The Potentials Theory on Denumerable Markov Chain

Boren(Wells) Guan

Date: April 29, 2024

1 Basic Definition

Definition 1.1. (Irreducible Markov Chain)

$x \leftrightarrow y$ if $x \to y$ and $y \to x$ and we call x and y communicate and easy to be checked an equivalence relation on X. Then we call an equivalence class w.r.t. $\leftrightarrow$ as an irreducible class. We call (X, P) to be irreducible if it is an irreducible class.

Definition 1.2. (Harmonic Functions)

Let (X, P) be a finite, irreducible Markov chain. We choose and fix a subset $X^\circ \subset X$ called the interior, and $\partial X = X - X^\circ$, we suppose X° is connected i.e. P_{X° is irreducible.

We call a function $h : X \to \mathbb{R}$ harmonic on X° if $h(x) = Ph(x)$ for every $x \in X^\circ$, where $Ph(x) = \sum_{y \in X} p(x, y)h(y)$, which is also called mean value property. We denote by $\mathcal{H}(X^\circ) = \mathcal{H}(X^\circ, P)$ is the linear space of all functions on X and harmonic on X°.

For a general finite Markov chain, we define the linear space of harmonic functions on X with

$$\mathcal{H} = \mathcal{H}(X, P) = \{h : X \to \mathbb{R}, h(x) = Ph(x), x \in X\}$$

Definition 1.3. (Hitting distribution)

Let $s = s^{\partial X}$, then $P_x(s^{\partial X} < \infty) = 1$ for any $x \in X$.

Then we may define

$$\nu_x(y) = P_x(s < \infty, Z_s = y), y \in \partial X$$

and then ν_x will become a probability distribution on ∂X, called the hitting distribution of ∂X.

Proof. Here we introduce $\tilde{P}$ which is defined by $\tilde{p}(x, y) = p(x, y)$ for $x \in X^\circ$ and $\tilde{p}(x, y) = \delta_x$ for $x \in \partial X$, then it is easy to check $h \in \mathcal{H}(X^\circ, P)$ iff $h \in \mathcal{H}(X^\circ, \tilde{P})$ and s is the same on (X, P) and $(X, \tilde{P})$. So consider s on $(X, \tilde{P})$, we know

$$P(s^{\partial X} < \infty) = 1$$

by corollary 1.3. $\square$

Definition 1.4. (Superharmonic functions)

All functions $f : X \to \mathbb{R}$ are assumed to be P-integrable (which is a subspace) i.e.

$$\sum_{y \in X} p(x, y)|f(y)| < \infty$$

for all $x \in X$.

A real function h on X is called harmonic if $h(x) = Ph(x)$ and superharmonic if $h(x) \geq Ph(x)$ for every $x \in X$.

Addition to $\mathcal{H}$, we define

$$\mathcal{H}^+ = \{h \in \mathcal{H}, h(x) \geq 0\} \quad \mathcal{H}^\infty = \{h \in \mathcal{H}, h \text{ is bounded on } X\}$$

and lett $\mathcal{S} = \mathcal{S}(X, P)$ the space of all superharmonic functions and similarly $\mathcal{S}^+, \mathcal{S}^\infty$

Definition 1.5. (Invariant and excessive measures)

Here we assume the invariant measure must satisfy nonnegative and

$$vP(y) = \sum_{x \in X} v(x)p(x, y) < \infty$$

Recall we call a measure on X is invariant or stationary if $v = vP$ and excessive or superinvariant $v = vP$. We denote $I^+ = I^+(X, P)$ and $E^+ = E^+(X, P)$ the cones of all invariant and excessive measures.

Definition 1.6. (Induced Markov chain)

Suppose (X, P) is irreducible and substochastic. Let $A \subset X$ and we may define

$$p^A(x, y) = P_x(t^A < \infty, Z_{t^A} = y)$$

where $p^A(x, y) = 0$ if $y \notin A$. Then we may know $P^A = (p^A(x, y))$ is substochastic and (A, P^A) is called the Markov chain induced by (X, P) on A.

Here the irreducibility of (X, P) implies irreducibility of the induced chain.

Proof. For $x, y \in A$ there are $n > 0$ and $x_1, \cdots, x_{n-1} \in X$ such that $p(x, x_1)p(x_1, x_2) \cdots p(x_{n-1}, y) > 0$ and let $x_{i_k} \in A$ and we know $p^A(x_{i_k}, x_{i_{k+1}}) \leq p^A(x_{i_k}, x_{i_{k+1}})$. □

Definition 1.7. If P^A is stochastic, then we call A is recurrent for (X, P).

Definition 1.8. For $A, B \subset X$, define the restriction of P to $A \times B$ by $P_{A,B} = (p(x, y))_{x \in A, y \in B}$.

Definition 1.9. (Potentials)

A G-integrable function $f : X \to \mathbb{R}$ is one that satisfies $\sum_y G(x, y)|f(y)| < \infty$ for each $x \in X$. In this case, $g(x) = Gf(x) = \sum_{y \in X} G(x, y)f(y)$ is called the potential of f, while f is called the charge of g. The support of f is $\{x \in X, f(x) \neq 0\}$.

We may know $(I - G)^{-1}$ convergent.

Definition 1.10. (F and L functions)

For $A \subset X, x, y \in X$, we define

$$F^A(x, y) = \sum_{n=0}^{\infty} P_x(Z_n = y, Z_j \notin A \text{ for } 0 \leq j < n)\chi_A(y)$$

and

$$L^A(x, y) = \sum_{n=0}^{\infty} P_x(Z_n = y, Z_j \notin A \text{ for } 0 < j \leq n)\chi_A(x)$$

And for P and an excessive measure v, define the v-reversal $\hat{P}$ of P as (to secure $\hat{p}$ is substochastic)

$$\hat{p}(x, y) = v(y)p(y, x)/v(x)$$

Definition 1.11. (Reduced measure)

Reduced measure on A is defined by

$$R^A[v](x) = \inf\{\mu \in E^+, \mu(y) \geq v(y), y \in A\}$$

Definition 1.12. (Potential of measures and Balayee)

Define the potential of an excessive measure v by vG.

If f is a non-negative G-integrable function on X, then the balayee of f is the function $f^A = L^A f$.

If μ is a non-negative, G-integrable measure on X, then the balayee of μ is the measure $\mu^A = \mu F^A$.

2 Solution of Dirichelet problem

Lemma 2.1. *We call a set $B \subset X$ convex if $x, y \in B$ and $x \to w \to y$ implies $w \in B$. For $B \subset X$ finite,convex set containing no essential elements. Then there is $\epsilon > 0$ such that for each $x \in B$ and all but finitely many $n \in \mathbb{N}$*

$$\sum_{y \in B} p^{(n)}(x, y) \leq (1 - \epsilon)^n$$

Proof. B is a disjoint union of finite nonessential irreducible classes $C(x_1), \cdots, C(X_k)$ and assume $C(x_1), C(x_2), \cdots, C(x_j)$ are the maximal elements in the partial order $\to$ restricted on $C(x_i), 1 \leq i \leq k$. We know there is $v_i \in X$ such that $x_i \to v_i$ but $v_i \nrightarrow x_i$ for $1 \leq i \leq j$ with $v_i \in X - B$. For $x \in B$, $x \to x_i$ for some i and hence $x \to v_i$ while $v_i \nrightarrow x$ for some i. So we may find m_x such that

$$\sum_{y \in B} p^{(m_x)}(x, y) < 1$$

Let $m = \max\{m_x, x \in B\}$ and $x \in B$, we know

$$\sum_{y \in B} p^{(m)}(x, y) = \sum_{y \in B} \sum_{\omega \in X} p^{(m_x)}(m_x)(x, \omega) p^{(m-m_x)}(\omega, y) < 1$$

since B is finite, there is $\kappa > 0$ such that

$$\sum_{y \in B} p^{(m)}(x, y) \leq 1 - \kappa$$

let $n \geq m$ and we assume $n = km + r$ and we know

$$\sum_{y \in B} p^{(n)}(x, y) = \sum_{w \in B} p^{(km)}(x, w) = \sum_{y \in B} p^{(k-1)m} \sum_{\omega \in B} p^{(m)}(y, \omega) \leq \cdots \leq (1 - \kappa)^k = (1 - \epsilon)^n$$

where $\epsilon = 1 - (1 - \kappa)^{1/2m}$. □

Lemma 2.2. *For C finite, non-essential irreducible class. The expected number of visits C starting from $x \in C$ is finite, i.e.*

$$E_x(v^C) \leq 1/\epsilon + M$$

Then we may know

$$P_x(\exists k, Z_n \in C \text{ for all } n > k) = 1$$

since $P(v^C = \infty) = 0$.

Lemma 2.3. *If the set of all non-essential states in X is finite, then the Markov chain reaches some essential class with probability one:*

$$P_x(s^{X_{ess}} < \infty) = 1$$

where X_{ess} is the union of all essential classes.

Lemma 2.4. *(Maximum principle) Let $h \in \mathcal{H}(X^\circ)$ and $M = \max_X h(x)$, then there is $y \in \partial X$ such that $h(y) = M$.*

If h is non-constant then $h(x) < M$ for every $x \in X^\circ$.

Proof. Here we may know if $x \in X^\circ$ and $h(x) = M$, then choose any $y \in X$ and we have

$$\begin{aligned} M = h(x) &= p^{(n)}(x,y)h(y) + \sum_{v \neq y} p^{(n)}(x,v)h(v) \\ &\leq p^{(n)}(x,y)h(y) + (1 - p^{(n)}(x,y))M \end{aligned}$$

where n such that $p^{(n)}(x,y) > 0$ and hence $h(y) = M$, which means h is then constant. And we are done. □

Theorem 2.5. *(Solution of the Dirichlet problem) For every function $g : \partial X \to \mathbb{R}$ there is a unique function $h \in \mathcal{H}(X^\circ, P)$ such that $h(y) = g(y)$ for all $y \in \partial(X)$ which is given by*

$$h(x) = \int_{\partial X} g d\nu_x$$

Proof. We firstly prove that the uniqueness of the solution, if $h, h' \in \mathcal{H}(X^\circ, P)$, then we know $h - h'$ should be the solution of the Dirichlet problem when $g = 0$ and by the maximum principle, we know $h - h' \leq 0$ and $h' - h \leq 0$ and we know $h = h'$.

Now we prove the existence of h, firstly we would like to show that $x \mapsto \nu_x(y)$ is harmonic, since

$$\begin{aligned} \sum_{v \in X} p(x,v)\nu_v(y) &= \sum_{v \in X} p(x,v)P_v(s < \infty, Z_s = y) \\ &= \sum_{v \in X} p(x,v)P_x(s < \infty, Z_s = y | Z_1 = v) \\ &= \sum_{v \in X} P_x(s < \infty, Z_s = y, Z_1 = v) \\ &= \nu_x(y) \end{aligned}$$

and hence $h = \int_{\partial x} g d\nu_x$ is actually a combination of harmonic functions with $h(y) = g(y)$ for $y \in \partial X$. □

Theorem 2.6. *Let (X, P) be a finite Markov chain, and denote its essential classes by $C_i, i \in I = \{1, \cdots, m\}$.*

a. If h is harmonic on X, then h is constant on each C_i.

b. For each function $g : I \to \mathbb{R}$ there is a unique function $h \in \mathcal{H}(X, P)$ such that for all $i \in I$ and $x \in C_i$ one has $h(x) = g(i)$.

Proof. a. We know for any $x \in C_i, x \to y$ iff $y \in C_i$ and then if $M_i = \max_{C_i} h = h(x), x \in C_i$, then for any $y \in C_i$, we know

$$h(x) = \sum_{y \in X} p^{(n)}(x, y)h(y) \le \sum_{v \in C_i, v \neq y} p^{(n)}(x, y)M + p^{(n)}(x, y)h(y)$$

for any $n, y \in C_i$ and we are done.

b. Let prove the uniqueness at first, if h, h' are harmonic functions on X, then assume $M = \max_X h$ and be obtained at $x \in X - X_{ess}$, then we know since $P_x(s < \infty)$ by corollary 1.3. where $s = s^{X_{ess}}$, then there will be an $y \in X_{ess}$ such that

$$M = h(x) \le p^{(n)}(x, y)h(y) + (1 - p^{(n)}(x, y))M$$

and hence the maximum has to be obtained at X^{ess} and the rest is easy to be checked.

Now we define $v_x(i) = P_x(s < \infty, Z_s \in C_i)$ which will be an harmonic function since

$$\sum_{y \in X} p(x, y)P_y(s < \infty, Z_s \in C_i) = v_x(i)$$

and it is easy to check that

$$h(x) = \sum_{i \in I} g(i)v_x(i)$$

will be a solution. □

3 Infinite cases

In the section we assume P is irreducible on X.

Lemma 3.1. *(Maximum principle) (Assume $|X| > 1$) If $h \in \mathcal{H}(X, P)$ and there is $x \in X$ such that $h(x) = M = \max_X h$, then h is constant, where P is substochatic. Furthermore, if $M \neq 0$ then P is stochastic.*

Proof. We still have

$$M \le \sum_{y \neq x'} p^{(n)}(x, y)M + p^{(n)}(x, x')h(x') \le (1 - p^{(n)}(x, x'))M + p^{(n)}(x, x')h(x')$$

and hence $h = M$, if $M \neq 0$. we know the equality has to be reached by P is stochastic. □

Lemma 3.2. *a. If $h \in \mathcal{S}^+$ then $P^n h \in \mathcal{S}^+$ for each n, and either $h = 0$ for $h > 0$.*

b. If $h_i, i \in I$ is a family of superharmonic functions and $h(x) = \inf_I h_i(x)$ defines a P-integrable function if I is finite or h_i is bounded below, then also h is superharmonic.

Proof. a. Firstly, the P-integrability of h implies that of Ph since

$$\sum_{y \in X} p(x, y)|Ph(y)| \le \sum_{y \in X, w \in X} p(x, y)|h(y)| < \infty$$

and by induction $P^n h \in \mathcal{S}^+$, and it is easy to check that $P^n h \le h$ by $f \ge g$ imples $Pf \ge Pg$, for each 0 and so if $h(x) = 0$ for some x, then h will be 0.

b. We know $Ph \le Ph_i \le h_i$ implies $Ph \le h$.

For the P-integrability, we may use the MCT for the first cases for h^- and Fatou for h^+. On the other case h^- is easier. □

Lemma 3.3. *If (X, P) is transient, then for each $y \in X$, the function $G(\cdot, y)$ is superharmonic and positive. There is at most one $y \in X$ for which $G(\cdot, y)$ is a constant function. If P is stochastic, then $G(\cdot, y)$ is non-constant for every y.*

Proof. We know

$$PG(x, y) = \sum_{w \in X} p(x, w)G(w, y) = G(x, y)$$

and

$$PG(y, y) = \sum_{w \in X} p(y, w)G(w, y) = G(y, y) - 1$$

and hence $G(\cdot, y) \in \mathcal{S}^+$. Suppose $y_1, y_2 \in X$ and $y_1 \neq y_2$ such that $G(\cdot, y_i)$ are constant, then

$$F(y_1, y_2) = G(y_1, y_2)/G(y_2, y_2) = 1, F(y_2, y_1) = 1$$

and then $F(y_1, y_1) \geq F(y_1, y_2)F(y_2, y_1) \geq 1 = 1$ and y_1 is recurrent, which is a contradiction.

If P is stochastic, since $G(\cdot, y)$ is strictly superharmonic and there will be a contradiction since constant function is harmonic. □

Lemma 3.4. *a. If $v \in E^+$ then $vP^n \in E^+$ for each n and either $v = 0$ or $v(x) > 0$ for every x.*

b. If $v_i, i \in I$ is a family of excessive measures, then also $v(x) = \inf_I v_i(x)$ is excessive.

c. If (X, P) is transient, then for each $x \in X$, the measure $G(x, \cdot)$ defined by $y \mapsto G(x, y)$ is excessive.

Proof. a. Here we know

$$vP^{(n)}(x) = \sum_{y \in X} p^{(n)}(y, x)v(y) \leq v(x)$$

and hence if $v(x) = 0$, then $v(y) = 0$ since (X, P) irreducible.

b. $vP \leq v_i P \leq v_i$.

c. We know

$$G(x, \cdot)P(y) = \sum_{w \in X} G(x, w)p(w, y) \leq G(x, y)$$

□

Lemma 3.5. *In the recurrent as well as in the transient case, for each $x \in X$, the measure $L(x, \cdot)$ defined by $y \mapsto L(x, y)$ is finite and excessive.*

Theorem 3.6. *(X, P) is recurrent iff every nonnegative superharmonic function is constant.*

Proof. (Here notice (X, P) is either transient or recurrent since it is irreducible).

a. Suppose that (X, P) is recurrent, we show that $\mathcal{S}^+ = \mathcal{H}^+$, let $h \in \mathcal{S}^+$, we have

$$g = h - Ph$$

is non-negative and P-integrable. We have

$$\sum_{k=0}^{n} P^k g = h - P^{n+1}(x)$$

If $g(y) > 0$ for some y, then

$$\sum_{k=0}^{n} p^{(k)}(x, y)g(y) \le \sum_{k=0}^{h} P^k g(x) \le h(x)$$

and then we have

$$G(y, y) \le h(y)/g(y) < \infty$$

which is a contradiction since y is recurrent. So $g = 0$ and hence h is harmonic.

Then consider for any $h \in \mathcal{S}^+ = \mathcal{H}^+$, let $x, y \in X$ and define $g(v) = \min_{h(v),h(x)}$, then we know

$$Pg(y) = \sum_{x \in X} p(y, x)g(x) \le Ph(y)$$

if $h(y) \le h(x)$ and the RHS is less than $h(x)$ since P is substochastic, so g is subharmonic and hence harmonic, then g should be constant and hence for any $y \ne x$ $h(y) \ge h(x)$ and then we know h is constant.

b. If (X, P) is transient, then since all the superharmonic functions are constant, then it has to be $|X| = 1$ which is a contradiction. □

Theorem 3.7. *Let (X, P) be substochastic and irreducible. Then (X, P) is recurrent iff there is a non-zero invariant measure ν such that each excessive measure is a multiple of ν. Then P must be stochastic.*

Proof. First, assume that P is recurrent. Then we know P must to be stochastic (since the constant functions are harmonic). We also know there us an excessive measure ν wuch that $\nu(y) > 0$ for all y. We consider ν-reveersal $\hat{P}$ of P and we know $\hat{P}$ is substochastic and $\hat{p}^{(n)}(x, y) = v(y)p^{(n)}(y, x)/v(x)$ and hence $\hat{P}$ is recurrent and hence stochastic, so ν must be invariant. If σ is any other excessive measure, we define $h(x) = \sigma(x)/\nu(x)$, then we know

$$\hat{P}h(y) = \sum_{x \in X} \hat{P}(x, y)\sigma(y)/\sigma(y) = \sum_{x \in X} p(y, x)\sigma(y)/\nu(x) \le h(y)$$

and we know h must be constant.

If (X, P) is transient, then $G(x, \cdot)$ is excessive but not invariant, which is contradiction. □

4 Induced Markov chains

Lemma 4.1. *If A is recurrent for (X, P) then*

$$P_x(t^A < \infty) = 1, \textit{ for all } x \in X$$

Proof. We know

$$P_x(t^A < \infty) = \sum_{y \in A} p(x, y) + \sum_{y \in X-A} p(x, y)P_y(t^A < \infty)$$

If we have $P_y(t^A < \infty) = 1$, then we know $h(x) = P_x(t^A < \infty)$ and hence to be a constant on (X, P). □

Lemma 4.2. $P^A = P_A + P_{A,X-A}G_{X-A}P_{X-A,A}$

Proof. Notice for $x, y \in A$, we have

$$p^A(x, y) = p(x, y) + \sum_{v \in X-A} p(x, v)P_v(t^A < \infty, Z_{t^A} = y)$$

and then

$$P_v(t^A < \infty, Z_{t^A} = y) = \sum_{w \in X-A} P_v(t^A < \infty, Z_{t^A-1} = \omega, Z_{t^A} = y)$$

$$\sum_{w \in X-A} \sum_{n \geq 1} P_v(t^A = n, Z_{n-1} = w, Z_n = y)$$

$$= \sum_{w \in X-A} G_{X-A}(v, w) p(w, y)$$

and we have

$$p^A(x, y) = p(x, y) + \sum_{v \in X-A} \sum_{w \in X-A} p(x, v) G_{X-A}(v, w) p(w, y)$$

□

Theorem 4.3. *If $A \subset B \subset X$, then $(P^B)^A = P^A$.*

Proof. We should give an interpretation of Z_n^B and define $w_N^B(\omega) = k$ if $n \leq v^B(\omega)$ and k is the instant of the n-th return visit to B, then $Z_n^B = Z_{w_n^B}$ if $n \leq v^B$.

Let t_B^A be the stopping time of the first visit of (Z_n^B) in A. Since $A \subset B$, we have for any $\omega \in \Omega$, $t^A(\omega) = \infty$ iff $t_B^A(\omega) = \infty$ and $t^A(\omega) \geq t^B(\omega)$. Hence, if $t^A(\omega) < \infty$, we know

$$Z_{t_B^A(\omega)^B}(\omega) = Z_{t^A(\omega)}(\omega)$$

so for $x, y \in A$, we have

$$(p^B)^A(x, y) = P_x(t_B^A < \infty, Z_{t_B^A}^B = y) = P_x(t^A < \infty, Z_{t^A} = y) = p^A(x, y).$$

by consider ω. □

Theorem 4.4. *Let $v \in E^+(X, P)$, $A \subset X$ and v_A the restriction of v to A. Then $v_A \in E^+(A, P^A)$.*

Proof. For $x \in A$, then

$$v_A(x) = v(x) \geq vP(x) = v_A P_A(x) + v_{X-A} P_{X-A,A}(x)$$

and hence

$$v_A \geq v_A P_A + v_{X-A} P_{X_A,A}$$

and similarly

$$v_{X-A} \geq v_{X-A} P_{X-A} + v_A P_{A,X-A}$$

and multiply $\sum_{k=0}^{n-1} P_{X-A}^k$ to RHS and we obtain

$$v_{X-A} \sum_{k=0}^{n-1} P_{X-A}^k \geq v_{X-A} P_{X-A}^n + v_A P_{A,X-A} (\sum_{k=0}^{n-1} P_{X-A}^k)$$

and hence

$$v_{X-A} \geq v_A P_{A,X-A} (\sum_{k=0}^{n-1} P_{X-A}^k)$$

for every $n \geq 1$. And we know

$$v_A P_{A,X-A} (\sum_{k=0}^{n-1} P_{X-A}^k) \to v_A P_{A,X-A} G(X - A)$$

since $I/(I - P_{X-A}) = G(X - A)$ and then

$$v_A \geq v_A P_A + v_A P_{A,X-A} G(X - A) P_{X-A,A} = v_A P^A$$

$\square$

5 Potentials, Riesz decomposition

For the rest part, we assume (X, P) is irreducible and transient, which means

$$0 < G(x, y) < \infty$$

for all $x, y \in X$.

Lemma 5.1. *a. If g is the potential of f, then $f = (I - P)g$. Furthermore, $P^n g \to 0$ pointwise.*

b. If f is non-negative, then $g = Gf \in \mathcal{S}^+$ and g is harmonic on $X - supp(f)$ that is $Pg(x) = g(x)$ for every $x \in X - supp(f)$.

Proof. a. Suppose that $f \geq 0$ firstly, then we know

$$PGf(x) = \sum_{y \in X} p(x, y) \sum_{w \in X} G(w, y) f(y) = GPf = \sum_{n \geq 1} P^n f = Gf - f$$

since

$$Gf = \sum_{y \in X} \sum_{n \geq 0} P^{(n)}(x, y) f(y) = \sum_{n \geq 0} P^n f$$

by MCT. And hence Gf is superharmonic and harmonic on $X - supp(f)$. Then notice

$$P^n g(x) = GP^n f(x) = \sum_{k=n}^{\infty} f(x)$$

has to be convergent to 0. For general f, decompose it as f^+ and f^- will be fine. $\square$

Theorem 5.2. *(Riesz decomposition theorem) If $u \in \mathcal{S}^+$ then there are a potential $g \in Gf$ and a function $h \in \mathcal{H}^+$ such that*

$$u = Gf + h$$

The decomposition is unique.

Proof. Since $u \geq 0$ and $u \geq u$, for every $x \in X$ and every $n \geq 0$, we know

$$P^n u(x) \geq P^{n+1} u(x) \geq 0$$

Therefore, there is the limit function

$$h(x) = \lim_{n \to \infty} P^n u(x)$$

where

$$Ph(x) = P(\lim_{n \to \infty} P^n u)(x) = \lim_{n \to \infty} P^{n+1} u(x) = h(x)$$

by DCT since u is P-integrable. Then let $f = u - Pu$ and then we know

$$u - h = Gf$$

Then let us prove the uniqueness, we consider $u = g_1 + h_1$ another decomposition, then $P^n = P^n g_1 + h_1$ and then we know $P^n u \to h_1$ since $P^n g_1 \to 0$ and we are done. $\square$

Corollary. *a. If g is a non-negative potential then the only funciton $h \in \mathcal{H}^+$ with $g \geq h$ is $h = 0$.*

b. If $u \in \mathcal{S}^+$ and there is a potential $g = Gf$ with $g \leq u$, then u is the potential of a non-negative function.

Proof. a. $h = P^n h \leq P^n g \to 0$ pointwise.

b. Trivial. □

Theorem 5.3. *(Approximation theorem) If $h \in \mathcal{S}^+(X, P)$ then there is a sequence of potentials $g_n = Gf_n, f_n \geq 0$ such that $g_n(x) \leq g_{n+1}(x)$ for x and n, and*

$$\lim_{n\to\infty} g_n(x) = h(x)$$

Notice here we do not use that h is G-integrable.

Proof. Define

$$R^A[h](x) = \inf\{u(x), u \in \mathcal{S}^+, u(y) \geq h(y) \text{ for all } y \in A\}$$

and $R^A[h] \leq h$. In particular, we have

$$R^A[h](x) = h(x)$$

for $x \in A$. And by lemma 2.3. we know $R^A[h](x) \in \mathcal{S}^+$. Let A be a finite subset X. Let $f_0(x) = h(x)$ if $x \in A$ and $f_0(x) = 0$. f_0 is non-negative and finitely supported. Then Gf_0 exists and finite on X, with $Gf_0 \geq f_0$. So Gf_0 is a superharmonic function since $PGf_0 = GPf_0 \leq Gf_0$ and with $Gf_0 \geq h$ on A. So we know $R^A[h] \leq Gf_0$.

So we know $R^A[h]$ has to be a potential and then let B be another finite subset of X containing A. Then $R^B[h] \geq R^A[h]$. Let A_n be an increasing sequence of finite subsets of X such that $X = \bigcup_n A_n$ and let $g_n = R^{A_n}[h]$ then we know $g_n \leq h$ but $g_n = h$ on A_n. □

6 Domination principle

Proposition 6.1. *a. We have*

$$\hat{L}^A(x, y) = \frac{v(y)F^A(y, x)}{v(x)}, \quad \hat{F}^A(x, y) = \frac{v(y)L^A(y, x)}{v(x)}$$

b. $x \in A \implies F^A(x, \cdot) = \delta_x, y \in A \implies L^A(\cdot, y) = 1_y$.

Proof. a. We have

$$\begin{aligned}
\hat{L}^A(x, y) &= \sum_{n\geq 0} \sum \hat{P}_x(Z_n = y, Z_j = x_j, 0 \leq j < n)\chi_A(x) \\
&= \sum_{n\geq 0} \sum v(y)p(y, \cdot) \cdots p(\cdot, x)/v(x) \\
&= v(y) \sum_{n\geq 0} P_y(Z_n = x, Z_j \notin A)\chi_A(x)/v(x) \\
&= v(y)F^A(y, x)/v(x)
\end{aligned}$$

and the rest is similar.

b. $x \in A$, then $F^A(x, y) = P_x(Z_0 = y)$. And the other one is similar. □

Lemma 6.2. *a.* $G = G_{X-A} + F^A G$.

b. $G = G_{X-A} + GL^A$.

c. $F^A G = GL^A = G - G_{X-A}$.

Proof. We know

$$p^{(n)}(x, y) = P_x(Z_n = y, s^A > n) + P_x(Z_n = y, s^A \leq n)$$
$$= p^{(n)}_{X-A}(x, y) + \sum_{v \in A} \sum_{k=0}^{n} P_x(Z_n = y, s^A = k, Z_k = v)$$
$$= p^{(n)}_{X-A}(x, y) + \sum_{v \in A} \sum_{k=0} P_x(s^A = k, Z_k = v) p^{(n-k)}(v, y)$$

then we have

$$G(x, y) = G_{X-A}(x, y) = \sum_{v \in A} (\sum_{k=0}^{\infty} P_x(s^A = k, Z_k = v))(\sum_{n=0}^{\infty} p^{(n)}(v, y))$$

and hence

$$G(x, y) = G_{X-A}(x, y) + \sum_{v \in X} F^A(x, v) G(v, y)$$

The rest is to enumerate the last time of visiting A. □

Lemma 6.3. $P^A = P_{A,X} F^A = L^A P_{X,A}$.

Proof. We know

$$p^A(x, y) = p(x, y) + \sum_{v \in X-A} p(x, v) P_v(s^A < |infty, Z_{s^A} = y)$$
$$= \sum_{v \in A} p(x, v) \delta_v(y) + \sum_{v \in X-A} p(x, v) F^A(v, y)$$
$$= \sum_{v \in X} p(x, v) F^A(v, y)$$

Then let $v = 1$ and we have

$$p^A(x, y) = \hat{p}(y, x) = \sum_{v \in X} \hat{p}(y, v) \hat{F}(v, x) = \sum_{v \in X} L(x, v) p(v, y)$$

and we are done.(Ensured by proposition 2.1. c) □

Lemma 6.4. *a. If $h \in \mathcal{S}^+(X, P)$, then $F^A h(x) = \sum_{y \in A} F^A(x, y) h(y)$ if finite and*

$$F^A h(x) \leq h(x)$$

b. If $v \in E^+(X, P)$, then $vL^A(y) = \sum_{x \in A} v(x) L^A(x, y)$ is finite and

$$vL^A(y) \leq v(y)$$

Proof. By approximation theorem, we may find $g_n = Gf_n$ such that $g_n \uparrow h$ on X. The f_n can be chosen to have finite support. So

$$F^A g_n = F^A G f_n = G f_n - G_{X-A} f_n \leq g_n \leq h$$

and hence $F^A h \leq h$ by MCT.

For the other conclusion, we know

$$vL^A(y) = \sum_{x \in A} v(x) L^A(x, y) = \sum_{x \in A} \hat{F}^A(y, x) v(y) \leq v(y)$$

$\square$

Theorem 6.5. *a. If $h \in \mathcal{S}^+$ then $R^A[h] = F^A h$. In particular, $R^A[h]$ is harmonic in every point of $X - A$ while $R^A[h] = h$ on A.*

b. If $v \in E^+$ then $R^A[v] = vL^A$. In particular, $R^A[v]$ is invariant in every point of $X - A$ while $R^A[v] = v$ on A.

Proof. a. For $x \in X - A$ and $y \in A$, we factorize and then

$$F^A(x, y) = p(x, y) + \sum_{v \in X-A} p(x, v)F^A(v, y) = \sum_{v \in X} p(x, v)F^A(v, y)$$

then

$$F^A h(x) = \sum_{y \in A} F^A(x, y)h(y) = \sum_{v \in X, y \in X} p(x, v)F^A(v, y)h(y) = P(F^A h)(x)$$

then for $x \in A$

$$P(F^A h)(x) = \sum PF^A(x, y)h(y) = P^A h(x) \leq h(x)$$

and it is easy to check $F^A h = h$ on A. sp we know $F^A \in \{u \in \mathcal{S}^+, u \geq h, y \in A\}$ then $R^A[h] \leq F^A h$. Then for $u \in Sar^+$ and $u \geq h$ on A, we know

$$u(x) \geq \sum_{y \in A} F^A(x, y)u(y) \geq F^A h(x)$$

and we are done.

b. For $x \in X$ we have $L^A(x, y) = 0$ and then

$$vL^A P(y) = \sum_{x \in A, w \in A} v(x)L^A(x, w)P(w, y) = \sum_{x \in A} v(x)L^A P(x, y) = vP^A \leq v(y)$$

for $y \in A$ and for $x \in X - A$, we have

$$vL^A P(x) = \sum_{y \in A, w \in A} v(y)L^A(y, w)P(w, x) = 0 = vL^A(x)$$

and then since $vL^A(y) = v(y)$ for all $y \in A$, so we are done.

$\square$

Theorem 6.6. *(Domination Principle) Let f be a non-negative, G-integrable function on X with support A. If $h \in \mathcal{S}^+$ is such that $h(x) \geq Gf(x)$ for every $x \in A$, then $h \geq Gf$ on the whole of X.*

Proof. We know

$$h(x) \geq F^A h(x) \geq \sum_{y \in A} F^A(x, y)Gf(y) = F^A Gf(x) = Gf^A(x) = Gf(x)$$

for every x since $f^A = f$. $\square$