Chapter 1

1.1 Stochastic Processes

We always assume the probability measure P is complete.
p.s. short for probability space.
r.v. short for random variables.

Definition 1.1

A $filtration$ on a p.s. $(\Omega, \mathcal{F}, P)$ is a collection of σ-fields $\{\mathcal{F}_t : t \in \mathbb{R}_+ = [0, \infty)\}$ such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset F \text{ for all } 0 \leq s < t < \infty$$

and we may define

$$\mathcal{F}_\infty = \sigma(\bigcup_{0 \leq t < \infty} \mathcal{F}_t)$$

naturally.

♣

Definition 1.2

To make $\mathcal{F}_t$ contains all $\mathcal{F}$-measurable P-null sets. We may complete $(\Omega, \mathcal{F}, P)$ and replace $\mathcal{F}_t$ with

$$\bar{\mathcal{F}}_t = \{B \in \mathcal{F}, \exists A \in \mathcal{F}_t \ s.t. \ P(A \triangle B) = 0\}$$

The filtration $\{\bar{\mathcal{F}}_t\}$ is called complete or augmented filtration.

♣

Proof

Actually, there is something we need to check.
Firstly, it is obviously $\mathcal{F}_t$ is a filtraiton in $(\Omega, \bar{\mathcal{F}}, P)$ where $\bar{F}$ is the complete σ-algebra of $\mathcal{F}$.
Then obviously $\bar{\mathcal{F}}_t$ contains $\mathcal{F}_t$ and will become a σ-algebra since for any $B \in \bar{\mathcal{F}}_t$, assume $A \in \mathcal{F}_t$ and $P(A \triangle B) = 0$

$$P(A^c \triangle B^c) = P(A \triangle B) = 0$$

and for $\{B_i\} \subset \bar{\mathcal{F}}_t$ with corresponding $\{A_i\} \in F_t$ we know

$$P((\bigcup_i B_i) \triangle (\bigcup_i A_i)) = P((\bigcup_i B_i) \cap (\bigcap_i A_i^c)) + P((\bigcap_i B_i^c) \cap (\bigcup_i A_i)) = 0$$

and hence $\bar{\mathcal{F}}_t$ is a σ-algebra, which obviously contains all the $\mathcal{F}$-measurable P-null sets and is a filtration.

Definition 1.3

A $stochastic\ processes$ is a collection of r.v.s $\{X_i, i \in I\}$ where I is most often $\mathbb{R}_+$ or its subset. If X_t take values in a space S, we call X is an S-valued process, where we treat S as a metric space so that there is $\mathcal{B}_S$ on it and S is most often $\mathbb{R}^n$.

♣

Definition 1.4

A process $X = \{X_t, t \in \mathbb{R}_+\}$ is adapted to the filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable for each $0 \leq t < \infty$.

♣

Proposition 1.1

The smallest filtration to which X is adapted is

$$F_t^X = \sigma\{X_s, 0 \leq s \leq t\}$$

♠

Proof

It is not hard to check F_t^X satisfy the requirement and it is easy to check X_s are $\mathcal{F}_t$-measurable if $s \leq t$, then we are done.

Definition 1.5

We call a process X is $measurable$ if $X : \mathbb{R}^+ \times \Omega \to S$ is $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$-measurable. We call X is $progressively\ measurable$ if the restriction of the function X to $[0,T]$ is $\mathcal{B}_{[0,T]} \otimes \mathcal{F}_T$-measurable for each T. ♣

Proposition 1.2

If X is progressively measurable then it is also adapted. ♠

Proof

Consider $s \le T$, then we know $X_s^{-1}(A) = X^{-1}(A)_s$ which is measurable for any $A \in \mathcal{B}_S$. Please refer to Folland's Real Analysis for the notation.

Definition 1.6

We call two stochastic processes X, Y are $indistinguishable$ if there exists an event Ω_0 such that $P(\Omega_0) = 1$ and $X_t = Y_t$ on Ω_0 for all $t \in \mathbb{R}_+$.
We call Y is a $modification$ or $version$ of X if $X_t = Y_t$ a.s. for each t. ♣

Definition 1.7

$Equality\ in\ distribution$ of processes X, Y means that $P(\{X \in A\}) = P(\{Y \in A\})$ for all measurable sets, which follows from the weaker equality of finite-dimensional distributions, i.e.

$$P(\{X_{t_1} \in B_1, X_{t_2} \in B_2, \cdots, X_{t_n} \in B_n\}) = P(\{Y_{t_1} \in B_1, Y_{t_2} \in B_2, \cdots, Y_{t_n} \in B_n\})$$

for all finite subsets $t_1, t_2, \cdots, t_n$. ♣

Proof

Recall what is a λ-system, we require a λ-system closed under complement and disjoint countable unions, and containing Ω. So it is easy to use π-λ theorem to show that $P(\{X \in A\}) = P(\{Y \in A\})$ for all sets in $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}$.

Proposition 1.3

A complete filtration may imply that Y is adapted if X is adapted and $X_t = Y_t$ a.s. ♠

Definition 1.8

A stopping time is a r.v. $\tau : \Omega \to [0,\infty]$ such that $\{\tau \le t\} \in \mathcal{F}_t$ for each $0 \le t < \infty$ for a filtration $\mathcal{F}_t$. ♣

Proposition 1.4

If σ and τ are stopping times for the same filtration, then so are $\min\{\sigma,\tau\}$ and $\max\{\sigma,\tau\}$. ♠

Definition 1.9

If τ is a stopping time, the σ-field of events known at time τ is defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F}, A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } 0 \le t < \infty\}$$ ♣

It is easy to check $\mathcal{F}_\tau$ is a σ-algebra and actually a deterministic time is a special case of a stopping time and $\mathcal{F}_\tau = \mathcal{F}_u$ if $\tau = u$ on Ω.

Definition 1.10

If $\{X_t\}$ is a process and τ is a stoppping time, X_τ denotes the value of the process at the random time τ,i.e. $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, where X_τ is defined on $\{\tau < \infty\}$. ♣

Lemma 1.1

Let σ and τ be stopping times, and X a process.
(i) For $A \in \mathcal{F}_\sigma$, the events $A \cap \{\sigma \leq \tau\}, A \cap \{\sigma < \tau\} \in \mathcal{F}_\tau$. In particular, $\sigma \leq \tau \implies \mathcal{F}_\sigma \subset \mathcal{F}_\tau$.
(ii) Both τ and $\min\{\sigma, \tau\}$ are $\mathcal{F}_\tau$-measurable. the events $\{\sigma \leq \tau\}, \{\sigma < \tau\}, \{\sigma = \tau\}$ lie in both $\mathcal{F}_\sigma$ and $\mathcal{F}_\tau$.
(iii) If the process X is progressively measurable then $X(\tau)$ is $\mathcal{F}_\tau$-measurable on the event $\{\tau < \infty\}$. ♡

Proof

a. Consider

$$A \cap \{\sigma \leq \tau\} \cap \{\tau \leq t\} = (A \cap \{\sigma \leq t\}) \cap \{\min\{\sigma, t\} \leq \min\{\tau, t\}\} \cap \{\tau \leq t\}$$

and notice $\min\{\sigma, t\}$ is $\mathcal{F}_t$-measurable, so we know

$$A \cap \{\sigma \leq \tau\} \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{R}_+$$

and notice $A \cap \{\sigma < \tau\} = \bigcup_{n \geq 1} (A \cap \{\sigma \leq \tau - n^{-1}\})$.

b. τ is obviously $\mathcal{F}_\tau$-measurable. Then we know $\min\{\sigma, \tau\}$ is $\mathcal{F}_{\min\{\sigma,\tau\}}$-measurable and by (a) we know it is $\mathcal{F}_\tau$-measurable. We only need to prove that $\{\sigma \leq \tau\}$ in both $\mathcal{F}_\sigma$ and $\mathcal{F}_\tau$, this can be implied from that

$$\{\sigma \leq \tau\} = \{\min\{\sigma, \tau\} = \sigma\}$$

which is $\mathcal{F}_\tau$-measurable and $\mathcal{F}_\tau$-measurable by (a).

c. We may consider $\{X_\tau \in B, \tau \leq t\}$ at first, and

$$\{X_\tau \in B, \tau \leq t\} = \{X_{\min\{\tau,t\}} \in B\} \cap \{\tau \leq t\}$$

which may encourage us consider

$$\omega \mapsto (\min\{t, \tau(\omega)\}, \omega) \mapsto X_{\min\{t,\tau\}}(\omega)$$

the preimage of the first map of rectangle is always $\mathcal{F}_t$-measurable and the second is true for X is $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$-measurable and hence $\{X_{\min\{t,\tau\}} \in B\}$ is always in $\mathcal{F}_t$ for any B Borel. Then we know $\{X_\tau \in B, \tau < \infty\}$ is in $\mathcal{F}_\tau$. notice where X_τ is only defined when $\tau < \infty$.

Definition 1.11

A stochastic process X is continuous if the path $t \mapsto X_t(\omega)$ is continuous for any $\omega \in \Omega$, and we may define $left$-$continuous$ and $right$-$continuous$ analogously.
Call an $\mathbb{R}^d$-valued process X is cadlag if it is right continuous with left limits and caglad if it is left continuous with right limits. ♣

Definition 1.12

X is a $finite\ variation\ process$ if the path $t \mapsto X_t(\omega)$ has bounded varation on each compact interval $[0, T]$ for any $\omega \in \Omega$. ♣

Lemma 1.2

Let X be adapted to the filtration $\{\mathcal{F}_t\}$ and suppose X is either left-continuous or right-continuous. Then X is progressively measurable. Recall when we say X is left-continuous or right-continuous, it is $\mathbb{R}^d$-valued. ♡

Proof

We assume X is right-continuous firstly and may consider

$$X_n(t, \omega) = X(0, \omega) \cdot \chi_{\{0\}}(t) + \sum_{k=0}^{2^n - 1} X(\frac{(k+1)T}{2^n}, \omega) \cdot \chi_{(kT2^{-n},(k+1)T2^{-n})}(t)$$

which is $\mathcal{B}_{[0,T]} \otimes \mathcal{F}_T$-measurable functions and $X_n \to X|_{[0,T]}$ since X is right-continuous, and hence $X|_{[0,T]} = \liminf X_n$ is $\mathcal{B}_{[0,T]} \otimes \mathcal{F}_T$-measurable. The case of left-continuity is similar.

Lemma 1.3

Suppose X and Y are right-continuous processes defined on the same probability space. Suppose $X_t = Y_t$ a.s. for all t in some dense countable subset of $\mathbb{R}_+$. Then X and Y are indistinguishable. The same conclusion holds under the assumption of left-continuity if 0 is in the dense subset above.

Proof

Denote the dense set as S, and let $X_t = Y_t$ on $\Omega_t \subset \Omega$, then we know $X_s = Y_s$ on $\bigcap_{t \in S} \Omega_t = \Omega_S$ for all $s \in S$. Then by the right-continuity of X, Y and hence $X_s = Y_s$ for any $s \in \Omega_S$. The case of the left-continuity is similar.

Definition 1.13

We may define the σ-fields

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$$

where $\mathcal{F}_{t+}$ is a new filtration and $\mathcal{F}_t \subset \mathcal{F}_{t+}$.
We call $\mathcal{F}_t$ is $right\text{-}continuous$ if $\mathcal{F}_t = \mathcal{F}_{t+}$. We may know that $\mathcal{F}_{t++} = \mathcal{F}_{t+}$.

Proof

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s \subset \bigcap_{s>t} \mathcal{F}_{s+} = \bigcap_{s>t, s'>s} \mathcal{F}_{s'} = \mathcal{F}_{t++}$$

and hence $\mathcal{F}_{t+} = \mathcal{F}_{t++}$.

Definition 1.14

Define

$$\mathcal{F}_{t-} = \sigma\Big(\bigcup_{s<t} \mathcal{F}_s\Big)$$

and $\mathcal{F}_{0-} = \mathcal{F}_0$, then call $\mathcal{F}_t$ is $left\text{-}continuous$ if $\mathcal{F}_t = \mathcal{F}_{t-}$ for all $t \in \mathbb{R}_+$.

Definition 1.15

We call $\mathcal{F}_t$ satisfies the $usual\ conditions$ if $\mathcal{F}_t$ is both complete and right-continuous.

Lemma 1.4

A $[0, \infty]$-valued r.v. τ is a stopping time with respect to $\mathcal{F}_{t+}$ iff $\{\tau < t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

Proof

To show the sufficiency, we know that $\{\tau \le t - n^{-1}\} \in \mathcal{F}_{(t-n^{-1})_+} \subset \mathcal{F}_t$ for all integer $n \ge 1$. And hence $\{\tau < t\} = \bigcup_{n \ge 1} \{\tau \le t - n^{-1}\} \in \mathcal{F}_t$.

To show the necessity, we know

$$\{\tau \le t\} = \bigcap_{s<t} \{\tau < s\} \in \mathcal{F}_{t+}$$

which is the required conclusion.

Definition 1.16

Given a set H, define

$$\tau_H(\omega) = \inf\{t \ge 0, X_t(\omega) \in H\}$$

which is the $hitting\ time$ of the set H. If the infimum is taken over $t > 0$, then call the above time the $first\ entry\ time$ into the set H.

Lemma 1.5

Let X be a process adapted to a filtration $\{\mathcal{F}_t\}$ and assume X is left- or right-continuous. If G is an open set, then τ_G is a stopping time with respect to $\{\mathcal{F}_{t_+}\}$. In particular, if $\{\mathcal{F}_t\}$ is right-continuous, τ_G is a stopping time with respect to $\{\mathcal{F}_t\}$.

Proof

We notice that

$$\{\tau_G < t\} = \{\omega, \inf\{s \geq 0, X_s(\omega) \in G\} < t\} = \{\omega, \exists q \in \mathbb{Q} \cap [0,t) \ s.t. \ X_q(\omega) \in G\} = \bigcup_{q \in \mathbb{Q} \cap [0,t)} X_q^{-1}(G) \in \mathcal{F}_t$$

Definition 1.17

For a process X, let $X_{[s,t]} = \{X(u), s \leq u \leq t\}$ with topological closure $\overline{X_{[s,t]}}$. For a set H define

$$\sigma_H = \inf\{t \geq 0 : \overline{X_{[0,t]}} \cap H \neq \emptyset\}$$

Lemma 1.6

Suppose X is a cadlag process adapted to $\{\mathcal{F}_t\}$ and H is a closed set. Then σ_H is a stopping time.

Proof

Firstly, we know

$$\overline{X_{[0,t]}(\omega)} = \overline{\{X_s(\omega), 0 \leq s \leq t\}} \supset \{X_s(\omega), 0 \leq s \leq t\} \cup \{X_{s-}(\omega), 0 < s \leq t\}$$

and for any $y \in \overline{X_{[0,t]}(\omega)}$, there exists $\{t_i\} \subset [0,t]$ such that $X_{t_i}(\omega) \to y$, and we only need to consider a convergent subsequence of $\{t_i\}$ is fine, then we know

$$\overline{X_{[0,t]}} = \{X_s, 0 \leq s \leq t\} \cup \{X_{s-}, 0 < s \leq t\}$$

then we consider

$$\{\sigma_H \leq t\} = \{\omega, \forall \epsilon > 0, \exists s < t + \epsilon \ s.t. \ X_s(\omega) \in H \text{ or } X_{s-}(\omega) \in H\} \supset \{X_0 \in H\} \cup \{X_s \in H \text{ or } X_{s-} \in H\}$$

and those may be not in the right set, we know there exists $s_i \downarrow t$ such that $X_{s_i}(\omega) \in H$ or $X_{s_i-}(\omega) \in H$, and we may find $s_i' < s_i$ such that $d(X_{s_i'}(\omega), X_{s_i-}(\omega)) < i^{-1}$ and we know $X_{s_i'}(\omega) \to X_t(\omega)$ and hence $X_{s_i-} \to X_t$, which means X_t has to be in H. So $\{\sigma_H \leq t\} = \{X_0 \in H\} \cup \{X_s \in H \text{ or } X_{s-} \in H\}$.

Then we claim that

$$\{\sigma_H \leq t\} = \bigcap_{n=1}^{\infty} \bigcup_{q \in \mathbb{Q} \cap [0,t] \cup \{t\}} \{X_q \in H_n\}$$

where $H_n = \{y, \exists x \in H \ s.t. \ d(x,y) < n^{-1}\}$. If $X_{s-}(\omega) \in H$, then we know there exists $s_i \uparrow s$ such that $X_{s_i}(\omega) \to X_{s-}(\omega)$ and then we know for any $n \in \mathbb{N}$, there exists $q \in \mathbb{Q} \cap [0,t]$ such that $X_q \in H_n$ and hence $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{q \in U} \{X_q \in H_n\}$. For any $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{q \in U} \{X_q \in H_n\}$, we know there exists $q_i \in \mathbb{Q} \cap [0,t] \cup \{t\}$ such that $X_{q_i}(\omega) \in H_i$ and it is easy to check $\omega \in \{\sigma_H \leq t\}$.

Corollary 1.1

Assume X is continuous and H is closed. Then τ_H is a stopping time.

Proof

We know $\sigma_H = \inf\{t \geq 0, X_{[0,t]} \cap H \neq \emptyset\} = \tau_H$ and hence the conclusion goes.

Theorem 1.1

Assume the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions, and X is a progressively measurable process with values in some metric space. Then τ_H or the infimum restricted to $t > 0$ are stopping times for every Borel set H.

Quadratic variation

Definition 1.18

The quadratic variation process $[Y] = \{[Y]_t : t \in \mathbb{R}_+\}$ *of a stochastic process* Y *is a process such that* $[Y]_0 = 0$, *the paths* $t \mapsto [Y]_t(\omega)$ *are nondecreasing for all* ω *and*

$$\lim_{|\pi|\to 0} \sum_{i\geq 0} (Y_{t_{i+1}} - Y_{t_i})^2 = [Y]_t \text{ in probability}$$

for all $t \geq 0$ *where* $|\pi|$ *is a partition of* $[0, t]$.

♣

Definition 1.19

Let X *and* Y *be two stochastic processes on the same p.s. The covariation process* $[X, Y] = \{[X, Y]_t, t \geq 0\}$ *where*

$$[X, Y] = [\frac{1}{2}(X + Y)] - [\frac{1}{2}(X - Y)]$$

i.e.

$$[X, Y]_t = \lim_{|\pi|\to 0} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \text{ in probability}$$

and also we have

$$[X, Y]_t = \frac{1}{2}([X + Y]_t - [X]_t - [Y]_t) a.s.$$
$$[X, Y]_t = \frac{1}{2}([X]_t + [Y]_t - [X - Y]_t) a.s.$$

♣

Definition 1.20

For any cadlag process Z, *the jump at* t *is denoted by*

$$\Delta Z(t) = Z(t) - Z(t-)$$

♣

Proposition 1.5

Suppose X *and* Y *are cadlag processes, and* $[X, Y]$ *exists as* $\frac{1}{2}([X]_t + [Y]_t - [X - Y]_t)$ *a.s.*

♠

Proof If we know $\Delta[X]_t = (\Delta X_t)^2$ a.s., then

$$\Delta[X, Y]_t = \frac{1}{2}(\Delta[X]_t + \Delta[Y]_t - \Delta[X - Y]_t) = (\Delta X_t)(\Delta Y_t) \text{ a.s.}$$

so it suffices to treat the case $X = Y$.

Pick $\delta, \epsilon > 0$, $t < u$ and $\eta > 0$ so that

$$P(|[X]_u - [X]_t - \sum_{i=0}^{m(\pi)-1} (X_{t_{i+1} - X_{t_i}})^2| < \epsilon) > 1 - \delta$$

for any partition of $[t, u]$ with $|\pi| < \eta$. Keep t_1 fixed, then refine π so that

$$P(|[X]_u - [X]_{t_1} - \sum_{i=1}^{m(\pi)-1} (X_{t_{i+1}} - X_{t_i})^2| < \epsilon) > 1 - \delta$$

and we have $1 - 2\delta$ for the intersection where

$$[X]_u - [X]_t \leq \sum_{i=0}^{m(\pi)-1} (X_{t_{i+1} - X_{t_i}})^2 \epsilon \leq (X_{t_1} - X_t)^2 + [X]_u - [X]_{t_1} + 2\epsilon$$

and hence

$$[X]_{t_1} \leq [X]_t + (X_{t_1} - X_t)^2 + 2\epsilon$$

which means $P([X]_t \leq [X]_{t+} \leq [X]_t + 3\epsilon) > 1 - 3\delta$ for any $\epsilon, \delta > 0$ and hence $[X]_{t+}$ a.s.

Similarly, we have

$$P(\Delta[X]_u \leq (\Delta X_u)^2 + 3\epsilon) > 1 - 3\delta$$

for any $\epsilon, \delta > 0$ and also

$$\Delta[X]_u \geq (\Delta X_u)^2 - 3\epsilon$$

with probability $\geq 1 - 3\delta$ and hence $\Delta[X]_u = (\Delta X_u)^2$ a.s.

Definition 1.21

An increasing process $A = \{A_t, 0 \leq t < \infty\}$ is an adapted process such that, for almost every ω, $A_0(\omega) = 0$ and $s \mapsto A_s(\omega)$ is nondecreasing and right-continuous, which is automatically cadlag. ♣

Lemma 1.7

Suppose the processes below exist. Then at a fixed t,

$$|[X,Y]_t| \leq [X]_t^{1/2}[Y]_t^{1/2} \quad a.s.$$

and more generally for $0 \leq s < t$

$$|[X,Y]_t - [X,Y]_s| \leq ([X]_t - [X]_s)^{1/2}([Y]_t - [Y]_s)^{1/2} \quad a.s.$$

furthermore,

$$|[X]_t - [Y]_t| \leq [X-Y]_t + 2[X-Y]_t^{1/2}[Y]_t^{1/2} \quad a.s.$$

In the cadlag case the inequalities are valid simultaneously at all $s < t \in \mathbb{R}^+$ with probability 1. ♡

Proof We know for any π a partition of $[0,t]$,

$$(\sum_{i=0}^{m(\pi)-1}(X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}))^2 \leq (\sum_{i=0}^{m(\pi)-1}(X_{t_{i+1}} - X_{t_i})^2)(\sum_{i=0}^{m(\pi)-1}(Y_{t_{i+1}} - Y_{t_i})^2)$$

then we have for any $\epsilon, \delta > 0$, there exists $\eta > 0$ for any $|\pi| < \delta$, the three distinctions are less than ϵ with probability $1 - \delta$, then we know

$$2\epsilon[X,Y]_t + [X,Y]_t^2 \leq [X]_t[Y]_t + \epsilon([X]_t + [Y]_t)$$