NOTES FOR RENORMALIZATION FLOW

Based on the paper by A.Dunlap and Cole

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Contents

1 Stochastic Integrals

1.1 Wiener Integral

Let T be a set and $X := \{X(t)\}_{t\in T}$ a T-indexed stochastic process. We recall that X is a Gaussian random field (process when $T \subset \mathbb{R}$) if $(X_{t_1}, \cdots, X_{t_m})$ is a Gaussian random vector for all $t_1, \cdots, t_m \in T$.

Definiton 1.1.1. Let $\mathscr{L}(\mathbb{R}^m)$ denote the collection of all Borel-measurable subsets of $\mathbb{R}^m$ that have finite Lebesgue measure. White noise on $\mathbb{R}^m$ is a mean-zero set-indexed Gaussian random field $\xi(A)_{A\in\mathscr{L}(\mathbb{R}^m)}$ with covariance function

$$E[\xi(A_1)\xi(A_2)] := |A_1 \cap A_2| \quad \text{for all } A_1, A_2 \in \mathscr{L}(\mathbb{R}^m),$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^m$ for every m.

2 Setup

2.1 Semilinear SHE

We consider the semilinear stochastic heat equation

$$du_t^\rho = \frac{1}{2}\Delta u_t^\rho dt + \gamma_\rho \sigma(u_t^\rho) dW_t^\rho, \quad t > 0, x \in \mathbb{R}^2$$

Here σ is a Lipschitz nonlinearity and $dW_t^\rho(x)$ is a Gaussian noise that is white in time and correlated in space at scale $\rho^{1/2} << 1$. We are interested in the pointwise behavior of $u_t^\rho(x)$ as $\rho \to 0$, which calls for an attenuation factor $\gamma_\rho \sim |\ln \rho|^{-1/2}$ due to critical scaling in two dimensions. In fact, we devote most of our attention to a variation on (2.1) in which we first multiply σ and then smooth the noise:

$$dv_t^\rho = \frac{1}{2}\Delta v_t^\rho dt + \gamma_\rho \mathcal{G}_\rho[\sigma(v_t^\rho) dW_t]$$

Definiton 2.1.1.

(Space-time White Noise)

Let $dW = (dW_t(x))_{t \in \mathbb{R}, x \in \mathbb{R}^2}$ be a standard $\mathbb{R}^m$-valued space-time white noise generating a temporal filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$. Writing $dW = (dW^1, \cdots, dW^m)$ in components, then

$$\mathbb{E}[dW_t^i(x) dW_{t'}^{i'}(x')] = \delta_{i,i'}\delta(t - t')\delta(x - x')$$

Proposition 2.1.1. Construction a space-time white noise.

Definiton 2.1.2.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a proability space and fix a target dimension $m \in \mathbb{N}$. The solution $v^\rho : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^m$ is a random vector-valued function parametrized by the correlastion parameter $\rho > 0$. We suppress the dependence of v^p on $\omega \in \Omega$.

Since v is vector-valued, our nonlinearity $\sigma : \mathbb{R}^m \to \mathbb{R}^{m \times m}$ is matrix-valued. Let $\mathcal{H}_+^m$ dentoe the set of nonnegative-definite symmetric real $m \times m$ matrices, equipped with the metric induced by the Frobenius norm

$$|A|_F^2 := \mathrm{tr}(AA^T) = \mathrm{tr}(A^2)$$

Let σ belong to the space $\mathrm{Lip}(\mathbb{R}^m, \mathcal{H}_+^m)$.

Definiton 2.1.3. Given $\tau \geq 0$, we define the heat operator

$$\mathcal{G}_\tau v = G_\tau * v$$

where $G_\tau = (2\pi\tau)^{-1} \exp\left(-\frac{|x|^2}{2\tau}\right)$ denotes the standard heat kernel. Define the spatially-smoothed noise $dW_t^\rho = G_\rho * dW_t$.

Proposition 2.1.2. We have

$$\mathbb{E}[dW_t^{\rho,i}(x) dW_{t'}^{\rho,i'}(x') = \delta_{i,i'}\delta(t - t') G_{2\rho}(x - x')$$

Proof. □

Definiton 2.1.4. Define

$$L(\tau) = \ln(1+\tau) \quad \text{for } \tau \geq 0$$

and set

$$\gamma_\rho = \sqrt{\frac{4\pi}{L(1/\rho)}}$$

Definiton 2.1.5.

(Mild Solution 1)

A mild solution for (2.1) is a predictable random field v^ρ such that for all $s < t$, we have

$$v_t^\rho(x) = \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \mathcal{G}_{t+\rho-r}[\sigma(v_r^\rho) dW_r](x)$$

which means

$$\begin{aligned}
v_t^\rho x &= \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \int G_{t+\rho-r}(y) \sigma(v_r^\rho)(x-y) dW_r(x-y) dy \\
&= \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int_s^t \int G_{t+\rho-r}(y) \sigma(v_r^\rho)(x-y) dW_r(x-y) dy \\
&= \mathcal{G}_{t-s} v_s^\rho(x) + \gamma_\rho \int \left(\int_s^t G_{t+\rho-r}(y) \sigma(v_r^\rho)(x-y) dW_r(x-y) \right) dy
\end{aligned}$$

which can be interpreted as an Ito integral. We only look for the solution v_t^ρ in the spaces $\mathscr{X}_t^l$ of $\mathbb{R}^m$-valued random fields z on $\mathbb{R}^2$ that are $\mathcal{F}_t$-measurable and

$$||z||_l := \sup_{x \in \mathbb{R}^2} (\mathbb{E}|z(x)|^l)^{1/l} < \infty$$

Proposition 2.1.3. For any $l \geq 2$, there is a family of random operators $(\mathcal{V}_{s,t}^{\sigma,\rho})_{s<t}$ such that if $v_s \in \mathscr{X}_s^l$, then $v_t^\rho = \mathcal{V}_{s,t}^{\sigma,\rho} v_s$ is a mild solution of (2.1) for $t \geq s$. We oftten write $\mathcal{V}_t^{\sigma,\rho} := \mathcal{V}_{0,t}^{\sigma,\rho}$.

Shown by some standard fixed-point arguments.

Definiton 2.1.6. (Forward-backward SDE)

The system of SDE:

$$\begin{aligned}
d\Gamma_{a,Q}^\sigma(q) &= J_\sigma(Q-q, \Gamma_{a,Q}^\sigma(q)) dB(q), && a \in \mathbb{R}^m, 0 < q < Q \\
\Gamma_{a,Q}^\sigma(0) &= a, && a \in \mathbb{R}^m, Q \geq 0 \\
J_\sigma(q,b) &= [\mathbb{E}\sigma^2(\Gamma_{a,Q}^\sigma(q))]^{1/2}, && q \geq 0, b \in \mathbb{R}^m
\end{aligned}$$

for B a standard $\mathbb{R}^m$-valued Brownian motion and $A^{1/2}$ is the unique positive-definite matrix square root of $A \in \mathcal{H}_+^m$.

2.2 Main Result