NOTES FOR ALGEBRAIC TOPOLOGY

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Contents

1 Fundamental Group

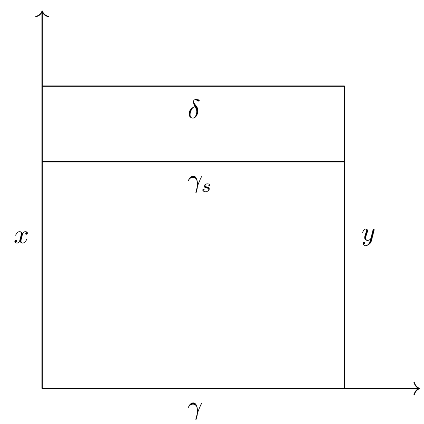
1.1 Concepts

Definiton 1.1.1. (Homotopy)

Two paths $\gamma, \delta \in \mathcal{P}(X, x, y) := \{\gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y, \gamma \text{ continuous}\}$ are called **homotopic**, denoted as $\gamma \sim \delta$, if there exists a continuous map

$$F : [0,1] \times [0,1] \to X$$

such that $F(\cdot, 0) = \gamma, F(\cdot, 1) = \delta, F(0, s) = x, F(1, s) = y$ for any $s \in [0,1]$ and call F a **homotopy** between γ and δ.



Lemma 1.1.1. The homotopy relation is an equivalence relation on the set $\mathcal{P}(X, x, y)$.

Proof.

Definiton 1.1.2. (Fundamental Group)

The **funadamental group** of X at the basepoint $x \in X$ is the set of equivalence classes of loops at x, i.e. $\Omega(X, x) := \mathcal{P}(X, x, x)$ under the homotopy relation.

Definiton 1.1.3. (Concatenation)

For $x, y, z \in X$, define a binary operation $*$ on paths:

$$\mathcal{P}(X, x, y) \times \mathcal{P}(X, y, z) \to \mathcal{P}(X, x, z)$$

by

$$(\gamma * \delta)(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Lemma 1.1.2. The concatenation is consistent with the homotopy relation, i.e. if $\gamma \sim \gamma', \delta \sim \delta'$, then $\gamma * \delta \sim \gamma' * \delta'$.

Proof.

Corollary 1.1.3. Conncatenation of paths induces a binaary law o nthe set $\pi_1(X, x)$ by

$$[\gamma] \cdot [\delta] := [\gamma * \delta]$$

Proof.

Theorem 1.1.4. $(\pi_1(X,x),\cdot)$ is a group.

Proof.

1.2 Basepoint Independence

Proposition 1.2.1. For $\delta \in \mathcal{P}(X,x,y)$, we may define $\delta_\# : \pi_1(X,x) \to \pi_1(X,y)$ by

$$[\gamma] \mapsto [\bar{\delta} * \gamma * \delta]$$

which is well-defined and an isomorphism.

Proof.

1.3 Functoriality

2 Classification of Compact Surfaces

2.1 Concepts

Definiton 2.1.1. An n-dimensional manifold with no boundary is a topological space X such that every $x \in X$ has a neighbourhood U_x homeomorphic to $\mathbb{R}^n$.

Definiton 2.1.2. A surface is a 2-dimensional manifold with no boundary.

Proposition 2.1.1. The identification space X obtained from a polugonal region P as above is Hausdorff and compact.

Definiton 2.1.3. Let M, N be surfaces. We define the connected sum of M and N denoted by $M\#N$ as

$$M\#N = (M - D_1) \sqcup (N - D_2)/(\partial D_1 \sim \partial D_2)$$

where D_1, D_2 are relatively disks in M, N.

Lemma 2.1.2. If L_1, L_2 are labeling schemes for M and N, then their concatenation L_1L_2 is a labeling scheme for $M\#N$.

Definiton 2.1.4. $T_n = T^2\#\cdots T^2$, and $P_n = \mathbb{R}P^2\#\cdots\mathbb{R}P^2$.

Theorem 2.1.3. Any compact surface is homeomorphic to S^2, T_n or P_n for some $n \in \mathbb{N}$.

2.2 Fundamental Group of a Labeling Scheme

Theorem 2.2.1. If X is the identification space of a labeling scheme

$$a_1^{\epsilon_1} a_2^{-\epsilon} \cdots a_n^{\epsilon^n}$$

with $\epsilon_i = \pm 1$ whose vertices are indentificed by the projection, then

$$\pi_1(X) = \langle a_1, \cdots, a_n | a_1^{\epsilon_1} a_2^{-\epsilon} \cdots a_n^{\epsilon^n} = 1\rangle$$

Proposition 2.2.2.

$$\pi_1(T_n) = \langle a_1, b_1, \cdots, a_n, b_n | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}\rangle$$
$$\pi_1(P_n) = \langle a_1, \cdots, a_n | a_1^2 \cdots a_n^2 = 1\rangle$$

Proposition 2.2.3. S^2, P_n, T_n have non isomorphic fundamental froups, hence they are not homotopu eqiovalent nor homeomorphic.

Theorem 2.2.4. Any surface is homeomorphic to one of S^2, T_n or P_n for some $n \in \mathbb{N}$.

Corollary 2.2.5. If X is a simply connected surface, the nit is homeomorphic to S^2.

2.3 Classification of Surfaces

Proposition 2.3.1. If P is a polygonal region with an even number of edges which are identified in pairs, then the quotient space X is a comapct 2-dimensional manifold.

Theorem 2.3.2. Every 2-dimensional compact surface is homeomorphic to the identification space of a regular labling scheme.

Theorem 2.3.3. Aa polygonal region of a regular labeling scheme is homromorphic to a standard labeling scheme.

3 Covering spaces

3.1 Concepts

Definiton 3.1.1. (Covering)

A map $p : E \to B$ is called a **covering** if

- p is continuous and onto.
- For any $b \in B$, there is $U \in \mathcal{N}(b)$ is evenly covered, i.e. $p^{-1}(U) = \sqcup_\alpha V_\alpha$ where V_α are disjoint open sets and $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphic for any α.

Definiton 3.1.2. (Equivalence of coverings)

Assume $p_1 : E_1 \to B, p_2 : E_2 \to B$ are coverings. We call them **equivalent** if there is an $f : E_1 \to E_2$ homeomorphism such that $p_2 \circ f = p_1$.

The equivalence of coverings is an equivalence relation.

Lemma 3.1.1. If $p : E \to B$ is a covering, $B_0 \subset B$ and $E_0 := p^{-1}(B_0)$, then $p|_{E_0}$ is a covering.

Theorem 3.1.2. (Path lifting property)

Let p be a covering, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. If $\gamma : I \to B$ is a path in B starting at b_0, then there is a unique lift $\widetilde{\gamma}_{e_0} : I \to E$ such that $\widetilde{\gamma}_{e_0} = e_0$.

Theorem 3.1.3. (Homotopy lifting property)

Let $p : E \to B$ be a covering, $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Let $F : I \times I \to B$ be a homotopy with with $F(0, s) = b_0$ for all $s \in I$. Then there is a unique lift $\widetilde{F} : I \times I \to E$ of F such that $\widetilde{F}(0, s) = e_0$ for any $s \in I$.

Corollary 3.1.4. If γ_1, γ_2 are paths in B starting at b_0 which are homotopic by F, then $\widetilde{\gamma}_{1e_0} \overset{\widetilde{F}}{\sim} \widetilde{\gamma}_{2e_0}$, where infer the same endpoints.

Proof.

We know $\widetilde{F}$ is a homotopy in E starting at e_0 and obviously an homotopy between $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ and hence they have the same endpoints.

Definiton 3.1.3. Let $b_0 \in B$, for $e_0 \in p^{-1}(b_0)$, define

$$\phi_{e_0} : \pi(B, b_0) \to p^{-1}(b_0)$$

by $[\gamma] \mapsto \widetilde{\gamma}_{e_0}(1)$.

Theorem 3.1.5. ϕ_{e_0} is onto if E is path-connected and injective if E is simply connected.

Proof.

If E is path-connected, then there exists γ from e_0 to any $e \in p^{-1}(b)$, then we know $[p \circ \gamma] \mapsto e$.

If E is simply-connected, we consider if $\phi_{e_0}([\gamma]) = \phi_{e_0}([\delta]) = e$, then since $\widetilde{\gamma} \sim \widetilde{\delta}$ by E is simply-connected, we know $\gamma \sim \delta$ and we are done.

Proposition 3.1.6. If $p : E \to B$ is a covering and B is path-connected, then for $b_0, b_1 \in B$ there is a bijection $p^{-1}(b_0) \to p^{-1}(b_1)$.

Proof.

We may consider γ a path from b_0, b_1, and then for any $e \in p^{-1}(b_0)$, there exists a unique lift $\widetilde{\gamma}_e$ of γ, and then we know the endpoint of $\widetilde{\gamma}_e$ is distinct and consider $\bar{\gamma}$, the conclusion is done.

Proposition 3.1.7. Let E be path connected, $p : E \to B$ a covering, and $p(e_0) = b_0$. Then $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. Further, if e_0 is changed to some other point $e_1 \in p^{-1}(b_0)$, then the images under p_* of the groups $\pi_1(E, e_0)$ and $\pi_1(E, e_1)$ are conjugate in $\pi_1(B, b_0)$.

Proof.

For $[\widetilde{\gamma}], [\widetilde{\delta}] \in \pi_1(E, e_0)$, if $\gamma \sim \delta$, then by the uniqueness of lift, we know $\widetilde{\gamma} \sim \widetilde{\delta}$.

There exists $\widetilde{l}$ a path from e_0, e_1, then for any $\widetilde{\gamma} \sim \widetilde{\delta}$ loops at e_0, then $l_\# : \pi_1(E, e_0) \to \pi_1(E, e_1)$ is an isomorphism. And then for any $[\widetilde{\gamma}] \in \pi_1(E, e_0)$, we know $p_*(\bar{l}) p_*(\widetilde{[(\gamma)]}) p_*(l) = p_*([\bar{l} * \gamma * l])$ which also induces a surjective from $p_*(\pi_1(E, e_0))$ to $p_*(\pi_1(E, e_1))$ and obviously injective, and hence they are conjugate.

Theorem 3.1.8. Let E be path-connected, $p : E \to B$ a covering map, $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Let $H := p_*(\pi_1(E, e_0)) \leq pi_1(B, b_0)$. Then

- A loop γ in B based at b_0 lifts to a loop in E at e_0 if and only if $[\gamma] \in H$.
- $\phi_{e_0} : H/\pi_1(B, b_0) \to p^{-1}(b_0), [\gamma] \mapsto \widetilde{\gamma}_{e_0}(1)$ is a bijection. In particular,

$$\# p^{-1}(b_0) = [\pi_1(B, b_0)] : p_*(\pi_1(E, e_0))$$

Proof.

We may show p_* to be a homomorphism, which can be shown by

$$p_*([\widetilde{\gamma}][\widetilde{\delta}]) = [\gamma * \delta] = [\gamma][\delta] = p_*([\widetilde{\gamma}]) p_*([\widetilde{\delta}])$$

and hence H is a subgroup. The first conclusion is trivial.

Theorem 3.1.9. (Lifting Lemma)

Let E, B, Y be path-connected and locally path-connected, i.e. there is a path-connected topology basis. Let $p : E \to B$ be a cover, $b_0 \in B, e_0 \in p^{-1}(b_0)$, and $f : Y \to B$ a continuous map such that $f(y_0) = b_0$. Then there exists a lift $\widetilde{f} : Y \to E$ of f such that $\widetilde{f}(y_0) = e_0$ if and only if $f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$.

$$
\begin{tikzcd}
 & (E, e_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & (B, b_0)
\end{tikzcd}
$$

Proof.
For the sufficiency, we know if so, then $p \circ \widetilde{f} = f$ and hence

$$f_*(\pi_1(Y, y_0)) = p_*(\widetilde{f}_*(\pi_1(Y, y_0))) \subset p_*(\pi_1(E, e_0))$$

For the necessity, firstly we may give the definition of $\widetilde{f}$ naturally by considering for any $y \in Y$, there exists γ a path from y_0 to y and hence $f \circ \gamma$ will become a path from b_0 to $f(y)$ in B. Then by path lifting property, we may define $\widetilde{f}(y)$ to be $\widetilde{f \circ \alpha}(1)$ and then $\widetilde{f}$ will be a lift if it is well-defined.

To see it is well-defined, we have to show that for any γ, δ from y_0 to y, $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \delta}(1)$. Notice $\gamma * \bar{\delta} \in \Omega(Y, y_0)$ and then $f \circ (\gamma * \bar{\delta}) \in \Omega(B, b_0)$ and hence

$$(f \circ \widetilde{(\gamma * \bar{\delta})})_{e_0} = (\widetilde{f \circ \alpha})_{e_0} * \overline{(\widetilde{f \circ \delta})}_{(\widetilde{f \circ \gamma})(1)} = (\widetilde{f \circ \alpha})_{e_0} * \overline{(\widetilde{f \circ \delta})_{e_0}}$$

and hence $\widetilde{f}$ is well-defined.

Now we only need to show that $\widetilde{f}$ is continuous. For any U open in E, we assume $\widetilde{y} \in U$ and we know for $f(y)$, there exists path-connected locally homeomorphism neighbourhood V of $f(y)$, such that $V' \subset U$ is homeomorphic to V by p. There exists W a path-connected neighbourhood of y such that $f(W) \subset V$ and then we show $f(W) \subset V'$. Since for any $w \in W$, there exists α a path from y to w and then we know $f \circ \alpha$ from $f(y)$ to $f(w)$ covered by V and use the homeomorphism we know $\widetilde{f}(w) \in V' \subset U$ and we are done.

Corollary 3.1.10. If Y is simply connected, then such a lift always exists.

Proposition 3.1.11. Let $p : E \to B$ be a cover, $b_0 \in B, e_0 \in p^{-1}(b_0)$, and $f : Y \to B$ a continuous map such that $f(y_0) = b_0$. If Y is connected and $\widetilde{f}_1, \widetilde{f}_2 : Y \to E$ are two lifts as in the previous theorem, then $\widetilde{f}_1 = \widetilde{f}_2$.

Proof.
Consider $A = \{\widetilde{f}_1 = \widetilde{f}_2\}$, we know A is nonempty.

We claim A is closed, since for any y such that $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, there exists U evenly covered neighbourhood of $f(y)$ such that $\widetilde{f}_i(y) \in U_i$ disjoint, since $\widetilde{f}_i$ are continuous and hence A is closed.

Similarly, we will have A is open by the locally homeomorphism.

3.2 Covering Transformations

For this subsection, all spaces are assumed to be path-connected and locally connected.

Definiton 3.2.1. If $p : E \to B, p' : E' \to B$ are coverings, a **homomorphism of coverings** $h : (E, p) \to (E', p')$ is a continuous map $h : E \to E'$ such that $p' \circ h = p$.

Definiton 3.2.2. An **isomorphism of coverings** is a homomorphism of coverings which is also a homeomorphism.

Theorem 3.2.1. Let $p : E \to B, p' : E' \to B$ be coverings of B with $p(e_0) = p'(e_0') = b_0 \in B$. Then there is an equivalence of coverings $h : E \to E'$, $h(e_0) = e_0'$ if and only if $H = p_*(\pi_1(E, e_0))$ and $H' = p_*'(\pi_1(E', e_0'))$ are equal as subgroups.

$$\begin{tikzcd}
 & (E', e_0') \arrow[d, "p'"] \\
(E, e_0) \arrow[ur, "\widetilde{h}"] \arrow[r, "p"] & (B, b_0)
\end{tikzcd}$$

Proof.

The sufficiency is trivial. It suffices to show the necessity, if $H = H'$, we know there exists $h : (E, e_0) \to (E', e_0')$ and $h' : (E', e_0') \to (E, e_0)$ such that $h \circ p' = p, h' \circ p = p'$. We have $(h \circ h') \circ p = p$, which is a lift of p and hence it has to be id_E and we are done.

Proposition 3.2.2. If $h, k : (E, p) \to (E', p')$ are homeomorphisms of coverings p, p' of B such that $h(e) = k(e)$ for some $e \in E$, then $h = k$.

Definiton 3.2.3. (Deck Transformation)

If $E = E', p = p'$ an equivalence of p interchanges points in the fiber over each $b \in B$, such a self-quivalence is called an automorphism of (E, p) or a **deck transformation**.

Definiton 3.2.4. (Deck Group)

The deck transformations form a group under composition of maps, called the **deck group** of (E, P) and denoted $\mathcal{D}(E, p)$.

Corollary 3.2.3. If $p : E \to B$ is a covering and $p(e_1) = p(e_2)$, then there is $h \in \mathcal{D}(E, p)$ with $h(e_1) = e_2$ if and only if $p_*(\pi_1(E, e_1)) = p_*(\pi_1(E, e_2))$.

Corollary 3.2.4. If $h \in \mathcal{D}(E, p)$ so that $h(x) = x$ for some x, then $h = id_E$.

Theorem 3.2.5. (Main Theorem)

Let $p : E \to B$ and $p' : E' \to B$ be covering maps. Let $p(e_0) = p'(e_0') = b_0$. The covering maps p and p' are equivalent if and only if the subgroups $H = p_*(E, e_0)$ and $H'(p_*'(E', e_0'))$ are conjugate in $\pi_1(B, b_0)$.

Notice this is a general case for Theorem 2.2.1.

Proof.

If there exists an equivalence h, and $h(e_0) = e_0''$, then we may have

$$p_*(\pi_1(E, e_0)) = p_*'(\pi_1(E', e_0''))$$

and there exists δ from e_0'' to e_0' and we know $\delta_\# : \pi_1(E', e_0'') \to \pi_1(E', e_0')$ an isomorphism, and the induced $p_*'(\pi_1(E', e_0''))$ is conjugate with $p_*'(\pi_1(E', e_0'))$ by $p_*'([\delta])$.

To show the necessity, we consider if H' is conjuate to H, then there exists $[\gamma] \in \pi_1(B, b_0)$ such that $[\gamma]^{-1} H' [\gamma] = H$, for γ we may consider $\widetilde{\gamma}_{e_0'}$ and denote $e_0'' = \widetilde{\gamma}_{e_0'}(1)$. Then we may know that $p_*(\pi_1(E', e_0'')) = H$ and there exists h such that $h(e_0) = h(e_0'')$ an equivalence.

3.3 Universal Covering Spaces

Definiton 3.3.1. (Universal Cover)

A covering $p : E \to B$ is called a universal covering map is E is simply connected, then call E a **universal cover**.

Corollary 3.3.1. If a universal cover of B exists, it is unique up to equivalence of coverings.

Definiton 3.3.2. (Semi-locally Simply Connectness)

A topological space B is semi-locally simply connected if for any $b \in B$, there is a neighborhood U_b of b such that the inclusion $\iota : U_b \hookrightarrow B$ induces a trivial homomorphism $\iota_* : \pi_1(U_b, b) \to \pi_1(B, b)$.

Theorem 3.3.2. A topological space B has a universal cover if and only if V is path connected, locally path connected and semi-locally connected. (Simply connectness infers path-connectness).

To show the theorem we need two conclusions.

Proposition 3.3.3. Let $p : E \to B$ be a covering map, $p(e_0) = b_0$. Assume E is simply-connected. Then there exists a neighborhood U of b_0 such that the inclusion $\iota : U \hookrightarrow B$ induces a trivial homomorphism $\iota_* : \pi_1(U, b_0) \to \pi_1(B, b_0)$.

Proof.

Only thing need to be paid attention is that find a neighbourhood such that the loops on it have fiber consisted of loops, that is there is always $U \in \mathcal{N}(b_0)$ such that U is evenly covered, and hence U will satisfy the requirement.

Theorem 3.3.4. Let B be path connected, locally path connected and semi-locally simply connected. Let $b_0 \in B$ and $H \subset \pi_1(B, b_0)$ a subgroup. Then there is a covering $p : E \to B$ and a point $e_0 \in p^{-1}(b_0)$ such that $p_*\pi_1(E, e_0) = H$.

Proof.

This theorem need a construction, like we consider $\mathcal{P}$ to be all the paths from b_0 in B and define a equivalent relation by $\alpha \sim \beta$ if $\alpha(1) = \beta(1)$ and $[\alpha * \bar{\beta}] \in H$, then denote $\alpha_\#$ to be its equivalence class, and E to be all the equivalence classes. Define $p : E \to B$ by $\alpha_\# \mapsto \alpha(1)$.

We may define $(U \in \mathcal{N}(\alpha(1)), \alpha_\#)$ by all $(\alpha * \gamma)_\#$ such that γ is covered in U and it is eays to check p is continuous under the topology generated by $(U, \alpha_\#)$. For any $b \in B$ and $p(\beta) = b$, we consider U to be a local simply connected set and then define $(U, \beta) \to U$ be $\beta * \alpha \mapsto \alpha(1)$, which is checked to be a bijection and equals the restriction of p on (U, β) and easy to be check a homeomorphism. If β, γ and there exists δ, δ' such that $\beta * \delta \sim \gamma * \delta'$ and they are both in $(U, \beta) \cap (U, \gamma)$ for some path-connected and locally simply connected U, then it can be checked that $\beta \sim \gamma$ and hence p is a covering.

Now we only need to check that there exists e_0 such that $p_*(\pi_1(E, e_0)) = H$, which is easy to be checked since for any $[\gamma] \in H$, there is a unique lift of γ and it has to be a loop at e_0, then we know π_* will be a surjection to H and we are done.

Now we may prove the Theorem 2.3.2.

Proof.

Only need to check the necessity, we may know let $H = e$ and we can find a covering such that $p_*(\pi_1(E, e_0))$ is trivial. We should consider the construction, and the simply connectness can be obtained fro the construction directly.

3.4 Group Actions and Covering maps

Assume all spaces path connected and locally path connected

Theorem 3.4.1. If $p : E \to B$ is a cover with

$$H = p_*(\pi_1(E, e)) \subset \pi_1(B, p(e))$$

then

$$\mathcal{D}(E, p) \cong N(H)/H$$

where $N(H) = \{g \in \pi_1(B, p(e)) | gHg^{-1}\}$ is the normalizer of H.

Proof.

We know $\phi_e : \pi_1(B, p(e)) \to F := p^{-1}(p(e))$ is surjective, and we may consider if $\phi_e([\gamma]) = \phi_e([\delta])$ then $p_*([\gamma])p_*([\delta])^{-1} \in H$ and hence it induce $\phi_e : \pi_1(B, p(e))/H \to F$ a bijection. And $\varphi_e : \mathcal{D}(E, p) \to F$ by $\varphi_e(h) = h(e)$.

Then consider for $e' \in Im\varphi_e$, consider α, β from e to e', we will have $p_*([\alpha])p_*([\beta])^{-1} \in H$ and $p_*([\alpha])Hp_*([\alpha])^{-1} = H$ and hence $p_*([\alpha]) \in N(H)$, and hence $Im\varphi_e \subset \phi_e(N(H)/H)$.

For any $e' \in \phi_e(N(H)/H)$, it is easy to check that $p_*(\pi_1(E, e')) = H$ and hence there exists $h \in \mathcal{D}(E, p)$ such that $h(e) = e'$.

Now we know $Im(\varphi_e) = \phi_e(N(H)/H)$ and hence $\phi_e^{-1} \circ \varphi_e$ is a injective and surjective, so an isomorphism and we are done.

Corollary 3.4.2. If $\pi_1(E, e) = 0$, then $\mathcal{D}(E, p) \cong \pi_1(B, p(e))$.

Definiton 3.4.1. A covering $p : E \to B$ is regular if p_* is a normal subgroup of $\pi_1(B, p(e))$ for any $e \in E$.

Proposition 3.4.3. A covering $p : E \to B$ is regular if and only if the deck group acts transitively on the fibers of p.

Proof.

The sufficiency can be obtained by consider the isomorphism between $\mathcal{D}(E, p)$ and $\pi_1(B, p(e))/H$.

To see the necessity, we may know that $N(H) = \pi_1(B, p(e))$ and we are done.

Corollary 3.4.4. If $p : E \to B$ is regular, then

$$\mathcal{D}(E, p) \cong \pi_1(B, p(e))/p_*\pi_1(E, e)$$

It is easy to know that a unversial cover is regular.

Example 3.4.1.

- $p : \mathbb{R} \to S^1$ by $t \mapsto \exp(2\pi it)$
- $\mathbb{R}^2 \to T^2$ naturally.
- $p : S^2 \to \mathbb{R}P^2$ quotient.

Definiton 3.4.2. We call G acts freely on X if $gx = x$ for some x implies that $g = e_G$.

Definiton 3.4.3. The group G acts properly discontinuous on X if for any $x \in X$, there is an open neighborhood U_x of x such that $gU_x \cap U_x =$ for any $g \neq e_G$.

Proposition 3.4.5. If X is Hausdorff and G is a finite group of homeomorphisms of X acting freely on X, the action of G is properly discontinuous.

Theorem 3.4.6. Let X be a path-connected, locally path-connected topological space, and $G \leq Homeo(X)$. Then $\pi : X \to X/G$ is a covering if and only if G acts properly discontinuous on X. Moreover, if this is the case, the deck group $\mathcal{D}(X, \pi)$ of the covering is isomorphic to G and the covering is regular.

Proof.

We know π is an open map. To see the necessity, for any $x \in X$, $x \in U$ such that $gU \cap U$ empty for any $g \neq e_G$. Then $\pi(U)$ is an evenly covered neighborhood of $[x]$ since it is an open continuous bijection.

To see the sufficiency, for $x \in X$, V_x is a neighborhood of $[x]$ which is evenly covered, $V_x \cong U$ containing x. Then if $y \in gU \cap U$, then $g^{-1}y, y$ are in U and they have the same image under π, which is a contradiction.

Now we prove that $\mathcal{D}(X, \pi) \cong G$. g is obviously in $\mathcal{D}(X, \pi)$, and for any $h \in \mathcal{D}(X, \pi)$, $h(x) = gx$, then h has to be g since π is a covering.

π is regular by proposition 2.4.3.

Corollary 3.4.7. If X is simply connected and G acts properperly discontinuously on X, then $\pi_1(X/G) \cong G$.

Proposition 3.4.8. If $p : E \to B$ is a cover, then $\mathcal{D}(E, p)$ acts properly discontinuous on E.

Proposition 3.4.9. Any regular cover of B is of the form E/G, where E is the universal cover of B and G acts properly discontinuous on E.

3.5 Exercises

Ex 3.1. Show that the map $p : S^1 \to S^1, p(z) = z^n$ is a covering.

Proof.

For $x = e^{i2\pi t}$, we have $y = e^{i2\pi t/n}$ such that $y^n = x$ and hence p is surjective. Obviously continuous and choose $B_(1/2n)$ which is evenly covered.

Ex 3.2. Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply-connected, then p is a homeomorphism.

Proof.

For $b \in B$, if there exists e_b, e'_b distinct in the fiber of b, then consider γ a path from e_b to e'_b and we know $p \circ \gamma$ is a trivial loop in B and hence $e_b = e'_b$ which is a contradiction. So p^{-1} is well defined and we know p is open by choosing a evenly covered neighborhood.

Ex 3.3.

- Show that if $n > 1$ then any continuous map $f : S^n \to S^1$ is nullhomotopic.
- Show that any continuous map $f : \mathbb{R}P^2 \to S^1$ is nullhomotopic.

Proof.

If $\widetilde{f}$ is a lift of a continuous map and it is nullhomotopic, then we know $p \circ F$ will be a homotopy from f to a constant. So since $\mathcal{R}$ is a cover of S^1 which is contractible, and we are done since $p_*(\pi_1(S^n, e))$ is trivial. For the second problem, notice that $p_*(\mathbb{R}\mathcal{P}^{\in}, e)$

has to be trivial since, $\mathbb{R}P^2 = S^3/\{0,1\}$ which means $\pi_1(\mathbb{R}P^2)$ is $\{0,1\}$ since S^3 is simply connected.

4 Homology

4.1 Singular Homology

Definiton 4.1.1. (Simplex)

The **standard** n-simplex is the set

$$\Delta^n := \left\{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \middle| \sum_{i=0}^{n} t_i = 1, t_i \geq 0\right\}$$

An n-simplex is the convex span in $\mathbb{R}^m$ of $n+1$ points $v_0, \cdots, v_n$ that do not lie in a hyperplane of dimension less than n.

We denote

$$[v_0, \cdots, v_n]$$

for the n-simplex generated by $\{v_i\}$, and there is a canonical linear homeomorphism from Δ^n to any n-simplex $[v_0, \cdots, v_n]$ given by

$$\Delta^n \to [v_0, \cdots, v_n] := (t_0, \cdots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i$$

If we delete one vertex, then remaining n vertices span a $(n-1)$-simplex, called a **face** of $[v_i]_{i=1}^n$ and the union of all faces is called the **bounday** of the simplex and $[v_0, \cdots, \hat{v_i}, \cdots, v_n]$ denotes that v_i is deleted.

Definiton 4.1.2. A **singular** n-simplex in a space X is a continuous map $\sigma : \Delta^n \to X$.

Definiton 4.1.3. (Homology)

Let $C_n(X)$ be the free abelian group with basis consisted of the singular n-simplices in X, i.e.

$$C_n(X) = \left\{\sum_i n_i\sigma_i | n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \text{ continuos}\right\}$$

where the formal sum $\sum\limits_{i=1} n_i\sigma_i$ is finite and we call an element of $C_n(X)$ an n-**chain** in X.

The **boundary maps** $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined as

$$\partial_n(\sigma) := \sum_{i=1}^{n} (-1)^i \sigma|_{[v_0,\cdots,\hat{v_i},\cdots,v_n]}$$

and we will know that $\partial_n \circ \partial_{n+1} = 0$.

We call $C_\bullet(X) = (C_n(X), \partial_n)n \in \mathbb{N}$ the **singular chain complex** of X.

The n-th singular homology group of X is defined by

$$H_n(X) := \ker(\partial_n)/\mathrm{Im}(\partial(n+1))$$

Proof.

We know that for $\sigma : \Delta^{n+1} \to X$

$$\begin{aligned}\partial_n(\partial_{n+1}(\sigma)) &= \partial_n\left(\sum_{i=1}^{n+1}(-1)^i\sigma|_{[v_0,\cdots,\hat{v_i},\cdots,v_{n+1}]}\right)\\ &= \sum_{i=1}^{n+1}\sum_{j\neq i}(-1)^i(-1)^{\delta(j,i)}\sigma|_{[v_0,\cdots,\hat{v_i},\cdots,\hat{v_j},\cdots,v_{n+1}]}\end{aligned}$$

where $\delta(j,i) = j$ if $j < i$ and it is $j-1$ if $j > i$, so we may get that for each not order 2-tuple (i,j), the coefficient will of $\sigma_{(i<j)}$ will always be $(-1)^{i+j-1} + (-1)^{j+i} = 0$.

By this, we may know that $\mathrm{Im}(\partial_{n+1})$ will be a subgroup of $\ker(\partial_n)$ the the definition goes.

Definiton 4.1.4.

- $Z_n := \ker(\partial_n)$ is the group of n-**cycles**.
- $Z_n := \mathrm{Im}(\partial_n)$ is the group of n-**boundaries**.

Proposition 4.1.1. Let x_0 be a point. Then

$$H_n(x_0) = \begin{cases}\mathbb{Z}, & n = 0\\ 0, & n > 0\end{cases}$$

Proof.

We may know $\partial_n(\sigma_n) = 0$ when n is odd and σ_{n-1} when n is even since there is only one kind of singular n-simplex and then we know $\ker(\partial_n) = \mathbb{Z}$ when n is odd and 0 when n is even and hence for all n even except for 0 σ_n have 0 kernel and for n odd it is $\mathbb{Z}/\mathbb{Z}$ and we are done.

Proposition 4.1.2. Suppose X is a space and $(X_\alpha)_{\alpha\in A}$ to be the path-connected components of X. Then, $H_n(X) \cong \bigoplus_{\alpha\in A} H_n(X_\alpha)$.

Proof.

Since Δ^n is path connected and we know $\mathrm{Im}(\sigma) \subset X_\alpha$ for some α, so we may construct an isomorphism between $C_n(X)$ and $\bigoplus_\alpha C_n(X_\alpha)$ by

$$(\sigma_\alpha) \mapsto \sigma_\alpha$$

for $\mathrm{Im}(\sigma_\alpha) \subset X_\alpha$ and span it to $\bigoplus_\alpha C_n(X_\alpha)$, since $\partial(C_n(X_\alpha)) \subset C_{n-1}(X_\alpha)$ we may know that $\ker(\partial_n), \mathrm{Im}(\partial_{n+1})$ can be also given a direct sum decomposition like this, and for some $\sigma + \mathrm{Im}(\partial_{n+1})$, we may maps it to $(0,\cdots,\sigma + \mathrm{Im}_\alpha(\partial_{n+1}),\cdots,0)$ if $\mathrm{Im}(\sigma) \subset X_\alpha$ and we are done by span it to $\bigoplus_\alpha H_n(X_\alpha)$.

Definiton 4.1.5. (Augmentation map)

$\epsilon : C_0(X) \to \mathbb{Z}$ by $\sum\limits_i n_i\sigma_i \mapsto \sum_i n_i$.

Proposition 4.1.3. If $X \neq \emptyset$ is path connected, then $H_0(X) \cong \mathbb{Z}$.

We know

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

and we claim $\ker(\epsilon) = \mathrm{Im}(\partial_1)$ for the augmentation map. If $\sum\limits_i n_i\sigma_i \in \ker(\epsilon)$, then $\sum\limits_i n_i = 0$ and we may assume that $\sigma_i : [v_0] \to X$ at p and for any p, q distinct in X, we may find a path from p to q which will satisfy that $\partial_1(\gamma) = (p) - (q)$ and hence we may obtained that $\ker(\epsilon) \subset \mathrm{Im}(\partial_1)$ by induction, and obviously $\mathrm{Im}(\partial_1) \in \ker(\epsilon)$.

Notice $\ker(\partial_0) = C_0(X)$ and then we know $C_0(X)/\ker(\epsilon) \cong H_0(X)$, where the former is isomorphic to $\mathbb{Z}$ and we are done.

Definiton 4.1.6. (Reduced Homology)

The **reduced homology** groups of X, $\widetilde{H}_n(X)$ are the homology groups of the augmented chain complex of X defined as

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

this complex is a chain complex since $\epsilon \circ \partial_1 = 0$. ϵ induces an onto map $C_0(X)/\mathrm{Im}(\partial_1) = H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$ and $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$ and $H_n(X) \cong \widetilde{H}_n(X)$ for $n \geq 1$.

4.2 Homotopy Invariance

Definiton 4.2.1. Let $f : X \to Y$ continuous and we have an induced homomorphism from $C_n(X) \to C_n(Y)$

$$f_\#(\sum_i n_i \sigma_i) = \sum_i n_i (f \circ \sigma_i)$$

Lemma 4.2.1. $f_\#$ is a chain map, i.e. $f_\# \partial_n = \partial_n f_\#$.

Proof.

Since

$$\begin{aligned} f_\#(\partial_n(\sigma)) &= f_\# \left(\sum_i (-1)^i \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} \right) \\ &= \sum_i (-1)^i f \circ \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} \\ &= \sum_i (-1)^i (f \circ \sigma)|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} \\ &= \partial_n(f_\#(\sigma)) \end{aligned}$$

Corollary 4.2.2. $f_\#$ takes n-cycles/boundaries to n-cycles/boundaries.

Corollary 4.2.3. The map $f : X \to Y$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.

Proposition 4.2.4.

- If $X \xrightarrow{g} Y \xrightarrow{f} Z$ are maps, then $(f \circ g)_* = f_* \circ g_*$.
- $(id_X)_* = id_{H_n(X)}$

Proof.

Notice

$$f_*(\sigma + \mathrm{Im}_X(\partial_n)) = f_\#(\sigma) + \mathrm{Im}_Y(\partial_n)$$

Theorem 4.2.5. If $f, g : X \to Y$ are homotopic maps, then they induce the same homomorphisms $f_* = g_* : H_n(X) \to H_n(Y)$ for every n.

Proof.

Corollary 4.2.6. If $f : X \to Y$ is a homotopy equivalence then $f_* : H_n(X) \to H_n(Y)$ are isomorphisms for every n.

Corollary 4.2.7. If X is contractible, then $\widetilde{H}_n(X) = 0$ for every n.

Definiton 4.2.2. (Chain Homotopy)

A map $P : C_n(X) \to C_{n+1}(X)$ satisfies

$$\partial P + P\partial = g_\# - f_\#$$

is called a **chain homotopy** between $g_\#, f_\#$.

More generally, if $(C_i, \partial_i), (D_i, \partial_i)$ are two chain complexs with two chain map $h, k : C_i \to D_i$ suh that there exists a map $P : C_n \to D_{n+1}$ such that $P\partial + \partial P = h - k$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[dl, "P"] & C_n \arrow[r] \arrow[d, "h \text{ or } k"] \arrow[dl, "P"] & C_{n-1} \arrow[r] \arrow[dl, "P"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

4.3 Homology of a pair

Definiton 4.3.1. Given a space X and a subspace $A \subset X$, define

$$C_n(X, A) := C_n(X)/C_n(A)$$

called the set of **relateive n-chains**.

$\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$ and induced maps $\partial : C_n(X, A) \to C_{n-1}(X, A)$, with $\partial^2 = 0$ and we get a chain complex $(C_i(X, A), \partial_i)$ whose homology is called the **relative homology** of the pair (X, A), denoted as $H_n(X, A)$.

Definiton 4.3.2. (Connecting Homomorphism)

We consider a short exact sequence of chain complexes

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

which means the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial"] & B_n \arrow[r, "j"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d] & B_{n-1} \arrow[r, "j"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

commutes and there is a map $\partial : H_n(C_i) \to H_{n-1}(A_i)$ called a **connecting homomorphism**.

Proof.

Consider $c \in \ker(\partial_n) \subset C_n$, since the sequence is exact and we may know j is surjective, there exists $b \in B_n$ such that $c = j(b)$ and hence

$$j(\partial(b)) = \partial(j(b)) = \partial(c) = 0$$

and hence $\partial(b) \in \ker j = \mathrm{Im(i)}$. So there exists $a \in A_{n-1}$ such that $\partial(b) = i(a)$ and hence $\partial(i(a)) = i(\partial(a)) = 0$, which means $\partial(a) = 0$ since $\ker i = 0$. Define $\partial(c) = [a] \in H_{n-1}(A)$. Let us check this will be come a homomorphism, that is for $[c] \in H_n(C_i)$, we have

$$\partial(c) = [a] \in H_{n-1}(A)$$

where there exists $b \in B_n$ such that $c = j(b)$ and $\partial(b) = i(a)$, if there exists b' such that $c = j(b')$ then $b - b' \in \ker(j) \in \mathrm{Im}(i)$ and there exists a' such that $i(a') = b - b'$ and

$$\partial(b') = i(a + \partial(a'))$$

which since means $[a] = [a + \partial(a')]$ and hence the homomorphism is well-defined.

And for $c + \partial(c')$ we know

$$c + \partial(c') = j(b + \partial(b'))$$

and hence $\partial(b)$ unchanged and it is well-defined on $H_n(C_i)$ and we are done.

Theorem 4.3.1. The sequence

$$\cdots \to H_n(A_\bullet) \stackrel{i_*}{\to} H_n(B_\bullet) \stackrel{j_*}{\to} H_n(C_\bullet) \stackrel{\partial}{\to} H_{n-1}(A_\bullet) \to \cdots$$

is exact.

Proof.

Recall

$$i([\alpha]) = [i(\alpha)]$$

which is well-defined because if $\alpha - \alpha' \in \mathrm{Im}(\partial) \subset A_n$, then there is $a \in A_{n+1}$ such that $\partial(a) = \alpha - \alpha'$

$$i(a) - i(a) = i(\alpha - \alpha') = i(\partial(a)) = \partial(i(a)) \in \mathrm{Im}(\partial)$$

and hence i_* is well-defined and similarly j_* is well defined and we have shown above that ∂ is well defined.

"$\mathrm{Im}(i_*) = \ker(j_*)$": for any $[\alpha] \in H_n(A_\bullet)$, we have $i_*([\alpha]) = [i(\alpha)]$ and then

$$j_*(i_*([\alpha])) = [j(i(\alpha))] = [0]$$

and if $j_*([\beta]) = 0$, then $[j(\beta)] = 0$, which means there exists $c \in C_{k+1}$ such that $j(\beta) = \partial(c)$. Since j is surjective and there exists $b \in B_{k+1}$ such that $c = j(b)$ and hence $j(\partial(b)) = j(\beta)$ and hence $\beta - \partial(b) \in \ker(j) = \mathrm{Im}(i)$ and hence there exists $\alpha \in A_k$ such that $i(a) = \beta - \partial(b)$

and hence $i_*([a]) = [i(a)] = [\beta]$ and we are done.

"$\mathrm{Im}(j_*) = \ker(\partial)$": for any $j_*([\beta]) = [j(\beta)]$, we know $\partial([j(\beta)]) = [a]$ where $i(a) = \partial(\beta), a \in A_{n-1}$ and hence

$$i(\partial(a)) = \partial(i(a)) = 0$$

and hence $a \in \ker(\partial)$. For the other side, if $\partial([\gamma]) = 0$, then there exists $a \in A_{n-1},, \beta \in B_n$ such that $j(\beta) = \gamma$ and $i(a) = \partial(\beta), a \in \mathrm{Im}(\partial)$, which means there exists $\alpha \in A_n$ such that $\partial(\alpha) = a$ and hence

$$\beta - i(\alpha) \in \ker(\partial)$$

then

$$j_*([\beta - i(\alpha)]) = [j(\beta) - j(i(\alpha))] = [\gamma]$$

and we are done.

"$\mathrm{Im}(\partial) = \ker(i_*)$": for $[\gamma] \in H_n(C_\bullet)$, we may know that $\partial([\gamma]) = [a]$ where $i(a) = \partial(\beta)$ such that $j(\beta) = \gamma$, and then $i_*([a]) = [i(a)] = [\partial(\beta)] = 0$. For the other side, if $[i(\alpha)] = i_*([\alpha]) = 0$, then there exists $b \in B_{n+1}$ such that $\partial(b) = i(\alpha)$ and then $\partial[j(b)] = [\alpha]$ and we are done.

Definiton 4.3.3. (Induced Homomorphism)

Consider $f : (X, A) \to (Y, B)$ such that $f(A) \subset (B)$, then we may know $f_\#(C_n(A)) \subset C_n(B)$ and hence $f_\# : C_n(X, A) \to C_n(Y, B)$ is well-defined.

Then $f_\#\partial = \partial f_\#$ and it can induce $f_* : H_n(X, A) \to H_n(Y, B)$ by for

$$f_*([\sigma]) = [f \circ \sigma]$$

and

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0$$

is exact and we may use the general theory on this sequence.

Proof.

Consider $[\sigma] - [\sigma'] = \partial([\gamma]) \in \mathrm{Im}_{C_{n+1}(X,A)}(\partial)$, then $f_\#([\sigma] - [\sigma']) = f_\#(\partial([\gamma])) = \partial(f_\#([\gamma])) \in \mathrm{Im}_{C_n(Y,B)}(\partial)$.

Theorem 4.3.2. Let X be a topological space and let A be a subspace of X. Then there is a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A)$$

Proof.

Since

$$\begin{tikzcd} C_n(X) \arrow[r, "\pi"] \arrow[d, "\partial"] & C_n(X, A) \arrow[d, "\partial"] \\ C_{n-1}(X) \arrow[r, "\pi"] & C_{n-1}(X, A) \end{tikzcd}$$

commutes and we are done.

Corollary 4.3.3. There is a long exact sequence

$$\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X,A) \to \widetilde{H}_{n-1}(A) \to \cdots$$

Proof.
Notice we have

$$\begin{tikzcd}
0 \arrow[r] & C_0(A) \arrow[r] \arrow[d, "\epsilon"] & C_0(X) \arrow[r] \arrow[d, "\epsilon"] & C_0(X,A) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

commutes.

Corollary 4.3.4. For $x_0 \in X$, we have $\widetilde{H}_n(X) \cong H_n(X, x_0)$ for all n.

Corollary 4.3.5. There is a long exact sequence for homology of $(X, A, B), B \subset A \subset X$

$$\cdots \to H_n(A,B) \to H_n(X,B) \to X_n(X,A) \to H_{n-1}(A,B)$$