Chapter 1

1.1 Brownian Motion

Definition 1.1

A real-valued stochastic process $B = (B_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}; P)$ is called a Brownian motion if it satisfies the following conditions:
a. Almost surely $B_0 = 0$.
b. For all $0 \leq t_1 < \cdots t_n$ the increments $B_{t_n} - B_{t_{n-1}}, \cdots, B_{t_2} - B_{t_1}$ are independent random variables.
c. If $0 \leq s < t$, the increment $B_t - B_s$ is a Gaussian random variable with mean zero and variance $t - s$.
d. With probability one, the map $t \to B_t$ is continuous.
A d-dimensional Brownian motion is defined as an $\mathbb{R}^d$-valued stochastic process $B = (B_t)_{t\geq 0}$, $B_t = (B_t^1, \cdots, B_t^d)$, where $B^1, \cdots, B^d$ are d independent Brownian motions.

♣

Proposition 1.1

Properties (a),(b),(c) are equivalent to that B is a Gaussian process,i.e. for any finite set of indices $t_1, \cdots, t_n$, $(B_{t_1}, \cdots, B_{t_n})$ is a multivariate Gaussian random variable, equivalently, any linear combination of B_{t_i} is normal distributed r.v., with mean zero and covariance function

$$\Gamma(s,t) = \min(s,t)$$

♠

Proof

Suppose (a),(b),(c) holds, then we know $(B_{t_1}, \cdots, B_{t_n})$ is normal for any finite indices and then

$$m(t) = E(B_t) = 0$$

$$\Gamma(s,t) = E(B_s B_t) = E(B^2_{\min(s,t)}) = \min(s,t)$$

Conversly, we know $E(B_0^2) = 0$ and hence $B_0 = 0$ a.s., then we know $E(B_s^2) = s$ and for any $0 < s < t$,

$$E(B_s(B_t - B_s)) = 0$$

and it is easy to check (c), and (b) is deduced by computing the covariance of the increments, notice that two r.v.s are independent iff $\phi_{(X_1, X_2, \cdots, X_n)} = \phi_{X_1}\phi_{X_2}\cdots\phi_{X_n}$ which implies that normal r.v.s are independent iff they have zero covariances.

Theorem 1.1

(Kolmogorov's continuity theorem) Suppose that $X = (X_t)_{t\in[0,T]}$ satisfies

$$E(|X_t - X_s|^\beta) \leq K|t-s|^{1+\alpha}$$

for all $s, t \in [0,T]$ and some constant $\beta, \alpha, K > 0$. Then there exists a version $\tilde{X}$ of X such that if

$$\gamma < \alpha/\beta$$

then

$$|\tilde{X}_t - \tilde{X}_s| \leq G_\gamma |t-s|^\gamma$$

for all $s, t \in [0,T]$ where G_γ is a random variable. The trajectories of $\tilde{X}$ are Holder continuous of order γ for any $\gamma < \alpha/\beta$.

♡

Proposition 1.2

There exists a version of B with Holder-continuous trajectories of order γ for any $\gamma < (k-1)/2k$ on any interval $[0,T]$.

♠

Proof

Since we know $B_t - B_s$ has the normal distribution $\mathcal{N}(0, t-s)$ and then we know

$$E\Big((B_t - B_s)^{2k}\Big) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} x^{2k} \exp^{-\frac{x^2}{2(t-s)}} dx = (2k-1)!!(t-s)^k = \frac{(2k)!}{2^k k!}(t-s)^k$$

and by the theorem 1.1, the proposition holds.

Proposition 1.3

Brownian motion are basic properties:

a. For any $a > 0$, the process $(a^{-1/2} B_{at})_{t \geq 0}$ is a Brownian motion.

b. For any $h > 0$, the process $(B_{t+h} - B_h)_{t \geq 0}$ is a Brownian motion.

c. The process $(-B_t)_{t \geq 0}$ is a Brownian motion.

d. Almost surely $\lim_{t \to \infty} B_t/t = 0$ and the process $X_t = tB_{1/t}$ for $t > 0$, $X_t = 0$ for $t = 0$ is a Brownian motion.

Proof

a. Consider $0 \leq t_1 < t_2 < \cdots < t_n$ and we may calculate the covariance matrix for

$$a^{-1/2} B_{at_n} - a^{-1/2} B_{at_{n-1}}, \cdots, a^{-1/2} B_{at_2} - a^{1/2} B_{at_1}$$

by

$$\begin{aligned}
&E[(a^{-1/2} B_{at_j} - a^{-1/2} B_{at_{j-1}})(a^{-1/2} B_{at_k} - a^{-1/2} B_{at_{k-1}})] \\
=&a^{-1}(at_j \wedge at_k) - a^{-1}(at_j \wedge at_{k-1}) - a^{-1}(at_{j-1} \wedge at_k) + a^{-1}(at_{j-1} \wedge at_{k-1}) \\
=&\begin{cases} t_j - t_{j-1} - t_{j-1} + t_{j-1} = t_j - t_{j-1} & \text{if } j = k \\ t_j - t_j - t_{j-1} + t_{j-1} = 0 & \text{if } j < k \\ 0 & \text{if } j > k \end{cases}
\end{aligned}$$

and hence $(a^{-1/2} B_{at})_{t \geq 0}$ satisfies the property (b) in definition 1.1, a,d are obvious and c is easy to be checked.

b. Obvious.

c. Obvious.

d. Notice B is Holder continuous. Now we only need to check that

$$E(tB_{1/t} sB_{1/s}) = ts(1/t \wedge 1/s) = (t \wedge s)$$

and the rest is easy to be checked.

Theorem 1.2

(The law of the iterated logarithm)

$$\limsup_{t \to s^+} \frac{|B_t - B_s|}{\sqrt{2|t-s| \ln \ln |t-s|}} = 1, \quad a.s.$$

Proposition 1.4

Fix a time interval $[0, t]$ and consider the following subvision π of this interval:

$$0 = t_0 < t_1 < \cdots < t_n = t$$

The norm of the subdivision π is defined as $|\pi| = \max_{0 \leq j \leq n-1}(t_{j+1} - t_j)$. Then

$$\lim_{|\pi| \to 0} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 = t$$

in $L^2(\Omega)$.

Proof

Consider let $\xi_j = (B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)$ and we know ξ_j are independent with mean 0 and hence

$$E\Big(\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 - t\Big)^2 = \sum_{j=0}^{n-1} E\xi_j^2 = \sum_{j=0}^{n-1}(3(t_{j+1}-t_j)^2 - 2(t_{j+1}-t_j)^2 + (t_{j+1}-t_j)^2)$$
$$= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2 \leq 2t|\pi| \to 0$$

Proposition 1.5

The total variation of Browian motion on an interval $[0, t]$ defined By

$$V = \sup_{\pi} \sum_{j=0}^{n-1} |B_{t_{j+1}} - B_{t_j}|$$

where $\pi = \{0 = t_0 < t_1 < \cdots < t_n\}$ is infinite with probability one.

Proof

Here we know

$$\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 \leq V \sup_j |B_{t_{j+1}} - B_{t_j}|$$

and hence if $V < \infty$, then

$$\lim_{|\pi|\to 0} \sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 = 0$$

which means $P(V < \infty) = 0$.