Chapter 1

m.s. for measure space
mrb. for measurable

1.1 L^p spaces

Definition 1.1

For a fixed m.s. $(X, \mathcal{M}, \mu)$*, if* f *is a measurable function on* X *and* $0 < p < \infty$*, we define*

$$||f||_p = \left[\int |f|^p d\mu\right]^{1/p}$$

and

$$L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C}, f \text{ mrb and } ||f||_p < \infty\}$$

♣

Lemma 1.1

(Yooung's inequality) If $a, b \geq 0$ *and* $0 < \lambda < 1$*, then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

with equality iff $a = b$*.*

♡

Proof

If $b = 0$, the inequality goes. Then assume $b > 0$, and it suffices to show that

$$\frac{a^\lambda}{b} \leq \lambda \frac{a}{b} + (1-\lambda)$$

and consider the function $f(x) = x^\lambda - \lambda x - (1-\lambda)$, we have $f'(x) = \lambda x^{1-\lambda} - \lambda$ which is less than zero if $x > 1$ and greater than zero if $x < 1$, so we know $f(x) \leq f(1) = 0$ and the inequality holds.

Theorem 1.1

(Holder Inequality) Suppose $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$*. If* f *and* g *are measurable functions on* X*, then*

$$||fg||_1 \leq ||f||_p ||g||_1$$

In particular, if $f \in L^p, g \in L^q$*, then* $fg \in L^1$ *and in this case equality holds iff* $\alpha|f|^p = \beta|g|^q$ *a.e. for some constants* α, β*.*

♡

Proof

Consider we should show that

$$\int |fg| d\mu \leq \int |f|^p d\mu \int |g|^q d\mu$$

and if $||f||_p = 0$ or $||g||_q = 0$, then the LHS equals to 0. Now we consider let replace f, g with $f/||f||_p, g/||g||_q$ and it is suffices to show

$$\int |fg| d\mu \leq 1$$

and notice we have

$$\int |fg| d\mu \leq \int \frac{1}{p}|f|^p + \frac{1}{q}|g|^q d\mu = 1$$

and the equality holds iff $|fg| = p^{-1}|f|^p + q^{-1}|g|^q$ a.e. which means $|f|^p = |g|^q$ a.e. for the replaced f, g.

Theorem 1.2

(Minkowski's Inequality) If $1 \leq p < \infty$ *and* $f, g \in L^p$*, then*

$$||f + g||_p \leq ||f||_p + ||g||_p$$

♡

Proof

Consider
$$\int |f+g|^p d\mu \le \int |f+g|^{p-1}(|f|+|g|) \le |||f+g|^{p-1}||_q(||f||_p+||g||_p) = ||f+g||_p^{(p-1)/p}$$
and the inequality holds.

Theorem 1.3

For $1 \le p < \infty$, L^p is a Banach space. ♡

Proof

It suffices to show that L^p is complete, which can be induced from any absolutely convergence series $S = \sum f_i$ converges. Let $S_n = \sum_{i=1}^{n} f_i$ and it is easy to check that S_n is Cauchy in L^p, then let $G = \sum |f_i|$ and we have $|G|_p = \lim |G_n|_p < \infty$ by the MCT where $G_n = \sum_{i=1}^{n} |f_i|$ and hence $G \in L^p$ which means S converges a.e. and consider
$$\lim ||S - S_n||_p = ||\lim S - S_n||_p = 0$$
by the DCT.

Proposition 1.1

For $1 \le p < \infty$, the set of simple functions $f = \sum_1^n a_j \chi_{E_j}$, where $\mu(E_j) < \infty$ for all j is dense in L^p. ♠

Proof

For $f \in L^p$, we may find $|f_j| \uparrow |f|$ and f_j converges to f pointwise, then we assume $f_j = \sum_1^n a_j \chi_{E_j}$ and then we have
$$\sum_1^n a_j^p \mu(E_j) = \int |f_j|^p d\mu \le \int |f|^p d\mu < \infty$$
and hence f_j is just in the required set, and by the DCT we know $||f - f_j||_p \to 0$.

Definition 1.2

$$||f||_\infty = \int \{a \ge 0 : \mu(\{x : |f(x)| > \alpha\}) = 0\}$$
with the convention that $\inf \emptyset = \infty$ *and then it is called the essential supremum of* $|f|$. *And define*
$$L^\infty = \{f : X \to \mathbb{C}, f \textit{ mrb and } ||f||_\infty < \infty\}$$
♣

Theorem 1.4

a. If f and g are measurable functions on X, then $||fg||_1 \le ||f||_1||g||_\infty$, if $f \in L^1$ and $g \in L^\infty$, $||fg||_1 = ||f||_1||g||_\infty$ iff $|g(x)| = ||g||_\infty$ a.e. on the set where $f(x) \ne 0$.
b. $||\cdot||_\infty$ is a norm on L^∞.
c. $||f_n - f||_\infty \to 0$ iff $f_n \to f$ uniformly a.e.
d. L^∞ is a Banach space.
e. The simple functions are dense in L^∞. ♡

Proof a. Let $E = \{|g| \le |g|_\infty\}$ and then E is conull, so
$$\int |fg| d\mu = \int_E |fg| d\mu \le ||g||_\infty \int_E |f| d\mu = \int |f| d\mu ||g||_\infty$$
where the equality can be reached when $g(x) = ||g||_\infty$ a.e. on E.

b. It suffices to show the triangle inequality where notice $|f| \le ||f||_\infty, g \le ||g||_\infty$ a.e. and hence $|f+g| \le ||f||_\infty + ||g||_\infty$ a.e.

c. Let $E_n = \{|f_n - f| \le ||f_n - f||_\infty\}$ and then let $E = \bigcap E_n$ conull and hence $f_n \to f$ on E uniformly.

d. If suffices to show that an absolutely convergent series $\sum f_i$ converges in L^∞ where we may know $f_i \leq ||f_i||_\infty$ a.e. on X for any integer i and hence the we will know $\sum |f_i| \leq \sum ||f_i||_\infty$ a.e. and hence $\sum f_i$ converges a.e. and we have $|\sum f_i - \sum_1^n f_i| \leq \sum_{n+1}^\infty ||f_i||_\infty \to 0$ a.e.

e. Let $f_j \to f$ be the simple functions converges to f uniformly where f is bounded and hence $f_j \to f$ uniformly a.e. and hence $||f_j - f||_\infty \to 0$.

Proposition 1.2

If $0 < p < q < r \leq \infty$, then $L^q \subset L^p + L^r$; that is, each $f \in L^q$ is the sum of a function in L^p and a function in L^r.

Proof

Considering $|f| > 1$ and $|f| \leq 1$ separately will be fine.

Proposition 1.3

If $0 < p < q < r \leq \infty$, then $L^p \cap L^r \subset L^q$ and $||f||_q \leq ||f||_p^\lambda ||f||_r^{1-\lambda}$ where $q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1}$.

Proof

Here we know

$$\int |f|^q d\mu = \int |f|^{\lambda q} |f|^{(1-\lambda)q} d\mu \leq |||f|^{\lambda q}||_{p/\lambda q} |||f|^{(1-\lambda)q}||_{r/(1-\lambda)q} = ||f||_p^{\lambda q} ||f||_r^{(1-\lambda)q}$$

by the Holder's inequality and the inequality holds.

Proposition 1.4

If A is any set and $0 < p < q \leq \infty$, then $l^p(A) \subset l^q(A)$ and $||f||_q \leq ||f||_p$.

Proof If $q = \infty$, then $||f||_\infty = \sup |f(\alpha)| \leq ||f||_p$. If $q < \infty$, then consider

$$||f||_q \leq ||f||_p^\lambda ||f||_\infty^{1-\lambda} \leq ||f||_p$$

Proposition 1.5

If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $||f||_p \leq ||f||_q \mu(X)^{(p^{-1} - q^{-1})}$.

Proof

Consider if $q = \infty$, then

$$\int |f|^p d\mu \leq \int |f|_\infty^p d\mu = ||f||_\infty^p \mu(X)$$

and if $q < \infty$, then

$$\int |f|^p d\mu = \int (|f|^q)^{p/q} (1)^{(q-p)/q} \leq |f^p|_{q/p} |1|_{q/(q-p)} = ||f||_q^p \mu(X)^{(1-p/q)}$$

by the Holder's inequality.

Proposition 1.6

Suppose that p and q are conjugate exponents and $1 \leq q < \infty$. If $g \in L^q$, then

$$||g||_q = ||\phi_g|| = \sup\{|\int fg|, ||f||_p = 1\}$$

If μ is semifinite, this result holds also for $q = \infty$, where define

$$\phi_g(f) = \int fg$$

Proof

It suffices to show that $||\phi_g|| \geq ||g||_q$. Let

$$f = \frac{|g|^{q-1}\overline{sgn(g)}}{||g||_q^{q-1}}$$

and we have

$$||f||_p = \frac{\int |g|^{(q-1)p}}{||g||_p^{q-1}} = 1$$

and $|\phi_g(f)| = \int fg = \dfrac{\int |g|^q}{||g||_q^{q-1}} = ||g||_q$.

If $q = \infty$, we know there exists $B \subset \{|g| > ||g||_\infty - \epsilon\}$ for any $\epsilon > 0$ such that $\mu(B) < \infty$, then let

$$f = \mu(B)^{-1}\chi_B \overline{sgn(g)}$$

and we have $||f||_1 = 1$ and

$$|\phi_g(f)| = \mu(B)^{-1}\int_B |g| \geq ||g||_\infty - \epsilon$$

and hence $||\phi_g|| = ||g||_\infty$.

Theorem 1.5

Let p and q be conjugate exponents. Suppose that g is a measurable function on X such that $fg \in L^1$ for all f in Σ which is the space of all simple functions with a finite measure support, and the quantity

$$M_q(g) = \sup\{|\int fg|, f \in \Sigma \text{ and } ||f||_p = 1\}$$

is finite. Also, suppose either that $S_g = \{x, g(x) \neq 0\}$ is σ-finite or that μ is semifinite. Then $g \in L^q$ and $M_q(g) = ||g||_q$.

Proof

Notice for any f bounded with a finite measure support and $||f||_p = 1$, we know $|f| \leq ||f||_\infty \chi_E$ where E is a finite support of f and consider f_n is simple function converge to f with $|f_n| \leq |f|$ and then we know

$$|\int fg| = \lim |\int f_n g| \leq M_q(g)$$

by the DCT.

Suppose $q < \infty$ and S_g is $\sigma - finite$, then we may find E_n increasing to S_g with $\mu(E_n) < \infty$, we may find $\phi_n \to g$ and let $g_n = \phi_n \chi_{E_n}$. Then $g_n \to g$ pointwise and let

$$f_n = \frac{g_n^{q-1}\overline{sgn(g)}}{||g_n||_q^{q-1}}$$

then we have

$$||f_n||_p = \frac{\int |g_n|^q}{||g_n||_q^q} = 1$$

and

$$|\int f_n g| = \int \frac{|g_n|^{q-1}|g|}{||g_n||_q^{q-1}} \geq ||g_n||_q$$

which means $M_q(g) \geq ||g_n||_q$ for any integer n and hence $M_q(g) \geq ||g||_q$ by the MCT, which means $g \in L^q$.

If μ is semifinite, then let $E = \{|g| > \epsilon\}$ and then we know there is $A \subset E$ with $\mu(A) < \infty$ if $\mu(E) > 0$, and we have

$$M_q(g) \geq |\int \mu(A)^{-p^{-1}}\chi_A \overline{sgn(g)}g| \geq \epsilon\mu(A)^{1-p^{-1}}$$

where $\mu(A)$ can be arbitrarily large if $\mu(E) = \infty$ and which is a contradiction. Therefore, μ is semifinite will imply that S_g is σ-finite.

If $q = \infty$, then let $A = \{|g| \geq M_\infty(g) + \epsilon\}$, if $\mu(A)$ is positive, then we let $f = \mu(A)^{-1}\chi_A sgn(g)$ and we know

$$|\int fg| \geq M_\infty(g) + \epsilon$$

which is a contradiction and hence $||g||_\infty \leq M_\infty(g)$.

Theorem 1.6

Let p and g be conjugate exponents. If $1 < p < \infty$, for each $\phi \in (L^p)^*$ there exists $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$ and hence L^q is isometrically isomorphic to $(L^p)^*$. The same conclusion holds for $p = 1$ if μ is σ-finite.

Proof

Firstly assume μ is finite, the all simple functions are in L^p, and then consider for disjoint sets E_j and $E = \bigcup_j E_j$, we have

$$||\chi_E - \sum_{i=1}^{n} \chi_{E_j}||_p = \mu(\bigcup_{n+1}^{\infty}) \to 0$$

then let $\nu(E) = \phi(\chi_E)$ and

$$\nu(E) = \phi(E) = \lim \phi(\sum_{i=1}^{n} \chi_{E_i}) = \lim \sum_{i=1}^{n} \nu(E_j)$$

and hence ν is a complex measure. Also if $\mu(E) = 0$, then $\nu(E) = \phi(\chi_E) = 0$, so there is an g measurable such that $\phi(\chi_E) = \nu(E) = \int_E g$ and notice

$$|\int fg| \leq ||\phi|| ||f||_p$$

for any simple function in L^p and hence $g \in L^q$ by theorem 1.5 and then we know $fg \in L^1$ for any $f \in L^p$ and hence $\phi(f) = \int fg$ for any $f \in L^p$.

If μ is σ-finite, let E_n increasing X, $\mu(E_n) > 0$ and then we know there is $g_n \in L^q(E_n)$ on E_n such that $\phi(f) = \int fg_n$ for any $f \in L^p(E_n)$ and $g_n = g_m$ on E_n a.e., then we define $g = g_n$ on E_n and we know $||g||_q = \lim ||g_n||_q \leq ||\phi||$ by the MCT, now we know

$$\int fg = \lim \int f\chi_{E_n} g = \lim \int fg_n = \lim \phi(f\chi_{E_n}) = \phi(f)$$

For general μ, for a σ-finite subset E, there is $g_E \in L^q(E)$ and $\phi(f) = \int fg_E$ for any $f \in L^p(E)$ and $||g_E||_q \leq ||\phi||$, so we may find E_n such that $||g_{E_n}||_q \to \sup ||g_E||_q$ and let $F = \bigcup_E$ which is σ-finite, then we know $||g_F||_q \geq ||g_{E_n}||_q$ for any integer n and hence $||g_F||_q = M$. Then for any A σ-finite, we will know

$$\int |g_F|^q + \int |g_{A-F}|^q = \int |g_{A\cup F}|^q \leq M = \int |g_F|^q$$

and hence $g_{A-F} = 0$ a.e. and hence $g_{A\cup F} = g_F$ a.e. for all A σ-finite subset. If $g \in L^p$, we know S_f is σ-finite and hence $\phi(f) = \int fg_{S_g \cup F} = \int fg_F$ for any $f \in L^p$.

Corollary 1.1

If $1 < p < \infty$, L^p is reflexive.

Theorem 1.7

(Chebyshev's Inequality) If $f \in L^p(0 < p < \infty)$, then for any $\alpha > 0$,

$$\mu(\{x : |f| > \alpha\}) \leq \left[\frac{||f||_p}{\alpha}\right]^p$$

Theorem 1.8

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let K be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$. Suppose that there exists $C > 0$ such that $\int |K(x, y) d\mu(x)| \leq C$ for a.e. $y \in Y$ and $\int |K(x, y) d\nu(y)| \leq C$ for a.e. $x \in X$ and that $1 \leq p \leq \infty$. If $f \in L^p(\nu)$, then the integral

$$Tf(x) = \int K(x, y) f(y) d\nu(y)$$

converges absolutely for a.e. $x \in X$, the function Tf thus defines is in $L^p(\mu)$ and $||Tf||_p \leq C||f||_p$. ♡

Proof Consider

$$\int |K(x,y)f(y)|d\nu(y) \leq ||K(x,\cdot)^{q^{-1}}||_q||K(x,y)^{p^{-1}}|f(y)|||_p \leq C^{q^{-1}}\left[\int |K(x,y)||f(y)|^p d\nu(y)\right]^{p^{-1}}$$

for a.e. $x \in X$, then we know

$$\begin{aligned}\int |Tf(x)|^p d\mu(x) &= \int |\int K(x,y)f(y)d\nu(y)|^p d\mu(x) \\ &\leq \int C^{p/q}\int |K(x,y)||f(y)|^p d\nu(y)d\mu(x) \\ &= C^{p/q}\int\int |K(x,y)|d\mu(x)|f(y)|^p d\nu(y) \\ &\leq C^{p/q+1}||f||_p^p < \infty\end{aligned}$$

and hence $Tf \in L^p(\mu)$ and $||Tf||_p \leq C||f||_p$.

Theorem 1.9

Suppose that $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ are σ-finite measure spaces, and let f be an $(\mathcal{M}\otimes\mathcal{N})$-measurable function on $X \times Y$.
a. If $f \geq 0$ and $1 \leq p < \infty$, then

$$\left[\int\left(\int f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int\left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y)$$

b. If $1 \leq p \leq \infty$, $f(\cdot,y) \in L^p(\mu)$ for a.e. y, and the function $y \mapsto ||f(\cdot,y)||_p$ is in $L^1(\nu)$, then $f(x,\cdot) \in L^1(\nu)$ for a.e. x, the function $x \mapsto \int f(x,y)d\nu(y)$ is in $L^p(\mu)$ and

$$||\int f(\cdot,y)d\nu(y)||_p \leq \int ||f(\cdot,y)||_p d\nu(y)$$

♡

Proof

a. Let $g \in L^q(\mu)$ and we have

$$\int\int f(x,y)d\nu(y)|g(x)|d\mu(x) \leq ||g||_q\int\left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y)$$

and hence $||\int f(x,y)d\nu(y)||_p \leq \int\left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y)$ by theorem 1.5.

b. This conclusion is obvious and by (a) if $p < \infty$ and it goes when $q = \infty$.

Theorem 1.10

Let K be a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that $K(\lambda x,\lambda y) = \lambda^{-1}K(x,y)$ for all $\lambda > 0$ and $\int_0^\infty |K(x,1)|x^{-1/p}dx \leq C < \infty$ for some $p \in [1,\infty]$, and let q be the conjuate exponent to p. For $f \in L^p$ and $g \in L^q$, let

$$Tf(y) = \int_0^\infty K(x,y)f(x)dx, \quad Sg(x) = \int_0^\infty K(x,y)g(y)dy$$

Then Tf and Sg are defined a.e. and $||Tf||_p \leq C||f||_p$ and $||Sg||_q \leq C||g||_q$. ♡

Proof Consider

$$\begin{aligned}\left(\int |Tf(y)|^p dy\right)^{1/p} = \left(\int |\int K(x,y)f(x)dx|^p dy\right)^{1/p} &\leq \left(\int\left(\int |K(x,y)f(x)|dx\right)^p dy\right)^{1/p} \\ &= \left(\int\left(\int |K(z,1)f(yz)|dz\right)^p dy\right)^{1/p} \\ &\leq \int ||f(\cdot z)||_p|K(z,1)|dz \\ &\leq C||f||_p\end{aligned}$$

by the Minkowski's inequaltiy for integral and $||f(yz)||_p = z^{-1/p}||f||_p$ and the other conclusion is the same since

$$\begin{aligned}\int_0^\infty |K(1,y)|y^{-1/q}dy &= \int_0^\infty |K(y^{-1},1)|y^{1-1/q}dy \\ &= -\int_0^\infty |K(u,1)|u^{1/q+1}(-u^{-2})du = \int_0^\infty |K(u,1)|u^{-1/p}du \le C\end{aligned}$$

Corollary 1.2

Let

$$Tf(y) = y^{-1}\int_0^y f(x)dx, \quad Sg(x) = \int_x^\infty y^{-1}g(y)dy$$

Then for $1 < p \le \infty$ and $1 \le q < \infty$,

$$||Tf||_p \le \frac{p}{p-1}||f||_p, \quad ||Sg||_q \le q||g||_q$$

♡

Proof

Let $K(x,y) = y^{-1}\chi_{(x<y)}$ and we know

$$\int |K(x,y)|x^{-1/p}dx = y^{-1}qx^{1/q}|_0^y = q = \frac{p}{p-1}$$

Definition 1.3

If f is a measurable function on $(X, \mathcal{M}, \mu)$, its $distribution\ function$ $\lambda_f : (0,\infty) \to [0,\infty]$ by

$$\lambda_f(\alpha) = \mu(|f| > \alpha)$$

♣

Proposition 1.7

a. λ_f is decreasing and right continuous.

b. If $|f| \le |g|$, then $\lambda_f \le \lambda_g$.

c. If $|f_n|$ increases to $|f|$, then λ_{f_n} increases to λ_f.

d. If $f = g + h$, then $\lambda_f(\alpha) \le \lambda_g(\frac{1}{2}\alpha) + \lambda_h(\frac{1}{2}\alpha)$.

♠

Proof

a. Trivial.

b. $\lambda_g(\alpha) = \mu(|g| > \alpha) \ge \mu(|f| > \alpha) = \lambda_f(\alpha)$.

c. $\{|f| > \alpha\} = \bigcup\{|f_n| > \alpha\}$.

d. $\{|f+g| > \alpha\} \subset \{|f| > \frac{1}{2}\alpha\}$ and $\{|g| > \frac{1}{2}\alpha\}$.

Proposition 1.8

If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and ϕ is a nonnegative Borel measurable function on $(0,\infty)$, then

$$\int_X \phi \circ |f| d\mu = -\int_0^\infty d\lambda_f(\alpha)$$

where $d\lambda_f = d\nu$, which is the negative Borel measure defined by λ_f.

♠

Proposition 1.9

Consider for a h-interval $(a,b]$, we have

$$\int_X \chi_{(a,b]}(|f|)d\mu = \mu(b \le |f| > a) = -\nu((a,b]) = -\int_0^\infty \chi_{(a,b]}d\lambda_f$$

and hence the equality holds for all Borel set E. The rest can be obtained by the MCT.

♠

Proposition 1.10

If $0 < p < \infty$, then

$$\int |f|^p d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha$$

Proof

If $\lambda_f(\alpha) = \infty$ for some α, then we know the both sides are infinity. Then we assume $\lambda_f < \infty$ and if f is simple, then λ_f should be bounded and vanish when $\alpha \to \infty$ and the integration by parts will show it immediately.

For general case, let $\{g_n\}$ be simple functions increase to $|f|^p$ and the MCT will guarantee the equality.

Definition 1.4

If f is a measurable function on X and $0 < p < \infty$, we define

$$[f]_p = (\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha))^{1/p}$$

and the weak L^p space is all f such that $[f]_p < \infty$.
We have

$$L^p \subset \text{weak } L^p, \quad [f]_p \le ||f||_p$$

Proposition 1.11

If f is a measurable function and $A > 0$, let $E(A) = \{x, |f| > A\}$ and set

$$h_A = f\chi_{X-E(A)} + A(sgn(f))\chi_{E(A)} \quad g_A = f - h_A = (sgn(f))(|f| - A)\chi_{E(A)}$$

then

$$\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A), \quad \lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \text{if } \alpha < A \\ 0 & \text{if } \alpha \ge A \end{cases}$$

Proof

Here we have

$$\lambda_{g_A}(\alpha) = \mu(\{|g_A| > \alpha\}) \le \mu(\{|f| > \alpha + A\})$$

and by the way

$$\lambda_f(\alpha + A) = \mu(\{|f| - A > \alpha\}) \le \mu(\{|g_A| > \alpha\})$$

Then we know

$$\lambda_{h_A}(\alpha) = \mu(\{|f||\chi_{X-E(A)}| > \alpha\}) + \mu(\{A|\chi_{E(A)}| > \alpha)\}) = \chi_{\alpha<A}(\lambda_f(\alpha) - \lambda_f(A) + \lambda_f(A)) = \chi_{\alpha<A}\lambda_f(\alpha)$$

Lemma 1.2

Let ϕ be a counded continuous function on the strip $0 \le Rez \le 1$ that is holomorphic on the interior of the strip. If $|\phi(z)| \le M_0$ for $Rez = 0$ and $|\phi(z)| \le M_1$ for $Rez = 1$, then $|\phi(z)| \le M_0^{1-t} M_1^t$ for $Rez = t, 0 < t < 1$.

Proof

Let $\phi_n(z) = \phi(z) M_0^{z-1} M_1^{-z} e^{n^{-1} z(z-1)}$ and we know $|\phi_n(0)|, |\phi_n(1)| \le 1$ when $Rez = 0, 1$ and notice $|\phi_n| \to 0$ when $|Imz| \to \infty$ since let $z = x + iy$ and

$$|\phi_n(z)| = |\phi(z)||M_0^{x-1}||M_1^{-x}|e^{n^{-1}(x(x-1)-y^2)}| \to 0, y \to \infty$$

and then we know $\phi_n(z) \le 1$ on the strip by the maximal modulus principle, then we have

$$|\phi(z)| M_0^{t-1} M_1^{-t} = \lim_{n \to \infty} |\phi_n(z)| \le 1$$

Theorem 1.11

(The Riesz-Thorin Interpolation Theorem)
Suppose that $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are mesure spaces and* $p_0, p_1, q_0, q_1 \in [1, \infty]$. *If* $q_0 = q_1 = \infty$, *suppose also that* ν *is semifinite. For* $0 < t < 1$, *define*

$$p_t^{-1} = (1-t)p_0^{-1} + tp_1^{-1}, \quad q_t^{-1} = (1-t)q_0^{-1} + tq_1^{-1}$$

If T *is a linear map from* $L^{p_0}(\mu) + L^{p_1}(\mu)$ *into* $L^{q_0}(\nu) + L^{q_1}(\nu)$ *such that* $||Tf||_{q_0} \leq M_0||f||_{p_0}$ *for* $f \in L^{p_0}(\mu)$ *and* $||Tf||_{q_1} \leq M_1||f||_{p_1}$ *for* $f \in L^{p_1}(\mu)$, *then* $||Tf||_{q_t} \leq M_0^{1-t}M_1^t||f||_{p_t}$ *for* $f \in L^{p_t}(\mu), 0 < t < 1$.

Proof

We know

$$||Tf||_{q_t} = \sup\{|\int (Tf)g|, g \in \Sigma_X, ||g||_{\tilde{q_t}} = 1\}$$

where $\tilde{q_t}$ is the conjugate of q_t and then we only need to show that

$$|\int (Tf)g| \leq M_0^{1-t}M_1^t$$

for any $g \in \Sigma_X$ and $||f||_{p_t} = 1$. We assume $f = \sum a_j \chi_{E_j}$ and $g = \sum b_k \chi_{F_k}$. Define

$$\alpha(z) = (1-t)p_0^{-1} + tp_1^{-1}, \quad \beta(z)(1-t)q_0^{-1} + tq_1^{-1}$$

and let

$$f_z = \sum |a_j|^{\alpha(z)/\alpha(t)} e^{i\theta_j} \chi_{E_j}$$
$$g_z = \sum |b_k|^{(1-\beta(z))/(1-\beta(t))} e^{i\varphi_k} \chi_{F_k}$$

where $\theta_j = Arg(a_j), \varphi_k = Arg(b_k)$ and

$$\phi(z) = \int (Tf_z)g_z$$

here we assume $\alpha(t) =\neq 0, \beta(t) \neq 1$ and hence $(p_0, p_1) \neq (\infty, \infty), (q_0, q_1) \neq (1, 1)$. Then we know

$$\phi(z) = \sum |a_j|^{\alpha(z)/\alpha(t)} |b_k|^{(1-\beta(z))/(1-\beta(t))} e^{i(\varphi_k + \theta_j)} \int (T\chi_{E_j})\chi_{F_k}$$

which is an entire function and we have

$$\begin{aligned} |\phi(ir)| \leq ||Tf_{ir}||_{q_0}||g_{ir}||_{\tilde{q_0}} &\leq M_0||f_{ir}||_{p_0}||g_{ir}||_{\tilde{q_0}} \\ &= M_0|\int |f|^{p_0 Re\alpha(ir)/\alpha(t)}|^{1/p_0}|\int |g|^{\tilde{q_0}Re(1-\beta(ir))/(1-\beta(t))}|^{1/\tilde{q_0}} \\ &= M_0 \end{aligned}$$

and

$$\begin{aligned} |\phi(1+ir)| \leq ||Tf_{1+ir}||_{q_1}||g_{ir}||_{\tilde{q_1}} &\leq M_1||f_{1+ir}||_{p_1}||g_{ir}||_{\tilde{q_0}} \\ &= M_1|\int |f|^{p_1 Re\alpha(1+ir)/\alpha(t)}|^{1/p_1}|\int |g|^{\tilde{q_1}Re(1-\beta(1+ir))/(1-\beta(t))}|^{1/\tilde{q_1}} \\ &= M_1 \end{aligned}$$

Therefore, we will know $|\int (Tf)g| = |\phi(t)| \leq M_0^{1-t}M_1^t$ by the lemma 1.2. When $p_0 = p_1 = \infty$, the inequality is trivial and when $q_0 = q_1 = 1$, let $g_z = g$ and the proof is fine.

Now we only need to prove that $Tf = \lim Tf_n$ for any $f \in L^{p_t}$ where $f_n \in \Sigma_X$ and $f_n \to f$ pointwise with $|f_n| \leq |f|$. Consider $g = f\chi_{|f|<1}$ and $h = f\chi_{|f|>1}$, then we know $g \in L^{p_0}$ and $h \in L^{p_1}$, then we know $||Tg_n - Tg||_{q_0} \leq M_0||g_n - g||_{p_0} \to 0$ and $||Th_n - Th||_{q_1} \leq M_1||h_n - h||_{p_1} \to 0$ by the DCT and hence there exists subsequence n_k such that $Tg_{n_k} \to Tg, Th_{n_k} \to Th$ pointwise and hence $Tf_{n_k} \to Tf$ pointwise, and

$$||Tf||_{q_t} \leq \liminf ||Tf_n||_{n_k} \leq \liminf M_0^{1-t}M_1^t||f_{n_k}||_{p_t} = M_0^{1-t}M_1^t||f||_{p_t}$$

and the problem goes.

Definition 1.5

For $T: X \to Y$ where X, Y are normed vector spaces and T is called sublinear if

$$|T(f+g)| \leq |Tf| + |Tg| \quad |T(cf)|c|Tf|$$

for any $f, g \in X, c > 0$.
Then we call a sublinear map T is strong type (p,q) if $L^p(\mu) \subset X$ and T maps $L^p(\mu)$ into $L^q(\nu)$, then there exists $C > 0$ such that $||Tf||_q \leq C||f||_p$ for all $f \in L^p(\mu)$ for any $1 \leq p, q \leq \infty$.
T is weak type (p,q) if $L^p(\mu) \subset X$ and T maps $L^p(\mu)$ into weak $L^q(\nu)$ and there exists $C > 0$ such that $[Tf]_q \leq C||f||_p$ for all $f \in L^p(\mu)$ for any $1 \leq p \leq \infty$ and $1 \leq q < \infty$.

♣

Theorem 1.12

(The Marcinkiewicz Interpolation Theorem)
Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are mesure spaces and $p_0, p_1, q_0, q_1 \in [1, \infty]$ such that $p_0 \leq q_0, p_1 \leq q_1$ and $q_0 \neq q_1$ and

$$p^{-1} = (1-t)p_0^{-1} + tp_1^{-1}, \quad q^{-1} = (1-t)q_0^{-1} + tq_1^{-1}$$

where $0 < t < 1$. If T is a sublinear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ to the space of measurable functions on Y that is weak types (p_0, q_0) and (p_1, q_1), then T is strong type (p,q). More precisely, if $[Tf]_{q_j} \leq C_j||f||_{p_j}$ for $j = 0, 1$, then $||Tf||_q \leq B_p||f||_p$ where B_p depends only on p_j, q_j, C_j in addition to p; and for $j = 0, 1$, $B_p|p - p_j|$ remains bounded as $p \to p_j$ if $p_j < \infty$.

♡

Proof

Assume $p_0 = p_1, q_0 < q_1$, then we know $q < \infty$ and

$$C_0||f||_{p_0} \geq [Tf]_{q_0}, \quad C_1||f||_{p_0} \geq [Tf]_{q_1}$$

and we know if $q_1 < \infty$ then for any f with $||f||_{p_0} = ||f||_{p_1} = 1$

$$\begin{aligned}
\int |Tf|^q = q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha &\leq q\Big[\int_0^1 \alpha^{q-1}\Big(\frac{C_0||f||_{p_0}}{\alpha})^{q_0} + \int_1^\infty \alpha^{q-1}\Big(\frac{C_1||f||_{p_1}}{\alpha})^{q_1}\Big]d\alpha \\
&= qC_0^{q_0}\int_0^1 \alpha^{q-q_0-1}d\alpha + qC_1^{q_1}\int_1^\infty \alpha^{q-q_1-1}d\alpha \\
&= \frac{q}{q-q_0}C_0^{q_0} + \frac{q}{q_1-q}C_1^{q_1} = B_p^q
\end{aligned}$$

If $q_1 = \infty$, then assume $||f||_{p_0} = 1$, we have

$$\int |Tf|^q = q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha \leq q\int_0^{C_1||f||_{p_0}} \alpha^{q-1}(\frac{C_0||f||_{p_0}}{\alpha})^{q_0}d\alpha = \frac{q}{q-q_0}C_0^{q_0}C_1^{q-q_0}$$

and hence

$$||Tf||_q = ||||f||_{p_0}T(f/||f_{p_0}||)||_q \leq B_p||f||_{p_0}$$

where

$$B_p = \Big(\Big(\frac{q}{q-q_0}C_0^{q_0}C_1^{q-q_0}\Big)^{1/q}\chi_{q_1=\infty} + \Big(\frac{q}{q-q_0}C_0^{q_0} + \frac{q}{q_1-q}C_1^{q_1}\Big)^{1/q}\chi_{q_1<\infty}\Big)$$

when $p_0 = p_1, q_0 < q_1$ and we know B_p is a constant respect to p and obviously we have $B_p|p - p_j|$ is bounded when $p \to p_j$. Then we assume $p_0 < p_1$, then we have for any $f \in L^p(\mu)$

$$\int |g_A|^{p_0} = p_0\int_0^\infty \alpha^{p_0-1}\lambda_{g_A}(\alpha)d\alpha \leq p_0\int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha)d\alpha$$
$$\int |h_A|^{p_1} = p_1\int_0^\infty \alpha^{p_1-1}\lambda_{h_A}(\alpha)d\alpha \leq p_1\int_0^A \alpha^{p_1-1}\lambda_f(\alpha)d\alpha$$

Let $A = A(\alpha)$ and

$$\int |Tf|^q = q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha \leq 2^q q\int_0^\infty \alpha^{q-1}(\lambda_{g_A}(\alpha) + \lambda_{h_A}(\alpha))d\alpha$$

and notice

$$\lambda_{g_A}(\alpha) \le \Big(\frac{C_0||g_A||_{p_0}}{\alpha}\Big)^{q_0}, \quad \lambda_{h_A}(\alpha) \le \Big(\frac{C_1||h_A||_{p_1}}{\alpha}\Big)^{q_1}$$

where we may see $g_A \in L^{p_0}, h_A \in L^{p_1}$ by consider $f' = f/A$, then $g_1' = g_A/A, h_1' = h_A/A$ and we have

$$\int |h_1'|^{p_1} \le \int |f'|^p, \quad \int |g_1'|^{p_0} \le \int (|g_1'|+1)^{p_0} \le \int |f'|^p$$

and hence $h_1' \in L^{p_1}, g_1' \in L^{p_0}$, which means the inequalities above holds for f and then we have

$$\begin{aligned}\int |Tf|^q &\le 2^q q \int_0^\infty \alpha^{q-1}\Big[\Big(\frac{C_0||g_A||_{p_0}}{\alpha}\Big)^{q_0} + \Big(\frac{C_1||h_A||_{p_1}}{\alpha}\Big)^{q_1}\Big]d\alpha \\ &= 2^q q\Big[C_0^{q_0} p_0^{q_0/p_0}\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0}d\alpha \\ &\quad + C_1^{q_1}p_1^{q_1/p_1}\int_0^\infty \alpha^{q-q_1-1}\Big(\int_0^{A(\alpha)}\beta^{p_1-1}\lambda_f(\beta)d\beta\Big)^{q_1/p_1}d\alpha\Big]\end{aligned}$$

where we have

$$\begin{aligned}\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0}d\alpha &\le \Big[\int_0^\infty \Big(\int_{A(\alpha)\le\beta}[\alpha^{p_0(q-q_0-1)/q_0}\beta^{p_0-1}\lambda_f(\beta)]^{q_0/p_0}d\alpha\Big)^{p_0/q_0}d\beta\Big]^{q_0/p_0} \\ &= \Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_{A(\alpha)\le\beta}\alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0}d\beta\Big]^{q_0/p_0}\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \alpha^{q-q_1-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_1-1}\lambda_f(\beta)d\beta\Big)^{q_1/p_1}d\alpha &\le \Big[\int_0^\infty \Big(\int_{A(\alpha)>\beta}[\alpha^{p_1(q-q_1-1)/q_1}\beta^{p_1-1}\lambda_f(\beta)]^{q_0/p_0}d\alpha\Big)^{p_1/q_1}d\beta\Big]^{q_1/p_1} \\ &= \Big[\int_0^\infty \beta^{p_1-1}\lambda_f(\beta)\Big(\int_{A(\alpha)>\beta}\alpha^{q-q_1-1}d\alpha\Big)^{p_1/q_1}d\beta\Big]^{q_1/p_1}\end{aligned}$$

then we may consider if $q_0 < q_1$ then let $A(\alpha) = \alpha^r$ and we have

$$\begin{aligned}\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0}d\alpha &\le \Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_0^{\beta^{1/r}}\alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0}d\beta\Big]^{q_0/p_0} \\ &= \frac{1}{q-q_0}\Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\beta^{p_0(q-q_0)/rq_0}\beta\Big]^{q_0/p_0}\end{aligned}$$

and let

$$r = \frac{p_0}{q_0}\frac{q-q_0}{p-p_0} = \frac{q_0^{-1}-q^{-1}}{q^{-1}}\frac{p^{-1}}{p_0^{-1}-p^{-1}} = \frac{q_0^{-1}-q_1^{-1}}{p_0^{-1}-p_1^{-1}}\frac{p^{-1}}{q^{-1}} = \frac{q_1^{-1}-q^{-1}}{p_1^{-1}-p^{-1}}\frac{p^{-1}}{q^{-1}} = \frac{p_1}{q_1}\frac{q-q_1}{p-p_1}$$

and we know if $||f||_p = 1$ then

$$\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0}d\alpha \le \frac{1}{q-q_0}\Big(\frac{||f||_p^p}{p}\Big)^{q_0/p_0} = |q-q_0|^{-1}p^{-q_0/p_0}$$

and similarly

$$\begin{aligned}\int_0^\infty \alpha^{q-q_1-1}\Big(\int_0^{A(\alpha)} \beta^{p_1-1}\lambda_f(\beta)d\beta\Big)^{q_1/p_1}d\alpha &\le \Big[\int_0^\infty \beta^{p_1-1}\lambda_f(\beta)\Big(\int_{\beta^{1/r}}^\infty\alpha^{q-q_1-1}d\alpha\Big)^{p_1/q_1}d\beta\Big]^{q_1/p_1} \\ &= \frac{1}{q_1-q}\Big[\int_0^\infty \beta^{p_1-1}\lambda_f(\beta)\beta^{p_1(q-q_1)/rq_1}\beta\Big]^{q_1/p_1}\end{aligned}$$

and then

$$\int_0^\infty \alpha^{q-q_1-1}\Big(\int_0^{A(\alpha)} \beta^{p_1-1}\lambda_f(\beta)d\beta\Big)^{q_1/p_1}d\alpha \le \frac{1}{q_1-q}\Big(\frac{||f||_p^p}{p}\Big)^{q_1/p_1} = |q-q_1|^{-1}p^{-q_1/p_1}$$

Therefore, we have

$$\int |Tf|^q \le 2^q q\Big[C_0^{q_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1} + C_1^{q_1}(p_1/p)^{q_1/p_1}|q-q_1|^{-1}\Big]$$

when $q_0 < q_1$ and if $q_0 > q_1$, let $A(\alpha) = \alpha^r$ and notice $r < 0$ so we have

$$\int_0^\infty \alpha^{q-q_0-1}\Big(\int_{A(\alpha)}^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\Big)^{q_0/p_0} d\alpha \le \Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_{\beta^{1/r}}^\infty \alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0}$$
$$= \frac{1}{q_0-q}\Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\beta^{p_0(q-q_0)/rq_0}\beta\Big]^{q_0/p_0}$$

and the rest calculation are similar, we can still get

$$\int |Tf|^q \le 2^q q\Big[C_0^{q_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1} + C_1^{q_1}(p_1/p)^{q_1/p_1}|q-q_1|^{-1}\Big] = B_t$$

and to show $B_p|p-p_j|$ is bounded when $p \to p_j, j = 0, 1$, it suffices to show that $|(p-p_j)/(q-q_j)|$ is bounded when $p \to p_j$ and which is easy to check by r.

For the rest conditions, we assume $p_1 = q_1 = \infty$ at first, we know

$$||Th_A||_\infty \le C_1||h_A||_\infty$$

and let $A(\alpha) = \alpha/C_1$ then $\lambda_{Th_A}(\alpha) = 0$ and then

$$\int |Tf|^q \le 2^q q C_0^{q_0} p_0^{q_0/p_0}\Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_0^{C_1\beta} \alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0}$$
$$= 2^q q C_0^{q_0} C_1^{q-q_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1}$$

when $||f||_p = 1$, and hence

$$B_p = 2\Big[C_0^{q_0}C_1^{q-q_0}(p_0/p)^{q_0/p_0}|q-q_0|^{-1}\Big]^{1/q}$$

at this considition, which is bounded when $p \to p_j, j = 0, 1$.

Then assume $q_0 < q_1 = \infty$, we have

$$||Th_A||_\infty \le C_1||h_A||_{p_1} \le C_1\Big(p_1\int_0^A \alpha^{p_1-1}\lambda_f(\alpha)d\alpha\Big)^{1/p_1} \le C_1 p_1^{1/p_1} A^{(p_1-p)/p_1}(||f||_p^p/p)^{1/p_1}$$

and let $A(\alpha) = [\alpha/[C_1(p_1||f||_p^p/p)^{1/p_1}]]^{\frac{p_1}{p_1-p}}$ and we get $||Th_{A(\alpha)}||_\infty \le \alpha$ and

$$\int |Tf|^q \le 2^q q C_0^{q_0} p_0^{q_0/p_0}\Big[\int_0^\infty \beta^{p_0-1}\lambda_f(\beta)\Big(\int_0^{d\beta^{(p_1-p)/p_1}} \alpha^{q-q_0-1}d\alpha\Big)^{p_0/q_0} d\beta\Big]^{q_0/p_0}$$
$$= 2^q q C_0^{q_0} d^{q-q_0} p_0^{q_0/p_0}|q-q_0|^{-1}\Big[\int_0^\infty \beta^{p_0-1+p_0(q-q_0)(p_1-p)/p_1q_0}\lambda_f(\beta)d\beta\Big]^{q_0/p_0}$$
$$= 2^q q C_0^{q_0}\Big(C_1(p_1||f||_p^p/p)^{1/p_1}\Big)^{q-q_0} p_0^{q_0/p_0}|q-q_0|^{-1}\Big(\frac{||f||_p^p}{p}\Big)^{q_0/p_0}$$

For $q_1 < q_0 = \infty$, we have

$$||Tg_A||_\infty \le C_0||g_A||_{p_0} \le C_0\Big(p_0\int_A^\infty \alpha^{p_0-1}\lambda_f(\alpha)d\alpha\Big)^{1/p_0} \le C_0 p_0^{1/p_0} A^{(p_0-p)/p_0}(||f||_p^p/p)^{1/p_0}$$

and let $A(\alpha) = [\alpha/[C_0(p_0||f||_p^p/p)^{1/p_0}]]^{\frac{p_0}{p_0-p}}$ and we get $||T_{g_{A(\alpha)}}||_\infty \le \alpha$ and then the rest are the same.

Definition 1.6

Suppose $X_n, n \ge 0$ is a submartingale. Let $a < b, N_0 = -1$ and for $k \ge 1$ let

$$N_{2k-1} = \inf\{m > N_{2k-2}, X_m \le a\}$$
$$N_{2k} = \inf\{m > N_{2k-1}, X_m \ge b\}$$

The N_j are stopping times so

$$H_m = \begin{cases} 1 & \text{if } N_{2k-1} < m \le N_{2k} \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

defines a predictable sequence. ♣

Proof

Notice

$$\{N_{2k-1} = n\} = \bigcup_{0\le m\le n-1} \{N_{2k-2} = m\} \cap (\bigcap_{n-1-m\ge k\ge 0} \{X_{m+k} > a\}) \cap \{X_n \le a\}$$

and

$$\{N_{2k} = n\} = \bigcup_{0\le m\le n-1} \{N_{2k-1} = m\} \cap (\bigcap_{n-1-m\ge k\ge 0} \{X_{m+k} < b\}) \cap \{X_n \ge b\}$$

and hence N_{2k-1}, N_{2k} are stopping times by induction.

And notice

$$\{N_{2k-1} < m \le N_{2k} \text{ for some } k\} = \bigcup_{k\ge 0} \{N_{2k-1} \le m-1\} \cap \{N_{2k} \ge m\} \in \mathcal{F}_{m-1}$$

and hence H_m is predictable.

Theorem 1.13

(Upcoming inequality) If $X_m, m \ge 0$, is a submartingale then

$$(b-a)EU_n \le E(X_n - a)^+ - E(X_0 - a)^+$$

where $U_n = \sup\{k, N_{2k} \le n\}$.

Proof Here we assume $Y_m = a + (X_m - a)^+$ and we have

$$(b-a)U_n \le (H \cdot Y)_n$$

let $K_m = 1 - H_m$ and then we know that $(K \cdot X)_n$ is a submartingale and then

$$E(K \cdot X)_n \ge E(K \cdot X)_0 = 0$$

so we know

$$E(H \cdot Y)_n \le E(Y_n - Y_0) = E(X_n - a)^+ - E(X_0 - a)^+$$

since $Y_n - Y_0 = (H \cdot Y)_n + (K \cdot Y)_n$

Theorem 1.14

(Martingale convergence theorem) If X_n is a submartingale with $\sup EX_n^+ < \infty$ then as $n \to \infty$, X_n converges a.s. to a limit X with $E|X| < \infty$.

Proof We know $(X-a)^+ \le X^+|a|$, then we know

$$EU_n \le (|a| + EX_n^+)/(b-a)$$

so $\sup X_n^+$ will imply than $EU < \infty$ where $U = \lim U_n$ and hence for all rational a, b, we know

$$P(\{\liminf X_n < a < b < \limsup X_n\}) = 0$$

and hence $\lim X_n$ exists a.s. and $EX^+ \le \liminf EX_n^+ < \ infty$ and hence $X < \infty$ a.s. and notice

$$EX_n^- = EX_n^+ - EX_n \le EX_n^+ - EX_0$$

and hence $EX^- \le \liminf EX_n^- \le \liminf EX_n^+ - EX_0 < \infty$ therefore $E|X| < \infty$.

Theorem 1.15

If $X_n \ge 0$ is a supermartingale then as $n \to \infty$, $X_n \to X$ a.s. and $EX \le EX_0$.

Proof Let $Y_n = -X_n$ and hence a submartingale with $EY_n^+ = 0$, then we know $X_n \to X$ a.s. and we also have

$$EX \le \liminf EX_n^+ \le EX_0$$

Proposition 1.12

The theorem 1.18. provide a method to show that a.s. convergence does not guarantee convergence in L^1.

Proof Let S_n be a symmetric simple random walk with $S_0 = 1$ and $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$, let $N = \inf\{n : S_n = 0\}$ and $X_n = S_{N \wedge n}$. Then we know X_n nonnegative and $EX_n = EX_0 = 1$ since X_n is a martingale, then we know $X_n \to X$ where X is some r.v. and hence $X = 0$, because there is no way to converge to others and hence X_n do not converge to X in L^1.

Proposition 1.13

Convergence in probability do not guarantee convergence a.s.

Proof Let $X_0 = 0$ and $P(X_k = 1|X_{k-1} = 0) = P(X_k = -1|X_{k-1} = 0) = \frac{1}{2k}, P(X_k = 0|X_{k-1} = 0) = 1 - \frac{1}{k}$ and $P(X_k = kX_{k-1}|X_{k-1} \neq 0) = \frac{1}{k}, P(X_k = 0|X_{k-1} \neq 0) = 1 - \frac{1}{k}$, then we know $X_k \to 0$ in probability, but $P(X_k = 0, k \geq K)$ and it picks discrete values and hence X_k can not converge to 0 a.s.