Homework0 - Paquette

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Exercise.1

Show there is a unique solition of the Dyson equation on $\mathbb{M}^+(n)$ given by

$$M = s(z)I \quad \text{where } s(z) := \frac{-z + \sqrt{z-2}\sqrt{z+2}}{2}$$

with $\sqrt{}$ the princial branch of the square root.

Proof. It is easy to see that $M = f(z)I$ for some f and also

$$f(z)(-f(z) - z) = 1$$

and hence $f(z)^2 - zf(z) + 1 = 0$ which means

$$(f(z) - z/2)^2 = (z-2)(z+2)/4$$

for all $z \in \mathbb{H}$. $\square$

Exercise.2

For the Newton flow, suppose that $s(t) = \frac{1}{n}tr(\mathcal{M})(t)$. Show that formally

$$\frac{ds(t)}{dt} = \frac{e^{-t}\frac{1}{n}tr(\xi)}{2s(t) + z}$$

Proof. We know

$$\frac{d}{dt}\mathcal{M}(t)(-tr(\mathcal{M}(t))/n - zI_n) = -\mathcal{M}(t)(-tr(\mathcal{M}(t))/n - zI_n)$$

and we take the trace of the both side we will have

$$\frac{d}{dt}(tr(\mathcal{M}(t))(-tr((M)(t) - nz))) = -tr((M)(t))(-tr(\mathcal{M}(t)) - nz)$$

and we will have

$$\frac{d}{dt}(s(t)(-s(t) - z)) = -s(t)(-s(t) - z)$$

and hence

$$s'(t) = -\frac{s(t)^2 - zs(t)}{2s(t) + z}$$

$\square$

Exercise.3

Conclude that there is an absolute constant $c > 0$ so that for any $z \in \mathbb{H}$ if $|2s(z) + z|^2 > c|\frac{1}{n}tr(\xi)|$ then the Newton flow is well-posed for all $t \in [0, \infty)$ and

$$|s(0) - s(\infty)| \leq \frac{|\frac{1}{n}tr(\xi)|}{\sqrt{|2s(z) + z|^2 - c|\frac{1}{n}tr(\xi)|}}$$

Proof. We have

$$d(2s(t) + z)^2/dt = 2e^{-t}\frac{1}{n}tr(\xi)$$

and hence

$$(2s(t) + z)^2|_0^\infty = -2\frac{1}{n}tr(\xi)e^{-t}|_0^\infty = \frac{2}{n}tr(\xi)$$

and we have $s(0) =$ □