Chapter 1

1.1 Basics of Stochastic Processes

We will refer X_t to be real or $\mathbb{R}^d$-valued continuous-time stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P)$. For every fixed $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is called a trajectory or sample path of the process.

For a real-valued stochastic process, let $- \leq t_1 < \cdots < t_n$ be fixed. Then we know

$$P_{t_1,\cdots,t_n} = P \circ (X_{t_1}, \cdots, X_{t_n})^{-1}$$

is a probablity distribtuion on $\mathbb{R}^n$, which is called the finite-dimensional marginal distribution of the process.

Theorem 1.1

(Kolmogorov's extension theorem) Consider a family of probablity measures

$$\{P_{t_1,\cdots,t_n}, t_1 < \cdots < t_n, n \geq 1, t_i \geq 0\}$$

such that

a. $P_{t_1,\cdots,t_n}$ is a probablity on $\mathbb{R}^n$.

b. For $\{t_{k_1} < \cdots < t_{k_m}\} \subset \{t_1 < t_2 < \cdots < t_n\}$,$P_{t_{k_1},\cdots,t_{k_m}}$ is required to be a marginal of $P_{t_1,\cdots,t_n}$, then there exists a real-valued stochastic process X_t owning finite-dimensional marginal distributions of $\{P_{t_1,\cdots,P_{t_n}}\}$. ♡

Definition 1.1

A real-valued process X_t is a second-order process iff $EX_t^2 < \infty, t \geq 0$, define

$$m_X(t) = EX_t, \Gamma_X(s,t) = cov(X_s, X_t)$$

♣

Definition 1.2

A real-valued process X_t is said to be Gaussian if its finite-dimensional marginal distributions are multidimensional Gaussian laws. ♣

Proposition 1.1

A Gaussian process is determined by m_X and Γ_X, conversely, for any $m : \mathbb{R}_+ \to \mathbb{R}$ and a symmetric $\Gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ which is nonnegative definite, there always exists a Gaussian process with mean m and covariance function Γ by Kolmogorov's extension theorem. ♠

Definition 1.3

We call two processes X, Y are equivalent if for all $t \geq 0$, $X_t = Y_t$ a.s. And we call they indistinguishable if $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$ and for all ω in some set with probability 1. ♣

Proposition 1.2

Two equivalent processes with right-continuous trajectories are indistinguishable. ♠

Proof Let $X_q = Y_q$ on Ω_q for $q \in \mathbb{Q}$ and let $\Omega' = \bigcap_{q\in\mathbb{Q}} \Omega_q$ and we know Ω' has the probability 1. And it is easy to check that $X_t = Y_t$ on Ω' for all t.

Theorem 1.2

(Kolmogorov's continuity theorem) Suppose that $X = X_t, t \in [0,T]$ satisfies

$$E(|X_t - X_s|^{\beta}) \leq K|t-s|^{1+\alpha}$$

for all $s, t \in [0,T]$ and for some constants $\beta, \alpha, K > 0$. Then there exists a version $\tilde{X}$ of X such that, if $\gamma < \alpha/\beta$, *then*

$$|\tilde{X}_t| - \tilde{X}_s \le G_\gamma |t-s|^\gamma$$

for all $s, t \in [0, T]$, *where* G_γ *is a random variable. The trajectories of* $\tilde{X}$ *are Holder continuous of* γ *for any* $\gamma < \alpha/\beta$.

Definition 1.4

$\mathcal{F}_t$ *is an increasing family of sub-*σ*-field of* $\mathcal{F}$. *A process* X_t *is* $\mathcal{F}_t$*-adapted if* X_t *is* $\mathcal{F}_t$*-measurable for all* $t \ge 0$.

Definition 1.5

An adapted process $X_t, t \ge 0$ *is a Markov process w.r.t. a filtration* $\mathcal{F}_t$ *if for any* $s \ge 0, t > 0$ *and any measurable and bounded function* $f : \mathbb{R} \to \mathbb{R}$,

$$E(f(X_{s+t})|\mathcal{F}_s) = E(f(X_{s+t})|X_s) \ a.s.$$

Proposition 1.3

A $\mathcal{F}_t$*-Markov process* X_t *is also a* $\mathcal{F}_t^X$*-Markov process where*

$$\mathcal{F}_t^X = \sigma\{X_u, 0 \le u \le t\}$$

Proof Notice

$$E(f(X_{s+t}|\mathcal{F}_s^X)) = E(E(f(X_{s+t}|\mathcal{F}_s))|\mathcal{F}_s^X) = E(E(f(X_{s+t}|X_s))|\mathcal{F}_s^X) = E(f(X_{s+t}|X_s))$$

since $\sigma(X_s) \subset F_s^X \subset F_t$.

Definition 1.6

Assume a filtration $\mathcal{F}_t$ *on* $(\Omega, \mathcal{F}, P)$ *satisfies that for any* $P(A) = 0$, $A \in \mathcal{F}$, $A \in \mathcal{F}_0$ *and it is right-continuous,i.e.*

$$\mathcal{F}_t = \cap_{n \ge 1} \mathcal{F}_{t+n^{-1}}$$

Then consider a r.v. $T : \Omega \to [0, \infty]$ *is a stopping time w.r.t. to the filtration if*

$$\{T \le t \in \mathcal{F}_t\}$$

for any $t \ge 0$.

Proposition 1.4

a. T *is a stopping time iff* $\{T < t\} \in \mathcal{F}_t$ *for all* $t \ge 0$.
b. $S \vee T$ *and* $S \wedge T$ *are stopping times.*
c. Givene a stopping time T,

$$\mathcal{F}_T = \{A, A \cap \{T \le t\} \in \mathcal{F}_t, \forall t \ge 0\}$$

is a σ*-algebra.*
d. If $S \le T$, *then* $\mathcal{F}_S \subset \mathcal{F}_T$.
e. Let $X_t, t \ge 0$ *be a continuous and adapted process. The hitting time of a set* $A \subset \mathbb{R}$ *is defined by*

$$T_A = \inf\{t \ge 0, X_t \in A\}$$

and whether A *is open or closed,* T_A *is a stopping time.*
f. Let X_t *be an adapted stochastic process with right-continuous paths and let* $T < \infty$ *be a stopping time. Then the random variable*

$$X_T(\omega) = X_{T(\omega)(\omega)}$$

is $\mathcal{F}_T$*-measurable.*

Definition 1.7

An adapted process $M = M_t, t \geq 0$ is called a martingale w.r.t. a filtration $\mathcal{F}_t, t \geq 0$ if
a. for all $t \geq 0$, $E(|M_t|) < \infty$
b. for each $s \leq t$, $E(M_t|\mathcal{F}_s) = M_s$

Proposition 1.5

a. For any integrable random varibale X, $E(X|\mathcal{F}_t)$ is a martingale.
b. If M_t is a submartingale then $t \to E(M_t)$ is nondecreasing.
c. If M_t is a martingale and φ is a convex function such that $E|\phi(M_t)| < \infty$ for all $t \geq 0$ then $\phi(M_t)$ is a submartingale.

Proof Only (c) is needed to be proved. Conside $ax + b \leq \phi(x)$ and we know

$$E(\phi(M_t)|\mathcal{F}_s) \geq aE(X_t|\mathcal{F}_s) + b$$

for any such a, b and hence

$$E(\phi(M_t)|\mathcal{F}_s) \geq \phi(M_t)$$

Definition 1.8

An adapted process $M_t, t \geq 0$ is called a local martingale if there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that, for any $n \geq 1$ $M_{t\wedge\tau_n}$ is a martingale.

Theorem 1.3

Let $M_t, t \geq 0$ be a continuous local martingale such that $M_0 = 0$. Let $\pi = \{0 = t_0 < t_1 < \cdots < T_n = t\}$ be a partition of $[0, t]$. Then we have

$$\sum_{j=0}^{n-1}(M_{t_{j+1}} - M_{t_j})^2 \to \langle M\rangle_t, |\pi| \to 0$$

in probability, where $\langle M\rangle_t, t \geq 0$ is called the quadratics variation of the local martingale. Moreover, if $M_t, t \geq 0$ is a martingale then the convergence holds in $L^1(\Omega)$.

Theorem 1.4

The quadratic variation is the unique continuous and increasing process satisfying $\langle M\rangle_0 = 0$ and

$$M_t^2 - \langle M\rangle_t$$

is a local martingale.

1.2 Brownian Motion

Definition 1.9

A real-valued stochastic process $B = (B_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}; P)$ is called a Brownian motion if it satisfies the following conditions:
a. Almost surely $B_0 = 0$.
b. For all $0 \leq t_1 < \cdots t_n$ the increments $B_{t_n} - B_{t_{n-1}}, \cdots, B_{t_2} - B_{t_1}$ are independent random variables.
c. If $0 \leq s < t$, the increment $B_t - B_s$ is a Gaussian random variable with mean zero and variance $t - s$.
d. With probability one, the map $t \to B_t$ is continuous.
A d-dimensional Brownian motion is defined as an $\mathbb{R}^d$-valued stochastic process $B = (B_t)_{t\geq 0}$, $B_t = (B_t^1, \cdots, B_t^d)$, where $B^1, \cdots, B^d$ are d independent Brownian motions.

Proposition 1.6

Properties (a),(b),(c) are equivalent to that B is a Gaussian process,i.e. for any finite set of indices $t_1, \cdots, t_n$, $(B_{t_1}, \cdots, B_{t_n})$ is a multivariate Gaussian random variable, equivalently, any linear combination of B_{t_i} is normal distributed r.v., with mean zero and covariance function

$$\Gamma(s,t) = \min(s,t)$$

Proof

Suppose (a),(b),(c) holds, then we know $(B_{t_1}, \cdots, B_{t_n})$ is normal for any finite indices and then

$$m(t) = E(B_t) = 0$$
$$\Gamma(s,t) = E(B_s B_t) = E(B^2_{\min(s,t)}) = \min(s,t)$$

Conversly, we know $E(B_0^2) = 0$ and hence $B_0 = 0$ a.s., then we know $E(B_s^2) = s$ and for any $0 < s < t$,

$$E(B_s(B_t - B_s)) = 0$$

and it is easy to check (c), and (b) is deduced by computing the covariance of the increments, notice that two r.v.s are independent iff $\phi_{(X_1, X_2, \cdots, X_n)} = \phi_{X_1}\phi_{X_2}\cdots\phi_{X_n}$ which implies that normal r.v.s are independent iff they have zero covariances.

Theorem 1.5

(Kolmogorov's continuity theorem) Suppose that $X = (X_t)_{t\in[0,T]}$ satisfies

$$E(|X_t - X_s|^\beta) \le K|t-s|^{1+\alpha}$$

for all $s, t \in [0,T]$ and some constant $\beta, \alpha, K > 0$. Then there exists a version $\tilde{X}$ of X such that if

$$\gamma < \alpha/\beta$$

then

$$|\tilde{X}_t - \tilde{X}_s| \le G_\gamma |t-s|^\gamma$$

for all $s, t \in [0,T]$ where G_γ is a random variable. The trajectories of $\tilde{X}$ are Holder continuous of order γ for any $\gamma < \alpha/\beta$.

Proposition 1.7

There exists a version of B with Holder-continuous trajectories of order γ for any $\gamma < (k-1)/2k$ on any interval $[0,T]$.

Proof

Since we know $B_t - B_s$ has the normal distribution $\mathcal{N}(0, t-s)$ and then we know

$$E\Big((B_t - B_s)^{2k}\Big) = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} x^{2k} \exp^{-\dfrac{x^2}{2(t-s)}}\, dx = (2k-1)!!(t-s)^k = \frac{(2k)!}{2^k k!}(t-s)^k$$

and by the theorem 1.1, the proposition holds.

Proposition 1.8

Brownian motion are basic properties:

a. For any $a > 0$, the process $(a^{-1/2}B_{at})_{t\ge 0}$ is a Brownian motion.

b. For any $h > 0$, the process $(B_{t+h} - B_h)_{t\ge 0}$ is a Brownian motion.

c. The process $(-B_t)_{t\ge 0}$ is a Brownian motion.

d. Almost surely $\lim_{t\to\infty} B_t/t = 0$ and the process $X_t = tB_{1/t}$ for $t > 0$, $X_t = 0$ for $t = 0$ is a Brownian motion.

Proof

a. Consider $0 \leq t_1 < t_2 < \cdots < t_n$ and we may calculate the covariance matrix for

$$a^{-1/2}B_{at_n} - a^{-1/2}B_{at_{n-1}}, \cdots, a^{-1/2}B_{at_2} - a^{1/2}B_{at_1}$$

by

$$\begin{aligned}
&E[(a^{-1/2}B_{at_j} - a^{-1/2}B_{at_{j-1}})(a^{-1/2}B_{at_k} - a^{-1/2}B_{at_{k-1}})] \\
=&a^{-1}(at_j \wedge at_k) - a^{-1}(at_j \wedge at_{k-1}) - a^{-1}(at_{j-1} \wedge at_k) + a^{-1}(at_{j-1} \wedge at_{k-1}) \\
=&\begin{cases} t_j - t_{j-1} - t_{j-1} + t_{j-1} = t_j - t_{j-1} & \text{if } j = k \\ t_j - t_j - t_{j-1} + t_{j-1} = 0 & \text{if } j < k \\ 0 & \text{if } j > k \end{cases}
\end{aligned}$$

and hence $(a^{-1/2}B_{at})_{t \geq 0}$ satisfies the property (b) in definition 1.1, a,d are obvious and c is easy to be checked.

b. Obvious.

c. Obvious.

d. Notice B is Holder continuous. Now we only need to check that

$$E(tB_{1/t}sB_{1/s}) = ts(1/t \wedge 1/s) = (t \wedge s)$$

and the rest is easy to be checked.

Theorem 1.6

(The law of the iterated logarithm)

$$\limsup_{t \to s^+} \frac{|B_t - B_s|}{\sqrt{2|t-s| \ln \ln |t-s|}} = 1, \quad a.s.$$

♡

Proposition 1.9

Fix a time interval $[0, t]$ and consider the following subvision π of this interval:

$$0 = t_0 < t_1 < \cdots < t_n = t$$

The norm of the subdivision π is defined as $|\pi| = \max_{0 \leq j \leq n-1}(t_{j+1} - t_j)$. Then

$$\lim_{|\pi| \to 0} \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 = t$$

in $L^2(\Omega)$.

♠

Proof

Consider let $\xi_j = (B_{t_{j+1}} - B_{t_j})^2 - (t_{j+1} - t_j)$ and we know ξ_j are independent with mean 0 and hence

$$\begin{aligned}
E\Big(\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 - t\Big)^2 = \sum_{j=0}^{n-1} E\xi_j^2 &= \sum_{j=0}^{n-1}(3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2) \\
&= 2\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2 \leq 2t|\pi| \to 0
\end{aligned}$$

Proposition 1.10

The total variation of Brownian morion on an interval $[0, t]$ defined by

$$V = \sup_{\pi} \sum_{i=1}^{n-1} [B_{t_{j+1} - B_{t_j}}]$$

where π is any partition of $[0, t]$, is infinite with probability 1.

♠

Proof

Here we know

$$\sum_{j=0}^{n-1}(B_{t_{j+1}}-B_{t_j})^2 \le V\sup_j |B_{t_{j+1}}-B_{t_j}|$$

and hence if $V<\infty$, then

$$\lim_{|\pi|\to 0}\sum_{j=0}^{n-1}(B_{t_{j+1}}-B_{t_j})^2=0$$

which means $P(V<\infty)=0$.

Definition 1.10

(Wiener integral) Let $\mathcal{E}_0$ be the set pf step functions in $\mathbb{R}_+$,i.e.

$$\phi(t)=\sum_{j=0}^{n-1}a_j\chi_{t_j,t_{j+1}}(t)$$

where $n\ge 1$ is an integer,$a_i\in\mathbb{R}$ and $0=t_0<\cdot<t_n$. And we may define Wiener integral of a step function by

$$\int_0^\infty \phi dB_t=\sum_{i=0}^{n-1}a_i(B_{t_{i+1}}-B_{t_i})$$

♣

Proposition 1.11

The Wiener integral is a linear isometry from $\epsilon_0\subset L^2(\mathbb{R}^+)$ to $L^2(\Omega)$.

♠

Proof Notice

$$E[(\int_0^\infty \phi dB_t)^2]=\sum_{i=0}^{\infty}a_i^2(t_{i+1}-t_i)=||\phi||_2$$

Definition 1.11

We have already know Wiener integral is a linear isometry from a dense subspace from $L^2(\mathbb{R}_+)$ to $L^2(\Omega)$, and hence we may call the extension of the linear isometry to be the Wiener integral and for any $\phi\in L^2(\mathbb{R}_+)$, denote

$$\int_0^\infty \phi dB_t$$

to be its image of the isometry.

♣

Definition 1.12

Let D be a Borel subset of $\mathbb{R}^m$, a white noise on D is a centered Gaussian family of random variables

$$\{W_A, A\subset\mathcal{B}(\mathbb{R}^m), A\subset D, m(A)<\infty\}$$

such that

$$E(W_AW_B)=m(A\cap B)$$

♣

Proposition 1.12

$\chi_A\to W_A$ is a linear isometry from $L^2(D)\to L^2(\Omega)$.

♠

Definition 1.13

Similarly, we may define the integral r.s.t. W of $\phi\in{}^2(D)$ denoted by

$$\phi\mapsto\int_D\phi W(dx)$$

by extending the linear isometry.

♣

Definition 1.14

Consider a Browian motion B defined on a probability space $(\Omega, \mathcal{F}, P)$. For any time $t \geq 0$, define $\mathcal{F}_t$ the σ-algebra by $B_s, 0 \leq s \leq t$ and the null events in $\mathcal{F}$., we call $\mathcal{F}_t$ the natural filtration of Browiabn motion on the probability space $(\Omega, \mathcal{F}, P)$.

♣

Lemma 1.1

Suppose X and Y

♡

Theorem 1.7

For any measurable and bounded (or nonnegative) function $f : \mathbb{R} \to \mathbb{R}$, $s \geq 0$ and $t \geq 0$, we have

$$E(f(B_{s+t})|\mathcal{F}_s) = (P_t f)(B_s)$$

where

$$(P_t f)(x) = \int_{\mathbb{R}} f(y) p_t(x-y) dy$$

where

$$p_t = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

♡

Check Durrett Theorem 7.2.1.

Proposition 1.13

The familty of operators P_t satifies the semigroup property $P_t \circ P_s = P_{t+s}$ and $P_0 = Id$.

♠

Proof

$$\begin{aligned}
P_t \circ P_s(f)(x) &= \int_{\mathbb{R}} P_s f(y) p_t(x-y) dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) p_s(y-z) p_t(x-y) dz dy \\
&= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{st}} e^{-\left(\frac{(y-z)^2}{2s} + \frac{(x-y)^2}{2t}\right)} dy dz \\
&= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{st}} e^{-\left(\frac{(\sqrt{s+t}y - (2tz+2sx)/\sqrt{s+t})^2 - (tz+sx)^2/(s+t) + tz^2 + sx^2}{2st}\right)} dy dz
\end{aligned}$$

and the rest is easy to be checked.

Theorem 1.8

The processes $B_t, (B_t^2 - t)$ and $e^{aB_t - a^2 t/2}, a \in \mathbb{R}$ are $\mathcal{F}_t$ martingales.

♡

Proof B_t is obviously a $\mathcal{F}_t$ martingale. Notice

$$E[(B_t^2 - t)|\mathcal{F}_s] = E[(B_t - B_s)^2 + B_s^2 + 2B_s(B_t - B_s) - t|\mathcal{F}_t] = t - s + B_s^2 - t = B_s^2 - s$$

and

$$E(e^{aB_t - a^2 t/2}|\mathcal{F}_s) = e^{-a^2 t/2} E(e^{a(B_t - B_s)} e^{aB_s}|\mathcal{F}_s) = e^{aB_s} E e^{a(B_t - B_s) - a^2 t/2} = e^{aB_s - a^2 s/2}$$

Definition 1.15

The Brownian hitting time is defined by

$$\tau_a = \inf\{t \geq 0, B_t = a\}$$

♣

Proposition 1.14

Fix $a > 0$. Then, for all $\alpha > 0$

$$E(e^{-\alpha\tau_a}) = e^{-\sqrt{2\alpha}a}$$

1.3 Stochastic Integrals

Definition 1.16

We say that a stochastic process u_t is progressively measurable if, for any $t \geq 0$ the restriction of u to $\Omega \times [0,t]$ is $\mathcal{F}_t \times B([0,t])$-measurable.

Definition 1.17

Let P be the σ-field of sets $A \subset \Omega \times \mathbb{R}^+$ such that χ_A is progressivelyy measurable. We denote by $L^2(P)$ the Hilbert space $L^2(\Omega \times \mathbb{R}^+, P, P \times m)$ equipped the norm

$$||u||^2 = E(\int_0^\infty u_s^2 ds) = \int_0^\infty Eu_s^2 ds$$

bu Fubini's theorem.

Definition 1.18

A process $u = u_t$ is called a simple process if it is of the form

$$u_t = \sum_{j=0}^{n-1} \phi_j \chi_{(t_j, t_{j+1}]}(t)$$

where $0 \leq t_0 < t_1 < \cdots < t_n$ and the ϕ_j are $\mathcal{F}_{t_j}$-measurable random variables such that $E\phi_j^2 < \infty$ and denote the space of simple processes as $\mathcal{E}$.
We define the stochastic integral of a process $u \in \mathcal{E}$ of u_t as

$$I(u) = \int_0^\infty u_t dB_t = \sum_{j=0}^{n-1} \phi_j (B_{t_{j+1}} - B_{t_j})$$

Here it is easy to see that $\phi_j \chi_{(t_j, t_{j+1}]}(t)$ is progressively measurable.

Proposition 1.15

Here are some properties of stochatic integrals.
a. For any $a, b \in \mathbb{R}$ and simple process $u, v \in \mathcal{E}$, we know

$$\int_0^\infty (au_t + bv_t) dB_t = a\int_0^\infty u_t dB_t + b\int_0^\infty v_t dB_t$$

b. For any $u \in \mathcal{E}$, we have

$$E(\int_0^\infty u_t dB_t) = 0$$

c. For any $u \in \mathcal{E}$, we know

$$E\Big(\Big(\int_0^\infty u_t dB_t\Big)^2\Big) = E\Big(\int_0^\infty u_t^2 dt\Big)$$

Proof (a) is trivial. (b) can be shown by the independence of ϕ_j and $B_{t_{j+1}} - B_{t_j}$. (c) Can be shown by

$$E\Big(\Big(\int_0^\infty u_t dB_t\Big)^2\Big) = E\Big(\Big(\sum_{j=0}^{n-1} \phi_j (B_{t_{j+1}} - B_{t_j})\Big)^2\Big) = \sum_{j=0}^{n-1} E\phi_j^2 (t_{j+1} - t_j)$$

Proposition 1.16

The space $\mathcal{E}$ of simple processes is dense in $L^2(P)$.

Proof For $u \in L^2(P)$, we define

$$u_t^{(n)} = n\int_{(t-1/n)\wedge 0}^{t} u_s ds$$

we know $u_t^{(n)}$ is continuous as $\mathbb{R} \to L^2(\Omega)$ and hence we also know

$$\lim_{n\to\infty} E\Big(\int_0^\infty |u_t - u_t^{(n)}|^2 dt\Big) = 0$$

since $\lim_{n\to\infty} u_t^{(n)} = u_t$ a.s. by Lebesgue differential, then we may know the limit above holds by the DCT since

$$\int_0^\infty |u_t^{(n)}(\omega)|^2 dt = \int_0^\infty n^2 |\int_{t-1/n}^{t} u_s(\omega) ds|^2 dt \le ||u_t||_2^2$$

where $f(s,t) = u_s\chi_{(t-1/n,t]}$ by the Minkowski's inequaility of integrals.

For $u \in L^2(P)$ continuous in $L^2(\Omega)$, we define

$$u_t^{(n,N)} = \sum_{j=0}^{(n,N)} u_{t_j}\chi_{(t_j,t_{j+1}]}(t)$$

where $t_j = jN/n$. The continuity in $L^2(\Omega)$ implies that

$$E\Big(\int_0^{\int} |u_t - u_t^{n,N}|^2 dt\Big) \le E\Big(\int_N^\infty u_t^2 dt\Big) + N \sup_{|t-s|\le N/n, t,s\le N} E(|u_t - u_s|^2)$$

and we let $N \to \infty$ and $n \to \infty$ we may get the conclusion.

Proposition 1.17

The stochastic integral can be extended to a linear isometry.

Proposition 1.18

Here are some properties, for any $u, v \in L^2(P)$, we know

$$E(I(u)) = 0, \quad E(I(u)I(v)) = E(\int_0^\infty u_s v_s ds)$$

and for any T we set

$$\int_0^T u_s dB_s = \int_0^\infty u_s\chi_{[0,T]}(s) dB_s$$

which is the indefinite integral of u with respect to B, where requiring $u \in L_T^2(P)$.

Definition 1.19

Define $L_\infty^2(P)$ the set of progressively processes such that

$$E\Big(\int_0^t u_s^2 ds\Big) < \infty$$

for each $t > 0$, for any process $u \in L_\infty^2(P)$, we can define the indefinite integral process

$$\int_0^t u_s dB_s$$

Proposition 1.19

Here are some properties of indefinite integral process.

a. For any $a \le b \le c$, we have

$$\int_a^b u_s dB_s + \int_b^c u_s dB_s = \int_a^c u_s dB_s$$

b. If $a<b$ and F is a bounded and $\mathcal{F}_a$-measurable random variable then

$$\int_a^b Fu_s dB_s = F\int_a^b u_s dB_s$$

c. Let $u \in L^2\infty(P)$, then the indefinite stochastic integral

$$M_t = \int_0^t u_s dB_s, t \geq 0$$

is a square integrable martingale w.r.t. the filtration $\mathcal{F}_t$ and admits a continuous version.

Proof (a) is trivial. For (b), we only need to consider $u_t^{(n)}$ simple and converging to u_t in $L^2(P)$, and we have

$$\int_a^b Fu_s^{(n)} dB_s = F\sum \phi_{t_j}(B_{t_{j+1}-B_{t_j}}) = F\int_a^b u_s^{(n)} dB_s$$

and we are done since F is bounded.

(c) For a simple process

$$u_t = \sum_{j=0}^{n-1} \phi_j \chi_{(t_j,t_{j+1}]}(t)$$

then we know

$$\begin{aligned} E\Big(\int_0^t u_v dB_v \Big| \mathcal{F}_s\Big) &= \sum_{j=0}^{n-1} E(\phi_j(B_{t_{j+1}\wedge t} - B_{t_j\wedge t})\Big|\mathcal{F}_s) \\ &= \sum_{j=0}^{n-1} E(E\Big(\phi_j(B_{t_{j+1}\wedge t} - B_{t_j\wedge t}\Big|\mathcal{F}_{t_j\wedge s}\Big))\Big|\mathcal{F}_s) \\ &= \int_0^s u_v dB_v \end{aligned}$$

and hence we know M_t is an $\mathcal{F}_t$-martingale if u is simple. For $T>0$, let $u^{(n)}$ converges to u in $L^2_T(P)$, then we know $t\in[0,T]$, we have

$$\int_0^t u_s^{(n)} dB_s \to \int_0^t u_s dB_s$$

in $L^2(\Omega)$ and we know the convergence of the conditional expectiations by $E(Z(X-E(X\mathcal{F})))=0$ for any $Z\in L^2(\mathcal{F})$ and hence $\int_0^t u_s dB_s$ is a martingale.

To show that the indefinite integral has a continuous version. Let $u\in L^2$ and fix $T>0$. Consider a sequence of simple processed $u^{(n)}$ which converegs to u in $L^2_T(P)$, by the continuity of the paths of Brownian motion, we know

$$M_t^{(n)} = \int_0^t u_s^{(n)} dB_s$$

has continuous trajectories. Then since $M^{(n)}$ is a martingale, so we know

$$\begin{aligned} P\Big(\sup_{0\leq t\leq T} |M_t^{(n)} - M_t^{(m)}| > \lambda\Big) &\leq \frac{1}{\lambda^2} E\Big(|M_t^{(n)} - M_t^{(m)}|^2\Big) \\ &= \frac{1}{\lambda^2} E\Big(\int_0^T |u_t^{(n)} - u_t^{(m)}|\Big) \to 0 \end{aligned}$$

as $n,m\to\infty$ and we may choose n_k such that

$$P\Big(\sup_{0\leq t\leq T} |M_t^{(n_{k+1})-M_t^{(n_k)}}| > 2^{-k}\Big) \leq 2^{-k}$$

and we may know that for any $\omega\in\Omega$, there is $k_1(\omega)$ such that

$$\sup_{0\leq t\leq T} |M_t^{(n_{k+1})}(\omega) - M_t^{(n_k)}(\omega)| \leq 2^{-k}$$

has probability of 1 and then we know $M_t^{(n_k)}(\omega)$ is uniformly convergent to a continuous function $J_t(\omega)$ a.s. and then we know $J_t(\omega) = \int_0^t u_s dB_s$ a.s. Then we may choose different t and construct a continuous version of $\int_0^t u_s dB_s$ inductively.

Proposition 1.20

For any $T, \lambda > 0$ and $u \in L^2_\infty(P)$, we know

$$P\Big(\sup_{t\in[0,T]} |M_t| > \lambda\Big) \le \frac{1}{\lambda^2} E\Big(\int_0^T u_t^2 dt\Big)$$

and

$$E\Big(\sup_{t\in[0,T]} |M_t|^2 \le 4E(\int_0^T u_t^2 dt)\Big)$$

by Doob's inequality and L^p maxmium inequalities.

Proposition 1.21

Let $u \in L^2_\infty(P)$. Consider a subdivision of the interval $[0,t]$

$$\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$$

then as $|\pi| \to 0$, we have

$$S^2_\pi(u) = \sum_{j=0}^{n-1} \Big(\int_{t_j}^{t_{j+1}} u_s dB_s\Big)^2 \to \int_0^t u_s^2 ds$$

in $L^1(\Omega)$.

1.4 Derivative and Divergence Operators

Definition 1.20

For this chapter, we will consider the probability space $(\Omega, \mathcal{F}, P)$ such that $\Omega = \mathbb{R}^n, \mathcal{F} = B(\mathbb{R}^n)$ to be the Borel σ-field of $\mathbb{R}^n$ and P to be the standard Gaussian probability with density

$$p(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$$

and we define the derivative operator for differentiable function $F : \mathbb{R}^n \to \mathbb{R}$ as

$$\nabla F = \Big(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}\Big)$$

and the divergence opertor for differentiable veector-valued functions $u : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\delta(u) = \sum_{i=1}^n \Big(u_i x_i - \frac{\partial u_i}{\partial x_i}\Big) = \langle u, x\rangle - divu$$

Proposition 1.22

The operator δ is the adjoint of ∇ that is

$$E(\langle u, \nabla F\rangle) = E(F\delta(u))$$

for $F : \mathbb{R}^n \to \mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable functions satisfying integral by parts.

Proof Since $\partial p/\partial x_i = -x_i p$ and we have

$$\begin{aligned}
\int \langle \nabla F, u\rangle p dx &= \sum_{i=1}^n \int \frac{\partial F}{\partial x_i} u_i p dx \\
&= \sum_{i=1}^n \Big(-\int F \frac{\partial u_i}{\partial x_i} p dx + \int F u_i x_i p dx\Big) \\
&= \int F\delta(u) p dx
\end{aligned}$$

Definition 1.21

Consider the Hilbert space $h \in H = L^2(\mathbb{R}^+)$ and the Wiener integral

$$B(h) = \int_0^\infty h(t)dB_t$$

and S the set of smooth and cylindrical random variables of the form

$$F = f(B(h_1), \cdots, B(h_n))$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $h_i \in H$.

Definition 1.22

For $F \in S$, we define the Malliavin derivative DF to be the H-valued random variable defined by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(h_1), \cdots, B(h_n))h_i(t)$$

which is well-defined and a linear but unbounded operator from S into $L^2(\Omega; H)$.
Let S_H to be the element $u = (u_t)$ with the form

$$u_t = \sum_{j=1}^n F_j h_j(t)$$

with $F_j \in S$ and $h_j \in H$. And the divergence of an element u in S_H will be given by

$$\delta(u) = \sum_{j=1}^n F_j B(h_j) - \sum_{j=1}^n \langle DF_j, h_j\rangle_H$$

Proposition 1.23

For $F \in S$ and $u \in S_H$, then

$$E(F\delta(u)) = E(\langle DF, u\rangle_H)$$

Proof Consider h_i, h_j orthonormal and then we may know that $B(h_i)$ are i.i.d., then for $F = f(B(h_1), \cdots, B(h_n))$ and

$$u = \sum_{j=1}^n g_j(B(h_1), \cdots, B(h_n))h_j$$

and

$$\begin{aligned}
E(\langle DF, u\rangle_H) &= \int \sum_{j=1}^n E(D_t F g_j(B(h_1), \cdots, B(h_n))h_j(t)) \\
&= \sum_{i,j=1}^n \int E\Big(\frac{\partial f}{\partial x_i}(B(h_1), \cdots, B(h_n))g_j(B(h_1), \cdots, B(h_n))h_i(t)h_j(t)\Big) \\
&= \sum_{i=1}^n E\Big(\frac{\partial f}{\partial x_i}(B(h_1), \cdots, B(h_n))g_j(B(h_1), \cdots, B(h_n))\Big) \\
&= \sum_{j=1}^n E\Big(F g_j(B(h_1), \cdots, B(h_n))B(h_j) - F\frac{\partial g_j}{\partial x_j}(B(h_1), \cdots, B(h_n))\Big) \\
&= E(F\delta(u))
\end{aligned}$$

since

$$\langle DF_j, h_j\rangle = \frac{\partial g_j}{\partial x_j}(B(h_1), \cdots, B(h_n))$$

Proposition 1.24

Suppose that $u, v \in S_H$, $F \in S$ and $h \in H$. Then for a complete orthonormal system in H denoted as e_i, we have

$$E(\delta(u)\delta(v)) = E(\langle u, v\rangle_H) + E(\sum_{i,j=1}^{\infty} D_{e_i}\langle u, e_j\rangle_H D_{e_j}\langle v, e_i\rangle_H)$$

$$D_h(\delta(u)) = \delta(D_h(u)) + \langle h, u\rangle_H$$

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_H$$

Proof We know

$$\begin{aligned}
D_h(\delta(u)) &= D_h\Big(\sum_{j=1}^{n} F_j B(h_j) - \sum_{j=1}^{n}\langle DF_j, h_j\rangle_H\Big) \\
&= \sum_{j=1}^{n} D_h F_j B(h_j) + \sum_{j=1}^{n} F_j\langle h, h_j\rangle_H - \sum_{j=1}^{n} D_h\Big\langle \sum_{i=1}^{n}\frac{\partial F_j}{\partial x_i}(B(h_1), \cdots, B(h_n))h_i, h_j\Big\rangle_H \\
&= \sum_{j=1}^{n} F_j\langle h, h_j\rangle_H + \sum_{j=1}^{n}\Big(D_h F_j B(h_j) - \langle D_h(DF_j), h_j\rangle_H\Big) \\
&= \langle u, h\rangle_H + \delta(D_h u)
\end{aligned}$$

by Fubini's Theorem.

Then

$$\begin{aligned}
E(\delta(u)\delta(v)) &= E\Big(\langle v, D(\delta(u))\rangle_H\Big) \\
&= \sum_{n\geq 1}\Big(\langle v, e_i\rangle_H D_{e_i}(\delta(u))\big) \\
&= E(\langle u, v\rangle)_H + \sum_{i=1}^{\infty} E(\langle v, e_i\rangle_H \delta(D_{e_i}u)) \\
&= E(\langle u, v\rangle)_H + \sum_{i=1}^{\infty} E(\langle D\langle v, e_i\rangle_H, D_{e_i}u\rangle_H)
\end{aligned}$$