NOTES FOR ODE

Based on lectures provided by Chanwoo Kim on MATH 716 2025 SPRING

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1 Introduction

1.1 Differential equations

Definiton 1.1.1. (General ODE)

Generally, we consider

$$y : \mathbb{R} \to N$$

where $t \mapsto y(t)$, N is a vector space such as $\mathbb{R}^d$. Then the general form ODE is

$$F(t, y, \frac{dy}{dt}, \cdots, \frac{d^n y}{dt^n}) = 0$$

Once we can solve it explicitly by

$$\frac{d^n y}{dt^n} = G(t, y, y', \cdots, y^{(n-1)})$$

and we may convert it into a system of first-order equation by

$$\begin{cases} \frac{dx_i}{dt} = x_{i+1} \quad i = 1, \cdots, n-1 \\ \frac{dx_n}{dt} = G(t, x_1, \cdots, x_{n-1}) \end{cases}$$

Definiton 1.1.2. (General ODE systems)

$$\frac{dx_i}{dt} = f_i(t, x_1, \cdots, x_n) i = 1, \cdots, n$$

or

$$\nabla x = f(t, x)$$

where $x : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ if $f(t, x) = f(x)$, we call it to be a **autonomous system**, else a **nonautonomous system**.

An **initial problem** is

$$\begin{cases} \nabla x = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

and the general solution

$$\begin{cases} \nabla x(t; c) = f(t, x(t; c)) \\ x(0; c) = c \end{cases}$$

Example 1.1.1. (Hamiltonian system)

Consider a d-bodies problem in $\mathbb{R}^3$, where denote q_i the locations $i = 1, \cdots, d$ and the potential energy to be $V(q) = V(q_1, \cdots, q_n)$ and we will have

$$m_i q_i'' = -\frac{\partial}{\partial q_i} V(q)$$

and for the momentum $p_i = m_i q_i'$ and we will have

$$\begin{cases} q_i' = \dfrac{p_i}{m_i} \\ p_i' = -\dfrac{\partial}{\partial q_i} V(q_i) \end{cases}$$

where ingeneral we define the Hamiltonian function

$$H(q, p) = T(p) + V(q)$$

where T for kinetic energy and V for potential energy, then

$$\begin{cases} q_i' = \dfrac{\partial H}{\partial p_i} \\ p_i' = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

and hence $H' = 0$.

1.2 One-dimensional Dynamics

Consider Autonomous initial value problem for $x(t) \in \mathbb{R}$

$$\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$$

then

$$\int_0^t \frac{x'(s)}{f(x)} ds = \int_0^t ds \implies \int_{x_0}^x \frac{ds}{f(u)} du = t$$

Example 1.2.1. The logistic equation

$$x' = rx(1 - x)$$

has equilibrium at $x \equiv 0$ and $x \equiv 1$, for the rest

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^{-rt}}$$

which implies $x \equiv 1$ to be an attractor.

1.3 Two-dimensional Dynamics

Consider $z = (x, y) \in \mathbb{R}^2, z' = [P(x, y), Q(x, y)]^T$ with equilibrium

$$S = \{(x, y), P(x, y) = Q(x, y) = 0\}$$

The nullclines, i.e. curves on which a single component vanishes

$$N_x = \{P = 0\}, \quad N_y = \{Q = 0\}$$

and hence $S = N_x \cap N_y$.

Definiton 1.3.1. Phase curves: determine solutions to

$$(x', y') = (P(x, y), Q(x, y))$$

as curves in the phase space.

Suppose $y = Y(x)$, an orbit is locally a graph, $y' = dY/dxx'$ and hence $dY/dx = y'/x' = Q/P = F(x, Y)$ which is a single, first order nonautonomous ODE.

Also the equation in the defintion can be viewed as

$$\begin{cases} dx = Pdt \\ dy = Qdt \end{cases} \implies \frac{dx}{P} = \frac{dy}{Q} = dt \implies -Qdx + Pdy = \alpha$$

Suppose $\alpha = F(x, y)dH = FH_x dx + FH_y dy$ and hence

$$\begin{cases} x' = F\frac{\partial H}{\partial x} \\ y' = -F\frac{\partial H}{\partial y} \end{cases}$$

2 Matrix ODEs

Consider $x' = f(x), x(t) \in \mathbb{R}^n$, if $f(x) = Ax$ where $A \in \mathbb{R}^{n\times n}$, then

$$\frac{dx}{dt} = Ax$$