RMT2024 at U of M – Paquette
Assignment 1

The semicircle law and the Newton flow. For Wigner matrices, the Dyson equation on $\mathbb{M}^+(n) = \{M \in \mathbb{M}(n) : \Im M \succ 0\}$ is given by

$$M(-\tfrac{1}{n}\operatorname{tr}(M) - z\operatorname{Id}_n) = \operatorname{Id}_n \quad z \in \mathbb{H}.$$

We let $F(M; z)$ be the mapping

$$F(M; z) \coloneqq M(-\tfrac{1}{n}\operatorname{tr}(M) - z\operatorname{Id}_n) - \operatorname{Id}_n.$$

We let $\mathcal{M}(t)$ be the solution of the *Newton flow*

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\mathcal{M}(t); z) = -F(\mathcal{M}(t); z) \quad F(\mathcal{M}(0); z) = \xi,$$

which may *a priori* not be well-defined nor exist for all time.

Exercises.

1. Show there is a unique solution of the Dyson equation on $\mathbb{M}^+(n)$, given by
$$M = s(z)\operatorname{Id}, \quad \text{where} \quad s(z) \coloneqq \frac{-z + \sqrt{z-2}\sqrt{z+2}}{2},$$
with $\sqrt{\cdot}$ the principal branch of the square-root.

2. For the Newton flow, suppose that $\mathfrak{s}(t) = \frac{1}{n}\operatorname{tr}(\mathcal{M}(t))$. Show that formally,
$$\dot{\mathfrak{s}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{s}(t) = \frac{e^{-t}\frac{1}{n}\operatorname{tr}(\xi)}{2\mathfrak{s}(t) + z}.$$

3. Conclude that there is an absolute constant $c > 0$ so that for any $z \in \mathbb{H}$ if $|2s(z) + z|^2 > c|\frac{1}{n}\operatorname{tr}(\xi)|$ then the Newton flow is well-posed for all $t \in [0, \infty)$ and
$$|\mathfrak{s}(0) - \mathfrak{s}(\infty)| \leq \frac{|\frac{1}{n}\operatorname{tr}(\xi)|}{\sqrt{|2s(z) + z|^2 - c|\frac{1}{n}\operatorname{tr}(\xi)|}}.$$
(*Note that $2s(z) + z$ only vanishes at the spectral edges.*) Conclude further that for any matrix A
$$|\operatorname{tr}(A\mathcal{M}(0)) - \operatorname{tr}(AM(z))| \leq \frac{|\frac{1}{n}\operatorname{tr}(\xi)| \times |\operatorname{tr}(AM(z))|}{\sqrt{|2s(z) + z|^2 - c|\frac{1}{n}\operatorname{tr}(\xi)|}} + |\operatorname{tr}(A\xi)|.$$

4. Complete the proof of the anisotropic semicircle law scale. Suppose X is a Wigner matrix (independent upper triangle, mean variance match GOE) having all moments (so there are finite $C(p)$ for all $p \in \mathbb{N}$ so that $\max_{ij}\mathbb{E}|X_{ij}|^p < C(p)$). Show that for any deterministic matrices

A with $\|A\|_* \leq 1$ (sum of singular values less than 1) and any fixed z with $\Im z > 0$,

$$|\operatorname{tr}(A\xi)| \xrightarrow[n\to\infty]{\mathbb{P}} 0, \quad \xi = \xi(z) = F(R(z;Y);z), \quad Y = \tfrac{1}{\sqrt{n}}X$$

They key is to represent ξ well. Starting from $\mathrm{Id}_n = R(z;Y)(Y - z\,\mathrm{Id}_n)$, use the Schur complement formula to represent each matrix vector product $R(z;Y)Y_j$ in terms of $R(z;Y^{[j]})$ and Y_j, with Y_j the j-th column of Y and $Y^{[j]}$ a matrix in which the j-th row/column was removed.