Chapter 1

1.1 06/17

Focus on the optimization of $||X\theta - \beta||_2$ where X is a $n \times d$ dimensional random matrix with each row iid.

Dyson Equation, suupose A is a $d \times d$ random matrix(symmetric and Gaussian). A deterministic equivalent is a deterministic matrix $M = M(z)$ such that for a test matrix we will have

$$|tr(R(z;A)B) - tr(M(z)B)| \overset{prob, d\to\infty}{\to} 0$$

E.g. $M(z) = s(z)I$, where $s(z) = \dfrac{\sqrt{z^2-4}-z}{2}$ is a det equiv for $A/\sqrt{d}$ for A GOE.

Dyson equation, suppose L is a Faussian matrix and then L^{-1} exusts a.s. and then

$$Id = E(Id) = E(L^{-1}L) = E(L^{-1}EL) + E(L^{-1}(L-EL)) = E(L^{-1}EL - L^{-1}(L'-EL)L^{-1})(L'-EL)$$

where L' is an i.i.d. copy of L.

How to find a deter equiv,

a. Find the dominant terms in the Dyson equations $M(EL - S(M)) = Id$ (S is a linear map on matrixs).

b. Show this has a unique solution.

c. Show $R(z;A)$ approxiamately satifies the Dyson equation.

d.Show stability an approximate solution of the Dyson quation is close to the above one.

1.2 Deformed MP law

Consider $(\dfrac{X^TX}{n} - zI)^{-1}$ and X has i.i.d. rows $EX_j = 0$ and the covariance matrix Σ and let

$$L = \begin{bmatrix} -zI & X^T/\sqrt{n} \\ X/\sqrt{n} & -I_n \end{bmatrix}$$

and we will have

$$L^{-1} = \begin{bmatrix} (\dfrac{X^TX}{n} - zI)^{-1} & X^T(I_n - XX^T/nz)^{-1}/\sqrt{n} \\ X(X^TX/n - zI_d)^{-1}/\sqrt{n} & -(I_n - XX^T/nz)^{-1} \end{bmatrix}$$

to L we apply to the Dyson equation

$$\begin{aligned} S(M) &= E\left[\begin{bmatrix} 0 & X^T/\sqrt{n} \\ X/\sqrt{n} & 0 \end{bmatrix}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} 0 & X^T/\sqrt{n} \\ X/\sqrt{n} & 0 \end{bmatrix}\right] \\ &= \frac{1}{n}\begin{bmatrix} E(X^TM_{22}X) & E(X^TM_{21}X^T) \\ E(XM_{12}X) & E(XM_{11}X^T) \end{bmatrix} \\ &= \frac{1}{n}\begin{bmatrix} \Sigma tr(M_{22}) & \Sigma M_{21}^T \\ M_{12}^T\Sigma & tr(M_{11}\Sigma)I_n \end{bmatrix} \end{aligned}$$

Suppose $||\Sigma||_{op}$ bounded and ind of dimension. And we take

$$S(M) = \frac{1}{n}\begin{bmatrix} \Sigma tr(M_{22}) & \\ s & tr(M_{11}\Sigma)I_n \end{bmatrix}$$

Then the related Dyson Equation is

$$\begin{bmatrix} M_{11} & M_{12}^T \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} -zI_d - \Sigma tr(M_{22})/n & \\ & -I_n(1 + tr(M_{11}\Sigma))/n \end{bmatrix} = I_{d+n}$$

Then $M_{12} = M_{21}^T = 0$ and $M_11 = (-zI_d - \Sigma tr(M_22)/n)^{-1}$ and $M_22 = -I_n/(1 + tr(M_11\Sigma))$. Let $m(z) = tr(M_22)/n$ and we will have

$$m(z) = \frac{-1}{1 + \frac{1}{n} tr(M_{11}\Sigma)} = \frac{-1}{1 + tr\left(\frac{\Sigma}{-zI_d - m(z)\Sigma}\right)}$$

So there exists M with $ImM_{11} > 0$.

1.3 Frostman A

Assumption: $\Sigma \subset \mathbb{C}, V : \Sigma \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. There is $\mu \in P(\Sigma)$ such that

$$I(\mu) = \int \log \frac{1}{|x-y|} d\mu(x) d\mu(y) < +\infty$$

and $\int V d\mu < \infty$, then $V(x) - \log(|x|^2 + 1) \to +\infty$ as $|x| \to \infty, x \in \Sigma$.

Theorem 1.1

There is a unique minimium $\mu \in P(\Sigma)$ of $I_V(\mu)$ say $\mu = \mu_V$ is compactly supported.

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Theorem 1.2

There is $l = l_V$ such that $2U^\mu + V \leq l$ on the support of μ_V and larger than l q.e. on Σ where

$$U^\mu(x) = \int \log \frac{1}{|x-y|} d\mu(y), (I(\mu)) = \int U^\mu d\mu$$

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