NOTES FOR PDE BY EVANS

Based on the Lecture Notes by Cole Graham

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Contents

1 Sobolev Spaces

1.1 Definitions

Definiton 1.1.1. (Sobolev Space)

The Sobolev space $W^{k,p}(\Omega)$ is the set of distributions on Ω whose weak partial derivatives up to order k are in $L^p(\Omega)$, i.e. for $f \in C_c^\infty(\Omega)$, there is always some $u_\alpha \in L^p, |\alpha| \leq k$ such that

$$\partial^\alpha u(f) = \int_\Omega u_\alpha f$$

Then $W^{k,p}$ is a Banach space under the norm

$$||u||_{W^{k,p}(\Omega)}^p := \sum_{|\alpha|\leq k} ||\partial^\alpha u||_{L^p(\Omega)}^p$$

where $||\partial_\alpha u||_{L^p(\Omega)} = ||u_\alpha||_{L^p(\Omega)}$ where u_α is the function satisfying the requirement above.

Definiton 1.1.2. The Sobolev space $\widetilde{W}^{k,p}(\Omega)$ is the completion in the $W^{k,p}$ norm of $C^k(\Omega)$ with finite norm.

Proposition 1.1.1. If $k \in \mathbb{N}_0$ and $p \in [1, \infty)$,then $W^{k,p} = \widetilde{W}^{k,p}$

Definiton 1.1.3. For $p = 2$, we can make $W^{k,2}$ a Hilbert space and use $H^k := W^{k,2}$. The inner product is defined by

$$\langle u, v\rangle_{H^k(\Omega)} := \sum_{|\alpha|\leq k} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega \partial^\alpha u \partial^\alpha v$$

Theorem 1.1.2. (Caccioppoli Inequality)

Let $u \in C^(B_R)$ satisfy $\Delta u = 0$. Then there is a constant $C(d) > 0$ such that for all $r \in [0, R)$,

$$\int_{B_r} |\nabla u|^2 \leq \frac{C}{(R-r)^2} \int_{B_R} u^2$$

Proof. Let $0 \leq \phi \leq 1$ be a smooth function that is supported in B_R and equals one on B_r. We know

$$\Delta(\phi u) = \phi\Delta u + 2\nabla\phi \cdot \nabla u + u\Delta\phi = 2\nabla\phi \cdot \nabla u + u\Delta\phi$$

and we may multiply both sides by $-\phi u$ and integrate by parts:

$$\begin{aligned}
\int_{B_R} -(\phi u)\Delta(\phi u) &= \int_{B_R} |\nabla(\phi u)|^2 + \int_{\partial B_R(\phi u)} (\phi u)\frac{\partial(\phi u)}{\partial \nu} dS \\
&= -\frac{1}{2}\int_{B^R} \nabla(\phi^2)\cdot\nabla(u^2) - \int_{B_R} u^2\phi\Delta\phi \\
&= \frac{1}{2}\int_{B_R} \Delta(\phi^2)u^2 - \int_{\partial B_R} u^2\frac{\partial(\phi^2)}{\partial\nu} dS - \int_{B_R} u^2\phi\Delta\phi
\end{aligned}$$

which means

$$\int_{B_R} |\nabla(\phi u)|^2 = \int_{B_R} \left(\frac{1}{2}\Delta(\phi^2) - \phi\Delta\phi\right) u^2 = \int_{B_R} |\nabla\phi|^2 u^2$$

and we may arrange $|\nabla\phi| \leq \dfrac{C(d)}{R-r}$ and hence we are done. $\square$

Corollary 1.1.3. For all $k \in \mathbb{N}$, if $u \in C^{k+1}(B_R)$ satisfies $\Delta u = 0$, then there is a constant $C(d,k) > 0$ such that for all $r \in [0,R)$,

$$\int_{B_r} |D^k u|^2 \leq \frac{C}{(R-r)^{2k}} \int_{B_R} u^2$$

Proposition 1.1.4. We may devide $[r,R]$ to k subintervals of $\dfrac{R-r}{k}$.

1.2 Sobolev Embedding

Theorem 1.2.1. There exists a constant $C(d) > 0$ such that for all $u \in W^{1,1}(\mathbb{R}^d)$, we have

$$||u||_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq C||\nabla u||_{L^1(\mathbb{R}^d)}$$

Corollary 1.2.2. For each $p \in [1,d)$, there exists $C(d,p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$, we have

$$||u||_{L^{p^*}(\mathbb{R}^d)} \leq C||\nabla u||_{L^p(\mathbb{R}^d)}$$

where $p^* = \dfrac{dp}{d-p}$.

Theorem 1.2.3. (Morrey's Inequality)

If $p \in (d,\infty]$, there exists $C(d,p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$, and

$$[u]_{C^{1-d/p}(\mathbb{R}^d)} \leq C||\nabla u||_{L^p(\mathbb{R}^d)}$$

where

$$[u]_{C^\alpha(\mathbb{R}^d)} := \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

and u is some "version" which means a function.

Proof. For all $x \in \mathbb{R}^d$ and $R > 0$, we have

$$\begin{aligned}
\left|\frac{1}{|\partial B_R|}\int_{\partial B_R(x)} (u-u(x))\right| &= \frac{1}{|S^{d-1}|}\int_{S^{d-1}} |u(x+R\theta)-u(x)|d\theta \\
&\leq \frac{1}{|S^{d-1}|}\int_{S^{d-1}}\int_0^R |\nabla u(x+r\theta)|drd\theta \\
&= \frac{1}{|S^{d-1}|}\int_{B_R(x)} |\nabla u(y)||y-x|^{-(d-1)}dy \\
&\leq \frac{1}{|S^{d-1}|}||\nabla u||_{L^p(B_R(x))}\left(\int_{B_R} r^{-(d-1)p/(p-1)}\right)^{(p-1)/p} \\
&= ||\nabla u||_{L^p(B_R(x))}\frac{p-d}{p-1}R^{(p-d)/p} \\
&= C(d,p)R^{1-d/p}||\nabla u||_{L^p(B_R(x))}
\end{aligned}$$

Suppose $|x-z| \leq R/2$ and we have

$$\left|\frac{1}{|\partial B_R|}\int_{\partial B_R(z)}(u-u(x))\right| = \frac{1}{|S^{d-1}|}\int_{S^{d-1}}|u(z+R\theta)-u(x)|d\theta$$
$$\leq \frac{1}{|S^{d-1}|}\int_{S^{d-1}}\int_0^{3R/2}|\nabla u(x+r\phi)|dr\left|\det\frac{D\theta}{D\phi}\right|d\phi$$
$$\leq C(d,p)R^{1-d/p}||\nabla u||_{L^p(B_{3R/2}(x))}$$

Now for $x \neq y$ in $\mathbb{R}^d$ and $R := |x-y|, z = (x+y)/2$ and then

$$|u(x)-u(y)| \leq |u(x) - \frac{1}{\partial B_R}\int_{\partial B_R(z)} u| + |u(y) - \frac{1}{\partial B_R}\int_{\partial B_R(z)} u| \leq CR^{1-d/p}||\nabla u||_{L^p(\mathbb{R}^d)}$$

□

Theorem 1.2.4. Let U be a bounded open subset of $\mathbb{R}^d$ and suppose ∂U is C^1. Assume $d < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^\gamma(\overline{U})$ for $\gamma = 1 - n/p$ with the estimate

$$||u^*||_{C^\gamma(\overline{U})} \leq C||u||_{W^{1,p}(U)}$$

where $C = C(p, n, U)$

Corollary 1.2.5. If $p \in (d, \infty)$, there exists $C(d,p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$ we have

$$||u||_{C^{1-d/p}(\mathbb{R}^d)} \leq C||u||_{W^{1,p}(\mathbb{R}^d)}$$

In particular, $W^{1,p} \hookrightarrow C^{1-d/p}$. The same holds for $p = \infty$ and then we replace $C^{0,1}$ the Lipschitz functions instead of C^1.

Proof. We know that

$$\left|\frac{1}{|B_R|}\int_{B_R(z)}(u-u(x))\right| \leq CR^{1-d/p}||\nabla u||_{L^p(B_R(x))}$$

and let $R = 1$ and we have

$$|u(x)| \leq C|\int_{B_1(x)} u| + C||\nabla u||_{L^p(B_1(x))} \leq C||u||_{W^{1,p}(\mathbb{R}^d)}$$

by Holder. □

Proposition 1.2.6. (General Sobolev Inequality)

Let Ω be a bounded C^1 domain. If $p \in [1, d)$, there exists a constant $C(d, p, \Omega) > 0$ such that for all $u \in W^{1,p}(\Omega)$,

$$||u||_{L^{p^*}(\Omega)} \leq C||u||_{W^{1,p}(\Omega)}$$

and if $p \in (d, \infty]$, we sinilarly have

$$||u||_{C^{1-d/p}(\Omega)} \leq C||u||_{W^{1,p}(\Omega)}$$

Theorem 1.2.7. (Rellich-Kondrachov)

Let Ω be a bounded C^1 domain. If $1 \leq p < d$ and $1 \leq q < p^*$, then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$. THat is, closed bounded sets in $W^{1,p}(\Omega)$ are cinoact in $L^q(\Omega)$.

Corollary 1.2.8. Let Ω be $\mathbb{R}^d$ or a bounded C^1 domain. Then there exists a constant $l(d) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $H^{k+l(\Omega)} \hookrightarrow C^k(\Omega)$.

Proof. If $u \in C^2(B_R)$ is harmonic, then for all $k \in \mathbb{N}$ and $r \in (0, R]$, there exists $C(d, k, r, R) > 0$ such that

$$||u||_{C^k(B_r)} \leq C||u||_{L^2(B_R)}$$

□