Homework0 - Kuijlaars

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Exercise 0.1

The semi-circle law is given by

$$d\mu_{sc} = \frac{1}{2\pi}\sqrt{4-x^2}dx \quad \text{on } [-2,2]$$

a. Compute its Stieltjes transform $F(z) := \int_{-2}^{2} \frac{d\mu_{sc}(x)}{z-x}$ for $z \in \mathbb{C} - [-2,2]$ by the means of contour integration.

Proof. a.We know

$$F(z) = \int_{-2}^{2} \frac{d\mu_{sc}(x)}{z-x} = \frac{1}{2\pi}\int_{-2}^{2} \frac{\sqrt{4-x^2}}{z-x}dx$$

and hence

$$F(z) = \frac{z+\sqrt{z^2-4}}{2}$$

b. We know

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \mathrm{Im}F(x-i\epsilon) &= \lim_{\epsilon\to 0} \mathrm{Im}\left(\frac{x-i\epsilon+\sqrt{x^2-4-\epsilon^2-2ix\epsilon}}{2}\right) \\
&= Im\left(\frac{\sqrt{x^2-4}}{2}\right) \\
&= \frac{\sqrt{}}{}
\end{aligned}$$

□

Exercise 0.2

Let μ be a measure on $\mathbb{C}$ with compact support. Show that U^μ is superharmonic on $\mathbb{C}$ and harmonic on $C - \mathrm{supp}(\mu)$

Proof. We know by Fubini's theorem

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} U^\mu(z_0+re^{i\theta})d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int \ln\frac{1}{|z_0+re^{i\theta}-s|}d\mu(s)d\theta \\
&= \int \frac{1}{2\pi}\int_0^{2\pi} \ln\frac{1}{|z_0+r^{i\theta}-s|}d\theta d\mu(s) \\
&\le \int \ln\frac{1}{|z_0-s|}d\mu(s) \\
&= U^\mu(z_0)
\end{aligned}$$

when $z_0 \in$ $\square$

Exercise 0.3

In our study of equilibrium measures we will associate with a function $V : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ a probablity measure μ on $\mathbb{R}$ and a constant l such that

$$2U^{\mu} + V = l, \quad \text{on the support of } \mu$$

$$2U^{\mu} + V \geq l, \quad \text{on } \mathbb{R}$$

Suppose that μ is a probability measure satisfying the equations above for some constant l. Let x_0 be a point where V assumes its minimum on $\mathbb{R}$, prove that $x_0 \in \operatorname{supp}(\mu)$.

Proof. If $x \notin \operatorname{supp}(\mu)$, we know there exists $\epsilon > 0$ such that $\mu((x_0 - \epsilon, x_0 + \epsilon)) = 0$ $\square$