NOTES FOR RENORMALIZATION FLOW

Based on the paper by A.Dunlap and Cole

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1 Setup

1.1 Semilinear SHE

We consider the semilinear stochastic heat equation

$$du_t^\rho = \frac{1}{2}\Delta u_t^\rho dt + \gamma_\rho \sigma(u_t^\rho)dW_t^\rho, \quad t > 0, x \in \mathbb{R}^2$$

Here σ is a Lipschitz nonlinearity and $dW_t^\rho(x)$ is a Gaussian noise that is white in time and correlated in space at scale $\rho^{1/2} << 1$. We are interested in the pointwise behavior of $u_t^\rho(x)$ as $\rho \to 0$, which calls for an attenuation factor $\gamma_\rho \sim |\ln\rho|^{-1/2}$ due to critical scaling in two dimensions. In fact, we devote most of our attention to a variation on (1.1) in which we first multiply σ and then smooth the noise:

$$dv_t^\rho = \frac{1}{2}\Delta v_t^\rho dt + \gamma_\rho \mathcal{G}_\rho[\sigma(v_t^\rho)dW_t]$$

Definiton 1.1.1.

   (Space-time White Noise)

   Let $dW = (dW_t(x))_{t\in\mathbb{R},x\in\mathbb{R}^2}$ be a standard $\mathbb{R}^m$-valued space-time white noise generating a temporal filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$. Writing $dW = (dW^1, \cdots, dW^m)$ in components, then

$$\mathbb{E}[dW_t^i(x)dW_{t'}^{i'}(x')] = \delta_{i,i'}\delta(t-t')\delta(x-x')$$

Proposition 1.1.1. Construction a space-time white noise.

Definiton 1.1.2.

   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a proability space and fix a target dimension $m \in \mathbb{N}$. The solution $v^\rho : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^m$ is a random vector-valued function parametrized by the correlastion parameter $\rho > 0$. We suppress the dependence of v^p on $\omega \in \Omega$.

   Since v is vector-valued, our nonlinearity $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is matrix-valued. Let $\mathcal{H}_+^m$ dentoe the set of nonnegative-definite symmetric real $m \times m$ matrices, equipped with the metric induced by the Frobenius norm

$$|A|_F^2 := \text{tr}(AA^T) = \text{tr}(A^2)$$

Let σ belong to the space $\text{Lip}(\mathbb{R}^m, \mathcal{H}_+^m)$.

Definiton 1.1.3. Given $\tau \geq 0$, we define the heat operator

$$\mathcal{G}_\tau v = G_\tau * v$$

where $G_\tau = (2\pi\tau)^{-1}\exp(-\frac{|x|^2}{2\tau})$ denotes the standard heat kernel. Define the spatially-smoothed noise $dW_t^\rho = G_\rho * dW_t$.

Proposition 1.1.2. We have

$$\mathbb{E}[dW_t^{\rho,i}(x)dW_{t'}^{\rho,i'}(x') = \delta_{i,i'}\delta(t-t')G_{2\rho}(x-x')$$

Proof. □

Definiton 1.1.4. Define

$$L(\tau) = \ln(1 + \tau) \quad \text{for } \tau \geq 0$$

and set

$$\gamma_\rho = \sqrt{\frac{4\pi}{L(1/\rho)}}$$

Definiton 1.1.5.

(Mild Solution 1)

A mild solution for (1.1) is a predictable random field v^ρ such that for all $s < t$, we have

$$v_t^\rho(x) = \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \mathcal{G}_{t+\rho-r}[\sigma(v_r^\rho) dW_r](x)$$

which means

$$v_t^\rho x = \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \int G_{t+\rho-r}(y) \sigma(v_r^\rho)(x-y) dW_r(x-y) dy$$

$$= \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \int G_{t+\rho-r}(y) \sigma(v_r^\rho)(x-y) dW_r(x-y) dy$$

$$= \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int \left(\int_s^t G_{t+\rho-r}(y) \sigma(v_r^\rho)(x-y) dW_r(x-y) \right) dy$$

which can be interpreted as an Ito integral. We only look for the solution v_t^ρ in the spaces $\mathscr{X}_t^l$ of $\mathbb{R}^m$-valued random fields z on $\mathbb{R}^2$ that are $\mathcal{F}_t$-measurable and

$$||z||_l := \sup_{x \in \mathbb{R}^2} \left(\mathbb{E}|z(x)|^l \right)^{1/l} < \infty$$

Proposition 1.1.3. For any $l \geq 2$, there is a family of random operators $(\mathcal{V}_{s,t}^{\sigma,\rho})_{s<t}$ such that if $v_s \in \mathscr{X}_s^l$, then $v_t^\rho = \mathcal{V}_{s,t}^{\sigma,\rho} v_s$ is a mild solution of (1.1) for $t \geq s$. We oftten write $\mathcal{V}_t^{\sigma,\rho} := \mathcal{V}_{0,t}^{\sigma,\rho}$.

Shown by some standard fixed-point arguments.

Definiton 1.1.6. (Forward-backward SDE)

The system of SDE:

$$d\Gamma_{a,Q}^\sigma(q) = J_\sigma(Q - q, \Gamma_{a,Q}^\sigma(q)) dB(q), \qquad a \in \mathbb{R}^m, 0 < q < Q$$
$$\Gamma_{a,Q}^\sigma(0) = a, \qquad\qquad\qquad\qquad a \in \mathbb{R}^m, Q \geq 0$$
$$J_\sigma(q, b) = [\mathbb{E}\sigma^2(\Gamma_{a,Q}^\sigma(q))]^{1/2}, \qquad\qquad q \geq 0, b \in \mathbb{R}^m$$

for B a standard $\mathbb{R}^m$-valued Brownian motion and $A^{1/2}$ is the unique positive-definite matrix square root of $A \in \mathcal{H}_+^m$.

1.2 Main Result