Potential theory for random matrices

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Lecture 1: exercises

Exercise 1.1. Prove that

$$|x - y| \leq \sqrt{|x|^2 + 1}\sqrt{|y|^2 + 1}$$

holds for every $x, y \in \mathbb{C}$. This equality was used in the proof of the existence of the equilibrium measure.

Exercise 1.2. Let $\Sigma \subset \mathbb{R}$ be a closed interval and V an admissible external field on Σ with equilibrium measure μ_V.

(a) Suppose V is a convex on Σ. Then prove that the support of μ_V is an interval.

(b) Suppose $\Sigma = [0, \infty)$, V is differentiable on $(0, \infty)$ and $x \mapsto xV'(x)$ is increasing on $(0, \infty)$. Prove that the support of μ_V is an interval containing 0.

Exercise 1.3. Let $V : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be an admissible external field on $\mathbb{R}$ that is even, i.e., $V(-x) = V(x)$.

(a) Show that the equilibrium measure μ_V is also even.

(b) Let μ_V^* be the pushforward of μ_V under the quadratic map $x \mapsto x^2$. That is, μ_V^* is the probability measure on $[0, \infty)$ satisfying $\int f d\mu_V^* = \int f(x^2) d\mu_V(x)$ for every bounded continuous function f on $[0, \infty)$.

Show that μ_V^* is the equilibrium measure in the external field $x \mapsto 2V(\sqrt{x})$ on $[0, \infty)$.

(c) Suppose that $x \mapsto V(\sqrt{x})$ is convex. Show that the support of μ_V is either an interval $[-a, a]$ with $a > 0$, or a union of two intervals $[-b, -a] \cup [a, b]$ with $0 < a < b$.

There are situations that give rise to equilibrium problems with a constraint. Here a constraint is a measure σ on a closed set Σ with $\int d\sigma > 1$. The set

$$\mathcal{P}^\sigma = \{\mu \in \mathcal{P}(\Sigma) \mid \mu \leq \sigma\}.$$

contains all probability measures on Σ that are constrained by σ. The inequality $\mu \leq \sigma$ means that $\mu(B) \leq \sigma(B)$ for all Borel subsets B of Σ.

In a constrained equilibrium problem one looks for the minimizer of $I_V(\mu) = I(\mu) + \int V d\mu$ within $\mathcal{P}^\sigma(\Sigma)$.

Exercise 1.4. Let $\Sigma \subset \mathbb{C}$ be closed and let $V : \Sigma \to \mathbb{R} \cup \{+\infty\}$ be admissible. Let σ be a measure with $\int d\sigma > 1$ (it could be $+\infty$) and $\mathrm{supp}(\sigma) = \Sigma$, with the additional property that that there exist $\mu \in \mathcal{P}^\sigma(\Sigma)$ for which $I(\mu)$ and $I_V(\mu)$ are finite.

(a) Prove that there is a unique minimizer of $I_V(\mu)$ within the class μ_V^σ. This is the equilibrium measure with external field V and constraint σ, and we denote it by μ_V^σ.

(b) Show that

$$\begin{aligned} 2U^{\mu_V^\sigma} + V(x) &\leq \ell, \quad \text{for q.e. } x \in \mathrm{supp}(\mu_V), \\ 2U^{\mu_V^\sigma} + V(x) &\geq \ell, \quad \text{for q.e. } x \in \mathrm{supp}(\sigma - \mu_V^\sigma)). \end{aligned}$$

Exercise 1.5 (optional)**.** Prove the following converse to Theorem 1.6.

If a probability measure $\tilde{\mu} \in \mathcal{P}(\Sigma)$ and a constant $\tilde{l}$ exist such that

$$\begin{aligned} 2U^{\tilde{\mu}}(x) + V(x) &\leq \tilde{\ell}, \quad \text{for } x \in \mathrm{supp}(\tilde{\mu}), \\ 2U^{\tilde{\mu}}(x) + V(x) &\geq \tilde{\ell}, \quad \text{for q.e. } x \in \Sigma, \end{aligned}$$

then $\tilde{\mu} = \mu_V$ and $\tilde{\ell} = \ell$.

Exercise 1.6 (optional)**.** The external field $V : \mathbb{R} \to \mathbb{R}$, $x \mapsto V(x) = \log(x^2 + 1)$ is only weakly admissible. Its equilibrium measure μ_V has unbounded support.

(a) Show that the equilibrium measure μ_V has the Cauchy density

$$\frac{d\mu_V(x)}{dx} = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}.$$

Hint: compute its Stieltjes transform $\int \frac{d\mu_V(x)}{z-x}$ for $z \in \mathbb{C} \setminus \mathbb{R}$, by means of contour integration, and then verify that $2U^{\mu_V} + V$ is constant on $\mathbb{R}$ from the general properties

$$\frac{d}{dx}U^{\mu}(x) = -\fint \frac{d\mu(s)}{x-s}$$

with $\fint$ denoting the Cauchy principal value, and

$$\lim_{\varepsilon \to 0+} \int \frac{d\mu(s)}{x \pm i\varepsilon - s} = \fint \frac{d\mu(s)}{x-s} \mp \pi i \frac{d\mu}{dx}, \qquad x \in \mathbb{R},$$

(b) Can you compute the equilibrium measure in external field V with constraint $d\sigma = a dx$ where $a > 0$ is a given positive number?