Correction des exercices de TD

Table des matières

1 TD/TP1 : Preuves en logique du premier ordre

Exercice 1 (Logique propositionnelle)

Démontrer les propositions suivantes dans LJ et LK :

1. $A \Rightarrow B \Rightarrow A$
2. $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$
3. $A \wedge B \Rightarrow B$
4. $B \Rightarrow A \vee B$
5. $(A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
6. $A \Rightarrow \bot \Rightarrow \neg A$
7. $\bot \Rightarrow A$
8. $(A \Leftrightarrow B) \Rightarrow A \Rightarrow B$
9. $(A \Leftrightarrow B) \Rightarrow B \Rightarrow A$
10. $(A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$

1.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A, B \vdash A$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$A \vdash B \Rightarrow A$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash A \Rightarrow B \Rightarrow A$}
\end{prooftree}$$

Ici, la preuve en LJ est la même que la preuve en LK.

2.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B, A \vdash B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$A \Rightarrow B, A \vdash B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$C, A \Rightarrow B, A \vdash C$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$B \Rightarrow C, A \Rightarrow B, A \vdash C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \Rightarrow B, A \vdash A$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$A \Rightarrow B \Rightarrow C, A \Rightarrow B, A \vdash C$}
\RightLabel{$\Rightarrow_{\text{right}} \times 3$}
\UnaryInfC{$\vdash (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$}
\end{prooftree}$$

Preuve dans LJ

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B, A \vdash B, C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \vdash A, B, C$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$A \Rightarrow B, A \vdash B, C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$C, A \Rightarrow B, A \vdash C$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$B \Rightarrow C, A \Rightarrow B, A \vdash C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \Rightarrow B, A \vdash C, A$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$A \Rightarrow B \Rightarrow C, A \Rightarrow B, A \vdash C$}
\RightLabel{$\Rightarrow_{\text{right}} \times 3$}
\UnaryInfC{$\vdash (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$}
\end{prooftree}$$

Preuve dans LK

3.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A, B \vdash B$}
\RightLabel{$\wedge_{\text{left}}$}
\UnaryInfC{$A \wedge B \vdash B$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash A \wedge B \Rightarrow B$}
\end{prooftree}$$

Ici, la preuve en LJ est la même que la preuve en LK.

4.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \vdash B$}
\RightLabel{$\vee_{\text{right}}$}
\UnaryInfC{$B \vdash A \vee B$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash B \Rightarrow A \vee B$}
\end{prooftree}$$

Preuve dans LJ

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \vdash A, B$}
\RightLabel{$\vee_{\text{right}}$}
\UnaryInfC{$B \vdash A \vee B$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash B \Rightarrow A \vee B$}
\end{prooftree}$$

Preuve dans LK

5.

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B, A \Rightarrow C, C \vdash C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B, A \Rightarrow C \vdash B$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$B, A \Rightarrow C, B \Rightarrow C \vdash C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A, C, B \Rightarrow C \vdash C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A, B \Rightarrow C \vdash A$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$A, A \Rightarrow C, B \Rightarrow C \vdash C$}
\RightLabel{$\vee_{\text{left}}$}
\BinaryInfC{$A \vee B, A \Rightarrow C, B \Rightarrow C \vdash C$}
\RightLabel{$\Rightarrow_{\text{right}} \times 3$}
\UnaryInfC{$\vdash (A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$}
\end{prooftree}$$

Preuve dans LJ

$$\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B, A \Rightarrow C, C \vdash C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B, A \Rightarrow C \vdash C, B$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$B, A \Rightarrow C, B \Rightarrow C \vdash C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A, C, B \Rightarrow C \vdash C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A, B \Rightarrow C \vdash C, A$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$A, A \Rightarrow C, B \Rightarrow C \vdash C$}
\RightLabel{$\vee_{\text{left}}$}
\BinaryInfC{$A \vee B, A \Rightarrow C, B \Rightarrow C \vdash C$}
\RightLabel{$\Rightarrow_{\text{right}} \times 3$}
\UnaryInfC{$\vdash (A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$}
\end{prooftree}$$

Preuve dans LK

6.	$\dfrac{\dfrac{}{A, \bot \vdash \neg A}\,\bot_{\text{left}}}{\vdash A \Rightarrow \bot \Rightarrow \neg A}\,\Rightarrow_{\text{right}} \times 2$	Ici, la preuve en LJ est la même que la preuve en LK.
7.	$\dfrac{\dfrac{}{\bot \vdash A}\,\bot_{\text{left}}}{\vdash \bot \Rightarrow A}\,\Rightarrow_{\text{right}}$	Ici, la preuve en LJ est la même que la preuve en LK.
8.	$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{ax} \quad \dfrac{}{B, A \vdash B}\,\text{ax}}{A \Leftrightarrow B, A \vdash B}\,\Leftrightarrow_{\text{left}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\,\Rightarrow_{\text{right}} \times 2$ Preuve dans LJ	$\dfrac{\dfrac{\dfrac{}{A \vdash A, B}\,\text{ax} \quad \dfrac{}{A, B \vdash B}\,\text{ax}}{A \Leftrightarrow B, A \vdash B}\,\Leftrightarrow_{\text{left}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\,\Rightarrow_{\text{right}} \times 2$ Preuve dans LK
9.	$\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{ax} \quad \dfrac{}{A, B \vdash A}\,\text{ax}}{A \Leftrightarrow B, B \vdash A}\,\Leftrightarrow_{\text{left}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}} \times 2$ Preuve dans LJ	$\dfrac{\dfrac{\dfrac{}{B \vdash A, B}\,\text{ax} \quad \dfrac{}{A, B \vdash A}\,\text{ax}}{A \Leftrightarrow B, B \vdash A}\,\Leftrightarrow_{\text{left}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}} \times 2$ Preuve dans LK
10.	$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, A, B \vdash A}\,\text{ax} \quad \dfrac{}{A \Rightarrow B, B \vdash B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\text{left}} \quad \dfrac{\dfrac{}{B \Rightarrow A, A \vdash A}\,\text{ax} \quad \dfrac{}{B \Rightarrow A, B, A \vdash B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\text{left}}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\text{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\text{right}} \times 2$ Preuve dans LJ	$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, A, B \vdash A}\,\text{ax} \quad \dfrac{}{A \Rightarrow B, B \vdash A, B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\text{left}} \quad \dfrac{\dfrac{}{B \Rightarrow A, A \vdash B, A}\,\text{ax} \quad \dfrac{}{B \Rightarrow A, B, A \vdash B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\text{left}}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\text{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\text{right}} \times 2$ Preuve dans LK

Exercice 2 (Logique du premier ordre)

Démontrer les propositions suivantes dans LJ et LK (si la proposition n'admet pas de preuve intuitionniste, démontrer la proposition dans $\text{LJ}_{(\text{em})}$) :

1. $\forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)$
2. $(\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$
3. $(\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)$
4. $(\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$
5. $(\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))$
6. $\neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$

1.	$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\,\text{ax}}{P(x) \vdash P(x) \vee Q(x)}\,\vee_{\text{right}}}{P(x) \vdash \exists y.P(y) \vee Q(y)}\,\exists_{\text{right}}}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\,\Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\,\forall_{\text{right}}$ Preuve dans LJ	$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), Q(x)}\,\text{ax}}{P(x) \vdash P(x) \vee Q(x)}\,\vee_{\text{right}}}{P(x) \vdash \exists y.P(y) \vee Q(y)}\,\exists_{\text{right}}}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\,\Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\,\forall_{\text{right}}$ Preuve dans LK

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q(x) \vdash Q(x)}\,\text{ax}}{Q(x) \vdash (\exists x.Q(x))}\,\exists_{\text{right}}}{Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\lor_{\text{right}} \qquad \dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\,\text{ax}}{P(x) \vdash (\exists x.P(x))}\,\exists_{\text{right}}}{P(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\lor_{\text{right}}}{P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\lor_{\text{left}}}{\exists x.P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \lor Q(x)) \Rightarrow (\exists x.P(x)) \lor (\exists x.Q(x))}\,\Rightarrow_{\text{right}}$$

Preuve dans LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q(x) \vdash P(x), Q(x)}\,\text{ax} \qquad \dfrac{}{P(x) \vdash P(x), Q(x)}\,\text{ax}}{P(x) \lor Q(x) \vdash P(x), Q(x)}\,\lor_{\text{left}}}{P(x) \lor Q(x) \vdash \exists x.P(x), \exists x.Q(x)}\,\exists_{\text{right}} \times 2}{P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\lor_{\text{right}}}{\exists x.P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \lor Q(x)) \Rightarrow (\exists x.P(x)) \lor (\exists x.Q(x))}\,\Rightarrow_{\text{right}}$$

Preuve dans LK

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x), Q(x) \vdash Q(x)}\,\text{ax} \qquad \dfrac{}{P(x), Q(x) \vdash P(x)}\,\text{ax}}{P(x), Q(x) \vdash P(x) \land Q(x)}\,\land_{\text{right}}}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \land Q(x)}\,\forall_{\text{left}} \times 2}{\forall x.P(x), \forall x.Q(x) \vdash \forall x.P(x) \land Q(x)}\,\forall_{\text{right}}}{(\forall x.P(x)) \land (\forall x.Q(x)) \vdash \forall x.P(x) \land Q(x)}\,\land_{\text{left}}}{\vdash (\forall x.P(x)) \land (\forall x.Q(x)) \Rightarrow \forall x.P(x) \land Q(x)}\,\Rightarrow_{\text{right}}$$

Ici, la preuve en LJ est la même que la preuve en LK.

4.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x), Q(x) \vdash Q(x)}\,\text{ax}}{P(x) \land Q(x) \vdash Q(x)}\,\land_{\text{left}}}{\forall x.P(x) \land Q(x) \vdash Q(x)}\,\forall_{\text{left}}}{\forall x.P(x) \land Q(x) \vdash \forall x.Q(x)}\,\forall_{\text{right}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P(x), Q(x) \vdash P(x)}\,\text{ax}}{P(x) \land Q(x) \vdash P(x)}\,\land_{\text{left}}}{\forall x.P(x) \land Q(x) \vdash P(x)}\,\forall_{\text{left}}}{\forall x.P(x) \land Q(x) \vdash \forall x.P(x)}\,\forall_{\text{right}}}{\forall x.P(x) \land Q(x) \vdash (\forall x.P(x)) \land (\forall x.Q(x))}\,\land_{\text{right}}}{\vdash (\forall x.P(x) \land Q(x)) \Rightarrow (\forall x.P(x)) \land (\forall x.Q(x))}\,\Rightarrow_{\text{right}}$$

Ici, la preuve en LJ est la même que la preuve en LK.

5.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\,\text{ax}}{\neg P(x), P(x) \vdash}\,\neg_{\text{left}}}{\forall x.\neg P(x), P(x) \vdash}\,\forall_{\text{left}}}{\forall x.\neg P(x), \exists x.P(x) \vdash}\,\exists_{\text{left}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\,\neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\,\Rightarrow_{\text{right}}$$

Ici, la preuve en LJ est la même que la preuve en LK.

6.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg P(x) \vdash \neg P(x)}\,\text{ax}}{\neg P(x) \vdash \exists x.\neg P(x)}\,\exists_{\text{right}}}{\neg\exists x.\neg P(x), \neg P(x) \vdash}\,\neg_{\text{left}}}{\neg\exists x.\neg P(x) \vdash \neg\neg P(x)}\,\neg_{\text{right}}}{\neg\exists x.\neg P(x) \vdash P(x)}\,\text{em}}{\neg\exists x.\neg P(x) \vdash \forall x.P(x)}\,\forall_{\text{right}}}{\neg(\forall x.P(x)), \neg\exists x.\neg P(x) \vdash}\,\neg_{\text{left}}}{\neg(\forall x.P(x)) \vdash \neg\neg\exists x.\neg P(x)}\,\neg_{\text{right}}}{\neg(\forall x.P(x)) \vdash \exists x.\neg P(x)}\,\text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\,\Rightarrow_{\text{right}}$$

Preuve dans $\text{LJ}_{(\text{em})}$
(pas de preuve dans LJ possible)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\,\text{ax}}{\vdash \neg P(x), P(x)}\,\neg_{\text{right}}}{\vdash \exists x.\neg P(x), P(x)}\,\exists_{\text{right}}}{\vdash \exists x.\neg P(x), \forall x.P(x)}\,\forall_{\text{right}}}{\neg(\forall x.P(x)) \vdash \exists x.\neg P(x)}\,\neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\,\Rightarrow_{\text{right}}$$

Preuve dans LK

Exercice 3 (Preuves en Coq)

Démontrer les propositions des exercices 1 et 2 en `Coq`.
On rappelle que pour lancer `Coq`, il suffit de se mettre dans un terminal et de taper la commande `coqide`, qui lance l'IDE de `Coq`.

► Voir TP1