NOTES FOR SMOOTH MANIFOLDS

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Contents

1 Smooth Manifolds

1.1 Topological Manifolds

Definiton 1.1.1. (Topological Manifolds)

We call M is a **topological manifold of dimension** n if

- M is a Hausdorff space
- M is second-countable
- M is locally Euclidean of dimension n, each point of M has a neighbourhood $U \cong V$ an open subset of $\mathbb{R}^n$.

Proposition 1.1.1. The third property is equivalent with that U is homeomorphic to some open ball in $\mathbb{R}^n$.

Theorem 1.1.2. (Topological Invariance of Dimension)

A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.

Definiton 1.1.2. (Coordinate Charts)

Let M be a topologiclal n-manifold. A **coordinate chart** on M is a pair (U, ϕ) for U open subset of M and $\phi : U \to \hat{U}$ an open subset of $\mathbb{R}^n$. ϕ is a **locall coordinate map** and the component functions $(x^1, \cdots, x^n)$ defined by $\phi(p) = (x^1(p), \cdots, x^n(p))$ are called **local coordinates** on U.

Here are some examples of topological manifolds.

Example 1.1.1. (Graphs of Continuous Functions)

Let $U \subset \mathbb{R}^n$ be an open subset, and let $f : U \to \mathbb{R}^k$ be a continuous function. The graph of f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ is $\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k, x \in U, y = f(x)\}$ with the subspace topology.

Proof.

Since $\mathbb{R}^n \times \mathbb{R}^k$ is Hausdorff and second-countable and we only need to check $\Gamma(f)$ is locally Euclidean. We may consider $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k$ which is obviously a bijection from $\Gamma(f) \to U$, the continuity of π_1^{-1} comes directly and that of π_1 comes for f is continuous.

Example 1.1.2. (Spheres)

The unit n-sphere S^n.

Proof.

Still only need to check the locally Euclidean property. Consider

$$U_i^+ = \{(x_1, \cdots, x_{n+1}), x_i > 0\}$$

and similarly defined U_i^-, then for D^n may define $x \mapsto (x_1, \cdots, x_{i-1}, 1 - |x|^2, x_i, \cdots, x_n)$ from $D^n \to U_i^+$ and similarly there is a homeomorphism from D^n to U_i^- and we are done.

Example 1.1.3. (Projective Spaces)

The n-**dimensial real projective space** denoted by $\mathbb{R}P^n$.

The charts is given by (U_i, ϕ_i), where $\widetilde{U}_i \subset \mathbb{R}^{n+1} - \{0\}$ and U_i is open, and

$$\phi_i([x_1, \cdots, x_{n+1}]) = \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right)$$

Proposition 1.1.3. $\mathbb{R}P^n$ is Hausdorff, second-countable and compact.

Definiton 1.1.3. (Product Manifolds)

Suppose $M_1, \cdots, M_k$ are topological manifolds of dimensions $n_1, \cdots, n_k$. Then the product space $M_1 \times \cdots \times M_k$ is Hausdorff and second-countable, for $(p_1, \cdots, p_k)$, we consider

$$\phi_1 \times \cdots \times \phi_k : U_1 \times \cdots \times U_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

will be a homeomorphism, which will make the product space a topological manifold.

Proof.

Waiting for adding.

Example 1.1.4. (Tori)

$T^n = S^1 \times \cdots \times S^1$.

Lemma 1.1.4. Every topological manifold has a countable basis of precompact coordinate balls.

Proof.

Waiting for adding.

Proposition 1.1.5. Let M be a topological manifold.

- M is locally path-connected.
- M is connected if and only if it is path-connected.
- The components of M are the same as its path components.
- M has coutably many components, each of which is an open subset of M and a connected topological manifold.

Proof.

Waiting for adding.

Proposition 1.1.6. Every topological manifold is locally compact.

Proof.

Waiting for adding.

Theorem 1.1.7. Every topological manifold is paracompact, i.e. for any topological manifold M, an open cover A of M and any basis B for the topology of M, then there exists a countable locally finite open refinement of A consisting of elements of B.

Proof.

Waiting for adding.

Proposition 1.1.8. The fundamental group of a topological manifold is countable.

1.2 Smooth Structures

Definiton 1.2.1. For M a topological n-manifold, if $(U,\phi),(V,\psi)$ are two charts such that $U\cap V$ nonempty, then if the **transition map** $\psi\circ\phi^{-1}$ is a diffeomorphism, then call the two charts **smoothly compatible**.

An **atlas** for M is a collection of charts whose domains cover M, and a **smooth atlas** is an atlas with any two charts in it are smoothly compatible.

A smooth atlas on M is **maximal** if it is not properly contained in a larger smooth atlas. Then a **smooth structure** on M is a maximal smooth atlas. A **smooth manifold** is a pair (M,A) where M to be a topological manifold and A a smooth structure on M.

Proposition 1.2.1. For M a topological manifold.

- Every smooth atlas A for M is contained in a unique maximal smooth atlas, called the smooth structure determined by A.
- Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof.

- We may consider

$$\mathcal{A}=\{(U,\phi),U\subset M \text{ open and smooth compatible with all charts in } A\}$$

.

- Easy to check.

Definiton 1.2.2. If M is a smooth manifold, any chart contained in the given smooth structure is a **smooth chart**. We call $B\subset M$ is a regular coordinate ball if there is a smooth cooredinate ball $B'\supset\overline{B}$ and a smooth coordinate map $\phi: B'\to\mathbb{R}^n$ such that $\phi(B)=B_r(0),\phi(\overline{B})=\overline{B}_r(0),\phi(B')=B_{r'}(0)$ with $r<r'$.

Proposition 1.2.2. Every smooth manifold has a countable basis of regular coordinate balls.

1.3 Examples of Smooth Manifolds

Example 1.3.1. (0-Dimensional Manifolds)

A topological manifold M of dimension 0 is just a countable discrete space.

Example 1.3.2. (Euclidean Spaces)

For $(\mathbb{R}^n, Id)$, we call this the **standard smooth structure**.

Example 1.3.3. (Finite-Dimensional Vector Spaces)

Since any notm on V induces the same topology, we may use assume it equips the 2-norms and consider $(E_1,\cdots,E_n)$ and define $E:\mathbb{R}^n\to V$

$$E(x)=\sum_{i=1}^n x_iE_i$$

this map is a homeomorphism, so (V, E^{-1}) is a chart. For any other basis we may check that $x \to \tilde{x}$ by an invertible linear map and we call this smooth structure as the standard smooth structure on V.

Example 1.3.4. (Spaces of Matrices)

Let $M(m \times n, \mathbb{R})$ denote the set of $m \times n$ matrices with real entries, and we identify it as $\mathbb{R}^{mn}$.

Example 1.3.5. (Open Submanifolds)

Let M be a smooth n-manifold and let $U \subset M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \phi) \text{ on } M \text{ such that } V \subset U\}$$

which will be a smooth structure on U and hence we may call any open subset on M an **open submanifold of** M.

Example 1.3.6. (The General Linear Group)

The **general linear group** $GL(n, \mathbb{R})$ is the set of intvertible $n \times n$ matrices. It is a smooth n^2-dimensional manifold as an open subset of n^2-dimensional vector space $M(n\mathbb{R})$.

Example 1.3.7. (Matrices of Full Rank)

Supoose $m < n$, we denote $M_m(m \times n, \mathbb{R})$ as the matrices of rank m. We may consider the nonsingular $m \times m$ submatrix and hence $M_m(m \times n, \mathbb{R})$ to be an open submanifold of $M(m \times n, \mathbb{R})$.

Example 1.3.8. (Space of Linear Maps)

Suppose V and W are finite-dimensional real vector spaces, then there will be a natural isomorphism between $L(V; W)$ and $M(m \times n, \mathbb{R})$.

Example 1.3.9. (Graphs of Smooth functions)

If $U \subset \mathbb{R}^n$ is an open subset and $f : U \to \mathbb{R}^k$ is a smooth function.

Example 1.3.10. (Spheres and Prjective Spaces)

Refer to the **standard smooth structure**.

Example 1.3.11. (Level Sets)

We will add this part later.

Example 1.3.12. (Smooth Product Manifolds) If $M_1, \cdots, M_k$ are smooth manifolds of dimensions $n_1, \cdots, n_k$ and we will induce the transition map

$$(\psi_1 \times \cdots \times \psi_k) \circ (\phi_1 \times \cdots \times \phi_k)^{-1} = (\psi_1 \circ \phi_1^{-1}) \times \cdots \times (\psi_k \circ \phi_k^{-1})$$

Lemma 1.3.1. (Smooth Manifold Chart Lemma) Let M be a set and suppose we are given a collection U_α of M with $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that

- For each α, ϕ_α is a bijection between U_α and an open subset $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$.
- For each α, β,$\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.
- If $U_\alpha \cap U_\beta$ is nonempty, then $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth.
- Countably many of the sets U_α cover M.
- Whenever p, q are distinct points in M, either there exists some U_α containing both p, q or there exists disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unqiue smooth manifold structure such that each (U_α, ϕ_α) is a smooth chart.

Definiton 1.3.1. (Grassmannian)

Let V be an n-dimensional real vector space. For any integer $0 \le k \le n$, we let $G_k(V)$ denote the set of all k-dimensional linear subspaces of V and $G_k(V)$ will be naturally given a structure of smooth manifold of dimension $k(n-k)$.

Proof.

We need to reply on the Smooth manifold chart lemma to construct the smooth structure on $G_k(V)$. Firstly, consider P a k-dimensional subspace of V and Q is complement with P, then for any $T \in L(P,Q)$, we consider $\gamma(T) = \{x + Tx, x \in P\}$ which is a k-dimensonal subspace of V and its intersection with Q is trivial. And for any X with trivial intersection with Q, for any $v \in X$, we consider $\pi_Q(v)$ to be the projection of v on Q, and $\pi_X(v)$ to be the projection of v on P. Then if $\pi_P(v) = \pi_P(w)$ for $v, w \in V$, then we know $\pi_Q(v) = \pi_Q(w)$ and hence $\pi_P(v)$ is an injective, and hence a surjective because of the dimension of V, which means $V \cong P$, so π_Q will induce a linear map from P to Q and we may check that the graph of this map is V. Then we will obtain a bijection from between $L(P,Q)$ and the k-dimensional subspaces with trivial intersection with Q.

So we may consider U_Q as all k-dim subspaces of V with intersecting Q trivially and we know U_Q has a bijection with $L(P,Q)$ and hence a bijection with an open subset $\phi_Q(U_Q) = \mathbb{R}^{k(n-k)}$. For any $K \in U_Q \cap U_{Q'}$, we may know $K \cap Q, K \cap Q'$ trivial and it will be identified to $L(P,Q), L(P',Q')$, then assume $I : U_Q \to L(P,Q), I' : U_{Q'} \to L(P',Q')$ the isomorphisms and denote $\psi_Q : L(P,Q) \to U_Q$ the isomorphism, then we may know any $X \in U_Q$, we have

$$\psi_Q^{-1}(X) = \pi_{Q,X} \circ \pi_{P,X}^{-1}$$

and hence

$$\psi_{Q'}^{-1}(X) = \pi_{Q',X} \circ \pi_{P',X}^{-1}$$

and then if we choose basis, we assume the transition matrix

$$T = \left(\begin{array}{c|c} A & B \\ C & D \end{array} \right)$$

and then

$$(\pi'_P \circ I_X)v = (A + BM)v \quad (\pi'_Q \circ I_X)v = (C + DM)v$$

and hence $N = (C+DM)(A+BM)^{-1}$ since $A+BM$ is full rank and we know the transition map is smooth. The countable cover is in fact finite by choosing a fixed basis.

Notice the conclusion that for any finite equal dimension subspaces, there is always a common $(n-k)$-dim Q to be their complement, then we are done. (A simple conclusion I have done before!)

1.4 Manifolds with Boundary

Definiton 1.4.1. (Closed upper half-space)

A **closed n-dimensional upper half space** $\mathbb{H}^n$ is

$$\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n, x_n \ge 0\}$$

and similarly we will have Int$\mathbb{H}^n$ to be the interior of the half-space and $\partial\mathbb{H}^n$.

Definiton 1.4.2. (Topological Manifold with boundary)

An n-**dimensional topological manifold with boundary** is a second-countable Hausdorff space M in which every point has a neighbourhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$. We will call (U, ϕ) an **interior chart** if $\phi(U)$ is an open subset.

A point $p \in M$ is called an **interior point of** M if it is in the domain of some interior chart. It is a **boundary point of** M if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$. The boundary point of M is denoted by ∂M and its interior can be denoted as Int$M$.

Theorem 1.4.1. (Topological Invariance of the Boundary)

If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both. Thus ∂M and Int$M$ are disjoint sets whose union is M.

Proposition 1.4.2. Let M be a topological n-manifold with boudary.

- Int$M$ is an open subset of M and a topological n-manifold without boundary.
- ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.
- M is a topological manifold if and only if $\partial M = \emptyset$.
- If $n = 0$ then $\partial M = \emptyset$ and M is a 0-manifold.

2 Smooth Maps

2.1 Smooth Functions and Smooth Maps

Definiton 2.1.1. (Smooth Function)

Suppose M is a smooth n-manifold, then $f : M \to \mathbb{R}^k$ is a **smooth function** if for any $p \in M$, there exists a smooth chart (U, ϕ) such that $p \in U$ and $f \circ \phi^{-1}$ is smooth on $\hat{U} = \phi(U)$.

$\hat{f} = f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k$ is a **coordinate representation of** f.

Definiton 2.1.2. For M, N smooth manifolds, $F : M \to N$ is a **smooth map** if for every $p \in M$, ther exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(P)$ such that $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

Proposition 2.1.1. Every smooth map is continuous.

Proof.

Waiting for add.

Proposition 2.1.2. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is smooth if and only if the following conditions is satisfied

- For every p, there exists smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U \cap F^{-1}(V))$ to $\psi(V)$.
- F is continuous and there exist smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ such that for each α and β, $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is smooth from $\phi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proposition 2.1.3. Let M and N be smooth manifolds with or without boudnary, and let $F : M \to N$ be a map.

- If every point $p \in M$ has a neighbourhood U such that the restriction $F|_U$ is smoot, then F is smooth.
- Conversely, if F is smooth, then its restriction to every open subset is smooth.

Corollary 2.1.4. (Gluing Lemma)

Let M and N be smooth manifolds with or without boundary, and let $(U_\alpha)_{\alpha \in A})$ be an open cover of M. Suppose that for each $\alpha \in$, we a re given a smooth map $F_\alpha U_\alpha \to N$ such that the maps agree on overlaps, then there exists a unique smooth map $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for any $\alpha \in A$.

Definiton 2.1.3. If $F : M \to N$ is a smooth map, and (U, ϕ) and (V, ψ) are any smooth charts for M and N, we call $\hat{F} : \psi \circ F \circ \phi^{-1}$ the coordinate representation of F.

Proposition 2.1.5. Let M, N, P be smooth manifolds with or without boundary.

- Every constant map $c : M \to N$ is smooth.
- The identity map of M is smooth.
- If $U \subset M$ is an open submanifold with or without boundary, then the inclusion $U \hookrightarrow M$ is smooth.

- If $F : M \to N$ and $G : N \to P$ are smooth, then so is $G \circ F : M \to P$.

Proposition 2.1.6. Suppose $M_1, \cdots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \cdots, M_k$ has nonempty boundary. For each i, let $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is the projection and then A map $F : N \to M_1 \times \cdots M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N \to M_i$ is smooth.

Example 2.1.1.

- Any map from a zero-dim manifold into a smooth manifold.
- If the circle S^1 is given the standard smooth structure, then $\epsilon\mathbb{R} \to S^1$ defined by $t \mapsto e^{2\pi it}$ is smooth.
- The map $\epsilon^n : \mathbb{R}^n \to T^n$.
- The inclusion map $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$.
- The quotient map $\pi : \mathbb{R}^{n+1}/\{0\} \to \mathbb{R}P^n$.
- $q : S^n \to \mathbb{R}P^n = \pi|S^n$ where π is the quotient map above.
- The projection maps from a product manifold to each component.

Definiton 2.1.4. A diffeomorphism from M to N is a smooth bijective map $F : M \to N$ that has a smooth inverse.

Example 2.1.2. Consider the maps $F : D^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to D^n$ given by

$$F(x) = \frac{x}{\sqrt{1 - |x|^2}} \quad G(x) = \frac{y}{\sqrt{1 + |y|^2}}$$

Proposition 2.1.7.

- Every composition of diffeomorphisms is a diffeomorphism.
- Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.
- Every diffeomorphism is a homeomorphism and an open map.
- The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.
- "Diffeomorphic" is an quivalence relation on the class of all smooth manifoldls with or without boundary.

Theorem 2.1.8. A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$.

Theorem 2.1.9. Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$ and F restricts to a diffeomorphism from $\mathrm{Int}M$ to $\mathrm{Int}N$.

2.2 Partitions of Unity

Definiton 2.2.1. Suppose M is a topological space, and let $X = (X_\alpha)_{\alpha in A}$ be an open cover of M. A **partition of unity subordinate to** X is an indexed family $\{\psi_\alpha\}_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with

- $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.
- $\text{supp}\psi_\alpha \subset X_\alpha$ for each $\alpha \in A$.
- The familty of supports is locally finite, i.e. that every point has a neighbourhood that intersects $\text{supp}\psi_\alpha$ for only finite indexes.
- $\sum\limits_{\alpha \in A} \psi_\alpha(X) = 1$ for all $x \in M$.

Theorem 2.2.1. (Existence of Partitions of Unity)

Suppose M is a smooth manifold with or without boundary, and X is any indexed open cover of M, then there exists a smooth partition of unity subordinate to X.

Definiton 2.2.2. If M is a topological space, $A \subset M$ is a closed subset, and $U \subset M$ is an open subset containing A, a continuous function $\psi : M \to \mathbb{R}$ is called **a bump function** for A supported in U if $0 \leq \psi \leq 1$ on M and $\psi \equiv 1$ on A and $\text{supp}\psi \subset U$.

Proposition 2.2.2. Let M be a smooth manifold with or without boudnary. For any closed subset $A \subset M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.

Lemma 2.2.3. (Extension Lemma for Smooth Functions)

Suppose M is a smooth manifold with or without boundary, $A \subset M$ is a closed subset and $f : A \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\text{supp}\tilde{f} \subset U$.

Definiton 2.2.3. If M is a topological space, an **exhaustion function for** M is a continuous function $f : M \to \mathbb{R}$ such that $f^{-1}((-\infty, c])$ is compact for each $c \in \mathbb{R}$.

Proposition 2.2.4. Every smooth manifold with or withoud boundary admits a smooth positive exhaustion function.

Theorem 2.2.5. Let M be a smooth manifold. If K is any closed subset of M, there is a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

3 Tangent Vector

3.1 Tangent Vectors

Definiton 3.1.1. (Derivation at p)

Let M be a smooth manifolds with or without boundary, and let p be a point of M. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a **derivation at** p if it satisfies

$$v(fg) = f(p)vg + g(p)vf$$

for all $f, g \in C^\infty(M)$.

The set of all derivations of $C^\infty(M)$ at p is the **tangent space of** M at p, denoted as T_pM.

Lemma 3.1.1. Suppose M is a smooth manifold with or without boundary, $p \in M, v \in T_pM$ and $f, g \in C^\infty(M)$.

- If f is a constant function, then $vf = 0$.
- If $f(p) = g(p) = 0$, then $v(fg) = 0$.

3.2 The Differential of a Smooth Map

Definiton 3.2.1. If M and N are smooth manifolds with or without boundary and $F : M \to N$ is a smooth map, for each $p \in M$ we define a map

$$dF_p : T_pM \to T_{F(p)}N$$

called the **differential of** F at p by

$$dF_p(v)(f) = v(f \circ F)$$

for $v \in T_pM$.

Proposition 3.2.1. Let M, N and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps and let $p \in M$.

- $dF_p : T_pM \to T_{F(p)}N$ is linear.
- $d(G \circ F) = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.
- $d(Id_M) = Id_{T_pM}$.
- If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proposition 3.2.2. Let M be a smooth manifold with or without a boundary, $p \in M$ and $v \in T_pM$. If $f, g \in C^\infty$ agree on some neighborhood of p, then $vf = vg$.

Proposition 3.2.3. Let M be a smooth manifold with or without boundary, let $U \subset M$ be an open subset, and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.

Proposition 3.2.4. If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.

Proof.

For $p \in M$, let (U, ϕ) be a smooth coordinate chart containing P, then we know $d\phi_p$ is an isomorphism from T_pU to $T_{\phi(p)}\hat{U}$ and since $T_pM \cong T_pU, T_{\phi(p)}\hat{U} \cong T_{\phi(p)}\mathbb{R}^n$ and we are done.

Lemma 3.2.5. Let $\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$. For any $a \in \partial\mathbb{H}^n$, the differential $d\iota_a : T_a\mathbb{H}^n \to T_a\mathbb{R}^n$ is an isomorphism.

Proposition 3.2.6. Suppose M is an n-dimensional smooth manifold with boundary. For each $p \in M$,T_pM is an n-dimensional vector space.

Proposition 3.2.7. Suppose V is a finite dimensional vector space with standard smooth manifold structure. For each point $a \in V$, the map $v \mapsto D_v|a$ where

$$D_v|_a f = \frac{d}{dt}|_{t=0} f(a + tv)$$

is a canonical isomorphism from V to T_aV such that for any linear map $L : V \to W$, we have

$$\begin{array}{ccc} V & \xrightarrow{\cong} & T_aV \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle dL_a} \\ W & \xrightarrow{\cong} & T_{L_a}W \end{array}$$

Proposition 3.2.8. Let $M_1, \cdots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection and for any $p \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \times \oplus T_{p_k}(M_k)$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \cdots, d(\pi_k)_p(v))$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

3.3 Computations in Coordinates

We denote

$$\frac{\partial}{\partial x_i}\Big|_p = (d\phi_p)^{-1}\left(\frac{\partial}{\partial x_i}\Big|_{\phi(p)}\right) = d(\phi^{-1})_{\phi(p)}\left(\frac{\partial}{\partial x_i}\Big|_{\phi(p)}\right)$$

which means

$$\frac{\partial}{\partial x_i}\Big|_p = \frac{\partial}{\partial x_i}\Big|_{\phi(p)}(f \circ \phi^{-1}) = \frac{\partial \hat{f}}{\partial x_i}(\hat{p})$$

Proposition 3.3.1. Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $(U, (x^i))$ containing p, the coordinate vectors $\partial/\partial_{x_i}\Big|_p$ form a basis for T_pM.

3.4 The Tangent Bundle

Definiton 3.4.1. The **tangent bundle** of M denoted by TM is defined by

$$TM = \bigsqcup_{p \in M} T_pM$$

then it comes a natural projection $\pi : TM \to M$.

Proposition 3.4.1. For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi : TM \to M$ is smooth. This smooth struture is called the **natural coordinates** on TM.

Proof.

For any smooth chart (U, ϕ) for M, we may consider $\pi^{-1}(U) \subset TM$ which induce a bijection

$$\tilde{\phi}\left(v_i \frac{\partial}{\partial x_i}\Big|_p\right) = (x_1(p), \cdots, x_n(p), v_1, \cdots, v_n)$$

from $\pi^{-1}(U)$ to $\phi(U) \times \mathbb{R}^n$.

Now it suffices to show that the transtion map is smooth since it is easy to check Hausdorff property. For any $(\pi^{-1}(U), \tilde{\phi}), (\pi^{-1}(V), \tilde{\psi})$ and we will know that

$$\tilde{\psi} \circ \tilde{\phi}^{-1} : \tilde{\phi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \phi(U \cap V) \times \mathbb{R}^n \to \tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \psi(U \cap V) \times \mathbb{R}^n$$

assume

$$\tilde{X}_X = \begin{pmatrix} \dfrac{d\widetilde{x_1}}{dx_1} & \cdots & \dfrac{d\widetilde{x_n}}{dx_1} \\ \vdots & \ddots & \vdots \\ \dfrac{d\widetilde{x_1}}{dx_n} & \cdots & \dfrac{d\widetilde{x_n}}{dx_n} \end{pmatrix}$$

and

$$\begin{aligned} \tilde{\psi} \circ \tilde{\phi}^{-1}(x_1, \cdots, x_n, v_1, \cdots, v_n) &= \tilde{\psi}\left(\sum_{i=1}^{n} v_i \frac{d}{dx_i}\Big|_{\phi^{-1}(x)}\right) \\ &= (\psi \circ \phi^{-1}(x), v\tilde{X}_X) \end{aligned}$$

is smooth.

Proposition 3.4.2. If M is a smooth n-manifold with or without boundary, and M can be covered by a single smooth chart, then TM is diffeomorphic to $M \times \mathbb{R}^n$.

Definiton 3.4.2. (Global Differential)

The **global differential** is denoted by $dF : TM \to TN$ defined by

$$dF|_{T_pM} = dF_p$$

Proposition 3.4.3. If $F : M \to N$ is a smooth map, then its global differential $dF : TM \to TN$ is a smooth map.

Proof.

Consider

$$\begin{aligned}\widetilde{dF}(x_1,\cdots,x_n,v_1,\cdots,v_n) &= \psi\left(dF(\sum_{i=1}^{n} v_i \frac{d}{dx_i}\Big|_{\phi^{-1}(x)})\right)\\ &= \sum_{i=1}^{n} v_i dF_{\phi^{-1}(x)}\left(\frac{d}{dx_i}\right)\\ &= \sum_{i=1}^{n} v_i\left(\sum_{j=1}^{n}\frac{d\left(\psi\circ F\circ\phi^{-1}\right)_j}{dx_i}\Big|_{\psi^{-1}(x)}\frac{d}{d\widetilde{x_j}}\right)\end{aligned}$$

is smooth.

Corollary 3.4.4. Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, then

- $d(G \circ F) = dG \circ dF$
- $d(Id_M) = Id_{TM}$
- If F is a diffeomorphism, then $dF : TM \to TN$ is also a diffeomorphism and $(dF)^{-1} = d(F^{-1})$

Proof.
We know

$$d(G\circ F)(v|_p) = (G\circ F)_p(v|_p) = dG_{F(p)}\circ dF_p(v|_p) = dG\circ dF(v|_p)$$

and similar for the rest conclusions.

3.5 Velocity Vectors of Curves

Definiton 3.5.1. A **curve** in M is a continuous map $\gamma : J \to M$ where J is an interval.

The **velocity** of γ at t_0 is the vector

$$\gamma'(t_0) = d\gamma(\frac{d}{dt}\Big|_{t_0}) \in T_{\gamma(t_0)}M$$

where

$$\gamma'(t_0) = d\gamma\left(\frac{d}{dt}\Big|_{t_0}\right) f = \frac{d}{dt}\Big|_{t_0}(f\circ\gamma) = (f\circ\gamma)'(t_0)$$

Proposition 3.5.1. Suppose M is a smooth manifold with or without boundary and $p \in M$. Every $v \in T_pM$ is the velocity of some smooth curve in M.

Proposition 3.5.2. Let $F : M \to N$ be a smooth map, and let $\gamma : J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma : J \to N$ is given by

$$(F\circ\gamma)'(t_0) = dF(\gamma'(t_0))$$

Proof.
We know

$$(F\circ\gamma)'(t_0) = d(F\circ\gamma)\left(\frac{d}{dt}\Big|_{t_0}\right) = d(F\circ\gamma)_{t_0}\left(\frac{d}{dt}\right) = dF_{\gamma(t_0)}\circ d\gamma_{t_0}\left(\frac{d}{dt}\right) = dF(\gamma'(t_0))$$

Corollary 3.5.3. Suppose $F : M \to N$ is a smooth map, $p \in M$ and $v \in T_pM$. Then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma : J \to M$ such that $0 \in J, \gamma(0) = p$ and $\gamma'(0) = v$.

4 Submersions, Immersions, and Embeddings

4.1 Maps of Constant Rank

Definiton 4.1.1. (Rank)

Given a smooth map $F : M \to N$ and a point $p \in M$, we define the *rank* of F at p to be the rank of linear map $dF_p : T_pM \to T_{F(p)N}$, which is the Jacobian matrix of F in any smooth chart.

If F has the same rank at every point, we call it has **constant rank**.

If the rank of dF_p reaches its upper bound $\min\{\dim M, \dim N\}$, we call F has **full rank** at p and if F has full rank every where, we say F has **full rank**.

Proof.

$$\begin{tikzcd}
\phi(U) \arrow[r, "\psi\circ F\circ\phi^{-1}"] \arrow[d, "\phi^{-1}"] & \psi(V) \arrow[d, "\psi^{-1}"] \\
U \arrow[r, "F"] \arrow[dr, "f\circ F"] & V \arrow[d, "f"] \\
& \mathbb{R}
\end{tikzcd}$$

and we have

$$\frac{\partial (f\circ F)\circ\phi^{-1}}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial f\circ\psi^{-1}}{d\tilde{x_i}} \frac{\partial \tilde{x_i}}{\partial x_j}$$

which implies that

$$dF_p\left(\frac{\partial}{\partial x_j}\right) = \sum_{i=1}^{n} \frac{\partial \tilde{x_i}}{\partial x_j}\frac{\partial}{\partial \tilde{x_i}}$$

and denote $\partial_M = \left(\frac{\partial}{\partial x_i}\right)_{i=1}^{n}$, $\partial_N = \left(\frac{\partial}{\partial \tilde{x_i}}\right)_{i=1}^{n}$, we have

$$dF_p\partial_M = \left(\frac{\partial(\psi\circ F\circ\phi^{-1})_i}{\partial x_j}\right)_{1\le i,j\le n}\partial_N$$

and hence the rank of dF_p is equals to the Jacobian matrix of F in any smooth chart because the transition between Jacobian matrices are induced by a diffeomorphism.

Definiton 4.1.2. (Submersion and Immersion)

A smooth map $F : M \to N$ is called a **smooth submersion** if its differetional is surjective ($\mathrm{rank}F = \dim N$) at each point and it is called a **smooth immersion** if its differential is injective ($\mathrm{rank}F = \dim M$) at each point.

Proposition 4.1.1. Suppose $F : M \to N$ is a smooth map and $p \in M$. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion and if dF_p is injective, then p has a neighborhood U such that $F|_U$ is an immersion.

Definiton 4.1.3. If M and N are smooth manifolds with or without boundary, a map $F : M \to N$ is called a **local diffeomorphism** if every point $p \in M$ has a neighbourhood U such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 4.1.2. (Inverse Function Theorem)

Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$

is apoint such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

Theorem 4.1.3. (Rank Theorem)

Suppose M and N are smooth manifolds of dimensions m and n, and $F : M \to N$ is a smooth map with constant rank r. For each $p \in M$ there exist smooth charts (U, ϕ) for M centereed at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subset V$, in which F has a coordinate representation of the form

$$\hat{F}(x_1, \cdots, x_r, x_{r+1}, \cdots, x_m) = (x_1, \cdots, x_r, 0, \cdots, 0)$$

Corollary 4.1.4. Let M and N be smooth manifolds, let $F : M \to N$ be a smooth map, and suppose M is conneceted. Then the following are equivalent:

- For each $p \in M$ there exist smooth chartss containing p and $F(p)$ in which the coordinate representation of F is linear.
- F has constant rank.

Proof.

Since the linear coordinate representation will induce a constant rank map on a neighbourhood, which means that F admits contant rank on a neighborhood for any point, and hence F has a contant rank on whole M because it is connected.

Conversely, it comes from the rank theorem.

Theorem 4.1.5. (Global Rank Theorem)

Let M and N be smooth manifolds, and suppose $F : M \to N$ is a smooth map of constant rank.

- If F is surjective m then it is a smooth submersion.
- If F is injective, then it is a smooth immersion.
- If F is bijective, then it is a diffeomorphism.

Theorem 4.1.6. Suppose M is a smooth m-manifold with boundary, N is a smooth n-manifold, and $F : M \to N$ is a smooth immersion. For any $p \in \partial M$, there exist a smooth boundary chart (U, ϕ) for M centered at p and a smooth coordinate chart (V, ψ) for N centered at $F(p)$ with $F(U) \subset V$, in which F has the coordinate representation

$$\hat{F}(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)$$

4.2 Embeddings

Definiton 4.2.1. (Smooth Embedding)

If M and N are smooth manifolds with or withour boundary, a **smooth embedding** of M into N is a smooth immersion $F : M \to N$ that is also a topological embedding, i.e., a homeomorphism onto its image.

Lemma 4.2.1. Suppose X and Y are topological spaces, and $F : X \to Y$ is a continuous map that is either open or closed.

- If F is surjective, then it is a quotient map.
- If F is injective, then it is a topological embedding.
- If F is bijective, then it is a homeomorphism.

Proof.

If F is surjective, then F is open is equivalent with F is closed, so for any $V \subset Y$, if $p^{-1}(V)$ open, then V is open. If V open, then then $p^{-1}(V)$ obviously open since F is continuous.

If F is injective, then we know F is a bijection between X to $F(X)$ and we may know it is continuous, and the inverse is also continuous if it is open or close.

The last conclusion is going on.

Proposition 4.2.2. Suppose M and N are smooth manifolds with or withour boundary, and $F : M \to N$ is an injective smooth immersion. If any of the following holds, then F is a smooth embedding.

- F is an open or closed map.
- F is a proper map.
- M is compact.
- M has empty boundary and $\dim M = \dim N$.

Proof.

The first condition makes $F : M \to F(M)$ a homeomorphism.

If F is a proper map, assume K is closed in M and y is a limit point of $F(K)$, consider V a precompact neighbourhood of y and we have $\overline{U}$ is compact, which means $F^{-1}(\overline{U})$ is compact in M and hence $K \cap \overline{U}$ compact, so $F(K) \cap \overline{U}$ compact and hence closed, y has to be a limit point of $F(K) \cap \overline{U}$ and we are done.

If M is compact, similarly we may know the proof above can be still used since F is still proper.

We may know F is a local diffeomorphism and hence a local diffeomorphism.

Theorem 4.2.3. Suppose M and N are smooth manifolds with orwithout boundary, and $F : M \to N$ is a smooth map. Then F is a smooth immersion if and only if every point in M has a neighbourhood $U \subset M$ such that $F|_U : U \to N$ is a smooth embedding.

Proof.

Firstly, if any point has a neighbourhood such that the restriction of F on it is an embedding, then it is full rank there and hence everywhere, then we are done.

If F is a smooth immersion, then by rank theorem or theorem 4.1.6. there exists U of p such that $F|_U$ is injective if $F(p) \in \partial N$, for those $F(p)$ on boundary, we may adopt the inclusion of half-upper space to $\mathbb{R}^n$.

Now we may assume for any $p \in M$, there exists a neighborhood U such that $F|_U$ is injective, then we choose $V \subset U$ precompact and $F|_{\overline{V}}$ is a smooth embedding by theorem 4.2.2.

4.3 Submersions

Definiton 4.3.1. (Section)

If $\pi : M \to N$ is any continuous map, a **section** of π is a continuous right inverse for π, a **local section** of π is a continuous map $\sigma : U \to M$ on some open subset of N such that $\pi \circ \sigma = Id_U$.

Theorem 4.3.1. (Local Section Theorem)

Suppose M and N are smooth manifolds and $\pi : M \to N$ is a smooth map. Then π is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.

Proof.

Firstly, if p is in the image of a smooth local section of π, then there exists a neighbourhood U of p such that $\pi|_{\sigma(U)} \circ \sigma = Id_U$ and hence $d\pi_p$ is surjective.

Conversely, we may know for p, there exists a chart (U, ϕ) such that the coordinate represenation is like that in rank theorem, and consider a small enough neighborhood of $(x_1(p), \cdots, x_k(p))$ and we are done.

5 Lie Groups

5.1 Basic Concepts

Definiton 5.1.1. (Lie Group)

A **Lie group** is a smooth manifold G that is a group, with multiplication $m : G \times G \to G$ and inversion map $i : G \to G$ is smooth.

Proposition 5.1.1. If G is a smooth manifold with a group structure such that the map $G \times G \to G$ is smooth, then G is a Lie group.

Proof.

Firstly we may obtained that the inversion is smooth since the inclusion $\iota : g \mapsto (e, g)$ from G to $G \times G$ is smooth and $m \circ \iota$ smooth and we are done.

The rest is to check $Id \otimes i$ is smooth.

Definiton 5.1.2. If G is a Lie group, any element $g \in G$ defines maps L_g, R_g by $L_g(h) = gh, R_g(h) = hg$, which is a diffeomorphism.

Definiton 5.1.3. If G and H are Lie groups, a **Lie group homomorphism** from G to H is a smooth map $F : G \to H$ is also a group homomorphism.

Theorem 5.1.2. Every Lie group homomorphism has constant rank.

6 Vector Fields

6.1 Vector Fields on Manifolds

Definiton 6.1.1. (Vector Fields)

If M is a smooth manifold with or without boundary, a **vector field** on M is a section of the map $\pi : TM \to M$, i.e. a continuous map $X : M \to TM$ with $\pi \circ X \ = Id_M$. **Smooth vector fields** are those smooth as maps from M to TM.

The **support** of X is define by the closure of

$$\{p \in M, X_p \neq 0\}$$

and **compactly supported** if it has a compact support. For a chart $(U, (x_i))$ if we write

$$X(p) = X_i(p) \frac{\partial}{\partial x_i}\Big|_p$$

then we call $X_i : U \to \mathbb{R}$ the component functions of X.

Proposition 6.1.1. Let M be a smooth manifold with or without boudnary, and let $X : M \to TM$ be a vector field, if $(U, (x_i))$ is any smooth coordinate chart on M, then the restriction of X to U is smooth if and only if its component functions w.r.t. this chart are smooth.

Proof.

Assume (x_i, v_i) to be the coordinates on $\pi^{-1}(U)$ and Then

$$\hat{X}(x) = (x_1, \cdots, x_n, \tilde{X}_1(x), \cdots, \tilde{X}_n(x))$$

and we are done.

Lemma 6.1.2. Let M be a smooth manifold with or without boundary, and let $A \subset M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset containing A, there exists a smooth global vector field $\tilde{X}$ on M such that $\tilde{X}|_A = X$ and $\text{supp}\tilde{X} \subset U$.

Proposition 6.1.3. Let M be a smooth manifold with or without boundary. Given $p \in M$ and $v \in T_pM$, there is a smooth global vector field X on M such that $X_p = v$.

Definiton 6.1.2. It is standard to use $\mathfrak{X}(M)$ to denote all smooth vector fields on M. With

$$(aX + bY)(p) = aX(p) + bY(p)$$

and we may define for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$

$$(fX)(p) = f(p)X_p$$

and we may see it is a smooth vector field.

Proposition 6.1.4. Let M be a smooth manifold with or without boundary.

- If X and Y are smooth vector fields on M and $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth vector field.

- $\mathfrak{X}(M)$ is a module over the ring $C^\infty(M)$.

Definiton 6.1.3. (Frame)

Suppose M a smooth n-manifold with or without boundary. An ordered k-tuple (X_i) defined on some subset A is **linear independent** if $(X_1(p), \cdots, X_k(p))$ is a linearly independent k-tuple in T_pM at each $p \in A$. It is called to **span the tangent bundle** if $(X_1(p), \cdots, X_k(p))$ spans T_pM at each $p \in A$.

A **local frame** for M is an ordered n-tuple of vector fields $(E_1, \cdots, E_n)$ defined on an open subset U that is linearly independent and spans the tangent bundle, and it is a **global frame** if $U = M$ and a **smooth frame** if E_i is smooth.

Definiton 6.1.4. If $X \in \mathfrak{X}(M)$ and f is a smooth function defined on an open subset $U \subset M$, we obtain a new function $Xf : U \to \mathbb{R}$ defined by

$$(Xf)(p) = X(p)f$$

Proof.

To see $Xf \in \mathfrak{X}(M)$, we may check for a chart (U, ϕ), we will have

$$\widetilde{(Xf)}(x) = \tilde{f}(x)\widetilde{X}(x) = \sum_{i=1}^{n} \tilde{f}(x)\tilde{X}_i(x)\frac{\partial}{\partial x_i}$$

Proposition 6.1.5. Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. The following are equivalent

- X is smooth.
- For every $f \in C^\infty(M)$, the function Xf is smooth on M.
- For every open subset $U \subset M$ and every $f \in C^\infty(U)$, the function Xf is smooth on U.

Proof.

We have proved (a) implies (b), and to see (b) implies (c), we may consider if $f \in C^\infty(U)$, then consider ψ a bump function which equals to 1 on some neighbourhood of p with supp$\psi \in U$ and we may know ψf can be extended to M and $X(\psi f)$ is smooth, which equals to Xf on some meighbourhood of p, and hence Xf is smooth in a neighbourhood of any point of U and we are done.

We may consider a local coordinates on U and then apply (c) to x_i and we may get $X(x_i) = X_i$ which is smooth on some neighborhood of any point.

Definiton 6.1.5. (Global Derivation)

A map $X : C^\infty \to C^\infty$ is a **derivation** if it is linear and

$$X(fg) = fX(g) + gX(f)$$

and we may know $\mathfrak{X}(M)$ is a subset of derivation.

Proposition 6.1.6. Let M be a smooth manifold with or without boundary. A map $D : C^\infty \to C^\infty$ is a derivation if and only if it is of the form $D(f) = X(f)$ for some smooth vector field $X \in \mathfrak{X}(M)$.

6.2 Vector Fields and Smooth Maps

Definiton 6.2.1. (F-related)

Suppose $F : M \to N$ is smooth and X is a vector field on M, and suppose there is a vector field Y on N such that

$$dF_p(X(p)) = Y(F(p))$$

for each $p \in M$, then we call X and Y are F-**related**.

Proposition 6.2.1. Suppose $F : M \to N$ is a smooth map between manifolds with or without boundary, $X \in \mathfrak{M}, Y \in \mathfrak{N}$. Then X and Y are F-related if and only if for every smooth function f define on an open subset of N, we have

$$X(f \circ F) = (Yf) \circ F$$

Proposition 6.2.2. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a diffeomorphism, For every $X \in \mathfrak{M}$, there is a unique smooth vector field on N that is F-related to X.

This vector field is called the **pushforward** of X by F.

Corollary 6.2.3. Suppose $F : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$, for any $f \in C^\infty(N)$

$$((F_*X)f) \circ F = X(f \circ F)$$

6.3 Lie Brackets

Definiton 6.3.1. For two smooth vector fields X, Y, we may define the **Lie Bracket** of X and Y by

$$[X, Y]f = XYf - YXf$$

Lemma 6.3.1. The Lie bracket of any pair of smooth vector fields is a smooth vector field.

Proposition 6.3.2. (Coordinate Formula for the Lie Bracket)

Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X_i \dfrac{\partial}{\partial x_i}$ and $Y = Y_j \dfrac{\partial}{\partial x_j}$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x_i) for M. Then $[X, Y]$ has the following coordinate expression

$$[X, Y] = \sum_{1 \le i,j \le n} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} (XY_j - YX_j) \frac{\partial}{\partial x_j}$$

Proposition 6.3.3. The Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$

- For $a, b \in \mathbb{R}$,$[aX + bY, Z] = a[X, Z] + b[Y, Z], [Z, aX + bY] = a[Z, X] + b[Z, Y]$
- $[X, Y] = -[Y, X]$
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$
- For $f, g \in C^\infty(M)$,$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$.

Proposition 6.3.4. Let $F : M \to N$ be a smooth map between manifolds with or withour boundary, and let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be vector fields such that X_i is F-related tp Y_i, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

6.4 The Lie Algebra of a Lie Group

Definiton 6.4.1. Suppose G is a Lie group. A vector field X on G is said to be **left-invariant** if it is invariant under all left translations, i.e.

$$d(L_g)'_g((g')) = X(gg')$$

which means X is L_g-related to itself and $(L_g)_*X = X$.

7 Integral Curves and Flows

7.1 Integral Curves

Definiton 7.1.1. (Integral Curve)

Suppose M is a smooth manifold with or without boundary. If V is a vector field on M, an **integral curve** of V is a differentiable curve $\gamma : J \to M$ such that

$$\gamma'(t) \ = V(\gamma(t))$$

for all $t \in J$.

Proposition 7.1.1. Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exist $\epsilon > 0$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ that is an integral curve of V starting at p.

Lemma 7.1.2. (Rescaling Lemma)

Let V be a smooth vector field on a smooth manifold M, let $J \subset \mathbb{R}$ be an interval, and let $\gamma : J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, the curve $\tilde{\gamma} \to M$ defined by $ga\tilde{m}ma(t) = \gamma(at)$ is an integral curve of the vector field aV, wjere $\tilde{J} = \{t, at \in J\}$.

Lemma 7.1.3. (Transition Lemma)

Let V, M, J and γ be as in the proceding lemma. For any $b \in \mathbb{R}$, the curve $\hat{\gamma} : \hat{J} \to M$ defined by $\hat{\gamma}(t) = \gamma(t + b)$ is also an integral curve of V, where $hatJ = \{t + b \in J\}$.

Proposition 7.1.4. Suppose M and N are smooth manifolds and $F : M \to N$ is a smooth map. Then $X \in \mathfrak{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y, meaning that for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.

7.2 Flows

Definiton 7.2.1. (Global Flow)

A **global flow** on M to be a continuous left $\mathbb{R}$-action on M, i.e. a continuous map $\theta : \mathbb{R} \times M \to M$ such that for all $s, t \in \mathbb{R}$ and $p \in M$

$$\theta(t, \theta(s, p)) = \theta(t + s, p), \quad \theta(0, p) = p$$

And we may care about continuous map $\theta_t : M \to M$

$$\theta_t(p) = \theta(t, p)$$

and for each $p \in M$, we may define $\theta^{(p)} : \mathbb{R} \to M$ by

$$\theta^{(p)}(t) = \theta(t, p)$$

Definiton 7.2.2. (Infinitesimal generator)

If $\theta : \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a tangent vector $V_p \in T_pM$ by

$$V_p = {\theta^{(p)}}'(0)$$

then $p \mapsto V_p$ is a vector field on M, which is called **infinitesimal generator** of θ.

Proposition 7.2.1. Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. The infinitesimal generator V of θ is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V.

Definiton 7.2.3. (Flow)

If M is a manifold, a **flow domain** for M is an open subset $D \subset \mathbb{R} \times M$ with the property that for each $p \in M$, $D^{(p)} = \{t, (t,p) \in D\}$ is an open interval containing 0.

A **flow** on M is a continuous map $\theta : D \to M$ where $D \subset \mathbb{R} \times M$ is a flow domain such that

$$\theta(0,p) = p$$

and for all $s \in D^{(p)}$ and $t \in D^{(\theta(s,p))}$ such that $s + t \in D^{(p)}$ we have

$$\theta(t, \theta(s,p)) = \theta(t+s, p)$$

If θ is a flow, we define $\theta_t(p) = \theta^{(p)}(t) = \theta(t,p)$ if $(t,p) \in D$. For each $t \in \mathbb{R}$, we also define

$$M_t = \{p, (t,p) \in D\}$$

If θ is smooth, the **infinitesimal generator** of θ is defined by $V_p = {\theta^{(p)}}'(0)$

Proposition 7.2.2. If $\theta : D \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ uis an integral curve.

Theorem 7.2.3. (Fundamental Theorem on Flows)

Let V be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\theta : D \to M$ whose infinitesimal generator is V. This flow has the following properties

- For each $p \in M$, the curve $\theta^{(p)} : D^{(p)} \to M$ is the unique maximal integral curve of V starting at p.
- If $s \in D^{(p)}$, then $D^{(\theta(s,p))}$ is the interval $D^{(p)}$.
- For each $t \in \mathbb{R}$, the set M_t is open in M and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

This unique flow is called the **flow generated** by V.

Proposition 7.2.4. Suppose M and N are smooth manifolds, $F : M \to N$ is a smooth map, $X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η the flow of Y. If X and Y are F-related, then for each $t \in \mathbb{R}$, $F(M_t) \subset N_t$ and $\eta_t \circ F = F \circ \theta_t$

$$\begin{tikzcd}
M_t \arrow[r, "F"] \arrow[d, "\theta_t"] & N_t \arrow[d, "\eta_t"] \\
M_{-t} \arrow[r, "F"] & N_{-t}
\end{tikzcd}$$

Corollary 7.2.5. Let $F : M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then the flow of F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$ with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

8 Vector Bundles

8.1 Vector Bundles

Definiton 8.1.1. (Vector Bundle)

Let M be a topological space. A **vecctor bundle** of rank k over M is a topological space E together with a surjective continuous map $\pi : E \to M$ such that

- For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is endowed with the structure of a k-dimensional real vector space.
- For each $p \in M$, there exist a neighborhood U of p in M and a homeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\pi_U \circ \phi = \pi$ and for each $q \in U$, the restriction of ϕ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

If M and E are smooth manifolds with or without boundary, π is a smooth map, and ϕ can be chosen to be diffeomorphisms, then E is called a **smooth vector bundle**.

A rank-1 vector bundle is called a **line bundle**. The space E is called the **total space of the bundle** and M is called its **base** and π to be its **projection**.

If there exists a local trivialization of E over all of M, then E is said to be a **trivial bundle** and if $E \to M$ is a smooth bundle that admits a smooth global trivialization, then we say that E is **smoothly trivial**.

Lemma 8.1.1. Let $\pi : E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of R trivializations of E with $U \cap V \neq \emptyset$. There exists a smooth map $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$$

where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$.

9 Differential Forms

9.1 Algebra of Alternating Tensors

Lemma 9.1.1. Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent

- α is alternating
- $\alpha(v_1, \cdots, v_k) = 0$ whenever the k-tuple $(v_1, \cdots v_k)$ is linear dependent
- α gives the value zero whenever two of its arguments are equal, i.e.

$$\alpha(v_1, \cdots, w, \cdots, w, \cdots, v_k) = 0$$

Definiton 9.1.1. (Alternation)

Alt : $T^k(V^*) \to \Lambda^k(V^*)$ is called the **alternation** defined by

$$\mathrm{Alt}\alpha = \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\sigma)(\alpha^\sigma)$$

Proposition 9.1.2. Let α be a covariant tensor on a finite-dimensional vector space

- Altα is alternating
- Alt$\alpha = \alpha$ if and only if α is alternating

Definiton 9.1.2. (Multi-index)

A **multi-index** of length k is an ordered k-tuple $I = (i_1, \cdots, i_k)$ and for $\sigma \in S_k$ define

$$I_\sigma = (i_{\sigma(1)}, \cdots, i_{\sigma(k)})$$