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# NOTES FOR DIRECTED POLYMER BY F. COMETS

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Based on the Lecture Notes by C.S.Z.

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# 1 Thermodynamics and Phase Transition

## 1.1 Useful Conclusions

**Definiton 1.1.1.**  $f : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is increasgin if  $f(x) < f(y)$  iff  $x_i < y_i$ .

**Definiton 1.1.2.** (Positively Associated)

A family  $X = (X_i)_{i=1}^k$  of real r.v.s on the same probability space are **postive associated** if for any  $f, g : \mathbb{R}^k \rightarrow \mathbb{R}$  bounded, increasing

$$Ef(X)g(X) \geq Ef(X)Eg(X)$$

**Proposition 1.1.1.** (FKG-Harris Inequality)

A family of independent, real random variables is positively associated.

## 1.2 Markov Property and the Partition Function

**Definiton 1.2.1.** (Partition Function)

For  $n, m \geq 1, x \in \mathbb{Z}^d$ , the r.v. on  $(\Omega = \mathbb{R}^{\mathbb{N} \times \mathbb{Z}^d}, \mathbb{P})$

$$Z_m^\beta \circ \theta_{n,x}(\omega) = Z_m(\theta_{n,x}\omega, \beta) = E_x \exp \left( \sum_{t=1}^m \beta \omega(t+n, S_t) \right) \quad (\text{finite and definitely positive})$$

is the partition function of the polymer of length  $m$  starting at  $x$  at time  $n$ .

**Proposition 1.2.1.**  $Z_m \circ \theta_{n,x} \stackrel{d}{=} Z_m$ .

**Proposition 1.2.2.** Let  $\mathcal{F}_n = \sigma\{S_t, t \leq n\}$  and we will have

$$Z_m \circ \theta_{n,x}(\omega) = E(\exp \beta(H_{n+m}(S) - H_n(S)) | \mathcal{F}_n)$$

on the event  $\{S_n = x\}$ , i.e.

$$\begin{aligned} Z_m \circ \theta_{n,x}(\omega) \chi_{S_n=x} &= E(\exp(\beta(H_{n+m}(S) - H_n(S))) \chi_{S_n=x} | \mathcal{F}_n) \\ &= E(\exp(\beta(H_{n+m}(S) - H_n(S))) | \mathcal{F}_n) \chi_{S_n=x} \end{aligned}$$

**Proposition 1.2.3.** We will have

$$Z_{n+m} = E(\exp \beta H_n(S) Z_m \circ \theta_{n,S_n})$$

which is referred to the Markov property, and we will have

$$Z_{n+m} = Z_n \times E_n^{\beta,\omega}(Z_m \circ \theta_{n,S_n})$$

where  $E_n^{\beta,\omega}$  refers the expectation under the polymer measure.

### 1.3 Markov Chain under the Polymer Measure

**Proposition 1.3.1.** For all  $\beta \in \mathcal{D} := \{\beta, p \text{ differentiable at } \beta\}$  and almost every environment  $\omega$ , we have

$$\lim_{n \rightarrow \infty} E_{P_n^{\beta, \omega}}(H_n(S)/n) = \lim_{n \rightarrow \infty} \mathbb{E}(E_{P_n^{\beta, \omega}}(H_n(S)/n)) = p'(\beta)$$

Moreover, for all  $\beta \in \mathbb{R}$  we have

$$p'(\beta-) \leq \liminf_{n \rightarrow \infty} E_{P_n^{\beta, \omega}}(H_n(S)/n) \leq \limsup_{n \rightarrow \infty} E_{P_n^{\beta, \omega}}(H_n(S)/n) \leq p'(\beta+)$$

*Proof.* Notice that we have already have  $p_n, p$  are convex and hence we know  $p'(\beta-), p'(\beta+)$  always exists. Let take a look of  $p'_n(\beta)$  again:

$$p'_n(\beta) = \frac{1}{nZ_n} \frac{\partial}{\partial \beta} \int \exp(\beta H_n(x)) P(dx) = \frac{1}{n} E_{P_n^{\beta, \omega}}(H_n(S))$$

since  $\int f(x)P(dx)$  is a finite summation of  $f(x)$ . Now consider

$$(\mathbb{E}p_n)'(\beta) = \frac{\partial}{\partial \beta} \mathbb{E}p_n = \mathbb{E}p'_n(\beta)$$

since  $\sum_x (2d)^{-n} \max\{1, \exp(TH_n(x))\}$  is  $L^1$  and we may apply the DCT for any  $\beta \in [0, T]$ . Notice  $E p_n \rightarrow p$  a.s. for all  $\beta$ , then we know

$$p'(\beta-) = \inf_{\epsilon > 0} \frac{p(\beta) - p(\beta - \epsilon)}{\epsilon} = \inf_{\epsilon > 0} \lim_{n \rightarrow \infty} \frac{\mathbb{E}p_n(\beta) - \mathbb{E}p_n(\beta - \epsilon)}{\epsilon} \leq \liminf_{n \rightarrow \infty} \mathbb{E}p'_n(\beta)$$

and we can obtain the second inequality similarly. Now, for somewhere  $p'(\beta)$  exists, we may know  $\lim_{n \rightarrow \infty} \mathbb{E}p'_n(\beta)$  exists and then we may replace  $\mathbb{E}p_n$  above with  $p_n$  in the view of  $\mathbb{P}$ -a.s. which means for almost every  $\omega$ .  $\square$

**Theorem 1.3.2.** The functions  $\beta \mapsto \lambda(\beta) - \mathbb{E}p_n$  and  $\beta \mapsto \lambda(\beta) - p(\beta)$  are non-decreasing on  $\mathbb{R}^+$  and non-increasing on  $\mathbb{R}^-$ .

*Proof.* We will compute

$$\begin{aligned} \frac{\partial}{\partial \beta} \mathbb{E} \ln Z_n &= \mathbb{E} E Z_n^{-1} H_n(S) \exp(\beta H_n(S)) \\ &= E \mathbb{E} Z_n^{-1} H_n(S) \exp(\beta H_n(S)) \end{aligned}$$

by Fubini, and we notice

$$\begin{aligned} Z_n &= \sum_x (2d)^{-n} \exp(\beta \sum_{t=1}^n \omega(t, x_t)) = f(\omega(t, y))_{1 \leq t \leq n, |y|_1 \leq n} \\ H_n(x) &= \sum_{t=1}^n \omega(t, x_t) = g(\omega(t, y))_{1 \leq t \leq n, |y|_1 \leq n} \\ \exp(\beta H_n(x)) &= \prod_{t=1}^n \exp(\beta \omega(t, x_t)) = h(\omega(t, y))_{1 \leq t \leq n, |y|_1 \leq n} \end{aligned}$$

and define

$$f_M = \text{sgn}(f) \min\{|f|, M\}, g_M = \text{sgn}(g) \min\{|g|, M\}, h_M = \text{sgn}(h) \min\{|h|, M\}$$

then for  $\beta \geq 0$ , we have  $f, g, h$  increasing and  $\beta \leq 0$   $f, h$  decreasing, and it is easy to check  $h/f$  is increasing with  $\beta \geq 0$ . Now we may use the FKG-Harris and we will have for fixed  $x, \beta \geq 0$ ,

$$\mathbb{E} Z_n^{-1} H_n(x) \exp(\beta H_n(x)) = \mathbb{E}(h/f) g h h^{-1} =$$

where  $f^{-1}gh$  is integrable, so we may use *DCT* and we will have

$$\mathbb{E} f^{-1} g h = \lim_{M \rightarrow \infty} f_M^{-1} g_M h_M \leq \lim_{M \rightarrow \infty} \mathbb{E} 1/h_M \mathbb{E} h_M / f_M \mathbb{E} g_M h_M = \mathbb{E} 1/h \mathbb{E} h / f \mathbb{E} g h$$

since  $1/h_M$  decreasing, and we will get an opposite inequality if  $\beta \leq 0$  since  $h/f$  decreasing and  $g_M h_M$  decreasing. Then

$$\frac{\partial}{\partial \beta} \mathbb{E} \ln Z_n \leq n \lambda'(\beta) \mathbb{E} \mathbb{E} Z_n^{-1} \exp(\beta H_n(S)) = n \lambda'(\beta)$$

and with the opposite inequality when  $\beta \leq 0$ , so we have

$$\mathbb{E} p'_n(\beta) \leq \lambda'(\beta)$$

on  $\mathbb{R}^+$  and the opposite on  $\mathbb{R}^-$ . The monocity of  $\lambda - p$  is induced by limit.  $\square$

**Theorem 1.3.3.** Suppose  $d \geq 3$  and the  $L_2$  condition holds, then

$$\lim_{n \rightarrow \infty} P_n^{\beta, \omega}$$

## 2 Martingale Approach and L2 Region

### 2.1 Checklist

- Proof of  $H_n/n - \lambda'(\beta)$  converges to 0 in  $L^1_{P_n^{\beta, \omega}}$ .
- Proof of  $W_n^\delta$  uniformly integrable
- Proof of  $\partial/\partial\beta \mathbb{E}W_n^\delta$
- Proof of  $s = \infty$
- compute  $\varphi, \psi$  in 3.3

### 2.2 Useful Conclusions

**Theorem 2.2.1.** (Martingale Convergence Theorem)

If  $X_n$  is a submartingale with  $\sup EX_n^+ < \infty$ , then  $X_n$  converges to some  $L^1$  limit  $X$  a.s. as  $n \rightarrow \infty$ .

**Theorem 2.2.2.** (Kolmogorov's 0-1 Law)

If  $X_1, X_2, \dots$  are independent and  $A \in \mathcal{T} := \cap \mathcal{F}'_n$  then  $P(A) = 0$  or  $1$ , where  $\mathcal{F}'_n = \sigma(X_n, X_{n+1}, \dots)$ .

*Proof.* We prove by some steps, first we show that for  $A \in \sigma(X_1, \dots, X_k)$  and  $B \in \sigma(X_{k+1}, X_{k+2}, \dots)$ , we have  $A, B$  independent. If  $B \in \sigma(X_{k+1}, \dots, X_{k+j})$ , we have  $A, B$  independent. And we know  $\cup_j \sigma(X_{k+1}, \dots, X_{k+j})$  is a  $\pi$ -system. And we only need to check  $\{B, P(A)P(B) = P(A)P(B)\}$  is a  $\lambda$ -system.  $\Omega$  is obviously in it firstly and for  $B \subset B'$ , we have

$$P(A(B' - B)) = P(AB') - P(AB) = P(A)(P(B') - P(B)) = P(A)P(B' - B)$$

and if  $B_n$  increases to  $B$ , we have

$$P(AB) = \lim_{n \rightarrow \infty} P(AB_n) = \lim_{n \rightarrow \infty} P(A)P(B_n) = P(A)P(B)$$

and we are done.

So we know  $A \in \sigma(X_1, \dots, X_k)$  is independent with  $B$  if  $B \in \mathcal{T}$  the tail algebra, and similarly we may check that for any  $A \in \sigma(X_1, \dots, X_n)$  is independent with  $B$  and hence  $B$  is independent with itself, which means  $P(B) = 0$  or  $1$ . □

**Proposition 2.2.3.** Every convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

*Proof.* For any  $x$ , let  $y < x$  and  $z > x$ , we know that for any  $t \in (y, z)$

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(x) - f(t)}{x - t} \geq \frac{f(x) - f(z)}{x - z}$$

and hence

$$|f(x) - f(t)| \leq \max \left\{ \left| \frac{f(x) - f(z)}{x - z} \right|, \left| \frac{f(x) - f(y)}{x - y} \right| \right\} |x - t|$$

and we are done. □

**Definiton 2.2.1.** (Uniformly Integrable)

A collection of r.v.s  $X_i, i \in I$  is **uniformly integrable** if

$$\lim_{M \rightarrow \infty} \left( \sup_{i \in I} E(|X_i|; |X_i| > M) \right) = 0$$

**Proposition 2.2.4.** Suppose that  $E|X_n| < \infty$  for all  $n$ . If  $X_n \rightarrow X$  in probability then the following are equivalent:

- $\{X_n\}_{n \geq 0}$  is uniformly integrable.
- $X_n \rightarrow X$  in  $L^1$ .
- $E|X_n| \rightarrow E|X| < \infty$ .

**Theorem 2.2.5.** ( $L^p$  convergence theorem)

If  $X_n$  is a martingale with  $\sup E|X_n|^p$  finite and  $p > 1$ , then  $X_n \rightarrow X$  a.s. and in  $L_p$ , where  $X$  is given by the martingale convergence theorem.

**Theorem 2.2.6.** (Skorokhod Representation Theorem)

For some distribution functions  $F_n$ , if  $F_n$  converges to some  $F_\infty$ , then there are random variables  $Y_n, 1 \leq n \leq \infty$  such that  $Y_n \rightarrow Y_\infty$  a.s.

*Proof.* Let  $\Omega = (0, 1)$  and  $P$  to be the Lebegues measure, and let  $Y_n(y) = \sup\{x, F_n(x) < y\}$  and we know  $Y_n$  is nondecreasing, then if  $Y_n(y) \leq a$ , notice  $F_n(Y_n(y)) \geq y$ , then we have  $y \leq F_n(a)$ , which means

$$P(Y_n \leq a) \leq F_n(a)$$

and if  $P(Y_n \leq a) < F_n(a)$ , then if  $Y_n(z) \leq a$ , we know  $z < F_n(a)$ , however  $Y_n(F_n(a)) = a$  which means  $F_n(a) < F_n(a)$  and hence a contradiction, so  $Y_n$  has the distribution function of  $F_n$ .

Let  $a_x = Y_\infty(x) = \sup\{y, F(y) < x\}$  and  $b_x = \inf\{y, F(y) > x\}$ , then we know  $a_x \leq b_x$  and define  $\Omega_0 = \{x, (a_x, b_x) \neq \emptyset\}$ , then  $\Omega_0$  is at most countable and we prove for any  $x \in \Omega - \Omega_0$ , we have  $Y_n(x) \rightarrow Y_\infty(x)$ . Firstly, for these  $x$  consider  $y < Y_\infty(x)$  for some  $y$  such that  $F$  is continuous at  $y$ , then  $F_n(y) \rightarrow F(y)$  and notice there is some  $y' > y$  such that  $F(y') < x$  and hence  $F(y) < x$ , so there is some  $N$  such that for any  $m \geq N$  we have  $F_m(y) < x$  and hence  $y \leq Y_m(x)$  and then

$$y \leq \liminf_{n \rightarrow \infty} Y_n(x)$$

for any  $y < Y_\infty(x)$ , which means

$$\liminf_{n \rightarrow \infty} Y_n(x) \geq Y_\infty(x)$$

Similarly for any  $z$  such that  $z > Y_\infty(x)$  and using the assumption that  $a_x = b_x$ . □

**Theorem 2.2.7.**  $X_n$  converges to  $X_\infty$  weakly if and only if

$$Eg(X_n) \rightarrow Eg(X_\infty)$$

for any bounded and continuous function  $g$ .

*Proof.* To see the sufficiency, we know we may find  $X_n \xrightarrow{d} Y_n$  for  $1 \leq n < \infty$  and  $Y_n \rightarrow Y_\infty$  a.s. in some probability space, then we have  $Eg(X_n) = Eg(Y_n)$  and by the DCT, we know  $Eg(Y_n) \rightarrow Eg(Y_\infty)$  for any bounded and continuous function  $g$ .

For the necessity, we know

$$P(X_n \leq a) = E(\chi_{(-\infty, a]}(X_n))$$

and we consider some slope continuous approaching  $\delta_\epsilon$  for  $\chi_{(-\infty, a]}$ , now we have

$$P(X_n \leq a) \leq E\delta_\epsilon(X_n) \leq P(X_n \leq a + \epsilon)$$

and if there is  $q > 0$  such that  $|F_n(a) - F_\infty(a)| > q$  infinitely often, then notice

$$|F_n(a) - F_\infty(a)| \leq |F_n(a) - F_n(a + \epsilon)| + |F_\infty(a) - F_\infty(a + \epsilon)| + |E\delta_\epsilon(X_n) - E\delta_\epsilon(X_\infty)|$$

and then there will be a contradiction and we are done.  $\square$

**Theorem 2.2.8.** ( $L^p$  maximum inequality)

If  $X_n$  is a submartingale then for  $1 < p < \infty$ , we have

$$E(\bar{X}_n^p) \leq \left(\frac{p}{p-1}\right) E(X_n^p)$$

where  $\bar{X}_n = \max_{0 \leq m \leq n} X_m^+$ .

## 2.3 Phase Transition of Weak Disorder and Strong Disorder Phase

**Definiton 2.3.1.** (Normalized Partition Function)

$$W_n = Z_n(\omega, \beta) \exp(-n\lambda(\beta))$$

where  $\lambda(\beta) = \mathbb{E} \exp(\beta \omega(n, x))$  which is not related to  $n, x$ .

**Theorem 2.3.1.** The limit

$$W_\infty = \lim_{n \rightarrow \infty} W_n$$

exists  $\mathbb{P}$ -a.s. and either the limit  $W_\infty$  is a.s. positive or it is a.s. zero.

*Remark.* We will show  $W_n$  is a martingale and use martingale convergence theorem and Kolmogorov's 0-1 law for  $W_\infty$ .

*Proof.* Firstly, notice for a fixed path  $x$ ,

$$\xi_n = \exp(\beta H_n(x) - n\lambda(\beta))$$

is a positive martingale w.r.t. the filtration  $G_n = \sigma\{\omega(j, x), j \leq n\}$ . And then we know

$$W_n = E \exp(\beta H_n(S) - n\lambda(\beta)) = \sum_{n\text{-length paths } x} (2d)^{-1} \xi_n(x)$$



is a positive martingale w.r.t. Also, consider

$$\mathbb{E}\xi_n = 1$$

and hence we may know  $\mathbb{E}W_n = 1$  and hence we may apply the martingale convergence theorem and get

$$W_\infty = \lim_{n \rightarrow \infty} W_n$$

exists and nonnegative  $\mathbb{P}$ -a.s. and  $\mathbb{E}W_\infty < \infty$ . Now assume  $\mathcal{F}'_n = \sigma\{\omega(j, x), j \geq n, |x|_1 \leq j\}$  and let  $\mathcal{F}' = \cap \mathcal{F}'_n$ . Then for any  $A \in \mathcal{F}'$ , we have  $\mathbb{P}(A) \in \{0, 1\}$  since we may apply the proof of Kolmogorov's 0-1 Law by replacing  $\sigma(X_k, \dots, X_{k+j})$  by some family of  $\sigma$ -algebras  $\mathcal{F}_i$  and consider  $\sigma(\mathcal{F}_k, \dots, \mathcal{F}_{k+j})$ . Now we only need to check  $\{W_\infty = 0\} \in \mathcal{F}$ , which comes from

$$\begin{aligned} W_\infty &= \lim_{m \rightarrow \infty} W_{n+m} \\ &= E(\xi_n(S) \times \lim_{m \rightarrow \infty} W_m \circ \theta_{x, S_n}) \\ &= \sum_x P(dx) \exp(\beta H_n(x) - n\lambda(\beta)) W_\infty \circ \theta_{n, x_n} \\ &= W_n \sum_x P_n^{\beta, \omega}(S_n = x) W_\infty \circ \theta_{n, x} \end{aligned}$$

and hence

$$\{W_\infty = 0\} = \cap_{P(S_n=x)>0} \{W_\infty \circ \theta_{n, x} = 0\} \in \mathcal{F}_n$$

for any  $n$  and we are done.  $\square$

**Definiton 2.3.2.** (Phase Transition)

The polymer is the **weak disorder** phase when  $\mathbb{P}(W_\infty > 0) = 1$  and the **strong disorder** phase when  $\mathbb{P}(W_\infty = 0) = 1$ .

**Proposition 2.3.2.** If  $W_\infty > 0$ , then  $p(\beta) = \lambda(\beta)$ , since

$$p(\beta) = \lambda(\beta) + \lim_{n \rightarrow \infty} n^{-1} \ln W_n.$$

Furthermore, we have

$$\lim_{n \rightarrow \infty} n^{-1} E_n^{\beta, \omega} H_n = \lambda'(\beta)$$

or

$$E_n^{\beta, \omega} |H_n/n - \lambda'(\beta)| \rightarrow 0$$

with  $n \rightarrow \infty$  expect an at most countable set.

*Proof.* Notice that we have

$$p(\beta) = \lim_{n \rightarrow \infty} p_n(\beta) = \lim_{n \rightarrow \infty} n^{-1} \ln W_n + \lambda(\beta) \quad \mathbb{P}\text{-a.s.}$$

so if  $W_\infty > 0$ , then we will have  $p(\beta) = \lambda(\beta)$  but there is no other arguments if  $W_\infty = 0$ .

Since we have already know  $p, \lambda$  convex, so continuous, and hence  $p(\beta) = \lambda(\beta)$  for all  $\beta, \mathbb{P}$ -a.s. for by choosing  $\beta$  rational and make the union. So we only need to find  $p'(\beta)$ .

Notice

$$p'_n(\beta) = \frac{\partial}{\partial \beta} \frac{1}{n} \ln \left( \sum_x (2d)^{-n} \exp(\beta H_n(x)) \right) = \frac{1}{n Z_n} E H_n \exp(\beta H_n(x)) = \frac{1}{n} E_n^{\beta, \omega} H_n$$

and we know for the region where  $p$  is differential, we have

$$\lim_{n \rightarrow \infty} n^{-1} E_n^{\beta, \omega} p'(\beta) = \lambda'(\beta)$$

□

**Proposition 2.3.3.** There exists  $\bar{\beta}_c(\mathbb{P}, d) \in [0, \infty]$  such that

$$\begin{cases} W_\infty > 0, \mathbb{P}\text{-a.s.} & \text{if } \beta \in [0, \bar{\beta}_c) \\ W_\infty = 0, \mathbb{P}\text{-a.s.} & \text{if } \beta > \bar{\beta}_c \end{cases}$$

*Proof.* We know  $W_n^\delta$  is uniformly integrable and hence we have

$$\lim_{n \rightarrow \infty} \mathbb{E} W_n^\delta = \mathbb{E} W_\infty^\delta$$

which is either 0 or strictly positive. Now consider

$$\begin{aligned} \frac{\partial}{\partial \beta} \mathbb{E} W_n^\delta &= \mathbb{E}(\delta W_n^{\delta-1} \frac{\partial}{\partial \beta} \sum_x (2d)^{-n} \exp(\beta H_n(x) - n\lambda(\beta))) \\ &= \mathbb{E}(\delta W_n^{\delta-1} E((H_n - n\lambda'(\beta))\xi_n)) \end{aligned}$$

□

## 2.4 $L^2$ Region

**Proposition 2.4.1.** The return probability

$$\pi_d := P(S_n = 0 \text{ for some } n \geq 1) \text{ is } \begin{cases} 1 & \text{if } d \leq 2 \\ < 1 & \text{if } d \geq 3 \end{cases}$$

and  $\pi_{d+1} < \pi_d$  for all  $d \geq 3$ .

**Theorem 2.4.2.** Suppose that  $d \geq 3$  and the  $L^2$  condition:

$$\lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_d)$$

holds, then  $W_\infty > 0$   $\mathbb{P}$ -a.s.

*Remark.* We will show  $W_n$  is a  $L^2$  martingale under the condition, and implies that  $W_\infty$  have a positive expectation.

*Proof.* We will use the  $L^2$  martingale to compute  $\mathbb{E} W_\infty^2$  and see if  $W_\infty = 0$   $\mathbb{P}$ -a.s. Since

$$W_n = \exp(-n\lambda(\beta)) E(\exp(\beta H_n(S)))$$

we may know that consider an independent copy of  $S$  and the product  $(\Omega^2, \mathcal{F}^{\otimes 2})$  and then

$$E_{P^{\otimes 2}} \exp(\beta [H_n(S) + H_n(S')] - 2n\lambda(\beta))$$

and then by Fubini

$$\begin{aligned} \mathbb{P}W_n^2 &= E_{P^{\otimes 2}} \mathbb{E} \prod_{t=1}^n \exp(\beta(\omega(t, S_t) + \omega(t, S'_t) - 2\lambda(\beta))) \\ &= E_{P^{\otimes 2}} \mathbb{E} \prod_{t=1}^n (\exp \lambda(2\beta) \chi_{(S_t=S'_t)} + \chi_{(S_t \neq S'_t)}) \\ &= E_{P^{\otimes 2}} \exp(\lambda_2(\beta) N_n) \end{aligned}$$

where  $N_n$  denotes the intersections of  $S, S'$  up to time  $n$ . Notice  $N_n$  increases to  $N_\infty$  and hence  $\mathbb{E}W_n^2$  will increase to  $E_{P^{\otimes 2}} \exp(\lambda_2(\beta) N_\infty)$ . Consider a simple symmetric random walk  $\tilde{S}$  with increment  $\tilde{s}_{2k+1} = s_{k+1}, \tilde{s}_{2k+2} = -s'_{k+1}$  and then we know that

$$\{\tilde{S} \text{ return}\} = \{S - S' \text{ return}\}$$

and hence  $N_\infty$  must have the geometrically distributed with  $p = \pi_d$ . Then

$$E_{P^{\otimes 2}} \exp(\lambda_2(\beta) N_\infty) = \sum_{k=0}^{\infty} (1 - \pi_d) \pi_d^k \exp(k\lambda_2(\beta))$$

which is

$$E_{P^{\otimes 2}} \exp(\lambda_2(\beta) N_\infty) = \begin{cases} \frac{1 - \pi_d}{1 - \pi_d \exp(\lambda_2(\beta))} & \text{if } \lambda_2(\beta) < -\ln \pi_d \\ \infty & \text{if } \lambda_2(\beta) \geq -\ln \pi_d \end{cases}$$

So we have  $\sup_n \mathbb{E}W_n^2$  is finite if and only if  $\lambda_2(\beta) < -\ln \pi_d$ , then we will have the convergence in  $L^2$  and hence

$$\mathbb{E}W_\infty^2 = \frac{1 - \pi_d}{1 - \pi_d \exp(\lambda_2(\beta))} > 0,$$

which means  $W_\infty > 0$   $\mathbb{P}$ -a.s. □

**Definiton 2.4.1.** ( $L_2$  Region)

The set of  $\beta$ 's defined by the  $L_2$  condition is called the  $L_2$  region. For  $d \geq 3$ , there will be a non-empty interval  $(0, \beta_{L_2})$  is in the  $L_2$  region.

*Proof.* Notice

$$\lambda'_2(\beta) = 2[\lambda'(2\beta) - \lambda'(\beta)]$$

which is nonnegative, and hence increasing on the postive axis and nonpositive, and hence decreasing on the negative axis. Notice

$$1/\pi_d > 1$$

iff  $d \geq 3$ , and  $\lambda_2(\beta) = 0$  at  $\beta = 0$ , so we may know for  $d \geq 3$  we have a nonnegative

$$\beta_{L_2} = \sup\{\beta \geq 0, \lambda_2(\beta) \leq (1/\pi_d)\} > 0$$

and we know  $p = \lambda$  when  $\beta \leq \beta_{L_2}$ . □

**Corollary 2.4.3.** Let  $s = \text{ess sup}_{\mathbb{P}} \omega(t, x)$ . We have

$$\lim_{\beta \rightarrow \infty} \lambda_2(\beta) = -\ln \mathbb{P}(\omega(t, x) = s)$$

where  $s = \infty$  makes the sense that  $\mathbb{P}(\omega(t, x) = \infty) = 0$ .

*Proof.* Let  $q$  be a measure defined by

$$q(A) = \mathbb{P}(\omega \in A)$$

for borel set  $A$ , and then assume  $s$  finite, we have

$$\mathbb{E}(e^{\beta\omega}; \omega = s) \leq \mathbb{E}(e^{\beta\omega}) = \mathbb{E}(e^{\beta\omega}; \omega \leq s - h) + \mathbb{E}(e^{\beta\omega}; \omega \in [s - h, s])$$

and hence

$$\begin{aligned} \beta s + \ln q(\{s\}) &\leq \lambda(\beta) \leq \ln(e^{\beta s} q([s - h, s]) + e^{\beta s - \beta h}) \\ &= \beta s + \ln q(\{s\}) + \ln \left( \frac{q([s - h, s]) + e^{-\beta h}}{q(\{s\})} \right) \end{aligned}$$

Therefore we have

$$-2 \ln \left( \frac{q([s - h, s]) + e^{-\beta h}}{q(\{s\})} \right) \leq \lambda_2(\beta) + \ln q(\{s\}) \leq \ln \left( \frac{q([s - h, s]) + e^{-2\beta h}}{q(\{s\})} \right)$$

and then

$$\begin{aligned} \limsup_{\beta \rightarrow \infty} \lambda_2(\beta) + \ln q(\{s\}) &\leq \inf_h \ln \left( \frac{q([s - h, s])}{q(\{s\})} \right) = 0 \\ \liminf_{\beta \rightarrow \infty} \lambda_2(\beta) + \ln q(\{s\}) &\geq -2 \inf_h \ln \left( \frac{q([s - h, s])}{q(\{s\})} \right) = 0 \end{aligned}$$

and we are done. □

**Theorem 2.4.4.** Under the assumptions that  $d \geq 3$  and the  $L_2$  condition holds, we have

$$\lim_{n \rightarrow \infty} E_{P_n^{\beta, \omega}} \frac{|S_n|^2}{n} = 1$$

for  $\mathbb{P}$ -a.s. and for all  $f \in C(\mathbb{R}^d)$  with at most polynomial growth at infinity

$$\lim_{n \rightarrow \infty} E_{P_n^{\beta, \omega}} f(S_n/\sqrt{n}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x/\sqrt{d}) \exp(-|x|^2/2) dx$$

for  $\mathbb{P}$ -a.s. and in particular, with  $Z$  a  $d$ -dimensional gaussian vector  $Z \sim \mathcal{N}_d(0, d^{-1}I_d)$ , we have

$$P_n^{\beta, \omega}(S_n/\sqrt{n} \in A) \rightarrow P(Z \in A)$$

for any borel set  $A$  in  $\mathbb{P}$ -a.s.

*Remark.* We introduce a family of martingales  $(M_n)_{n \geq 1}$  on  $(\Omega, \mathcal{G}, \mathbb{P})$  of the form

$$M_n = E\varphi(n, S_n) \exp(\beta H_n(S) - n\lambda(\beta))$$

for a path  $x$  and  $\varphi : \mathbb{N} \times \mathbb{Z}^d \rightarrow \mathbb{R}$  is a function for which we assume

- there are constants  $C_i, p \in \mathbb{N}, i = 0, 1, 2$  such that

$$|\varphi(n, x)| \leq C_0 + C_1|x|^p + C_2n^{p/2}$$

for all  $(n, x) \in \mathbb{N} \times \mathbb{Z}^d$

- $\Phi_n := \varphi(n, S_n)$  is a martingale on  $(\Omega_{traj}, \mathcal{F}, P)$  w.r.t the filtration

$$\mathcal{F}_n = \sigma(S_j; j \leq n)$$

Now consider

$$\begin{aligned} \mathbb{E}(M_{n+1}|\mathcal{G}_n) &= \mathbb{E}(E\varphi(n+1, S_{n+1}) \exp(\beta H_{n+1}(S) - (n+1)\lambda(\beta))|\mathcal{G}_n) \\ &= E\varphi(n+1, S_{n+1}) \exp(\beta H_n(S) - n\lambda(\beta)) \\ &= EE(\varphi(n+1, S_{n+1}) \exp(\beta H_n(S) - n\lambda(\beta))|\mathcal{F}_n) \\ &= M_n \end{aligned}$$

by  $\Phi_n$  is a martingale.

Also we will have a proposition

**Proposition 2.4.5.** Suppose that  $d \geq 3$  and  $L_2$  condition holds, and we have the martingales above with the two properties hold, then there exists  $\kappa \in [0, p/2)$  such that

$$\max_{0 \leq j \leq n} |M_j| = O(n^\kappa)$$

with  $n \rightarrow \infty, \mathbb{P}$ -a.s. In addition,  $p < \frac{1}{2}d - 1$ , then

$$\lim_{n \rightarrow \infty} M_n \text{ exists } \mathbb{P}\text{-a.s. and in } L^2(\mathbb{P})$$

if the second property above does not hold, we will have the sequence  $M_n$  have a larger bound

$$M_n = O(n^{p/2})$$

for  $n \rightarrow \infty, \mathbb{P}$ -a.s.

*Proof.* Let  $\varphi(n, x) = |x|^2 - n$  and then we may know  $p = 2$  and then by the proposition above, there exist  $0 \leq \kappa < 1$  such that

$$\max_{0 \leq j \leq n} |M_j| = O(n^\kappa) = o(n)$$

and notice

$$M_n = E(|S_n|^2 - n) \exp(\beta H_n(S) - n\lambda(\beta)) = E_{P^{\beta, \omega}}(|S_n|^2 - n) W_n$$

and hence

$$E_{P^{\beta, \omega}}|S_n|^2 - n = M_n/W_n = o(n)$$

for  $\mathbb{P}$ -a.s. and hence we have proved the first conclusion.

For the further conclusion, we consider the multi-index  $a$  and prove for  $f(x) = x^a$  with using the induction on  $|a|_1$ . Denote

$$\begin{aligned}\varphi(n, x) &= \left( \frac{\partial}{\partial \theta} \right)^a \exp(\theta \cdot x - n\rho(\theta)) \big|_{\theta=0} \\ \psi(n, x) &= \left( \frac{\partial}{\partial \theta} \right)^a \exp\left(\theta \cdot x - n \frac{|\theta|^2}{2d}\right) \big|_{\theta=0}\end{aligned}$$

where  $\rho(\theta) = \ln\left(\frac{1}{d} \sum_{j=1}^d \cosh(\theta_j)\right)$ , we have  $\varphi(n, x) = x^a + \varphi_0(n, x)$  and  $\psi(n, x) = x^a + \psi_0(n, x)$  where

$$\varphi_0(n, x) = \sum_{j \geq 1, |b|_1 + 2j \leq |a|_1} A_a(b, j) x^b n^j, \quad \psi_0(n, x) = \sum_{j \geq 1, |b|_1 + 2j = |a|_1} A_a(b, j) x^b n^j$$

for some  $A_a(b, j) \in \mathbb{R}$  and hence

$$\begin{aligned}(x/\sqrt{n})^a &= \varphi(n, x) n^{-|a|_1/2} - \varphi_0(n, x) n^{-|a|_1/2} + \psi_0(n, x) n^{-|a|_1/2} - \psi_0(n, x) n^{-|a|_1/2} \\ &= \varphi(n, x) n^{-|a|_1/2} - \psi_0(1, x/\sqrt{n}) + (\psi_0(n, x) - \varphi_0(n, x)) n^{-|a|_1/2}\end{aligned}$$

since where we have

$$\begin{aligned}\psi_0(n, x) n^{-|a|_1/2} &= \sum_{j \geq 1, |b|_1 + 2j = |a|_1} A_a(b, j) x^b n^j n^{-(|b|_1/2 - j)} \\ &= \sum_{j \geq 1, |b|_1 + 2j = |a|_1} A_a(b, j) (x/\sqrt{n})^b \\ &= \psi_0(1, x/\sqrt{n}).\end{aligned}$$

To sum up, we have

$$\begin{aligned}E_{P_n^{\beta, \omega}}(S_n/\sqrt{n})^a &= E_{P_n^{\beta, \omega}} \varphi(n, S_n) n^{-|a|_1/2} - E_{P_n^{\beta, \omega}}(\psi_0(1, S_n/\sqrt{n})) \\ &\quad + E_{P_n^{\beta, \omega}}(\psi_0(n, S_n) - \phi_0(n, S_n)) n^{-|a|_1/2} \\ &= \frac{1}{W_n} E \varphi(n, S_n) \xi_n n^{-|a|_1/2} - \frac{1}{W_n} E(\psi_0(1, S_n/\sqrt{n}) \xi_n) \\ &\quad + \frac{1}{W_n} E(\psi_0(n, S_n) - \phi_0(n, S_n)) \xi_n n^{-|a|_1/2}\end{aligned}$$

where  $\xi_n = \exp(\beta H_n(S) - n\lambda(\beta))$  and the first term and the third term will vanish for  $n \rightarrow \infty$ , since we may check that  $\varphi, \psi$  satisfies the first condition in the above proposition with  $p = |a|_1$  and then we use the last conclusion in the proposition 2.4.5 and we will see that the third term vanishes. By induction hypothesis, we will know that the second term

converges to

$$(2\pi)^{-d/2} \int (x/\sqrt{d})^a e^{-|x|^2/2} dx$$

□

Now we will go through the proof of the proposition 2.4.5.

*Proof.* Firstly, we assume that we have

$$\mathbb{E}M_n^2 = O(b_n), \quad b_n = \sum_{j=1}^n j^p - d/2$$

and setting  $M_n^* = \max_{0 \leq j \leq n} |M_j|$ , and it is sufficient to show that for any  $\delta > 0$ , we have

$$M_n^* = O(n^\delta \sqrt{b_n})$$

for  $n \rightarrow \infty$ ,  $\mathbb{P}$ -a.s., for  $k > 1/\delta$ , we have

$$\begin{aligned} \mathbb{P}(M_{n^k}^* > n^{k\delta} \sqrt{b_{n^k}}) &\leq \mathbb{P}(M_{n^k}^* > n \sqrt{b_n^k}) \\ &\leq \mathbb{E}(M_{n^k}^*)^2 / n^2 b_{n^k} \\ &\leq 4\mathbb{E}M_{n^k}^2 / (n^2 b_{n^k}) \leq Cn^{-2} \end{aligned}$$

and hence we know by the BC lemma that

$$M_{n^k}^* \leq n^{k\delta} \sqrt{b_{n^k}} \text{ for large enough } n$$

is almost sure

□

### 3 Semimartingale Approach

#### 3.1 Checklist

- The theorem that square integrabl martingale converges a.s. on the event  $\langle M \rangle < \infty$ .
- $M_n / \langle M \rangle_n \rightarrow 0$ .

#### 3.2 Useful Conclusions

**Theorem 3.2.1.** (Helly's selection Theorem)

For every sequence  $F_n$  of distribution functions, there is a subsequence  $F_{n_k}$  and a right continuous nondecreasing function  $F$  so that  $F_n$  converges to  $F$  vaguely, i.e.  $\lim_{k \rightarrow \infty} F_{n_k}(y) = F(y)$  at all continuity points  $y$  of  $F$ .

*Proof.* Consider  $q_i$  to be all the rational numbers and then we know there has to be a subsequence of  $F_n$  such that  $F_n(q_1)$  converge to some value, denoted with  $F(q)$  and by recursive constructing we will have a function  $F$  such that there is a subsequence  $F_{n_k}(q_i) \rightarrow F(q_i)$  for all the rational numbers. It is easy to check for any  $q_i < q_j$ , since  $F_{n_k}(q_i) \leq F_{n_k}(q_j)$ , we know

$$F(q_i) = \lim_{k \rightarrow \infty} F_{n_k}(q_i) \leq \lim_{k \rightarrow \infty} F_{n_k}(q_j) \leq F(q_j)$$

and hence we may contruct  $F$  by choose

$$F(x) = \inf\{F(q), q \in \mathbb{Q}, q > x\}$$

which is easy to be checked nondecreasing and right continuous.

For any point  $y$  such that  $F$  is continuous at  $y$ , then notice for any  $\epsilon > 0$ , we have  $q_1, q_2$  rational numbers such that  $q_1 < y < q_2$  and

$$F(y) - \epsilon < F(q_1) \leq F(x) \leq F(q_2) < F(y) + \epsilon$$

and let  $n_k$  large enough we may have

$$F(y) - \epsilon < F_{n_k}(q_1) \leq F_{n_k}(y) \leq F_{n_k}(q_2) < F(y) + \epsilon$$

and we are done. □

**Theorem 3.2.2.** Every subsequential limit is the distribution function of a probability measure if and only if the sequence  $F_n$  is **tight**, i.e. for all  $\epsilon > 0$  there is an  $M_\epsilon$  so that

$$\limsup_{n \rightarrow \infty} (1 - F_n(M_\epsilon) + F_n(-M_\epsilon)) \leq \epsilon$$

*Proof.* To see the sufficiency, assume  $F_n$  is tight and  $F_{n_k} \xrightarrow{v} F$  for some  $F$ , let  $r < -M_\epsilon, s > M_\epsilon$  be continuity point of  $F$  and then we know

$$1 - F(s) + F(r) \leq \limsup_{k \rightarrow \infty} 1 - (F_{n_k}(M_\epsilon) - F_{n_k}(-M_\epsilon)) \leq \epsilon$$

whcih means  $\limsup_{x \rightarrow \infty} F(x) - F(-x) = 1$  and hence  $F$  is a distribution function.



To see the necessity, we may see if  $F_n$  not tight, there is an  $\epsilon > 0$  and a subsequence  $n_k \rightarrow \infty$  such that

$$1 - F_{n_k}(k) + F_{n_k}(-k) \geq \epsilon$$

for all  $k$ , assume  $F_{n_{k_j}}$  converges to  $F$  a distribution function weakly, and let  $r < 0 < s$  continuity points of  $F$ , then

$$1 - F(s) + F(r) = \lim_{j \rightarrow \infty} 1 - F_{n_{k_j}}(s) + F_{n_{k_j}}(r) \geq \liminf_{j \rightarrow \infty} 1 - F_{n_{k_j}}(k_j) + F_{n_{k_j}}(k_j) \geq \epsilon$$

and let  $-r, s \rightarrow \infty$  will induce a contradiction.  $\square$

**Theorem 3.2.3.** Consider a sequence of random variables  $X_n, 0 \leq n \leq \infty$ , if for any  $n$  integer  $EX_k^n \rightarrow EX_\infty^n$ , then  $X_n$  converges weakly in  $X_\infty$ .

*Proof.* We know  $EX_k^2 \rightarrow EX^\infty = T$  finite, we have

$$1 - P(-M \leq X_k \leq M) \leq EX_k^2/M^2 \rightarrow T/M^2$$

and hence let  $M \geq \sqrt{T/\epsilon}$

$$\limsup_{k \rightarrow \infty} (1 - P(-M \leq X_k \leq M)) \leq \lim_{k \rightarrow \infty} EX_k^2/M^2 \leq \epsilon$$

which means  $F_k$  is tight where  $F_k$  is the distribution function of  $X_k$ . Then for any bounded  $g$ , we may consider  $\delta > 0$  and let  $M$  such that

$$\limsup_{k \rightarrow \infty} (1 - P(-M \leq X_k \leq M)) \leq \delta/|g|_{L^\infty}$$

and we may find polynomials  $p_n$  converges to  $g$  uniformly on  $[-M, M]$ , where we know

$$\lim_{k \rightarrow \infty} Ep_n(X_k; |X_k| \leq M) = Ep_n(X_\infty; |X_\infty| \leq M)$$

and hence

$$\lim_{k \rightarrow \infty} Eg(X_k; |X_k| \leq M) = Eg(X_\infty; |X_\infty| \leq M)$$

since  $p_n \rightarrow g$  uniformly and then

$$|\liminf_{k \rightarrow \infty} Eg(X_k) - Eg(X_\infty)| < 2\delta$$

for any  $\delta > 0$  and hence  $\liminf_{k \rightarrow \infty} Eg(X_k) = E(g)(X_\infty)$  for any bounded and continuous function  $g$ , so as for  $\limsup$  and we are done.  $\square$

**Theorem 3.2.4.** (Doob's Decomposition)

Any  $\mathcal{G}_n$ -adapted process  $X = \{X_n\}_{n \geq 0} \subset L^1(\mathbb{P})$  can be decomposed in a unique way as

$$X_n = M_n(X) + A_n(X), \quad n \geq 1$$

where  $M(X)$  is an  $\mathcal{G}_n$ -martingale and  $A(X)$  is predictable, i.e.  $A_n(X)$  is  $\mathcal{G}_{n-1}$  measurable with  $A_0 = 0$ .

*Proof.* We know if this decomposition exists, then

$$\Delta A_n = \mathbb{E}(\Delta X_n | \mathcal{G}_{n-1})$$

and

$$\Delta M_n = \Delta X_n - \mathbb{E}(\Delta X_n | \mathcal{G}_{n-1})$$

and then

$$A_n = \sum_{i=1}^n \mathbb{E}(\Delta X_i | \mathcal{G}_{i-1}), \quad M_n = X_n - \sum_{i=1}^n \mathbb{E}(\Delta X_i | \mathcal{G}_{i-1})$$

□

**Proposition 3.2.5.** If  $N$  is a square integrable martingale, then the compensator  $A(N^2)$  is denoted by  $\langle N \rangle_n$  and is given by

$$\Delta \langle N \rangle_n = E(N_n^2 - N_{n-1}^2 | \mathcal{G}_{n-1}) = E((\Delta N_n)^2 | \mathcal{G}_{n-1})$$

**Theorem 3.2.6.**  $\lim X_n$  exists and finite on  $\{A_\infty < \infty\}$ .

**Theorem 3.2.7.** Let  $f \geq 1$  be increasing with  $\int_0^\infty f(t)^{-2} dt < \infty$ . Then  $X_n/f(A_n) \rightarrow 0$  a.s. on  $\{A_\infty = \infty\}$ .

### 3.3 Semimartingale Decomposition

**Definiton 3.3.1.**

We care about the Doob's decomposition of  $X_n = -\ln W_n = M_n + A_n$ . Then  $-\ln W_n$  is a submartingale and  $A_n$  is increasing about  $n$ .

*Proof.* We know  $-\ln$  is convex and then

$$\mathbb{E}(-\ln W_n | \mathcal{G}_{n-1}) = \mathbb{E}(\sup\{aW_n + b\} | \mathcal{G}_{n-1}) \geq -\ln(W_{n-1})$$

and hence a submartingale, then

$$\mathbb{E}(M_n + A_n | \mathcal{G}_{n-1}) = M_{n-1} + A_n \geq M_{n-1} + A_{n-1}$$

and hence  $A_n$  increasing. □

**Definiton 3.3.2.**

We introduce

$$U_n = E_{n-1}^{\beta, \omega} \exp(\beta \omega(n, S_n) - \lambda(\beta)) - 1$$

and we will have

$$U_n + 1 = W_n / W_{n-1}$$

and then

$$W_n = \prod_{t=1}^n (1 + U_t)$$

and hence

$$\begin{aligned}\triangle A_n &= -\mathbb{E}(\ln(1 + U_n)|\mathcal{G}_{n-1}) \\ \triangle M_n &= -\ln(1 + U_n) + \mathbb{E}(\ln(1 + U_n)|\mathcal{G}_{n-1})\end{aligned}$$

*Proof.* We have

$$\begin{aligned}W_n &= E \exp(\beta H_n(S) - n\lambda(\beta)) \\ &= E \exp(\beta H_{n-1}(S) - (n-1)\lambda(\beta)) \exp(\beta\omega(n, S_n) - \lambda(\beta)) \\ &= E_{n-1}^{\beta, \omega} \exp(\beta\omega(n, S_n) - \lambda(\beta)) W_{n-1}\end{aligned}$$

□

**Definiton 3.3.3.**

Define

$$I_n = \sum_{x \in \mathbb{Z}^d} P_{n-1}^{\beta, \omega}(S_n = x)^2$$

and then consider  $\tilde{S}$  an independent copy of  $S$ , where  $S$  and  $\tilde{S}$  are called **replica** and then

$$I_n = (P_{n-1}^{\beta, \omega})^{\otimes 2}(S_n = \tilde{S}_n)$$

**Theorem 3.3.1.** Let  $\beta \neq 0$ . Then

$$\{W_\infty = 0\} = \left\{ \sum_{n \geq 1} I_n = \infty \right\}, \quad \mathbb{P}\text{-a.s.}$$

Moreover, if  $\mathbb{P}(W_\infty = 0) = 1$ , there exists  $c_1, c_2 \in (0, \infty)$  depending on  $\beta, \mathbb{P}$  such that for  $\mathbb{P}$ -a.s.

$$c_1 \sum_{k=1}^n I_k \leq -\ln W_n \leq c_2 \sum_{k=1}^n I_k \quad \text{for large enough } n$$

and also

$$\lim_{n \rightarrow \infty} \frac{-\ln W_n}{A_n} = 1 \text{ a.s.}$$

**Lemma 3.3.2.** Let  $e_i, 1 \leq i \leq m$  be positive, nonconstant i.i.d. random variables on a probability space such that

$$\mathbb{P}(e_1) = 1, \quad \mathbb{P}(e_1^3 + \ln^2 e_1) < \infty$$

For  $\{\alpha_i\}_{i=1}^m$  nonnegative such that  $\sum_{i=1}^m \alpha_i = 1$ , define a centered random variable  $U > -1$  by

$U = \sum_{i=1}^m \alpha_i e_i - 1$ . Then, there exists a constant  $c \in (0, \infty)$  independent of  $m$  and of  $\{\alpha_i\}_{i=1}^m$  such that

$$\begin{aligned}\frac{1}{c} \sum_{i=1}^m \alpha_i^2 &\leq \mathbb{E} \left( \frac{U^2}{2+U} \right) \\ \frac{1}{c} \sum_{i=1}^m \alpha_i^2 &\leq -\mathbb{E}(\ln(1+U)) \leq c \sum_{i=1}^m \alpha_i^2 \\ \mathbb{E}(\ln^2(1+U)) &\leq c \sum_{i=1}^m \alpha_i^2\end{aligned}$$

*Proof.* Notice

$$\mathbb{E}(U^2) = \mathbb{E}\left(\left(\sum_{i=1}^m (a_i e_i)\right)^2 - 1\right) = \text{var}(e_1) \sum_{i=1}^m \alpha_i^2$$

and

$$\begin{aligned} \mathbb{E}(U^3) &= \mathbb{E}\left(\sum_{i=1}^m a_i e_i\right)^2 \left(\sum_{i=1}^m a_i e_i - 1\right) \\ &= \mathbb{E}\left(\sum_{i=1}^m a_i e_i\right)^3 - \text{var}(e_1) \sum_{i=1}^m \alpha_i^2 \\ &\leq (\mathbb{E}(e_1^3) + 4) \sum_{i=1}^m \alpha_i^2 \end{aligned}$$

and then

$$\begin{aligned} c_1 \sum_{i=1}^m \alpha_i^2 &= \mathbb{E}\left(\frac{U}{\sqrt{2+U}} U \sqrt{2+U}\right) \\ &\leq \mathbb{E}\left(\frac{U^2}{2+U}\right)^{1/2} \mathbb{E}(2U^2 + U^3)^{1/2} \\ &\leq c_3 \left(\sum_{i=1}^m \alpha_i^2\right)^{1/2} \mathbb{E}\left(\frac{U^2}{2+U}\right)^{1/2}. \end{aligned}$$

Define  $\phi(u) = u - \ln(1+u)$  and then

$$\mathbb{E} \ln(1+U) = -\mathbb{E}(\phi(U))$$

for all  $u > -1$ . Notice

$$\left(\phi(u) - \frac{1}{4} \frac{u^2}{2+u}\right)' = \frac{3}{4} - \left(\frac{1}{(u+1)(u+2)^2} + \frac{1}{u+2}\right)$$

and we know  $\phi(u) \geq \frac{1}{4} \frac{u^2}{2+u}$  which implies the LHS of the second inequality. For the RHS, notice

$$\begin{aligned} \mathbb{E}(\phi(U)) &= \mathbb{E}(\phi(U); 1+U \geq \epsilon) + \mathbb{E}(\phi(U); 1+U < \epsilon) \\ &= \mathbb{E}(\phi(U); 1+U \geq \epsilon) - \mathbb{E}(\ln(1+U); 1+U < \epsilon) + \mathbb{E}(U; 1+U < \epsilon) \\ &\leq \mathbb{E}(\phi(U); 1+U \geq \epsilon) - \mathbb{E}(\ln(1+U); 1+U < \epsilon) \end{aligned}$$

for  $\epsilon \in (0, 1)$ , notice  $\phi(u) \leq \frac{1}{2}(u/\epsilon)^2$  for  $1+u \geq \epsilon$  and then

$$\mathbb{E}(\phi(U); 1+U \geq \epsilon) \leq \frac{1}{2} \epsilon^{-2} \mathbb{E} U^2 = \frac{1}{2} \epsilon^{-2} c_1 \sum_{i=1}^m \alpha_i^2$$

Let  $\gamma = -\mathbb{E} \ln(e_1) \geq 0$  (which is by the Chebyshev's inequality) and choose  $\epsilon$  such that  $\ln(1/\epsilon) - \gamma \geq 1$ . Define

$$V = \sum_{i=1}^m \alpha_i (\ln e_i + \gamma)$$

and by Chebyshev, we have

$$V - \gamma \leq \ln(1 + U) \leq \ln \epsilon$$

and hence

$$-\mathbb{E}(\ln(1 + U); 1 + U \leq \epsilon) \leq \mathbb{E}(-V; -V \geq 1) + \gamma \mathbb{P}(-V \geq 1) \leq (1 + \gamma) \mathbb{E}(V^2)$$

where

$$\mathbb{E}V^2 = \mathbb{E}(\ln e_1 + \gamma)^2 \sum_{i=1}^m \alpha_i^2$$

similarly, we have

$$\mathbb{E}(\ln^2(1 + U); 1 + U \leq \epsilon) \leq (2 + 2\gamma^2) \mathbb{E}(V^2)$$

and it is easy to check  $|\ln(1 + U)| \leq \frac{-\ln \epsilon}{\epsilon} |u|$  if  $\epsilon |u| \leq 1 + u$  and hence

$$\mathbb{E}(\ln^2(1 + U); \epsilon \leq 1 + U) \leq \epsilon^{-2} \ln^2 \epsilon^{-1} \mathbb{E}(U^2)$$

and we are done.  $\square$

*Proof.* Use the Lemma 3.3.2. and consider  $\alpha_x^n = P_{n-1}^{\beta, \omega}(S_n = x)$  and  $e_x^n = \exp(\beta \omega(n, x) - \lambda(\beta))$  which is independent with  $\mathcal{G}_{n-1}$  and  $\alpha_x^n$  is measurable in  $\mathcal{G}_{n-1}$  and hence we may use  $\mathbb{P}(\mathcal{G}_{n-1})$  in the lemma. Notice

$$\begin{aligned} U_n &= \sum a_x^n e_x^n - 1 \\ \frac{1}{c} I_n &\leq \Delta A_n = -\mathbb{E}(\ln(1 + U_n) | \mathcal{G}_{n-1}) \leq c I_n \\ \mathbb{E}(\ln^2(1 + U_n) | \mathcal{G}_{n-1}) &\leq c I_n \end{aligned}$$

Then if  $\sum_{n \geq 1} I_n < \infty$ , then we know  $\sum \ln^2(1 + U_n)$  is integrable and hence  $M_n^2$  is integrable by

$$\mathbb{E}(\Delta M_n)^2 \leq \mathbb{E} \ln(1 + U_n)^2$$

by the projection property of conditional expectation. Then we will have

$$\Delta \langle M \rangle_n \leq c I_n$$

and hence we have  $A_\infty < \infty$  and  $\langle M \rangle_\infty < \infty$ , which means  $\lim_{n \rightarrow \infty} M_n$  exists and finite, which implies  $\lim \ln W_n$  exists and finite, so  $W_\infty > 0$ .

By the approximation above, we have

$$\{\sum I_n = \infty\} = \{A_\infty = \infty\}$$

then if  $\langle M \rangle_\infty < \infty$ , then we know  $M_\infty$  exists and finite. If  $\langle M \rangle_\infty = \infty$ , then we may know  $M_n / \langle M \rangle_n \rightarrow 0$  a.s. and it is easy to check that for both cases we have

$$-\frac{\ln W_n}{A_n} \rightarrow 1$$

for  $\mathbb{P}$ -a.s. and we are done.  $\square$

**Corollary 3.3.3.** We have  $\mathbb{P}$ -a.s.

$$p(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \mathbb{E}(\ln E_{t-1}^{\beta, \omega} \exp(\beta \omega(t, S_t)) | \mathcal{G}_{t-1})$$

*Proof.* We have

$$p(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} (\ln(W_n) + n\lambda(\beta)) = \lim_{n \rightarrow \infty} \frac{1}{n} (-A_n + n\lambda(\beta))$$

and we are done.  $\square$

### 3.4 Size-Biasing Bounds

**Definiton 3.4.1.** Define

$$\beta_{sb} = \sup\{\beta \geq 0 : E^{\otimes 2}(\exp(\lambda_2(\beta)N_\infty(S, \tilde{S})) | \tilde{S}) < \infty \text{ for } \tilde{S}\text{-a.s.}\}$$

where the event  $\{E^{\otimes 2}(\exp(\lambda_2(\beta)N_\infty(S, \tilde{S})) | \tilde{S}) < \infty\}$  belongs to the tail  $\sigma$ -field of  $\tilde{S}$ , and therefore it has probability 0 or 1.

We also have

$$\beta_{sb} \geq \beta_{L^2}$$

**Proposition 3.4.1.** Consider  $P, \tilde{P}, \mathbb{P}, \tilde{\mathbb{P}}$  to be independent, where

$$\hat{P}(\hat{\omega}(i, x) \in \cdot) = \mathbb{E}(e(i, x); \omega(i, x) \in \cdot)$$

and let  $\hat{\omega}$  be an i.i.d. environment and  $\hat{e}(i, x) = \exp(\beta \hat{\omega}(i, x) - \lambda(\beta))$  and similarly define  $e(i, x)$ . Now we define

$$\hat{e}_{\tilde{S}}(i, x) = \begin{cases} \hat{e}(i, x) & \text{if } \tilde{S}_i = x, \\ e(i, x) & \text{if } \tilde{S}_i \neq x \end{cases}$$

and

$$\hat{W}_n^{e, \hat{e}, \tilde{S}} = E \prod_{i=1}^n \hat{e}_{\tilde{S}}(i, S_i)$$

Then for  $f : [0, \infty) \rightarrow \mathbb{R}$  bounded measurable,

$$\mathbb{E}W_n f(W_n) = \mathbb{E}\hat{\mathbb{E}}\tilde{E}f(\hat{W}_n^{e, \hat{e}, \tilde{S}})$$

*Proof.* We have

$$\begin{aligned}
\mathbb{E}W_n f(W_n) &= \mathbb{E} \left( \tilde{E} \left( \prod_{i=1}^n e(i, \tilde{S}_i) \right) f \left( E \prod_{i=1}^n e(i, S_i) \right) \right) \\
&= \tilde{E} \left( \mathbb{E} \left( \prod_{i=1}^n e(i, \tilde{S}_i) f \left( E \prod_{i=1}^n e(i, S_i) \right) \right) \right) \\
&= \tilde{E} \left( \mathbb{E} \left( \prod_{i=1}^n e(i, \tilde{S}_i) f \left( \sum_x P(x) \prod_{i=1}^n e(i, x_i) \right) \right) \right) \\
&= \hat{\mathbb{E}} f \left( \sum_x P(x) \prod_{i=1}^n \hat{e}_{\tilde{S}}(i, x_i) \right)
\end{aligned}$$

□

**Theorem 3.4.2.**  $W_\infty > 0$  when  $\beta < \beta_{sb}$  and hence

$$\beta_{sb} \leq \bar{\beta}_c \leq \beta_c$$

### 3.5 Localization v.s. Delocalization

**Definiton 3.5.1.** (The probability of the favourite endpoint)

$$J_n = \max_{x \in \mathbb{Z}} P_{n-1}^{\beta, \omega} \{S_n = x\}$$

and we have

$$J_n^2 \leq I_n \leq J_n$$

**Definiton 3.5.2.** We call the polymer is **localized** if

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n J_t > 0, \mathbb{P}\text{-a.s.}$$

and **delocalized** if

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n J_t = 0, \mathbb{P}\text{-a.s.}$$

**Theorem 3.5.1.** let  $\beta \neq 0$ . The polymer is localized iff  $p < \lambda$  and delocalized iff  $p = \lambda$ .

*Proof.* We have

$$\left( \frac{1}{n} \sum_{t=1}^n I_t \right)^2 \leq \left( \frac{1}{n} \sum_{t=1}^n J_t \right)^2 \leq \frac{1}{n} \sum_{t=1}^n J_t^2 \leq \frac{1}{n} \sum_{t=1}^n I_t \leq \frac{1}{n} \sum_{t=1}^n J_t$$

which implies that

$$c_1 \left( \frac{\ln W_n}{n} \right)^2 \leq \left( \frac{1}{n} \sum_{t=1}^n J_t \right)^2 \leq \frac{1}{n} \sum_{t=1}^n J_t^2 \leq c_2 \frac{\ln W_n}{n} \leq c_3 \frac{1}{n} \sum_{t=1}^n J_t$$

and we are done. □

## 4 The Localized Phase

### 4.1 Checklist

- Prove the integration by parts for Gaussian.

### 4.2 Useful Conclusions

**Lemma 4.2.1.** (Integration by Part)

If  $X$  is centered normal, and  $f$  is smooth with

$$\lim_{|x| \rightarrow \infty} f(x) \exp(-x^2/(2EX^2)) = 0,$$

then

$$E(Xf(X)) = E(X^2)E(f'(X))$$

**Corollary 4.2.2.** (Integration by Part for Gaussian Vectors)

If  $(X, X-1, \dots, X_n)$  is a centered, gaussian vector, and  $F$  is smooth with

$$\lim_{||x|| \rightarrow \infty} F(x) \exp(-ax^2) = 0$$

for all  $a > 0$ , then

$$E(XF(X_1, \dots, X_n)) = \sum_{i=1}^n E(XX_i)E(F_{x_i}(X_1, \dots, X_n))$$

**Theorem 4.2.3.** (Chernoff's bound)

For a r.v., assume all the required moment exist, then

$$P(X \geq a) \leq \inf_{t>0} E(e^{tX})e^{-ta}$$

**Definiton 4.2.1.** (Legendre-Fenchel Transform)

Consider a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we define

$$f^*(k) = \sup_{x \in \mathbb{R}} (kx - f(x))$$

**Definiton 4.2.2.** (Supporting Line)

We call  $f : \mathbb{R} \rightarrow \mathbb{R}$  has a supporting line at  $x$  if there exists  $\alpha \in \mathbb{R}$

**Proposition 4.2.4.**

### 4.3 Path Localization

**Definiton 4.3.1.** In this chapter, we consider Gaussian environment

$$\omega(t, x) \sim \mathcal{N}(0, 1)$$



and for  $y : \mathbb{N} \rightarrow \mathbb{Z}^d$  and  $S$  a path, we define

$$N_n(S, y) = \sum_{t=1}^n \chi_{\{S_t = y_t\}}$$

and

$$\mathcal{F} = \{\beta > 0, p \text{ is differentiable at } \beta, p'(\beta) < \lambda(\beta)\}$$

**Theorem 4.3.1.** Assume that the environment is Gaussian. There exists  $y^{(n)} : [0, n] \rightarrow \mathbb{Z}^d$  such that

$$\liminf_{n \rightarrow \infty} \mathbb{E} E_n^{\beta, \omega} \left( \frac{N_n(S, y^{(n)})}{n} \right) \geq 1 - \frac{p'}{\lambda'}(\beta) > 0$$

for all  $\beta \in \mathcal{F}$ . Moreover,

$$\lim_{\beta \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{E} E_n^{\beta, \omega} \left( \frac{N_n(S, y^{(n)})}{n} \right) = 1$$

*Proof.* We have

$$\begin{aligned} \frac{d}{d\beta} \mathbb{E} p_n(\omega, \beta) &= \frac{1}{n} \frac{d}{d\beta} \mathbb{E} \ln \left( \sum_x P(S^{(n)} = x) \exp(\beta H_n(x)) \right) \\ &= \frac{1}{n} \mathbb{E} \frac{1}{Z_n} \sum_x P(S^{(n)} = x) \exp(\beta H_n(x)) H_n(x) \\ &= \frac{1}{n} \sum_{t \leq n, x} \mathbb{E} (P_n^{\beta, \omega}(S_t = x) \omega(t, x)) \end{aligned}$$

which has a uniform  $L_1$  bound.

Let

$$\begin{aligned} F(\omega(t, x))_{t \leq n, x} &= P_n^{\beta, \omega}(S_t = x) \\ &= \sum_{S_t = x} P(S) \exp(\beta \sum_{i=1}^n \omega(i, S_i)) \end{aligned}$$

and

$$\lim_{x \rightarrow \infty} \exp(\beta \sum_{i=1}^n x_{i, S_i} - a|x|^2) \leq \lim_{x \rightarrow \infty} \exp(|x|(\beta n - a|x|)) = 0$$

which means we may use the integration by parts. Notice

$$\begin{aligned} P_n^{\beta, \omega}(S_t = x) &= \frac{1}{Z_n} \sum_{x, x_t = x} P(S^{(n)} = x) \exp(\beta H_n(x)) \\ &= \frac{Z_n P_n^{\beta, \omega}(S_t = x)}{Z_n P_n^{\beta, \omega}(S_t = x) + (Z_n - Z_n P_n^{\beta, \omega}(S_t = x))} \end{aligned}$$

and hence

$$\begin{aligned} \frac{dP_n^{\beta, \omega}(S_t = x)}{d\beta} &= \frac{1}{Z_n^2} (\beta Z_n P_n^{\beta, \omega}(S_t = x) Z_n - \beta Z_n^2 P_n^{\beta, \omega}(S_t = x)^2) \\ &= \beta (P_n^{\beta, \omega}(S_t = x) - P_n^{\beta, \omega}(S_t = x)^2) \end{aligned}$$

so we have

$$\begin{aligned}
\frac{d}{d\beta} \mathbb{E} p_n(\omega, \beta) &= \frac{\beta}{n} \sum_{t \leq n, x} \mathbb{E} (P_n^{\beta, \omega}(S_t = x) - P_n^{\beta, \omega}(S_t = x)^2) \\
&= \beta \frac{1}{n} \mathbb{E} \left( \sum_{t \leq n} P_n^{\beta, \omega^{\otimes 2}}(S, \tilde{S} \text{ do not coincide at time } t) \right) \\
&= \beta \left( 1 - \frac{1}{n} \mathbb{E} \sum_{t \leq n} P_n^{\beta, \omega^{\otimes 2}}(S, \tilde{S} \text{ coincide at time } t) \right)
\end{aligned}$$

and consider how many contributions to  $P_n^{\beta, \omega^{\otimes 2}}(S, \tilde{S} \text{ coincide at } t_1, \dots, t_k)$  is  $k$  times and hence

$$\beta \left( 1 - \mathbb{E} E_n^{\beta, \omega^{\otimes 2}} \left( \frac{N_n(S, \tilde{S})}{n} \right) \right)$$

Since  $\lambda(\beta) = \beta^2/2$  and we have

$$\lim_{n \rightarrow \infty} \mathbb{E} E_n^{\beta, \omega^{\otimes 2}} \left( \frac{N_n(S, \tilde{S})}{n} \right) = 1 - p'(\beta)/\beta = 1 - \frac{p'}{\lambda'}(\beta)$$

For fixed  $n, \beta, \omega$ , we may define

$$y^{(n)}(t) = \arg \max_x P_n^{\beta, \omega}(S_t = x)$$

and then

$$P_n^{\beta, \omega^{\otimes 2}}(S_t = \tilde{S}_t) \leq P_n^{\beta, \omega}(S_t = y_t^{(n)})$$

and then

$$\mathbb{E} E_n^{\beta, \omega^{\otimes 2}} \left( \frac{N_n(S, \tilde{S})}{n} \right) \leq \mathbb{E} P_n^{\beta, \omega} \left( \frac{N_n(S, y^{(n)})}{n} \right)$$

Recall that  $p$  is convex and  $p$  is almost linear by

$$p(\beta) \leq \beta \inf_{b \in (0, \beta]} \frac{\lambda(b) + \ln(2c)}{b} - \ln(2d)$$

and hence  $p'(\beta)$  should be bounded when  $\beta$  is large and hence

$$1 \geq \lim_{\beta \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{E} E_n^{\beta, \omega} \left( \frac{N_n(S, y^{(n)})}{n} \right) \geq \lim_{\beta \rightarrow \infty} 1 - \frac{C}{\beta}$$

for some constant  $C$ . □

**Theorem 4.3.2.** Assume  $d = 1$  or  $d = 2$ . Then for all  $\beta \neq 0, p(\beta) < \lambda(\beta)$  and therefore  $W_\infty = 0$ .

*Proof.* Notice for all  $z \in \mathbb{Z}^d$ , we have

$$P_{t-1}^{\beta, \omega^{\otimes 2}}(S_t = \tilde{S}_t + z) = \sum_x P_{t-1}^{\beta, \omega}(S_t = x) P_{t-1}^{\beta, \omega}(S_t = x + z) \leq P_{t-1}^{\beta, \omega^{\otimes 2}}(S_t = \tilde{S}_t) = I_t$$

Notice when  $d = 1$ ,

$$1 = \sum_{z, 2|z, |z| \leq 2t} P_{t-1}^{\beta, \omega^{\otimes 2}}(S_t = \tilde{S}_t + z) \leq 2t + 1I_t$$

and hence  $\sum I_t$  diverge, by

$$-\ln W_n \sim \sum I_t$$

and we know  $W_\infty = 0$ .

For  $d = 2$ , if  $W_\infty > 0$  a.s., then let

$$A_n = \{|S_n^{(1)}| \leq K\sqrt{n \ln n}, |S_n^{(2)}| \leq K\sqrt{n \ln n}\}$$

and

$$X_n = E(\exp(\beta H_{n-1} - (n-1)\lambda(\beta)); A_n^c).$$

Then

$$\mathbb{P}(X_n \geq \exp(-K^2 n \ln n/4)) \leq e^{K^2 n \ln n/4} \mathbb{E}(X_n) = e^{K^2 n \ln n/4} P(A_n^c),$$

and by the Chernov's bound:

$$\begin{aligned} P(\pm S_n^{(1)} > K\sqrt{n \ln n}) &\leq \inf_{t>0} E(\exp(tS_n^{(1)}))e^{-tK(\sqrt{n \ln n})} \\ &= \exp \left[ \inf_{t \geq 0} \left( \ln E \exp(tS_n^{(1)}) - tK(\sqrt{n \ln n}) \right) \right] \\ &= \exp \left( -\gamma^*(K\sqrt{n \ln n}) \right) \end{aligned}$$

and  $\gamma^*$  is the LF transform of

$$\gamma(u) := \ln E \exp(uS_n^{(1)}) = \ln \frac{1 + \cosh u}{2} \leq u^2/2$$

and hence

$$\gamma^*(v) \geq v^2/2$$

and then

$$P(A_n^c) \leq \exp(-K^2 n \ln n)$$

and

$$\mathbb{P}(X_n \geq \exp(-K^2 n \ln n/4)) \leq e^{-3K^2 n \ln n/4}.$$

By BC-lemma, we have  $X_n \rightarrow 0$   $\mathbb{P}$ -a.s., which implies

$$Y_n := P_{n-1}^{\beta, \omega}(A_n^c) = X_n/W_n \rightarrow 0/W_\infty = 0$$

for  $\mathbb{P}$ -a.s. Denote  $\mathcal{C}(n, K) = [-K\sqrt{n \ln n}, K\sqrt{n \ln n}]^2$  and

$$\begin{aligned} (1 - Y_n)^2 &= \sum_{x, y \in \mathcal{C}(n, K)} P_{n-1}^{\beta, \omega^{\otimes 2}}(S_n = x, \tilde{S}_n = y) \\ &\leq \sum_{z \in \mathcal{C}(n, 2K)} P_{n-1}^{\beta, \omega^{\otimes 2}}(S_n = \tilde{S}_n + z) \\ &\leq (4K\sqrt{n \ln n})^2 I_n \end{aligned}$$

and hence  $\sum I_n$  diverge and we are done.  $\square$

**Theorem 4.3.3.** Assume  $\omega(t, x)$  has mean 0 and variance 1. For  $d = 1$ , as  $\beta \rightarrow 0$  we have

$$\lambda(\beta) - p(\beta) = O(\beta^4)$$

and  $d = 2, \beta \rightarrow 0$ , we have

$$\lambda(\beta) - p(\beta) = \exp(-\pi\beta^{-2}(1 + o(1)))$$

## 5 KPZ Equation and Universality

### 5.1 Checklist

- Construction of Gaussian field

### 5.2 Useful Conclusions

### 5.3 KPZ Equation

**Definiton 5.3.1.** (KPZ Equation)

$$\frac{\partial h}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 h}{\partial x^2}(t, x) + \frac{1}{2} \left( \frac{\partial h}{\partial x}(t, x) \right)^2 + \beta W(t, x)$$

where  $W$  is a space-time gaussian white noise ( $t \geq 0, x \in \mathbb{R}$ ). The noise is a distribution-valued Gaussian field on  $\mathbb{R}^+ \times \mathbb{R}$  with mean 0 and covariance

$$\mathbb{E}[W(t, x)W(s, y)] = \delta(t - s)\delta(x - y)$$

which means the random variables

$$\left\{ \int W(t, x)(f) dt dx \mid f \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}) \right\}$$

is a Gaussian family with mean 0 and

$$\mathbb{E} \left( \int_{\mathbb{R}^+ \times \mathbb{R}} W(f) \int_{\mathbb{R}^+ \times \mathbb{R}} W(g) \right) = \int_{\mathbb{R}^+ \times \mathbb{R}} fg$$