Chapter 1

1.1 Brownian Motion

Definition 1.1

A real-valued stochastic process $B = (B_t)_{t\geq 0}$ *defined on a probability space* $(\Omega, \mathcal{F}; P)$ *is called a Brownian motion if it satisfies the following conditions:*

a. Almost surely $B_0 = 0$.

b. For all $0 \leq t_1 < \cdots t_n$ *the increments* $B_{t_n} - B_{t_{n-1}}, \cdots, B_{t_2} - B_{t_1}$ *are independent random variables.*

c. If $0 \leq s < t$, *the increment* $B_t - B_s$ *is a Gaussian random variable with mean zero and variance* $t - s$.

d. With probability one, the map $t \to B_t$ *is continuous.*

A d-dimensional Brownian motion is defined as an $\mathbb{R}^d$*-valued stochastic process* $B = (B_t)_{t\geq 0}$, $B_t = (B_t^1, \cdots, B_t^d)$, *where* $B^1, \cdots, B^d$ *are* d *independent Brownian motions.*

♣

Proposition 1.1

Properties (a),(b),(c) are equivalent to that B *is a Gaussian process,i.e. for any finite set of indices* $t_1, \cdots, t_n$, $(B_{t_1}, \cdots, B_{t_n})$ *is a multivariate Gaussian random variable, equivalently, any linear combination of* B_{t_i} *is normal distributed r.v., with mean zero and covariance function*

$$\Gamma(s,t) = \min(s,t)$$

♠

Proof

Suppose (a),(b),(c) holds, then we know $(B_{t_1}, \cdots, B_{t_n})$ is normal for any finite indices and then

$$m(t) = E(B_t) = 0$$
$$\Gamma(s,t) = E(B_s B_t) = E(B^2_{\min(s,t)}) = \min(s,t)$$

Conversly, we know $E(B_0^2) = 0$ and hence $B_0 = 0$ a.s., then we know $E(B_s^2) = s$ and for any $0 < s < t$,

$$E(B_s(B_t - B_s)) = 0$$

and it is easy to check (c), and (b) is deduced by computing the covariance of the increments, notice that two r.v.s are independent iff $\phi_{(X_1, X_2, \cdots, X_n)} = \phi_{X_1}\phi_{X_2}\cdots\phi_{X_n}$ which implies that normal r.v.s are independent iff they have zero covariances.