10

Let f be a function on the open set U in \mathbb{C} and consider if


$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists then we say that f has a complex derivative at z_0 . We denote the complex derivative by $f'(z_0)$. 

Theorem 1.3

Let $U \subset \mathbb{C}$ be an open set and let f be holomorphic on U . Then f' exists at each point of U and

$$f'(z) = \frac{\partial}{\partial z} f$$

for all $z \in U$. 

Proof

Consider

$$\gamma(t) = (1 - t)z_0 + tz$$

and then we know


$$f(z) - f(z_0) = f(\gamma(1)) - f(\gamma(0)) = \int_{\gamma} \frac{\partial}{\partial z} f dz = (z - z_0) \int_0^1 \frac{\partial}{\partial z} f(\gamma(t)) dt = \frac{\partial}{\partial z} f(z_0) + \int_0^1 \left(\frac{\partial}{\partial z} f(\gamma(t)) - \frac{\partial}{\partial z} f(z_0) \right) dt$$

and hence

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \frac{\partial}{\partial z} f(z_0) \right| \leq \int_0^1 \left| \frac{\partial}{\partial z} f(\gamma(t)) - \frac{\partial}{\partial z} f(z_0) \right| dt \rightarrow 0$$

when $z \rightarrow z_0$.

Theorem 1.4

If $f \in C^1(U)$ and f has a complex derivative at each point of U , then f is holomorphic on U . In particular, if a continuous, complex-valued function f on U has a complex derivative at each point and if f' is continuous on U , then f is holomorphic on U . 

Proof

It is easy to check

$$\lim_{h \rightarrow 0, h \in \mathbb{R}} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial x} v(x_0, y_0)$$

and

$$\lim_{h \rightarrow 0, h \in \mathbb{R}} \frac{f(z_0 + ih) - f(z_0)}{h} = -i \frac{\partial}{\partial y} u(x_0, y_0) + \frac{\partial}{\partial y} v(x_0, y_0)$$

and hence f satisfies the C-R equations so holomorphic.

Notice the continuity of f' may implies that $f \in C^1(U)$ and hence the problem goes.

Theorem 1.5

Let f be holomorphic in a neighborhood of $P \in \mathbb{C}$. Let ω_1, ω_2 be complex numbers of unit modulus. Consider the directional derivatives


$$D_{\omega_1} f(P) = \lim_{t \rightarrow 0} \frac{f(P + t\omega_1) - f(P)}{t}$$

and

$$D_{\omega_2} f(P) = \lim_{t \rightarrow 0} \frac{f(P + t\omega_2) - f(P)}{t}$$

then

a. $|D_{\omega_1} f(P)| = |D_{\omega_2} f(P)|$

b. If $f'(P) \neq 0$, then the directed angle from ω_1 to ω_2 equals the directed angle from $D_{\omega_1} f(P)$ to $D_{\omega_2} f(P)$. 

Proof

Notice that

$$D_{\omega_j} = f'(P)\omega_j, j = 1, 2$$

and then the conclusions go.

Lemma 1.2

Let $(\alpha, \beta) \subset \mathbb{R}$ be an open interval and let $H : (\alpha, \beta) \rightarrow \mathbb{R}, F : (\alpha, \beta) \rightarrow \mathbb{R}$ be continuous functions. Let $p \in (\alpha, \beta)$ and suppose that dH/dx exists and equals $F(x)$ for all $x \in (\alpha, \beta) - \{p\}$. Then $(dH/dx)(p)$ exists and $(dH/dx)(x) = F(x)$ for all $x \in (\alpha, \beta)$.



Proof

Assume $[a, b] \subset (\alpha, \beta)$ and then $K(x) = H(a) + \int_a^x F(t)dt$ on $[a, b]$, so we know $K - H$ is continuous on $[a, b]$ and constant on $[a, p] \cup (p, b]$, which means $K = H$ on $[a, b]$.

Theorem 1.6

Let $U \subset \mathbb{C}$ be either an open rectangle or an open disc and let $P \in U$. Let f and g be continuous, real-valued functions on U which are continuously differentiable on $U - \{P\}$. Suppose further that

$$\frac{\partial}{\partial y}f = \frac{\partial}{\partial x}g \text{ on } U - \{P\}$$

Then there exists a C^1 function $h : U \rightarrow \mathbb{R}$ such that

$$\frac{\partial}{\partial x}h = f, \frac{\partial}{\partial y}h = g$$

at every point of U .



Proof

Consider a closed rectangle containing p inside in U and define $h(x, y) = \int_a^x f(t, b)dt + \int_b^y g(x, s)ds$ and we know that $\frac{\partial}{\partial y}h = g(x, y)$ and $\frac{\partial}{\partial x}h = f(x, y)$ for any $x \neq P_x$, then for a fixed y , we know $dh(x, y)/dx = f(x, y)$ exists for all points in U except for (p_x, y) and hence $dh(x, y)/dx = f(x, y)$ at (p_x, y) . Then we know $\frac{\partial}{\partial x}h = f, \frac{\partial}{\partial y}h = g$ on U .

Theorem 1.7

Let $U \subset \mathbb{C}$ be either an open rectangle or an open disc. Let $P \in U$ be fixed. Suppose that F is continuous on U and holomorphic on $U - \{P\}$. Then there is a holomorphic H on U such that $\frac{\partial}{\partial z}H = F$.



Proof

Consider $F = u + iv$, then we have

$$\frac{\partial}{\partial y}v = \frac{\partial}{\partial x}u \text{ and } \frac{\partial}{\partial y}u = \frac{\partial}{\partial x}(-v)$$

on $U - \{P\}$, then we know there exists h_1, h_2 on U such that $\frac{\partial}{\partial x}h_1 = u, \frac{\partial}{\partial y}h_1 = (-v), \frac{\partial}{\partial x}h_2 = v, \frac{\partial}{\partial y}h_2 = u$ and let $H = h_1 + ih_2$, we have

$$\frac{\partial}{\partial z}H = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(h_1 + ih_2) = (u + u) + i(v + v) = F$$

Definition 1.11

The boundary $\partial D(P, r)$ of the disc $D(P, r)$ can be parametrized as a simple closed curve $\gamma : [0, 1] \rightarrow \mathbb{C}$ by setting

$$\gamma(t) = P + re^{2\pi it}$$

we call it counterclockwise orientation.



Lemma 1.3

Let γ be the boundary of a disc $D(z_0, r)$ in the complex plane, equipped with counterclockwise orientation. Let z be a point inside the circle $\partial D(z_0, r)$. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{\xi - z} d\xi = 1$$

**Proof**

Consider $I(z) = \int_{\gamma} \frac{1}{\xi - z} d\xi = \int_0^1 \frac{1}{(z_0 + e^{2\pi i t}) - z} (2\pi i) e^{2\pi i t} dt$ and since

$$\frac{\partial}{\partial x} \frac{1}{\xi - z} = \frac{1}{(\xi - z)^2}, \quad \frac{\partial}{\partial y} \frac{1}{\xi - z} = i \frac{1}{(\xi - z)^2}$$

and hence we have

$$\frac{\partial}{\partial \bar{z}} I(z) = \int_{\gamma} \frac{\partial}{\partial \bar{z}} \left(\frac{1}{\xi - z} \right) d\xi = 0 \quad \frac{\partial}{\partial z} I(z) = \int_{\gamma} \frac{\partial}{\partial z} \left(\frac{1}{\xi - z} \right) d\xi = \int_{\gamma} \frac{1}{(\xi - z)^2} d\xi$$

where $\frac{1}{(\xi - z)^2}$ is the complex derivative of the holomorphic function $\frac{-1}{\xi - z}$ and hence

$$\frac{\partial}{\partial z} I(z) = \int_{\gamma} \frac{1}{(\xi - z)^2} d\xi = 0$$

Therefore, $I(z)$ is holomorphic on $D(z_0, r)$ and $\frac{\partial}{\partial z} I = 0$ which means I is constant on $D(z_0, r)$ and notice

$$I(z_0) = 2\pi i$$

and hence the equation holds.

Theorem 1.8

(The Cauchy integral formula) Suppose that U is an open set in \mathbb{C} and that f is a holomorphic function on U . Let $z_0 \in U$ and let $r > 0$ be such that $\overline{D}(z_0, r) \subset U$. Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be the C^1 curve $\gamma(t) = z_0 + r \cos(2\pi t) + ir \sin(2\pi t)$. Then for each $z \in D(z_0, r)$,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi$$



Proof By theorem 1.7, there is H such that

$$\frac{\partial}{\partial z} H = \frac{f(\xi) - f(z)}{\xi - z}$$

if $\xi \neq z$ and $\frac{\partial}{\partial z} H(z) = f'(z)$ holomorphic on $D(z_0, r + \epsilon)$ and hence

$$\int_{\gamma} \frac{f(\xi) - f(z)}{\xi - z} d\xi = 0$$

and the equation holds by the lemma 1.3.

Theorem 1.9

(The Cauchy integral theorem) If f is a holomorphic function on an open disc U in the complex plane, and if $\gamma : [a, b] \rightarrow U$ is a C^1 curve in U with $\gamma(a) = \gamma(b)$, then

$$\int_{\gamma} f(z) dz = 0$$



Proof Only need to pick G such that $\frac{\partial}{\partial z} G = f$ on U is fine.

Definition 1.12

A piecewise C^1 curve $\gamma : [a, b] \rightarrow \mathbb{C}$, $a < b$, $a, b \in \mathbb{R}$ is a continuous function such that there exists a finite set of numbers $a_1 \leq a_2 \leq \dots \leq a_k$ satisfying $a_1 = a$ and $a_k = b$ and with the property that for every $1 \leq j \leq k - 1$,

$\gamma|_{[a_j, a_{j+1}]}$ is a C^1 curve. As before, γ is a piecewise C^1 curve in an open set U if $\gamma_{[a, b]} \subset U$.



Definition 1.13

If $U \subset \mathbb{C}$ is open and $\gamma : [a, b] \rightarrow U$ is a piecewise C^1 curve in U and if $f : U \rightarrow \mathbb{C}$ is a continuous, complex-valued function on U , then

$$\int_{\gamma} f(z) dz = \sum_{j=1}^k \int_{\gamma|_{[a_j, a_{j+1}]}} f(z) dz$$

and the definition is well-defined.



Proof

We need to show for any $\{a_j\}_1^k, \{b_i\}_1^m$, the RHS determined by the chosen sequence is the same. Assume $a_{j_t} = b_{i_t}, 1 \leq t \leq q$, with $\{a_j\}_{j_t+1}^{j_{t+1}-1} \cap \{b_i\}_{i_t+1}^{i_{t+1}-1} = \emptyset$, then we know $\gamma|_{[a_{j_t}, a_{j_{t+1}}]}$ is a C^1 curve and hence the integral over the curve is the same.

Lemma 1.4

Let $\gamma : [a, b] \rightarrow U$ open in \mathbb{C} to be a piecewise C^1 curve. Let $\phi : [c, d] \rightarrow [a, b]$ be a piecewise C^1 strictly monotone increasing function with $\phi(c) = a, \phi(d) = b$. Let $f : U \rightarrow \mathbb{C}$ be a continuous function on U . Then the function $\gamma \circ \phi : [c, d] \rightarrow U$ is a piecewise C^1 curve and

$$\int_{\gamma} f(z) dz = \int_{\gamma \circ \phi} f(z) dz$$



Proof Use the proposition 1.11.

Lemma 1.5

If $f : U \rightarrow \mathbb{C}$ is a holomorphic function and if $\gamma : [a, b] \rightarrow U$ is a piecewise C^1 curve, then

$$f(\gamma(b)) - f(\gamma(a)) = \int_{\gamma} f'(z) dz$$



Proof Use the proposition 1.7.

Proposition 1.12

If $f : \mathbb{C} - \{0\} \rightarrow \mathbb{C}$ is a holomorphic function, and if γ_r describes the circle of radius r around 0, traversed once around counter-clockwise, then, for any two positive numbers $r_1 < r_2$,

$$\int_{\gamma_{r_1}} f(z) dz = \int_{\gamma_{r_2}} f(z) dz$$



Proposition 1.13

Let $0 < r < R < \infty$ and define the annulus $\mathcal{A} = \{z \in \mathbb{C} : r < |z| < R\}$. Let $f : \mathcal{A} \rightarrow \mathbb{C}$ be a holomorphic function. If $r < r_1 < r_2 < R$ and if for each j the curve γ_{r_j} describes the circle of radius r_j around 0, traversed once counter clockwise, then we have

$$\int_{\gamma_{r_1}} f dz = \int_{\gamma_{r_2}} f dz$$



Applications of the Cauchy integral

Theorem 1.10

Let $U \subset \mathbb{C}$ be an open set and let f be holomorphic on U . Then $f \in C^\infty(U)$. Moreover, if $\overline{D}(P, r) \subset U$ and $z \in D(P, r)$, then

$$\left(\frac{\partial}{\partial z}\right)^k f(z) = \frac{k!}{2\pi i} \int_{|\xi-P|=r} \frac{f(\xi)}{(\xi-z)^{k+1}} d\xi$$

for any integer k .



Proof

Use the induction to f , assume

$$\left(\frac{\partial}{\partial z}\right)^k f(z) = \frac{k!}{2\pi i} \int_{|\xi-P|=r} \frac{f(\xi)}{(\xi-z)^{k+1}} d\xi$$

and $\left(\frac{\partial}{\partial z}\right)^k f(z)$ is holomorphic, then we gonna prove that

$$\left(\frac{\partial}{\partial z}\right)^{k+1} f(z) = \frac{(k+1)!}{2\pi i} \int_{|\xi-P|=r} \frac{f(\xi)}{(\xi-z)^{k+2}} d\xi$$

and $\left(\frac{\partial}{\partial z}\right)^{k+1} f(z)$ is holomorphic. Consider

$$\begin{aligned} \left| \frac{f(\xi)}{(\xi-\omega)^{k+1}} - \frac{f(\xi)}{(\xi-z)^{k+1}} \right| &\leq \sup_{\xi \in \partial D(P, r)} |f(\xi)| \epsilon^{-2k-2} \left| \sum_{i=1}^{k+1} C_{k+1}^i (2r)^{k+1-i} (\omega-z)^i \right| \\ &\leq |\omega-z|(k+1) \left(\sup_{\xi \in \partial D(P, r)} |f(\xi)| \epsilon^{-2k-2} \left| \sum_{i=0}^k C_k^i (2r)^{k-i} (\omega-z)^i \right| \right) \\ &\leq |\omega-z|(k+1) \left(\sup_{\xi \in \partial D(P, r)} |f(\xi)| \epsilon^{-2k-2} (2r+1)^k \right) \end{aligned}$$

for all $|\omega-z|$ small enough and hence

$$\frac{f(\xi)}{(\xi-\omega)^{k+1}} \rightarrow \frac{f(\xi)}{(\xi-z)^{k+1}}$$

uniformly when $\omega \rightarrow z$, so may know

$$\lim_{\omega \rightarrow z} \frac{\left(\frac{\partial}{\partial z}\right)^{k+1} f(\omega) - \left(\frac{\partial}{\partial z}\right)^{k+1} f(z)}{\omega - z} = \lim_{\omega \rightarrow z} \frac{k!}{2\pi i} \int_{|\xi-P|=r} \frac{\frac{f(\xi)}{(\xi-\omega)^{k+1}} - \frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega - z} d\xi$$

and we know that

$$\lim_{\omega \rightarrow z} \frac{k!}{2\pi i} \int_{|\xi-P|=r} \frac{\frac{f(\xi)}{(\xi-z)^{k+1}} - \frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega - z} d\xi = \frac{k!}{2\pi i} \int_{|\xi-P|=r} \lim_{\omega \rightarrow z} \frac{\frac{f(\xi)}{(\xi-\omega)^{k+1}} - \frac{f(\xi)}{(\xi-z)^{k+1}}}{\omega - z} d\xi$$

by the DCT and hence

$$\lim_{\omega \rightarrow z} \frac{\left(\frac{\partial}{\partial z}\right)^{k+1} f(\omega) - \left(\frac{\partial}{\partial z}\right)^{k+1} f(z)}{\omega - z} = \frac{(k+1)!}{2\pi i} \int_{|\xi-P|=r} \frac{f(\xi)}{(\xi-z)^{k+2}} d\xi$$

which means $\left(\frac{\partial}{\partial z}\right)^k f(z)$ is holomorphic and the equality holds. Then we use the induction, and the conclusion goes.

Corollary 1.3

If $f : U \rightarrow \mathbb{C}$ is holomorphic, then $f' : U \rightarrow \mathbb{C}$ is holomorphic.



Theorem 1.11

If ϕ is a continuous function on $\{\xi : |\xi - P| = r\}$, then the function f given by

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - P| = r} \frac{\phi(\xi)}{\xi - z} d\xi$$

is defined and holomorphic on $D(P, r)$.

**Theorem 1.12**

(Morera) Suppose that $f : U \rightarrow \mathbb{C}$ is a continuous function on a connected open subset U of \mathbb{C} . Assume that for every closed, piecewise C^1 curve $\gamma : [0, 1] \rightarrow U$, $\gamma(0) = \gamma(1)$, it holds that

$$\int_{\gamma} f(\xi) d\xi = 0$$

Then f is holomorphic on U .



Proof Consider $x \in U$ and define $F(y) = \int_{\phi} f dz$ for any $y \in U$ where ϕ is a piecewise C^1 curve from x to y , where we know the integral is well-defined since any integral of f on a closed, piecewise C^1 curve is 0. Then for any $y \in U$, consider a segment from $y + h$ where $|h|$ is small enough and we know

$$\lim_{|h| \rightarrow 0} \frac{F(y+h) - F(y)}{h} = \lim_{|h| \rightarrow 0} \frac{1}{h} \int_0^h f(y+z) dz = f(y)$$

which means F is holomorphic on U and $F' = f$ on U , and hence f is holomorphic on U .

Definition 1.14

let $P \in \mathbb{C}$ be fixed. A complex power series centered at P is an expression of the form

$$\sum a_k (z - P)^k$$

where a_k is complex valued.

**Lemma 1.6**

(Abel) If $\sum a_k (z - P)^k$ converges at some z , then the series converges at each $\omega \in D(P, r)$, where $r = |z - P|$.

**Proof**

Since $\sum a_k (z - P)^k$ converges, we know $a_k (z - P)^k \rightarrow 0$ and hence bounded, then we know

$$|a_k| \leq M r^{-k}$$

for some $M > 0$ and then for any $\omega \in D(P, r)$, assume $|\omega - P| = \delta < r$, then we know

$$|a_k (\omega - P)^k| \leq |a_k| \delta^k \leq M (\delta/r)^{-k}$$

and hence

$$\sum |a_k (\omega - P)^k| \leq M \sum (\delta/r)^{-k} < \infty$$

which means $\sum a_k (\omega - P)^k$ converges.

Definition 1.15

Let $\sum a_k (z - P)^k$ be a power series. Then

$$r = \sup\{|\omega - P| : \sum a_k (\omega - P)^k \text{ converges}\}$$

is called the radius of convergence of the power series.

**Lemma 1.7**

If $\sum a_k (z - P)^k$ is a power series with radius of convergence r , then the series converges for each $\omega \in D(P, r)$ and diverges for each ω such that $|\omega - P| > r$.



Lemma 1.8

(The root test) The radius of convergence of the power series $\sum a_k(z - P)^k$ is

$$\frac{1}{\limsup |a_k|^{1/k}}$$

if $\limsup |a_k|^{1/k} > 0$ or

$$\infty$$

if $\limsup |a_k|^{1/k} = 0$.

**Proof**

Assume $\alpha = \limsup |a_k|^{1/k}$, if $|\omega - P| > 1/\alpha$, then denote $|\omega - P| = c/\alpha$, $c > 1$ and we know

$$|a_k(z - P)^k| = (c|a_k|^{1/k}/\alpha)^k$$

and we know there are infinitely many a_k such that $|a_k|^{1/k}/\alpha > 1/c$ and hence the series diverge.

For $|\omega - P| < 1/\alpha$, we denote $|\omega - P| = d/\alpha$, $d < 1 - \epsilon$ for some $\epsilon > 0$ and we have

$$|a_k(\omega - P)^k| \leq (|a_k|^{1/k}d/\alpha)^k \leq (1 - \epsilon)^k$$

when k is sufficiently large and hence the series is absolutely convergent and the condition for $\alpha = 0$ is similar.

Definition 1.16

Let $\sum f_k(z)$ be a series of functions on a set E . The series is said to be uniformly Cauchy if for any $\epsilon > 0$, there is an integer N such that

$$\left| \sum_{k=m}^n f_k(z) \right| < \epsilon$$

on E for any $n \geq m \geq N$.

**Proposition 1.14**

Let $\sum a_k(z - P)^k$ be a power series with radius of convergence r . Then, for any number R with $0 \leq R < r$, the series $\sum |a_k(z - P)|^k$ converges uniformly on $\overline{D}(P, R)$ and hence $\sum a_k(z - P)^k$ converges uniformly and absolutely on $\overline{D}(P, R)$.



Proof We know

$$\lim_{k \rightarrow \infty} |a_k r^k| \rightarrow 0$$

and hence there exists $M > 0$ such that

$$|a_k| \leq \frac{M}{r^k}$$

then we know

$$\sum_{k=0}^n |a_k(z - P)^k| \leq \sum_{k=0}^n M(r/R)^k$$

on $\overline{D}(P, R)$ and hence the series converges uniformly.

Lemma 1.9


If a power series

$$\sum_{j=0}^{\infty} a_j(z - P)^j$$

has radius of convergence $r > 0$, then the series defines a C^∞ function $f(z)$ on $D(P, r)$. The function f is

holomorphic on $D(P, r)$. The series obtained by termwise differentiation k times of the original power series,

$$\sum_{j=k}^{\infty} \frac{j!}{k!} a_j (z - P)^{j-k}$$

converges on $D(P, r)$ and its sum is $(\partial/\partial z)^k f(z)$ for each $z \in D(P, r)$. 

Proof

For any $z \in D(P, r)$, we know the series is absolutely convergent at z , and hence


$$D_h f(z) = \lim_{d \rightarrow 0} \sum_{j=0}^{\infty} a_j \frac{(z + dh - P)^j - (z - P)^j}{d} = \sum_{j=0}^{\infty} a_j j (z - P)^{j-1}$$

since

$$\sum_{j=0}^{\infty} j |a_j r'^{j-1}| \leq C + \sum_{j=m}^{\infty} |a_j (r' + \epsilon)^{j-1} / (r' + \epsilon)^j|$$

for some $\epsilon > 0$ and integer m big sufficiently, and hence we may exchange the summation and the limit. Then we know f is holomorphic and hence in C^∞ and we may use the induction to $\frac{\partial^k}{\partial z^k} f$.

Proposition 1.15

If both series $\sum_{j=0}^{\infty} a_j (z - P)^j$ and $\sum_{j=0}^{\infty} b_j (z - P)^j$ converge on a disc $D(P, r)$, $r > 0$ and if $\sum_{j=0}^{\infty} a_j (z - P)^j = \sum_{j=0}^{\infty} b_j (z - P)^j$ on $D(P, r)$, then $a_j = b_j$ for every j . 


Proof

Use the lemma 1.9. directly.

Theorem 1.13

Let $U \subset \mathbb{C}$ be an open set and let f be holomorphic on U . Let $P \in U$ and suppose that $D(P, r) \subset U$. Then the complex power series

$$\sum_{k=0}^{\infty} \frac{(\frac{\partial}{\partial z})^k f(P)}{k!} (z - P)^k$$

has radius of convergence at least r . It converges to $f(z)$ on $D(P, r)$. 

Proof

For $z \in D(P, r)$, we know

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - P|=r'} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{|\xi - P|=r'} \frac{f(\xi)}{\xi - P} \sum_{n \geq 0} ((z - P)(\xi - P)^{-1})^n d\xi$$

for $r' > |z - P|$ and $D(z, r') \subset D(P, r)$ and then we know

$$f(z) = \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{|\xi - P|=r'} \frac{f(\xi)}{\xi - P} \sum_{n=0}^N ((z - P)(\xi - P)^{-1})^n d\xi$$

since the series converges uniformly. Then

$$f(z) = \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{1}{2\pi i} \int_{|\xi - P|=r'} \frac{f(\xi)}{(\xi - P)^{n+1}} (z - P)^n d\xi = \sum_{k=0}^{\infty} \frac{(\frac{\partial}{\partial z})^k f(P)}{k!} (z - P)^k$$

Theorem 1.14

(The Cauchy estimates) Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function on an open set U , $P \in U$ and assume that the

closed disc $\overline{D}(P, r), r > 0$ is contained in U . Set $M = \sup_{z \in \overline{D}(P, r)} |f(z)|$, then for $k \geq 1$ we have

$$\left| \frac{\partial^k f}{\partial z^k}(P) \right| \leq \frac{Mk!}{r^k}$$



Proof

We know

$$\left| \frac{\partial^k f}{\partial z^k}(P) \right| = \left| \frac{k!}{2\pi i} \int_{|\xi-P|=r} \frac{f(\xi)}{(\xi-z)^{k+1}} d\xi \right| \leq \frac{Mk!}{r^k}$$

Lemma 1.10

Suppose that f is a holomorphic function on a connected open set $U \subset \mathbb{C}$. If $\partial f / \partial \bar{z} = 0$ on U , then f is constant on U .



Proof Notice $\frac{\partial}{\partial x} f = \frac{\partial}{\partial y} f = 0$ on U .

Definition 1.17

A function f is said to be entire if it is defined and holomorphic on all of \mathbb{C} , that is, $f : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic.



Theorem 1.15

(Liouville's theorem) A bounded entire function is constant.



Proof For any $P \in \mathbb{C}$, we may know

$$\left| \frac{\partial}{\partial \bar{z}} f(P) \right| \leq M/r$$

for any $r > 0$ and hence $\frac{\partial}{\partial \bar{z}} f = 0$ on \mathbb{C} and hence it is a constant on \mathbb{C} .

Theorem 1.16

If $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function and if for some real number C and some positive integer k it holds that

$$|f(z)| \leq C|z|^k$$

for all z with $|z| > 1$, then f is a polynomial in z of degree at most k .



Proof We know

$$\left| \left(\frac{\partial}{\partial \bar{z}} \right)^{k+l} f(0) \right| \leq C(k+l)!/r^l$$

for any $r \geq 0$ and hence $\left(\frac{\partial}{\partial \bar{z}} \right)^{k+l} f(0) = 0$.

Theorem 1.17

Let $p(z)$ be a nonconstant polynomial. Then p has a root.



Proof

If not, we know $g(z) = 1/p(z)$ is holomorphic on \mathbb{C} and bounded since $|p(z)| \rightarrow \infty, |z| \rightarrow \infty$, so by the Liouville's theorem, we know $p(z)$ is constant and hence a contradiction.

Corollary 1.4

If $p(z)$ is a holomorphic polynomial of deg k , then there are k complex numbers $\alpha_1, \dots, \alpha_k$ and a constant C such

that

$$p(z) = C \prod_{i=1}^k (z - \alpha_i)$$



Theorem 1.18

Let $f_j : U \rightarrow \mathbb{C}, j \geq 1$ be a sequence of holomorphic functions on an open set U in \mathbb{C} . Suppose that there is a function $f : U \rightarrow \mathbb{C}$ such that, for each compact subset E of U , the sequence $f_j|_E$ converges uniformly to $f|_E$. Then f is holomorphic on U .



Proof

Firstly, it is easy to check f is continuous on U .

For $z \in U$, we may consider $D_z = \overline{D}(z, r) \subset U$ is a compact set, and we know $f_j \rightarrow f$ uniformly on D_z , then

$$f(z+d) = \lim_{n \rightarrow \infty} f_n(z+d) = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{|\xi-z|=r} \frac{f_n(\xi)}{\xi - (z+d)} d\xi = \frac{1}{2\pi i} \int_{|\xi-z|=r} \frac{f(\xi)}{\xi - (z+d)} d\xi$$

then we know

$$f'(z) = \lim_{|d| \rightarrow 0} \frac{1}{2\pi i} \int_{|\xi-z|=r} f(\xi) \left(\left| \frac{1}{\xi - (z+d)} - \frac{1}{\xi - z} \right| / d \right) d\xi = \frac{1}{2\pi i} \int_{|\xi-z|=r} \frac{f(\xi)}{(\xi - z)^2} d\xi$$

and hence f is holomorphic on U .

Corollary 1.5

If f_j, f, U are as in the theorem above, then for any integer $k \geq 0$, we have

$$\left(\frac{\partial}{\partial z} \right)^k f_j(z) \rightarrow \left(\frac{\partial}{\partial z} \right)^k f(z)$$

uniformly on compact sets.



Proof We have

$$\left| \left(\frac{\partial}{\partial z} \right)^k f_j(z) - \left(\frac{\partial}{\partial z} \right)^k f(z) \right| \leq \frac{k!}{r^k} \sup_{\overline{D}(z,r)} |f(z) - f_j(z)|$$

if $\overline{D}(z, r) \subset U$ and the rest is easy to be checked.

Theorem 1.19

Let $U \subset \mathbb{C}$ be a connected open set and let $f : U \rightarrow \mathbb{C}$ be holomorphic. Let $Z = \{z \in U, f(z) = 0\}$. If there are a $z_0 \in Z$ and $\{z_j\}_{j=1}^\infty \in Z - \{z_0\}$ such that $z_j \rightarrow z_0$, then $f = 0$ on U .



Proof

Consider

$$E = \{z, \left(\frac{\partial}{\partial z} \right)^k f(z) = 0 \text{ for any integer } k \geq 0\}$$

and we claim $z_0 \in E$, if not there exists n_0 such that

$$\left(\frac{\partial}{\partial z} \right)_{z_0}^{n_0} f(z_0) \neq 0$$

and hence

$$g(z) = \sum_{i=n_0}^{\infty} \left(\frac{\partial}{\partial z} \right)^i f(z) \frac{(z - z_0)^{i-n_0}}{i!}$$

is not 0 at z_0 but $g(z_j) = 0$ for any z_j , and hence $g(z_0) = 0$ by the continuity, which is a contradiction and hence $z_0 \in E$.

Now it is easy to check E is closed about U and also E is open since for any $z \in E$, we know

$$f(z+d) = \sum \partial^j f(z) / j! d^j$$

for any d in some open call centered at z and hence the ball is in E . Notice U is connected and we know $E = U$ and theorem is proved.

Corollary 1.6

Let $U \subset \mathbb{C}$ be a connected open set and $D(P, r) \subset U$. If f is holomorphic on U and $f|_{D(P, r)} = 0$, then $f = 0$ on U .

**Corollary 1.7**

Let $U \subset \mathbb{C}$ be a connected open set and $D(P, r) \subset U$. Let f, g be holomorphic on U . If $\{z, f(z) = g(z)\}$ has an accumulation in U , then $f = g$ on U .

**Corollary 1.8**

Let $U \subset \mathbb{C}$ be a connected open set and $D(P, r) \subset U$. Let f, g be holomorphic on U . If $fg = 0$ on U , then either $f = 0$ on U or $g = 0$ on U .



Proof Choose a point $z, f(z) \neq 0$ is fine.

Corollary 1.9

Let $U \subset \mathbb{C}$ be connected and open and let f be holomorphic on U . If there is a $P \in U$ such that

$$\left(\frac{\partial}{\partial z}\right)^j f(P) = 0$$

for every j , then $f = 0$ on U .

**Corollary 1.10**

If f and g are entire holomorphic functions and if $f = g$ for all $x \in \mathbb{R} \subset \mathbb{C}$, then $f = g$.

**Definition 1.18**

Let $U \subset \mathbb{C}$ be an open set and $P \in U$. Suppose that $f : U - \{P\} \rightarrow \mathbb{C}$ is holomorphic. In this situation we say that f has an isolated singular point at P .

**Definition 1.19**

If $\lim_{z \rightarrow P} |f(z)| = +\infty$, then we call f has a pole at P . If P is not a pole or a removable singularity, we call f has an essential singularity at P .

**Theorem 1.20**

(The Riemann removable singularities theorem) Let $f : D(P, r) - \{P\} \rightarrow \mathbb{C}$ be holomorphic and bounded. Then

- $\lim_{z \rightarrow P} f(z)$ exists
- the function $\hat{f} : D(P, r) \rightarrow \mathbb{C}$ defined by

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \neq P \\ \lim_{\xi \rightarrow P} f(\xi) & \text{if } z = P \end{cases}$$

**Proof**

Consider

$$g(z) = \begin{cases} (z - P)^2 f(z) & \text{if } z \in D(P, r) - \{P\} \\ 0 & \text{if } z = P \end{cases}$$

we claim that $g \in C^1(D(P, r))$. Since we know

$$\frac{\partial g}{\partial \bar{z}} = 0$$

on $D(P, r) - \{P\}$ and if $g \in C^1(D(P, r))$, then g is holomorphic. Notice

$$g'(z) = 2(z - P)f(z) + (z - P)^2 f'(z)$$

on $D(P, r) \rightarrow \mathbb{C}$ and

$$\frac{\partial g}{\partial x}(P) = \lim_{h \rightarrow 0} hf(P+h) = 0$$

and similarly $\partial g / \partial y(P) = 0$ and it suffices to show

$$\lim_{z \rightarrow P} g'(z) = \lim_{z \rightarrow P} 2(z-P)f(z) + (z-P)^2 f'(z)$$

equals to 0, which can be implied by the Cauchy estimation. Now we know g is C^1 and hence holomorphic on $D(P, r)$. Then let

$$H(z) = \sum_{n=2}^{\infty} \left(\frac{\partial}{\partial z} \right)^n g(P) / n! (z-P)^2$$

which has radius convergence at least r and holomorphic on $D(P, r)$, which equals to $f(z)$ on $D(P, r) - \{P\}$ and satisfies the requirements.

Theorem 1.21

(Casorati-Weierstrass) If $f : D(P, r_0) - \{P\}$ is holomorphic and P is an essential singularity of f , then $f(D(P, r) - \{P\})$ is dense in \mathbb{C} for any $0 < r < r_0$.



Proof

It suffices to show $r = r_0$, then there is $\lambda \in \mathbb{C}$ and an $\epsilon > 0$ such that

$$|f(z) - \lambda| > \epsilon$$

for all $z \in D(P, r_0) - \{P\}$. Consider the function $g : D(P, r_0) - \{P\} \rightarrow \mathbb{C}$ defined by

$$g(z) = \frac{1}{f(z) - \lambda}$$

then P is a removable singularity for g with

$$f(z) = \lambda + \frac{1}{\hat{g}(z)}$$

on $D(P, r) - \{P\}$ where $\hat{g} \neq 0$, so if $\hat{g}(P) = 0$, then it is easy to check that P is a pole of f , which is a contradiction, so $\lim_{z \rightarrow P} f(z)$ exists and finite, which means P is a removable singularity of f and hence contradictory.

Definition 1.20

A Laurent series on $D(P, r)$ is a expression of the form

$$\sum_{j=-\infty}^{\infty} a_j (z-P)^j$$

and when we say a series with double infinities converges, we mean $\sum_{n \geq 0} \alpha_n$ and $\sum_{n \leq 0} \alpha_n$ converge both.



Lemma 1.11

If $\sum_{j=-\infty}^{\infty} a_j (z-P)^j$ converges at $z_1 \neq P$ and $z_2 \neq P$ with $|z_1 - P| < |z_2 - P|$, then the series converges for all z such that $|z_1 - P| < |z - P| < |z_2 - P|$.



Proof

Assume $S_n(z) = \sum_{j=0}^n a_j (z-P)^j$ and $W_n(z) = \sum_{j=-n}^{-1} a_j (z-P)^j$ and we know $S_n(z_2), W_n(z_1)$ converges and hence there exists M such that

$$|a_{-j}| |z_1 - P|^{-j}, |a_j| |z_2 - P|^j < M$$

then for any z in the annulus, we know

$$\sum_{j=0}^n |a_j| |z - P|^j \leq M \sum_{j=0}^n \left(\frac{|z - P|}{|z_2 - P|} \right)^j$$

and

$$\sum_{j=1}^n |a_{-j}| |z - P|^{-j} \leq M \sum_{j=1}^n \left(\frac{|z - P|}{|z_1 - P|} \right)^{-j}$$

which means $S_n(z)$, $W_n(z)$ are both absolutely convergent and the conclusion holds.

Proposition 1.16

Let $0 \leq r_1 < r_2 \leq \infty$. If the Laurent series $\sum_{-\infty}^{\infty} a_j (z - P)^j$ converges on $D(P, r_2) - \overline{D}(P, r_1)$ to a function f , then for any r satisfying $r_1 < r < r_2$, and each $j \in \mathbb{Z}$,

$$a_j = \frac{1}{2\pi i} \int_{|\xi - P|=r} \frac{f(\xi)}{(\xi - P)^{j+1}} d\xi$$

Proof

We know since the series converges uniformly on the circle $|z - P| = r$, then

$$\int_{|\xi - P|=r} \frac{f(\xi)}{(\xi - P)^{j+1}} d\xi = \int_{|\xi - P|=r} \sum_{-\infty}^{\infty} a_k (\xi - P)^{k-j-1} d\xi = \sum_{-\infty}^{\infty} a_k \int_{|\xi - P|=r} (\xi - P)^{k-j-1} d\xi$$

and then we know

$$\int_{|\xi - P|=r} (\xi - P)^{k-j-1} d\xi = \begin{cases} 0 & \text{if } k - j - 1 \neq -1 \\ 2\pi i & \text{if } k - j - 1 = -1 \end{cases}$$

and hence

$$\int_{|\xi - P|=r} \frac{f(\xi)}{(\xi - P)^{j+1}} d\xi = 2\pi i a_j$$

Theorem 1.22

(The Cauchy integral formula for an annulus) Suppose that $0 \leq r_1 < r_2 \leq +\infty$ and that $f : D(P, r_2) - \overline{D}(P, r_1) \rightarrow \mathbb{C}$ is holomorphic. Then for each s_1, s_2 such that $r_1 < s_1 < s_2 < r_2$ and each $z \in D(P, s_2) - \overline{D}(P, r_1)$, it holds that

$$f(z) = \frac{1}{2\pi i} \int_{|\xi - P|=s_2} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{|\xi - P|=s_1} \frac{f(\xi)}{\xi - z} d\xi$$

Theorem 1.23

(The existence of Laurent expansions) If $0 \leq r_1 < r_2 \leq \infty$ and $f : D(P, r_2) - \overline{D}(P, r_1) \rightarrow \mathbb{C}$ is holomorphic, then there exist complex numbers a_j such that

$$\sum_{-\infty}^{\infty} a_j (z - P)^j$$

converges on $D(P, r_2) - \overline{D}(P, r_1)$ to f . If $r_1 < s_1 < s_2 < r_2$, then the series converges absolutely and uniformly on $D(P, s_2) - \overline{D}(P, s_1)$.

Proof

Notice

$$\int_{|\xi - P|=s_2} \frac{f(\xi)}{\xi - z} d\xi = \int_{|\xi - P|=s_2} \frac{f(\xi)}{\xi - P} \frac{1}{1 - \frac{z - P}{\xi - P}} d\xi = \frac{1}{2\pi i} \int_{|\xi - P|=s_2} \sum_{n \geq 0} \left(\frac{f(\xi)(z - P)^n}{(\xi - P)^{n+1}} \right) d\xi$$

and notice the series converge uniformly on $|\xi - P| = s_2$, so we know

$$\int_{|\xi - P|=s_2} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n \geq 0} \left(\int_{|\xi - P|=s_2} \frac{f(\xi)}{(\xi - P)^{n+1}} d\xi \right) (z - P)^n$$

and similarly, we may know

$$\int_{|\xi-P|=s_1} \frac{f(\xi)}{\xi-z} d\xi = \sum_{n \geq 1} \left(\int_{|\xi-P|=s_1} \frac{f(\xi)}{(\xi-P)^{-n+1}} d\xi \right) (z-P)^{-n}$$

and the rest is by the theorem 1.22.

Proposition 1.17

If $f : D(P, r) - \{P\} \rightarrow \mathbb{C}$ is holomorphic, then f has a unique Laurent series expansion

$$f(z) = \sum_{-\infty}^{\infty} a_j (z-P)^j$$

which converges absolutely for $z \in D(P, r) - \{P\}$. The convergence is uniform on compact subsets of $D(P, r) - \{P\}$.

Proposition 1.18

There are three possibilities for the Laurent series of a holomorphic function f ,

- a. $a_j = 0$ for all $j < 0$;
- b. for some $k > 0$, $a_j = 0$ for all $-\infty < j < -k$;
- c. neither (a) or (b).

Proof

(a) implies P is removable is obviously, conversely, consider the series expansion of the holomorphic expansion \hat{f} .

(b) implies P is a pole can be seen by

$$|f(z)| \geq (z-P)^{-k} \left(|a_{-k}| - \sum_{j=-k+1}^{\infty} |a_j| (z-P)^{j+k} \right)$$

and hence $f(z) \rightarrow \infty, z \rightarrow P$.

For the other direction, we may consider there exists $D(P, r)$ such that $f(z)$ is nonzero there and let $g(z) = 1/f(z)$ which is holomorphic on $D(P, r)$ and P is a removable singularity of g , hence we may find \hat{g} is holomorphic on $D(P, r)$ and hence

$$H(z) = (z-P)^m Q(z)$$

for some integer Q and some function Q nonzero at P , which means $Q(z)$ is nonzero on $D(P, r)$ and we may find $1/Q(z)$ holomorphic on $D(P, r)$, then we will find a series of f .

Definition 1.21

If a function f has a Laurent expansion

$$f(z) = \sum_{j=-k}^{\infty} a_j (z-P)^j$$

for some $k > 0$ and $a_{-k} \neq 0$, then we say that f has a pole of order k at P .

Proposition 1.19

Let f be holomorphic on $D(P, r) - \{P\}$ and suppose that f has a pole of order k at P . Then the Laurent series coefficients a_j of f expanded about the point P , for $j = -k, -k+1, -k+2, \dots$ are given by the formula

$$a_j = \frac{1}{(k+j)!} \left(\frac{\partial}{\partial z} \right)^{k+j} ((z-P)^k f) |_{z=P}$$

Definition 1.22

An open set $U \subset \mathbb{C}$ is holomorphically simply connected if U is connected and if, for each holomorphic function $f : U \rightarrow \mathbb{C}$, there is a holomorphic function $F : U \rightarrow \mathbb{C}$ such that $F' = f$.

Lemma 1.12

A connected open set U is holomorphically simply connected if and only if for each holomorphic function $f : U \rightarrow \mathbb{C}$ and each piecewise C^1 closed curve γ in U ,

$$\int_{\gamma} f = 0$$

**Definition 1.23**

If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a piecewise C^1 curve and if $P \notin \tilde{\gamma} = \gamma([a, b])$, then the index of γ with respect to P , written $\text{Ind}_{\gamma}(P)$ is defined to be the number

$$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{\xi - P} d\xi$$

**Lemma 1.13**

If $\gamma : [a, b] \rightarrow \mathbb{C} - \{P\}$ is a piecewise C^1 closed curve and if P is a point not on that curve then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{\xi - P} = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - P} dt$$

is an integer.

**Proof**

Consider

$$g(t) = (\gamma(t) - P) \exp \left(- \int_a^t \gamma'(s) / [\gamma(s) - P] ds \right)$$

then g is continuous and we also have

$$g'(t) = \gamma'(t) \exp \left(- \int_a^t \frac{\gamma'(s)}{\gamma(s) - P} ds \right) + (\gamma(t) - P) \frac{-\gamma'(t)}{\gamma(t) - P} \exp \left(- \int_a^t \frac{\gamma'(s)}{\gamma(s) - P} ds \right) = 0$$

and it is easy to check $g(a) = g(b)$ and hence

$$- \int_a^b \frac{\gamma'(s)}{\gamma(s) - P} ds$$

must be an integer multiple of $2\pi i$.

Definition 1.24

The residue of f at P , written as $\text{Res}_f(P)$ is defined by the coefficient of $(z - P)^{-1}$ in the Laurent expansion of f about P .

**Theorem 1.24**

(The residue theorem) Suppose that $U \subset \mathbb{C}$ is an h.s.c. open set in \mathbb{C} and that P_1, \dots, P_n are distinct points of U . Suppose that $f : U - \{P_1, \dots, P_n\} \rightarrow \mathbb{C}$ is a holomorphic function and γ is a closed, piecewise C^1 curve in $U - \{P_1, \dots, P_n\}$. Then

$$\int_{\gamma} f = \sum_{j=1}^n \text{Res}_f(P_j) \left(\int_{\gamma} \frac{1}{\xi - P_j} d\xi \right)$$

**Proof**

Let s_j be the negative part of the Laurent series of f at P_j and we know $f - s_j$ is holomorphic on some neighbourhood of P_j and hence we may know

$$\int_{\gamma} (f - \sum s_j) = 0$$

and hence

$$\int_{\gamma} f = \int_{\gamma} \sum s_j$$

where

$$\int_{\gamma} s_j(\xi) d\xi = \sum_{k=1}^{\infty} a_{-k}^{(j)} \int_{\gamma} (\xi - P_j)^{-k} d\xi = 2\pi i a_{-1}^{(j)} \text{Ind}_{\gamma}(P_j)$$

Proposition 1.20

Let f be a function with a pole of order k at P . Then

$$\text{Res}_f(P) = \frac{1}{(k-1)!} \left(\frac{\partial}{\partial z} \right)^{k-1} ((z-P)^k f(z))|_{z=P}$$



Definition 1.25

A set S in \mathbb{C} is discrete iff for each $z \in S$ there is a positive number r such that $S \cap D(z, r) = \{z\}$



Definition 1.26

A meromorphic function f on an open set U with singular set S is a function $f : U - \{S\} \rightarrow \mathbb{C}$ such that

- the set S is closed in U and is discrete.
- the function F is holomorphic on $U - \{S\}$.
- for each $z \in S$ and $r > 0$ such that $D(z, r) \subset U$ and $S \cap D(z, r) = \{z\}$, the function $f|_{D(z, r) - \{z\}}$ has a pole at z .



Lemma 1.14

If U is a connected open set in \mathbb{C} and if $f : U \rightarrow \mathbb{C}$ is a holomorphic function with $f \neq 0$, then the function

$$F : U - \{z : f(z) = 0\} \rightarrow \mathbb{C}$$

define by $F(z) = 1/f(z), z \in U - \{z, f(z) = 0\}$ is a meromorphic function on U with singular set equal to $\{z \in U, f(z) = 0\}$.



Proof It is easy to check $S = \{z, f(z) = 0\}$ is closed and discrete by theorem 1.19 and F is obviously holomorphic on $U - S$. The rest is easy to check.(connected open set is for the theorem 1.19)

Definition 1.27

Suppose that $f : U \rightarrow \mathbb{C}$ is a holomorphic function on an open set $U \subset \mathbb{C}$ and that for some $R > 0, U \supset \{z : |z| > R\}$. Define $G : \{z : 0 < |z| < 1/R\} \rightarrow \mathbb{C}$ by $G(z) = f(1/z)$, then we say that

- f has a removable singularity at ∞ if G has a removable singularity at 0.
- f has a pole at ∞ if G has a pole at 0.
- f has an essential singularity at ∞ if G has an essential singularity at 0.



Theorem 1.25

Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function. Then $\lim_{|z| \rightarrow +\infty} f(z)$ is finite iff f is a non constant polynomial.



Proof


Consider the series expansion of f which is exactly the Laurent expansion of G and we are done.

Definition 1.28

Suppose that f is a meromorphic function defined on an open set $U \subset \mathbb{C}$ such that for some $R > 0$, we have $U \supset \{z, |z| > R\}$. We say that f is a meromorphic at ∞ if the function $G(z) = f(1/z)$ is meromorphic in the usual sense on $\{z, |z| < 1/R\}$.



Theorem 1.26

A meromorphic function f on \mathbb{C} which is also meromorphic at ∞ must be a rational function, i.e. a quotient of polynomials in z . Conversely, every rational function is meromorphic on \mathbb{C} and at ∞ . 


Proof

We know there has to be $R > 0$ such that all finite poles of f is in $\overline{D}(0, R)$, denoted as P_1, P_2, \dots, P_k and we may know

$$F(z) = (z - P_1)^{n_1} \cdots (z - P_k)^{n_k} f(z)$$

has removable singularities on \mathbb{C} and then it suffices to show that F is rational. If ∞ is a removable singularity or pole of F , then the problem goes. If not, we know $F(1/z)$ has an essential singularity at 0 and then we may find $G(1/z)$ has infinitely many negative terms, which is a contradiction.

Definition 1.29

Let $f : U \rightarrow \mathbb{C}$ be holomorphic and has zeros but not identically zero, then we know f has the series expansion and call the first nonzero term determined by the least positive integer n as the order of z_0 as a zero of f . 

Lemma 1.15

If f is holomorphic on a neighborhood of a disc $\overline{D}(z_0, r)$ and has a zero of order n at z_0 and no other zeros in the closed disc, then

$$\frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f'(\xi)}{f(\xi)} d\xi = n$$



Proof We know

$$f(z) = (z - z_0)^n \sum_{j=n}^{\infty} \left[\sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0) (z - z_0)^{j-n} \right]$$

where we define

$$H(z) = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{\partial^j f}{\partial z^j}(z_0) (z - z_0)^{j-n}$$

which is holomorphic on an open disc containing $\overline{D}(z_0, r)$ and nonzero on the closed disc, so we may know that H'/H is holomorphic on some neighbourhood of the closed disc and since

$$\frac{f'(\xi)}{f(\xi)} = \frac{H'(\xi)}{H(\xi)} + \frac{n}{\xi - z_0}$$

we may know that the integral equals to n .

Proposition 1.21

Suppose that $f : U \rightarrow \mathbb{C}$ is a holomorphic on an open set $U \subset \mathbb{C}$ and that $\overline{D}(P, r) \subset U$. Suppose that f is nonvanishing on $\partial D(P, r)$ and that z_1, z_2, \dots, z_k are the zeros of f in the interior of the disc. Let n_l be the order of the zero of f at z_l , then

$$\frac{1}{2\pi i} \int_{|\xi - P|=r} \frac{f'(\xi)}{f(\xi)} d\xi = \sum_{l=1}^k n_l$$



Proof Let

$$H(z) = \frac{f(z)}{(z - z_1)^{n_1} (z - z_2)^{n_2} \cdots (z - z_k)^{n_k}}$$

and the rest is easy to be checked.

Lemma 1.16

If $f : U - \{Q\} \rightarrow \mathbb{C}$ is a nowhere zero holomorphic function on $U - \{Q\}$ with a pole of order n at Q and if $\overline{D}(Q, r) \subset U$, then

$$\frac{1}{2\pi i} \int_{\partial D(Q, r)} \frac{f'(\xi)}{f(\xi)} d\xi = -n$$



Proof We know $H(z) = (z - Q)^n f(z)$ has a removable singularity at Q where

$$\frac{H'(\xi)}{H(\xi)} = \frac{n}{\xi - Q} + \frac{f'(\xi)}{f(\xi)}$$

and the rest is easy to be checked.

Theorem 1.27

Suppose that f is a meromorphic function on an open set $U \subset \mathbb{C}$, that $\overline{D}(P, r) \subset U$ and that f has neither poles nor zeros on $\partial D(P, r)$. Then

$$\frac{1}{2\pi i} \int_{\partial D(P, r)} \frac{f'(\xi)}{f(\xi)} d\xi = \sum_{j=1}^p n_j - \sum_{k=1}^q m_k$$

where n_1, n_2, \dots, n_p are the multiplicities of the zeros z_1, z_2, \dots, z_p of f in $D(P, r)$ and m_1, m_2, \dots, m_q are the orders of the poles w_1, w_2, \dots, w_q of f in $D(P, r)$.



Proof Multiplying $(z - P_k)^{m_k}$ for the poles and dividing $(z - z_i)^{n_i}$ for the zeros.

Theorem 1.28

(The open mapping theorem) If $f : U \rightarrow \mathbb{C}$ is a nonconstant holomorphic function on a connected open set U , then $f(U)$ is an open set in \mathbb{C} .



Proof It suffices to show for any $Q \in f(U)$, there exists $\epsilon > 0$ such that $D(Q, \epsilon) \subset f(U)$. Assume that $f(P) = Q$ let $g(z) = f(z) - Q$ and we know there exists an $r > 0$ such that g can not be zero on $\overline{D}(P, r) - \{P\}$ by considering the series expansion and we know

$$\frac{1}{2\pi i} \int_{\partial D(P, r)} \frac{f'(\xi)}{f(\xi) - Q} d\xi = n$$

where n is the order of P as a zero of g , so we know there exists $\epsilon > 0$ such that $|g(\xi)| > \epsilon$ on $\partial D(P, r)$ by the compactness and we claim that $D(Q, \epsilon)$ is in $f(U)$. Define

$$N(z) = \frac{1}{2\pi i} \int_{\partial D(P, r)} \frac{f'(\xi)}{f(\xi) - z} d\xi$$

for $z \in D(Q, \epsilon)$ and it is well-defined since

$$|f(\xi) - z| \geq |g(\xi)| - |z - Q| > \epsilon - |z - Q| > 0$$

and then it is easy to check N is continuous on $D(Q, \epsilon)$, but it is integer-valued and hence it has to be n on $D(Q, \epsilon)$, which means there exists zeros for $f(\xi) - z$ inside the $D(P, r)$ and hence $D(Q, \epsilon) \subset f(D(P, r)) \subset f(U)$.

Lemma 1.17

Let $f : U \rightarrow \mathbb{C}$ be a nonconstant holomorphic function on a connected open set $U \subset \mathbb{C}$. Then the multiple points of f in U are isolated.



Proof Since f is nonconstant, the holomorphic function f' is not identically zero, and then we know the zeros of f' is isolated by theorem.1.19. And any multiple point of f is a zero of f' and hence the lemma holds.

Theorem 1.29

Suppose that $f : U \rightarrow \mathbb{C}$ be a nonconstant holomorphic function on a connected open set U such that $P \in U$ and $f(P) = Q$ with order k . Then there are numbers $\delta, \epsilon > 0$ such that each $q \in D(Q, \epsilon) - \{Q\}$ has exactly k distinct

preimages in $D(P, \delta)$ and each preimage is a simple point of f .



Proof There exist δ_1 such that $D(P, \delta_1) - \{P\}$ is a simple point of f . Then let $\delta, \epsilon > 0$ such that $Q \in f(D(P, \delta) - \{P\})$ and $D(Q, \epsilon) \subset f(D(P, \delta))$ without meeting $f(\partial D(P, \delta))$ since $f(D(P, \delta))$ is open, then for any $q \in D(Q, \epsilon) - \{Q\}$, we know

$$\frac{1}{2\pi i} \int_{\partial D(P, \delta)} \frac{f'(\xi)}{f(\xi) - q} d\xi = k$$

since the integral is continuous as a function of q and the problem goes.

Theorem 1.30

(Rouche's theorem) Suppose that $f, g : U \rightarrow \mathbb{C}$ are holomorphic functions on an open set $U \subset \mathbb{C}$. Suppose also that $\overline{D}(P, r) \subset U$ and that, for each $\xi \in \partial D(P, r)$,

$$|f(\xi) - g(\xi)| < |f(\xi)| + |g(\xi)|$$

Then

$$\frac{1}{2\pi i} \int_{\partial D(P, r)} \frac{f'(\xi)}{f(\xi)} d\xi = \frac{1}{2\pi i} \int_{\partial D(P, r)} \frac{g'(\xi)}{g(\xi)} d\xi$$



Proof Define

$$f_t : U \rightarrow \mathbb{C}$$

by $f_t = tf + (1 - t)g$ and we know f_t is nonzero for $t \in [0, 1]$, $\xi \in \partial D(P, r)$ since if $f_t(\xi) = 0$ for some t , we have

$$t(f(\xi) - g(\xi)) = -g(\xi), (1 - t)(f(\xi) - g(\xi)) = f(\xi)$$

and hence

$$|f(\xi) - g(\xi)| = |f(\xi)| + |g(\xi)|$$

which is a contradiction, then

$$I_t = \frac{1}{2\pi i} \int_{\partial D(P, r)} \frac{f'_t(\xi)}{f_t(\xi)} d\xi, t \in [0, 1]$$

will be continuous to t , however, it takes values in the integers, so I_t has to be constant and hence $I_0 = I_1$.

Theorem 1.31

(Hurwitz's theorem) Suppose that $U \subset \mathbb{C}$ is a connected open set and that $\{f_j\}$ is a sequence of nowhere vanishing holomorphic functions on U . If the sequence of nowhere vanishing holomorphic functions on U . If the sequence $\{f_j\}$ converges uniformly on compact subsets of U to a limit function f_0 , then either f_0 is nowhere vanishing or $f_0 = 0$.



Proof Consider a contradiction, we suppose f_0 is not constant 0 but $f_0(P) = 0$ for some $P \in U$. Then there has to be an $r > 0$ such f_0 is nonzero on $\overline{D}(P, r) - \{P\}$, then

$$I = \frac{1}{2\pi i} \int_{|\xi - P| = r} \frac{f'_0(\xi)}{f_0(\xi)} d\xi$$

is a positive integer, equal to the order of the zero of f at P .

However,

$$I_j = \frac{1}{2\pi i} \int_{|\xi - P| = r} \frac{f'_j(\xi)}{f_j(\xi)} d\xi$$

equals to 0 since f_j is nonzero everywhere, but $I_j \rightarrow I$ and there will be a contradiction.

Definition 1.30

A domain in \mathbb{C} is a connected open set. A bounded domain is a connected open set U such that there is an $R > 0$ with $|z| < R$ for all $z \in U$.



Theorem 1.32

(The maximum modulus principle) Let $U \subset \mathbb{C}$ be a domain. Let f be a holomorphic function on U . If there is a point $P \in U$ such that $|f(P)| \geq |f(z)|$ for all $z \in U$, then f is constant.



Proof Assume there is such a P and then if f is not constant, we know $f(U)$ is open and hence there have to be some $\zeta \in f(U)$ such that $|\zeta| \geq |f(P)|$.

Corollary 1.11

(Maximum modulus theorem) Let $U \subset \mathbb{C}$ be a bounded domain. Let f be a continuous function on \bar{U} that is holomorphic on U . Then the maximum value of $|f|$ on \bar{U} must occur on ∂U .

**Theorem 1.33**

Let $U \subset \mathbb{C}$ be a domain and let f be a holomorphic function on U . If there is a point $P \in U$ at which $|f|$ has a local maximum, then f is constant.



Proof If $|f|$ do have a local maximum, then we know f is constant on an open ball in U and hence f has to be constant since U is connected.

Proposition 1.22

Let f be holomorphic on a domain $U \subset \mathbb{C}$. Assume that f never vanishes. If there is a point $P \in U$ such that $|f(P)| \leq |f(z)|$ for all $z \in U$, then f is constant.

**Proposition 1.23**

(Schwarz's lemma) Let f be holomorphic on the unit disc, assume that $f(z) \leq 1$ on \mathbb{C} and $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If either $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation, i.e. $f(z) = \alpha z$ for some complex constant α of unit modulus.



Proof Consider $g(z) = f(z)/z$, where g have a removable singularity at 0 and hence we may consider it as a holomorphic function on unit disc, then notice $g(z) \leq 1/(1-\epsilon)$ on $D(0, 1-\epsilon)$ by the maximum modulus theorem and hence we know $g \leq 1$ on D , and hence $|f'(0)| \leq 1$.

If $|f(z)| = |z|$ for some $z \neq 0$, we know $|g(z)| = 1$ with g constant on D , and it is the same when $|g(0)| = 1$.

Theorem 1.34

(Schwarz-Pick) Let f be holomorphic on the unit disc with $|f(z)| \leq 1$ for all $z \in D$. Then for any $a \in D(0, 1)$ and with $f(a) = b$, we have

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}$$

Moreover, if $f(a_1) = b_1, f(a_2) = b_2$, then

$$\left| \frac{b_2 - b_1}{1 - \bar{b}_1 b_2} \right| \leq \left| \frac{a_2 - a_1}{1 - \bar{a}_1 a_2} \right|$$

if either the equalities above can be reached, then f is a conformal self-mapping to itself.



Proof If $c \in D(0, 1)$, define

$$\phi_c(z) = \frac{z - c}{1 - \bar{c}z}$$

and notice that

$$\left| \frac{z - c}{1 - \bar{c}z} \right| < 1$$

, iff $|z| < 1$ and it follows that ϕ_c is a holomorphic mapping from $D(0, 1)$ to itself with inverse ϕ_{-c} , then consider

$$g(z) = \phi_b \circ f \circ \phi_{-a}$$

and we know

$$|g(z)| \leq 1, g(0) = 0$$

and hence

$$g'(0) \leq 1, |g(z)| \leq z$$

For the second conclusion, let

$$h = \phi_{b_1} \circ f \circ \phi_{-a_1}(z) \leq |z|$$

and then we know

$$\phi_{b_1}(b_2) \leq |\phi_{a_1}(a_2)|$$

Definition 1.31

A real-valued function $u : U \rightarrow \mathbb{R}$ on an open set $U \subset \mathbb{C}$ is harmonic if it is C^2 on U and

$$\Delta u = 0$$



Lemma 1.18

If $u : D \rightarrow \mathbb{R}$ is a harmonic function on a disc D , then there is a holomorphic function $F : D \rightarrow \mathbb{C}$ such that $\operatorname{Re} F = u$ on D .



Definition 1.32

Any function v such that $u + iv$ is holomorphic is called a harmonic conjugate of u .



Corollary 1.12

If $u : U \rightarrow \mathbb{R}$ is a harmonic function on an open set $U \subset \mathbb{C}$, then $u \in C^\infty$.



Lemma 1.19

If U is a holomorphically simply connected open set and if $u : U \rightarrow \mathbb{R}$ is a harmonic function, then there is a C^2 harmonic function v such that $u + iv : U \rightarrow \mathbb{C}$ is holomorphic.



Proof Let

$$H = \frac{\partial}{\partial x}u - i \frac{\partial}{\partial y}u$$

and we know H is holomorphic on U and hence there exists F' such that $F' = H$, and it is easy to check that $(\operatorname{Re} F)' = u + c$ for some constant c and then we know $F - C = u + iv'$ is the required function.

Theorem 1.35

(The maximum principle for harmonic functions) If $u : U \rightarrow \mathbb{R}$ is harmonic on a connected open set U and if there is a point $P_0 \in U$ with the property that $u(P_0) = \sup_{Q \in U} u(Q)$, then u is constant on U .



Proof We may show that $\{z, u(z) = u(P_0)\}$ is open by considering $\operatorname{Re} H = u$ on $D(P_0, r)$ for H holomorphic and use the maximum modulus principle to e^H . And the set is closed in U is easy to be checked.

Corollary 1.13

(Minimum principle for harmonic functions) If $u : U \rightarrow \mathbb{R}$ is harmonic on a connected open set U and if there is a point $P_0 \in U$ with the property that $u(P_0) = \inf_{Q \in U} u(Q)$, then u is constant on U .



Corollary 1.14

Let $U \subset \mathbb{C}$ be a bounded, connected open set and let u be a continuous, real-valued function on the closure \overline{U} of

U that is harmonic on U . Then

$$\max_{\bar{U}} u = \max_{\partial U} u (\min_{\bar{U}} u = \min_{\partial U} u)$$



Theorem 1.36

(The mean value property) Suppose that $u : U \rightarrow \mathbb{R}$ is a harmonic function on an open set $U \subset \mathbb{C}$ and that $\bar{D}(P, r) \subset U$ for some $r > 0$. Then

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta}) d\theta$$



Proof Choose $s > r$ such that $D(P, s) \subset U$ and we may find $H : D(P, s) \rightarrow \mathbb{C}$ such that $H = u + iv$ holomorphic and we know

$$H(P) = \frac{1}{2\pi i} \int_{\gamma} \frac{H(\xi)}{\xi - P} = \frac{1}{2\pi i} \int_0^{2\pi} H(P + re^{i\theta}) d\theta$$

and the rest is easy to be checked.

Lemma 1.20

Let $a \in D(0, 1)$, then the holomorphic function $\phi_a(z) = \frac{z - a}{1 - \bar{a}(z)}$ has the following properties:

- a. ϕ_a is holomorphic and invertible on a neighborhood of $\bar{D}(0, 1)$
- b. $\phi_a : D(0, 1) \rightarrow D(0, 1)$ is one-to-one and onto
- c. $(\phi_a)^{-1} = \phi_{-a}$
- d. $\phi_a(a) = 0$



Lemma 1.21

If $u : U \rightarrow \mathbb{R}$ is harmonic and if $H : V \rightarrow U$ is holomorphic, then $u \circ H$ is harmonic.



Proof We prove a chain rule for complex functions:

$$\begin{aligned} \frac{\partial}{\partial z}(f \circ g) &= \frac{1}{2} \left(\frac{\partial}{\partial x} f(g_1, g_2) \frac{\partial}{\partial x} g_1 + \frac{\partial}{\partial y} f(g_1, g_2) \frac{\partial}{\partial x} g_2 \right) - \frac{i}{2} \left(\frac{\partial}{\partial x} f(g_1, g_2) \frac{\partial}{\partial y} g_1 + \frac{\partial}{\partial y} f(g_1, g_2) \frac{\partial}{\partial y} g_2 \right) \\ &= \frac{1}{2} \left(\frac{\partial}{\partial z} f(g_1, g_2) \frac{\partial}{\partial x} g_1 + \frac{\partial}{\partial \bar{z}} f(g_1, g_2) \frac{\partial}{\partial x} g_1 - i^{-1} \frac{\partial}{\partial z} f(g_1, g_2) \frac{\partial}{\partial x} g_2 + i^{-1} \frac{\partial}{\partial \bar{z}} f(g_1, g_2) \frac{\partial}{\partial x} g_2 \right) \\ &\quad - \frac{1}{2} \left(i \frac{\partial}{\partial z} f(g_1, g_2) \frac{\partial}{\partial y} g_1 + i \frac{\partial}{\partial \bar{z}} f(g_1, g_2) \frac{\partial}{\partial y} g_1 - \frac{\partial}{\partial z} f(g_1, g_2) \frac{\partial}{\partial y} g_2 + \frac{\partial}{\partial \bar{z}} f(g_1, g_2) \frac{\partial}{\partial y} g_2 \right) \\ &= \frac{\partial}{\partial z} f(g) \frac{\partial}{\partial \bar{z}} g_1 + \frac{\partial}{\partial \bar{z}} f(g) \frac{\partial}{\partial z} g_1 - i^{-1} \frac{\partial}{\partial z} f(g) \frac{\partial}{\partial z} g_2 + i^{-1} \frac{\partial}{\partial \bar{z}} f(g) \frac{\partial}{\partial z} g_2 \\ &= \frac{\partial}{\partial z} f(g) \frac{\partial}{\partial \bar{z}} g + \frac{\partial}{\partial \bar{z}} f(g) \frac{\partial}{\partial z} \bar{g} \end{aligned}$$

and similarly we know

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}(f \circ g) &= \frac{1}{2} \left(\frac{\partial}{\partial x} f(g_1, g_2) \frac{\partial}{\partial x} g_1 + \frac{\partial}{\partial y} f(g_1, g_2) \frac{\partial}{\partial x} g_2 \right) + \frac{i}{2} \left(\frac{\partial}{\partial x} f(g_1, g_2) \frac{\partial}{\partial y} g_1 + \frac{\partial}{\partial y} f(g_1, g_2) \frac{\partial}{\partial y} g_2 \right) \\ &= \frac{1}{2} \left(\frac{\partial}{\partial z} f(g_1, g_2) \frac{\partial}{\partial x} g_1 + \frac{\partial}{\partial \bar{z}} f(g_1, g_2) \frac{\partial}{\partial x} g_1 - i^{-1} \frac{\partial}{\partial z} f(g_1, g_2) \frac{\partial}{\partial x} g_2 + i^{-1} \frac{\partial}{\partial \bar{z}} f(g_1, g_2) \frac{\partial}{\partial x} g_2 \right) \\ &\quad + \frac{1}{2} \left(i \frac{\partial}{\partial z} f(g_1, g_2) \frac{\partial}{\partial y} g_1 + i \frac{\partial}{\partial \bar{z}} f(g_1, g_2) \frac{\partial}{\partial y} g_1 - \frac{\partial}{\partial z} f(g_1, g_2) \frac{\partial}{\partial y} g_2 + \frac{\partial}{\partial \bar{z}} f(g_1, g_2) \frac{\partial}{\partial y} g_2 \right) \\ &= \frac{\partial}{\partial z} f(g) \frac{\partial}{\partial \bar{z}} g_1 + \frac{\partial}{\partial \bar{z}} f(g) \frac{\partial}{\partial z} g_1 - i^{-1} \frac{\partial}{\partial z} f(g) \frac{\partial}{\partial \bar{z}} g_2 + i^{-1} \frac{\partial}{\partial \bar{z}} f(g) \frac{\partial}{\partial \bar{z}} g_2 \\ &= \frac{\partial}{\partial z} f(g) \frac{\partial}{\partial \bar{z}} g + \frac{\partial}{\partial \bar{z}} f(g) \frac{\partial}{\partial z} \bar{g} \end{aligned}$$

so

$$\Delta u(H) = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} u(H) = 4 \frac{\partial}{\partial z} \left(\frac{\partial}{\partial \bar{z}} u(H) \frac{\partial}{\partial \bar{z}} \bar{H} \right) = \frac{\partial}{\partial \bar{z}} (u(H)) \Delta \bar{H} = 0$$

Theorem 1.37

(The Poisson integral formula) Let $u : U \rightarrow \mathbb{R}$ be a harmonic function on a neighbourhood of $\overline{D(0,1)}$. Then for any $a \in D(0,1)$,

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) \frac{1 - |a|^2}{|a - e^{i\theta}|^2} d\phi$$



Proof Consider $u(\phi_a)$ is fine.

Theorem 1.38

Let f be a continuous function on $\partial D(0,1)$. Define

$$u(z) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta & \text{if } z \in D(0,1) \\ f(z) & \text{if } z \in \partial D(0,1) \end{cases}$$

then u is continuous on $\overline{D(0,1)}$ and harmonic on $D(0,1)$.

**Definition 1.33**

A continuous function $h : U \rightarrow \mathbb{R}$ on an open set $U \subset \mathbb{C}$ has the small circle mean value property if, for each point $P \in U$ there is an $\epsilon_P > 0$ such that $\overline{D(P, \epsilon_P)} \subset U$ and for every $0 < \epsilon < \epsilon_P$

$$h(P) = \frac{1}{2\pi} \int_0^{2\pi} h(P + \epsilon e^{i\theta}) d\theta$$

**Theorem 1.39**

If $h : U \rightarrow \mathbb{R}$ is a continuous function on an open set U with the SCMV property, then h is harmonic.

**Corollary 1.15**

If $\{h_j\}$ is a sequence of real-valued harmonic functions that converges uniformly on compact subsets of U to a function $h : U \rightarrow \mathbb{R}$, then h is harmonic on U .



Proof By theorem 7.4.2, it is easy to check that h has the SCMV property.