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# NOTES FOR RENORMALIZATION FLOW

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Based on the paper by A.Dunlap and Cole

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# 1 Setup

## 1.1 Semilinear SHE

We consider the semilinear stochastic heat equation

$$du_t^\rho = \frac{1}{2}\Delta u_t^\rho dt + \gamma_\rho \sigma(u_t^\rho) dW_t^\rho, \quad t > 0, x \in \mathbb{R}^2$$

Here  $\sigma$  is a Lipschitz nonlinearity and  $dW_t^\rho(x)$  is a Gaussian noise that is white in time and correlated in space at scale  $\rho^{1/2} \ll 1$ . We are interested in the pointwise behavior of  $u_t^\rho(x)$  as  $\rho \rightarrow 0$ , which calls for an attenuation factor  $\gamma_\rho \sim |\ln \rho|^{-1/2}$  due to critical scaling in two dimensions. In fact, we devote most of our attention to a variation on (??) in which we first multiply  $\sigma$  and then smooth the noise:

$$dv_t^\rho = \frac{1}{2}\Delta v_t^\rho dt + \gamma_\rho \mathcal{G}_\rho[\sigma(v_t^\rho) dW_t]$$

### Definiton 1.1.1.

(Space-time White Noise)

Let  $dW = (dW_t(x))_{t \in \mathbb{R}, x \in \mathbb{R}^2}$  be a standard  $\mathbb{R}^m$ -valued space-time white noise generating a temporal filtration  $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ . Writing  $dW = (dW^1, \dots, dW^m)$  in components, then

$$\mathbb{E}[dW_t^i(x) dW_{t'}^{i'}(x')] = \delta_{i,i'} \delta(t - t') \delta(x - x')$$

**Proposition 1.1.1.** Construct a space-time white noise.

### Definiton 1.1.2.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and fix a target dimension  $m \in \mathbb{N}$ . The solution  $v^\rho : \Omega \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^m$  is a random vector-valued function parametrized by the correlation parameter  $\rho > 0$ . We suppress the dependence of  $v^\rho$  on  $\omega \in \Omega$ .

Since  $v$  is vector-valued, our nonlinearity  $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$  is matrix-valued. Let  $\mathcal{H}_+^m$  denote the set of nonnegative-definite symmetric real  $m \times m$  matrices, equipped with the metric induced by the Frobenius norm

$$|A|_F^2 := \text{tr}(AA^T) = \text{tr}(A^2)$$

Let  $\sigma$  belong to the space  $\text{Lip}(\mathbb{R}^m, \mathcal{H}_+^m)$ .

**Proposition 1.1.2.** Show that  $|\cdot|_F$  is a norm on  $\mathcal{H}_+^m$ .

**Definiton 1.1.3.** Given  $\tau \geq 0$ , we define the heat operator

$$\mathcal{G}_\tau v = G_\tau * v$$

where  $G_\tau = (2\pi\tau)^{-1} \exp(-\frac{|x|^2}{2\tau})$  denotes the standard heat kernel. Define the spatially-smoothed noise  $dW_t^\rho = G_\rho * dW_t$ .

**Proposition 1.1.3.** We have

$$\mathbb{E}[dW_t^{\rho,i}(x) dW_{t'}^{\rho,i'}(x')] = \delta_{i,i'} \delta(t - t') G_{2\rho}(x - x')$$

*Proof.*

□

**Definiton 1.1.4.** Define

$$L(\tau) = \ln(1 + \tau) \quad \text{for } \tau \geq 0$$

and set

$$\gamma_\rho = \sqrt{\frac{4\pi}{L(1/\rho)}}$$

**Definiton 1.1.5.**

(Mild Solution 1)

A mild solution for (??) is a predictable random field  $v^\rho$  such that for all  $s < t$ , we have

$$v_t^\rho(x) = \mathcal{G}_{t-s}v_s^\rho(x) + \gamma_\rho \int_s^t \mathcal{G}_{t+\rho-r}[\sigma(v_r^\rho)dW_r](x)$$

which means

$$\begin{aligned} v_t^\rho x &= \mathcal{G}_{t-s}v_s^\rho(x) + \gamma_\rho \int_s^t \int G_{t+\rho-r}(y)\sigma(v_r^\rho)(x-y)dW_r(x-y)dy \\ &= \mathcal{G}_{t-s}v_s^\rho(x) + \gamma_\rho \int_s^t \int G_{t+\rho-r}(y)\sigma(v_r^\rho)(x-y)dW_r(x-y)dy \\ &= \mathcal{G}_{t-s}v_s^\rho(x) + \gamma_\rho \int \left( \int_s^t G_{t+\rho-r}(y)\sigma(v_r^\rho)(x-y)dW_r(x-y) \right) dy \end{aligned}$$

which can be interpreted as an Ito integral. We only look for the solution  $v_t^\rho$  in the spaces  $\mathcal{X}_t^l$  of  $\mathbb{R}^m$ -valued random fields  $z$  on  $\mathbb{R}^2$  that are  $\mathcal{F}_t$ -measurable and

$$||z||_l := \sup_{x \in \mathbb{R}^2} (\mathbb{E}|z(x)|^l)^{1/l} < \infty$$

**Proposition 1.1.4.** For any  $l \geq 2$ , there is a family of random operators  $(\mathcal{V}_{s,t}^{\sigma,\rho})_{s < t}$  such that if  $v_s \in \mathcal{X}_s^l$ , then  $v_t^\rho = \mathcal{V}_{s,t}^{\sigma,\rho}v_s$  is a mild solution of (??) for  $t \geq s$ . We often write  $\mathcal{V}_t^{\sigma,\rho} := \mathcal{V}_{0,t}^{\sigma,\rho}$ .

Shown by some standard fixed-point arguments.

**Definiton 1.1.6.** (Forward-backward SDE)

The system of SDE:

$$\begin{aligned} d\Gamma_{a,Q}^\sigma(q) &= J_\sigma(Q - q, \Gamma_{a,Q}^\sigma(q))dB(q), & a \in \mathbb{R}^m, 0 < q < Q \\ \Gamma_{a,Q}^\sigma(0) &= a, & a \in \mathbb{R}^m, Q \geq 0 \\ J_\sigma(q, b) &= [\mathbb{E}\sigma^2(\Gamma_{a,Q}^\sigma(q))]^{1/2}, & q \geq 0, b \in \mathbb{R}^m \end{aligned}$$

for  $B$  a standard  $\mathbb{R}^m$ -valued Brownian motion and  $A^{1/2}$  is the unique positive-definite matrix square root of  $A \in \mathcal{H}_+^m$ .

**Definiton 1.1.7.** Given  $\sigma \in \text{Lip}(\mathbb{R}^m, \mathcal{H}_+^m)$ , let  $\overline{Q}_{\text{FBSDE}}(\sigma) \in [0, \infty]$  be the supremum of all  $Q \geq 0$  such that there is a continuous function  $J_\sigma : [0, Q] \times \mathbb{R}^m \rightarrow \mathcal{H}_+^m$  satisfying the FBSDE and

$$\sup_{q \in [0, \overline{Q}]} \text{Lip}(J_\sigma(q, \cdot)) < \infty$$

and define for  $M, \beta \in (0, \infty)$

$$\Sigma(M, \beta) := \{\sigma \in \text{Lip}(\mathbb{R}^m, \mathcal{H}_+^m) : |\sigma(u)|_F^2 \leq M + \beta^2 |u|^2 \text{ for all } u \in \mathbb{R}^m\}$$

**Definition 1.1.8.**

( $L^2$ -subcritical)

A nonlinearity  $\sigma$  is  $L^2$ -subcritical if  $\sigma \in \Sigma(M, \beta)$  for some  $M \in (0, \infty)$  and  $\beta \in (0, 1)$  and  $\bar{Q}_{\text{FBSDE}}(\sigma) > 1$ .

## 1.2 Main Result

**Theorem 1.2.1.** Let  $\sigma$  be  $L^2$ -subcritical. Fix  $Q \in (0, 1)$  and define  $\tilde{\sigma} := (1 - Q)^{1/2} J_\sigma(Q, \cdot)$  and  $\tilde{\rho} := \rho^{1-Q}$ . Then there is a new white noise  $d\tilde{W}$  such that  $\mathcal{G}_{\tilde{\rho}v}$  is an approximate mild solution of (??) with  $(\tilde{\rho}, \tilde{\sigma}, d\tilde{W})$  in place of  $(\rho, \sigma, dW)$ .

**Definition 1.2.1.** Denote  $\mathcal{W}_2$  the Wasserstein-2 metric: for any two probability distributions  $\mu, \nu$

$$\mathcal{W}_2(\mu, \nu) = \inf_{\pi} \left( \int |x - y|^2 \pi(dx, dy) \right)$$

where  $\pi$  owns marginal distributions of  $\mu$  and  $\nu$ .

Denote  $\langle a \rangle := (|a|^2 + 1)^{1/2}$  the Japanese bracket.

**Theorem 1.2.2.** For each  $L^2$ -subcritical  $\sigma$  and  $\bar{T} \in [1, \infty)$ , there is a constant  $C(\sigma, \bar{T}) \in (e, \infty)$  such that for all  $v_0 \in L^\infty(\mathbb{R}^2, \mathbb{R}^m)$ ,  $t \in [\bar{T}^{-1}, \bar{T}]$ , and  $\rho \in (0, C^1)$ , the solution  $v^\rho$  of (??) satisfies

$$\mathcal{W}_2(v_t^{\rho(x)}, \Gamma_{a,1}^\sigma(1)) \leq C \langle \|v_0\|_{L^\infty} \rangle \sqrt{\frac{\ln |\ln \rho|}{|\ln \rho|}}$$

where  $(\Gamma_{a,1}^\sigma(q))_{q \in [0,1]}$  solves the FBSDE with  $Q = 1$  and  $a = \mathcal{G}_t v_0(x)$ .

## 1.3 Decoupling Flow

**Definition 1.3.1** (Parabolic Equation).

The following parabolic equation

$$\begin{aligned} \partial_q H(q, b) &= \frac{1}{2} [H(q, b) : \nabla_b^2] H(q, b) \\ H(0, b) &= \sigma^2(b) \end{aligned}$$

here for  $A, B$  matrices, we denote

$$A : B = \text{tr}[AB]$$

and there is an explicit formula

$$([H(q, b) : \nabla_b^2] H(q, b))_{ij} = \text{tr}[H(q, b) \nabla_b^2] H_{ij}(q, b) = \sum_{k,l=1}^m H_{kl}(q, b) \frac{\partial^2 H_{ij}}{\partial b_k \partial b_l}(q, b)$$

*Proof.* We know

$$\mathrm{tr}[H(q, b) \nabla_b^2] = \sum_{k=1}^n \sum_{l=1}^n H_{kl}(q, b) \frac{\partial^2}{\partial b_k \partial b_l}$$

□

## 2 Well-posedness of the FBSDE

### 2.1 Main Goal

The main result is

**Theorem 2.1.1.** Let  $\sigma \in \text{Lip}(\mathbb{R}^m, \mathcal{H}_+^m)$ . For any  $Q \in [0, \bar{Q}_{\text{FBSDE}}(\sigma))$ , we have

$$\bar{Q}_{\text{FBSDE}}(\sigma) \geq Q + \text{Lip}(J_\sigma(Q, \cdot))^{-2}$$

**Definiton 2.1.1.**  $\mathcal{X}$  is the Banach space of  $\mathcal{H}_+^m$ -valued continuous functions on  $\mathbb{R}^m$  with the norm (and this norm is finite)

$$\|\sigma\|_{\mathcal{X}} := \sup_{x \in \mathbb{R}^m} \frac{|\sigma(x)|_F}{\langle x \rangle}$$

**Proposition 2.1.2.** Prove  $\mathcal{X}$  is a Banach space.

**Proposition 2.1.3.**  $\text{Lip}(\mathbb{R}^m; \mathcal{H}_+^m) \subset \mathcal{X}$ .

### 2.2 SDE Solution Theory

**Definiton 2.2.1.** Given a  $\mathbb{R}^m$ -valued Brownian motion  $(B(q))_{q \geq 0}$  adapted to a filtration  $\{\mathcal{G}_q\}_{q \geq 0}$ . For an adapted process  $Y$  on  $[0, Q]$ , a function  $g : [0, Q] \times \mathbb{R}^m \rightarrow \mathcal{H}_+^m$  and a constant  $a \in \mathbb{R}^m$ , we define a new adapted process  $\mathcal{R}_{a,Q}^g Y$  on  $[0, Q]$  by

$$\mathcal{R}_{a,Q}^g Y(q) := a + \int_0^q g(Q - p, Y(p)) dB(p)$$

whenever this stochastic integral is defined. For  $Q > 0$ , define

$$\mathcal{A}_Q := \left\{ J : [0, Q] \times \mathbb{R}^m \rightarrow \mathcal{H}_+^m \text{ continuous} : \sup_{q \in [0, Q]} \text{Lip}(J(q, \cdot)) < \infty \right\}$$

**Proposition 2.2.1.** Fix  $L < \infty$  and  $Q \in (0, \infty)$  and suppose that  $g \in \mathcal{A}_Q$  satisfies

$$\sup_{q \in [0, Q]} \text{Lip}(g(q, \cdot)) \leq L$$

Then, for any  $a \in \mathbb{R}^m$ , there is a unique strong solution  $\Theta_{a,Q}^g$  to the SDE

$$\begin{aligned} d\Theta_{a,Q}^g(q) &= g(Q - q, \Theta_{a,Q}^g(q)) dB(q), \quad q \in [0, Q] \\ \Theta_{a,Q}^g(0) &= a \end{aligned}$$

The solution  $\Theta_{a,Q}^g$  satisfies the moment bound

$$\sup_{q \in [0, Q]} \mathbb{E} |\Theta_{a,Q}^g(q)|^l < \infty \quad \text{for all } l \in [1, \infty)$$

Moreover, there exists a constant  $C = C(L, Q)$  such that for any  $Q' \in [0, Q]$  and any adapted

process  $\Gamma$  on  $[0, Q']$ , we have

$$\sup_{q \in [0, Q']} \mathbb{E} |\Gamma(q) - \Theta_{a, Q}^g(q)|^2 \leq C \sup_{[0, Q']} \mathbb{E} |\Gamma(q) - \mathcal{R}_{a, Q}^g \Gamma(q)|^2$$

and

$$\mathbb{E} |\Theta_{a, Q}^g(q) - \Theta_{\tilde{a}, Q}^g(q)|^2 \leq C |a - \tilde{a}|^2$$

*Proof.* Define  $\mathcal{K}_{L, Q'}$  on adapted processes on  $[0, Q']$  by

$$\sup_{q \in [0, Q']} e^{-L^2 q} (\mathbb{E} |\Gamma(q)|^2)^{1/2},$$

which makes  $\mathcal{R}_{a, Q}^g$  a contraction.  $\square$

### 2.3 Local Solution

**Definiton 2.3.1.** Let  $Q > 0$  and let  $\sigma \in \text{Lip}(\mathbb{R}^m; \mathcal{H}_+^m)$ . We say that  $J \in \mathcal{A}_Q$  is a *root decoupling function* for FBSDE on  $[0, Q]$  if, for all  $q \in [0, Q]$  and all  $a \in \mathbb{R}^m$ , we have

$$J(q, a) = [\mathbb{E} \sigma(\Theta_{a, q}^J(q))]^{1/2}$$

In this case, we also call  $J^2$  the decoupling function. In the equation above,  $\Theta_{a, q}^J$  is as in (??).

**Definiton 2.3.2.** For  $\lambda \in (0, \infty)$ , we define the set of functions

$$\Lambda(\lambda) := \{\sigma \in \text{Lip}(\mathbb{R}^m; \mathcal{H}_+^m) : \text{Lip}(\sigma) \leq \lambda\}$$

**Proposition 2.3.1.** Suppose that  $\lambda \in (0, \infty)$ ,  $\sigma \in \Lambda(\lambda)$ ,  $Q_0 > 0$  and that  $J$  is a root decoupling function for FBSDE on  $[0, Q_0]$ . Then for all  $Q \in [0, Q_0 \wedge \lambda^{-2}]$ , we have

$$\text{Lip}(J(Q, \cdot)) \leq (\lambda^{-2} - Q)^{-1/2}.$$

*Proof.*  $\square$

**Lemma 2.3.2.** Suppose that  $c \in (0, \infty)$ ,  $\overline{Q} < c^{-2}$  and  $f : [0, \overline{Q}] \rightarrow [c^2, \infty)$  satisfies

$$f(Q) \leq c^2 \exp \left\{ \int_0^Q f(q) dq \right\}$$

for all  $Q \in [0, \overline{Q}]$ . Then

$$f(Q) \leq (c^{-2} - Q)^{-1} \quad \text{for all } Q \in [0, \overline{Q}]$$

*Proof.* Define  $g(Q) = \int_0^Q f(q) dq$  and we have

$$g'(q) = f(q),$$

then

$$(1 - e^{-g(q)})' = g'(q) e^{-g(q)} = f(q) e^{-g(q)}$$



and hence

$$1 - e^{-g(Q)} = \int_0^Q f(q)e^{-g(q)} dq \leq c^2 Q$$

Therefore, we have

$$f(Q) \leq c^2 e^{g(Q)} \leq \frac{c^2}{1 - c^2 Q}$$

Here is another proof by (??), consider  $h$  defined on  $[0, \overline{Q}]$  non-negative defined by

$$h(t) = \ln(f(t)/c^2),$$

then we have

$$h(t) \leq \int_0^t c^2 \exp(h(s)) ds.$$

Define

$$G(x) = -e^{-x} + C$$

and we have

$$h(q) \leq -\ln(1 - c^2 q)$$

for any  $q \in [0, \overline{Q}]$ , which means

$$f(Q) \leq c^2 / (1 - c^2 Q) = (c^{-2} - Q)^{-1}.$$

□

**Definiton 2.3.3.** Define

$$\mathcal{Q}_\sigma g(Q, a) = [\mathbb{E} \sigma^2(\Theta_{a,Q}^g(Q))]^{1/2},$$

where  $(\Theta_{a,Q}^g(q))_{q \in [0, Q]}$  from (??). We note that a fixed point of  $\mathcal{Q}_\sigma$  is a root decoupling function for (??). We also note that

$$|\mathcal{Q}_\sigma g(Q, a)|_F^2 = |\mathbb{E} \sigma^2(\Theta_{a,Q}^g(Q))|_F^2 = \mathbb{E} \text{tr} \sigma^2(\Theta_{a,Q}^g(Q)) = \mathbb{E} |\sigma(\Theta_{a,Q}^g(Q))|_F^2$$

Define the set of functions

$$\Lambda(M, \lambda) := \{\sigma \in \text{Lip}(\mathbb{R}^m; \mathcal{H}_+^m) : \text{Lip}(\sigma) \leq \lambda \text{ and } |\sigma(u)|_F^2 \leq M + \lambda^2 |u|^2 \text{ for all } u \in \mathbb{R}^m\}$$

For  $Q_0 < \lambda^{-2}$ , define the set of functions  $\mathcal{Z}_{Q_0, M, \lambda}$  by

$$\begin{aligned} \mathcal{Z}_{Q_0, M, \lambda} = \{ & g : [0, Q_0] \times \mathbb{R}^m \rightarrow \mathcal{H}_+^m \text{ continuous,} \\ & g(q, \cdot) \in \Lambda((1 - \lambda^2 q)^{-2} M, (\lambda^{-2} - q)^{-1/2}) \text{ for all } q \in [0, Q_0] \} \end{aligned}$$

We will construct the root decoupling function  $J_\sigma$  as a fixed point of  $\mathcal{Q}_\sigma$  in a certain  $\mathcal{Z}_{Q_0, M, \lambda}$ .

**Proposition 2.3.3.** Fix  $\lambda, M \in (0, \infty)$  and  $Q_0 \in [0, \lambda^{-2})$ . For any  $\sigma \in \Lambda(M, \lambda)$ , there is a unique root decoupling function  $J_\sigma \in \mathcal{A}_{Q_0}$ . In particular, we have

$$\overline{Q}_{\text{FBSDE}}(\sigma) \geq \text{Lip}(\sigma)^{-2}.$$

Moreover, there is a  $C = C(Q_0, M, \lambda) < \infty$  such that for any  $g \in \mathcal{Z}_{Q_0, M, \lambda}$ , we have

$$\sup_{q \in [0, Q_0]} \|(g - J_\sigma)(q, \cdot)\|_{\mathcal{X}} \leq C \sup_{q \in [0, Q_0]} \|(g - \mathcal{Q}_\sigma g)(q, \cdot)\|_{\mathcal{X}}$$

and indeed

$$\lim_{n \rightarrow \infty} \sup_{q \in [0, Q_0]} \|(\mathcal{Q}_\sigma^n g - J_\sigma)(q, \cdot)\|_{\mathcal{X}} = 0$$

where  $\mathcal{Q}_\sigma^n$  denotes the  $n$ -fold iterated application of  $\mathcal{Q}_\sigma$ .

*Proof.* We know any decoupling function  $J \in \mathcal{A}_{Q_0}$  has to be in  $\mathcal{Z}_{Q_0, M, \lambda}$  for some  $M', \lambda \in (0, \infty)$   $\square$

## 2.4 Extension of the Solution

**Lemma 2.4.1.** Let  $\sigma \in \text{Lip}(\mathbb{R}^m; \mathcal{H}_+^m)$ , whenever  $0 \leq Q_1 \leq Q_2 < \overline{Q}_{\text{FBSDE}}(\sigma)$ , we have for any  $b \in \mathbb{R}^m$ ,

$$J_\sigma^2(Q_2, b) = \mathbb{E}[J_\sigma^2(Q_1, \Gamma_{b, Q_2}^\sigma(Q_2 - Q_1))]$$

*Proof.* Remain.  $\square$

**Lemma 2.4.2.** Let  $\sigma \in \text{Lip}(\mathbb{R}^m; \mathcal{H}_+^m)$ , whenever  $0 \leq Q_1 \leq Q_2 < \overline{Q}_{\text{FBSDE}}(\sigma)$  and if

$$Q_2 - Q_1 < \overline{Q}_{\text{FBSDE}}(J_\sigma(Q_1, \cdot)),$$

we have

$$J_\sigma(Q_2, b) = J_{J_\sigma(Q_1, \cdot)}(Q_2 - Q_1, b) \quad \text{for all } b \in \mathbb{R}^m.$$

*Proof.* We have by (??) that, for any  $Q \in [Q_1, \overline{Q}_{\text{FBSDE}}(\sigma))$  and any  $b \in \mathbb{R}^m$ ,

$$J_\sigma^2(Q, b) = \mathbb{E}[J_\sigma^2(Q_1, \Gamma_{b, Q}^\sigma(Q - Q_1))]$$

By assumption, there is a unique solution to the following FBSDE problem for  $Q \in [0, Q_2 - Q_1]$ :

$$\begin{aligned} d\Gamma_{b, Q}^{J_\sigma(Q_1, \cdot)}(q) &= J_{J_\sigma(Q_1, \cdot)}(Q - q, \Gamma_{b, Q}^{J_\sigma(Q_1, \cdot)}(q))dB(q), \quad q \in (0, Q) \\ \Gamma_{b, Q}^{J_\sigma(Q_1, \cdot)}(0) &= b \\ J_{J_\sigma(Q_1, \cdot)}(Q, b) &= (\mathbb{E}[J_\sigma^2(Q_1, \Gamma_{b, Q}^{J_\sigma(Q_1, \cdot)}(Q))])^{1/2} \end{aligned}$$

Also, if given  $Q \in [0, \overline{Q}_{\text{FBSDE}}(\sigma) - Q_1]$ , (??) and (??) with  $Q$  replaced by  $Q_1 + Q$  will yield

$$\begin{aligned} d\Gamma_{b, Q+Q_1}^\sigma(q) &= J_\sigma(Q + Q_1 - q, \Gamma_{b, Q+Q_1}^\sigma(q))dB(q), \quad q \in (0, Q) \\ \Gamma_{b, Q+Q_1}^\sigma(0) &= b \\ J_\sigma(Q + Q_1, b) &= (\mathbb{E}[\sigma^2(\Gamma_{b, Q+Q_1}^\sigma(Q))])^{1/2} = (\mathbb{E}[J_\sigma^2(Q_1, \Gamma_{b, Q+Q_1}^\sigma(Q))])^{1/2} \end{aligned}$$

which means  $(\Gamma_{b, Q+Q_1}^\sigma(q)$  and  $J_\sigma(Q + Q_1, b)$ ) will solve the previous FBSDE system, and by the uniqueness of FBSDE problem, we have

$$J_{J_\sigma(Q_1, \cdot)}(Q, b) = J_\sigma(Q + Q_1, b) \quad \text{for all } b \in \mathbb{R}^m \text{ and } Q \in [0, Q_2 - Q_1]$$

□

**Proposition 2.4.3.** For any  $Q' \in [0, \overline{Q}_{\text{FBSDE}}(\sigma))$ , we have

$$\overline{Q}_{\text{FBSDE}}(\sigma) \geq Q' + \overline{Q}_{\text{FBSDE}}(J_\sigma(Q', \cdot))$$

*Proof.* For  $Q' < \overline{Q}_{\text{FBSDE}}\sigma$ , there is a unique root decoupling function  $J_\sigma \in \mathcal{A}_{Q'}$  for (??) on  $[0, Q']$ . Let  $P \in [0, \overline{Q}_{\text{FBSDE}}(J_\sigma(Q', \cdot))]$ , there is a unique RDF  $J_{J_\sigma(Q', \cdot)} \in \mathcal{A}_P$  in (??) with  $\sigma$  replaced by  $J_\sigma(Q', \cdot)$ .

We wish to extend the function  $J_\sigma$  to the time interval by outting

□

### 3 Parabolic Equation for the decoupling function

#### 3.1 Main Theorem

*Remark.* The space

$$L^1(\mathcal{K}; W^{2,\infty}(U; \mathcal{H}_+^m))$$

means some functions which are  $L^1$  with fixed space point and time varying and  $W^{2,\infty}(U; \mathcal{H}_+^m)$  with fixed time point and space varying, with the norm

$$\|f\| = \int \|f(t, \cdot)\|_{W^{2,\infty}} dt.$$

For two normed spaces, their intersection will be equipped with a norm obtained by adding both the norms together.

**Definiton 3.1.1.** An *almost classical solution* to (??) on a time interval  $[0, Q_0]$  is a continuous function  $H : [0, Q_0] \times \mathbb{R}^m \rightarrow \mathcal{H}_+^m$  such that the following conditions hold:

1. For every compact  $\mathcal{K} \subset (0, Q_0)$  and bounded open  $U \subset \mathbb{R}^m$ ,

$$H_{\mathcal{K} \times U} \in L^1(\mathcal{K}; W^{2,\infty}(U; \mathcal{H}_+^m)) \cap C^1(\mathcal{K} \times U; \mathcal{H}_+^m).$$

where  $W^{2,\infty}(U; \mathcal{H}_+^m)$  is the Sobolev space of functions on  $U$  taking values in  $\mathcal{H}_+^m$  with weak second derivative measurable and essentially bounded.

2. We have  $H(0, b) = \sigma^2(b)$  for all  $b \in \mathbb{R}^m$ ;
3. We have

$$\partial_q H(q, b) = \frac{1}{2} [H(q, b) : \nabla_b^2] H(q, b) \quad \text{for almost all } (q, b) \in (0, Q_0) \times \mathbb{R}^m.$$

**Proposition 3.1.1.** Fix  $Q_0 \in (0, \overline{Q}_{\text{FBSDE}}(\sigma))$ . Suppose  $H$  is an almost classical solution of (??) on  $[0, Q_0]$  such that  $\sup_{q \in [0, Q_0]} \text{Lip}(\sqrt{H}(q, \cdot)) < \infty$ . Also assume that for each compact  $\mathcal{K} \subset (0, Q_0)$ , there exists a constant  $C(\mathcal{K})$  such that for all  $R > 0$ ,  $U_R = \{b : |b| < R\}$  the open ball in  $\mathbb{R}^m$  of radius  $R$ ,

$$\|H|_{\mathcal{K} \times U_R}\|_{L^1(\mathcal{K}; W^{2,\infty}(U_R; \mathcal{H}_+^m)) \cap C^1(\mathcal{K} \times U_R; \mathcal{H}_+^m)} \leq CR^C$$

then  $\sqrt{H} = J_\sigma$  on  $[0, Q_0] \times \mathbb{R}^m$ .

## 4 Open Problem

One possible question is that if  $\sigma$  is bounded, then in term of the solution  $u_\epsilon$  of the stochastic pde

$$\partial_t u_\epsilon = \frac{1}{2} \Delta u_\epsilon + \sigma(u) dW^\epsilon,$$

what will  $u_\epsilon$  converges to as  $\epsilon \rightarrow 0$ .

## A Fundamentals

### A.1 Wiener Integral

Let  $T$  be a set and  $X := \{X(t)\}_{t \in T}$  a  $T$ -indexed stochastic process. We recall that  $X$  is a Gaussian random field (process when  $T \subset \mathbb{R}$ ) if  $(X_{t_1}, \dots, X_{t_m})$  is a Gaussian random vector for all  $t_1, \dots, t_m \in T$ .

**Definiton A.1.1.** Let  $\mathcal{L}(\mathbb{R}^m)$  denote the collection of all Borel-measurable subsets of  $\mathbb{R}^m$  that have finite Lebesgue measure. White noise on  $\mathbb{R}^m$  is a mean-zero set-indexed Gaussian random field  $\xi(A)_{A \in \mathcal{L}(\mathbb{R}^m)}$  with covariance function

$$E[\xi(A_1)\xi(A_2)] := |A_1 \cap A_2| \quad \text{for all } A_1, A_2 \in \mathcal{L}(\mathbb{R}^m),$$

where  $|\cdot|$  denotes the Lebesgue measure on  $\mathbb{R}^m$  for every  $m$ .

### A.2 Useful Approximations

**Proposition A.2.1** (Bihari-LaSalle Inequality).

Let  $u$  and  $f$  be non-negative continuous functions defined on the half-infinite ray  $[0, \infty)$  and let  $w$  be a continuous non-decreasing function defined on  $[0, \infty)$  and  $w(u) > 0$  on  $(0, \infty)$ . For  $u$  we have

$$u(t) \leq \alpha + \int_0^t f(s)w(u(s))ds, \quad t \in [0, \infty),$$

where  $\alpha$  is a non-negative constant, then

$$u(t) \leq G^{-1} \left( G(\alpha) + \int_0^t f(s)ds \right), \quad t \in [0, T]$$

where the function  $G$  is defined by

$$G(x) = \int_{x_0}^x \frac{dy}{w(y)}, \quad x \geq 0, x_0 \geq 0,$$

and  $G^{-1}$  is the inverse function of  $G$  and  $T$  is chosen so that

$$G(\alpha) + \int_0^t f(s)ds \in \text{Dom}(G^{-1}), \quad \forall t \in [0, T]$$

*Proof.* Notice  $G(x) = \int_{x_0}^x \frac{dy}{w(y)}$  is increasing and

$$\text{Dom}(G^{-1}) = \left[ G(0) = - \int_0^{x_0} \frac{dy}{w(y)}, G(0) + \int_0^\infty \frac{dy}{w(y)} \right]$$

then since  $u \geq 0$ , then we have

$$G(u(t)) \leq G \left( \alpha + \int_0^t f(s)\omega(u(s))ds \right) := H(t)$$

and we have

$$H'(t) = \frac{f(t)\omega(u(t))}{\omega(\alpha + \int_0^t f(s)\omega(u(s))ds)} \leq f(t)$$

so we have

$$H(t) \leq G(\alpha) + \int_0^t f(s)ds.$$

Therefore, we may have the requested inequality for all  $0 \leq t \leq T$  where

$$T := \sup_t \left\{ \int_0^t f(s)ds \leq G(0) - G(\alpha) + \int_0^\infty \frac{dy}{\omega(y)} \right\}$$

□

### A.3 Matrix Analysis

**Theorem A.3.1** (Spectral Theorem for Real Symmetric Matrices).

Let  $A \in \mathbb{R}^{n \times n}$  be a real symmetric matrix, that is,  $A^T = A$ . Then:

1. All eigenvalues of  $A$  are real.
2. Eigenspaces corresponding to distinct eigenvalues are orthogonal.
3. The space  $\mathbb{R}^n$  decomposes as an orthogonal direct sum of eigenspaces:

$$\mathbb{R}^n = \bigoplus_{\lambda \in \sigma(A)} \ker(A - \lambda I).$$

4. Equivalently, there exists an orthogonal matrix  $Q$  and a real diagonal matrix  $\Lambda$  such that

$$A = Q\Lambda Q^T.$$