Chapter 1

m.s. for measure space
mrb. for measurable
t.v.s. for a topological vector space

1.1 Topological Vector space

Definition 1.1

A vector space X is said to be a normed space if to every $x \in X$ there is associated a nonnegative real number $||x||$ such that
a. $||x+y|| \leq ||x|| + ||y||, x, y \in X$
b. $||\alpha x|| = |\alpha| ||x||$ *if* $x \in X$ *and* α *is scalar*
c. $||x|| > 0$ *if* $x \neq 0$.
A Banach space is a complete normed space.

♣

Definition 1.2

Suppose τ is a topology on a vector space X such that
a. every point of X is a closed set, and
b. the vector space operations are continuous w.r.t. τ

♣

Proposition 1.1

Let X be a topological vector space. For $a \in X, \lambda \neq 0$, define the translation operator T_a and the multiplication operator M_λ by

$$T_a(x) = a + x, M_\lambda(x) = \lambda x$$

then T_a and M_λ are homeomorphisms of X onto X.

♠

It can be induced by the continuity of addition and multiplication, also that of inverse.

Definition 1.3

By the proposition above, we know every vector space topology τ is translation-invariant, i.e. a set $E \subset X$ is open iff $a + E$ is open for any $a \in X$.
The local base means a collection $\mathscr{B}$ of neighbourhoods of 0 such that every neighborhood of 0 contains a member of $\mathscr{B}$, so the open sets of X will be the unions of translates of members of $\mathscr{B}$.
A metric d on a vector space X will be called invariant if $d(x+z, y+z) = d(x, y)$ for any $x, y, z \in X$.
A subset E of a topological space is said to be bounded if to every neighborhood V of 0 in X, there is a number $s > 0$ such that $E \subset tV$ for any $t > s$.

♣

Definition 1.4

In the following definitions, X always denotes a topological vector space, with topology τ.
a. X is locally convex if there is a local base $\mathscr{B}$ whose members are convex.
b. X is locally bounded if 0 has a bounded neighbourhood.
c. X is locally compact if 0 has a neighborhood whose closure is compact.
d. X is metrizable if τ is compatible with some metric d.
e. X is an F-space if its topology τ is induced by a complete invariant metric d.
f. X is a Frechet space if X is a locally convex F-space.
g. X is normable if a norm exists on X such that the metric induced by the norm is compatible with τ.

h. X has the Heine-Borel property if every closed and bounded subset of X is compact.

♣

Theorem 1.1

Suppose K and C are subsets of a topological vector space X, K is compact, C is closed and $K \cap C = \emptyset$. Then 0 has a neighbor hood V such that

$$(K+V) \cap (C+V) = \emptyset$$

♡

Proof For any W a neighbourhood of 0, we may find U a neighbourhood of 0 such that $U = -U$ and $U + U = W$, by consider there are V_1, V_2 neighbourhoods of 0 such that $V_1 + V_2 \subset W$, then let $U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$, and then we may find V symmetric such that $V + V \subset U$, then $V + V + V + V \subset W$, now we assume K is nonempty, and then for any $x \in K$, we may find V_x such that $x + V_x + V_x + V_x \cap C = \emptyset$ and then $X + V_x + V_x \cap C + V_x$ is empty since V_x is symmetric, then the rest is easy to be checked.

Theorem 1.2

If $\mathscr{B}$ is a local base for a topological vector sapce X, then every member of $\mathscr{B}$ contains the closure of some member of $\mathscr{B}$.

♡

Proof For $V \in \mathscr{B}$, we may find $U \in \mathscr{B}$ such that $U + U \subset V$ and hence $\overline{U} \subset V$.

Theorem 1.3

Every topological vector space is a Hausdorff space.

♡

Can be induced by theorem 1.1. directly.

Theorem 1.4

Let X be a t.v.s.

a. If $A \subset X$ then $\overline{A} = \cap(A+V)$ where V runs through all neighbourhoods of 0.

b. If $A \subset X$ and $B \subset X$, then $\overline{A} + \overline{B} \subset \overline{A+B}$.

c. If Y is a subspace of X, so is $\overline{Y}$.

d. If C is a convex subset of X, so are $\overline{C}$ and C°.

e. If B is a balanced subset of X, so is $\overline{B}$; if also $0 \in B^\circ$ then B° is balanced.

f. If E is a bounded subset of X, so is $\overline{E}$.

♡

Proof a. It suffices to show that $\cap(A+V)$ is closed, if for any V, $x + V \cap A$ nonempty, then if $x \notin A + U$ then $x - U \cap A$ empty and hence a contradiction. So $x \in \cap(A+V)$ and we are done.

b. For any $x \in \overline{A}, y \in \overline{B}, V$ a neighbourhood of 0, we know there exists U_1, U_2 neighbourhood of 0 such that $x + U_1 + y + U_2 \subset x + y + V$ and hence $x + y + V \cap A + B \supset (x + U_1 + y + U_2) \cap A + B$ is always nonempty and we are done.

c. If $x, y \in Y$ is an accumulation, then for any $U, V \in \beta$, we know there will be $x_0, y_0 \in Y \cap U, Y \cap V$ then we know $\lambda x_0 + y_0 \in \lambda x_0 + y_0 + \lambda U + V$, and since hence the problem goes by choosing $U + V$.

d. We may know that

$$tC^\circ + (1-t)C^\circ \subset C$$

and since the left side is open, so it is easy to check that $tC^\circ + (1-t)C^\circ \subset C^\circ$, and we are done.

Notice $\alpha\overline{A} = \overline{\alpha A}$ and we may know

$$t\overline{C} + (1-t)\overline{C} \subset \overline{C}$$

and we are done.

e. For $0 \leq |\alpha| \leq 1$, we know $\alpha B \subset B$ and hence $\alpha\overline{B} \subset \overline{\alpha B} \subset \overline{B}$. For $0 < |\alpha| \leq 1$, we know $\alpha B^\circ \subset (\alpha B)^\circ$ and $\alpha^{-1}(\alpha B)^\circ \subset B$ and hence $\alpha^{-1}(\alpha B)^\circ \subset B^\circ$ so we know $\alpha B^\circ = (\alpha B)^\circ$. And then $\alpha B^\circ \subset B^\circ$ and if $0 \in B^\circ$, the equality holds for $\alpha = 0$.

f.For any V, there exists $s>0$ such that $t>s$ implies $E\subset tV$, then we know there exists W such that $\overline{W}\subset V$ and then there exists s' such that $\overline{E}\subset t'\overline{W}\subset t'V$ for any $t'>s'$ and we are done.

Theorem 1.5

In a topological vector space X

a. every neighbourhood of 0 contains a balanced neighborhood of 0

b. every convex neighborhood of 0 contains a balanced convex neighbourhood of 0.

Proof a. Suppose U is a neighbourhood of 0 in X. We know there exists V and $\delta>0$ such that $\alpha V\subset U$ if $|\alpha|<\delta$, then let $W=\bigcup_{|\alpha|<\delta}\alpha V$ and we are done.

b. Suppose U is a convex neighborhood, consider $V=\cap_{|\alpha=1|}\alpha U$ and let W be as in (a), then we may check that $W\subset V$, then we know V is balanced by choose $0\le r\le 1, |\beta|=1$ and we have

$$r\beta V=\cap_{|\alpha|=1}r\beta\alpha U=\cap_{|\alpha|=1}r\alpha U\subset V$$

and hence V balanced, and so is V° since which containing W and hence 0, and it is convex, we are done.

Corollary 1.1

a. Every topological vector space has a balanced local base.

b. Every locally convex space has a blanced convex local base.

Theorem 1.6

Suppose V is a neighbourhood of 0 in a topological vector space X.

a. If $0<r_1<r_2<\cdots$ and $r_n\to\infty$, then

$$X=\bigcup_{n=1}^{\infty}r_nV$$

b. Every compact subset K of X is bounded.

c. If $\delta_1>\delta_2>\cdots$ and $\delta_n\to 0$, and if V is bounded, then the collection

$$\{\delta_nV\}$$

is a local base for X.

Proof a. For $x\in X$ and V a neighbourhood of 0, since $\alpha\mapsto\alpha x$ is continuous, then we know $\{\alpha,\alpha x\in V\}$ is open and containing 0, so we may know $(1/r_n)x\in V$ for large n and hence $x\in r_nV$ for some n.

b. We know for any V neighbourhood of 0, there are finite r_n such that $K\subset\bigcup_{i=1}^n r_iV$.

c. Let U be a neighbourhood of 0, then if V is bounded, there exists $s>0$ such that $V\subset tU$ for all $t>s$. Then we know there exists n such that $s\delta_n<1$ and hence $V\subset(1/\delta_n)U$ and we are done.

Theorem 1.7

Let X and Y be topological vector spaces. If $\Lambda:X\to Y$ is linear and continuous at 0, then Λ is continuous. In fact, Λ is uniformly continuous, i.e. to each neighborhood W of 0 in Y corresponds a neighborhood V of 0 in X such that

$$y-x\in V\implies\Lambda y-\Lambda x\in W$$

Theorem 1.8

Let Λ be a linear functional on a topological vector space X. Assume $\Lambda\neq 0$ for some $x\in X$. Then each of the following four properties implies the other three

a. Λ is continuous.

b. $\mathcal{N}(\Lambda)$ is closed.

c. $\mathcal{N}(\Lambda)$ is not dense in X.

d. Λ is bounded in some neighborhood V of 0.

Proof (a) implies (b) is trivial. (b) implies (c) is trivial. Now we consider (c) implies (d), we know there exists x and V balanced such that

$$(x+V)\cap\mathcal{N}(\Lambda)=\emptyset$$

since ΛV is a balanced subset, so ΛV is bounded or $\Lambda V = K$ since it is balanced. Then we know if $\Lambda V = K$, there is y such that $\Lambda y = -\Lambda x$ and then $x+y\in\mathcal{N}(\Lambda)$, which is a contradiction.

(c) implies (d) is trivial.

Lemma 1.1

If X is a complex topological vector space and $f:\mathbb{C}^n\to X$ is linear, then f is continuous. ♡

Proof Let e_i be the standard basis of $\mathbb{C}^n$ and let $u_i = f(e_i)$, then for $z=(z_i)\in C^n$ we know $f(z)=z_1u_1+\cdots z_nu_n$. And then the continuity is secured by that of addition and scalar multiplication.

Theorem 1.9

If n is a positive integer and Y is an n-dimensional subspace of a complex topological vector space X, then

a. every isomorphism of $\mathbb{C}^n$ onto Y is a homeomorphism and

b. Y is closed. ♡

Proof a. Suppose $f:\mathbb{C}^n\to Y$ is an isomorphism. This means that f is linear, one-to-one, and $f(\mathbb{C}^n)=Y$. Put $K=f(\partial D)$, then we know K is compact since f is continuous, and $f(0)=0$. So there is a balanced neighborhood V of 0 in X which does not intersect K. Then $f^{-1}(V)$ is therefore disjoint from S. Since f is linear, E is balanced and henced connected. So $E\subset D$ and we know $f^{-1}(V\cap Y)\subset D$, so we know f^{-1} is continuous by theorem 1.8.d.

b. Let $p\in\overline{Y}$ and we know for some $t>0$, $p\in tV$ and then p is in the closure of

$$Y\cap(tV)\subset f(tB)\subset f(t\overline{B})$$

and then $p\in f(t\overline{B})\subset Y$.

Theorem 1.10

Every locally compact topological vector space X has finite dimension. ♡

Proof We consider V is a neighbourhood of 0 and $\overline{V}$ is compact, then we know there exists x_i such that

$$\overline{V}\subset\sum(x_i+2^{-1}V)$$

and let Y be the subspace generated by x_i. We know

$$V\subset Y+\frac{1}{2}V$$

and then we know $\frac{1}{2}V\subset Y+\frac{1}{4}V$ since Y is a subspace and by induction we know

$$V\subset\cap_n(Y+2^{-n}V)$$

since V is bounded and we know $2^{-n}V$ is a local base and hence $V\subset\overline{Y}=Y$, so then we know $X=\bigcup_n nV=Y$ and hence X is finite dimensional.

Theorem 1.11

If X is a locally bounded topological vector space with the Heine-Borel property, then X has finite dimention. ♡

Proof We know there is a neighbourhood V of 0 is bounded, then we know $\overline{V}$ is bounded and hence compact. So X is locally compact and we are done.

Theorem 1.12

If X is a topological vector space with a countable local base, then there is a metric d on X such that

a. d is compatible with the topology of X

b. the open balls centered at 0 are balanced and

c. d is invariant
If X is locally convex, then d can be chosen to satisfy
d. all open balls are convex.

Proof a. We know we may choose balanced local base V_n such that

$$V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n$$

when X is locally convex, this local base can be chosen to be convex.

Let D be the set of all 2-adic rational numbers r with finite positions to be 1, let $A(r) = X$ for $r \geq 1$ and

$$A(r) = r_1 V_1 + \cdots$$

for $0 \leq r < 1$ and define $f(x) = \inf\{r, x \in A(r)\}$ with $d(x, y) = f(x - y)$, then we know d is a metric by

$$A(r) + A(s) \subset A(r + s)$$

and then we may know $f(x + y) \leq f(x) + f(y)$. Notice for $x \neq 0$ obviously, there is a V_n not containing x, and then $f(x) > 0$ and $f(0) = 0$. For $\delta < 2^{-n}$, we may know $B_\delta(0) \subset V_n$ and hence $B_\delta(0)$ will be a local base of (x). Notice $A(r)$ is balanced, so we know B is balanced and we are done.

d. If V_n convex, then $A(r)$ convex and we are done.

Definition 1.5

a. Suppose d is a metric on a set X. x_n is a Cauchy sequence if it is Cauchy under d.
b. For a topological vector space, x_n is Cauchy means for a local base $\mathcal{B}$ and $V \in \mathcal{B}$, there always exists a N such that $x_n - x_m \in V$ if $n, m > N$.
c. It is easy to check if τ is compatible to an invariant metric d, then a seq is d-Cauchy iff it is τ-Cauchy. With corollary thatt d_1, d_2 invaiant metrics on a vector space X, we know d_1, d_2 have the same Cauchy seqs and d_1 complete iff d_2 complete.

Theorem 1.13

Suppose that (X, d_1) and (Y, d_2) are metric spaces, and (X, d_1) is complete. If E is a closed set in X, $f : E \to Y$ is continuous and

$$d_2(f(x'), f(x'')) \geq d_1(x', x'')$$

for all $x', x'' \in E$, then $f(E)$ is closed.

Proof Choose any accumulation is fine.

Theorem 1.14

Suppose Y is a subspace of a topological vector space X, and Y is an F-space. Then Y is a closed subspace of X.

Proof Let $B_n = \{y : y \in Y, d(y, 0) < n^{-1}\}$ amd U_n be a neighbourhood of 0 in X, such that $Y \cap U_n = B_n$, and choose symmetric neighrboods V_n of 0 in X such that $V_n + V_n \subset U_n$ and $V_{n+1} \subset U_n$. Then suppose $x \in \overline{Y}$ and $E_n = Y \cap (x + V_n)$, then if $y_1, y_2 \in E_n$, we know $y_1 - y_2 \in U_n$ and hence in B_n. Then we know $\cap E_n$ is a singelton $\{y_0\}$. By the way, we may consider

$$F_n = Y \cap (x + W \cap Y_n)$$

and hence $\cap F_n$ is a singleton y_0 and then y_0 is in all $x + W$, so $y_0 = x$ and we are done.

Theorem 1.15

a. If d is a translation invariant metricd on a vector space X, then

$$(nx, 0) \leq nd(x, 0)$$

b. If $\{x_n\}$ is a seq in a metrizable tvs X and if $x_n \to 0$ as $n \to \infty$, then there are positive scalars $\gamma_n \to \infty$ and $\gamma_n x_n \to 0$.

Proof We only prove (b) by considering $n_k \leq n < n_{k+1}$ such that $d(x_n, 0) < k^{-2}$.

Proposition 1.2

Any Cauchy seq is bounded.

Proof For any W, consider V balanced with $V + V \subset W$, then we may find N such that $x_n - X_N \in V$ and $s > 0$ such that $X_N \in sV$, and then $x_n \in sV + V \subset \max s, 1V \subset \max s, 1W$ and we are done.

Theorem 1.16

The following two properties of E in a tvs are equivalent

a. E is bounded

b. If x_n is a seq in E and α_n is a seq of scalars such that $\alpha_n \to 0$ as $n \to \infty$, then $\alpha_n x_n \to 0$ as $n \to \infty$.

Proof (a) implies (b) is trivial.

(b) implies (a) find r_n, V such that $r_n V$ does not contain E and there will be a contradiction.

Definition 1.6

Suppose X and Y are tvss and $\Lambda : X \to Y$ is linear. Then Λ is bounded if Λ maps bounded sets into bounded sets.

Theorem 1.17

Suppose X and Y are topological vector spaces and Λ is linear. Among the following four properties of Λ, we have (a) $\implies$ (b) $\implies$ (c) and if X is metrizable, then also (c) $\implies$ (d) $\implies$ (a).

a. Λ is continuous

b. Λ is bounded

c. If $x_n \to 0$, then Λx_n is bounded.

d. If $x_n \to 0$ then $\Lambda x_n \to 0$.

Proof (a) $\implies$ (b), we know that for any bounded E, we know for any $V \subset Y$, we have $f^{-1}(V)$ is an open neighbourhood of 0 in X and there exists s such that $E \subset tf^{-1}V$ for any $t > s$ and then $f(E) \subset tV$ for any $t > s$ and hence $f(E)$ is bounded.

(b) $\implies$ (c), we know $x_n \to 0$ and hence x_n is bounded, and we are done.

Now we assume that X is metrizable, then we know since $x_n \to 0$, then we may find $\gamma_n \to \infty$ such that $\gamma_n x_n \to 0$ and then $\Lambda(\gamma_n x_n)$ bounded, so $\Lambda x_n \to 0$.

(d) $\implies$ (a), we know for V an neighbourhood of 0 open in Y, then if there exists $x_n \in f^{-1}(V)^c$ such that $x_n \to 0$, then there will be a contradiction and hence there exists an neighborhood U in X such that $f(U) \subset V$ and we are done by use a union.

Definition 1.7

A seminorm on a vector space X is a real-valued function p on X such that

a. $p(x + y) \leq p(x) + p(y)$ and

b. $p(\alpha x) = |\alpha| p(x)$

for all x and y in X and all scalars $\alpha = \alpha$.

A family P of seminorms on X is said to be separating if to each $x \neq 0$ corresponds at least one $p \in P$ with $p(x) \neq 0$.

Then considering a convex set $A \subset X$ which is absorbing, i.e. for any x there exists some $t = t(x) > 0$ such that $x \in tA$.

The Minkowski functional μ_A of A is defined by

$$\mu_A(x) = \inf\{t > 0, t^{-1}x \in A\}$$

♣

Theorem 1.18

Suppose p is a seminorm on a vector space X. Then

a. $p(0) = 0$.

b. $|p(x) - p(y)| \leq p(x - y)$.

c. $p(x) \geq 0$.

d. $p(x) = 0$ is a subspace of X.

e. The set $B = \{x, p(x) < 1\}$ is convex, balanced, absorbing, and $p = \mu_B$.

♡

Proof It suffices to show (e). It is obviously B is balanced, absorbing. And since for any $t > p(x)$, we know $p(t^{-1}x) = t^{-1}p(x) < 1$ and hence $\mu_B(x) \leq p(x)$ and similarly we know $\mu_B(x) \geq p(x)$ are we are done.

Theorem 1.19

Suppose A is a convex absorbing set in a vector space X. Then

a. $\mu_A(x + y) \leq \mu_A(x) + \mu_B(y)$.

b. $\mu_A(tx) = t\mu_A(x)$ if $t \geq 0$.

c. μ_A is a seminorm if A is balanced.

d. If $B = \{x, \mu_A(x) < 1\}$ and $C = \{x : \mu_A(x) \leq 1\}$, then $B \subset A \subset C$ and $\mu_B = \mu_A = \mu_C$.

♡

Proof There is no need to check (a),(b),(c).

Notice $0 \in A$ and we know for any $x \in X$ and $t > \mu_A(x)$, $t^{-1} \in A$ and and hence $B \subset A \subset C$ and then we know $\mu_B \geq \mu_A \geq \mu_C$. For $x \in X$, choose s, t such that $\mu_C(x) < s < t$ and we know $x/s \in C$ and $\mu_A(x/t) < 1$ and we know $x/t \in B$, so $\mu_B(x) \leq t$ and then $\mu_B(x) \leq \mu_C(x)$ and we are done.

Theorem 1.20

Suppose β is a convex balanced local base in a topological vector space X. Associagte to every $V \in \beta$ denote its Minkowski functional μ_V and then

a. $V = \{x \in X, \mu_V(x) < 1\}$ for every $V \in \beta$ and

b. $\{\mu_V, V \in \beta\}$ is a separating family of continuous seminorms on X.

♡

Theorem 1.21

a. We know $\{x \in X, \mu_V(x) < 1\} \subset V$ and for any $x \in V$, $x/t \in V$ for some $t \in 1$ since V is open, and then we know $\mu_V(x) < 1$.

b. We have already know that μ_V are seminorms and separating since for $x \neq y$, we may find V such that $x - y \notin V$ and then $\mu_V(x - y) \geq 1$. For $r > 0$, we know $|\mu_V(x) - \mu_V(y) < r$ if $x - y \in rV$ and hence μ_V is continuous.

♡

Theorem 1.22

Suppose P is a separating famuly of seminorms on a vector space X. Associate to each $p \in P$ and to each positive number n the set

$$V(p, n) = \{x, p(x) < 1/n\}$$

Let β be the collection of all finite intersections of the sets $V(p, N)$. Then β is a convex balanced local baase for a topology τ on X, which turns X into a locally convex space such that

a. every $p \in P$ is continuous and

b. a set $E \subset X$ is bounded iff every $p \in P$ is bounded on E.

♡

Proof Consider the topology to be all unions of translates of members in β.

For $x \neq 0$, we know $p(x) > 0$ for some $p \in P$ and hence $\{0\}$ is a closed set and hence all singelton. For U a neighborhood of 0, we may find $\cap V(p_i, n_i) \subset U$ and hence $V + V \subset U$ where $V = \cap V(p_i, 2n_i)$. Also $x \in sV$ for some $s > 0$, let $y \in x + tV$ and $|\beta - \alpha| < 1/s$ where $\alpha x \in U$ and $t = s/(1 + |\alpha|s)$, we know

$$|\beta y - \alpha x| \subset |\beta| tV + |\beta - \alpha| sV \subset V + V \subset U$$

since $|\beta| t \leq 1$ and V balanced. Now we know (X, τ) is a topological vector space.

We know that p is continuous by $V(p, n)$ open.

Now we prove (b), it suffices to show the necessity, which is obvious.

Theorem 1.23

A tvs X is normable iff its origin has a convex bounded neighborhood.

Proof It suffices to show the necessity, V is a convex bounded neighborhood of 0. Then V containes a convex balanced neighborhood U of 0 and U is also bounded, define $||x|| = \mu(x)$ where μ is the Minkowski functional of U, then rU form a local base for the topology of X. If $x \neq 0$, then $x \in rU$ for some $r > 0$ and hence $||x|| > 0$, then we know $|| \cdot ||$ is a norm with $\{x, ||x|| < 1\} = U$ and we are done.

Now we use a proposition to summarize the chapter.

Proposition 1.3

Here is a list of some relations between these properties of a topological vector space X.

a. If X is locally bounded, then X has a countable local base.

b. X is metrizable iff X has a countable local base.

c. X is normable iff X is locally convex and locally bounded.

d. X has finite dimension iff X is locally compact.

e. If a locally bounded space X has the Heine-Borel property, then X has finite dimension.

Proof a. δV will be a local base.

b. Consider theorem 1.12.

c. Consider theorem 1.3.

d. Consider $|a_i| \leq 1$.

e. Consider theorem 1.11.

Definition 1.8

(The spaces $C(\Omega)$) If Ω is a nonempty open set in $\mathbb{R}^n$, then Ω is the union of countably many compact sets $K_n \neq \emptyset$ which can be chosen so that K_n lies in the interior of K_{n+1}. Then define the topology on $C(\Omega)$ by the seminorms

$$p_n(f) = \sup |f(x)|, x \in K_n$$

Proposition 1.4

$C(\Omega)$ is a Frechet space. And $E \subset C(\Omega)$ is bounded iff there are numbers $M_n < \infty$ such that $p_n(f) \leq M_n$ for all $f \in E$. $C(\Omega)$ is not loacally bounded.

Proof We may define

$$d(f, g) = \max_n \frac{2^{-n} p_n(f - g)}{1 + p_n(f - g)}$$

and it is easy to check that f_i converges uniformly on K_n to $f \in C(\Omega)$ and easy to check that $d(f, f_i) \to 0$. And notice $V_n = \{f \in C(\Omega), p_n(f) < n^{-1}\}$.

A set E is bounded iff there are M_n such that $|f(x)| \leq M_n, x \in K_n$ and since V_n contains f such that $p_{n+1}(f)$ large arbitrarily and we know $C(\Omega)$ is not locally bounded.

Chapter 2 Completeness

Definition 2.1

Let S be a topological space, $E \subset S$ is said to be nowhere dense if its closure $\overline{E}$ has an empty interior. The sets of the first category in S are those countable unions of nowhere dense sets. ♣

Theorem 2.1

If S is either

a. a complete metric space, or

b. a locally compact Hausdorff space.

then the intersection of every countable collection of dense open subsets of S is dense in S. ♡

Definition 2.2

Suppose X, Y are tvs and Γ is a collection of linear mappings from X to Y, we say Γ is equicontinuous if to every neighbourhood W of 0 in Y there corresponds a neighborhood V of 0 in X such that $\Lambda(V) \subset W$ for all $\Lambda \in \Gamma$ ♣

Theorem 2.2

Suppose X and Y are topological vector spaces, Γ is an equicontinuous collection of linear mappings from X into Y, and E is a bounded subset of X. Then Y has a bounded subset F such that $\Lambda(E) \subset F$ for every $\Lambda \in \Gamma$. ♡

Proof Let F be the unions of the sets $\Lambda(E)$ for $\Lambda \in \Gamma$. Let W be a neighborhood of 0 in Y, we know there is V neighborhood of 0 in$X$ such that $\Lambda(V) \subset W$, then we know F is bounded by E is bounded.

Theorem 2.3

Suppose X and Y are topological vector spaces, Γ is a collection of continuous linear mappings from X into Y, and B is the set of all $x \in X$ whose orbits

$$\Gamma(x) = \{\Lambda x, \Lambda \in \Gamma\}$$

are bounded in Y.

If B is of the second category in X, then $B = X$ and Γ is equicontinuous. ♡

Proof Choose balanced neighborhoods W and U of 0 in Y such that $\overline{U} + \overline{U} \subset W$. Let $E = \bigcap_{\Lambda \in \Gamma} \Lambda^{-1}(\overline{U})$. If $x \in B$, then $\Gamma(x) \subset nU$ for some n, and hence $x \in nE$ and hence

$$B \subset \bigcup_{n=1}^{\infty} nE$$

so we know E is closed and has an interior point x, then we know $x - E$ contains a neighborhood V of 0 in X and then

$$\Lambda(V) \subset \Lambda x - \Lambda(E) \subset W$$

and hence Γ is equicontinuous.

Then we know Γ is uniformly bounded, which means Γx is bounded in Y and hence $B = X$.

Chapter 3 Distribution theory

Definition 3.1

(The space $\mathscr{D}(\Omega)$)
Let $\mathscr{D}(\Omega) = \bigcup_{K\subset\Omega, K \text{ compact}} \mathscr{D}_K$. ♣

Consider the norms

$$||\phi||_N = \max\{|D^\alpha\phi(x)|, \in \Omega, |\alpha| \le N\}$$

for $\phi \in \mathscr{D}(\Omega)$ and we claim the restriction on these norms to any fixed $\mathscr{D}_K$ induce the same topology on $\mathscr{D}_K$ by the seminorms p_N. Here we know

$$||\phi_N|| \le ||\phi_{N+1}|| \quad p_N(\phi) \le p_{N+1}(\phi)$$

and for suffient large N, $||\cdot||_N = p_N$ on $\mathscr{D}_K$ and we are done.

For the topology induced by this norms, we know the induced metric is not compact, since consider any $\phi \in \mathscr{D}(\mathbb{R})$ and let

$$\varphi_m = \sum_{k=1}^{m} \frac{1}{2^k}\phi(x-m)$$

we know the limit exists but does not have compact support, and also the sequence is Cauchy under the metric.

Definition 3.2

Let Ω be a nonempty open set in $\mathbb{R}^n$
a. For every compact $K \subset \Omega$, τ_K denotes the Frechet topology of $\mathscr{D}_K$ induced by the norms.
b. β is the collection of all convex balanced sets $W \subset \mathscr{D}(\Omega)$ such that $\mathscr{D}_K \cap W \in \tau_K$ for every compact $\subset \Omega$.
c. τ is the collection of all unions of sets of the form $\phi + W$ with $\phi \in \mathscr{D}(\Omega)$ and $W \in \beta$. ♣

Theorem 3.1

a. τ is a topology in $\mathscr{D}(\Omega)$ and β is a local base for τ.
b. τ makes $\mathscr{D}(\Omega)$ into a locally convex topological vector space. ♡

Proof a. It suffices to show that for $V_1, V_2 \in \tau$ and $\phi \in V_1 \cap V_2$, there is $W \in \beta$ such that

$$\phi + W \subset V_1 \cap V_2$$

We know there is $\phi_i \in \mathscr{D}(\Omega)$ and $W_i \in \beta$ such that

$$\phi \in \phi_i + W_i \subset V_i$$

Choose K so that D_K contains ϕ_1, ϕ_2 and ϕ. Since $\mathscr{D}_K \cap W_i$ is open, so we know $\phi - \phi_i \in (1-\delta_i)W_i$ for some $\delta_i > 0$. The convexity of W_i implies therefore that

$$\phi - \phi_i + \delta_i W_i \subset (1-\delta_i)W_i + \delta_i W_i = W_i$$

so that $\phi + \delta_i W_i \subset \phi_i + W_i \subset V_i$ hence $W = (\delta_1 W_1) \cap (\delta_2 W_2)$ will satisfiy the requirement.

Suppose next that ϕ_1, ϕ_2 are distinct elements of $\mathscr{D}(\Omega)$ and put

$$W = \{\phi \in \mathscr{D}(\Omega), ||\phi||_0 < ||\phi_1 - \phi_2||_0\}$$

then we know $W \in \beta$ and ϕ_1 is not in W, so we know singelton will be closed. And $(\phi_1 + \frac{1}{2}W) + (\phi_2 + \frac{1}{2}W) = (\phi_1 + \phi_2) + W$ will show the continuity of addition under τ.

For any $\phi_0 \in \mathscr{D}(\Omega), W \in \beta$, we know there is always some $\mathscr{D}_K$ containing ϕ_0 and $W \cap \mathscr{D}_K$ is open in $\mathscr{D}_K$, so there exists $\delta > 0$ such that $\delta\phi_0 \in 2^{-1}W$.

For

$$\alpha\phi - \alpha_0\phi_0 = \alpha(\phi - \phi_0) + (\alpha - \alpha_0)\phi_0$$

we know

$$\alpha\phi - \alpha_0\phi_0 \in \alpha\alpha cW + 2^{-1}W$$

for any $|\alpha - \alpha_0| < \delta$ and $\phi - \phi_0 \in cW$, then we know let $c = 1/2(|\alpha_0| + \delta)$ will be fine.

Theorem 3.2

a. A convex balanced subset V of $\mathscr{D}(\Omega)$ is open iff $V \subset \beta$.
b. The topology τ_K of any $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ coincides with the subspace topology that $\mathscr{D}_K$ inherits from $\mathscr{D}(\Omega)$.
c. If E is a bounded subset of $\mathscr{D}(\Omega)$, then $E \subset \mathscr{D}_K$ for some $K \subset \Omega$ and there are numbers $M_N < \infty$ such that every $\phi \in E$ satisfies the inequalities

$$||\phi||_N \leq M_N$$

d. $\mathscr{D}(\Omega)$ has the Heine-Borel property.
e. If ϕ_i is a Cauchy sequence in $\mathscr{D}(\Omega)$, then $\{\phi_i\} \subset \mathscr{D}_K$ for some compact $K \subset \Omega$ and

$$\lim_{i,j\to\infty} ||\phi_i - \phi_j||_N = 0$$

f. If $\phi_i \to 0$ in the topology of $\mathscr{D}(\Omega)$, then there is a compact $K \subset \Omega$ which contains the support of every ϕ_i and $D^\alpha\phi_i \to 0$ uniformly.
g. In $\mathscr{D}(\Omega)$, every Cauchy sequence converges. ♡

Proof a. If $V \in \tau$, for $\phi \in \mathscr{D}_K \cap V$, we know $\phi + W \in V$ for some $W \in \beta$ and then

$$\phi + (\mathscr{D}_K \cap W) \subset \mathscr{D}_K \cap V$$

and hence $\mathscr{D}_K \cap V \in \tau_K$. The opposite direction is trivial.

b. For any $B = ||\phi||_N < \delta$, we know it is convex and balanced in $\mathscr{D}(\Omega)$ with $B \cap \mathscr{D}_K$ is open in $\mathscr{D}_K$, so we know τ_K is a subtopology of the subspacce topology inherited from $\mathscr{D}(\Omega)$.

For any W convex and balanced, we know $W \cap \mathscr{D}_K$ is always open and hence the subspace topology is a subset of τ_K and we are done.

c. Consider E bounded but not in $\mathscr{D}_K$ for any K, then there are $\phi_m \in E$ wuth $x_m \in \Omega$ with no limit point in Ω and $\phi_m(x_m) \neq 0$. Let W be the set of ϕ such that

$$|\phi(x_m) < m^{-1}|\phi_m(x_m)|$$

we know $\mathscr{D}_K \cap W \in \tau_K$ and hence $W \in \beta$, but $\phi_m \in mW$ and hence there is no rW containing E.

Then every bounded E of $\mathscr{D}(\Omega)$ lies in some $\mathscr{D}_K$ and hence for each norm, there exists M_N such that $||\phi||_N \leq M_N$ on E.

d. Follows from C.

e. Since every ϕ_i is bounded, we know $\phi \in \mathscr{D}_K$ for some K and hence they are also Cauchy in $\mathscr{D}_K$.

f. Follows from e.

g. Follows from (b), (e) and the completeness of $\mathscr{D}_K$.

Theorem 3.3

Suppose Λ is a linear mapping of $\mathscr{D}(\Omega)$ into a locally convex space Y. Then each of the following four properties implies the others
a. Λ is continuous.
b. Λ is bounded .
c. If $\phi_i \to 0$ in $\mathscr{D}(\Omega)$, then $\Lambda\phi_i \to 0$ in Y.
d. The restrictions of Λ to every $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ are continuous. ♡

Proof (a) implies (b) follows the conclusion in tvs.

(b) implies (c), we know ϕ_i will be in some $\mathscr{D}_K$ and $\Lambda|_{\mathscr{D}_K}$ is bounded , so $\Lambda\phi_i \to 0$ in Y.

(c) implies (d), for $\phi_i \to 0$ in $\mathscr{D}_K$, we know $\phi_i \to 0$ in $\mathscr{D}(\Omega)$ and then we know Λ is continuous by metrilizing $\mathscr{D}_K$.

(d) implies (a), for any V convex balanced neighborhood of Y, we know $\Lambda^{-1}(V)$ is convex and balanced with $\mathscr{D}_K \cap \Lambda^{-1}(V)$ is open, so $\Lambda^{-1}(V)$ is open in τ and we are done.

Corollary 3.1

D^α is a continuous mapping of $\mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$. ♡

Definition 3.3

A linear functional on $\mathscr{D}(\Omega)$ which is continuous is called a distribution. ♣

Theorem 3.4

If Λ is a linear functional on $\mathscr{D}(\Omega)$, the following two conditions are equivalent
a. $\Lambda \in \mathscr{D}'(\Omega)$.
b. To every $K \subset \Omega$, corresponds a nonnegative ineger N and $C < \infty$ such that

$$|\Lambda\phi| \leq C||\phi_N||$$

on $\mathscr{D}_K$. ♡

Proof (a) implies (b), if all images of $\{||\phi_N|| < 1\}$ is unbounded on $\mathscr{D}_K$, then we know any images of open set is unbounded, which is a contradiction.

(b) implies (a), we know Λ is continuous on any $\mathscr{D}_K$ by consider the preimage of $(-1, 1)$.

Definition 3.4

If Λ is such that one$N$ will do for all K, then the smallest N is called the order of Λ. Else, call Λ to have infinite order.
Then we may define $\delta_x(\phi) = \phi(x)$, and we know δ_x is a distribution of order 0. ♣

Definition 3.5

We define the differentiation of distributions

$$(D^{\alpha\Lambda})(\phi) = (-1)^{|\alpha|}\Lambda(D^\alpha\phi)$$

And the multiplication by functions, for $f \in C^\infty(\Omega)$ define

$$(f\Lambda)(\phi) = \Lambda(f\phi)$$

by the Leibniz formula

$$D^\alpha(fg) = \sum_{\beta\leq\alpha}(D^{\alpha-\beta}f)(D^\beta g)$$

♣

Definition 3.6

For $\mathscr{D}'(\Omega)$, define the topology on it to be the weak-topology.* ♣

Theorem 3.5

Suppose $\Lambda_i \in \mathscr{D}'(\Omega)$, and $\Lambda\phi = \lim \Lambda_i\phi$ exists for every $\phi \in \mathscr{D}(\Omega)$, then $\Lambda \in \mathscr{D}'(\Omega)$ and $D^\alpha\Lambda_i \to D^\alpha\Lambda$ in $\mathscr{D}'(\Omega)$. ♡

Proof It suffices to show $\Lambda \in \mathscr{D}'(\Omega)$. We only need to check that Λ is continuous on $\mathscr{D}_K$, since $\mathscr{D}_K$ is a complete metric space, then we know Λ_i is uniformly bounded and hence Λ is bounded, so it is continuous.

For the second conclusion, only need to check that there will be an N and C such that $|D^\alpha\Lambda\phi| \leq C||\phi||_{N+|\alpha|}$ for any K compact and hence D^α will be continuous as well.

Theorem 3.6

If $\Lambda_i \to \Lambda$ in $\mathscr{D}'(\Omega)$ and $g_i \to g$ in $C^\infty(\Omega)$, then $g_i\Lambda_i \to g\Lambda$ in $\mathscr{D}'(\Omega)$. ♡

Proof We need to show that for any ϕ, $\Lambda_i(g_i\phi) \to \Lambda(g\phi)$, which can be seen by

$$|\Lambda_i(g_i\phi) - \Lambda(g\phi)| \leq |\Lambda_i((g_i - g)\phi)| + |\Lambda_i(g\phi) - \Lambda(g\phi)|$$

since Λ_i is uniformly bounded as a map from $\mathscr{D}_K \to R$, and then we know for any $\epsilon > 0$, there exists W a neighbourhood of 0 such that $\Lambda_i(W) \in (-\epsilon, \epsilon)$ for any i, so let i large enough let $(g_i - g)\phi \in W$ and we are done.

Definition 3.7

Suppose $\Lambda_i \in \mathscr{D}'(\Omega)$ and ω is an open subset of Ω, then $\Lambda_1 = \Lambda_2$ on ω means $\Lambda_1\phi = \Lambda_2\phi$ for every $\phi \in \mathscr{D}(\omega)$. ♣

Theorem 3.7

If Γ is a collection of open sets in $\mathbb{R}^n$ whose union is Ω then there exists a sequence $\phi_i \in \mathscr{D}(\Omega)$ with $\phi_i \geq 0$, such that

a. each ϕ_i has its support in some member of Γ

b. $\sum_{i=1}^{\infty} \phi_i = 1$ for every $x \in \Omega$

c. to every compact $K \subset \Omega$ correspond an integer m and an open set $W \supset K$ such that

$$\phi_1(x) + \cdots \phi_m(x) = 1$$

for all $x \in W$.

Such ϕ_i is called a locally finite partition of unity in Ω. ♡

Proof Let S be a countable dense subset of Ω. Let B_i be all the closed ball B_i with center in S and with rational radius such that it will lie in some member of Γ. Let V_i be the open ball with center p_i and radius $r_i/2$ where r_i was assume to be sufficent close to $\max d(p_i, \omega^c)$, such that $\bigcup_i V_i = \Omega$.

Then we know there are $\phi_i \in \mathscr{D}(\Omega)$ such that $0 \leq \phi 1$ and $\phi_i = 1$ in V_i and 0 outside of B_i define $\varphi_1 = \phi_1$ and inductively

$$\varphi_{i=1} = (1 - \phi_1) \cdots (1 - \phi_i)\phi_{i+1}$$

and then we know $\varphi_i = 0$ outside of B_i and

$$\varphi_1 + \cdots \varphi_i = 1 - (1 - \phi_1) \cdots (1 - \phi_i)$$

which equals to 1 for $x \in V_1 \cup \cdots \cup V_m$. For compact set, we know $K \subset \bigcup_{1 \leq i \leq m} V_i$ for some m.

Theorem 3.8

Suppose Γ is an open cover of an open set $\Omega \subset \mathbb{R}^n$ and suppose that to each $\omega \in \Gamma$ corresponds a distribution $\Lambda_\omega \in \mathscr{D}'(\omega)$ such that

$$\Lambda_{\omega_1} = \Lambda_{\omega_2}$$

on $\omega_1 \cap \omega_2$ for $\omega_1, \omega_2 \in \Gamma$ with nonemptyset disjoint. Then there exists a unique $\Lambda \in \mathscr{D}'(\Omega)$ such that $\Lambda = \Lambda_\omega$ on ω for every $\omega \in \Gamma$. ♡

Proof Let ϕ_i be a locally finite partition of unity w.r.t. Γ and we know there exists ω_i containing the suppose of ϕ_i, if $f \in \mathscr{D}(\Omega)$, then $f = \sum_{n \geq 0} \phi_n f$ and define

$$\Lambda f = \sum_{n \geq 0} \Lambda_{\omega_i}(\phi_i f)$$

then we know $\Lambda \in \mathscr{D}'(\Omega)$ easily. For any $h \in \mathscr{D}(\omega)$, we know $\phi_i h \in \mathscr{D}(\omega_i \cap \omega)$ so

$$\Lambda h = \sum \Lambda_{\omega_i}(\phi_i h) = \Lambda_\omega(\sum \phi_i h) = \Lambda_\omega(h)$$

and we are done. The uniqueness is easy to be checked.

Definition 3.8

Suppose $\Lambda \in \mathscr{D}'(\Omega)$, if ω is a open subset of Ω and if $\Lambda\phi = 0$ for every $\phi \in \mathscr{D}(\Omega)$, we say Λ vanished in ω. Let W be the union of all open subset of Ω where Λ vanished on, and W^c is the support of Λ.
It is easy to check Λ vanished in W.

♣

Theorem 3.9

Suppose $\Lambda \in \mathscr{D}'(\Omega)$ and S_Λ is the support of Λ.
a. If the support of some $\phi \in \mathscr{D}(\Omega)$ does not intersect S_Λ, then $\Lambda\phi = 0$.
b. If S_Λ is empty, the n $\Lambda = 0$.
c. If $\varphi \in C^\infty(\Omega)$ and $\varphi = 1$ in some open set V containing $S\Lambda$, then $\varphi\Lambda = \Lambda$.
d. If S_Λ is a compact subset of Ω, then Λ has finite order i.e. there is a constant $C < \infty$ and a nonnegative integer N such that

$$|\Lambda\phi| \leq C||\phi||_N$$

for any $\phi \in \mathscr{D}(\Omega)$. Then Λ extends in a unique way to a continuous linear functional on $C^\infty(\Omega)$.

♡

Proof (a),(b),(c) trivial.

(d) If S_Λ is compact, then we know there exists $\varphi \in \mathscr{D}(\Omega)$ such that $\varphi\Lambda = \Lambda$ and let the support of φ to be K. Then we know there exists c_1, N such that $|\Lambda\phi| \leq c_1||\phi||_N$ for all $\phi \in \mathscr{D}_K$. And c_2 such that $||\varphi\phi|| \leq c_2||\phi||_N$ for every $\phi \in \mathscr{D}(\Omega)$. Then

$$|\Lambda\phi| = |\Lambda(\varphi\phi)| \leq c_1c_2||\phi||_N$$

for every $\phi \in \mathscr{D}(\Omega)$, then for $f \in C^\infty(\Omega)$, define $\Lambda f = \Lambda(\varphi f)$ to be the extension and we know the extension is continuous. However, notice $\mathscr{D}(\Omega)$ is dense in $C^\infty(\Omega)$ and then the extension should be unique.

Theorem 3.10

Suppose $\Lambda \in \mathscr{D}'(\Omega), p \in \Omega$, $\{p\}$ is the support of λ and λ has order N. Then there are constants c_α such that

$$\Lambda = \sum_{|\alpha| \leq N} c_\alpha D^\alpha \delta_p$$

Conversly, it is easy to check that the distribution of the form (1) has $\{p\}$ for its support.

♡

Proof Assume $p = 0$ and $\phi \in \mathscr{D}(\Omega)$ such that

$$(D^\alpha)(0) = 0, |\alpha| \leq N$$

If $\eta > 0$, there is a compact ball $K \subset \Omega$ centered at 0 such that

$$|D^\alpha\phi| \leq \eta$$

on K if $|\alpha| = N$, then we claim that

$$|D^\alpha\phi(x)| \leq \eta n^{N-|\alpha|}|x|^{N-|\alpha|}$$

we know

$$|\nabla D^\beta| \leq n \cdot \eta n^{N-i}|x|^{N-i}$$

by induction and we are done.

Choose $\varphi \in \mathscr{D}(\mathbb{R}^n)$ which is 1 in some neighbourhood of 0 and whose support is in the unit ball B of $\mathbb{R}^n$, define

$$\varphi_r(x) = \varphi(x/r)$$

and we know

$$||\varphi_r||_N \leq \eta C||\varphi||_N$$

for r small enough since Λ has order N, there is C_1 such that $|\Lambda\varphi| \leq C_1||\varphi||_N$ for all $\varphi \in \mathscr{D}_k$ and we know

$$|\Lambda\phi|$$

Theorem 3.11

Suppose $\Lambda \in \mathscr{D}'(\Omega)$ and K is a compact subset of Ω, then there is a continuous function f in Ω and α such that

$$\Lambda\phi = (-1)^{|\alpha|} \int_{\Omega} f(x)(D^{\alpha}\phi)(x)dx$$

for every $\phi \in \mathscr{D}_K$.

♡

Proof Firstly assume $K \subset Q$ the unit cube in $\mathbb{R}^n$ and we know

$$|\phi| \le \max_{x\in Q} |(D_i\phi)(x)|$$

for $\phi \in \mathscr{D}_Q$, let $T = D_1 D_2 \cdots D_n$ and we know

$$\phi(y) = \int_{x\le y} (T\phi)(x)dx$$

and we know

$$||\phi||_N \le \max_{x\in Q} |(T^N\phi)| \le \int_Q |(T^{N+1}\phi)|$$

Since $\Lambda \in \mathscr{D}'(\Omega)$, there exists N and C such that

$$|\Lambda\phi| \le C||\phi||_N$$

and hence

$$|\Lambda| \le C \int_K |(T^{N+1}\phi)(x)|dx$$

Since T is one-to-one on $\mathscr{D}_Q$ and hence $\mathscr{D}_K$, we know $T^{N+1} : D_K \leftrightarrow D_K$ is one-to-one, so we can let $\Lambda_1 T^{N+1}\phi = \Lambda\phi$ for $\phi \in \mathscr{D}_K$ and a linear functional of $\mathscr{D}_K$ with

$$|\Lambda_1\phi| \le C \int_K |\phi|$$

for y in the range of T^{N+1} and then we may use the Hahn-Banach to extends Λ_1 to a bounded linear functional on $L^1(K)$. In other words, there is a bounded Borel function g on K such that

$$\Lambda\phi = \Lambda_1 T^{N+1}\phi = \int_K g(x)(T^{N+1}\phi)(x)dx$$

Define $g(x) = 0$ outside K and let

$$f(y) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} g(x)dx$$

then f is continuous and use the integrations by parts we know

$$\Lambda\phi = (-1)^n \int_{\Omega} f(x)(T^{N+2}\phi)(x)$$

Theorem 3.12

Suppose K is compact, V and Ω are open in $\mathbb{R}^n$ and $K \subset V \subset \Omega$. Suppose also that $\Lambda \in \mathscr{D}'(\Omega)$, that K is the support of Λ, and that Λ has order N. Then there exists finitely many continuous functions f_β in Ω with supports in V such that

$$\Lambda = \sum_{\beta} D^{\beta} f_{\beta}$$

♡

Proof Choose an open W with compact closure, such that $K \subset W, \overline{W} \subset V$, then we lmpw there is a continuous function f in Ω such that

$$\Lambda\phi = (-1)^{|\alpha|} \int_{\Omega} f(x)(D^{\alpha}\phi)(x)dx$$

We may multiply f with a continuous function equaling 1 on $\overline{W}$ with support in V.

Fix $\varphi \in \mathscr{D}(\Omega)$, with support in W such that $\varphi = 1$ on some open set containing K, then

$$\Lambda\phi = \Lambda(\varphi\phi) = (-1)^{|\alpha|} \int_{\Omega} f \sum_{\beta\le\alpha} c_{\alpha\beta} D^{\alpha-\beta}\varphi D^{\beta}\phi$$

and let $f_\beta = (-1)^{\alpha-\beta||}c_{\alpha\beta}f \cdot D^{\alpha-\beta}\varphi$.

Theorem 3.13

Suppose $\Lambda \in \mathscr{D}'(\Omega)$ There exists continuous functions g_α in Ω, for each multi-index α.
a. each compact $K \subset \Omega$ intersects the supports of only finitely many g_α
b. $\Lambda = \sum D^\alpha = \sum_\alpha D^\alpha g_\alpha$.
If Λ has finite order, the n the functions g_α can be chosen so that only finitely many are nonzero.

♡

Definition 3.9

For $u \in \mathscr{D}$, define

$$(\tau_x u)(y) = u(y - x), \check{u}(y) = u(-y)$$

and for $u \in \mathscr{D}'$ Define

$$(u * \phi)(x) = u(\tau_x \check{\phi})$$

and $\tau_x u(\phi) = u(\tau_{-x}\phi)$ for $u \in \mathscr{D}'$.

♣

Theorem 3.14

Suppose $u \in \mathscr{D}', \phi, \varphi \in \mathscr{D}$, then
a. $\tau_x(u * \phi) = (\tau_x u) * \phi = u * (\tau_x \phi)$ for all $x \in \mathbb{R}^n$.
b. $u * \phi \in C^\infty$ and

$$D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha \phi)$$

c. $u * (\phi * \varphi) = (u * \phi) * \varphi$.

♡

Definition 3.10

The term approximate identity on $\mathbb{R}^n$ will denote a sequence of functions h_j of the form

$$h_j(x) = j^n h(jx)$$

for $h \in \mathscr{D}$ and $\int h = 1$.

♣

Theorem 3.15

Suppose h_j is an approximate identity on $\mathbb{R}^n, \phi \in \mathscr{D}$ and $u \in \mathscr{D}'$, then
a. $\lim_{j\to\infty} \phi * j_j = h$ in $\mathscr{D}$.
b. $\lim_{j\to\infty} u * h_j = u$ in $\mathscr{D}'$.

♡

Proof We know

$$|f - f * h_j|(x) \leq \int |f(x)h_j(t) - f(x-t)h_j(t)|dt \leq \max_{t \in j^{-1}K} |f(x) - f(x-t)|$$

and hence $f * h_j \to f$ uniformly on compact sets, then it is easy to check that $D^\alpha(\phi * h_j) \to D^{\alpha\phi}$ uniformly on compact sets.

It is easy to verify (b) and then any distribution i s a limit in the topology of $\mathscr{D}'$ is a seq of infinitely differentiable functions.

Theorem 3.16

a. If $u \in \mathscr{D}'$ and

$$L\phi = u * \phi$$

for $\phi \in \mathscr{D}$, then L is a continuous linear mapping of $\mathscr{D}$ into C^∞ which satisfies

$$\tau_x L = L\tau_x$$

b. Conversly, if L is a continuous linear mapping of $\mathscr{D}$ into $C(\mathbb{R}^n)$ and if L satisfies $\tau_x L = L\tau_x$, then there is a *unique $u \in \mathscr{D}'$ such that $L\phi = u * \phi$.* ♡

Proof a. The second equality holds automatically, to prove L is continuous, we only need to show that $L|_{\mathscr{D}_K}$ is continuous, assume $\phi_i \to \phi$ in $\mathscr{D}_K$ and then we know

$$|(u * \phi_i) - (u * \phi)|(x) = |u(\tau_x \phi_i - \tau_x \phi)| \to 0$$

and

$$|D^\alpha(u * \phi_i - u * \phi)|(x) = |u * (D^\alpha \phi_i - D^\alpha \phi)|(x) \to 0$$

b. Define $u(\phi) = (L\check{\phi})(0)$ and the rest is easy to be checked.

Definition 3.11

The convolution of u with compact support and any $\phi \in C^\infty$ is define by

$$(u * \phi)(x) = u(\tau_x \check{\phi})$$

♣

Theorem 3.17

Suppose $u \in \mathscr{D}'$ has compact support, and $\phi \in C^\infty$. Then

a. $\tau_x(u * \phi) = (\tau_x u) * \phi = u * (\tau_x \phi)$ if $x \in \mathbb{R}^n$.

b. $u * \phi \in C^\infty$ and

$$D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha \phi)$$

If $\varphi \in \mathscr{D}$, then

c. $u * \varphi \in \mathscr{D}$

d. $u * (\phi * \varphi) = (u * \phi) * \varphi = (u * \varphi) * \phi$. ♡

Definition 3.12

If $u, v \in \mathscr{D}'$ and at least one of there two distributions has compact support, define

$$L\phi = u * (v * \phi)$$

for $\phi \in \mathscr{D}$, we have $\tau_x L = L\tau_x$ and we will denote $u * v$ to be this distribution. ♣

Theorem 3.18

Suppose $u \in \mathscr{D}', v \in \mathscr{D}', w \in \mathscr{D}'$

a. If at leasst one of u, v has compact support, then $u * v = v * u$.

b. If S_u, S_v are the supports of u and v, and if at least one of these is compact, then

$$S_{u*v} \subset S_u + S_v$$

c. If at least two of the supports S_u, S_v, S_w are compact, then

$$(u * v) * w = u * (v * w)$$

d. If δ is the Dirac measure, then

$$D^\alpha u = (D^\alpha \delta) * u$$

e. If at least one of the sets S_u, S_v is compact, then

$$D^\alpha(u * v) = (D^\alpha u) * v = u * (D^\alpha v)$$

♡

Chapter 4 Fourier Transform

Theorem 4.1

Suppose $f, g \in L^1(\mathbb{R}^n), x \in \mathbb{R}^n$, *then*

a. $(\tau_x f)^\wedge = e_{-x}\hat{f}$.

b. $(e_x f)^\wedge = \tau_x \hat{f}$.

c. $(f * g)^\wedge = \hat{f}\hat{g}$.

d. If $\lambda > 0$ *and* $h(x) = f(x/\lambda)$, *then* $\hat{h}(t) = \lambda^n \hat{f}(\lambda t)$.

♡

Definition 4.1

(Rapidly decreasing functions)

The functions $f \in \mathbb{C}^\infty$ *such that*

$$\sup_{|\alpha| \leq N} \sup (1 + |x|^2)^N |(D_\alpha f)(x)| < \infty$$

for any N, *the space is denoted by* $\mathcal{S}_n$ *and the norms defines a locally convex topology.*

♣

Theorem 4.2

a. $\mathcal{S}_n$ *is a Frechet space.*

b. If P *is a polynomial,* $g \in \mathcal{S}_n$, *then*

$$f \mapsto Pf, \quad f \mapsto gf, \quad f \mapsto D^\alpha f$$

is a continuous linear mapping of $\mathcal{S}_n$ *to* $\mathcal{S}_n$.

c. If $f \in \mathcal{S}_n$ *and* P *is a polynomial, then*

$$(P(D)f)^\wedge = P\hat{f}, \quad (Pf)^\wedge = P(-D)\hat{f}$$

d. The Fourier transform is a continuous linear mapping of $\mathcal{S}_n$ *to* $\mathcal{S}_n$.

♡