NOTES FOR RIEMANNIAN MANIFOLDS

Based on the MATH 765 by A.Waldron 2026, GTM 275 by Loring W.Tu, GTM 94 by F.Warner

Author
Wells Guan

Contents

1 Preliminary

1.1 Manifolds

Definiton 1.1.1. A topological space M is locally Euclidean of dimension n if for every point p in M, there is a homeomorphism ϕ of a neighborhood U of p with an open subset of $\mathbb{R}^n$. Such a pair $(U, \phi : U \to \mathbb{R}^n)$ is called a coordinate chart or simply a chart. If $p \in U$, then we say that (U, ϕ) is a chart about p. A collection of charts $\{(U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^n)\}$ is C^∞ compatible if for every α and β, the transition function

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is C^∞. A collection of C^∞ compatible charts $\{(U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^n)\}$ that cover M is called a C^∞ atlas. A C^∞ atlas is said to be maximal if it contains every chart that is C^∞ compatible with all the charts in the atlas.

Definiton 1.1.2. A topological manifold is a Hausdorff, second countable, locally Euclidean topological space. A smooth manifold is a pair consisting of a topological manifold M and a maximal C^∞ atlas $\{(U_\infty, \phi_\alpha)\}$ on M.

Definiton 1.1.3. A function $f : M \to \mathbb{R}^n$ on a manifold M is said to be smooth if there is a chart (U, ϕ) about p in the maximal atlas of M such that the function

$$f \circ \phi^{-1} : \mathbb{R}^m \supset \phi(U) \to \mathbb{R}^n$$

is smooth. The function $f : M \to \mathbb{R}$ is said to be smooth on M if it is smooth at every point of M. Recall that an algebra over $\mathbb{R}$ is a vector space A together with a bilinear map $\mu : A \times A \to A$, called multiplication, such that under addition and multiplication, A becomes a ring. Under addition, multiplication, and scalar multiplication, the set of all smooth functions $f : M \to R$ is an algebra over $\mathbb{R}$, denoted by $C^\infty(M)$.

Definiton 1.1.4. A map $F : N \to M$ between two manifolds is smooth at $p \in N$ if there is a chart (U, ϕ) about p in N and a chart (V, ψ) about $F(p)$ in M with $V \supset F(U)$ such that the composite map

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \psi(V) \subset \mathbb{R}^n$$

is smooth at $\phi(p)$. It is smooth on N if it is smooth at every point of N. A smooth map $F : N \to M$ is called a diffeomorphism if it has a smooth inverse, i.e., a smooth map $G : M \to N$ such that $F \circ G = \mathbb{1}_M$ and $G \circ F = \mathbb{1}_N$.

1.2 Tangent Vectors

Definiton 1.2.1. For two C^∞ functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ defined on neighborhoods U and V of p to be equivalent if there is a neighborhood W of p contained in both U and V such that f agrees with g on W. The equivalence class of $f : U \to R$ is called the germ of f at p.

The set $C_p^\infty(M)$ of germs of C^∞ real-valued functions at p in M is an algebra over $\mathbb{R}$.

Definiton 1.2.2. A tangent vector (point-derivation) at a point p of a manifold M is a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that for any $f, g \in C_p^\infty(M)$

$$D(fg) = (Df)g(p) + f(p)Dg.$$

The set of all tangent vectors at p is a vector space $T_p(M)$ called the tangent space of M at p.

Definiton 1.2.3. At a point p in a coordinate chart $(U, \phi) = (U, x^1, \cdots, x^n)$ where $x^i = r^i \circ \phi$ is the ith component of ϕ, we define the coordinate vectors $\partial/\partial x^i|_p \in T_pM$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f \circ \phi^{-1}$$

for each $f \in C_p^\infty(M)$.

Proposition 1.2.1. The coordinate vectors $\partial/\partial x^i|_p$ form a basis of the tangent space T_pM.

Definiton 1.2.4. If $F : N \to\ M$ is a smooth map, then at each point $p \in N$ its differential

$$F_{*,p} : T_pN \to T_{F(p)}M$$

is the linear map defined by

$$(F_(*, p)X_p)(h) = X_p(h \circ F)$$

for $X_p \in T_pN$ and $h \in C_{F(p)}^\infty(M)$.

Proposition 1.2.2. If $F : N \to M$ abd $G : M \to P$ are C^∞ maps, then for any $p \in N$,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$$

Proof. For any $X_p \in T_pN, h \in C_{G\circ F(p)}^\infty(M)$, we have

$$\begin{aligned}(G \circ F)_{*,p}(X_p)(h) =& X_p(h \circ (G \circ F)) \\ =& X_p((h \circ G) \circ F) = F_{*,p}X_p(h \circ G) \\ =& (G_{*,p} \circ F_{*,p})X_p(h)\end{aligned}$$

□

Definiton 1.2.5. Let $\phi : M \to N$ be a smooth map from smooth manifold M to N, then

(a) ϕ is an immersion if $d\phi_m$ is injective for each $m \in M$.

(b) The pair (M, ϕ) is submanifold of N if ϕ is an injective immersion.

(c) ϕ is an imbedding if ϕ is an injective immesrsion which is also a homeomorphism into $\phi(M)$, that is ϕ is open with $\phi(M)$ equipped with the relative topology.

(d) ϕ is a diffeomorphism if ϕ maps M injectively onto N and ϕ^{-1} is smooth.

Definiton 1.2.6. A set $f_1, \cdots, f_j$ of smooth functions defined on some neighborhood of m in M is called an independent set at m if the differentials $df_1, \cdots, df_j$ form an independent set in T_mM^*.

Theorem 1.2.3. (Inverse Function Theorem) Let $U \subset \mathbb{R}^d$ be open, and let $f : U \to \mathbb{R}^d$ be smooth. If the Jacobian matrix is nonsingular at $p \in U$, then there exists an open set V with $p \in V \subset U$ such that $f|V$ maps V injectively onto the open set $f(V)$ and $(f|V)^{-1}$ is smooth.

Corollary 1.2.4. Assume that $\phi : M \to N$ is smooth, that $m \in M$, and $d\phi : T_m M \to T_{\phi(m)} N$ is an isomorphism. Then there is a neighbourhood U of m such that $\phi : U \to \phi(U)$ is a diffeomorphism onto the open set $\phi(U)$ in N.

Proof. Since $d\phi$ is an isomorphism, we know dim M = dim N, Consider (U, ψ) a chart containing m and (V, τ) a chart containing $\phi(m)$, then we know $\psi : U \to \psi(U), \tau : V \to \tau(V)$ are both diffeomorphisms and hence $(\tau \circ \phi \circ \psi^{-1})_{*,m} : T_{\psi(m)}\psi(U) \to T_{\tau(\phi(m))}\tau(V)$ is an isomorphism and hence the Jacobian matrix is non-singular, so there is an open set $W \subset \psi(U)$ such that $\tau \circ \phi \circ \psi^{-1} : W \to \tau \circ \phi \circ \psi^{-1}(W)$ is a diffeomorphism and hence induce a map $\psi^{-1}(W) \to \tau^{-1}(\tau \circ \phi \circ \psi^{-1}(W)) = \phi(\psi^{-1}(W))$ is a diffeomorphism. $\square$

Corollary 1.2.5. Suppose that dim $M = d$ and that $f_1, \cdots, f_d$ is an independent set of functions at $m_0 \in M$. Then the functions $f_1, \cdots, f_d$ form a coordinate system on a neighborhood of m_0.

1.3 Vector Fields

Definiton 1.3.1. A vector field X on a manifold M is the assignment of a tangent vector $X_p \in T_p M$ to each point p, then we can have

$$X_p = a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{with } a^i(p) \in \mathbb{R}$$

and X is said to be smooth if M has a smooth atlas such that on each chart (U, x^i) a^i are smooth. We denote the set of all C^∞ vector fields on M by $\mathscr{X}(M)$.

A frame of vector fields on an open set $U \subset M$ is a collection of vector fields $X_1, \cdots, X_n$ on U such that at each point $p \in U$, the vectors $(X_i)_p$ form a basis for $T_p M$.

Proposition 1.3.1. For some $f \in C^\infty(M)$, we have the induced function on M by

$$(Xf)(p) = X_p f$$

which is still in $C^\infty(M)$.

Proof. For a chart (U, x^i), we have

$$(Xf)(p) = a^i(p) \partial f / \partial x_i|_p$$

which is smooth on U. $\square$

Definiton 1.3.2. The Lie bracket of two vector fields $X, Y \in \mathscr{X}(M)$ is the vector field $[X, Y]$ defined by

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf) \quad \text{for } p \in M \text{ and } f \in C_p^\infty(M)$$

which is still in $\mathscr{X}(M)$.

1.4 Differential Forms

2 Riemann Metrics

2.1 Definitions

Definiton 2.1.1.

(Riemannian Metric)

Let M be a smooth manifold. g is a smoothly real inner product on the tangent spaces of M in the sense that if X and Y are smooth vector fields on M, then $p \mapsto \langle X_p, Y_p \rangle_p$ is a smooth function on M.

A smooth manifold endowed with a Riemannian metric is called a Riemannian manifold.

Definiton 2.1.2.

(Length and Angle)

Given a Riemannian metric g on M, we can speak about the length

$$|v| = |v|_g = \sqrt{g_x(v,v)}$$

of a tangent vector $v \in T_xM$, and about the angle between two nonzero tangent vectors $v, w \in T_xM$, we have

$$\theta = \arccos g_x(\frac{v}{|v|}, \frac{w}{|w|})$$

Proposition 2.1.1. Since we have the coordinate frame $\{\partial/\partial x^i\}_{i=1}^n$ for TM, then let

$$g_{ij} = g(\partial/\partial_{x^i}, \partial/\partial x^j)$$

be local components, which are n^2 smooth functions on the coordinate patch, for two vector fields $X = X^i\partial/\partial x^i, Y = Y^j\partial/\partial x^j$ the inner -product is given by

$$g(X,Y) = X^iY^jg_{ij}$$