
NOTES FOR PDE BY EVANS

Based on the Lecture Notes by Cole Graham

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1 Sobolev Spaces

1.1 Definitions

Definiton 1.1.1. (Sobolev Space)

The Sobolev space $W^{k,p}(\Omega)$ is the set of distributions on Ω whose weak partial derivatives up to order k are in $L^p(\Omega)$, i.e. for $f \in C_c^\infty(\Omega)$, there is always some $u_\alpha \in L^p$, $|\alpha| \leq k$ such that

$$\partial^\alpha u(f) = \int_{\Omega} u_\alpha f$$

Then $W^{k,p}$ is a Banach space under the norm

$$\|u\|_{W^{k,p}(\Omega)}^p := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega)}^p$$

where $\|\partial^\alpha u\|_{L^p(\Omega)} = \|u_\alpha\|_{L^p(\Omega)}$ where u_α is the function satisfying the requirement above.

Definiton 1.1.2. The Sobolev space $\widetilde{W}^{k,p}(\Omega)$ is the completion in the $W^{k,p}$ norm of $C^k(\Omega)$ with finite norm.

Proposition 1.1.1. If $k \in \mathbb{N}_0$ and $p \in [1, \infty)$, then $W^{k,p} = \widetilde{W}^{k,p}$

Definiton 1.1.3. For $p = 2$, we can make $W^{k,2}$ a Hilbert space and use $H^k := W^{k,2}$. The inner product is defined by

$$\langle u, v \rangle_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)} = \sum_{|\alpha| \leq k} \int_{\Omega} \partial^\alpha u \partial^\alpha v$$

Theorem 1.1.2. (Caccioppoli Inequality)

Let $u \in C(B_R)$ satisfy $\Delta u = 0$. Then there is a constant $C(d) > 0$ such that for all $r \in [0, R)$,

$$\int_{B_r} |\nabla u|^2 \leq \frac{C}{(R-r)^2} \int_{B_R} u^2$$

Proof. Let $0 \leq \phi \leq 1$ be a smooth function that is supported in B_R and equals one on B_r . We know

$$\Delta(\phi u) = \phi \Delta u + 2 \nabla \phi \cdot \nabla u + u \Delta \phi = 2 \nabla \phi \cdot \nabla u + u \Delta \phi$$

and we may multiply both sides by $-\phi u$ and integrate by parts:

$$\begin{aligned} \int_{B_R} -(\phi u) \Delta(\phi u) &= \int_{B_R} |\nabla(\phi u)|^2 + \int_{\partial B_R(\phi u)} (\phi u) \frac{\partial(\phi u)}{\partial \nu} dS \\ &= -\frac{1}{2} \int_{B_R} \nabla(\phi^2) \cdot \nabla(u^2) - \int_{B_R} u^2 \phi \Delta \phi \\ &= \frac{1}{2} \int_{B_R} \Delta(\phi^2) u^2 - \int_{\partial B_R} u^2 \frac{\partial(\phi^2)}{\partial \nu} dS - \int_{B_R} u^2 \phi \Delta \phi \end{aligned}$$

which means

$$\int_{B_R} |\nabla(\phi u)|^2 = \int_{B_R} \left(\frac{1}{2} \Delta(\phi^2) - \phi \Delta \phi \right) u^2 = \int_{B_R} |\nabla \phi|^2 u^2$$

and we may arrange $|\nabla\phi| \leq \frac{C(d)}{R-r}$ and hence we are done. \square

Corollary 1.1.3. For all $k \in \mathbb{N}$, if $u \in C^{k+1}(B_R)$ satisfies $\Delta u = 0$, then there is a constant $C(d, k) > 0$ such that for all $r \in [0, R)$,

$$\int_{B_r} |D^k u|^2 \leq \frac{C}{(R-r)^{2k}} \int_{B_R} u^2$$

Proposition 1.1.4. We may divide $[r, R]$ to k subintervals of $\frac{R-r}{k}$.

1.2 Sobolev Embedding

Theorem 1.2.1. There exists a constant $C(d) > 0$ such that for all $u \in W^{1,1}(\mathbb{R}^d)$, we have

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^1(\mathbb{R}^d)}$$

Corollary 1.2.2. For each $p \in [1, d)$, there exists $C(d, p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$, we have

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

where $p^* = \frac{dp}{d-p}$.

Theorem 1.2.3. (Morrey's Inequality)

If $p \in (d, \infty]$, there exists $C(d, p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$, and

$$[u]_{C^{1-d/p}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

where

$$[u]_{C^\alpha(\mathbb{R}^d)} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

and u is some "version" which means a function.

Proof. For all $x \in \mathbb{R}^d$ and $R > 0$, we have

$$\begin{aligned} \left| \frac{1}{|\partial B_R|} \int_{\partial B_R(x)} (u - u(x)) \right| &= \frac{1}{|S^{d-1}|} \int_{S^{d-1}} |u(x + R\theta) - u(x)| d\theta \\ &\leq \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \int_0^R |\nabla u(x + r\theta)| dr d\theta \\ &= \frac{1}{|S^{d-1}|} \int_{B_R(x)} |\nabla u(y)| |y - x|^{-(d-1)} dy \\ &\leq \frac{1}{|S^{d-1}|} \|\nabla u\|_{L^p(B_R(x))} \left(\int_{B_R} r^{-(d-1)p/(p-1)} \right)^{(p-1)/p} \\ &= \|\nabla u\|_{L^p(B_R(x))} \frac{p-d}{p-1} R^{(p-d)/p} \\ &= C(d, p) R^{1-d/p} \|\nabla u\|_{L^p(B_R(x))} \end{aligned}$$

Suppose $|x - z| \leq R/2$ and we have

$$\begin{aligned} \left| \frac{1}{|\partial B_R|} \int_{\partial B_R(z)} (u - u(x)) \right| &= \frac{1}{|S^{d-1}|} \int_{S^{d-1}} |u(z + R\theta) - u(x)| d\theta \\ &\leq \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \int_0^{3R/2} |\nabla u(x + r\phi)| dr \left| \det \frac{D\theta}{D\phi} \right| d\phi \\ &\leq C(d, p) R^{1-d/p} \|\nabla u\|_{L^p(B_{3R/2}(x))} \end{aligned}$$

Now for $x \neq y$ in \mathbb{R}^d and $R := |x - y|$, $z = (x + y)/2$ and then

$$|u(x) - u(y)| \leq |u(x) - \frac{1}{|\partial B_R|} \int_{\partial B_R(z)} u| + |u(y) - \frac{1}{|\partial B_R|} \int_{\partial B_R(z)} u| \leq C R^{1-d/p} \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

□

Theorem 1.2.4. Let U be a bounded open subset of \mathbb{R}^d and suppose ∂U is C^1 . Assume $d < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^\gamma(\bar{U})$ for $\gamma = 1 - n/p$ with the estimate

$$\|u^*\|_{C^\gamma(\bar{U})} \leq C \|u\|_{W^{1,p}(U)}$$

where $C = C(p, n, U)$

Corollary 1.2.5. If $p \in (d, \infty)$, there exists $C(d, p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^d)$ we have

$$\|u\|_{C^{1-d/p}(\mathbb{R}^d)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^d)}$$

In particular, $W^{1,p} \hookrightarrow C^{1-d/p}$. The same holds for $p = \infty$ and then we replace $C^{0,1}$ the Lipschitz functions instead of C^1 .

Proof. We know that

$$\left| \frac{1}{|B_R|} \int_{B_R(z)} (u - u(x)) \right| \leq C R^{1-d/p} \|\nabla u\|_{L^p(B_R(x))}$$

and let $R = 1$ and we have

$$|u(x)| \leq C \left| \int_{B_1(x)} u \right| + C \|\nabla u\|_{L^p(B_1(x))} \leq C \|u\|_{W^{1,p}(\mathbb{R}^d)}$$

by Holder. □

Proposition 1.2.6. (General Sobolev Inequality)

Let Ω be a bounded C^1 domain. If $p \in [1, d)$, there exists a constant $C(d, p, \Omega) > 0$ such that for all $u \in W^{1,p}(\Omega)$,

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$$

and if $p \in (d, \infty]$, we similarly have

$$\|u\|_{C^{1-d/p}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$$

Theorem 1.2.7. (Rellich-Kondrachov)

Let Ω be a bounded C^1 domain. If $1 \leq p < d$ and $1 \leq q < p^*$, then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$. That is, closed bounded sets in $W^{1,p}(\Omega)$ are cinoact in $L^q(\Omega)$.

Corollary 1.2.8. Let Ω be \mathbb{R}^d or a bounded C^1 domain. Then there exists a constant $l(d) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $H^{k+l(\Omega)} \hookrightarrow C^k(\Omega)$.

Proposition 1.2.9. If $u \in C^2(B_R)$ is harmonic, then for all $k \in \mathbb{N}$ and $r \in (0, R]$, there exists

$$C(d, k, r, R) > 0$$

such that

$$\|u\|_{C^k(B_r)} \leq C \|u\|_{L^2(B_R)}$$

Proof. Assume $u \in C^\infty(B_R)$ and fix k and take $l(d)$, then since

$$\|u\|_{H^{k+l}(B_r)} \leq C(d, k, R, r) \|u\|_{L^2(B_R)}$$

and we are done by $H^{k+l}(\Omega)$ is compactly embedded in $C^k(\Omega)$. \square

Proposition 1.2.10. (Weyl's Lemma)

If $u \in L^2(B_1)$ is harmonic in a distributional sense, then $u \in C_{loc}^\infty(B_1)$ and $\Delta u = 0$ classically.

1.3 Traces

Proposition 1.3.1. Let $\mathbb{H} = \mathbb{R} \times \mathbb{R}^+$. There is a bounded operator $\text{Tr} : H^1(\mathbb{H}) \rightarrow L^2(\partial\mathbb{H})$ such that for all $u \in H^1(\mathbb{H}) \cap C(\overline{\mathbb{H}})$, $\text{Tr}u = u|_{\partial\mathbb{H}}$.

Proof. By density, we may only prove the bound on $H^1(\mathbb{H}) \cap C(\overline{\mathbb{H}})$. Given $x \in \mathbb{R}$ and $y \in [0, 1]$, we have

$$u(x, 0) = u(x, y) - \int_0^y \partial_y u(x, z) dz$$

and we will have

$$u(x, 0)^2 \leq 2u(x, y)^2 + 2y \int_0^y |\partial_y u(x, z)|^2 dz$$

by Cauchy and then

$$\|u\|_{L^2(\partial\mathbb{H})}^2 \leq 2 \int_{\mathbb{R} \times [0, 1]} u^2 + 2 \int_{\mathbb{R} \times [0, 1]} |\nabla u|^2 \leq 2 \|u\|_{H^1(\mathbb{H})}^2$$

\square

Theorem 1.3.2. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there is a well-defined trace operator $\text{Tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ as above.

Definiton 1.3.1. Let Ω be open, $p \in [1, \infty]$ and k some integer. Then $W_0^{k,p}(\Omega)$ is the completion in the $W^{k,p}$ norm of $C_c^k(\Omega)$.

Proof. Let Ω be a bounded Lipschitz domain and $p \in [1, \infty]$. Then $W_0^{1,p}(\Omega)$ is the kernel of the trace operator. \square

Theorem 1.3.3. (Poincare Inequality)

Let Ω be bounded and Lipschitz and $p \in [1, \infty]$. Then there exists a constant $C(p, \Omega) > 0$ such that if $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

Theorem 1.3.4. Let Ω be bounded and Lipschitz and $p \in [1, \infty]$. Then there exists a constant $C(p, \Omega) > 0$ such that if $u \in W^{1,p}(\Omega)$, then

$$\|u - \frac{1}{|\Omega|} \int_{\Omega} u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$