# Bonus 06 - MATH 722

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Let $\overline{D}(a_j, r_j)$ be pairwise disjoint closed disc closed discs in $D(0,1)$ such that the union of discs $\bigcup_{j=1}^{\infty} \overline{D}(a_j, r_j)$ is dense in $\overline{D}(0,1)$. Let $K = \overline{D}(0,1) - (\bigcup_{j=1}^{\infty} D(a_j, r_j))$. Prove that such discs can be chosen so that $\sum_j r_j < 1$ and that in this case the conclusion of Mergelyan's theorem fails on $K$.

**Sol.**

The first part is the proof of the existence of $a_j, r_j$ such that $\sum_{r_j} < 1, \overline{a_j, r_j}$ disjoint and $\bigcup_{j=1}^{\infty} \overline{D}(a_j, r_j)$ are dense. Consider the partition $A_n$ of $\overline{D}(0,1)$ by

$$A_{n,k,s} = \{e^{x+iy}, e^x \in [k2^{-n}, (k+1)2^{-n}), y \in [s2\pi/2^n, (k+1)2\pi/2^n), 0 \le k < 2^n, 0 \le s < 2^n\}$$

then we may construct $P_n$ by choose $a_{n,k,s} \in A^{\circ}_{n,k,s} - \bigcup_{m<n} \bigcup_{0\le k,s<2^m} \overline{D}(a_{m,k,s}, r_{m,k,s})$ and

$$r_{n,k,s} < \min\{\min\{8^{-n}, d(a_{n,k}, \overline{D} - A^{\circ}_{n,k})\}, \min\{|a_{n,k,s} - a_{m,p,q}|, m < n, 0 \le p, q < 2^m\}\},$$

then let $P = \bigcup_{n\ge 1} P_n$ and we know $P$ is countable and dense in $\overline{D}$ and hence $\bigcup_{n\ge 0, 0\le s,t\le 2^n} \overline{D}(a_{n,s,t}, r_{n,s,t})$ will be dense and it is easy to check that

$$\sum_{n,s,t} r_{n,s,t} < 1$$

and we are done by resorting $a_{n,s,t}$.

We consider $f(z) = Rez$, and then we know $f$ is continuous on $K$ and holomorphic on $K^{\circ}$ since it is empty, so we know for any $\epsilon > 0$, there exists $p(z)$ polynomial such that

$$|p(z) - f(z)| < \epsilon$$

on $K$ and hence we know let $\gamma$ be $\partial D(0,1)$ with counterclockwise orientation. So we know

$$2\pi\epsilon > \int_\gamma |p(z) - f(z)| dz \ge |\int_\gamma p(z) - \int_\gamma f(z)| = \pi$$

since

$$\int_\gamma f(z) = \pi i$$

and $p$ is holomorphic on $\mathbb{C}$, however, let $\epsilon < 1/2$ and there will be a contradiction, and hence Mergelyan's thoerem fails on $K$.