ERGODIC THEORY AND DYNAMICS - NOTES, WORKSHEETS, AND PROBLEM SETS

Contents

Description of the contents. The following is the course material for Math 758 taught at the University of Wisconsin - Madison in Spring 2024. The structure of each 75 minute class was $5-10$ minutes of review, $40-45$ minutes of lecture, and 20 minutes of collaboration on an in-class worksheet, whose solutions were discussed at the end of class. Each of the sections correspond to a single class, beginning with lecture notes, and ending with the in-class worksheet. The problem sets that were assigned are included after even-numbered lectures.

Prerequisite knowledge from measure theory and functional analysis. The monotone class theorem, the Caratheodory extension theorem, the monotone convergence theorem, the dominated convergence theorem, the Radon-Nikodym derivative, Hahn-Banach, Stone-Weierstrass, Tychonoff's theorem, Lebesgue density, the use of the Fourier transform to study functions in L^2 of tori, Fubini's theorem, L^1.

1. Examples and Poincare recurrence

A *measurable space* X is a set together with a *sigma algebra* of "measurable sets" (i.e. a collection of sets containing the empty set and closed under complements and countable intersections and unions). If X is a topological space, then the *Borel sigma algebra* is the smallest sigma algebra containing all open sets. A measurable (resp. topological) *dynamical system* is a measurable (resp. topological) space X and a measurable (resp. continuous) map $f : X \longrightarrow X$ (i.e. preimages of measurable (resp. open) sets are measurable (resp. open)). A system is called *measure-preserving* if there is a measure μ on X so that, for any set U, $\mu(f^{-1}(U)) = \mu(U)$. The data (X, f, μ) is called a *measure-preserving system (m.p.s)* and, if $\mu(X) = 1$, a *p.m.p.s.*

Before examples, we recall the following (a vast generalization of which we'll see later on). Suppose that G is a *topological group*, i.e. a topological space that is also a group so that group multiplication $m : G\times G \longrightarrow G$ and group inversion $\iota : G \longrightarrow G$ given by $m(g, h) = gh$ and $\iota(g) = g^{-1}$ are continuous. A measure μ on G is called *(left) translation-invariant* if for any measurable subset A and any $g \in G$, $\mu(gA) = \mu(A)$. We will use the following.

Proposition 1. *Up to scaling, volume (i.e. Lebesgue measure) is the only translation invariant measure on* $\mathbb{R}^n$ *and* $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$.

The proposition shows that the following dynamical systems preserve Lebesgue measure.

Examples of Measure Preserving Systems.

(1) Circle rotations, i.e. $f(x) = x + \alpha \bmod 1$ where $\alpha \in \mathbb{R}$.
(2) Translations on tori, i.e. $f(x_1, \dots, x_n) = (x_1 + \alpha_1, \dots, x_n + \alpha_n) \bmod 1$ where $\alpha_i \in \mathbb{R}$.
(3) Translations on $\mathbb{R}^n$, i.e. $f(x) = x + v$ where $v \in \mathbb{R}^n$.

More Examples of Measure Preserving Systems.

(1) Circle doubling, e.g. $f(x) = 2x \bmod 1$.
(2) Toral Automorphisms, i.e. the self-map on $\mathbb{R}^n/\mathbb{Z}^n$ induced by a linear map $A \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$.
(3) Linear maps of $\mathbb{R}^n$ to itself under matrices with determinant 1.

We will explain now why the first two maps preserve Lebesgue measure. Recall that if $f : X \longrightarrow Y$ is measurable, then the *pushforward* of a measure μ on X is defined by $f_*\mu(U) := \mu(f^{-1}(U))$ for any measurable subset U of Y.

Lemma 2. *A measure ν on Y is $f_*\mu$ if and only if for any function g in $L^1(Y, f_*\mu)$*

$$\int_X g \circ f d\mu = \int_Y g d\nu$$

Proof. The reverse direction is immediate since it holds for any indicator function. For the forward direction, by assumption, the formula holds when g is an indicator function. So it holds for finite sums of indicator functions. These are dense in $L^1(Y, f_*\mu)$ so we use dominated convergence to conclude. □

Note that if $f : X \longrightarrow X$ is measurable, then a measure μ is f-invariant if and only if $f_*\mu = \mu$.

Corollary 3. *If $A \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ has nonzero determinant and $f : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ is the map it induces on the torus, the f preserves Lebesgue measure μ.*

Proof. We must show that $f_*\mu = \mu$, so it suffices to that $f_*\mu$ is translation-invariant. Let $v \in \mathbb{T}^n$. Since A has nonzero determinant, f is a surjection and so there is some $w \in \mathbb{T}^n$ so that $f(w) = v$. If U is any measurable subset of $\mathbb{T}^n$, then

$$f_*\mu(U + v) = \mu(f^{-1}(U + v)) = \mu(f^{-1}(U) + w) = \mu(f^{-1}(U)) = f_*\mu(U).$$

□

Given a measurable dynamical system, (X, f) and a measurable set U a point p in U *recurs to* U if it returns to U infinitely often under iterates of f. Given a topological dynamical system (X, f), a point p is called *recurrent* if p recurs to any open set containing it.

Theorem 4 (Measurable Poincare Recurrence)**.** *If (X, f, μ) is a p.m.p.s. and U is a measurable set, then almost every point in U is U-recurrent.*

Proof. Let B be the set of all points in U that never return to U. Formally, $B = U \cap \bigcap_{n\geq 1} f^{-n}(X\backslash U)$. So $f^{-n}(B)$ consists of points p so that $f^n(p) \in U$ but $f^k(p) \notin U$ for $k > n$. So $f^{-n}(B) \cap f^{-m}(B) = \emptyset$ for $n \neq m$. Since X contains $\bigcup_{n>0} f^{-n}(B)$, each summand of which has equal measure, B has measure zero. Therefore, the set of points F_k in U that return to U under some iterate of f^k is full measure in U. Therefore, $\bigcap_{k\geq 0} F_k$ is too. This set consists of points that return to U infinitely often. □

Corollary 5 (Topological Poincare Recurrence)**.** *Almost every point in a second countable topological p.m.p.s. is recurrent.*

Proof. Let (U_n) be a countable basis and let B_n be the measure zero subset of U_n consisting of non-recurrent points. The union $\bigcup_{n\geq 0} B_n$ is measure zero and so its complement is full measure and consists of recurrent points. $\square$

Worksheet for Lecture 1

Problem 1. Let X be the open unit disk in $\mathbb{C}$ and let $f : X \longrightarrow X$ be given by $f(z) = z^2$. Use Poincare recurrence to show that the only f-invariant probability measure is the delta measure at 0.

Problem 2. Let α be a real number. Show that the collection of points in $[0,1)$ given by $\{n\alpha \bmod 1\}_{n\geq 0}$ is dense if and only if α is irrational. (Hint: For α irrational, it suffices to show that the sequence gets arbitrarily close to 0. Use the fact that, for any $\epsilon > 0$, there must be two distinct integers n and m so that $n\alpha$ mod 1 and $m\alpha$ mod 1 are ϵ-close.)

Problem 3. Let $X = \mathbb{R}$ and let $f(x) = x+1$. Use Poincare recurrence to show that there are no finite f-invariant probability measures. Show that f induces a map $\widetilde{f}$ from the one point compacfication of $\mathbb{R}$ (i.e. the circle S^1) to itself and find all $\widetilde{f}$-invariant measures on S^1.

Problem 4. Let X be the unit circle in $\mathbb{R}^2$. Let A be a 2×2 matrix with real entries, determinant 1, and which does not have finite order. Define $f : X \longrightarrow X$ by $f(v) = \frac{A(v)}{|A(v)|}$. Find all f-invariant measures. (Hint: any 2×2 real matrix with determinant 1 is similar to one of the following: a rotation matrix, a diagonal matrix, or $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for some $t \in \mathbb{R}$. In the rotation matrix case, you may use the fact (which follows from Problem 2) that any measure on the circle that is invariant under an irrational rotation is invariant under *every* rotation.)

2. Ergodicity

An m.p.s (X, T, μ) is called *ergodic* if the only T-invariant measurable sets are null or conull. Equivalently, the only almost-invariant measurable sets are null or conull (*almost-invariant* means that the symmetric difference of a set and its preimage is measure zero). Non-examples include rational rotations of the circle and the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{R}^2/\mathbb{Z}^2$.

Lemma 6. *A p.m.p.s. (X, T, μ) is ergodic if and only if for any two positive measure sets A and B there is some n so that $T^{-n}(A) \cap B$ has positive measure.*

Proof. For the forward direction, we note that $\bigcup_n T^{-n}(A)$ is almost-invariant (it is contained in itself under preimages and the preimage has the same measure) and hence full measure. For the reverse direction, if A is an invariant positive measure set, then it never intersects $X - A$, so $X - A$ must be null. □

Lemma 7. *Let $T : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ be given by $T(x) = x + v$ where $v \in \mathbb{T}^n$. If $\{mv\}_{m \geq 0}$ is dense in $\mathbb{T}^n$, then T is ergodic with respect to Lebesgue measure.*

Proof. Let A and B be two positive measure subsets. Find density points $a \in A$ and $b \in B$, these are points so that for some $\epsilon > 0$, the ball B_a (resp. B_b) of radius ϵ around a (resp. b) has the property that 90% of the points in it are contained in A (resp. B). There is $\delta(n, \epsilon) > 0$ so that if two balls of radius ϵ in $\mathbb{R}^n$ have δ-close centers, then their intersection contains at least 99% of the points in both balls. By assumption, there is some m so that $a + mv$ is δ-close to b. So $T^m(B_a) \cap B_b$ contains 99% of points in both balls. In particular, at least 89% of its points belong to $T^m(A)$ and at least 89% belong to B. Therefore, at least 78% of points in the intersection belong to $T^m(A) \cap B$, so $\mu(T^m(A) \cap B) \geq (.78)(.99)\mu(B_a) > 0$. □

Remark 8. A good exercise is showing that $\{mv\}_{m \geq 0}$ is dense in $\mathbb{T}^n$ if and only if the smallest closed subgroup of $\mathbb{T}^n$ containing v is $\mathbb{T}^n$ itself. The classification of closed subgroups of $\mathbb{T}^n$ then shows that $\{mv\}_{m \geq 0}$ is dense if and only if there is no nonzero $w \in \mathbb{Z}^n$ so that $w \cdot v = 0$ mod 1.

Lemma 9. *Fix $p \in [1, \infty]$. A p.m.p.s. (X, T, μ) is ergodic if and only if the only T-invariant function in $L^p(X, \mu)$ is the constant function.*

Proof. For the reverse direction, if $A \subseteq X$ is T-invariant, then so is 1_A, which must be constant a.e. and hence equal a.e. to 0 or 1. For the forward direction, if $f : X \longrightarrow \mathbb{R}$ is nonconstant and invariant then there must be some number c so that $\{x : f(x) > c\}$ and $\{x : f(x) < c\}$ have positive measure. Both must be invariant contradicting ergodicity. □

Lemma 10. *Let T be the action induced by an invertible integral $n \times n$ matrix A on $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. Let μ be Lebesgue measure on $\mathbb{T}^n$. Then T is ergodic if and only if A does not have a root of unity as an eigenvalue.*

Proof. Note that the characters of X are the continuous homomorphisms $\chi : X \longrightarrow \mathbb{C}^\times$. These are all of the form $\chi_m := \chi(x_1, \ldots, x_n) = \exp\left(2\pi i \left(m_1 x_1 + \ldots + m_n x_n\right)\right)$ where $m := (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Fourier theory tells us that if f belongs to $L^2(X)$ then $f = \sum_{m \in \mathbb{Z}^n} c_m \chi_m$ where $c_m := \int_X f \overline{\chi_m}$ and that $\|f\|_2^2 = \sum_m |c_m|^2$. T is ergodic if and only if the only T-invariant function in L^2 is the constant function. Note that

$$\chi_m(Ax) = \exp(2\pi i m \cdot Ax) = \exp(2\pi i (A^T m) \cdot x) = \chi_{A^T m}(x).$$

Therefore, a function f is T-invariant means that $c_m = c_{mA^T} = c_{m(A^T)^2} = \ldots$. Therefore T is ergodic if and only if 0 is the only point in $\mathbb{Z}^n$ that is periodic under the action of A^T. This occurs if and only if $(A^T)^k - I$ has no kernel for any $k > 0$. □

Worksheet for Lecture 2

Problem 1. Show that for almost every number $x \in [0,1)$ with a decimal expansion $0.a_0a_1a_2\ldots$ the asymptotic fraction of the a_i that are equal to 7 is $\frac{1}{10}$. (Hint: Let $X = [0,1)$ and consider $T : X \longrightarrow X$ given by $T(x) = 10x$ (mod) 1. Now apply the ergodic theorem to an appropriately chosen indicator function.)

Problem 2. The following is a version of the law of large numbers. Suppose that we flip a fair coin and record a 0 whenever heads appears and a 1 whenever tails appears. If S_n is the average of all numbers we have recorded from flip 1 to flip n, then S_n converges to $\frac{1}{2}$ almost surely. Show that this is immediate from the ergodic theorem. (Hint: Adapt the method considered in the previous problem by interpreting the space of outcomes $\{0,1\}^{\mathbb{N}}$ as the set of binary expansions of numbers in $[0,1)$.).

Problem 3. Let $T : S^1 \longrightarrow S^1$ be an irrational rotation of the circle and let μ be Lebesgue measure (i.e. length). Show that for *every* (not just almost-every) point $x \in S^1$ and every continuous function $f : S^1 \longrightarrow \mathbb{R}$, $\frac{f(x)+\ldots+f(T^{n-1}x)}{n} \longrightarrow \int_X f d\mu$.

Problem 4. Let $T : S^1 \longrightarrow S^1$ be the rotation of the circle by 90 degrees. Find an ergodic T-invariant measure (we have seen that this measure will not be Lebesgue). Similarly, find an ergodic invariant measure for the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{T}^2$ and, for style points, find a such a measure that is not just a finite linear combination of delta measures.

Problem 5. Let (X, T, μ) be a pmps. Show that the only T-invariant measurable sets are null or conull if and only if the same is true of T-almost invariant measurable sets.

Problem Set 1: Due after Lecture 4

Problem 1. (Invertible extensions) Einsiedler War Exercises 2.1.7 and 2.1.8

Problem 2. (Solenoids) Einsiedler Ward Exercises 2.1.9

Problem 3. (von Neumann's mean ergodic theorem) Let V be a real *Hilbert space*, i.e. a real vector space together with a symmetric positive definite bilinear form $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$ that is also complete, i.e. defining a norm $\|v\| := \sqrt{\langle v, v \rangle}$, any Cauchy sequence in V with respect to the norm converges to some vector v. Let $A : V \longrightarrow V$ be a *unitary map*, i.e. a linear map so that $\langle Av, Aw \rangle = \langle v, w \rangle$ for all $v, w \in V$. The *invariant subspace* is $V^A := \{v \in V : Av = v\}$.

(1) Consider the continuous linear map $B : V \longrightarrow V$ where $B(v) = v - Av$. Its kernel is V^A. Show that $V = V^A \oplus \overline{\operatorname{im}(B)}$ where $\oplus$ indicates that V is the direct sum of two orthogonal subspaces.
(2) Let $\pi : V \longrightarrow V^A$ be the projection onto the first summand in the decomposition $V = V^A \oplus \overline{\operatorname{im}(B)}$. Let $\mathrm{Av}_n : V \longrightarrow V$ be given by $\mathrm{Av}_n(v) := \frac{1}{n} \sum_{k=0}^{n-1} A^k v$. Show that $\mathrm{Av}_n(v) \longrightarrow \pi(v)$ for every $v \in V$. (Hint: it suffices to consider v of the form $v = w - Aw$.)
(3) Now suppose that (X, T, μ) is an ergodic pmps. Let $V = L^2(X, \mu)$. Let $A : V \longrightarrow V$ be given by $A(f) = f \circ T$. Given $f \in V$ let $c := \int_X f d\mu$. Show that $\| \frac{f(T^{n-1}) + f(T^{n-2}) + \ldots + f}{n} - c \|_{L^2} \longrightarrow 0$.

Problem 4. (Gauss map and Gauss measure) Show that the measure on $X := (0, 1)$ defined by $\mu((a, b)) = \int_a^b \frac{dx}{1+x}$ is invariant under the map $T : X \longrightarrow X$ that sends x to $\frac{1}{x}$ mod 1.

Problem 5. (Lecture 1 Worksheet Problems) Do all the problems from the worksheet from Lecture 1.

Problem 6. (Lecture 2 Worksheet Problems) Do all the problems from the worksheet from Lecture 2.

3. The Birkhoff Ergodic Theorem

Given a p.m.p.s. (X, T, μ) and $f : X \longrightarrow \mathbb{R}$ a function in L^1, set $S_0(f) := 0$,

$$S_n(f) := \sum_{k=0}^{n-1} f(T^k) \quad \text{and} \quad \mathrm{Av}_n(f) := \frac{S_n(f)}{n}.$$

Theorem 11 (The Maximal Ergodic Theorem)**.** *For $\alpha \in \mathbb{R}$, let E_α be the points in X so that $\mathrm{Av}_n(f) > \alpha$ for some n. Then $\alpha\mu(E_\alpha) \leq \int_{E_\alpha} f$.*

Proof. By replacing f with $f - \alpha$ we can suppose that $\alpha = 0$. Note

$$S_n(f) \circ T + f = S_{n+1}(f).$$

If $M_n(f) := \max_{0 \leq k \leq n}(S_n(f))$, then for $0 \leq k \leq n$,

$$M_n(f) \circ T + f \geq S_k(f) \circ T + f \geq S_{k+1}(f).$$

For x in $P_n := \{x : M_n(f)(x) > 0\}$, some $S_k(f)(x) > S_0(f)(x) = 0$, so

$$M_n(f) \circ T + f \geq M_n(f).$$

Therefore, since $M_n(f) = 0$ on $X - P_n$ and since $M_n(f) \geq 0$.

$$\int_{P_n} f \geq \int_{P_n} M_n(f) d\mu - \int_{P_n} M_n(f) \circ T d\mu \geq \int_X M_n(f) d\mu - \int_X M_n(f) \circ T d\mu = 0.$$

Notice that (P_n) is an ascending chain of sets whose union is E_0. By the dominated convergence theorem,

$$\int_{E_0} f d\mu = \lim_{n \longrightarrow \infty} \int_X f \chi_{P_n} d\mu \geq 0.$$

$\square$

Remark 12. By replacing f with $-f$, the maximal ergodic theorem also implies that for any $\beta \in \mathbb{R}$, if F_β is the set of points in X so that $\mathrm{Av}_n(f) < \beta$ for some n, then $\beta\mu(F_\beta) \geq \int_{F_\beta} f$.

Theorem 13 (The Birkhoff Ergodic Theorem, 1931)**.** *If $f^*(x) = \limsup_n \mathrm{Av}_n(f)$ and $f_*(x) = \liminf_n \mathrm{Av}_n(f)$, then $f_* = f^*$, these functions are invariant, and $\int_X f^* d\mu = \int_X f d\mu$. In particular, if (X, μ, T) is ergodic, $\mathrm{Av}_n(f)$ converges pointwise almost everywhere to $\int_X f d\mu$.*

Proof. The following idea shows invariance for f^* and f_*. Note that

$$\frac{n}{n+1}\mathrm{Av}_n(f)(T(x)) + \frac{1}{n+1} f(x) = \mathrm{Av}_{n+1}(f)(x)$$

Fixing x and applying limsup to both sides gives $f^* \circ T = f^*$,

Let $a < b$ be two rational numbers and let $E(a, b)$ be the set of x so that $f_*(x) \leq a < b \leq f^*(x)$. This is a T-invariant set and we claim that it has measure zero. If not, then after replacing X with $E(a,b)$ and μ with $\frac{\mu}{\mu(E(a,b))}$, the maximal ergodic theorem yields $b < \int_{E(a,b)} f \frac{d\mu}{\mu(E(a,b))} < a$, a contradiction. The set where $f^* \neq f_*$ is $\bigcup_{(a,b)\in\mathbb{Q}} E(a,b)$, which is null. This shows that $\mathrm{Av}_n(f)$ converges pointwise almost everywhere to f^*.

Now suppose that f is bounded. It follows that $(\mathrm{Av}_n(f))$ are all bounded by the same constant and so the dominated convergence theorem yields that

$$\int_X f^* d\mu = \lim_{n\longrightarrow\infty} \int_X \mathrm{Av}_n(f) d\mu = \int_X f d\mu$$

where the second equality follows from the T-invariance of μ.

Now suppose that f is unbounded. It suffices to show that $(\mathrm{Av}_n(f))$ converges to f^* in $L^1(X,\mu)$. Since this sequence converges pointwise a.e. to f^*, it suffices to show that $(\mathrm{Av}_n(f))$ is Cauchy in $L^1(X,\mu)$. Let $\epsilon > 0$. Choose a bounded function $g : X \longrightarrow \mathbb{R}$ so that $\|f - g\|_1 < \frac{\epsilon}{3}$ (the existence of g follows by dominated convergence) and note that we have already seen that $(\mathrm{Av}_k(g))_{k\geq 0}$ converges pointwise and hence in L^1 to g^* (going from pointwise to L^1 convergence uses the boundedness $(\mathrm{Av}_n(g))$). Choose N so that $k, m > N$ implies that $\|\mathrm{Av}_k(g) - \mathrm{Av}_m(g)\|_1 < \frac{\epsilon}{3}$. Then for $k, m > N$, by the triangle inequality, we bound $\|\mathrm{Av}_k(f) - \mathrm{Av}_m(f)\|_1$ by

$$\|\mathrm{Av}_k(f) - \mathrm{Av}_k(g)\|_1 + \|\mathrm{Av}_k(g) - \mathrm{Av}_m(g)\|_1 + \|\mathrm{Av}_m(g) - \mathrm{Av}_m(f)\|_1 < \epsilon.$$

$\square$

Worksheet for Lecture 3

Problem 1. Suppose that (X, T, μ) is an ergodic topological p.m.p.s on a compact metric space X. A point x is called *μ-generic* if for every continuous function $f : X \longrightarrow \mathbb{R}$, $\frac{f(x)+f(Tx)...+f(T^{n-1}x)}{n} \longrightarrow \int_X f d\mu$. Show that μ-a.e. point in X is generic. (Hint: Use the fact that if X is a compact metric space, then a consequence of the Stone-Weierstrass theorem is that there is a countable dense set of functions in $C(X)$.)

Problem 2. Suppose that (X, T, μ) is a p.m.p.s. and that T is invertible. Show that $\lim_n \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \frac{1}{n} \lim_n \sum_{k=0}^{n-1} f(T^{-k}x)$ almost surely. (Hint: Consider the invariant set where one limit is bigger than the other. Integrate both functions over this set.)

4. The Riesz Representation Theorem

Let X be a compact metric space. Let $C(X)$ be the space of continuous functions from X to $\mathbb{R}$. Let $\mathcal{M}(X)$ be the space of finite Borel probability measures. A *(real) Banach space* $(V, \|\cdot\|)$ is a (real) vector space V with a *norm* $\|\cdot\|$ (i.e. a definite function $\|\cdot\| : V \longrightarrow [0, \infty)$ satisfying the triangle inequality and so $\|av\| = |a|\|v\|$ for $a \in \mathbb{R}$ and $v \in V$) that is *complete*, i.e. any Cauchy sequence converges. The space $C(X)$ is a Banach space with $\|f\| := \max_{x \in X} |f(x)|$. The *dual space* V^* is the set of *linear functionals* $\phi : V \longrightarrow \mathbb{R}$, i.e. continuous linear maps.

Lemma 14. *Suppose that X is a compact metric space. If K is a closed subset and μ is a finite measure, then*

$$\mu(K) = \inf\{\int_X f d\mu : 1_K \leq f \in C(X)\}.$$

Proof. The lefthand side is less than the righthand side. Let U_n be the points of distance at most $\frac{1}{n}$ from K. This is an open set and $\bigcap_n U_n = K$. By Ursyohn's lemma, we can find $f_n : X \longrightarrow [0, 1]$ so that $1_K \leq f_n \leq 1_{U_n}$. By dominated convergence, $\mu(K) = \int_X 1_K d\mu = \lim_{n \longrightarrow \infty} \int_X f_n d\mu$. □

Lemma 15. *If X is a compact metric space, then $\mathcal{M}(X)$ injects into $C(X)^*$.*

Proof. If $\|f - g\| < \epsilon$, then

$$|\mu(f) - \mu(g)| = \left|\int_X (f - g) d\mu\right| \leq \int_X |f - g| d\mu \leq \epsilon \int_X d\mu = \epsilon\mu(X).$$

So we have continuity as desired. The injectivity of the map from $\mathcal{M}(X)$ to $C(X)^*$ follows since the measure of any closed subset is determined by the image of μ in $C(X)^*$. Any finite Borel measure is determined by its values on closed sets. □

Recall that the *Cantor set* is homeomorphic to $\{0, 1\}^{\mathbb{N}}$.

Lemma 16. *If X is a compact metric space, then there is a continuous surjection from the Cantor set C to X.*

Proof. Cover X be a collection of n_1 balls U_i of diameter at most 1. For each i, cover U_i by a collection of n_2 balls U_{ij} of diameter at most $\frac{1}{2}$. Repeat. Elements of $D := \prod_i \{1 \ldots n_i\}$ determine a nested sequence of compact sets whose intersection is a singleton. The map sending D to X given by sending a sequence to this singleton is continuous (for any sequence d and any $\epsilon > \frac{1}{m}$ the (open) set of sequences in D that agree with d in the first m places are sent to points that are within $\frac{1}{m}$ of the image of d). Suppose without loss of generality that $n_i = 2^{q_i}$ for some integer q_i. Then D can be expressed as $\prod_{i\geq 0} \text{Maps}(\{1,\dots,q_i\} \longrightarrow \{0,1\})$. This is homeomorphic to the Cantor, i.e. functions from $\mathbb{N}$ to $\{0,1\}$, by thinking of these as functions from $\{1,\dots,q_1|q_1+1,\dots,q_1+q_2|q_1+q_2+1,\dots q_1+q_2+q_3|\dots\}$ to $\{0,1\}$. □

We will say that a functional $\mu : C(X) \longrightarrow \mathbb{R}$ is *positive* if $\mu(f) \geq 0$ for any $f : X \longrightarrow (0,\infty)$. This forms a *cone*, i.e. a subset of a vector space that is closed under addition and positive scalar multiplication. Note that if $\phi \in C(X)^*$ is positive and $f \geq g \in C(X)$, then $\phi(f) \geq \phi(g)$.

Lemma 17. *Suppose that X is the Cantor set. Then the cone of positive linear functionals in $C(X)^*$ can be identified with $\mathcal{M}(X)$.*

Proof. Let $\phi \in C(X)^*$ be a positive linear functional. Let $\mathcal{B}$ be the subsets of $\{0,1\}^{\mathbb{N}}$ that are preimages of subsets of $\{0,1\}^n$ under the restriction maps $\pi_n : \{0,1\}^{\mathbb{N}} \longrightarrow \{0,1\}^n$. Any subset of $\{0,1\}^n$ is closed and open so the same holds for the subsets of $\mathcal{B}$. In particular, the indicator functions $(\chi_B)_{B\in\mathbb{B}}$ are continuous. Moreover, $\mathcal{B}$ forms a *ring of sets*, i.e. it is closed under finite union and relative complement. By compactness, any countable disjoint union of nonempty elements of $\mathcal{B}$ belonging to $\mathcal{B}$ is actually a finite disjoint union. Therefore, ϕ determines a countably additive function $\mu : \mathcal{B} \longrightarrow [0,\infty)$ by $\mu(B) := \phi(\chi_B)$. By the Caratheodory extension theorem, μ may be extended to a measure on X. Notice that for finite linear combinations of elements of $(\chi_B)_{B\in\mathcal{B}}$, μ and ϕ agree, i.e. for constants a_B

$$\mu(\sum_{B\in\mathcal{B}} a_B\chi_B) = \sum_{B\in\mathcal{B}} a_B\mu(\chi_B) = \sum_{B\in\mathcal{B}} a_B\phi(\chi_B) = \phi(\sum_{B\in\mathcal{B}} a_B\chi_B).$$

Such functions are dense in $C(X)$ so since μ and ϕ are continuous and agree on a dense set, they are equal. □

Recall that the *(dual) norm* $\|\mu\|$ of a functional $\mu \in C(X)^*$ is the smallest constant C so that $|\mu(f)| \leq C\|f\|$ for all $f \in C(X)$.

Lemma 18. *A nonzero linear functional $\mu \in C(X)^*$ is positive if and only if $\mu(\chi_X) = \|\mu\|$.*

Proof. For the forward direction note that $\langle f, g\rangle := \mu(fg)$ defines a nonnegative semidefinite bilinear form and hence Cauchy-Schwarz applies. For any $f \in C(X)$,

$$|\mu(f)|^2 = |\mu(f\cdot\chi_X)|^2 \leq \mu(f^2)\mu(\chi_X) \leq \mu(\|f\|^2\cdot\chi_X)\mu(\chi)_X) = \|f\|^2\mu(\chi_X)^2.$$

Since equality holds when $f = \chi_X$, $\|\mu\| = \mu(\chi_X)$.

For the reverse direction suppose without loss of generality that $\mu(\chi_X) = \|\mu\| = 1$. Let $f : X \longrightarrow [m,1]$ be continuous with $m > 0$. Noting that $\frac{1+m}{2}$ is the midpoint of this interval,

$$|\mu(f) - \frac{1+m}{2}| = |\mu(f) - \mu(\frac{1+m}{2}\chi_X)| \leq \|f - \frac{1+m}{2}\| \leq \frac{1-m}{2},$$

i.e. $\mu(f) \in [m,1]$. $\square$

Theorem 19 (Riesz Representation Theorem, 1941)**.** *If X is any compact metric space, then the cone of positive linear functionals in $C(X)^*$ can be identified with $\mathcal{M}(X)$.*

Proof. Let C be the Cantor set and let $p : C \longrightarrow X$ be a continuous surjection. Since p is a surjection, $p^* : C(X) \longrightarrow C(C)$ given by $p^*(f) = f \circ p$ is a linear isometry (and hence an injection). Let $\phi \in C(X)^*$ be a positive linear functional and define $p^*\phi : p^*(C(X)) \longrightarrow \mathbb{R}$ to be the function that sends $f \circ p$ to $\phi(f)$. By Hahn-Banach, $p^*\phi$ can be extended to a functional $\psi : C(C) \longrightarrow \mathbb{R}$ of the same norm. The previous two lemmas imply that the extension is positive and hence given by integration against a measure μ on C. For $f \in C(X)$,

$$\phi(f) = \psi(p \circ f) = \int_C f(p)d\mu = \int_X f dp_*\mu.$$

$\square$

Worksheet for Lecture 4

Problem 1. Let X be a compact metric space. Suppose that $T_n : X \longrightarrow X$ is a sequence of continuous functions that uniformly converge to $T : X \longrightarrow X$. Suppose that μ is a Borel probability measure on X that is invariant under T_n for each n. Show that μ is also T-invariant. (Hint: Apply the dominated convergence theorem and the Riesz representation theorem). Conclude that the only measure on the circle that is invariant under an irrational rotation is Lebesgue. (Hint: recall that if θ is irrational, then $\{n\theta \bmod 1\}_{n\geq 0}$ is dense in $[0,1)$.)

Problem 2. Let $\pi_n : \{0,1\}^{\mathbb{N}} \longrightarrow \{0,1\}^n$ be the restriction map. Show that functions of the form $f \circ \pi_n$ (where f and n are arbitrary) are dense in $C(\{0,1\}^{\mathbb{N}})$.

Problem set: Due after Lecture 6

Problem 1 (Another L^2 characterization of ergodicity): Einsiedler-Ward Exercise 2.5.1.

Problem 2 (Souped-Up Poincare Recurrence): Einsiedler-Ward Exercise 2.5.4 and 2.5.5.

Problem 3 (Metrizability of $\mathcal{M}(X)$): Let X be a compact metric space. Show that the set of contracting linear functionals in $C(X)^*$, i.e. functionals $\phi : C(X) \longrightarrow \mathbb{R}$ so that $|\phi(f)| \leq \|f\|$ for all $f \in C(X)$, is a compact metric space with respect to the weak* topology. (Hint: By the Stone-Weierstrass theorem, there is a countable collection of continuous functions $f_n : X \longrightarrow \mathbb{R}$ that are dense in $C(X)$. Define $d(\mu_1, \mu_2) := \sum_{n \geq 0} 2^{-n} |\mu_1(f_n) - \mu_2(f_n)|$.)

Problem 4 (The Jordan Decomposition): Let X be a compact metric space. Let $\phi : C(X) \longrightarrow \mathbb{R}$ be a bounded linear functional. Recall that given a continuous function $f : X \longrightarrow \mathbb{R}$ we define its positive part $f^+ := \max(0, f)$ and its negative part $f^- := \max(0, -f)$ so that $f = f^+ - f^-$. Let $N(X) \subseteq C(X)$ be the collection of continuous maps from X to $[0, \infty)$. Define $\phi^+ : N(X) \longrightarrow \mathbb{R}$ by $\phi^+(f) := \sup\{\phi(g) : g \leq f, g \in N(X)\}$.

(1) Show that ϕ^+ is a continuous linear map. (Hint: To show linearity note that $\phi^+(f_1 + f_2) \geq \phi^+(f_1) + \phi^+(f_2)$ is clear. For the reverse inequality, suppose that $0 \leq g \leq f_1 + f_2$ and define $g_1 = \min(g, f_1)$ and $g_2 = g - g_1$.)
(2) Extend ϕ^+ to $C(X)$ by defining $\mu_1 : C(X) \longrightarrow \mathbb{R}$ so that $\mu_1(f) := \phi^+(f^+) - \phi^+(f^-)$. Show that μ_1 is a positive continuous linear map and hence defines a measure. (Hint: To show linearity, it suffices to show that if $f = g - h$ where $g, h \geq 0$, then $\mu_1(f) = \phi^+(g) - \phi^+(h)$. Write $f^+ - f^- = g - h$, rearrange, and use the linearity of ϕ^+.)
(3) Conclude that there are two finite Borel measures μ_1 and μ_2 so that $\phi(f) = \mu_1(f) - \mu_2(f)$ for any $f \in C(X)$.

Problem 5 (Choquet's theory of barycenters): Let V be a locally convex topological vector space (this just means that we can apply the Hahn-Banach theorem, which says that for any two distinct vectors $x, y \in V$ there is a linear functional $f \in V^*$ so that $f(x) \neq f(y)$). Let $C \subseteq V$ be a convex compact subset. Let μ be a probability measure on C. A point $c \in C$ is a *barycenter* if for every continuous linear map

$f : C \longrightarrow \mathbb{R}$, $f(c) = \mu(f)$, i.e. the value of f on c is the μ-average of f over C.

(1) Show that there is a barycenter if and only if for every finite collection $T = (f_1, \dots, f_n)$ of continuous maps from V to $\mathbb{R}^n$ the point $p := (\mu(f_1), \dots, \mu(f_n))$ is contained in $T(C)$.
(2) Show that C has a barycenter. (Hint: If not, then find a hyperplane in $\mathbb{R}^n$ that separates p from $T(C)$.)
(3) Use the Hahn-Banach theorem to show that barycenters are unique.
(4) Let $b : \mathcal{M}(C) \longrightarrow C$ be the map that associates a probability measure on C to its barycenter. Show that when $\mathcal{M}(C)$ is equipped with the weak* topology that this map is continuous and G-equivariant.

Problem 6: Do Lecture 3 worksheet Problems 1 and 2 and Lecture 4 worksheet Problem 1.

5. Banach-Alaoglu, Markov-Kakutani and Krylov-Bogolyubov

Let X be a compact metric space. Let $\mathcal{M}^1(X)$ be the space of Borel probability measures. The space of all functions from $C(X)$ to $\mathbb{R}$ equipped with the topology of pointwise convergence is $\prod_{f\in C(X)}\mathbb{R}$ equipped with the product topology. Since $C(X)^*$ is a subspace the induced topology on it is called the *weak** topology.

Theorem 20 (Banach-Alaoglu, 1940). *The set of contracting functionals in $C(X)^*$, i.e. those for which $|\mu(f)|\le\|f\|$ for all $f\in C(X)$ is compact in the weak* topology.*

Proof. The contracting linear functionals are a subset of $C := \prod_{f\in C(X)}[-\|f\|,\|f\|]$, which is compact by Tychonoff's theorem. More exactly, it is the intersection of C and $C(X)$, so it is a closed subset of a compact set and hence compact. $\square$

Corollary 21. *$\mathcal{M}^1(X)$ is compact and convex in the weak* topology.*

Proof. These are precisely the contracting linear functionals in $C(X)^*$ that (1) are positive and (2) send χ_X to 1. The two numbered conditions are closed conditions. $\square$

For any continuous map $g : X \longrightarrow X$ we note that $g_* : \mathcal{M}^1(X) \longrightarrow \mathcal{M}^1(X)$ is continuous since for any sequence (μ_n) that weak* converges to μ, $(g_*\mu_n)$ weak* converges to $g_*\mu$ since for any $f\in C(X)$,

$$g_*\mu_n(f) = \mu_n(f\circ g) \longrightarrow \mu(f\circ g) = g_*\mu(f).$$

A *topological semigroup* is a group G together with a topology so that multiplication $m : G\times G \longrightarrow G$ given by $m(g,h) = gh$ is continuous. The main examples for us are $\mathbb{N}$ and $\mathbb{Z}$ equipped with the discrete topology and $[0,\infty)$ and $\mathbb{R}$ equipped with the usual topology. An action of G on a topological space X is a continuous (semi)group action $G\times X \longrightarrow X$. In the sequel we will suppress the words "topological" and "continuous". An $\mathbb{N}$-action on a space, is just a continuous map $f : X \longrightarrow X$. An $\mathbb{R}$-action, is just a flow $\phi_t : X \longrightarrow X$. A semigroup G is *amenable* if every continuous action of G on a compact metric space X admits a G-invariant measure, i.e. a measure μ on X so that $g_*\mu = \mu$ for all $g\in G$.

Lemma 22 (Markov-Kakutani, 1936). *If G is abelian then it is amenable.*

Proof. Let G act continuously on a compact metric space X. Set $\mathcal{M} := \mathcal{M}^1(X)$. For each $g\in G$, let $A_{n,g}(\mu) = \frac{1}{n}\sum_{i=1}^{n}(g^i)_*\mu$. Let S be the set of $\{A_{n,g}\}_{n\ge 0; g\in G}$ after closing under composition. This is an abelian semigroup since G is abelian. We note that if $\phi_1, \dots, \phi_n$ are transformations in S then $\bigcap_i \phi_i(\mathcal{M})$ is nonempty since it contains $\phi(\mathcal{M})$ where $\phi = \phi_1 \dots \phi_n$ (since S is abelian). Since $\mathcal{M}$ is compact, the finite-intersection property implies that $\bigcap_{s\in S} s(\mathcal{M})$ is nonempty. Let μ be an element of this set. For each $g \in G$, there is some μ_n so $A_{n,g}(\mu_n) = \mu$. Therefore,

$$\|\mu - g_*\mu\| = \frac{1}{n}\left\|\sum_{i=1}^{n}(g_i)_*\mu_n - \sum_{j=2}^{n+1}(g^j)_*\mu_n\right\| \leq \frac{2}{n} \longrightarrow 0$$

□

Corollary 23 (Krylov-Bogolyubov, 1937)**.** *If X is a compact metric space and $T : X \longrightarrow X$ is continuous then there is an T-invariant measure.*

Later we will see that every compact Hausdorff group G admits a G-invariant probability measure called *Haar measure*.

Corollary 24. *Compact groups are amenable.*

Proof. Let G act on a compact space X and let μ be the Haar measure on G. Let $x \in X$ be any point and let $\phi : G \longrightarrow X$ be the map that sends g to $g \cdot x$. Note $\phi(g \cdot h) = g \cdot \phi(h)$. Let $\nu := \phi_*\mu$. Then for any measurable set $A \subseteq X$

$$\nu(g^{-1}A) = \phi_*\mu(g^{-1}A) = \mu(\phi^{-1}g^{-1}A) = \mu(g^{-1}\phi^{-1}A) = \mu(\phi^{-1}A) = \nu(A).$$

□

Other than tori the quintessential examples of compact groups are the *orthogonal groups* $O(n)$ and the *unitary groups* $U(n)$, which consist of matrices A in $\mathrm{GL}(n,\mathbb{R})$ (resp. $\mathrm{GL}(n,\mathbb{C})$) that preserve the dot product, i.e. $Av \cdot Aw = v \cdot w$ (resp. Hermitian product, i.e. $Av \cdot \overline{Aw} = v \cdot \overline{w}$) for all v and w. There is also the *compact symplectic group* $\mathrm{Sp}(n)$ which consists of matrices in $\mathrm{GL}(n,\mathbb{H})$ that preserve the standard Hermitian product on $\mathbb{H}^n$. Up to taking finite products, finite covers, finite index subgroups, and finite extensions (and adding in five exceptional groups) $O(n), U(n)$, and $\mathrm{Sp}(n)$ (for $n \geq 1$) account for all compact groups that are also manifolds.

5.1. **Bonus Section: Continuous Semigroup Actions and their induced representations.** The *compact-open topology* on the space $C(X,X)$ of continuous self-maps of X is that of uniform convergence, i.e. the one metrized by

$$d_{C(X,X)}(f,g) := \max_{x\in X} d_X(f(x),g(x)).$$

Lemma 25. *If G acts continuously on X, then the homomorphism $G \longrightarrow C(X, X)$ is a continuous.*

Proof. Fix $\epsilon > 0$ and $h \in G$. For each $x \in X$, there is neighborhood U_x of x and a neighborhood W_x of h in G so that $W_x \cdot U_x \subseteq B(h(x), \frac{\epsilon}{2})$. By compactness, we can find a finite collection (x_i) of points so that $\bigcup_i U_{x_i}$ covers X. Let $W := \bigcap_i W_{x_i}$. If $g \in W$ and $x \in X$, then $x \in U_{x_i}$, so $|g(x) - h(x)| < \epsilon$ since both points are $\frac{\epsilon}{2}$-close to $h(x_i)$. $\square$

Given a Banach space V, the vector space $B(V)$ of bounded linear maps from V to itself is topologized with the topology of pointwise convergence, called the *strong operator topology*. Given a topological semigroup G acting continuously on a metric space X, each element $g \in G$ defines a linear contraction $g^* : C(X) \longrightarrow C(X)$ where $g^*(f) = f \circ g$. This is homomorphism from the opposite semigroup G^{op} to $B(C(X))$.

Lemma 26. *The homomorphism from G^{op} to $B(C(X))$ that sends g to g^* is continuous. Moreover, G acts continuously on $\mathcal{M}^1(X)$*

Proof. We must show that for any continuous function $f : X \longrightarrow \mathbb{R}$, any $\epsilon > 0$, and any $g \in G$, there is a neighborhood U of g so that $h \in U$ implies that $\|f \circ h - f \circ g\| < \epsilon$. Since f is uniformly continuous, there is a $\delta > 0$ so that if $d(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon$. By the previous lemma, there is a neighborhood U of g so that if $h \in U$, then $d(h(x), g(x)) < \delta$ for all $x \in X$, which implies that $\|f \circ h - f \circ g\| < \epsilon$. The second claim is an exercise. $\square$

Worksheet for Lecture 5

Problem 1. (Nonabelian free groups are not amenable) Consider the action of $\mathrm{GL}(2,\mathbb{R})$ on the unit circle X where a matrix A sends v to $\frac{A(v)}{|A(v)|}$. Show that there are no invariant measures on X for the action of the group generated by $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $B := r_{-\frac{\pi}{4}} A r_{\frac{\pi}{4}}$ where r_θ is the rotation by θ matrix. Conclude that any nonabelian free group is not amenable. (Adapting this argument will show that $\mathrm{SL}(2,\mathbb{Z})$ is also not amenable.)

Problem 2. Consider the measure μ_r on $\mathbb{R}^2$ that averages a function over the circle of radius r. Show that (μ_r) weak* converges to the delta measure supported at the origin as $r \longrightarrow 0$.

Problem 3. Suppose that (μ_n) are Borel probability measures on a compact metric space X that weak* converge to μ. Show that for any compact subset K, $\limsup_n \mu_n(K) \leq \mu(K)$. (Hint: recall that there is a collection of continuous functions $f_n : X \longrightarrow [0,1]$ so that $1_K \leq f_n \leq f_{n-1}$ and so that, for any Borel probability measure ν, $\nu(K) = \lim_n \int_X f_n d\nu$.) Use the previous problem to show that equality need not always hold.

Problem 4. (Equidistribution of generic points) Under the same assumptions as the previous problem, show that for any open set U with μ-measure zero boundary $\lim_n \mu_n(U) = \mu(U)$. (Hint: Note that the previous problem implies that $\liminf_n \mu_n(U) \geq \mu(U)$ and use that $\nu(U) \leq \nu(\overline{U})$ for any measure ν). Finally, show that if (X, T, μ) is an ergodic topological pmps and $x \in X$ is generic, then for any open set U with measure zero boundary

$$\lim_{n \longrightarrow \infty} \frac{\#\{1 \leq k \leq n : T^k(x) \in U\}}{n} = \mu(U).$$

(Hint: Note that genericity of x is equivalent to the statement that $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k x}$ weak* converges to μ.)

Problem 5. (Compactness of Orthogonal Groups) Recall that an $n \times n$ matrix A with real entries is orthogonal if and only if $A^T A = I$ or, equivalently, if its columns form an orthonormal basis of $\mathbb{R}^n$. Use these two characterizations to explain why $O(n)$ is compact.

Problem 6. ($O(n)$-invariant measures on spheres) Note that $O(n)$ acts continuously on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. Show that the standard (normalized) $(n-1)$-dimensional volume (called simply

Lebesgue measure m) is the only $O(n)$-invariant probability measure. (Hint: If μ is another invariant probability measure, then argue that it is absolutely continuous with respect to Lebesgue and use the Radon-Nikodym theorem to write $d\mu = f dm$. Conclude that f must be invariant and hence constant.)

6. Grassmanians, Furstenberg's Theorem on Projective Actions, and Borel Density

Given a topological space X and an equivalence relation $\sim$, the *quotient topology* on $X/\sim$ is the one for which $A \subseteq X/\sim$ is open if and only if $p^{-1}(A)$ is open where $p : X \longrightarrow X/\sim$ is the *projection map*. In other words, $f : X/\sim \longrightarrow Y$ is continuous if and only if $f \circ p$ is. If a group G acts on X, then X/G is the space $X/\sim$ where $x \sim y$ if there is an element $g \in G$ so that $g \cdot x = y$.

Lemma 27. *Let G be a compact group. If G acts continuously and transitively on a Hausdorff space X with point stabilizer H, then X is homeomorphic to G/H.*

Proof. Suppose that H stabilizes $x \in X$. Then the map sending G to X by sending g to $g \cdot x$ factors through a map $\phi : G/H \longrightarrow X$, which is a continuous surjection by assumption. It is injective since $gH \cdot x = g'H \cdot x$ if and only if $g^{-1}g' \in H$, i.e. $g' \in gH$. Continuous bijections from a compact space to a Hausdorff one are homeomorphisms. □

In fact the conclusion holds if G is a locally compact Hausdorff group that is σ-compact and X is a locally compact Hausdorff space. Recall that the *Grassmannian* $\mathrm{Gr}_d(\mathbb{R}^n)$ *of d-dimensional subspaces in* $\mathbb{R}^n$ is the set of all d-dimensional subspaces of $\mathbb{R}^n$. If V_n is a sequence of subspaces, then we say that V_n converges to V if a basis of V_n converges to a basis of B. We remark that the universal Grassmannian $\mathrm{Gr}_d(\mathbb{R}^\infty)$ is the classifying space for rank d vector bundles.

Corollary 28. *Since $O(n)$ is compact and acts transitively on the Grassmannian, $\mathrm{Gr}_d(\mathbb{R}^n)$ is compact for all d and n. In particular, it is homeomorphic to $O(n)/O(d) \times O(n-d)$.*

Proof. If $(e_1, \ldots, e_n)$ is the standard basis, then the stabilizer of the span of $(e_1, \ldots, e_d)$ is $O(d) \times O(n-d)$. □

When $d = 1$, the Grassmannian is called *projective space* $\mathbb{P}^{n-1} := \mathbb{P}(\mathbb{R}^n)$, which is the space of lines through the origin of $\mathbb{R}^n$. We note that $\mathrm{GL}(n, \mathbb{R})$ acts on this space with the matrices of the form λI acting trivially. The resulting quotient group is denoted $\mathrm{PGL}(n, \mathbb{R})$. We are interested in the dynamics of matrices in $\mathrm{GL}(n, \mathbb{R})$ acting on the space of lines (more or less equivalently, after taking a $\mathbb{Z}/2$ cover, on spheres). Since the Plücker embedding sends $\mathrm{Gr}_d(\mathbb{R}^n)$ into $\mathbb{P}(\Lambda^d\mathbb{R}^n)$ the dynamics of matrix groups on Grassmannians is included in the study of their dynamics on projective space.

Lemma 29 (Furstenberg's Lemma, 1963)**.** *Suppose that (g_m) is a sequence of matrices in $\mathrm{SL}(n,\mathbb{R})$ with unbounded entries. Suppose that for $\mu,\nu \in \mathcal{M}^1(\mathbb{P}(\mathbb{R}^n))$, $(g_m)_*\mu$ weak* converges to ν. Then there are proper subspaces R and V of $\mathbb{R}^n$ so that ν is supported on $\mathbb{P}(R)\cup\mathbb{P}(V)$.*

As an example if $g_m = \begin{pmatrix} m & \\ & \frac{1}{m} \end{pmatrix}$ then ν will be supported on the projectivization of the x and y axes.

Proof. Let $\|g_m\|_\infty$ be the maximum absolute value of an entry in g_m. Passing to a subsequence, suppose that $\frac{g_m}{\|g_m\|_\infty}$ converges to a matrix g. Then

$$\det(g) = \lim_m \frac{\det g_m}{\|g_m\|_\infty^n} = 0.$$

Let N and R be the kernel and image of g respectively. Passing to a subsequence, suppose that $g_m \cdot N$ converges in the Grassmannian to a subspace V. If ℓ is a line in $\mathbb{R}^n$, then $g_m\ell$ converges to a line in V (resp. R) provided that ℓ is contained in N (resp. not contained in N). Write $\mu = \mu_1 + \mu_2$ where $\mu_1(A) := \mu(A \cap \mathbb{P}(N))$ and $\mu_2(A) = \mu(A \backslash \mathbb{P}(N))$ for any measurable set A. Passing to a subsequence, suppose that $(g_m)_*\mu_1$ and $(g_m)_*\mu_2$ weak* converge to measures ν_1 and ν_2 respectively. We claim that ν_1 is supported on $\mathbb{P}(V)$ and ν_2 on $\mathbb{P}(R)$. Since the arguments are identical we will just prove the first claim. We want to show that if $f : \mathbb{P}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is continuous and vanishes on $\mathbb{P}(V)$, then $\nu_1(f) = 0$. We have seen that $f \circ g_m \restriction_{\mathbb{P}(N)}$ pointwise converges to 0. By the dominated convergence theorem and the fact that μ_1 is supported on $\mathbb{P}(N)$ we have the following.

$$\int_{\mathbb{P}(\mathbb{R}^n)} f d\nu_1 := \lim_m \int_{\mathbb{P}(\mathbb{R}^n)} f(g_m) d\mu_1 = \lim_m \int_{\mathbb{P}(N)} f(g_m) d\mu_1 = \int_{\mathbb{P}(N)} \lim_m f(g_m) d\mu_1 = 0.$$

□

Corollary 30. *Let G be a noncompact subgroup of $\mathrm{SL}(n,\mathbb{R})$ for which there is a G-invariant ergodic probability measure μ on $\mathbb{P}(\mathbb{R}^n)$. Then there is a finite collection $V_1,\ldots,V_m$ of equidimensional subspaces that G permutes transitively and so μ is supported on $\mathbb{P}(V_1)\cup\ldots\cup\mathbb{P}(V_m)$. Moreover, if G_i is the subgroup of matrices that fix V_i, then G_i acts on $\mathbb{P}(V_i)$ via a compact group, i.e. one that conjugates into $O(V_i)\times\mathbb{R}^\times$.*

Proof. Since G is not compact there is a sequence (g_m) with unbounded entries. Since these fix μ, μ is supported on the union of two invariant subspaces by Furstenberg's Lemma. Let V be a smallest dimensional subspace to which μ assigns positive measure. If $g \in G$ and $\mu((g\cdot V)\cap V) > 0$, then the minimality assumption implies that $g \cdot V = V$. So up to null sets $\{g \cdot V\}_{g\in G}$ is a disjoint union of subspaces, each of which has measure $\mu(V)$. Since μ is a probability measure, this set is finite. Since μ is ergodic, G permutes these subspaces transitively. If H is the subgroup that fixes V, then $\mu \restriction_{\mathbb{P}(V)}$ is fully supported and so H must act by a compact group, i.e. one conjugated into $O(V)$. $\square$

6.1. **Bonus Material: Borel Density.** A closed subgroup H of G is said to have *cofinite volume* if there is a G-invariant probability measure on G/H. An example is $\mathrm{SL}(n,\mathbb{Z})$ in $\mathrm{SL}(n,\mathbb{R})$. The *Zariski closure* of H is the largest subgroup L of G so that if $p(A) := p(a_{11},\dots,a_{nn})$ is a polynomial in the entries of a matrix $A = (a_{ij})$ and $p(A) = 0$ for all $A \in H$, then $p(A) = 0$ for all $A \in L$. When $G = \mathrm{SL}(n,\mathbb{R})$, Chevalley showed that one can always find a vector space V that G acts irreducibly on and so that L is the stabilizer of a line.

Corollary 31 (Borel Density)**.** *If H is a closed cofinite volume subgroup of $G := \mathrm{SL}(n,\mathbb{R})$, then its Zariski closure L is $\mathrm{SL}(n,\mathbb{R})$ itself.*

Proof. Push forward the invariant measure on G/H to $G/L \subseteq \mathbb{P}(V)$. Since G acts irreducibly, the measure is not supported on a proper subspace and so G acts via a compact group by Furstenberg's theorem. But since $\mathrm{SL}(n,\mathbb{R})$ is simple it does not map to compact Lie groups except for the trivial group. So G acts trivially on V and hence G/L is a point, i.e. $G = L$. $\square$

Worksheet for Lecture 6

Problem 1: (Upper triangular matrices acting on S^2) Let P be the group of 3×3 upper triangular matrices. Let $g \in P$ act on the sphere S^2 by sending v to $\frac{gv}{|gv|}$. Find all P-invariant measures on S^2.

Problem 2: (Flag manifolds) Let $0 < d_1 < \ldots < d_k$ be any increasing sequence of positive integers. Let $d = (d_1, \ldots, d_k)$. The *flag manifold* $F_d(\mathbb{R}^n)$ is the set of subspaces $V_1 \subseteq \ldots \subseteq V_k \subseteq \mathbb{R}^n$ so that $\dim V_i = d_i$. This space is topologized similarly to the Grassmannian. Prove that $O(n)$ acts transitively on F_d find a closed subgroup H so that $O(n)/H$ is homeomorphic to F_d. In particular, this shows that flag manifolds are compact.

Problem 3: (Solvable Matrix Groups and Representation Theory) Let G be a solvable subgroup of $\mathrm{SL}_n(\mathbb{R})$ for $n > 1$ that contains some sequence of matrices with unbounded entries. (The group in the previous problem is an example). Show that there is some finite index subgroup H of G whose action on $\mathbb{R}^n$ is not irreducible (i.e. for which there is a proper nonzero H-invariant subspace). Recall that solvable groups are amenable.

Problem 4: (Matrix Groups acting on S^2) Let G be a subgroup of 3×3 invertible matrices. Let $g \in G$ act on the sphere S^2 by sending v to $\frac{gv}{|gv|}$. Suppose that G preserves a probability measure μ on the sphere. Show that one of the following occurs:

(1) A finite index subgroup of G fixes a point or stabilizes a great circle.
(2) There is some $A \in \mathrm{GL}(3, \mathbb{R})$ so that AGA^{-1} acts on the sphere via the (projective) orthogonal group $PO(3)$ (in which case G preserves $A_* m$ where m is the area measure on the sphere).

Problem 5: (Amenable Matrix Groups) Let G be a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. Let N be the maximal closed normal solvable subgroup of G. Levi's Theorem states that G/N is either compact or admits an irreducible linear action by a noncompact group on $\mathbb{R}^m$ for some m. Show that G is amenable if and only if G/N is compact. (Hint: On the next homework you will show that a group G with a normal subgroup N is amenable if N and G/N both are).

Problem 6: (Refresher on group quotients and topology) Let G be a locally compact σ-compact Hausdorff topological group acting transitively on a locally compact Hausdorff space X. If H is the stabilizer of $x_0 \in X$, then show that X is homeomorphic to G/H.

(Hint: (1) Show that it suffices to show that the continuous bijection $\phi : G/H \longrightarrow X$ (equivalently from G to X) is open where $\phi(gH) = g{\cdot}x$, i.e. that open sets are sent to open sets. (2) Then show that it suffices to show that open subsets of the identity in G with compact closure are sent to open subsets of X. (3) If U is such a subset, then use σ-compactness to show that there is a countable set (g_n) of points in G so that $\bigcup_n g_n U$ covers G. Therefore, $X = \bigcup_n g_n \phi(\overline{U})$. Use the Baire category theorem to conclude that $\phi(U)$ has interior. (4) If $\phi(h)$ is the interior point, then $\phi(e)$ is an interior point of $h^{-1}\phi(U)$. Use this to conclude.)

Problem Set 3: Due after Lecture 8

Problem 1 (Solvable groups are amenable): Prove the following:

(1) Show that if G is an amenable group that acts continuously and linearly on a compact convex subset C of a locally convex topological vector space, then G has a fixed point. (Hint: Use barycenters (see last week's problem set).)
(2) Show that if N is an amenable normal subgroup of G and G/N is amenable, then so is G. (Hint: If X is a compact metric space on which G acts, consider the compact convex subset of $\mathcal{M}^1(X)$ consisting of N-invariant measures. Show that this set has a G/N-fixed point, which must be a G-invariant measure).
(3) Conclude that solvable groups are amenable.

Problem 2. (Unique ergodicity) Let $T : X \longrightarrow X$ be a continuous self-map of a compact metric space. T is called *uniquely ergodic* if there is only one T-invariant Borel probability measure μ on X. We have seen that irrational rotations on the circle are examples.

(1) Show that μ is ergodic.
(2) Show that *every* (not just almost every) point is μ-generic.
(3) Conversely, show that if every point is μ-generic for some T-invariant measure μ, then μ is uniquely ergodic.

Problem 3: (Amenability and Folner Sequences) Let G be a σ-compact locally compact Hausdorff topological group and let m be its (left) Haar measure. For any compact subset K of G and any $\epsilon > 0$ say that a finite-measure set F is (K, ϵ)- invariant if $m(F \Delta KF) < \epsilon m(F)$. Suppose that (F_n) is a sequence of compact finite measure subsets so that for any compact K and any ϵ there is an N so that $n > N$ implies that F_n is (K, ϵ)-invariant. Show that G is amenable. (Hint: If G acts on a compact metric space X let ν be any Borel probability measure on X and, for any measurable subset A of X, define $\mu_n(A) := \frac{1}{m(F_n)} \int_{F_n} \nu(g^{-1}A) dm(g)$. Let μ be any weak* limit of μ_n and show that μ is G-invariant.) A set (F_n) is called a *Folner sequence*. Explain why $\mathbb{Z}^d$ has a Folner sequence.

Problem 4: Do problems 3 and 4 from Worksheet 5.

Problem 5: Do problems 1 and 3 from Worksheet 6.

Problem 6: Do problems 4 and 5 from Worksheet 6.

7. Circle homeomorphisms, minimality, rotation number, and Poincare's classification

The action of G on a topological space X is called *minimal* if the smallest closed nonempty G-invariant set is X itself. (A *minimal set*, i.e. a closed invariant set with no proper closed invariant subsets always exists by Zorn's lemma). Equivalently the action is minimal if, for any $x \in X$, $G \cdot x$ is dense in X. (Birkhoff defined "minimality" for the first time in 1912).

Examples:

(1) For a rational circle rotation, minimal sets all have cardinality the order of the rotation. In contrast, irrational circle rotations are minimal.
(2) For the map on the open unit disk in $\mathbb{C}$ given by $T(z) = z^2$, the only minimal set is $\{0\}$.

Lemma 32 (Classification of minimal sets on the circle)**.** *If G does not act minimally then it has a minimal set that is either finite or a Cantor set (i.e. a perfect nowhere dense set).*

Proof. Suppose that G has no finite invariant sets. Let C be a minimal set. It must be perfect since its set of limit points is G-invariant and closed. We will show that C must also be nowhere dense. If not, then C contains a closed interval and the union of all closed intervals contained in C is invariant. The closure of the endpoints of these intervals is a proper nonempty invariant subset contained in C contradicting minimality. $\square$

On the homework you will show Denjoy's theorem (1932) which says that the Cantor set is not a minimal set if the homeomorphism is C^2.

Lemma 33 (Fekete's Lemma, 1923)**.** *A sequence of positive real numbers is subadditive if $a_{n+m} \leq a_n + a_m$ for all n, m. If (a_n) is such a sequence then $\lim_n \frac{a_n}{n}$ exists and equals $\inf_{n\geq 0} \frac{a_n}{n}$.*

Proof. Fix k and for any $n > k$ write $n = mk + r$ for $r \in \{0, \ldots, k-1\}$. Note that $a_{mk} \leq ma_k$. Therefore, $\frac{a_n}{n} \leq \frac{ma_k}{mk+r} + \frac{\max(a_0,\ldots,a_r)}{n}$. Therefore, $\limsup_n \frac{a_n}{n} \leq \frac{a_k}{k}$ for all k. So $\limsup_n \frac{a_n}{n} \leq \inf_n \frac{a_n}{n} \leq \liminf_n \frac{a_n}{n}$. $\square$

Let $f : S^1 \longrightarrow S^1$ be a homeomorphism. Lift the homeomorphism to a map $F : \mathbb{R} \longrightarrow \mathbb{R}$ so that $F(x+1) = F(x) + 1$. The *escape rate of a point x_0 (aka rotation number $\tau(f)$)* is $\lim_{n\longrightarrow\infty} \frac{d(F^n x_0, x_0)}{n}$ (we will show now that limit exists). Given a lift F, $G(x) = F(x) + m$ is another lift for any $m \in \mathbb{Z}$. Note that $G^k(x) = F^k(x) + km$. Therefore, rotation number is defined in $\mathbb{R}/\mathbb{Z}$. Alternatively, we could choose F so that $d(F(x), x) < 1$ for all $x \in \mathbb{R}$.

Lemma 34. *Rotation number is well-defined and independent of the point x_0.*

Proof. Let $G_n := \max_{x\in[0,1]} d(F^n x, x)$. Then $G_{n+m} \leq G_n + G_m$. We note too that $d(F^n x, F^n y) < 1$ if $d(x, y) < 1$. So if $g_n := \min_{x\in[0,1]} d(F^n x, x)$, then $g_n \leq G_n \leq g_n + 2$. We're done now by Fekete. $\square$

Corollary 35 (The Monotonicity Lemma)**.** *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be any lift of a circle homeomorphism f with rotation number τ. Let x_0 be any point in $\mathbb{R}$. Then $F^{n_1}(x_0) + m_1 < F^{n_2}(x_0) + m_2$ implies that $n_1\tau + m_1 < n_2\tau + m_2$.*

Proof. Set $y_0 := F^{n_2}(x_0)$. By Fekete's Lemma,

$$\tau = \inf_{n\geq 0} \frac{F^n(y_0) - y_0}{n} \leq \frac{F^{n_1-n_2}(y_0) - y_0}{n_1 - n_2} \leq \frac{m_2 - m_1}{n_1 - n_2}$$

$\square$

Lemma 36. *Rotation number is* 0 *if and only if f has a fixed point.*

Proof. If f has a fixed point, then there is a lift F that does as well and hence the rotation number is zero by definition. Conversely, suppose that the rotation number is zero and choose a lift so that $d(Fx, x) < 1$ for all x. Set $\delta = \min_{x\in[0,1]} d(F(x), x)$. So $n > d(F^n(x), x) \geq n\delta$. So rotation number vanishes in $\mathbb{R}/\mathbb{Z}$ if and only if $\delta = 0$. $\square$

Corollary 37. *Rotation number is rational if and only if f has a periodic point.*

Proof. We note that $\tau(f^n) = n\tau(f) \bmod 1$ and that f has a periodic point with period n if and only if f^n has a fixed point. $\square$

A measurable (resp. topological) dynamical system $T : X \longrightarrow X$ is said to be *semiconjugate* to a dynamical system $G : Y \longrightarrow Y$ if there is a measurable (resp. continuous) surjection h so that the following diagram commutes,

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "h"] & X \arrow[d, "h"] \\ Y \arrow[r, "G"] & Y \end{tikzcd}$$

The systems are *conjugate* if h is invertible and the inverse is measurable (resp. continuous).

Theorem 38 (Poincare's classification of circle homeomorphisms, 1885)**.** *If $f : S^1 \longrightarrow S^1$ is a homeomorphism that has irrational rotation number, then f is conjugate to a rotation by $\tau(f)$ if it is minimal. Otherwise it is semiconjugate to one.*

Proof. Let x_0 be a point on the circle that lifts to 0. Let $\mathcal{O}$ be the lift of $\{f^n x_0\}_{n\geq 0}$ to $\mathbb{R}$. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a lift of f. Define a map $H : \mathcal{O} \longrightarrow \mathbb{R}$ that sends $F^n(0) + m$ to $n\tau + m$ for any $n, m \in \mathbb{N}$. (Since τ is irrational, f has no periodic orbits so each point in $\mathcal{O}$ can be written uniquely as $F^n(0) + m$.) By the monotonicity lemma, H is increasing and its image is dense in $\mathbb{R}$. Therefore H extends to a continuous increasing bijection from $\overline{\mathcal{O}}$ to $\mathbb{R}$ (note that $\{f^n x_0\}$ and hence $\mathcal{O}$ is perfect and the limit of a sequence of points approaching a number from above and below must agree by density of the image of H). H therefore extends to a surjection from $\mathbb{R}$ to $\mathbb{R}$ and fits into a commutative diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\ \mathbb{R} & \xrightarrow{x \longrightarrow x+\tau} & \mathbb{R} \end{array}$$

Note that $F(x+1) = F(x)+1$ and similarly for the bottom horizontal arrow. So taking a quotient by $\mathbb{Z}$ gives us the desired semiconjugacy. $\square$

7.1. **Bonus Section: Semiconjugacy for group actions, Compact Subgroups of** $\mathrm{Homeo}(S^1)$**, and the Tits alternative.**

Lemma 39 (Semiconjugacy of G-actions to minimal actions)**.** *If G leaves a minimal Cantor set C invariant, then there is a continuous surjection $\phi : S^1 \longrightarrow S^1$ so that $\phi(C) = S^1$ and so there is a minimal G action on S^1 so that ϕ is equivariant. Moreover, there is an G-invariant measure on C if and only if there is one on $\phi(S^1)$.*

Proof. We have already seen that there is a continuous surjection from C to S^1. We extend this over the complement of C by demanding that the function be constant on intervals in the complement of C. As in the construction of the Cantor function, we may demand that this map is "non-decreasing", which shows us that $S^1/\sim$ is homeomorphic to S^1, where $a \sim b$ if a and b are in the closure of the same interval in the complement of the Cantor set. Therefore, the action of G descends to an action on $S^1/\sim$, which is a circle. For the second claim, let $E \subseteq C$ be the countable subset of interval endpoints. Then $\phi : C - E \longrightarrow$

$S^1 - \phi(E)$ is a G-equivariant bijection. So there is an invariant measure on one if and only if there is an invariant measure on the other. $\square$

Theorem 40 (Tits alternative for $\mathrm{Homeo}(S^1)$, Margulis 2000)**.** *If G is a subgroup of* $\mathrm{Homeo}(S^1)$ *then one of the following occurs:*

(1) There is a finite G-invariant subset of the circle
(2) G is isomorphic to a subgroup of $O(2)$.
(3) G contains a nonabelian free group.

In particular, G satisfies a version of the *Tits alternative*, i.e. either G has an invariant measure or contains a nonabelian free subgroup. The usual Tits alternative says that a finitely generated matrix group is either virtually solvable (and hence is amenable) or contains a nonabelian free subgroup (and hence is not amenable). We have already seen that we may suppose that G acts minimally on the circle.

The topology on $\mathrm{Homeo}(S^1)$ is the *compact-open topology*, i.e. the one metrized by $d(f, g) = \max_{x\in S^1} d(f(x), g(x))$.

Lemma 41. *If G is a subgroup of* $\mathrm{Homeo}(S^1)$ *with compact closure, then G is conjugate to a subgroup of the rotation group $O(2)$.*

Proof. If G is not compact, then we replace it with its compact closure and let m be its Haar measure. Let λ be the usual length measure on the circle. Define a new measure on the circle by

$$\mu(A) := \int_G \lambda(g^{-1}A)dm(g)$$

Note that if $h \in G$, then

$$\mu(h^{-1}A) = \int_G \lambda(g^{-1}h^{-1}A)dm(g) = \int_G \lambda(g^{-1}A)d(h_*m) = \int_G \lambda(g^{-1}A)dm = \mu(A).$$

This measure is nonatomic and nonzero on any open set (take a small neighborhood of the origin, this neighborhood has positive measure and only changes the length of any interval by ϵ). These properties imply that $h(x) = \mu([0, x])$ is a strictly increasing function from 0 to 1 and hence defines a circle homeomorphism. Its inverse is then the function that sends $x \in [0, 1)$ to y so that $\mu([0, y]) = x$. So

$$h_*^{-1}\mu([0, x]) = \mu([0, y]) = x$$

which shows us that $h_*^{-1}\mu = \lambda$. Therefore, $h^{-1}Gh$ preserves length and hence belongs to $O(2)$. $\square$

Worksheet for Lecture 7

Problem 1: Show that if $f : S^1 \longrightarrow S^1$ is a homeomorphism with rational rotation number $\frac{p}{q}$ where p and q are coprime positive integers, then any periodic point has period q. (Hint: Show that any periodic point has period a multiple of q, then explain why the only periodic points of f^q are fixed points).

Problem 2: Show that if two circle homeomorphisms are in the same conjugacy class then they have the same rotation number.

Problem 3: Explain why $G := \text{Homeo}(S^1)$ is a topological group, i.e. explain why $m : G \times G \longrightarrow G$ so that $m(g, h) = gh$ and $\iota(g) = g^{-1}$ are continuous. The topology on $\text{Homeo}(S^1)$ is the *compact-open topology*, i.e. the one metrized by $d(f, g) = \max_{x \in S^1} d(f(x), g(x))$.

Problem 4: Show that the map from $\text{Homeo}(S^1)$ to $\mathbb{R}/\mathbb{Z}$ given by rotation number is continuous.

8. Conditional expectation and conditional measures

Let $(X, \mathcal{B}, \mu)$ be a probability measure space. Let $i : \mathcal{A} \longrightarrow \mathcal{B}$ be an inclusion of σ-algebras. The typical $\mathcal{B}$-measurable function won't be $\mathcal{A}$-measurable, but what's the best approximation of a function in $L^1(X, \mathcal{B}, \mu)$ by a function in $L^1(X, \mathcal{A}, \mu)$, i.e. can we find a "reasonable map" $i^* : L^1(X, \mathcal{B}, \mu) \longrightarrow L^1(X, \mathcal{A}, \mu)$? Among other things it should be a one-sided inverse to the inclusion map $\iota : L^1(X, \mathcal{A}, \mu) \longrightarrow L^1(X, \mathcal{B}, \mu)$ that sends an $\mathcal{A}$-measurable function f to f. First we develop some intuition.

Examples of i^*.

(1) The smallest σ-algebra is $\{\emptyset, X\}$. Given $f \in L^1(X, \mathcal{B}, \mu)$, $i^*(f)$ must be constant and we choose $i^*(f) := \int f d\mu$.
(2) The next smallest σ-algebra is $\{\emptyset, A, X \backslash A, X\}$ for some nonempty subset A. Given $f \in L^1(X, \mathcal{B}, \mu)$, $i^*(f)$ should return the average over A if $x \in A$ and the average over $X \setminus A$ for $x \notin A$.

Theorem 42 (Conditional Expectation)**.** *There is a linear contraction $i^* : L^1(X, \mathcal{B}, \mu) \longrightarrow L^1(X, \mathcal{A}, \mu)$ that is a one-sided inverse to $\iota : L^1(X, \mathcal{A}, \mu) \longrightarrow L^1(X, \mathcal{B}, \mu)$, i.e. $i^* \circ \iota = \mathrm{id}$. It is functorial in the sense that if $j : \mathcal{B} \longrightarrow \mathcal{C}$ is an inclusion of σ-algebras then $(j \circ i)^* = i^* \circ j^*$ and $(\mathrm{id})^* = \mathrm{id}$. Moreover, $i^*(f)$ is the uniquely characterized (up to its definition on sets of measure zero) as the $\mathcal{A}$-measurable function such that $\int_A i^*(f) d\mu = \int_A f d\mu$ for all $A \in \mathcal{A}$.*

The standard notation is $E[f|\mathcal{A}] := i^*(f)$.

Proof. Start with a nonnegative function $f \in L^1(X, \mathcal{B}, \mu)$. Define a measure $\mu_f : \mathcal{A} \longrightarrow [0, \infty)$ by $\mu_f(A) = \int_A f d\mu$. This is absolutely continuous with respect to μ so the Radon-Nikodym theorem provides a unique (up to its definition on sets of measure zero) $\mathcal{A}$-measurable function $i^*(f)$ so that $\mu_f = i^*(f)\mu$ which is characterized by the property that $\int_A f d\mu = \int_A i^*(f) d\mu$ for any $A \in \mathcal{A}$. With this definition define $i^*(f) = i^*(f^+) - i^*(f^-)$ where f^+ and f^- are functions so that $f = f^+ - f^-$. Functoriality and linearity follow from the characteristic property. $\square$

Corollary 43 (Restatement of the Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a pmps. Let $f \in L^1$. Let $\mathcal{A}$ be the σ-subalgebra of T-invariant sets. The functions $\mathrm{Av}_n(f) := \frac{f(x)+f(Tx)+\ldots+f(T^{n-1}x)}{n}$ converge pointwise almost everywhere and in L^1 to $E[f|\mathcal{A}]$.*

Proof. By the ergodic theorem, the averages converge pointwise almost everywhere and in L^1 to f^*, which is T-invariant and satisfies $\int_A f^* d\mu =$

$\int_A f d\mu$ for any T-invariant set. This is the characterization of $E[f|\mathcal{A}]$. $\square$

Theorem 44 (Disintegration of measures - continuous version)**.** *Let $(X, \mathcal{B}, \mu)$ be a Borel probability measure space with X a compact metric space. Let $\mathcal{A}$ be a σ subalgebra of $\mathcal{B}$. For almost every $x \in X$ there is a probability measure $\mu_x^{\mathcal{A}}$ which is characterized by the property that for any $f \in C(X)$, $E[f|\mathcal{A}](x) = \int_X f d\mu_x^{\mathcal{A}}$.*

In the context of the ergodic theorem, $\mu_x^{\mathcal{A}}$, with $\mathcal{A}$ the σ-algebra of invariant sets, will be a choice of a weak* limit of $\frac{\delta_x + \delta_{Tx} + \ldots + \delta_{T^{n-1}x}}{n}$

Proof. The map $i^* : C(X) \longrightarrow L^1(X, \mathcal{A}, \mu)$ that sends f to $E[f|\mathcal{A}]$ is a linear contraction with the property that $E[f|\mathcal{A}] \geq 0$ if $f \geq 0$. So $(i^*)_x : C(X) \longrightarrow \mathbb{R}$ that sends f to $(i^*(f))(x)$ is a positive linear functional and hence determines a measure $\mu_x^{\mathcal{A}}$ by the Riesz representation theorem. This measure is characterized by the fact that $\mu_x^{\mathcal{A}}(f) = E[f|\mathcal{A}](x)$ for all $f \in C(X)$ and all $x \in X$. $\square$

Remark 45. In fact the above proof is incomplete in that $E[f|\mathcal{A}]$ is only uniquely determined up to its values on sets of measure zero. To fix this, one should start with a countable dense subspace $V \subseteq C(X)$ and define $E[f|\mathcal{A}]$ for $f \in V$ as outputting an actual function (not an equivalence class of functions). This will require deleting a set of measure zero to ensure that the desired linearity, contraction, and positivity properties hold on the nose and not just up to sets of measure zero. Then the functionals $(i^*(f))(x)$ can be extended by linearity from V to the subspace W of real linear combinations of elements of V and then to $C(X)$ by Hahn-Banach.

Theorem 46 (Disintegration of measures - L^1 version)**.** *The conclusion of the previous theorem holds if f is in $L^1(X, \mathcal{B}, \mu)$.*

Proof. We must show that for any $f \in L^1(X, \mathcal{B}, \mu)$, $\int_X f d\mu_x$ is $\mathcal{A}$-measurable and that for any $A \in \mathcal{A}$ $\int_A \int_X f d\mu_x d\mu = \int_A f d\mu$. It suffices to show this fact for nonnegative functions. By taking monotone increasing limits of simple functions, it suffices to prove the claim for simple functions and hence for indicator functions. By dominated convergence, the claim for continuous functions implies the claim for indicator functions of open (and hence closed) sets. By monotone convergence it also holds for countable increasing unions of sets. Therefore, the sets for which the claim holds are closed under countable increasing union, countable nested intersection, and complements. This is a monotone class including all the open sets and hence coincides with the Borel σ-algebra by the monotone class theorem. $\square$

Homework 4: Due after lecture 10

Problem 1: (Flows on the circle) A *topological flow* is a continuous $\mathbb{R}$-action on a topological space X. Show that every topological flow on the circle either has a fixed point or is conjugate to a flow where, for some fixed λ, and for all $t \in \mathbb{R}$, t acts by rotating the circle by $t\lambda$ radians. (Hint: Distinguish between the cases where the action is transitive and where it is not.)

Problem 2: ($\mathrm{Homeo}^+(S^1)$ **is perfect)** Let $\mathrm{Homeo}^+(S^1)$ be the group of orientation preserving homeomorphisms of the circle. The goal of this problem is to show that the only homomorphism from this group to an abelian group is the trivial one.

(1) Consider the group G of increasing bijections from $[0, 1]$ to itself. Show that if $f, g \in G$ and $f(x) > x$ and $g(x) > x$ for all $x \in (0, 1)$ then f and g are conjugate in G.
(2) Now suppose that $f : S^1 \longrightarrow S^1$ is an orientation preserving homeomorphism that has a fixed point. Using the previous problem, show that f and f^2 are conjugate in $\mathrm{Homeo}^+(S^1)$. Conclude that f can be written as a commutator.
(3) Recall that any 2×2 rotation matrix can be written as a commutator of elements in $\mathrm{SL}(2, \mathbb{R})$. Conclude that any element of $\mathrm{Homeo}^+(S^1)$ can be written as a product of two commutators.

Problem 3: (Rotation number is a class function) Show that rotation number is a class function on $\mathrm{Homeo}(S^1)$, i.e. two homeomorphisms have the same rotation number if they are in the same conjugacy class.

Problem 4: ($\mathrm{Homeo}^+(S^1)$ **is topologically a circle)** Show that $\mathrm{Homeo}(S^1)$ deformation retracts to $O(2)$. (Hint: It is easier to work with lifts of (orientation-preserving) circle homeomorphisms to $\mathbb{R}$ since these are just strictly increasing functions $F : \mathbb{R} \longrightarrow \mathbb{R}$ that are 1-periodic, i.e. so that $F(x+1) = F(x)+1$. The lift of a rotation by α is just $F(x) = x + \alpha$. So you need to exhibit a deformation retract from the space of increasing 1-periodic functions to the space of functions of the form $x + \alpha$. A “straight-line homotopy” should do the trick.)

Problem 5: (Rotation number is a continuous map) Problems 3 and 4 from Worksheet 7.

Problem 6: (Expansive circle maps) Let G be a group of orientation preserving circle homeomorphisms. Show that either G is abelian or G is *expansive*, i.e. there is a sequence of intervals I_n on the circle whose length goes to 0 and a sequence of elements $g_n \in G$ so that $g_n(I_n)$ has length bounded away from zero.

9. The ergodic decomposition and disintegration

Let X be a compact metric space with Borel σ-algebra $\mathcal{B}$ with a σ-subalgebra $\mathcal{A}$ and a probability measure μ. Let $T : X \longrightarrow X$ be a measurable map that preserves μ. Suppose that $\mathcal{E}$ is the σ subalgebra of T-almost-invariant sets.

Lemma 47. *The map $X \longrightarrow \mathcal{M}(X)$ that sends x to $\mu_x^{\mathcal{A}}$ is $\mathcal{A}$-measurable.*

Proof. A basis of the σ-algebra is given by sets of the form $U_{f,r,\epsilon} := \{\mu : |\mu(f) - r| < \epsilon\}$ where $f \in C(X)$ is a fixed continuous function and r and ϵ are fixed real numbers. The preimage is

$$\{x \in X : |\mu_x(f) - r| < \epsilon\} = \{x \in X : E[f|\mathcal{A}](x) \in (r - \epsilon, r + \epsilon)\},$$

which is just $E[f|\mathcal{A}]^{-1}((r - \epsilon, r + \epsilon))$. □

Lemma 48. *There is a countably generated σ-algebra that agrees with $\mathcal{A}$ up to sets of measure zero.*

Proof. Since $C(X)$ is separable and dense in $L^1(X, \mu)$, $L^1(X, \mu)$ is also separable. Therefore, there is a countable collection (A_i) of sets in $\mathcal{A}$ whose indicator functions are dense in $\{\chi_A\}_{A \in \mathcal{A}}$. Let $\widetilde{\mathcal{A}} \subseteq \mathcal{A}$ be the σ-algebra they generate. Let $A \in \mathcal{A}$. There is a sequence $\chi_{A_{n_i}}$ that converges to χ_A in $L^1(X, \mathcal{A}, \mu)$. This sequence is Cauchy in $L^1(X, \widetilde{\mathcal{A}}, \mu)$ and hence has a limit f which is $\widetilde{\mathcal{A}}$-measurable and so $\|f - \chi_A\|_1 = 0$. So $f^{-1}(1)$ is in $\widetilde{\mathcal{A}}$ and agrees with A up to sets of measure zero. □

If $\mathcal{A}$ is countably generated by sets $(A_i)_{i=1}^{\infty}$, then the *atom (in $\mathcal{A}$)* of $x \in X$ is the smallest subset of $\mathcal{A}$ containing x. More precisely, for each i define B_i to be A_i (resp. $X \setminus A_i$) if x is in (resp. not in) A_i. The atom is $[x]_{\mathcal{A}} := \bigcap_{i=1}^{n} B_i$. To justify this, let $\mathcal{S}$ be the closure of the set $\{\chi_{A_i}\}_{i=1}^{\infty}$ under products, increasing limits, the operation of sending f to $1 - f$, and addition of functions f and g provided that $\{x : f \neq 0\}$ and $\{x : g \neq 0\}$ are disjoint. If x and y are two points in an atom then they are sent to the same value under the generating functions and hence under every function in $\mathcal{S}$ as desired. We have therefore also shown that $y \in [x]_{\mathcal{A}}$ implies that $x \in [y]_{\mathcal{A}}$.

Corollary 49. *If $\mathcal{A}$ is countably generated, $[x]_{\mathcal{A}}$ has full $\mu_x^{\mathcal{A}}$-measure almost surely. It follows that $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$ if $[x]_{\mathcal{A}} = [y]_{\mathcal{A}}$.*

Proof. If x is in the full measure set where $\mu_x^{\mathcal{A}}(A_i) = E[\chi_{A_i}|\mathcal{A}] = \chi_{A_i}(x)$, then μ_x assigns full measure to each B_i so the atom has full measure. □

Lemma 50. *Let $\mathcal{F}$ be a dense collection of functions in $C(X)$. Then (X, T, μ) is ergodic if and only if for each $f \in \mathcal{F}$, $\mathrm{Av}_n(f)$ converges to $\int_X f d\mu$ pointwise almost-everywhere for each $f \in \mathcal{F}$.*

Proof. The forward direction is Birkhoff. For the reverse direction, von Neumann's ergodic theorem implies that $\mathrm{Av}_n(f)$ converges in L^2 to the projection of f onto the subspace of T-invariant functions. The image of $\mathcal{F}$ is onto the subspace of constant functions, so the only T-invariant functions are constant and the system is ergodic. $\square$

Lemma 51. *Given a pmps $(X, T, \mathcal{B}, \mu)$ of a compact metric space, let $\mathcal{E}$ be the σ-algebra of almost-invariant sets. Then $\mu_x^{\mathcal{E}}$ is almost surely an ergodic invariant probability measure.*

Proof. Fix $f \in L^1$. Note that $E[f \circ T|\mathcal{E}] = E[f|\mathcal{E}]$ since $\int_A f \circ T d\mu = \int_A f d\mu$ for any T-invariant A. Therefore,

$$\int_X f dT_* \mu_x^{\mathcal{E}} = \int_X f \circ T d\mu_x^{\mathcal{E}} = E[f \circ T|\mathcal{E}](x) = E[f|\mathcal{E}](x) = \int_X f d\mu_X^{\mathcal{E}}$$

for almost every x. So there is a full measure set where this equality holds on a countable dense set of $C(X)$ and hence on all of $C(X)$ and hence the desired invariance follows from the Riesz representation theorem. For ergodicity, choose a countable dense collection of function (f_n) in $C(X)$. For almost every $x \in X$ and every m, $\mathrm{Av}_n(f_m)(x) \longrightarrow E[f_m|\mathcal{E}](x) = E[f_m|\mathcal{E}](x)$. The limit is unchanged if we replaced x with $y \in [x]_{\mathcal{E}}$. This implies that $\mu_x^{\mathcal{E}}$ is ergodic. $\square$

Theorem 52 (The ergodic decomposition, 1959)**.** *For any T-invariant $\mu \in \mathcal{M}^1(X)$, there is a measure ν supported on the T-invariant ergodic measures of $\mathcal{M}^1(X)$ so that for any $f \in L^1$, $\int_X f d\mu = \int_{\mathcal{M}^1(X)} \int_X f d\tau d\nu(\tau)$.*

This is usually just written as $\mu = \int_{\mathcal{M}^1(X)} d\nu$. As a historical note, this is a consequence of work of Choquet-Bishop-de-Leeuw in 1959.

Proof. Pushforward the measure μ to a measure ν on $\mathcal{M}(X)$ under the map sending x to μ_x. Then for any $f \in L^1(X, \mu)$

$$\int_X f d\mu = \int_X E[f|\mathcal{E}](x) d\mu = \int_X \int_X f d\mu_x^{\mathcal{E}} d\mu = \int_{\mathcal{M}^1(X)} \int_X f d\tau d\nu(\tau).$$

$\square$

Remark 53. A more intricate proof that follows a similar argument is available for G-invariant probability measures where G is any σ-compact metrizable group. Note that such a measure is *ergodic* if the only G-almost-invariant sets are null or conull.

Worksheet for Lecture 9

Problem 1. (The Rokhlin Disintegration Theorem, 1952). Let $T : X \longrightarrow Y$ be a continuous map between compact metric spaces. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the Borel σ-algebra on X (resp. Y). Let μ be a probability measure on X and let ν be its pushforward to Y. Show that for almost every $y \in Y$ there is a probability measure ν_y supported on $T^{-1}(y)$ with the property that for any map $f \in L^1(X, \mu)$,

$$\int_X f d\mu = \int_Y d\nu(y) \int_{T^{-1}(y)} f d\nu_y$$

(Hint: Let $\mathcal{C}$ be the pullback of $\mathcal{B}$. Show that $\mu_x^{\mathcal{C}} = \mu_y^{\mathcal{C}}$ almost surely if and only if $T(x) = T(y)$.).

Problem 2. (An example with disintegration) Let $T : [0,1]^2 \longrightarrow [0,1]$ be the projection onto the first factor. Let μ be a measure on $[0,1]^2$ whose pushforward is ν. Find ν_x when μ is Lebesgue measure and when $\mu = f(x,y)dxdy$ for some bounded measurable function f.

Problem 3. In the notation of the last lecture, argue that $\mu_x^{\mathcal{E}} = \mu_{Tx}^{\mathcal{E}}$ almost surely. (Hint: Show that $E[f|\mathcal{E}] \circ T = E[f|\mathcal{E}]$ for any function f in L^1.)

Problem 4 (Lifts) Let B and X be compact metric spaces and suppose that we have the following commutative diagram,

$$\begin{array}{ccc} B \times X & \xrightarrow{S} & B \times X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{T} & B \end{array}$$

Suppose that S and T are measurable, invertible, and that μ is a T-invariant probability measure. Suppose that λ is a measure on $B \times X$ whose pushforward to B is μ. Show that λ is S-invariant if and only if the disintegrations $(\mu_b)_{b \in B}$ have the property that $S_* \mu_b = \mu_{S(b)}$ almost surely. (Hint: For the forward direction, in the notation of Problem 1, imitate the proof that $\mu_x^{\mathcal{E}}$ is T-invariant that we saw in class)

10. The Birkhoff Ergodic Theorem for flows and flows on translation surfaces

A *flow* is a measurable action of $\mathbb{R}$ on a measurable space $(X, \mathcal{B}, \mu)$, i.e. for each $t \in \mathbb{R}$ there is a map $g_t : X \longrightarrow X$ so that $g_t \circ g_s = g_{t+s}$. The flow is *measure-preserving* if $(g_t)_*\mu = \mu$ for all t.

Example (Straight-line flow on translation surfaces). A *translation surface*, i.e. a closed orientable surface that is composed of finitely many polygons in $\mathbb{C}$ with sides identified by translation (by an element of $\mathbb{C}$). The prototypical example is the square torus. Since side identifications are by translation, translation surfaces have a well-defined measure, μ, which comes from the area-measure on $\mathbb{C}$. A point p on a translation surface is said to have *order* k, if the circle of points of distance exactly ϵ from it has circumference $2\pi\epsilon(k+1)$ for sufficiently small ϵ. There are finitely many such points called *cone points*. Since edge identifications are by translation, it makes sense to talk about lines, their lengths, and their slopes. Given an angle θ and a translation surface X, the *straight line flow* (in direction θ on X) is the flow where g_t sends a point x, t units further along the line of slope θ through x. This transformation is manifestly area-preserving. One wrinkle is that it is undefined on the finite set of lines that have endpoints at cone points (at any cone point of order k, there are $k+1$ choices of lines of slope θ that a point may continue along).

Let $\mathcal{E}$ be the σ-algebra of measurable sets that are g_t-almost-invariant for all t. For any $f \in L^1(X, \mu)$, set $\frac{1}{T}(\mathrm{Av}_T(f))(x) = \int_0^T f(g_t x)dt$.

Theorem 54 (The Birkhoff ergodic theorem for flows)**.** *For any $f \in L^1(X, \mu)$, $\mathrm{Av}_T(f)$ converges pointwise a.e. and in L^1 to $E[f|\mathcal{E}]$ as $T \longrightarrow \infty$.*

We will break the proof into a few lemmas.

Lemma 55. *For positive integers n, $\mathrm{Av}_n(f)$ converges pointwise almost everywhere and in L^1.*

Proof. Let $F(x) := \int_0^1 f(g_t x)dt$. This function is in L^1 since, by Fubini,

$$\int_X |F| d\mu \leq \int_X \int_0^1 |f(g_t x)| dt d\mu = \int_0^1 \int_X |f(g_t x)| d\mu dt = \int_X |f(x)| d\mu < \infty$$

We notice that $g_1^n = g_n$ and that

$$F(g_s x) = \int_0^1 f(g_t g_s x)dt = \int_0^1 f(g_{t+s} x)dt = \int_s^{s+1} f(g_t x)dt.$$

By the Birkhoff ergodic theorem,

$$\mathrm{Av}_n(f) := \frac{1}{n}\int_0^n f(g_t x)dt = \frac{F(x) + F(g_1 x) + \ldots + F(g_{n-1}x)}{n}$$

converges pointwise almost everywhere and in L^1. $\square$

Lemma 56. *If f^* is the limit of $\mathrm{Av}_n(f)$, then for any flow invariant A, $\int_A f^* d\mu = \int_A f d\mu$.*

Proof. By the previous lemma, $\mathrm{Av}_n(\chi_A f) = \chi_A \mathrm{Av}_n(f)$ converges to $\chi_A f^*$ in L^1. Therefore,

$$\int_A f^* d\mu = \int_X \chi_A f^* d\mu = \lim_{n\longrightarrow\infty} \int_X \chi_A \frac{1}{n}\int_0^n f(g_t x)dtd\mu$$

where the final equality is by L^1-convergence (which allows us to apply the conclusion of dominated convergence). Now applying Fubini,

$$\int_A f^* d\mu = \lim_{n\longrightarrow\infty} \frac{1}{n}\int_0^n \int_X \chi_A(g_t x) f(g_t x) d\mu dt = \int_A f(x)d\mu$$

where the final equality is by g_t-invariance of μ. $\square$

Lemma 57. *Fix a positive real number τ, then $\frac{1}{t}\int_t^{t+\tau} |f(g_s x)|ds$ converges pointwise almost everywhere and in L^1 to 0.*

Proof. The previous lemma implies that $\frac{1}{n}\int_0^n |f(g_t x)|dt$ converges pointwise almost everywhere and in L^1. This tells us that

$$\frac{1}{n}\int_n^{n+1} |f(g_t x)|dt = \frac{1}{n}\int_0^n |f(g_t x)|dt - \frac{1}{n}\int_0^{n+1} |f(g_t x)|dt$$

converges to 0 pointwise almost everywhere and in L^1. The result follows from the estimate

$$\frac{1}{t}\int_t^{t+\tau} |f(g_s x)|ds \le \frac{1}{t}\int_{\lfloor t\rfloor}^{\lfloor t\rfloor+1} |f(g_s dx)|ds + \ldots + \frac{1}{t}\int_{\lceil t+\tau\rceil-1}^{\lceil t+\tau\rceil} |f(g_s dx)|ds$$

$\square$

Corollary 58. *The limit of $\mathrm{Av}_n(f)$ is $E[f|\mathcal{E}]$*

Proof. By Lemma 56, it suffices to show that f^* is flow invariant. Fix a real number τ.

$$|f^*(g_\tau x) - f^*(x)| = \lim_{t\longrightarrow\infty} |\mathrm{Av}_t(g_\tau x) - \mathrm{Av}_t(x)|$$

This is bounded above by

$$\lim_{t\longrightarrow\infty} \frac{1}{t}\int_0^t |f(g_s g_\tau x)| - |f(g_s x)|ds = \lim_{t\longrightarrow\infty} \frac{1}{t}\left(\int_0^\tau |f(g_s x)|dx + \int_t^{t+\tau} |f(g_s x)|ds\right)$$

The first integral is bounded for almost every x by Fubini, so the first term tends to zero. The second term tends to zero by the previous lemma. $\square$

Proof of the ergodic theorem for flows: Notice that $\frac{\lceil s \rceil}{s}\mathrm{Av}_{\lceil s \rceil}(f)$ converges pointwise a.e. and in L^1 to $E[f|\mathcal{E}]$ as $s \longrightarrow \infty$. Since

$$|\mathrm{Av}_s(f) - \frac{\lceil s \rceil}{s}\mathrm{Av}_{\lceil s \rceil}(f)| \leq \frac{1}{s}\int_s^{s+1} |f(g_t x)| dt$$

converges to 0 pointwise almost everywhere and in L^1, we are done. $\square$

Worksheet for Lecture 10

Problem 1 (Straight line flow on the flat torus). Show that on the square-torus, straight line flow is either periodic or uniquely ergodic. (Hint: Have we already shown this claim for the time one flow?)

Problem 2 (Geodesic flow on the flat torus). Let X be the square torus, i.e. a unit square in $\mathbb{R}^2$ with opposite sides identified. The *unit tangent bundle* T^1X is the set of pairs (x, v) where $x \in X$ and $v \in T_xX$ has unit length (in the Euclidean metric on $\mathbb{R}^2$). Geodesic flow on T^1X is the family of maps $g_t : T^1X \longrightarrow T^1X$ that sends a point (x, v) to $(x + tv, v)$. Show that T^1X is homeomorphic to $X \times S^1$. Prove that Lebesgue measure on $X \times S^1$ (i.e. the product measure of area on the torus and length on the circle) is flow invariant. Is it ergodic? Is it minimal?

Problem 3 (Geodesic flow on the round sphere). Let $S^2 \subseteq \mathbb{R}^3$ be the sphere of radius one. Let $T^1S^2 \subseteq \mathbb{R}^3 \times \mathbb{R}^3$ be its unit tangent bundle. Geodesic flow $g_t : T^1S^2X \longrightarrow T^1S^2$ sends a point (x, v) to the combination point/tangent vector to the great circle through x in the direction of v that is t units further along the great circle. As before, Lebesgue measure on T^1S^2 is flow invariant. Find all ergodic invariant measures for geodesic flow.

Problem 4 (Dynamics and topology for T^1S^2). Note that $\mathrm{SO}(3)$ acts simply transitively on T^1S^2 and conclude that both spaces are homeomorphic. (Note that the aforementioned Lebesgue measure on T^1S^2 can be identified with Haar measure on $\mathrm{SO}(3)$.) Under this identification explain why $g_t : T^1S^2 \longrightarrow T^1S^2$ can be identified with the map $g_t : \mathrm{SO}(3) \longrightarrow \mathrm{SO}(3)$ that sends g to gr_t where $r_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Recall now that $\mathbb{P}(\mathbb{R}^4)$ is the space of lines in $\mathbb{R}^4$ and remember that it can be written as a closed disk in $\mathbb{R}^3$ of radius π with boundary identified by the relation $v \sim -v$. For each element of $\mathrm{SO}(3)$ there is a unit vector v that it fixes and so that the linear map is given by rotation of angle θ in $[-\pi, \pi]$ about v. Note that the map sending such a matrix to $v\theta \in \mathbb{P}(\mathbb{R}^4)$ is a continuous bijection and hence a homeomorphism. Deduce that T^1S^2 is not homeomorphic to $S^1 \times S^2$.

Homework 5: Due after Lecture 12

Problem 1 (Disintegration). Prove the Rokhlin disintegration theorem. More specifically, do problems 1 and 2 on Worksheet 9.

Problem 2 (Orbit closures and topological groups). Suppose that G is a compact Hausdorff topological group, which, for our purposes means that it has a Haar probability measure μ, i.e. a probability measure μ so that $\mu(gA) = \mu(A)$ for all Borel measurable $A \subseteq G$ and all $g \in G$. The goal of this problem is to show that $\{g^n\}_{n>0}$ and $\{g^n\}_{n<0}$ have the same closures for every $g \in G$.

(1) Show that if X is a compact space and $T : X \longrightarrow X$ is a homeomorphism so that μ is a T-invariant measure, then $\{T^n x\}_{n\geq 0}$ and $\{T^n x\}_{n\leq 0}$ have identical closures for almost every $x \in X$.
(2) Now prove the claim that for every $g \in G$, $\{g^n\}_{n\geq 0}$ and $\{g^n\}_{n\leq 0}$ have the same closures

Problem 3 (Geodesic flow on constant non-negatively curved surfaces). Do problems 2, 3, and 4 on Worksheet 10 about geodesic flow on flat tori and round spheres.

Problem 4 (Fibered Systems 1). Do problem 4 on Worksheet 9.

Problem 5 (Suspension flows on mapping tori). Suppose that $T : X \longrightarrow X$ is any homeomorphism of a compact topological space. The *suspension flow* associated to this system is the following. Consider the product space $X \times \mathbb{R}$ on which $\mathbb{Z}$ acts by $n \cdot (x, r) = (T^n x, r + n)$. The *mapping torus* E is the quotient space $(X \times \mathbb{R})/\mathbb{Z}$. The mapping torus admits a map $E \longrightarrow \mathbb{R}/\mathbb{Z}$ by projecting onto the second factor. The fiber of this map is X. Consider the flow $g_t : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ where $g_t(x, r) = (x, r + t)$. Show that this descends to a flow on E. Show that there is a bijection between the flow invariant Borel probability measures on E and the T-invariant Borel probability measures on X.

Problem 6 (Fibered Systems 2). Let B and X be compact metric spaces and suppose that we have the following commutative diagram,

$$\begin{tikzcd} B \times X \arrow[r, "S"] \arrow[d, "\pi"] & B \times X \arrow[d, "\pi"] \\ B \arrow[r, "T"] & B \end{tikzcd}$$

Suppose that S and T are measurable (but not necessarily invertible) and that μ is a T-invariant probability measure. Let $\mathcal{B}$ and $\mathcal{X}$ be the Borel σ-algebras on B and X. Write $S(b, x) = (Tb, \rho_b x)$ and suppose that ρ_b is a homeomorphism for all b. We have seen that if ν_b is a family of measures on X so that $(\rho_b)_*\nu_b = \nu_{T(b)}$ for all $b \in B$, then the measure defined by sending $f \in C(B \times X)$ to

$$\lambda(f) := \int_B \int_X f(b,x)d\nu_b(x)d\mu(b)$$

is S-invariant. Let $\mu_b^{T^{-1}\mathcal{B}}$ and $\lambda_{(b,x)}^{S^{-1}(\mathcal{B}\otimes\mathcal{X})}$ be conditional measures coming from conditioning μ and λ on the σ-subalgebra in the superscript. Define a measure $\tau_{(b,x)}$ by declaring that it sends $f \in C(B \times X)$ to

$$\tau_{(b,x)}(f) := \int_B f(b', \rho_{b'}^{-1}\rho_b x)d\mu_b^{T^{-1}\mathcal{B}}(b').$$

Show that for λ-a.e. (b,x), $\tau_{(b,x)} = \lambda_{(b,x)}^{S^{-1}(\mathcal{B}\otimes\mathcal{X})}$. (Hint: It suffices to show that $\tau_{(b,x)}$ has the characteristic property of conditional measures, i.e. that for any $f \in L^1(B \times X, \lambda)$

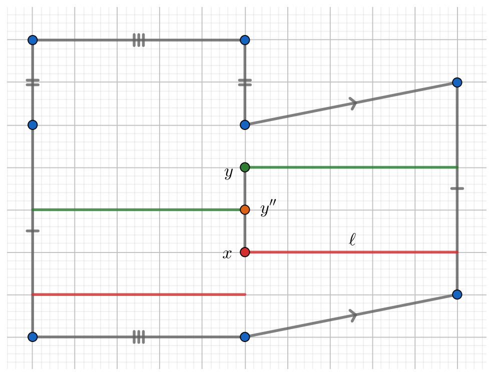
$$\lambda(f) = \int_{B\times X}\int_{B\times X} f(b',x')d\tau_{(b,x)}(b',x')d\lambda(b,x)$$

This problem can be solved by repeatedly using Fubini, the characteristic property of conditional measures, the identity $(\rho_b)_*\nu_b = \nu_{T(b)}$, and the fact that the conditional measures $\mu_b^{T^{-1}\mathcal{B}}$ give full measure to the smallest set in $T^{-1}\mathcal{B}$ containing b.)

11. Minimal flows on translation surfaces

Let X be a translation surface and let g_t be straight line flow in the (positive) horizontal direction. A *(horizontal) leaf* is a maximal horizontal line segment in the surface. The leaf may be finite-length, infinite in both directions, or infinite in just one.

Lemma 59. *Suppose that $I = [x, y]$ is a transverse line segment to a leaf ℓ through a point x that has infinite length in the positive horizontal direction . There is some $t > 0$ so that $g_t(x) \in I$.*



Proof. There are finitely many points in I that flow forward into a cone point without returning to I first. Choose $y' \in I$ so that $I' := [x, y']$ has positive length and no point in $[x, y']$ hits a cone point before returning to I. If I' does not return to I after flowing by time t, then there is an embedded cylinder of length $t|I'|$ in the surface. Since the surface has finite area, there is a time τ so that $g_\tau(I') \cap I$ has positive length. If $g_\tau(x)$ is not in the interior of I, then we are in the situation shown in the figure, i.e. there is some $y'' \in I'$ so that $g_\tau(y'') = x$. Let $I'' := [x, y'']$. As before, there is some time $\tau' > \tau$ so that $g_{\tau'}(I'') \cap I$ has positive length. Either $g_{\tau'}(x)$ or $g_{\tau'}(y'') = g_{\tau'-\tau} \circ g_\tau(x)$ belongs to the interior of I''. □

A *saddle connection* on a translation surface X is a straight line segment on X whose endpoints are cone points and which contains no cone points in its interior.

Corollary 60 (Maier, 1943)**.** *On a translation surface, straight line flow in any direction that does not contain a saddle connection is minimal.*

Proof. Consider any leaf in a saddle connection free direction and suppose to a contradiction that its limit set C is not all of X. Therefore, choose a point x in the boundary of C (which is nonempty since $C \neq X$). Let $I = [x, y]$ be a line segment perpendicular to the flow direction and so that I is not a subset of C (such a segment exists since x is not an interior point of C). Let $I' := [x', y']$ be a maximum sub-interval of I whose interior does not contain points in C. By maximality, $x' \in C$. Since the flow-direction is saddle connection free, the leaf through x' is infinite (in at least one direction) so the previous lemma implies that $g_t(x') \in \text{int}(I')$ for some $t \in \mathbb{R}$, a contradiction (note that C is flow-invariant). □

Lemma 61. *No ergodic probability measure is a nontrivial linear combination of two distinct invariant probability measures.*

Proof. Suppose not and write $\mu = a\mu_1 + b\mu_2$ where μ is ergodic and a and b are positive numbers so that $a + b = 1$. By the Radon-Nikodym theorem, $\mu_1 = f\mu$ where $f \in L^1(X, \mu)$. Invariance implies that f is invariant and hence constant μ-a.e. Since μ_1 is a probability measure, $f \equiv 1$ μ-a.e. So $\mu = \mu_1$ and so $b = 0$, a contradiction. □

Worksheet for Lecture 11.

Problem 1. (Countability of Saddle Connections) Show that given a translation surface the number of saddle connections is countable. Deduce that the directions for which straight-line flow is minimal is cocountable. (Hint: Each polygon in $\mathbb{C}$ is equipped with a 1-form dz, which, since polygons are glued by translation to form a translation surface, X, can be assembled to given a closed differential 1-form ω on X. Note that if γ is a path connecting points $a+ib$ to $c+id$ in a polygon in $\mathbb{C}$ used to form X, then $\int_\gamma \omega = (c+id)-(a+ib)$. Use this to conclude that the possible slopes of saddle connections is countable.).

Problem 2. (IETs) An *n-interval exchange transformation (n-IET)* is a map from $[0,1]$ to $[0,1]$ where we divide $[0,1]$ into intervals $[0,a_1],[a_1,a_2],\ldots,[a_{n-1},1]$ and then we rearrange them in a length-preserving way. (This map is undefined at endpoints of these intervals). For instance a non-identity 2-IET sends $[0,a_1]$ isometrically to $[1-a_1,1]$ and $[a_1,1]$ to $[0,1-a_1]$. Suppose that X is a translation surface with no horizontal saddle connection. Let I be a vertical line segment on X whose endpoints flow into cone points under forward horizontal flow. Show that the map from I to itself that sends a point to its first return to I is an IET. (We could think about X as the suspension of this IET.)

Problem 3. (IET conjugates) Let $T:[0,1]\longrightarrow[0,1]$ be a map that is well-defined and injective on the complement of finitely many points. Suppose that μ is a measure that is mutually absolutely continuous with respect to Lebesgue measure and T-invariant. Show that this dynamical system is conjugate to an IET and that the conjugacy sends μ to Lebesgue measure. (Hint: Use the function $g(x)=\mu([0,x])$ to establish the conjugacy.)

Problem 4. (Continuous flows on surfaces) Let X be a surface. Let $g_t : X \longrightarrow X$ be a flow on the surface with finitely many fixed points and finitely many flow lines that lead to or away from fixed points. Suppose that the flow preserves a measure μ that assigns positive measure to each nonempty open set. Let τ be a simple closed curve that is nowhere tangent to the vector field that defines the flow. Show that the first return map to τ is topologically conjugate to an IET.

Note: This problem can be used to show that, for minimal area-preserving flows on surfaces, there is a straight-line flow on a translation surface with the same leaves, although the flow along the leaves may happen at a different speed.

12. Haar measure

Let G be a locally compact Hausdorff topological group. The *(left) Haar measure* is a unique (up-to-scaling) measure μ_G on G so that if $S \subseteq G$ then $\mu(g \cdot S) = \mu(S)$ for all $g \in G$. We, moreover, demand that compact sets have finite measure and that open sets have positive measure. Before showing existence and uniqueness we give a few examples.

Examples.

(1) The usual volume on $\mathbb{R}^n$.
(2) The measure $\frac{dx}{x}$ on $\mathbb{R}^\times$, i.e. the measure μ where $\mu(a, b) = \log(b) - \log(a) = \log\left(\frac{b}{a}\right)$, which is the same as the measure of (ca, cb).
(3) Arclength is the Haar measure on S^1.
(4) The usual volume is the Haar measure on the torus. More generally, the Haar measure on $G \times H$ is $\mu_G \times \mu_H$.

Theorem 62. *There is a Haar measure μ_G.*

Proof Idea: Fix a compact set K_0 with nonempty interior. For any compact set K and any set V, let $[K : V]$ be the minimum number of (left) translates of V needed to cover K. If V is open, then this number is finite. Let V_n be a nested sequence of open neighborhoods of the identity where $\bigcap_n V_n = \{e\}$. For compact K define,

$$\mu_n(K) := \frac{[K : V_n]}{[K_0 : V_n]}.$$

Note that this function is (left) G-invariant.

Lemma 63. *For any compact K, $\frac{1}{[K_0 : K]} \le \mu_n(K) \le [K : K_0]$.*

Proof. Let $K_0 = \bigcup_{i=1}^m g_i V_n$ be a covering of K_0 by the smallest number of (left) translates of V_n. We have that

$$[K : V_n] \le [K : K_0][K_0 : V_n]$$

(note that $[K : K_0] < \infty$ since K_0 has nonempty interior). This establishes the upper bound. The lower bound follows by symmetry. $\square$

Let $\mathcal{K}$ be the collection of all compact subsets of G. Then μ_n is a function from $\mathcal{K}$ to $[0, \infty)$. More specifically, it belongs to $\prod_{K \in \mathcal{K}} \left[\frac{1}{[K_0:K]}, [K : K_0]\right]$, which is compact by Tychonoff. Let μ be any accumulation point of (μ_n) in this space, which, as an accumulation point of G-invariant function is G-invariant.

Lemma 64. *If K_1 and K_2 are disjoint compact sets, then $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$.*

Proof. Fix $\epsilon > 0$. We will show that the difference between the left and right hand sides is at most 3ϵ. In the space of functions from $\mathcal{K}$ to $\mathbb{R}$ the set of functions $f : \mathcal{K} \longrightarrow \mathbb{R}$ so that $|f(K_1) - \mu(K_1)| < \epsilon$, $|f(K_2) - \mu(K_2)| < \epsilon$, and $|f(K_1 \cup K_2) - \mu(K_1 \cup K_2)| < \epsilon$ is open. Therefore, there is an infinite sequence of μ_n in this open set. For sufficiently large n, we have that if $gV_n \cap K_1 \neq \emptyset$, then $gV_n \cap K_2 = \emptyset$. Therefore, for sufficiently large n, $\mu_n(K_1 \cup K_2) = \mu_n(K_1) + \mu_n(K_2)$ so when μ_n is in U for such n we have

$$|\mu(K_1 \cup K_2) - \mu(K_1) - \mu(K_2)| \leq 3\epsilon + |\mu_n(K_1 \cup K_2) - \mu_n(K_1) - \mu_n(K_2)| = 3\epsilon$$

□

Now proceed as in the proof of the Riesz representation theorem for noncompact spaces to argue that for an open set U if we define $m_G(U)$ to be the sup of $\mu(K)$ ranging over all compact $K \subseteq U$ and $m_G^*(B)$ to be the inf of $m_G(U)$ for any open U containing B, then m_G^* determines an outer measure that restricts to the desired measure on G. □

Proposition 65. *Haar measure is unique up to scaling.*

Proof. Suppose that m_1 and m_2 are two Haar measures. Let $m := m_1 + m_2$. By the Radon Nikodym theorem, there are function $f_i \in L^1(X, m)$ so that $dm_i = f_i dm$. But then by invariance, for every $g \in G$, $f_i(gx) = f_i(x)$ for almost every $x \in G$. This implies that f_i is constant almost everywhere as desired. (Note: Since the equality only holds almost everywhere, there is actually a subtle issue that is handled in Einsiedler-Ward to deduce that f_i is constant). □

Now we will end with one more example.

Lemma 66. *The Haar measure on $G = \mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is $\frac{dm(A)}{|\det(A)|^n}$ where dm is Lebesgue measure on $\mathbb{R}^{n^2}$.*

Proof. We will stick to $n = 2$. Think about an element $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}(2, \mathbb{R})$ as specified by two column vectors $v_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$, i.e. $B = (v_1, v_2)$. We will write $\mathrm{GL}(2, \mathbb{R}) \subseteq \mathbb{R}^2 \times \mathbb{R}^2$. Notice that $AB = (Av_1, Av_2)$. Therefore, we have that if $S = S_1 \times S_2 \subseteq \mathrm{GL}(2, \mathbb{R}) \subseteq \mathbb{R}^2 \times \mathbb{R}^2$, then $m(A \cdot S_1 \times S_2) = m(A \cdot S_1) m(A \cdot S_2) = |\det A|^2 m(S_1 \times S_2)$.

Let $f(B)$ be a compactly supported function on $\mathrm{GL}(2, \mathbb{R})$, then

$$\int_G f(A \cdot B) \frac{dm(B)}{|\det B|^2} = \int_G f(A \cdot B) \frac{|\det A|^2 dm(B)}{|\det AB|^2} = \int_G f(C) \frac{dm(C)}{|\det C|^2}$$

□

Exercise. *Let N denote the subgroup of upper triangular $n \times n$ matrices with 1s on the diagonal. Adapt the proof to show that Haar measure on N is just Lebesgue measure.*

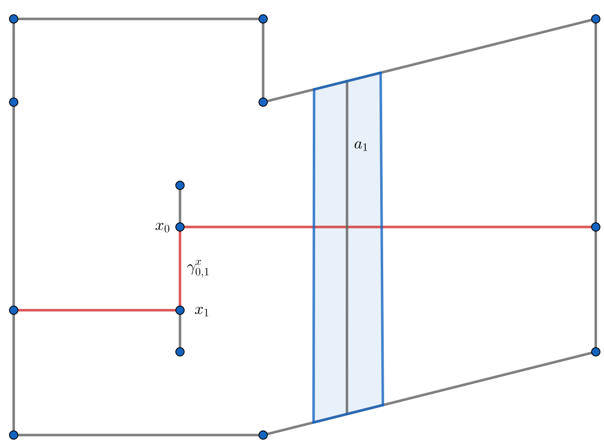
Since left and right Haar measure are unique up to scaling, it follows that if μ is right-Haar measure, then there is a constant $\Delta(g)$ so that the measure $\mu_g(S) = \mu(g^{-1} \cdot S) = \Delta(g)\mu(S)$. The map $\Delta : G \longrightarrow \mathbb{R}$ is a group homomorphism. G is called *unimodular* if Δ is the trivial homomorphism. This occurs if G is abelian, compact, or perfect (i.e. has trivial abelianization).

13. The asymptotic cycle for flows on translation surfaces and Masur's criterion

The following *asymptotic cycle* argument is due to Schwartzman (1957).

Theorem 67. *In a saddle-connection-free direction on a genus g translation surface, the number of flow invariant ergodic probability measures is at most $2g$.*

Suppose without loss of generality that the flow direction is horizontal. Fix a (finite-length) vertical line segment I. Fix a point $x = x_0 \in I$ and let x_1 be its *first (point of) return* in I under (forward) horizontal flow. Let x_2 be its second return, etc. By minimality, $(x_i)_{i=1}^{\infty}$ is dense in I and so $I^x_{ij} := (x_i, x_j)$ forms a basis of the topology of I. For $i < j$, let γ^x_{ij} be the closed curve formed by following a horizontal leaf from x_i to x_j and then closing the line segment with the interval $[x_j, x_i]$. Let μ be an ergodic flow invariant measure.



Lemma 68. *For μ-a.e. point $y \in I$, $s_n := \frac{\gamma^y_{0n}}{|\gamma^y_{0n}|}$ converges in $H^1(X, \mathbb{R})$. Here $|\gamma^y_{0n}|$ is the length of γ^y_{0n}. The limit is called the* asymptotic cycle.

Proof. Fix a basis of integral homology $(a_1, \dots, a_{2g})$ where each a_i is a concatenation of paths that travel upward vertically and paths that travel horizontally. By Poincare duality, s_n converges if and only if $(s_n \cdot a_i)_{n \geq 0}$ is a convergent sequence for all $i \in \{1, \dots, 2g\}$. For each a_i thicken each vertical segment that comprises it into a union of open sets U_i covered by embedded horizontal line segments of length ϵ. For μ-a.e. point in I, the percent of time that s_n spends in U_i tends to

$\mu(U_i)$. (This requires that the boundary has μ-measure zero, which it does since the flow direction is minimal and hence any leaf has infinite length and hence μ-measure zero). This time is exactly $\epsilon(s_n \cdot a_i)$, so we have the desired convergence. $\square$

Lemma 69. *μ is completely determined by its asymptotic cycle.*

Proof. Thicken I to an open set $(-\epsilon, \epsilon) \times I$ by adding in horizontal line segments of length ϵ. Disintegrating the measure we see that $\mu = m \times \nu$ where ν is a measure on I and m is Lebesgue measure on horizontal line segments. We note that ν completely determines μ and that ν is completely determined by $\nu(I^x_{ij})$. Choose a generic point $y \in I$, i.e. one for which the asymptotic percent of time that horizontal flow through y spends in $(-\epsilon, \epsilon) \times I^x_{ij}$ is $\epsilon\nu(I^x_{ij})$. As in the previous proof, this fraction of time (for segments from y_1 to y_n at least) is

$$s_n \cdot I^x_{ij} = s_n \cdot \gamma^x_{ij} - \frac{\gamma^x_{ij} \cdot [y_0, y_n]}{|\gamma^y_{0n}|}$$

Since the term being subtracted on the right has bounded numerator and denominator tending to infinity we see that $s \cdot \gamma^x_{ij} = \nu(I^x_{ij})$. $\square$

Proof of Theorem 67: The map from ergodic measures to $H^1(X, \mathbb{R})$ is injective by Lemma 69. The image of the ergodic measures is a linearly independent set by Lemma 61, so it has at most $2g$ many elements. $\square$

Given a translation surface X we can form a new translation surface $g_t \cdot X$ by applying the matrix $\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$ to the polygons comprising X. We say that a sequence of translation surfaces X_n converges to a translation surface X if there is a triangulation τ of X and a triangulation τ_n of X_n for all sufficiently large n so that τ_n converges to τ. Finally, we say that $\{g_t \cdot X\}_{t \geq 0}$ *recurs* if there is a sequence of times t_n, tending to infinity as n does, so that $g_{t_n} \cdot X$ converges.

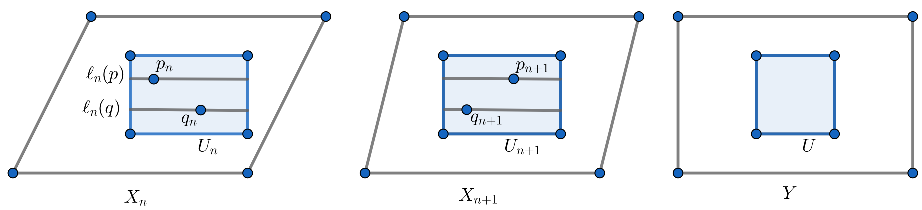
Theorem 70 (Masur's criterion, 1992)**.** *If $\{g_t \cdot X\}_{t \geq 0}$ recurs, then the horizontal flow on X is uniquely ergodic.*

Proof. Suppose that $X_n := g_{t_n} \cdot X$ converges to Y. Let $\mu_1, \ldots, \mu_m$ be the ergodic (horizontal) flow invariant measures on X. Let E_i be the generic points for μ_i. Let C_i be the points on Y that are accumulation points of $g_{t_n} \cdot p$ for $p \in E_i$. It suffices to show that, for each rectangle U on Y, that

1. U contains some element of C_i for some i (see the worksheet), and
2. U cannot contain points in C_i and C_j for $i \neq j$.

Connectedness will then imply that there is only one ergodic flow-invariant measure.

Let $U_n \subseteq X_n$ be rectangles converging to U. Let p be a generic point for μ_1 so that $g_{t_n} \cdot p$ belongs to U_n for all n. Suppose to a contradiction that this is also true for a generic point q of μ_2. Write $p_n := g_{t_n} p$ and $q_n = g_{t_n} q$. Let $\ell_n(p)$ and $\ell_n(q)$ be the horizontal line segments through p_n and q_n in U_n.



Since μ_1 and μ_2 are distinct ergodic measures there is a vertical line segment I on x so that $(-\epsilon, \epsilon) \times I$ has distinct μ_1 and μ_2 measure. These measures disintegrate to $m \times \nu_i$ on I where m is Lebesgue measure and by assumption $\nu_1(I) \neq \nu_2(I)$. Since p and q are generic,

$$\frac{\ell_n(p) \cdot g_{t_n}(I)}{|g_{-t_n}(\ell_n(p))|} = \frac{g_{-t_n}(\ell_n(p)) \cdot I}{|g_{-t_n}(\ell_n(p))|} \longrightarrow \nu_1(I)$$

and similarly for $\ell_n(q)$. However, we notice that $\ell_n(p)$ and $\ell_n(q)$ intersect any vertical line segment the same number of times (up to adjusting by ± 1, or ± 2). So $\nu_1(I) = \nu_2(I)$, which is a contradiction. Now cover each X_n be a finite collection of these converging boxes to conclude. $\square$

Worksheet for Lecture 13

Problem 1. Show that if μ and ν are two distinct ergodic invariant measures for horizontal flow on a genus g translation surface, then their asymptotic cycles have zero intersection number with one another. Conclude that the number of flow invariant ergodic probability measures is at most g.

Problem 2. (Warning: don't try this problem unless you've thought about differential forms on manifolds before!) Let X be a C^1 vector field on a manifold M equipped with a measure μ. Consider the linear map ρ_X from differential 1-forms to $\mathbb{R}$ defined by $\rho_X(\omega) := \int_M \omega(X) d\mu$. Show that closed forms are sent to zero and hence that ρ_X defines an element of $H_1(M, \mathbb{R}) = \mathrm{Hom}(H^1(M, \mathbb{R}), \mathbb{R})$. Show that when M is a translation surface, X is the horizontal vector field, and μ is an ergodic invariant measure, this is exactly the asymptotic cycle of μ.

Problem 3. Suppose that you fix 21 (possibly overlapping, not necessarily independent) subsets of the population of Madison. For instance, one subset may include all college graduates; another may include all people born in Wisconsin. Each subset includes at least 10% of the population. Show that there is a person in Madison that belongs to at least three subsets.

Problem 4. Use the argument in the previous problem to show that if (X, μ) is a probability measure space and (X_n) is an infinite sequence of subsets so that, for some $\epsilon > 0$, $\mu(X_n) > \epsilon$, then there is a positive measure subset of X contained in X_n for infinitely many n. Use this to prove the claim that cites the worksheet in the proof of Masur's criterion.

Problem 5. Let X be a genus g translation surface with one cone point. Show that the horizontal flow on X can be divided into p cylinders and m *minimal components*, i.e. subsurfaces with boundaries a union of saddle connections so that the horizontal flow in the interior of the subsurface is minimal. Show that $p + m \leq g$.

14. Hyperbolic space and the hyperbolic metric

Define *complex projective space* $\mathbb{P}^1(\mathbb{C})$, or simply $\mathbb{P}^1$, to be the space of complex lines in $\mathbb{C}^2$. Its points can be written as $[a : b]$ where $a, b \in \mathbb{C}$ are not both zero and so $[a : b] = [ca : cb]$ for any $c \in \mathbb{C}^\times$. If $b \neq 0$, then $[a : b] = [\frac{a}{b} : 1]$, which shows that $\mathbb{P}^1 = \mathbb{C} \cup \{[1 : 0]\}$. Since $\mathbb{P}^1$ is compact, it is the one-point compactification of $\mathbb{C}$, i.e. a sphere. Finally, $\mathbb{P}^1$ contains an embedded copy of real projective space, given by sending $[a : b]$, for a and b real, to the corresponding point in $\mathbb{P}^1$. Since $\mathrm{SL}(2, \mathbb{C})$ acts on lines in $\mathbb{C}^2$, it acts on $\mathbb{P}^1$ with $\mathrm{SL}(2, \mathbb{R})$ preserving $\mathbb{P}^1(\mathbb{R})$. Since the elements of $\mathrm{SL}(2, \mathbb{R})$ act continuously, they preserve the two disks that are the complement of $\mathbb{P}^1(\mathbb{R})$ in $\mathbb{P}^1(\mathbb{C})$. These disks are $\{[a : b] \in \mathbb{P}^1 : \mathrm{Im}(\frac{a}{b} > 0\}$ and $\{[a : b] \in \mathbb{P}^1 : \mathrm{Im}(\frac{a}{b} < 0\}$.

The *upper half plane* is $\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$ can be identified with one of these disks and is preserved by $G := \mathrm{SL}(2, \mathbb{R})$. The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on a point $z \in \mathbb{H}$ is by sending it to $\frac{az+b}{cz+d}$. The group $\mathrm{SL}(2, \mathbb{R})$ has some special subgroups:

(1) Let $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, so $g_t(z) = e^t z$. Set $A = \{g_t\}_{t \in \mathbb{R}}$.
(2) Let $n_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, so $n_u(z) = z + u$. Set $N = \{n_u\}_{u \in \mathbb{R}}$.
(3) Let $P := AN$, which consists of real affine transformations of $\mathbb{H}$, i.e. those of the form $az + b$ where $a, b \in \mathbb{R}$ and $a > 0$.
(4) Let $r_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and set $K = \{r_\theta\}_{\theta \in [0, 2\pi)}$.

Lemma 71. *The map from $\mathrm{SL}(2, \mathbb{R})$ to $\mathbb{H}$ that sends g to $g \cdot i$ induces a bijection from G/K to $\mathbb{H}$.*

Proof. It suffices to show that K is the stabilizer of i. This stabilizer consists of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $ad - bc = 1$ and $ai + b = i(ci + d)$, which implies that $a = d$, $b = -c$, and $a^2 + b^2 = 1$. So $a = \cos\theta$ and $b = \sin\theta$ for some θ as desired. $\square$

The upper half plane is biholomorphic to the unit disk $\mathbb{D}$ by $f(z) = \frac{z-i}{z+i}$.

Lemma 72. *If rot_θ denotes the action on $\mathbb{C}$ by rotation by θ, then we have the following commutative diagram,*

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{r_\theta} & \mathbb{H} \\ \downarrow f & & \downarrow f \\ \mathbb{D} & \xrightarrow{\mathrm{rot}_{2\theta}} & \mathbb{D} \end{array}$$

Proof. This follows since

$$\frac{1}{2i}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

□

Given $z = x + iy$ and two tangent vectors $v, w \in T_z\mathbb{H}$, which we identify as elements of $\mathbb{R}^2$, define $\langle v, w\rangle_z := \frac{v \cdot w}{y^2}$. If we instead interpret v and w as complex numbers we may write $\langle v, w\rangle_z := \frac{v\overline{w}}{y^2}$. This is the *hyperbolic inner product.* We say that the inner product is g-invariant for $g \in \mathrm{SL}(2, \mathbb{R})$ if $\langle g_* v, g_* w\rangle_{g\cdot z} = \langle v, w\rangle_z$ for all $z \in \mathbb{H}$ and $v, w \in T_z\mathbb{H}$ where $g_* v$ is shorthand for $dg_z(v)$.

Lemma 73. *The hyperbolic inner product is invariant for all $g \in \mathrm{SL}(2, \mathbb{R})$.*

Proof. We note that if $g(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$, then

$$g'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{1}{(cz + d)^2}$$

and

$$\mathrm{Im}(g(z)) = \frac{g(z) - \overline{g(z)}}{2i} = \frac{(az + b)(c\overline{z} + d) - (a\overline{z} + b)(cz + d)}{2i|cz + d|^2} = \frac{(ad - bc)(z - \overline{z})}{2i|cz + d|^2} = \frac{\mathrm{Im}(z)}{|cz + d|^2}$$

Therefore,

$$\langle g_* v, g_* w\rangle_{g\cdot z} = \frac{v\overline{w}}{|cz + d|^4}\frac{|cz + d|^2}{\mathrm{Im}(z)^2} = \frac{v\overline{w}}{\mathrm{Im}(z)^2} = \langle v, w\rangle_z.$$

□

The unit tangent bundle $T^1\mathbb{H}$ is the subset of $T\mathbb{H}$ consisting of pairs (z, v) such that $v \in T_z\mathbb{H}$ has unit norm in the hyperbolic inner product. Since $\{\pm I\}$ acts trivially on $T^1\mathbb{H}$, define $\mathrm{PSL}(2, \mathbb{R}) := \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$.

Lemma 74. *The map from $\mathrm{PSL}(2, \mathbb{R})$ to $T^1\mathbb{H}$ that sends g to $(g\cdot i, g_* i)$ is a bijection. Moreover, $\mathrm{PSL}(2, \mathbb{R}) = KAN$.*

Proof. The stabilizer of i is K, so the stabilizer of a tangent vector at i is trivial in $\mathrm{PSL}(2, \mathbb{R})$ since r_θ acts on the tangent space of i by rotation by 2θ. Therefore, it suffices to show that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $T^1\mathbb{H}$. We are done since

$$(x+iy, iye^{-2i\theta}) \xrightarrow{n_{-x}} (iy, iye^{-2i\theta}) \xrightarrow{a_{-\log y}} (i, ie^{-2i\theta}) \xrightarrow{r_\theta} (i,i).$$

$\square$

Given a smooth curve $\gamma : [0,1] \longrightarrow \mathbb{H}$, its *hyperbolic length* if $\ell(\gamma) := \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t)\rangle_{\gamma(t)}} dt$. The *hyperbolic distance* between two points z and w in $\mathbb{H}$ is the infinimum of the length of all curves joining z to w. Both hyperbolic length and distance are $\mathrm{PSL}(2,\mathbb{R})$ invariant.

Worksheet Lecture 14

Problem 1. (The Polar Decomposition) Imitate the proof of the KAN decomposition to show that $\mathrm{SL}(2,\mathbb{R}) = KAK$.

Problem 2. (Visualizing $\mathrm{SL}(2,\mathbb{R})$**)** Show that $\mathrm{PSL}(2,\mathbb{R})$ can be identified with a solid torus $\mathbb{D} \times S^1$. Notice that this implies that it has the same homotopy and homology groups as the circle.

Problem 3. (Mautner phenomenon) Recall that AN acts on $\mathbb{H}$ by the (real) affine group. Show that AN acts simply transitively and so we may identify $\mathbb{H}$ with AN. Use this identification to prove that $n_{e^{-t}u} = g_{-t}n_u g_t$ for all u and t. The unenlightening proof of this involves multiplying three 2×2 matrices. The enlightening proof just uses the identification of AN and $\mathbb{H}$ and involves drawing a trapezoid.

15. Geodesic flow on hyperbolic surfaces

A *geodesic segment* is a curve $\gamma : [0,1] \longrightarrow \mathbb{H}$ whose length is equal to the distance between its endpoints. The segment is *unit-speed* if additionally $d(\gamma(t), \gamma(0)) = t$ for all t. Note that $\mathrm{PSL}(2,\mathbb{R})$ takes (unit-speed) geodesic segments to (unit-speed) geodesic segments.

Lemma 75. *There is a unique unit-speed geodesic segment connecting any two points in $\mathbb{H}$. The one connecting i to $e^T i$ is $\gamma : [0,T] \longrightarrow \mathbb{H}$ where $\gamma(t) = e^t i$.*

Proof. Since we can apply $\mathrm{PSL}(2,\mathbb{R})$ to any pair of points to move them to the points i and $e^T i$ for some i, the first claim follows from the second. We see that $\gamma(t)$ is unit speed since $\langle \gamma'(t), \gamma'(t)\rangle_{\gamma(t)} = 1$. If $\eta : [0,1] \longrightarrow \mathbb{H}$ is another curve joining i to $e^T i$, then, writing $\eta(t) = x(t) + iy(t)$,

$$\ell(\eta) = \int_0^1 \frac{|\eta'(t)|}{|\eta(t)|} \geq \int_0^1 \frac{|y'(t)|}{y(t)} dt \geq \int_0^1 \frac{y'(t)}{y(t)} dt = T.$$

Clearly equality holds if and only if $x'(t) = 0$ and $y'(t) > 0$. So the unique geodesic segment joining i to $e^T i$ is the segment of the imaginary line joining them. The claim follows since there is a unique unit-speed parameterization of any curve η for which $\eta'(t) \neq 0$ for any t. □

The lemma implies that for each $(z, v) \in \mathbb{T}^1\mathbb{H}$ there is a unique unit speed geodesic $\gamma(t)$ so that $\gamma(0) = z$ and $\gamma'(0) = v$. *Geodesic flow on* $T^1\mathbb{H}$ is the flow $h_t : T^1\mathbb{H} \longrightarrow T^1\mathbb{H}$ that sends (z, v) to $(\gamma(t), \gamma'(t))$. The lemma implies that $h_t(i, i) = (e^t i, e^t i) = g_t(i, i)$. So if $g \cdot (i, i) = (z, v)$, then $h_t(z, v) = gg_t \cdot (i, i)$. To summarize we have the following commutative diagram,

$$\begin{tikzcd}
\mathrm{PSL}(2,\mathbb{R}) \arrow[r, "g \longrightarrow gg_t"] \arrow[d, "\cong"] & \mathrm{PSL}(2,\mathbb{R}) \arrow[d, "\cong"] \\
T^1\mathbb{H} \arrow[r, "h_t"] & T^1\mathbb{H}
\end{tikzcd}$$

A *(geodesic) polygon in* $\mathbb{H}$ is a polygon whose sides are geodesic segments. In analogy with the definition we made for translation surfaces, a *(finite area) hyperbolic cone surface* is a finite collection of geodesic polygons in $\mathbb{H}$ with equal length edges glued together by elements of $\mathrm{PSL}(2,\mathbb{R})$. If the cone surface has no cone points, then this is called a *(finite area) hyperbolic surface.* Note that hyperbolic cone surfaces inherit a metric and a notion of geodesic flow from $\mathbb{H}$. Examples include: gluing opposite sides of an ideal symmetric quadrilateral together to form a once-punctured torus; gluing together the fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$; and forming pants to put a hyperbolic structure on a genus two surface.

Lemma 76. *If θ is an angle that a unit tangent vector makes with the horizontal then $\frac{dx dy d\theta}{y^2}$ is a $G = \mathrm{PSL}(2,\mathbb{R})$ invariant measure on $T^1\mathbb{H}$.*

Proof. We have already seen that $\frac{dx dy}{y^2}$ is a G-invariant measure. Since G acts by holomorphic, i.e. angle-preserving, maps the claim follows. $\square$

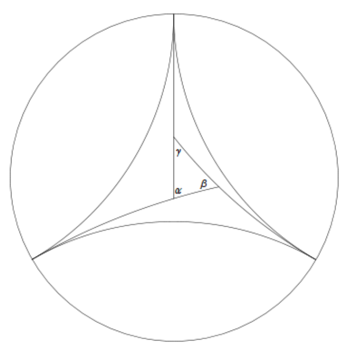
The measure constructed in the previous lemma is called the *Liouville measure* and can be identified with left Haar measure, m, on $\mathrm{PSL}(2,\mathbb{R})$. By uniqueness of left Haar measure, there is a homomorphism $c : G \longrightarrow \mathbb{R}_{>0}$ so that $c(g)m(U) = m(Ug^{-1})$ for every measurable U. Since $\mathrm{PSL}(2,\mathbb{R})$ is perfect, i.e. has no nontrivial abelian quotients, left Haar measure is automatically also a right Haar measure.

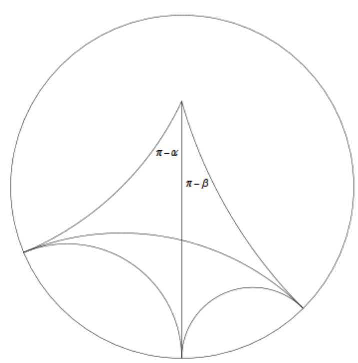
Since Haar measure is given by a top-dimensional differential form that is preserved by G, this differential form defines a top-dimensional differential form on T^1X for any hyperbolic surface X. This is called the *Liouville measure on* T^1X .

Worksheet for Lecture 15

Problem 1. (Hyperbolic area) Show that $\frac{dxdy}{y^2}$ defines an area form, called the *hyperbolic area form*, on $\mathbb{H}$ that is invariant under $\mathrm{PSL}(2,\mathbb{R})$. If T is the geodesic triangle in $\mathbb{H}$ whose edges are the lines $\{x=-1\}$, $\{x=1\}$, and the upper half of the unit circle centered at 0, show that the hyperbolic area of T is π. Show that any geodesic triangle with vertices on the boundary of $\mathbb{H}$ has area π.

Problem 2. (Hyperbolic Gauss-Bonnet 1) The following proof and figure are due to Danny Calegari [1]. Show that the area $f(\alpha)$ of geodesic triangle in $\mathbb{H}$ with two vertices on the boundary of $\mathbb{H}$ and one angle $\pi-\alpha$ has area only depending on α. Use the following figure to conclude that a hyperbolic triangle has area that only depends on its angles. Moreover, if these angles are α, β, γ, then the area is $\pi - f(\alpha) - f(\beta) - f(\gamma)$.





[1] See https://lamington.wordpress.com/2010/04/10/hyperbolic-geometry-notes-2-triangles-and-gauss-bonnet/

Problem 3. (Hyperbolic Gauss-Bonnet 2) Use the previous figure to prove that $f(\alpha) + f(\beta) = f(\alpha + \beta - \pi) + \pi$. Argue that $f(0) = 0$ and $f(\pi) = \pi$. Now argue by induction that $f(\alpha) = \alpha$ whenever α is an integral multiple of $\frac{\pi}{2^n}$ for some integer n. Conclude that $f(\alpha) = \alpha$ and hence that

$$\text{area}(\Delta(\alpha, \beta, \gamma)) = \pi - \alpha - \beta - \gamma.$$

16. The Hopf argument

Lemma 77. *The group of orientation-preserving isometries, i.e. distance-preserving maps, of* $\mathbb{H}$ *is* $\mathrm{PSL}(2,\mathbb{R})$.

Proof. Let g be an isometry that sends $(i,i) \in T^1\mathbb{H}$ to (z,v). Let h be the element of $\mathrm{PSL}(2,\mathbb{R})$ that sends (z,v) to (i,i). So $h \circ g$ is an isometry that fixes i and fixes every tangent vector to i. Since $h \circ g$ sends geodesics to geodesics and since a (unit-speed) geodesic is determined by an element of $T^1\mathbb{H}$, it follows that $h \circ g$ fixes all geodesics that start at i. The map from $T_i\mathbb{H}$ to $\mathbb{H}$ that sends $v \in T_i\mathbb{H}$ to $\gamma(1)$ where γ is the constant-speed geodesic so that $\gamma(0) = i$ and $\gamma'(0) = v$ is a homeomorphism. Therefore, $h \circ g$ fixes every point in $\mathbb{H}$ as desired. □

Given a group G, a *lattice* is a discrete subgroup Γ so that $\Gamma \backslash G$ has a finite G-invariant measure.

Lemma 78. *If X is a finite area hyperbolic surface, then there is a lattice Γ in $G = \mathrm{PSL}(2,\mathbb{R})$ so that $X = \Gamma\backslash\mathbb{H}$, $T^1X = \Gamma\backslash G$, and geodesic flow is given by sending Γg to $\Gamma g g_t$.*

Proof. Given a hyperbolic surface X, we lift the metric on it to its universal cover $\widetilde{X}$. Since this is a simply connected space with a complete metric of constant negative curvature, the universal cover is $\mathbb{H}$. Moreover, the action of $\pi_1(X)$ is by a discrete group of orientation-preserving isometries Γ, which must be a subgroup of G by the previous lemma. This shows that $X = \Gamma\backslash\mathbb{H}$. By pulling back the tangent bundle we see that $T^1X = \Gamma\backslash T^1\mathbb{H}$, which we identify with $\Gamma\backslash G$. The expression for geodesic flow comes from the expression for geodesic flow on $T^1\mathbb{H}$. Since Liouville measure is determined by a differential form on G that is both left and right G-invariant, Liouville measure is flow-invariant since it is invariant under right multiplication by g_t. Finally, since X is a finite area hyperbolic surface, the Liouville measure of T^1X is $2\pi \mathrm{area}_{hyp}(X)$, which is finite. □

Fix a left-invariant metric $\widetilde{d}$ on $\mathrm{PSL}(2,\mathbb{R})$ (for instance, by pushing around an inner product that we specify at $T_e\mathrm{PSL}(2,\mathbb{R})$ by left-translation). This descends to a metric on $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ by defining $d(\Gamma g, \Gamma h) := \min_{(\gamma_1,\gamma_2)\in\Gamma\times\Gamma} \widetilde{d}(\gamma_1 g, \gamma_2 h)$. We notice that since $n_{e^{-t}u} = g_{-t} n_u g_t$ we have that

$$\widetilde{d}(gg_t, gn_ug_t) = \widetilde{d}(e, n_{e^{-t}u}) \xrightarrow{t\longrightarrow\infty} 0.$$

We say that $\{gn_u\}_{u\in\mathbb{R}}$ is the *stable foliation* W^+, i.e. the *horocycle* of points through g that approach one another under forward geodesic flow. A similar computation, shows that $\{gm_u\}_{u\in\mathbb{R}}$ is the *unstable* *foliation* W^- where $m_u := \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$, i.e. the horocycle of points through g that approach one another under backwards geodesic flow. The *center foliation* is the one given by $\{gg_t\}_{t\in\mathbb{R}}$.

Lemma 79. *At any point $g \in \mathrm{PSL}(2,\mathbb{R})$ the tangent vectors to the stable, unstable, and center foliations form a basis of the tangent space.*

Proof. It suffices to show the claim at the identity in $\mathrm{SL}(2,\mathbb{R})$ where the tangent vectors are $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to the unstable, center, and stable foliations respectively. □

Remark 80. A general fact (from the theory of manifolds) about foliations satisfying the conclusion of Lemma 79 is that at any point $g \in \mathrm{PSL}(2,\mathbb{R})$ there are local coordinates that identify a neighborhood of g with an open subset of $\mathbb{R}^3$ so that lines parallel to the x (resp. y, resp. z) axes correspond to the leaves of the stable (resp. unstable, resp. center) foliation.

Theorem 81 (The Hopf argument, 1939)**.** *Geodesic flow on the unit tangent bundle of a finite area hyperbolic surface is ergodic.*

Proof. Let X be the surface and let μ be Liouville measure. By the von Neumann ergodic theorem, it suffices to show that the only flow-invariant functions in $L^2(T^1X,\mu)$ are constant. Since the the compactly supported continuous functions are dense in $L^2(T^1X,\mu)$ it suffices to show that if $f : T^1X \longrightarrow \mathbb{R}$ is continuous and compactly supported then its projection to the subspace of flow-invariant functions is constant. This projection is the Birkhoff average, i.e. $\overline{f}(x) := \lim_{T\longrightarrow\infty} \frac{1}{T}\int_0^T f(g_t x)dt$. Since f is compactly supported, for any $\epsilon > 0$ there is a $\delta > 0$ so that if $d(x,y) < \delta$, then $|f(x) - f(y)| < \epsilon$ (d refers to the distance defined after Lemma 78). If x and y are on the same leaf of the stable foliation, then there is some $T_0 > 0$ so that it $t > T_0$, then $d(xg_t, yg_t) < \delta$. Since Birkhoff averages do not depend on initial segments we see that

$$|\overline{f}(x) - \overline{f}(y)| \leq \lim_{T\longrightarrow\infty} \frac{1}{T}\int_{T_0}^{T_0+T} |f(xg_t) - f(yg_t)|dt < \epsilon.$$

Since ϵ is arbitrary, $\overline{f}$ is constant on stable leaves. Since forward and backward Birkhoff averages agree almost surely, $\overline{f}$ is also constant on unstable leaves. Finally the Birkhoff averages are constant on center leaves. Since these foliations provide local coordinates on T^1X, $\overline{f}$ is constant as desired. □

17. Weak and strong mixing

Given a sequence of real numbers (a_n) its *Cesaro averages* are $A_n := \frac{1}{N}\sum_{i=1}^{N} a_i$. The sequence *Cesaro converges to* L if (A_n) converges to L. (Note that the usual notion of convergence implies Cesaro-convergence). We say that the sequence *Cesaro absolutely converges to* L if $\frac{1}{N}\sum_{i=1}^{N} |a_i - L| \longrightarrow 0$.

Lemma 82. *Suppose that (X,T,μ) is a pmps. The system is ergodic if and only if for any $f,g \in L^2$, $\int_X f(T^n x)g(x)d\mu$ Cesaro-converges to $\int_X f \int_X g$.*

Proof. By the von-Neumann ergodic theorem, ergodicity is equivalent to $\frac{1}{N}\sum_{k=0}^{N-1} f(T^k)$ converging in L^2 to $\int_X f d\mu$. □

Lemma 83 (The Cesaro criterion for ergodicity)**.** *Suppose that (X,T,μ) is a pmps. The system is ergodic if and only if for any measurable A and B, $\mu(T^{-n}A \cap B)$ Cesaro-converges to $\mu(A)\mu(B)$.*

Proof. The forward direction is the previous lemma. The reverse direction is the following. Suppose that A and $X\backslash A$ are T-invariant. Then the sequence 0 Cesaro-converges to $\mu(A)\mu(X \setminus A)$ implying that one of the two is zero. □

A pmps (X,T,μ) is *strong (resp. weak) mixing* if for any measurable A and B, $\mu(T^{-n}A \cap B)$ converges (resp. Cesaro absolutely converges) to $\mu(A)\mu(B)$. Clearly, strong mixing implies weak mixing, which implies ergodicity. Mixing was first defined by Koopman and von Neumann in 1932.

A circle rotation is not strong mixing since, letting A be any interval of length $\ell < \frac{\pi}{3}$ we see that $T^{-n}A \cap A$ can be empty for infinitely many n.

Lemma 84 (Ergodic group automorphisms are mixing)**.** *Suppose that $A \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ is an integer matrix with nonzero determinant and no eigenvalues equal on the unit circle in $\mathbb{C}$. This determines a strong-mixing action, which we call T, on $X := \mathbb{R}^n/\mathbb{Z}^n$.*

Proof. Recall that the characters of X are the continuous homomorphisms $\chi : X \longrightarrow \mathbb{C}^\times$, which are all of the form $\chi_m := \chi(x_1,\ldots,x_n) = \exp(2\pi i\,(m_1x_1 + \ldots + m_nx_n))$ where $m := (m_1,\ldots,m_n) \in \mathbb{Z}^n$. It suffices to prove that $\langle f \circ T^n, g\rangle \longrightarrow \langle f, g\rangle$ for any $f,g \in L^2$. Fourier theory tells us that if f belongs to $L^2(X)$ then $f = \sum_{m\in\mathbb{Z}^n} c_m\chi_m$ where $c_m := \int_X f\overline{\chi_m}$ and that $\|f\|_2^2 = \sum_m |c_m|^2$. It suffices to show that $\langle \chi_m(T^n), \chi_k\rangle \longrightarrow 0$ as n goes to ∞ if m and k are not zero. We have already see that $\chi_m(T^n) = \chi_{(A^T)^n m}$. No eigenvalues on the unit circle means that A^T has no periodic points other than 0, so the result follows. ☐

A subset of J of the positive integers has density d, if $\frac{|J\cap\{1,\ldots,N\}|}{N} \longrightarrow d$ as $N \longrightarrow \infty$.

Lemma 85. *Let (a_n) be a bounded sequence of nonnegative real numbers. The following are equivalent:*

(1) (a_n) Cesaro-converges to 0.
(2) (a_n^2) Cesaro-converges to 0.
(3) There is a density zero set of integers J so that a_n converges to 0 once we throw out the elements with indices in J.

Proof. It suffices to show the equivalence of the first and final property. Suppose the final property holds. Let p_N be the percent of integers in $\{1,\ldots,N\}$ in J. Suppose that $a_n \le L$ for all n. Then

$$A_N \le p_N L + \frac{(1-p_N)N}{N} A'_N$$

where A'_N is the Cesaro average of the elements of (a_n) with $1 \le n \le N$ that are not in J. The righthand side converges to zero.

Now suppose that (a_n) Cesaro-converges to 0. Let J_k be the indices n where $a_n > \frac{1}{k}$. This is a density zero set since the limit of Cesaro averages are at least $\frac{d}{k}$ where d is the density of J_k (note that the elements of the sequence are nonnegative). Let N_k be an increasing sequence of numbers so that percent of numbers in $[1,n] \cap J_k$ is less than $\frac{1}{k}$ if $n > N_k$. Let $J := \bigcup_k J_k \cap [N_k, N_{k+1}]$. Tossing out J removes every a_n where $a_n > \frac{1}{k}$ if $n \in [N_k, N_{k+1}]$. So a_n converges to 0 away from J. Finally, we note that the percent of elements of J in $[0,N]$ with $N \in [N_k, N_{k+1}]$ is at most the percent of J_k in this range which is at most $\frac{1}{k}$. So J has density zero. ☐

Theorem 86. *Suppose that (X,T,μ) is a pmps. The following are equivalent:*

(1) T is weak mixing
(2) For any ergodic pmps (Y,S,ν), $(X\times Y, T\times S, \mu\times\nu)$ is ergodic.
(3) $(X\times X, T\times T, \mu\times\mu)$ is ergodic.

Proof. Suppose first that T is weak-mixing. Let A_1, B_1 be measurable subsets of X and A_2, B_2 measurable subsets of Y. We must consider the sequence

$$(\mu\times\nu)((A_1\times A_2)\cap(T\times S)^{-n}(B_1\times B_2)) = \mu(T^{-n}A_1\cap B_1)\nu(S^{-n}A_2\cap B_2).$$

So the Nth Cesaro-average of this sequence is

$$\frac{1}{N}\sum_k \mu(A_1)\mu(B_1)\nu(S^{-n}A_2\cap B_2)+\frac{1}{N}\sum_k(\mu(T^{-n}A_1\cap B_1)-\mu(A_1)\mu(B_1))\nu(S^{-n}A_2\cap B_2).$$

The first term converges to the $\mu(A_1)\nu(B_1)\mu(A_2)\nu(B_2)$ (by ergodicity of ν) and the second converges to 0 (by weak mixing of μ).

The second condition clearly implies the final one, so it suffices to show that the final condition implies the first. Let A and B be measurable. We observe that by ergodicity

$$\lim \frac{1}{N}\sum_{k=0}^{N-1}\mu(T^{-k}A\cap B)^2 = \lim \frac{1}{N}\sum_{k=0}^{N-1}(\mu\times\mu)((T\times T)^{-k}(A\times A)\cap B\times B) = \mu(A)^2\mu(B)^2.$$

By Lemma 85, we must show that the limit of the following is 0,

$$\frac{1}{N}\sum_{k=0}^{N-1}(\mu(T^{-k}A\cap B)-\mu(A)\mu(B))^2$$

Expanding, this sequence becomes,

$$\frac{1}{N}\sum_{k=0}^{N-1}\mu(T^{-k}A\cap B)^2+\mu(A)^2\mu(B)^2-2\mu(A)\mu(B)\frac{1}{N}\sum_{k=0}^{N-1}(\mu(T^{-k}A\cap B)$$

which indeed converges to 0 by ergodicity. $\square$

Worksheet for Lecture 17

Problem 1 (Weak Mixing Implies Weak Mixing of All Orders): Show that if (X, T, μ) and (Y, S, ν) are weak-mixing, then so is $(X \times Y, T \times S, \mu \times \nu)$. Conclude that $(X \times X \times \ldots \times X, T \times T \times \ldots \times T, \mu \times \mu \times \ldots \times \mu)$ is weak-mixing.

Problem 2 (Group translations are never mixing): Let G be a compact group and let $g \in G$ be any element and let μ be the left Haar measure. Let $T : G \longrightarrow G$ be the map $T(x) = gx$. Show that (X, T, μ) is not weak-mixing.

Problem 3 (Circle homeomorphisms are never mixing for non-boring measures): Let μ be a measure on the circle that is not supported at one or two points. Let $T : S^1 \longrightarrow S^1$ be a circle homeomorphism that preserves μ. Show that T is not strong mixing. (Hint: Write the circle as a union of three positive-measure intervals A, B, C that only overlap at endpoints (and so the endpoints have measure zero). If T is mixing, then there is an N so that $T^n(A)$ intersects A, B, and C for all $n > N$. Show that this means that $T^n(A)$ must entirely contain one of the three intervals and show that this produces a contradiction.)

Problem 4 (Furstenberg's Characterization of Weak Mixing): Let (X, T, μ) be a pmps. Recall that the system is ergodic if and only if for any two positive measures sets A and B there is some n so that $T^{-n}(A) \cap B$ is nonempty. Show that the system is weak-mixing if and only if for any three positive measure sets A, B, C there is some n so that both $T^{-n}(A) \cap B$ and $T^{-n}(A) \cap C$ is are nonempty. Conclude finally that if the system is weak-mixing and $A_1, \ldots, A_k, B_1, \ldots, B_k$ are positive measure sets then there is some n so that $T^{-n}(A_i) \cap B_i$ is nonempty for all $1 \leq i \leq k$.

18. The Howe-Moore argument for the mixing of geodesic and horocycle flow

For today, $G = \mathrm{SL}(2, \mathbb{R})$ unless otherwise stated. If G acts continuously on a topological space X and preserves a measure μ, then G induces an unitary action on $L^2(X, \mu)$, i.e. $\langle f \circ g, h \circ g \rangle = \langle f, h \rangle$ for any $g \in G$. If $V := L^2_0(X, \mu)$ is the (Hilbert) subspace of mean-zero functions, i.e. ones for which $\int f d\mu = 0$, then ergodicity is equivalent to the non-existence of nonzero G-invariant vectors in V. A sequence (h_n) is mixing if $\langle h_n \cdot v, w \rangle \longrightarrow 0$ for any $v, w \in V$. With this motivation, we make the following assumption.

Standing Assumption. *G acts by unitary maps on the Hilbert space V and has no nonzero invariant vectors.*

Lemma 87 (Horocycle flow is ergodic)**.** *There are no nonzero N-invariant vectors in V.*

Proof. Suppose not and let $0 \neq v \in V$ be N-invariant. Consider the map $f : G \longrightarrow \mathbb{R}$ given by $f(g) := \langle g \cdot v, v \rangle$. This map factors through $G/N = \mathbb{R}^2 - \{(0, 0)\}$. Since the action is unitary

$$f(ng) = \langle ngv, v \rangle = \langle gv, n^{-1}v \rangle = \langle gv, v \rangle = f(g)$$

So $f : \mathbb{R}^2 - \{(0, 0)\} \longrightarrow \mathbb{R}$ is constant on horizontal lines, excluding possibly the x-axis. However, by continuity, the function must also be constant on x-axis. However, the x-axis is identified with P/N. Since the action is unitary and $\langle pv, v \rangle = \langle v, v \rangle$ for any $p \in P$, it follows that $Pv = v$. So f factors through G/P, which is a circle. The P-action on G/P acts by north-south dynamics so, so f induces a P-invariant map from G/P to $\mathbb{R}$, it is constant and hence equal to 1. As before, since the action is unitary $Gv = v$, which is a contradiction. ☐

Given a lattice Γ in $G = \mathrm{SL}(2, \mathbb{R})$, *horocycle flow* is the map from $\Gamma \backslash G$ to itself given by sending Γg to $\Gamma g n_t$ where $n_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

Corollary 88. *Horocycle flow is ergodic.*

Proof. Since G acts transitively on $\Gamma \backslash G$ there are no nonzero G-invariant vectors in $L^2_0(\Gamma \backslash G)$, so the result follows from the previous lemma. ☐

Recall that on a Hilbert space V the *weak* topology* is one for which $v_n \longrightarrow v$ means that for any $w \in V$, $\lim_{n \longrightarrow \infty} \langle v_n, w \rangle = \langle v, w \rangle$. By Banach-Alaoglu, the unit disk is compact in the weak* topology.

Corollary 89 (Geodesic flow is mixing)**.** *The action of A is mixing i.e. $\langle g_t v, w \rangle \longrightarrow 0$ as $t \longrightarrow \infty$ for any $v, w \in V$.*

Proof. Let t_m be any sequence tending to infinity and, for simplicity, write g_m instead of g_{t_m}. Suppose in order to derive a contradiction that there are unit vectors v and w and a constant ϵ so that $\langle g_m v, w\rangle > \epsilon$ (possibly after passing to a subsequence). Passing to a further subsequence, assume that $g_m v$ converges in the weak* topology to a vector $u \in V$. This vector is necessarily nonzero. For any vector $n \in N$ we have that

$$|\langle nu, u\rangle - \langle u, u\rangle| = \lim_{m\longrightarrow\infty} |\langle n g_m v, u\rangle - \langle g_m v, u\rangle| = \lim_{m\longrightarrow\infty} |\langle g_{-m} n g_m v, g_{-m} u\rangle - \langle v, g_{-m} u\rangle|$$

By Cauchy-Schwarz (and the fact that g_m is unitary), this limit is bounded above by

$$\lim_{m\longrightarrow\infty} \|(g_{-m} n g_m - \mathrm{id})v\| \|u\|$$

we have seen that $g_{-m} n g_m$ converges to 0 so the limit converges to 0 and so $\langle nu, u\rangle = \langle u, u\rangle$, which implies (since n is unitary) that u is a nonzero N-invariant vector, which is a contradiction. □

Corollary 90 (Howe-Moore vanishing, 1979)**.** *For any $v, w \in V$ and any sequence h_n in G tending to ∞, $\langle h_n \cdot v, w\rangle \longrightarrow 0$. In particular, horocycle flow is mixing.*

Proof. By the KAK decomposition, $h_n = k_n g_{t_n} k_n'$ where t_n goes to infinity and k_n and k_n' belong to the compact subgroup K. Suppose to a contradiction that $\langle h_n v, w\rangle$ is bounded away from 0 (after perhaps passing to a subsequence). Passing to a further subsequence $k_n \longrightarrow k$ and $k_n' \longrightarrow k'$. Then,

$$|\langle h_n v, w\rangle| = |\langle g_{t_n} k_n' v, k_n^{-1} w\rangle| \leq \|k^{-1} - k_n^{-1}\| + \|k_n' - k'\| + |\langle g_{t_n} k' v, k^{-1} w\rangle \longrightarrow 0.$$

which is a contradiction. □

Our predominant examples of *simple Lie groups with finite center* are matrix groups $\mathrm{SL}(n, \mathbb{R})$ and its subgroups $\mathrm{SO}(p, q)$ that preserve a symmetric bilinear form of signature (p, q) on $\mathbb{R}^{p+q}$.

Theorem 91 (Moore's Ergodicity Theorem, 1966)**.** *Let G be a simple Lie group with finite center and let H be a noncompact subgroup. Let X be a topological space with a continuous ergodic G-action that preserves a finite measure. Then the action of H is ergodic and mixing.*

Essentially the general case reduces to using that G is generated by copies of $\mathrm{SL}(2, \mathbb{R})$ and then reducing to the $\mathrm{SL}(2, \mathbb{R})$ case.

Worksheet for Lecture 18

Problem 1 (Transfer principle 1): Let G be a topological group acting on a space X and preserving a measure μ. Let H be a closed subgroup of G. Show that if H acts ergodically on X, then G acts ergodically on $X \times G/H$ with respect to $\mu \times m$ where m is the push-forward of Haar measure. The measure m may not be finite but ergodicity still makes sense in that it means that G-invariant measurable sets are either null or conull. (Hint: If A is G-invariant and neither null nor conull then consider the map $\pi : X \times G/H \longrightarrow G/H$ and set $A_{gH} := \pi^{-1}(gH) \cap A$. We see that $h \cdot A_{gH} = A_{hgH}$ so all the sets A_{gH} have the same μ-measure and so by Fubini A_{eH} is neither null nor conull. Now use that H acts ergodically on X.)

Problem 2 (Transfer principle 2): Prove the converse of the previous problem, i.e. show that if G acts ergodically on $X \times G/H$ then H acts ergodically on X. (Hint: You may use the fact that there is a measurable section $s : G/H \longrightarrow G$ of the quotient map $q : G \longrightarrow G/H$. Now if there is an H-invariant subset of $B \subseteq X$ produce a G-invariant subset A of $X \times G/H$. The idea should be to reverse engineer the previous construction by setting $A_{eH} := B$ and then setting $A_{gH} := s(gH) \cdot A_{eH}$).

Problem 3 (Transfer principle 3): If H_1 and H_2 are two subgroups of G, then H_1 is ergodic on G/H_2 if and only if H_2 is ergodic on G/H_1. Conclude that if Γ is a lattice in $\mathrm{SL}(2, \mathbb{R})$ then the only measurable functions on $\partial\mathbb{D}$ that are Γ-invariant are constant functions. Show that the same is true when considering Γ-invariant functions on $\partial\mathbb{D} \times \partial\mathbb{D}$.