# Chapter 1

## 1.1 Basics of Stochastic Processes

We will refer $X_t$ to be real or $\mathbb{R}^d$-valued continuous-time stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P)$. For every fixed $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is called a trajectory or sample path of the process.

For a real-valued stochastic process, let $- \leq t_1 < \cdots < t_n$ be fixed. Then we know

$$P_{t_1, \cdots, t_n} = P \circ (X_{t_1}, \cdots, X_{t_n})^{-1}$$

is a probablity distribtuion on $\mathbb{R}^n$, which is called the finite-dimensional marginal distribution of the process.

**Theorem 1.1**

*(Kolmogorov's extension theorem) Consider a family of probablity measures*

$$\{P_{t_1, \cdots, t_n}, t_1 < \cdots < t_n, n \geq 1, t_i \geq 0\}$$

*such that*

*a. $P_{t_1, \cdots, t_n}$ is a probablity on $\mathbb{R}^n$.*

*b. For $\{t_{k_1} < \cdots < t_{k_m}\} \subset \{t_1 < t_2 < \cdots < t_n\}$, $P_{t_{k_1}, \cdots, t_{k_m}}$ is required to be a marginal of $P_{t_1, \cdots, t_n}$, then there exists a real-valued stochastic process $X_t$ owning finite-dimensional marginal distributions of $\{P_{t_1, \cdots, P_{t_n}}\}$.* ♡

**Definition 1.1**

*A real-valued process $X_t$ is a second-order process iff $EX_t^2 < \infty, t \geq 0$, define*

$$m_X(t) = EX_t, \Gamma_X(s,t) = cov(X_s, X_t)$$

♣

**Definition 1.2**

*A real-valued process $X_t$ is said to be Gaussian if its finite-dimensional marginal distributions are multidimensional Gaussian laws.* ♣

**Proposition 1.1**

*A Gaussian process is determined by $m_X$ and $\Gamma_X$, conversely, for any $m : \mathbb{R}_+ \to \mathbb{R}$ and a symmetric $\Gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ which is nonnegative definite, there always exists a Gaussian process with mean $m$ and covariance function $\Gamma$ by Kolmogorov's extension theorem.* ♠

**Definition 1.3**

*We call two processes $X, Y$ are equivalent if for all $t \geq 0$, $X_t = Y_t$ a.s. And we call they indistinguishable if $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$ and for all $\omega$ in some set with probability $1$.* ♣

**Proposition 1.2**

*Two equivalent processes with right-continuous trajectories are indistinguishable.* ♠

**Proof** Let $X_q = Y_q$ on $\Omega_q$ for $q \in \mathbb{Q}$ and let $\Omega' = \bigcap_{q \in \mathbb{Q}} \Omega_q$ and we know $\Omega'$ has the probability $1$. And it is easy to check that $X_t = Y_t$ on $\Omega'$ for all $t$.

**Theorem 1.2**

*(Kolmogorov's continuity theorem) Suppose that $X = X_t, t \in [0,T]$ satisfies*

$$E(|X_t - X_s|^\beta) \leq K|t-s|^{1+\alpha}$$

*for all $s, t \in [0,T]$ and for some constants $\beta, \alpha, K > 0$. Then there exists a version $\tilde{X}$ of $X$ such that, if $\gamma < \alpha/\beta$,* *then*

$$|\tilde{X_t}| - \tilde{X_s} \leq G_\gamma |t-s|^\gamma$$

*for all* $s, t \in [0, T]$, *where* $G_\gamma$ *is a random variable. The trajectories of* $\tilde{X}$ *are Holder continuous of* $\gamma$ *for any* $\gamma < \alpha/\beta$.

**Definition 1.4**

$\mathcal{F}_t$ *is an increasing family of sub-*$\sigma$*-field of* $\mathcal{F}$. *A process* $X_t$ *is* $\mathcal{F}_t$*-adapted if* $X_t$ *is* $\mathcal{F}_t$*-measurable for all* $t \geq 0$.

**Definition 1.5**

*An adapted process* $X_t, t \geq 0$ *is a Markov process w.r.t. a filtration* $\mathcal{F}_t$ *if for any* $s \geq 0, t > 0$ *and any measurable and bounded function* $f : \mathbb{R} \to \mathbb{R}$,

$$E(f(X_{s+t})|\mathcal{F}_s) = E(f(X_{s+t})|X_s)\ a.s.$$

**Proposition 1.3**

*A* $\mathcal{F}_t$*-Markov process* $X_t$ *is also a* $\mathcal{F}_t^X$*-Markov process where*

$$\mathcal{F}_t^X = \sigma\{X_u, 0 \leq u \leq t\}$$

**Proof** Notice

$$E(f(X_{s+t}|\mathcal{F}_s^X)) = E(E(f(X_{s+t}|\mathcal{F}_s))|\mathcal{F}_s^X) = E(E(f(X_{s+t}|X_s))|\mathcal{F}_s^X) = E(f(X_{s+t}|X_s))$$

since $\sigma(X_s) \subset F_s^X \subset F_t$.

**Definition 1.6**

*Assume a filtration* $\mathcal{F}_t$ *on* $(\Omega, \mathcal{F}, P)$ *satisfies that for any* $P(A) = 0$, $A \in \mathcal{F}$, $A \in \mathcal{F}_0$ *and it is right-continuous,i.e.*

$$\mathcal{F}_t = \cap_{n \geq 1} \mathcal{F}_{t+n^{-1}}$$

*Then consider a r.v.* $T : \Omega \to [0, \infty]$ *is a stopping time w.r.t. to the filtration if*

$$\{T \leq t \in \mathcal{F}_t\}$$

*for any* $t \geq 0$.

**Proposition 1.4**

*a.* $T$ *is a stopping time iff* $\{T < t\} \in \mathcal{F}_t$ *for all* $t \geq 0$.
*b.* $S \vee T$ *and* $S \wedge T$ *are stopping times.*
*c. Givene a stopping time* $T$,

$$\mathcal{F}_T = \{A, A \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$$

*is a* $\sigma$*-algebra.*
*d. If* $S \leq T$, *then* $\mathcal{F}_S \subset \mathcal{F}_T$.
*e. Let* $X_t, t \geq 0$ *be a continuous and adapted process. The hitting time of a set* $A \subset \mathbb{R}$ *is defined by*

$$T_A = \inf\{t \geq 0, X_t \in A\}$$

*and whether* $A$ *is open or closed,* $T_A$ *is a stopping time.*
*f. Let* $X_t$ *be an adapted stochastic process with right-continuous paths and let* $T < \infty$ *be a stopping time. Then the random variable*

$$X_T(\omega) = X_{T(\omega)(\omega)}$$

*is* $\mathcal{F}_T$*-measurable.*

**Definition 1.7**

*An adapted process $M = M_t, t \geq 0$ is called a martingale w.r.t. a filtration $\mathcal{F}_t, t \geq 0$ if*
*a. for all $t \geq 0$, $E(|M_t|) < \infty$*
*b. for each $s \leq t$, $E(M_t|\mathcal{F}_s) = M_s$*

**Proposition 1.5**

*a. For any integrable random varibale $X$, $E(X|\mathcal{F}_t)$ is a martingale.*
*b. If $M_t$ is a submartingale then $t \to E(M_t)$ is nondecreasing.*
*c. If $M_t$ is a martingale and $\varphi$ is a convex function such that $E|\phi(M_t)| < \infty$ for all $t \geq 0$ then $\phi(M_t)$ is a submartingale.*

**Proof** Only (c) is needed to be proved. Conside $ax + b \leq \phi(x)$ and we know

$$E(\phi(M_t)|\mathcal{F}_s) \geq aE(X_t|\mathcal{F}_s) + b$$

for any such $a, b$ and hence

$$E(\phi(M_t)|\mathcal{F}_s) \geq \phi(M_t)$$

**Definition 1.8**

*An adapted process $M_t, t \geq 0$ is called a local martingale if there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that, for any $n \geq 1$ $M_{t \wedge \tau_n}$ is a martingale.*

**Theorem 1.3**

*Let $M_t, t \geq 0$ be a continuous local martingale such that $M_0 = 0$. Let $\pi = \{0 = t_0 < t_1 < \cdots < T_n = t\}$ be a partition of $[0, t]$. Then we have*

$$\sum_{j=0}^{n-1} (M_{t_{j+1}} - M_{t_j})^2 \to \langle M \rangle_t, |\pi| \to 0$$

*in probability, where $\langle M \rangle_t, t \geq 0$ is called the quadratics variation of the local martingale. Moreover, if $M_t, t \geq 0$ is a martingale then the convergence holds in $L^1(\Omega)$.*

**Theorem 1.4**

*The quadratic variation is the unique continuous and increasing process satisfying $\langle M \rangle_0 = 0$ and*

$$M_t^2 - \langle M \rangle_t$$

*is a local martingale.*

## 1.2 Brownian Motion

**Definition 1.9**

*A real-valued stochastic process $B = (B_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}; P)$ is called a Brownian motion if it satisfies the following conditions:*
*a. Almost surely $B_0 = 0$.*
*b. For all $0 \leq t_1 < \cdots t_n$ the increments $B_{t_n} - B_{t_{n-1}}, \cdots, B_{t_2} - B_{t_1}$ are independent random variables.*
*c. If $0 \leq s < t$, the increment $B_t - B_s$ is a Gaussian random variable with mean zero and variance $t - s$.*
*d. With probability one, the map $t \to B_t$ is continuous.*
*A d-dimensional Brownian motion is defined as an $\mathbb{R}^d$-valued stochastic process $B = (B_t)_{t \geq 0}$, $B_t = (B_t^1, \cdots, B_t^d)$, where $B^1, \cdots, B^d$ are d independent Brownian motions.*

**Proposition 1.6**

*Properties (a),(b),(c) are equivalent to that $B$ is a Gaussian process,i.e. for any finite set of indices $t_1, \cdots, t_n$, $(B_{t_1}, \cdots, B_{t_n})$ is a multivariate Gaussian random variable, equivalently, any linear combination of $B_{t_i}$ is normal distributed r.v., with mean zero and covariance function*

$$\Gamma(s,t) = \min(s,t)$$

**Proof**

Suppose (a),(b),(c) holds, then we know $(B_{t_1}, \cdots, B_{t_n})$ is normal for any finite indices and then

$$m(t) = E(B_t) = 0$$
$$\Gamma(s,t) = E(B_s B_t) = E(B^2_{\min(s,t)}) = \min(s,t)$$

Conversly, we know $E(B_0^2) = 0$ and hence $B_0 = 0$ a.s., then we know $E(B_s^2) = s$ and for any $0 < s < t$,

$$E(B_s(B_t - B_s)) = 0$$

and it is easy to check (c), and (b) is deduced by computing the covariance of the increments, notice that two r.v.s are independent iff $\phi_{(X_1,X_2,\cdots,X_n)} = \phi_{X_1}\phi_{X_2}\cdots\phi_{X_n}$ which implies that normal r.v.s are independent iff they have zero covariances.

**Theorem 1.5**

*(Kolmogorov's continuity theorem) Suppose that $X = (X_t)_{t\in[0,T]}$ satisfies*

$$E(|X_t - X_s|^\beta) \le K|t-s|^{1+\alpha}$$

*for all $s, t \in [0,T]$ and some constant $\beta, \alpha, K > 0$. Then there exists a version $\tilde{X}$ of $X$ such that if*

$$\gamma < \alpha/\beta$$

*then*

$$|\tilde{X}_t - \tilde{X}_s| \le G_\gamma |t-s|^\gamma$$

*for all $s, t \in [0,T]$ where $G_\gamma$ is a random variable. The trajectories of $\tilde{X}$ are Holder continuous of order $\gamma$ for any $\gamma < \alpha/\beta$.*

**Proposition 1.7**

*There exists a version of $B$ with Holder-continuous trajectories of order $\gamma$ for any $\gamma < (k-1)/2k$ on any interval $[0,T]$.*

**Proof**

Since we know $B_t - B_s$ has the normal distribution $\mathcal{N}(0, t-s)$ and then we know

$$E\Big((B_t - B_s)^{2k}\Big) = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} x^{2k}\exp^{-\frac{x^2}{2(t-s)}}\,dx = (2k-1)!!(t-s)^k = \frac{(2k)!}{2^k k!}(t-s)^k$$

and by the theorem 1.1, the proposition holds.

**Proposition 1.8**

*Brownian motion are basic properties:*

*a. For any $a > 0$, the process $(a^{-1/2}B_{at})_{t\ge 0}$ is a Brownian motion.*

*b. For any $h > 0$, the process $(B_{t+h} - B_h)_{t\ge 0}$ is a Brownian motion.*

*c. The process $(-B_t)_{t\ge 0}$ is a Brownian motion.*

*d. Almost surely $\lim_{t\to\infty} B_t/t = 0$ and the process $X_t = tB_{1/t}$ for $t > 0$, $X_t = 0$ for $t = 0$ is a Brownian motion.*

**Proof**

a. Consider $0 \le t_1 < t_2 < \cdots < t_n$ and we may calculate the covariance matrix for

$$a^{-1/2}B_{at_n} - a^{-1/2}B_{at_{n-1}}, \cdots, a^{-1/2}B_{at_2} - a^{1/2}B_{at_1}$$

by

$$\begin{aligned}
&E[(a^{-1/2}B_{at_j} - a^{-1/2}B_{at_{j-1}})(a^{-1/2}B_{at_k} - a^{-1/2}B_{at_{k-1}})] \\
=&a^{-1}(at_j \wedge at_k) - a^{-1}(at_j \wedge at_{k-1}) - a^{-1}(at_{j-1} \wedge at_k) + a^{-1}(at_{j-1} \wedge at_{k-1}) \\
=&\begin{cases} t_j - t_{j-1} - t_{j-1} + t_{j-1} = t_j - t_{j-1} & \text{if } j = k \\ t_j - t_j - t_{j-1} + t_{j-1} = 0 & \text{if } j < k \\ 0 & \text{if } j > k \end{cases}
\end{aligned}$$

and hence $(a^{-1/2}B_{at})_{t\ge 0}$ satisfies the property (b) in definition 1.1, a,d are obvious and c is easy to be checked.

b. Obvious.

c. Obvious.

d. Notice $B$ is Holder continuous. Now we only need to check that

$$E(tB_{1/t}sB_{1/s}) = ts(1/t \wedge 1/s) = (t \wedge s)$$

and the rest is easy to be checked.

**Theorem 1.6**

*(The law of the iterated logarithm)*

$$\limsup_{t\to s^+} \frac{|B_t - B_s|}{\sqrt{2|t-s| \ln\ln|t-s|}} = 1, \quad a.s.$$

**Proposition 1.9**

*Fix a time interval $[0, t]$ and consider the following subvision $\pi$ of this interval:*

$$0 = t_0 < t_1 < \cdots < t_n = t$$

*The norm of the subdivision $\pi$ is defined as $|\pi| = \max_{0\le j\le n-1}(t_{j+1} - t_j)$. Then*

$$\lim_{|\pi|\to 0} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 = t$$

*in $L^2(\Omega)$.*

**Proof**

Consider let $\xi_j = (B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)$ and we know $\xi_j$ are independent with mean $0$ and hence

$$\begin{aligned}
E\Big(\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 - t\Big)^2 &= \sum_{j=0}^{n-1} E\xi_j^2 = \sum_{j=0}^{n-1}(3(t_{j+1}-t_j)^2 - 2(t_{j+1}-t_j)^2 + (t_{j+1}-t_j)^2) \\
&= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2 \le 2t|\pi| \to 0
\end{aligned}$$

**Proposition 1.10**

*The total variation of Brownian morion on an interval $[0, t]$ defined by*

$$V = \sup_{\pi} \sum_{i=1}^{n-1} [B_{t_{j+1} - B_{t_j}}]$$

*where $\pi$ is any partition of $[0, t]$, is infinite with probability $1$.*

**Proof**

Here we know

$$\sum_{j=0}^{n-1}(B_{t_{j+1}}-B_{t_j})^2 \le V\sup_j|B_{t_{j+1}}-B_{t_j}|$$

and hence if $V<\infty$, then

$$\lim_{|\pi|\to 0}\sum_{j=0}^{n-1}(B_{t_{j+1}}-B_{t_j})^2=0$$

which means $P(V<\infty)=0$.

**Definition 1.10**

*(Wiener integral) Let $\mathcal{E}_0$ be the set pf step functions in $\mathbb{R}_+$,i.e.*

$$\phi(t)=\sum_{j=0}^{n-1}a_j\chi_{t_j,t_{j+1}}(t)$$

*where $n\ge 1$ is an integer,$a_i\in\mathbb{R}$ and $0=t_0<\cdot<t_n$. And we may define Wiener integral of a step function by*

$$\int_0^\infty \phi dB_t=\sum_{i=0}^{n-1}a_i(B_{t_{i+1}}-B_{t_i})$$

♣

**Proposition 1.11**

*The Wiener integral is a linear isometry from $\epsilon_0\subset L^2(\mathbb{R}^+)$ to $L^2(\Omega)$.*

♠

**Proof** Notice

$$E[(\int_0^\infty \phi dB_t)^2]=\sum_{i=0}^{\infty}a_i^2(t_{i+1}-t_i)=||\phi||_2$$

**Definition 1.11**

*We have already know Wiener integral is a linear isometry from a dense subspace from $L^2(\mathbb{R}_+)$ to $L^2(\Omega)$, and hence we may call the extension of the linear isometry to be the Wiener integral and for any $\phi\in L^2(\mathbb{R}_+)$, denote*

$$\int_0^\infty \phi dB_t$$

*to be its image of the isometry.*

♣

**Definition 1.12**

*Let $D$ be a Borel subset of $\mathbb{R}^m$, a white noise on $D$ is a centered Gaussian family of random variables*

$$\{W_A, A\subset\mathcal{B}(\mathbb{R}^m), A\subset D, m(A)<\infty\}$$

*such that*

$$E(W_AW_B)=m(A\cap B)$$

♣

**Proposition 1.12**

*$\chi_A\to W_A$ is a linear isometry from $L^2(D)\to L^2(\Omega)$.*

♠

**Definition 1.13**

*Similarly, we may define the integral r.s.t. $W$ of $\phi\in{}^2(D)$ denoted by*

$$\phi\mapsto\int_D\phi W(dx)$$

*by extending the linear isometry.*

♣

**Definition 1.14**
*Consider a Browian motion $B$ defined on a probability space $(\Omega, \mathcal{F}, P)$. For any time $t \geq 0$, define $\mathcal{F}_t$ the $\sigma$-algebra by $B_s, 0 \leq s \leq t$ and the null events in $\mathcal{F}$., we call $\mathcal{F}_t$ the natural filtration of Browiabn motion on the probability space $(\Omega, \mathcal{F}, P)$.* ♣

**Lemma 1.1**
*Suppose $X$ and $Y$* ♡

**Theorem 1.7**
*For any measurable and bounded (or nonnegative) function $f : \mathbb{R} \to \mathbb{R}$, $s \geq 0$ and $t \geq 0$, we have*

$$E(f(B_{s+t})|\mathcal{F}_s) = (P_t f)(B_s)$$

*where*

$$(P_t f)(x) = \int_{\mathbb{R}} f(y) p_t(x-y) dy$$

*where*

$$p_t = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

♡

Check Durrett Theorem 7.2.1.

**Proposition 1.13**
*The familty of operators $P_t$ satifies the semigroup property $P_t \circ P_s = P_{t+s}$ and $P_0 = Id$.* ♠

**Proof**

$$\begin{aligned} P_t \circ P_s(f)(x) &= \int_{\mathbb{R}} P_s f(y) p_t(x-y) dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) p_s(y-z) p_t(x-y) dz dy \\ &= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{st}} e^{-\left(\frac{(y-z)^2}{2s} + \frac{(x-y)^2}{2t}\right)} dy dz \\ &= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{st}} e^{-\left(\frac{(\sqrt{s+t}y - (2tz+2sx)/\sqrt{s+t})^2 - (tz+sx)^2/(s+t) + tz^2 + sx^2}{2st}\right)} dy dz \end{aligned}$$

and the rest is easy to be checked.

**Theorem 1.8**
*The processes $B_t, (B_t^2 - t)$ and $e^{aB_t - a^2 t/2}, a \in \mathbb{R}$ are $\mathcal{F}_t$ martingales.* ♡

**Proof** $B_t$ is obviously a $\mathcal{F}_t$ martingale. Notice

$$E[(B_t^2 - t)|\mathcal{F}_s] = E[(B_t - B_s)^2 + B_s^2 + 2B_s(B_t - B_s) - t|\mathcal{F}_t] = t - s + B_s^2 - t = B_s^2 - s$$

and

$$E(e^{aB_t - a^2 t/2}|\mathcal{F}_s) = e^{-a^2 t/2} E(e^{a(B_t - B_s)} e^{aB_s}|\mathcal{F}_s) = e^{aB_s} E e^{a(B_t - B_s) - a^2 t/2} = e^{aB_s - a^2 s/2}$$

**Definition 1.15**
*The Brownian hitting time is defined by*

$$\tau_a = \inf\{t \geq 0, B_t = a\}$$

♣

**Proposition 1.14**

*Fix $a > 0$. Then, for all $\alpha > 0$*

$$E(e^{-\alpha \tau_a}) = e^{-\sqrt{2\alpha} a}$$

♠

**Theorem 1.9**

♡