Math 734 – Spring 2024

# Homework 1

**Due: Wed, Feb. 14, 10PM CT.** Homework should be submitted as a single PDF file via Canvas. Please read the additional instructions in the syllabus. Late homework will **not** be accepted.

[Durrett] refers to the course textbook Richard Durrett: Probability: Theory and Examples, 5th edition, 2019

**Exercise 1.** Let $(\Omega, \mathscr{F}_0, \mathbb{P})$ be a probability space and let $X, X' : \Omega \to \mathbb{R}$ be $(\mathscr{F}_0 - \mathscr{B})$-measurable RVs that are absolutely integrable. Suppose that $\mathbb{P}(X\mathbf{1}_B = X'\mathbf{1}_B) = 1$ for all $B \in \mathscr{F}$, where $\mathscr{F} \subseteq \mathscr{F}_0$ is a $\sigma$-algebra on $\Omega$. Show that $\mathbb{E}[X\,|\,\mathscr{F}] = \mathbb{E}[X'\,|\,\mathscr{F}]$.

**Exercise 2.** Suppose we have a stick of length $L$. Break it into two pieces at a uniformly chosen point and let $X_1$ be the length of the longer piece. Break this longer piece into two pieces at a uniformly chosen point and let $X_2$ be the length of the longer one. Define $X_3, X_4, \cdots$ in a similar way.

**(i)** Let $U \sim \mathrm{Uniform}([0, L])$. Show that $X_1$ takes values from $[L/2, L]$, and that $X_1 = \max(U, L-U)$.

**(ii)** From (i), deduce that for any $L/2 \le x \le L$, we have

$$\mathbb{P}(X_1 \ge x) = \mathbb{P}(U \ge x \text{ or } L-U \ge x) = \mathbb{P}(U \ge x) + \mathbb{P}(U \le L-x) = \frac{2(L-x)}{L}.$$

Conclude that $X_1 \sim \mathrm{Uniform}([L/2, L])$. What is $\mathbb{E}[X_1]$?

**(iii)** Show that $X_2 \sim \mathrm{Uniform}([x_1/2, x_1])$ conditional on $X_1 = x_1$. That is,

$$\mathbb{P}(X_2 \ge x\,|\,X_1) = \frac{2(X_1-x)}{X_1} \quad \text{for } X_1/2 \le x \le X_1.$$

(*Hint:* Use the results in Ex. 5.1.12.) Using iterated expectation, show that $\mathbb{E}[X_2] = (3/4)^2 L$.

**(iv)** In general, show that $X_{n+1}\,|\,X_n \sim \mathrm{Uniform}([X_n/2, X_n])$. Conclude that $\mathbb{E}[X_n] = (3/4)^n L$.

**Exercise 3** (Markov's inequality)**.** Let $X$ be a RV on $(\Omega, \mathscr{F}_0, \mathbb{P})$ with $X \ge 0$ and let $\mathscr{F} \subseteq \mathscr{F}_0$ be a sub-$\sigma$-algebra. Show that for each $a > 0$,

$$\mathbb{P}(X \ge a\,|\,\mathscr{F}) \le a^{-1}\mathbb{E}[X\,|\,\mathscr{F}].$$

**Exercise 4** (Chebyshev's inequality)**.** Let $X$ be a RV on $(\Omega, \mathscr{F}_0, \mathbb{P})$ with $X \ge 0$ and let $\mathscr{F} \subseteq \mathscr{F}_0$ be a sub-$\sigma$-algebra. Show that for each $a > 0$,

$$\mathbb{P}(|X| \ge a\,|\,\mathscr{F}) \le a^{-2}\mathbb{E}[X^2\,|\,\mathscr{F}].$$

**Exercise 5** (Cauchy-Schwarz inequality)**.** Let $X, Y$ be RVs on $(\Omega, \mathscr{F}_0, \mathbb{P})$ with $X \ge 0$ and let $\mathscr{F} \subseteq \mathscr{F}_0$ be a sub-$\sigma$-algebra. Show that

$$\mathbb{E}[XY\,|\,\mathscr{F}]^2 \le \mathbb{E}[X^2\,|\,\mathscr{F}]\,\mathbb{E}[Y^2\,|\,\mathscr{F}].$$

**Exercise 6** (Bias-Variance decomposition)**.** Let $X, Y$ be RVs on $(\Omega, \mathscr{F}_0, \mathbb{P})$ with $X \ge 0$ and let $\mathscr{G} \subseteq \mathscr{F} \subseteq \mathscr{F}_0$ be sub-$\sigma$-algebras. Show that

$$\mathbb{E}\left[(X - \mathbb{E}[X\,|\,\mathscr{G}])^2\right] = \mathbb{E}\left[(\mathbb{E}[X\,|\,\mathscr{F}] - \mathbb{E}[X\,|\,\mathscr{G}])^2\right] + \mathbb{E}\left[(X - \mathbb{E}[X\,|\,\mathscr{F}])^2\right].$$

In particular, if $\mathscr{F} = \mathscr{F}_0$, then

$$\underbrace{\mathbb{E}\left[(X - \mathbb{E}[X\,|\,\mathscr{G}])^2\right]}_{\text{MSE}} = \underbrace{\mathbb{E}\left[(\mathbb{E}[X] - \mathbb{E}[X\,|\,\mathscr{G}])^2\right]}_{\text{bias}} + \underbrace{\mathbb{E}\left[(X - \mathbb{E}[X])^2\right]}_{\text{variance}}. \tag{1}$$

To put in context, suppose we are estimating a random response $X$ by $\hat{X} = \mathbb{E}[X\,|\,\mathscr{G}]$. The error of such estimate, measured by the mean squared error (MSE), is the LHS of (1). This is decomposed into the two terms in the RHS, the bias and the variance.

**Exercise 7** (Law of total variance)**.** Let $X$ be RVs on $(\Omega, \mathscr{F}_0, \mathbb{P})$ with $X \geq 0$ and let $\mathscr{F} \subseteq \mathscr{F}_0$ be sub-$\sigma$-algebra. $\mathrm{Var}(X \,|\, \mathscr{F}) = \mathbb{E}\left[(X - \mathbb{E}[X \,|\, \mathscr{F}])^2 \,|\, \mathscr{F}\right]$. Show that

$$\mathrm{Var}(X \,|\, \mathscr{F}) = \mathbb{E}[X^2 \,|\, \mathscr{F}] - \mathbb{E}[X \,|\, \mathscr{F}]^2.$$

Furthermore, show that

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X \,|\, \mathscr{F})] + \mathrm{Var}(\mathbb{E}[X \,|\, \mathscr{F}]).$$

**Exercise 8** (Durrett)**.** 4.2.2, 4.2.3, 4.2.5, 4.2.9, 4.2.10