# Chapter 1

## Fundamental Concepts

**Definition 1.1**

*If $U \subset \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$ is a continuous function, then $f$ is called $C^1$ on $U$ if $\partial f/\partial x, \partial f/\partial y$ exist and are continous on $U$.*

♣

**Definition 1.2**

*We define for $f = u + iv : U \to \mathbb{C}$ a $C_1$ function*

$$\frac{\partial}{\partial z} f := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f$$
$$\frac{\partial}{\partial \bar{z}} f := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f$$

*which is easy to be checked linear and the chain rules.*

♣

where we may check let $z = x + iy, \bar{z} = x - iy$, we have

$$\frac{\partial}{\partial z} z = 1, \quad \frac{\partial}{\partial z} \bar{z} = 0$$
$$\frac{\partial}{\partial \bar{z}} z = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{z} = 1$$

**Proposition 1.1**

*(The Leibniz Rules) We have for any $F, G \in C^1$*

$$\frac{\partial}{\partial z}(F \cdot G) = \frac{\partial F}{\partial z} \cdot G + F \cdot \frac{\partial G}{\partial z}$$
$$\frac{\partial}{\partial \bar{z}}(F \cdot G) = \frac{\partial F}{\partial \bar{z}} \cdot G + F \cdot \frac{\partial G}{\partial \bar{z}}$$

♠

**Proposition 1.2**

*We have for $l \leq j, m \leq k$ nonnegative integers and then*

$$(\frac{\partial^l}{\partial z^l})(\frac{\partial^m}{\partial \bar{z}^m})(z^j \bar{z}^k) = \frac{j!}{l!}\frac{k!}{m!} z^{j-l} \bar{z}^{k-m}$$

♠

**Proposition 1.3**

*If $p(z, \bar{z}) = \sum a_{lm} z^l \bar{z}^m$ is a polynomaial, then $p$ contains no term with $m > 0$ iff $\frac{\partial p}{\partial \bar{z}} \equiv 0$.*

♠

**Corollary 1.1**

*If $p(z, \bar{z}) = qz, \bar{z}$ are polynomials, then they have same coefficients.*

♡

**Definition 1.3**

*A $C_1$ function $f : U \mapsto \mathbb{C}$ is said to be holomorphic if*

$$\frac{\partial f}{\partial \bar{z}} = 0$$

*at every point of $U$.*

♣

**Definition 1.4**

*A $C^1$ function $f = u(x,y) + iv(x,y) : U \to \mathbb{C}$ is holomorphic if*

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

*at every point of $U$, which is called the $Cauchy$-$Riemann$ equations.* ♣

**Proposition 1.4**

*If $f : U \to \mathbb{C}$ is $C^1$ and if $f$ satisfies the C-R equations, then*

$$\frac{\partial}{\partial z} f = \frac{\partial}{\partial x} f = -i\frac{\partial}{\partial y} f$$

*on $U$.* ♠

**Proof**

We have

$$\begin{aligned} \frac{\partial}{\partial x} f &= \frac{\partial}{\partial x} u + i\frac{\partial}{\partial x} v = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})u = 2\frac{\partial}{\partial z} u \\ \frac{\partial}{\partial x} f &= \frac{\partial}{\partial x} u + i\frac{\partial}{\partial x} v = i(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})v = 2\frac{\partial}{\partial z} iv \\ -i\frac{\partial}{\partial y} f &= -i\frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})u = 2\frac{\partial}{\partial z} u \\ -i\frac{\partial}{\partial y} f &= -i\frac{\partial}{\partial y} u + \frac{\partial}{\partial y} v = i(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})v = 2\frac{\partial}{\partial z} iv \end{aligned}$$

on $U$.

**Definition 1.5**

*If $U \subset \mathbb{C}$ is open and $u \in C^2(U)$, then $u$ is called $harmonic$ if*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

*where we also denote it as*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

*where the operator is called the $Laplace$ $operator$.* ♣

Here we have

$$4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} u = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \Delta u$$

**Proposition 1.5**

*The real and imaginary parts of a holomorphic $C^2$ function are harmonic.* ♠

**Proof**

Assume $f = u + iv$ and then according to C-R equations, we have

$$\frac{\partial^2}{\partial x^2} u = \frac{\partial^2}{\partial x \partial y} v = \frac{\partial^2}{\partial y \partial x} v = -\frac{\partial^2}{\partial y^2} u$$

and

$$\frac{\partial^2}{\partial x^2} v = -\frac{\partial^2}{\partial x \partial y} u = -\frac{\partial^2}{\partial y \partial x} u = -\frac{\partial^2}{\partial y^2} v$$

**Lemma 1.1**

*It $u(x,y)$ is a real-valued polynomial with $\Delta u = 0$, then there exists a (holomorphic) $Q(z)$ such that $ReQ = u$.* ♡

**Proof**

Consider $u(x,y) = u(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}) = P(z,\bar{z}) = \sum a_{lm} z^l \bar{z}^m$, we know $\Delta u = 0$ and hence

$$P(z,\bar{z}) = a_0 0 + \sum^m a_k z^k + \sum^n b_k \bar{z}^k$$

$P$ is real-valued and we know

$$a_0 0 + \sum^m a_k z^k + \sum^n b_k \bar{z}^k = \bar{a_0} 0 + \sum^m \bar{a_k} \bar{z}^k + \sum^n \bar{b_k} z^k$$

and hence $a_0 0 \in \mathbb{R}$, $a_k = \bar{b_k}$ and hence

$$u(z) = c + \sum^n a_k z^k + \sum^n \bar{a_k} \bar{z}^k = Re(c + 2\sum^n a_k z^k) = Re(Q)$$

where $Q$ is obviously holomorphic.

**Theorem 1.1**

*If $f, g$ are $C^1$ functions on the rectangle*

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : |x-a| < \delta, |y-b| < \epsilon\}$$

*and if*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \text{ on } \mathcal{R}$$

*then there is a function $h \in C^(\mathcal{R})$ such that*

$$\frac{\partial}{\partial x} h = f, \frac{\partial}{\partial y} h = g$$

*on $\mathcal{R}$. If $f, g$ are real-valuedd, the nwe may take $h$ to be real-valued also.* ♡

**Proof**

For $(x,y) \in \mathcal{R}$, define

$$h(x,y) = \int_a^x f(t,b)dt + \int_b^y g(x,s)ds$$

and we know

$$\frac{\partial}{\partial y} h(x,y) = g(x,y)$$

and

$$\frac{\partial}{\partial x} h(x,y) = f(x,b) + \frac{\partial}{\partial x}\int_b^y g(x,s)ds = f(x,b) + \int_b^y \frac{\partial}{\partial y} f(x,s) = f(x,b) + f(x,y) - f(x,b) = f(x,y)$$

and hence $h \in C^2(\mathcal{R})$ and real-valued if $f, g$ are.

**Corollary 1.2**

*If $\mathcal{R}$ is an open rectangle (or open disc) and if $u$ is a real-valued harmonic function on $\mathbb{R}$, then there is a holomorphic function $F$ on $\mathbb{R}$ such that $ReF = u$.* ♡

**Proof**

We know

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$

and hence there exists $v$ real-valued such taht

$$\frac{\partial}{\partial x} v = -\frac{\partial}{\partial y} u, \frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u$$

and hence $F = u + iv$ is a holomorphic function with $Re(F) = u$.

**Theorem 1.2**

*If $U \subset \mathbb{C}$ is either an open rectangle or an open disc and if $F$ is holomorphic on $U$, then there is a holomorphic function $H$ on $U$ such that $\partial H/\partial z = F$ on $U$.*

**Proof**

Consider $H = h_1 + ih_2$ and we have $F = u(z) + iv(z)$, then we let $f = u, g = -v$ and we will have

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial x} g$$

and hence we have a real $C^2$ function $h_1$ such that

$$\frac{\partial}{\partial x} h_1 = u, \frac{\partial}{\partial y} h_1 = -v$$

and $h_2 \in C^2$ with

$$\frac{\partial}{\partial x} h_2 = v, \frac{\partial}{\partial y} h_2 = u$$

Then

$$\frac{\partial}{\partial z} H = \frac{1}{2}(\frac{\partial}{\partial x} h_1 + \frac{\partial}{\partial y} h_2) + \frac{i}{2}(\frac{\partial}{\partial x} h_2 - \frac{\partial}{\partial y} h_1) = u + iv = F$$

**Definition 1.6**

*A function $\phi : [a, b] \to \mathbb{R}$ is called $continuously\ differentiable$ and we write $\phi \in C^1([]a, b)$ if*

*(a) $\phi$ is continous on $[a, b]$*

*(b) $\phi'$ exists on $(a, b)$*

*(c) $\phi'$ has a continuous extension to $[a, b]$, i.e.*

$$\lim_{t \to a^+} \phi'(t) \text{ and } \lim_{t \to b^-} \phi'(t)$$

*both exists. Then $\phi(b) - \phi(a) = \int_a^b \phi'(t)dt$.*

**Proof**

Here notice that $\phi$ is absolutely continuous on $[a, b]$ respect to $m$, then we know $\phi(b - \epsilon) - \phi(a + \epsilon) = \int_{a+\epsilon}^{b-\epsilon} \phi'(t)dt$ for any $epsilon > 0$, and hence

$$\phi(b) - \phi(a) = \int_a^b \phi'(t)dt$$

**Definition 1.7**

*A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be continuous on $[a, b]$ if both $\gamma_1$ and $\gamma_2$ are, $\gamma = \gamma_1 + i\gamma_2$. The curve is $C_1$ on $[a, b]$ if $\gamma_1, \gamma_2$ are $C_1$ on $[a, b]$ and then we may denote*

$$\frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt}$$

**Definition 1.8**

*Let $\varphi : [a, b] \to \mathbb{C}$ be continuous on $[a, b]$. Write $\varphi(t) = \varphi_1(t) + i\varphi_2(t)$. Then we define*

$$\int_a^b \varphi(t)dt = \int_a^b \varphi_1(t)dt + i\int_a^b \varphi_2(t)dt$$

**Proposition 1.6**

*Let $U \subset \mathbb{C}$ be open and let $\gamma : [a, b] \to U$ be a $C_1$ curve. If $f : U \to \mathbb{R}$ and $f \in C^1(U)$, then*

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \left(\frac{\partial}{\partial x} f(\gamma(t))\frac{d\gamma_1}{dt} + \frac{\partial}{\partial y} f(\gamma(t))\frac{d\gamma_2}{dt}\right)dt$$

This is due to the chain rule.

**Proposition 1.7**

*Repalce $f$ above as complex-valued and holomorphic, then we have*

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial}{\partial z} f(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)dt$$

**Proof**

Notice

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &= \int_a^b \Big(\frac{\partial}{\partial x} u(\gamma(t)) \frac{d\gamma_1}{dt}(t) + \frac{\partial}{\partial y} u(\gamma(t)) \frac{d\gamma_2}{dt}(t)\Big) + i\Big(\frac{\partial}{\partial x} v(\gamma(t)) \frac{d\gamma_1}{dt}(t) + \frac{\partial}{\partial y} v(\gamma(t)) \frac{d\gamma_2}{dt}(t)\Big) dt \\ &= \frac{\partial}{\partial x} f(\gamma(t)) \frac{d\gamma}{dt}(t) = \int_a^b \frac{\partial}{\partial z} f(\gamma(t)) \frac{d\gamma}{dt}(t) dt \end{aligned}$$

**Definition 1.9**

*If $U \subset \mathbb{C}$ open and $F : U \to \mathbb{C}$ is continuous on $U$ and $\gamma : [a,b] \to U$ is a $C_1$ curve, then we define the complex line integral*

$$\int_\gamma F(z)dz = \int_a^b F(\gamma(t)) \frac{d\gamma}{dt} dt$$

**Proposition 1.8**

*Let $U \subset \mathbb{C}$ be open and let $\gamma : [a,b] \to U$ be a $C^1$ curve. If $f$ is a holomorphic function on $U$, then*

$$f(\gamma(b)) - f(\gamma(a)) = \int_\gamma \frac{\partial}{\partial z} f(z) dz$$

**Proposition 1.9**

*If $\phi : [a,b] \to \mathbb{C}$ is continuous, then*

$$|\int_a^b \phi(t)dt| \le \int_a^b |\phi(t)|dt$$

**Proposition 1.10**

*Let $U \subset \mathbb{C}$ be open and $f \in C^0(U)$. If $\gamma : [a,b] \to U$ is a $C^1$ curve, then*

$$|\int_\gamma f(z)dz| \le (\sup_{t\in[a,b]} |f(\gamma(t))|) \cdot l(\gamma)$$

*where*

$$l(\gamma) = \int_a^b |\frac{d\gamma}{dt}(t)|dt$$

**Proposition 1.11**

*Let $U \subset \mathbb{C}$ be an open set and $F : U \to \mathbb{C}$ a continuous function. Let $\gamma : [a,b] \to U$ be a $C^1$ curve. Suppose that $\theta : [c,d] \to [a,b]$ is a one-to-one, onto, increasing $C^1$ function with a $C^1$ inverse. Let $\tilde{\gamma} = \gamma \circ \phi$. Then*

$$\int_{\tilde{\gamma}} f dz = \int_\gamma f dz$$

**Proof**

We have

$$\int_{\tilde{\gamma}} f dz = \int_c^d f(\gamma(\phi(t))) \frac{d\gamma(\phi(t))}{dt} dt = \int_a^b f(\gamma(s)) \frac{\gamma(s)}{ds} \phi'(\phi^{-1}(s))(\phi^{-1})'(s) ds = \int_\gamma f dz$$

since $\phi'(\phi^{-1}(s))(\phi^{-1})' = 1$.

**Definition 1.10**

*Let $f$ be a function on the open set $U$ in $\mathbb{C}$ and consider if*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

*exists then we say that $f$ has a complex derivative at $z_0$. We denote the complex derivative by $f'(z_0)$.* ♣

**Theorem 1.3**

*Let $U \subset \mathbb{C}$ be an open set and let $f$ be holomorphic on $U$. Then $f'$ exists at each point of $U$ and*

$$f'(z) = \frac{\partial}{\partial z} f$$

*for all $z \in U$.* ♡

**Proof**

Consider

$$\gamma(t) = (1-t)z_0 + tz$$

and then we know

$$f(z) - f(z_0) = f(\gamma(1)) - f(\gamma_0) = \int_\gamma \frac{\partial}{\partial z} f dz = (z - z_0) \int_0^1 \frac{\partial}{\partial z} f(\gamma(t)) dt = \frac{\partial}{\partial z} f(z_0) + \int_0^1 (\frac{\partial}{\partial z} f(\gamma(t)) - \frac{\partial}{\partial z} f(z_0)) dt$$

and hence

$$|\frac{f(z) - f(z_0)}{z - z_0} - \frac{\partial}{\partial z} f(z_0)| \leq \int |\frac{\partial}{\partial z}(\gamma(t)) - \frac{\partial}{\partial z} f(z_0)| dt \to 0$$

when $z \to z_0$.

**Theorem 1.4**

*If $f \in C^1(U)$ and $f$ has a complex derivative at each point of $U$, then $f$ is holomorphic on $U$. In particular, if a continuous, complex-valued function $f$ on $U$ has a complex derivative at each point and if $f'$ is continuous on $U$, then $f$ is holomorphic on $U$.* ♡

**Proof**

It is easy to check

$$\lim_{h \to 0, h \in \mathbb{R}} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial x} v(x_0, y_0)$$

and

$$\lim_{h \to 0, h \in \mathbb{R}} \frac{f(z_0 + h) - f(z_0)}{h} = -i \frac{\partial}{\partial y} u(x_0, y_0) + \frac{\partial}{\partial y} v(x_0, y_0)$$

and hence $f$ satisfies the C-R equations so holomorphic.

Notice the continuity of $f'$ may implies that $f \in C^1(U)$ and hence the problem goes.

**Theorem 1.5**

*Let $f$ be holomorphic in a neighborhood of $P \in \mathbb{C}$. Let $\omega_1, \omega_2$ be complex numbers of unit modulus. Consider the directional derivatives*

$$D_{\omega_1} f(P) = \lim_{t \to 0} \frac{f(P + t\omega_1) - f(P)}{t}$$

*and*

$$D_{\omega_2} f(P) = \lim_{t \to 0} \frac{f(P + t\omega_2) - f(P)}{t}$$

*then*

*a.* $|D_{\omega_1} f(P)| = |D_{\omega_2} f(P)|$

*b. If $f'(P) \neq 0$, then the directed angle from $\omega_1$ to $\omega_2$ equals the directed angle from $D_{\omega_1} f(P)$ to $D_{\omega_2} f(P)$.* ♡

**Proof**

Notice that

$$D_{\omega_j} = f'(P)\omega_j, j = 1, 2$$

and then the conclusions go.

**Lemma 1.2**

*Let $(\alpha, \beta) \subset \mathbb{R}$ be an open interval and let $H : (\alpha, \beta) \to \mathbb{R}, F : (\alpha, \beta) \to \mathbb{R}$ be continuous functions. Let $p \in (\alpha, \beta)$ and suppose that $dH/dx$ exists and equals $F(x)$ for all $x \in (\alpha, \beta)\{p\}$. Then $(dH/dx)(p)$ exists and $(dH/dx)(x) = F(x)$ for all $x \in (\alpha, \beta)$.*

**Proof**

Assume $[a, b] \subset (\alpha, \beta)$ and then $K(x) = H(a) + \int_a^x F(t)dt$ on $[a, b]$, so we know $K - H$ is continuous on $[a, b]$ and constant on $[a, p) \cup (p, b]$, which means $K = H$ on $[a, b]$.

**Theorem 1.6**

*Let $U \subset \mathbb{C}$ be either an open rectangle or an open disc and let $P \in U$. Let $f$ and $g$ be continuous, real-valued functions on $U$ which are continuously differentiable on $U - \{P\}$. Suppose further that*

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial x} g \text{ on } U = \{P\}$$

*Then there exists a $C^1$ function $h : U \to \mathbb{R}$ such that*

$$\frac{\partial}{\partial x} = f, \frac{\partial}{\partial y} = g$$

*at every point of $U$.*

**Proof**

Consider a closed rectangle containing $p$ inside in $U$ and define $h(x, y) = \int_a^x f(t, b)dt + \int_b^y g(x, s)ds$ and we know that $\frac{\partial}{\partial y} h = g(x, y)$ and $\frac{\partial}{\partial x} h = f(x, y)$ for any $x \neq P_x$, then for a fixed $y$, we know $dh(x, y)/dx = f(x, y)$ exists for all points in $U$ except for $(p_x, y)$ and hence $dh(x, y)/dx = f(x, y)$ at $(p_x, y)$. Then we know $\frac{\partial}{\partial x} h = f, \frac{\partial}{\partial y} h = g$ on $U$.

**Theorem 1.7**

*Let $U \subset \mathbb{C}$ be either an open rectangle or an open disc. Let $P \in U$ be fixed. Suppose that $F$ is continuous on $U$ and holomorphic on $U - \{P\}$. Then there is a holomorphic $H$ on $U$ such that $U$ such that $\frac{\partial}{\partial z} H = F$.*

**Proof**

Consider $F = u + iv$, then we have

$$\frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u \text{ and } \frac{\partial}{\partial y} u = \frac{\partial}{\partial x}(-v)$$

on $U - \{P\}$, then we know there exists $h_1, h_2$ on $U$ such that $\frac{\partial}{\partial x} h_1 = u$, $\frac{\partial}{\partial y} h_1 = (-v)$, $\frac{\partial}{\partial x} h_2 = v$, $\frac{\partial}{\partial y} h_2 = u$ and let $H = h_1 + ih_2$, we have

$$\frac{\partial}{\partial z} H = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(h_1 + ih_2) = (u + u) + i(v + v) = F$$

**Definition 1.11**

*The boundary $\partial D(P, r)$ of the disc $D(P, r)$ can be parametrized as a simple closed curve $\gamma : [0, 1] \to \mathbb{C}$ by setting*

$$\gamma(t) = P + re^{2\pi it}$$

*we call it counterclockwise orientation.*

**Lemma 1.3**

*Let $\gamma$ be the boudnary of a disc $D(z_0, r)$ in the complex plane, equiipped with counterclockwise orientation. Let $z$ be a point inside the circle $\partial D(z_0, \gamma)$ . Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\xi - z}d\xi = 1$$

♡

**Proof**

Consider $I(z) = \int_\gamma \frac{1}{\xi - z}d\xi = \int_0^1 \frac{1}{(z_0 + e^{2\pi i t}) - z}(2\pi i)e^{2\pi i t}dt$ and since

$$\frac{\partial}{\partial x}\frac{1}{\xi - z} = \frac{1}{(\xi - z)^2}, \frac{\partial}{\partial y}\frac{1}{\xi - z} = i\frac{1}{(\xi - z)^2}$$

and hence we have

$$\frac{\partial}{\partial \bar{z}}I(z) = \int_\gamma \frac{\partial}{\partial \bar{z}}(\frac{1}{\xi - z})d\xi = 0 \quad \frac{\partial}{\partial z}I(z) = \int_\gamma \frac{\partial}{\partial z}(\frac{1}{\xi - z})d\xi = \int_\gamma \frac{1}{(\xi - z)^2}d\xi$$

where $\frac{1}{(\xi - z)^2}$ is the complex derivative of the holomorphic function $\frac{-1}{\xi - z}$ and hence

$$\frac{\partial}{\partial z}I(z) = \int_\gamma \frac{1}{(\xi - z)^2}d\xi = 0$$

Therefore, $I(z)$ is holomorphic on $D(z_0, r)$ and $\frac{\partial}{\partial z}I = 0$ which means $I$ is constant on $D(z_0, r)$ and notice

$$I(z_0) = 2\pi i$$

and hence the equation holds.

**Theorem 1.8**

*(The Cauchy integral fomula) Suppose that $U$ is an open set in $\mathbb{C}$ and that $f$ is a holomorphic function on $U$. Let $z_0 \in U$ and let $r > 0$ be such that $\overline{D}(z_0, r) \subset U$. Let $\gamma : [0, 1] \to \mathbb{C}$ be the $C^1$ curve $\gamma(t) = z_0 + r\cos(2\pi t) + ir\sin(2\pi t)$. Then for each $z \in D(z_0, r)$,*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi$$

♡

**Proof** By theorem 1.7, there is $H$ such that

$$\frac{\partial}{\partial z}H = \frac{f(\xi) - f(z)}{\xi - z}$$

if $\xi \neq z$ and $\frac{\partial}{\partial z}H(z) = f'(z)$ holomorphic on $D(z_0, r + \epsilon)$ and hence

$$\int_\gamma \frac{f(\xi) - f(z)}{\xi - z}d\xi = 0$$

and the equation holds by the lemma 1.3.

**Theorem 1.9**

*(The Cauchy integral theorem) If $f$ is a holomorphic function on an open disc $U$ in the complex plane, and if $\gamma : [a, b] \to U$ is a $C^1$ curve in $U$ with $\gamma(a) = \gamma(b)$, then*

$$\int_\gamma f(z)dz = 0$$

♡

**Proof** Only need to pick $G$ such that $\frac{\partial}{\partial z}G = f$ on $U$ is fine.