Bonus 05 - MATH 722

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Problem

If $f : U \to \mathbb{R}$ is merely continuous, we might call f strictly subharmonic if whenever $\overline{D}(P,r) \subset U$, then

$$f(P) < \frac{1}{2\pi}\int_0^{2\pi} f(P+re^{i\theta})d\theta$$

For C^2 functions, is this equivalent to the assertion that $\Delta f > 0$? Does one definition imply the other? Can you think of a definition that applies to continuous functions and is equivalent to $\Delta f > 0$ when f is C^2?

Sol.

We claim that $\Delta f > 0$ may implies that whenever $\overline{D}(P,r) \subset U$, then we have

$$f(P) < \frac{1}{2\pi}\int_0^{2\pi} f(P+re^{i\theta}d\theta)$$

Notice that $\inf\{z \in \overline{D}(P,r), \Delta f(z)\} = \delta > 0$, then we know

$$\Delta(f - \delta|z|^2) \geq 0$$

on some neighbourhood of $\overline{D}(P,r)$ and hence

$$f(P) - \delta|P|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} [f(P+re^{i\theta}) - \delta(|P|^2 + \bar{P}re^{i\theta} + Pre^{-i\theta} + r^2)]d\theta$$

which means

$$f(P) \leq \frac{1}{2\pi}\int_0^{2\pi} f(P+re^{i\theta}) - \delta r^2 < \frac{1}{2\pi}\int_0^{2\pi} f(P+re^{i\theta})$$

for any $\overline{D}(P,r) \in U$. Now notice that $\Delta f > 0$ iff for any $\overline{D}(P,r) \subset U$, there exists $\delta > 0$ such that $\Delta(f - \delta|z|^2) \geq 0$ on a neighbourhood of $\overline{D}(P,r)$ iff for any $\overline{D}(P,r) \subset U$, there exists $\delta_{P,r} > 0$ such that

$$f(Q) \leq \frac{1}{2\pi}\int_0^{2\pi} f(Q+\rho e^{i\theta})d\theta - \delta_{P,r}\rho^2$$

for any $\overline{D}(Q,\rho) \subset V$, which is a neighborhood of $\overline{D}(P,r)$. This conclusion can be expressed more generally: for any K a compact subset of U, there exists δ_K and a neighbourhood V of K such that for any $\overline{D}(P,r) \subset V$, we have

$$f(P) \leq \frac{1}{2\pi} f(P+re^{i\theta}) - \delta_K r^2$$

which is obviously a definition applies to continuous functions.

Now we claim that the two statements in the problem is not equivalent, consider $f(z) = e^{|z|^2} - |z|^2$ and it is easy to check that $\Delta f(z) > 0$ for all $z \neq 0$ and $\Delta f(0) = 0$. And by the proof above, it is easy to check that the inequality holds for any $z \neq 0$, and

$$\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta = e^{r^2} - r^2 > 1 = f(0)$$

for any $r > 0$ and hence it is a counter-example for the suffiency of the equivalence.