Homework04 - MATH 722

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Before Reading:

To make the proof more readable, I will miss or gap some natural or not important facts or notations during my writing. If you feel it hard to see, you can refer the appendix after the proof, where I will try to explain some simple conclusions (will be marked) more clearly. In case that you misunderstand the mark, I will add the mark just after those formulas between \$ and before those between \$\$.

And I have to claim that the appendix is of course a part of my assignment, so the reference of it is required. Enjoy your grading!

Chapter.5 Ex.18

Let $p_t(Z) = a_0(t) + a_1(t)z + \cdots + a_n(t)z^n$ be a polynomial in which the coefficients depend continuously on $t \in (-1, 1)$. Prove that if the roots of p_{t_0} are distinct, for some fixed value of the parameter, then the same is true for p_t when t is sufficiently close to t_0 - provided that the degree of p_t remains the same as the degree of p_{t_0}.

Sol.

Without loss of the generality, we may assume that $a_n(t) \neq 0$ for all $t \in (-1, 1)$ since the degree of p_t does not change. We may consider

$$I_{(t,z,r)} = \frac{1}{2\pi i} \int_{\partial B(z,r)} \frac{p_t'(\xi)}{p_t(\xi)} d\xi$$

and since p_{t_0} has isolated zeros $z_1, z_2, \cdots, z_n$, then we may find r_i small enough such that $p_{t_0}(\xi) \neq 0$ on $\partial B(z_i, r_i)$ and we may also find δ small enough such that $p_t(\xi) \neq 0$ on $\partial B(z_i, r_i)$ for any $t \in (t_0 - \delta, t_0 + \delta)$, so we may know

$$I_{(t,z_i,r_i)} = I_{(t_0,z_i,r_i)} = 1$$

for any $t \in (t_0 - \delta, t_0 + \delta)$ and since the degree of p_t is always n, then we know p_t has also n distinct zeros if t is sufficiently close to t_0.

Chapter.6 Ex.1

Does there exist a holomorphic mapping of the disc onto $\mathbb{C}$?

Sol.

Consider

$$f(z) = (\frac{2}{z-i} - 3i)^2$$

will be a required map.

Chapter.6 Ex.2

Prove that if f is entire and one-to-one, then f must be linear.

Sol.

Consider $g(z) = f(1/z)$ is a meromorphic function on $\mathbb{C} - \{0\}$ and 0 is a pole of g since, if 0 is removable, then f is constant which is a contradiction. If 0 is an essential singularity, then we know there exists a small ball D centered at 0 such that $g(D - \{0\})$ is dense in $\mathbb{C}$, but $g(\mathbb{C} - \overline{D})$ is open and hence g is not one-to-one, which is a contradiction, and consider f has a series expansion at 0 by

$$f(z) = \sum_{n \geq 0} a_n z^n$$

and hence

$$g(z) = \sum_{n \geq 0} a_n z^{-n}$$

is a Laurent expansion of g, and hence f will become a polynomial on $\mathbb{C}$, however, f cannot have degree greater than 2, we may always find more than 2 zeros points or more than 2 distinct points with the same image around the unique zero of f. Therefore, f has to be linear.

Chapter.6 Ex.5

Let $\Omega = \mathbb{C} - \{0\}$. Give an explicit description of all the biholomorphic self-maps of Ω. Now let Ω be $\mathbb{C} - \{P_1, \cdots, P_k\}$. Give an explicit description of all the biholomorphic self-maps of Ω.

Sol.

Firstly, assume f is a conformal map from $\mathbb{C} - \{0\}$ to itself, we refer the proof of Ex.2 and we know f can not have an essential singularity at 0, similarly f can not have an essential singularity at ∞ by considering $f(1/z)$. If f has a removable singularity at 0, then $\hat{f}(0) \in \{0, \infty\}$ or $f^{-1} \circ \hat{f}(0) = 0$ which is a contradiction. If $\hat{f}(0) = 0$, then we may consider the expansion of $\hat{f}$ at 0 and hence it becomes a polynomial, with degree less than 1 according to the proof of Ex.2. If $\hat{f}(0) = \infty$, then consider $g(z) = f^{-1}(1/z)$ and the problem goes. Therefore, $f(z) = az$ or a/z for some $a \in \mathbb{C} - \{0\}$.

Similarly, for $\Omega = \mathbb{C} - \{P_1, \cdots, P_k\}$, we know f have removable singularity or poles at P_i, and if P_i is a removable singularity, we know $\hat{f}(P_i)$ will be some P_j. Without loss of the generality, we assume $P_1, \cdots, P_m$ are removable singularities with $\hat{f}(P_i) = P_{\phi(i)}$, and $P_{m+1}, \cdots, P_k$ are poles, however there can not be more than 2 pole by consider the preimage of a neighbourhood of ∞, and hence we may have only less than 1 pole in $\{P_1, \cdots, P_k\}$ and the problem is the same with the first part, and hence $f(z) = a(z - P_j)$ or $a/(z - P_j)$ for some j.

Chapter.6 Ex.6

Let $\Omega = \mathbb{C} - \{z : |z| \le 1\}$. Determine all biholomorphic self-maps of Ω.

Sol.

Consider f is a biholomorphic self-map of Ω and we know $1/f(1/z)$ is a biholomorphic self-map of $D - \{0\}$. If g is a biholomorphic self-map of $D - \{0\}$, we know 0 has to be a removable singularity with $\hat{g} = 0$ and hence $\hat{(g)}(z) = \omega z$ for some $\omega \in \mathbb{C}, |\omega| = 1$. Therefore we know

$$f(z) = \omega z$$

for some $|\omega| = 1$.

Addition

Find conformal maps:

a. $D - \{z \in \mathbb{R}, |z| \ge 1/2\}$ onto D.

b. $\mathbb{C}_+ - \{z \in i\mathbb{R}, |z| \ge 1\}$ onto $\mathbb{C}_+$.

Sol.

a. We consider
$$f_1(z) = i\frac{1-z}{1+z}, f_2(z) = z^2 + 1/9, f_3(re^{i\theta}) = \sqrt{r}e^{i\theta/2}, f_4(z) = \frac{i-z}{i+z}$$
and $f_4 \circ f_3 \circ f_2 \circ f_1$ meets the requirement.

b. Consider
$$f_1(z) = z^{-2} + 1, f_2(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$$
and $f_1 \circ f_2$ meets the requirement.