NOTES FOR ODE

Based on the James D Meiss

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Contents

1 Existence and Uniqueness Theorems

1.1 Basic definitions

Definiton 1.1.1. (Lipschitz continuity)

Suppose $(X, \rho_X), (Y, \rho_Y)$ are metric spaces. A function $f : X \to Y$ is **Lipschitz** if for all $x_1, x_2 \in X$, there is a K such that $\rho_Y(f(x_1), f(x_2)) \leq K\rho_X(x_1, x_2)$, then the mallest such K is called the **Lipschitz constant** for f on X.

f is **locally Lipschitz** if for every $x \in X$, there is a neighborhood $N(x)$ such that f is Lipschitz on $N(x)$.

Lemma 1.1.1. A Lipschitz function is uniformly continuous. If f is locally Lipschitz, then it is Lipschitz on any compact set.

Lemma 1.1.2. Suppose that $A \subset \mathbb{R}^n$ is compact and convex and $f \in C^1(A, \mathbb{R}^n)$. Then f is Lipschitz with constant $K = \max_{x\in A} ||Df||$.

Proof.

For any $x, y \in A$, we know $\xi(s) = x + s(y - x) \in A$ for any $0 \leq s \leq 1$. Then

$$f(y) - f(x) = \int_0^1 \frac{d}{ds}(f(\xi(s)))ds = \int_0^1 Df(\xi(s))(y - x)ds$$

(since

$$\frac{d}{ds}(f(\xi(s))) = \lim_{t\to 0} \frac{f(\xi(s) + t\sum_{i=1}^{n}(y - x)_i) - f(\xi(s))}{t}$$

and then) we have

$$|f(y) - f(x)| \leq \int_0^1 ||Df(\xi(s))|||y - x|ds \leq K|y - x|$$

and we are done.

Corollary 1.1.3. If $E \subset \mathbb{R}^n$ is open and $f \in C^1(E, \mathbb{R}^n)$, then f is locally Lipschitz.

Lemma 1.1.4. Suppose $f \in C^k(E, \mathbb{R}^n)$ where E is some subset E of $\mathbb{R}^n$ for $k \geq 0$ and $x \in C^0(J, E), J = [t_0 - a, t_0 + a]$ is a solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(\tau))d\tau$$

Then $x \in C^{k+1}(J, E)$ and a solution to

$$x' = f(x), x(t_0) = x_0$$

Proof.