

# Do higher-order interactions promote synchronization?

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Understanding how nonpairwise interactions alter dynamical processes in networks is of fundamental importance to the characterization and control of many coupled systems. Recent discoveries of hyperedge-enhanced synchronization under various settings raised speculations that such enhancements might be a general phenomenon. Here, we demonstrate that even for simple systems such as Kuramoto oscillators, the effects of higher-order interactions are highly representation-dependent. Specifically, we show numerically and analytically that hyperedges typically enhance synchronization in random hypergraphs, but have the opposite effect in simplicial complexes. As an explanation, we identify higher-order degree heterogeneity as the key structural determinant of synchronization stability in systems with a fixed coupling budget. Our findings highlight the importance of appropriate representations in describing higher-order interactions. In particular, the choice of simplicial complexes or hypergraphs has significant ramifications and should not be purely motivated by technical conveniences.

Synchronization, the emergence of order in populations of interacting entities, is a widespread phenomenon that has been observed in many natural and man-made systems [1], from circadian rhythms [2] and vascular networks [3] to the brain [4]. The relationships of interdependence between these entities has been typically modeled as a network, where links encode pairwise interactions among nodes [5]. Yet, from ecosystems to the human brain, growing evidence suggests that in many cases a node may feel the influences of multiple other nodes at the same time, and that such interactions cannot be decomposed into pairwise ones [6]. The presence of these higher-order interactions has been associated with novel collective phenomena in a variety of dynamical processes [7, 8], including diffusion [9, 10], spreading [11, 12], and evolution [13].

In the case of synchronization, nonpairwise couplings arise naturally from phase reductions of oscillators with pairwise couplings [14–17]. They have been linked to the emergence of abrupt transitions [18–20], multistability [21], heteroclinic dynamics [22, 23], chimeras [24], and chaos [25]. These findings have happened hand in hand with analytical frameworks for analyzing coupled oscillators with higher-order interactions, such as low dimensional descriptions [26], Laplacian operators [27, 28], and Hodge decomposition [29, 30].

A natural question is whether introducing higher-order interactions tends to promote or impede synchronization. Recently, hyperedge-enhanced synchronization has been observed for a range of node dynamics [27, 28, 31–33]. It is thus tempting to conjecture that nonpairwise interactions synchronize oscillators more efficiently than pairwise ones. This seems physically plausible given that higher-order interactions enable more nodes to exchange information simultaneously, thus allowing more efficient communication and ultimately leading to enhanced synchronization performance.

In this Letter, we show that whether higher-order interactions promote or impede synchronization strongly depends on the overall organization of the underlying higher-order network. In particular, through a rich-get-richer effect, higher-order interactions consistently destabilize synchronization in simplicial complexes. On the other hand, through a homogenizing mechanism, higher-order interactions tend to stabilize synchronization in random hypergraphs. There, depending on the densities of connections, a sweet spot can emerge, where a mixture of pairwise and nonpairwise interactions maximizes synchronization stability. This alludes to a synergy between pairwise and nonpairwise interactions, where combined influences outperform either type of interactions alone.

To highlight the effect of higher-order interactions, we consider a simple system consisting of  $n$  identical phase oscillators [27], whose states  $\theta = (\theta_1, \dots, \theta_n)$  evolve according to

$$\begin{aligned} \dot{\theta}_i = & \omega + \frac{\gamma_1}{\langle k^{(1)} \rangle} \sum_{j=1}^n A_{ij} \sin(\theta_j - \theta_i) \\ & + \frac{\gamma_2}{2! \langle k^{(2)} \rangle} \sum_{j,k=1}^n B_{ijk} \frac{1}{2} \sin(\theta_j + \theta_k - 2\theta_i). \end{aligned} \quad (1)$$

System (1) is a natural generalization of the Kuramoto model [34] that includes interactions up to order two (i.e., three-body interactions) [35]. The oscillators have natural frequency  $\omega$  and the coupling strengths at each order are  $\gamma_1$  and  $\gamma_2$ , respectively. The adjacency tensors determine which oscillators interact:  $A_{ij} = 1$  if nodes  $i$  and  $j$  have a first-order interaction, and zero otherwise. Similarly,  $B_{ijk} = 1$  if and only if nodes  $i$ ,  $j$  and  $k$  have a second-order interaction. All interactions are assumed to be unweighted and undirected. Following Refs. [31, 36], we set

$$\gamma_1 = 1 - \alpha, \quad \gamma_2 = \alpha, \quad \alpha \in [0, 1]. \quad (2)$$

The parameter  $\alpha$  controls the relative strength of the first- and second-order interactions, from all first-order ( $\alpha = 0$ ) to all second-order ( $\alpha = 1$ ), allowing us to keep the total coupling budget constant and fairly compare the effects of pairwise and nonpairwise interactions. In addition, we normalize each coupling strength by the average degree of the corresponding order,  $\langle k^{(i)} \rangle$ , and further divide  $\gamma_2$  by two to avoid counting triangles twice. Finally, we normalize the second-order coupling function by an additional factor of two so that each interaction contributes to the dynamics with an equal weight regardless of the number of oscillators involved.

Synchronization,  $\theta_i = \theta_j$  for all  $i \neq j$ , is a solution of Eq. (1) and we are interested in the effect of  $\alpha$  on its stability. The system allows analytical treatment following the multiorder Laplacian approach introduced in Ref. [27]. We define the second-order Laplacian as

$$L_{ij}^{(2)} = k_i^{(2)} \delta_{ij} - A_{ij}^{(2)}, \quad (3)$$

which is a natural generalization of the graph Laplacian  $L_{ij}^{(1)} \equiv L_{ij} = k_i \delta_{ij} - A_{ij}$ . Here, we used the generalized degree  $k_i^{(2)} = \frac{1}{2} \sum_{j,k=1}^n B_{ijk}$  and the second-order adjacency matrix  $A_{ij}^{(2)} = \sum_{k=1}^n B_{ijk}$ .

Using the standard linearization technique, the evolution of a generic small perturbation  $\delta\theta = (\delta\theta_1, \dots, \delta\theta_n)$  to the synchronization state can now be written as

$$\delta\dot{\theta}_i = - \sum_{j=1}^n L_{ij}^{(\text{mul})} \delta\theta_j, \quad (4)$$

in which the multiorder Laplacian

$$L_{ij}^{(\text{mul})} = \frac{1-\alpha}{\langle k^{(1)} \rangle} L_{ij}^{(1)} + \frac{\alpha}{\langle k^{(2)} \rangle} L_{ij}^{(2)}. \quad (5)$$

We then sort the eigenvalues of the multiorder Laplacian  $\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_{n-1} \geq \Lambda_n = 0$ . The Lyapunov exponents of Eq. (4) are simply the opposite of those eigenvalues. We set  $\lambda_i = -\Lambda_{n+1-i}$  so that  $0 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . The second Lyapunov exponent  $\lambda_2 = -\Lambda_{n-1}$  determines synchronization stability:  $\lambda_2 < 0$  indicates stable synchrony, and larger absolute values indicate a quicker recovery from perturbations.

We start by showing numerically the effect of  $\alpha$  (the proportion of coupling strength assigned to second-order interactions). By considering the two scenarios shown in Fig. 1, random hypergraphs and simplicial complexes, we find that these two canonical constructions exhibit opposite trends.

The construction of random hypergraphs is determined by two wiring probabilities  $p_1 = p$  and  $p_2 = p_\Delta$ : a  $d$ -hyperedge is created between any  $d+1$  of the  $n$  nodes with probability  $p_d$  [37]. Simplicial complexes are special cases of hypergraphs and have the additional requirement that if a second-order interaction  $(i, j, k)$  exists, then the

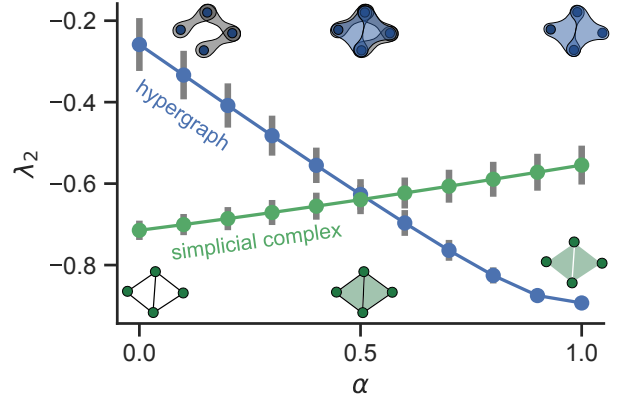


FIG. 1. Synchronization is enhanced by higher-order interactions in random hypergraphs but is impeded in simplicial complexes. The maximum transverse Lyapunov exponent  $\lambda_2$  is plotted against  $\alpha$  for random hypergraphs (blue) and simplicial complexes (green). As  $\alpha$  is increased, the coupling goes from first-order-only ( $\alpha = 0$ ) to second-order-only ( $\alpha = 1$ ). Each point represents the average over 50 independent hypergraphs or simplicial complexes with  $n = 100$  nodes. The error bars represent standard deviations. We set  $p = p_\Delta = 0.1$  for random hypergraphs and  $p = 0.5$  for simplicial complexes.

three corresponding first-order interactions  $(i, j)$ ,  $(i, k)$ , and  $(j, k)$  must also exist. We construct simplicial complexes by first generating an Erdős-Rényi graph with wiring probability  $p$ , and then adding a three-body interaction to every three-node clique in the graph [38]. One can also construct simplicial complexes from structures other than Erdős-Rényi graphs, such as small-world networks [39]. The results below are robust to the choice of different network structures.

Figure 1 shows that higher-order interactions impede synchronization in simplicial complexes, but improve it in random hypergraphs. For simplicial complexes, maximum transverse Lyapunov exponent  $\lambda_2$  increases with  $\alpha$  for all  $p$  (data shown for  $p = 0.5$  in Fig. 1). For random hypergraphs, the monotonic trend holds for  $p \simeq p_\Delta$ . For  $p$  significantly larger than  $p_\Delta$ , the curve becomes U-shaped, with a minimum at an optimal  $0 < \alpha^* < 1$ , as shown in Fig. 2.

The extreme values of the spectrum of a Laplacian can be related to the extreme values of the degrees of the associated graph:  $\lambda_n$  can be bounded by the maximum degree  $k_{\max}$  from both directions,  $\frac{n}{n-1} k_{\max} \leq |\lambda_n| \leq 2k_{\max}$  [40]; and  $\lambda_2$  can be bounded by the minimum degree  $k_{\min}$  from both directions,  $2k_{\min} - n + 2 \leq |\lambda_2| \leq \frac{n}{n-1} k_{\min}$  [41]. For the multiorder Laplacian, the degree  $k_i^{(\text{mul})}$  is given by the weighted average of degrees of different orders, in this case  $k_i^{(\text{mul})} = \frac{1-\alpha}{\langle k^{(1)} \rangle} k_i^{(1)} + \frac{\alpha}{\langle k^{(2)} \rangle} k_i^{(2)} = L_{ii}^{(\text{mul})}$ . In Fig. 2, we show that  $\frac{n}{n-1} k_{\min}$  is a good approximation for  $|\lambda_2|$  in random hypergraphs and is able to explain the U-shape observed for  $\lambda_2(\alpha)$ .

These degree-based bounds allow us to understand

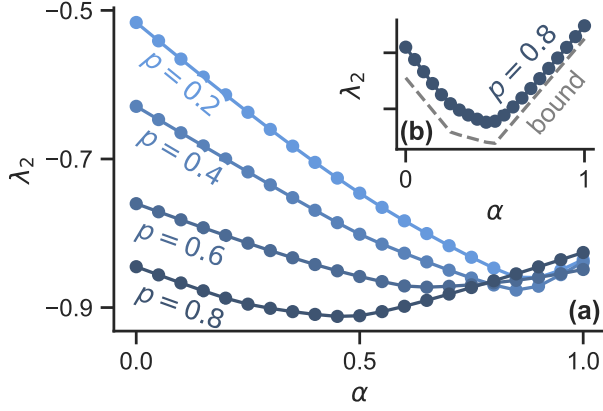


FIG. 2. Pairwise and non-pairwise interactions synergize to optimize synchronization. (a) U-shaped curves are observed for  $\lambda_2(\alpha)$  corresponding to random hypergraphs over a wide range of  $p$  values. (b) Degree-based bound  $|\lambda_2| \leq \frac{n}{n-1} k_{\min}$  predicts the non-monotonic dependence on  $\alpha$ . Each data point represents a 100-node random hypergraph and the three-body connection probability is set to  $p_{\Delta} = 0.05$ .

the opposite dependence on  $\alpha$  for random hypergraphs and simplicial complexes. For simplicial complexes, the reason for the deterioration of synchronization stability is the following: Adding 2-simplices to triangles makes the network more heterogeneous (degree-rich nodes get richer; well-connected parts of the network become even more highly connected), thus making the eigenvalues more spread out.

To quantify this rich-get-richer effect, we focus on simplicial complexes constructed from Erdős-Rényi graphs  $G(n, p)$ . In this case, we can derive the relationship between the first-order degrees  $k^{(1)}$  and second-order degrees  $k^{(2)}$ . If node  $i$  has first-order degree  $k^{(1)}$ , then there are at most  $\binom{k^{(1)}}{2}$  2-simplices that can potentially be attached to it. For example, when node  $i$  is connected to nodes  $j$  and  $k$ , then the 2-simplex  $\Delta_{ijk}$  is present if and only if node  $j$  is also connected to node  $k$ . Because the edges are independent in  $G(n, p)$ , we should expect about  $p \binom{k^{(1)}}{2}$  2-simplices attached to node  $i$ :

$$k^{(2)} \approx p \binom{k^{(1)}}{2} = p k^{(1)} (k^{(1)} - 1) / 2. \quad (6)$$

This quadratic dependence of  $k^{(2)}$  on  $k^{(1)}$  provides a foundation for the rich-get-richer effect. To further quantify how the degree heterogeneity changes going from the first-order interaction to the second-order interaction, we calculate the following heterogeneity ratio

$$r = \frac{k_{\max}^{(2)} / k_{\min}^{(2)}}{k_{\max}^{(1)} / k_{\min}^{(1)}}. \quad (7)$$

If  $r > 1$ , it means there is higher degree heterogeneity among 2-simplices than in the pairwise network, which

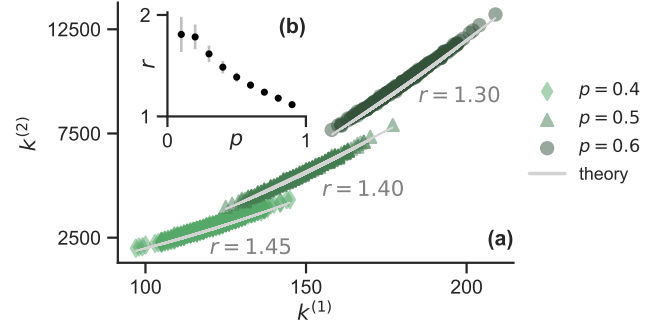


FIG. 3. Higher-order interactions increase degree heterogeneity in simplicial complexes. (a) First-order degrees  $k^{(1)}$  and second-order degrees  $k^{(2)}$  follow Eq. (6) for simplicial complexes constructed from Erdős-Rényi graphs. The heterogeneity ratio  $r$  is well approximated by Eq. (8). (b) Degree heterogeneity of the second-order Laplacian is larger than that of the first-order Laplacian in simplicial complexes for all values of  $p$ .

translates into worse synchronization stability in the presence of higher-order interactions. Plugging Eq. (6) into Eq. (7), we obtain

$$r \approx k_{\max}^{(1)} / k_{\min}^{(1)} \geq 1. \quad (8)$$

This shows that the coupling structure of 2-simplices is always more heterogeneous than 1-simplices for simplicial complexes constructed from Erdős-Rényi graphs. (We verified numerically that the same holds for simplicial complexes constructed from other network structures, including small-world and scale-free networks.) Moreover, the more heterogeneous is the pairwise network, the bigger the difference between first-order and second-order couplings in terms of heterogeneity. Specifically, because Erdős-Rényi graphs are more heterogeneous for smaller  $p$ , the heterogeneity ratio  $r$  is bound to decrease with  $p$ .

Figure 3(a) shows  $k^{(1)}$  vs.  $k^{(2)}$  for three simplicial complexes with  $n = 300$  and various values of  $p$ . The relationship between  $k^{(1)}$  and  $k^{(2)}$  is well predicted by Eq. (6). The heterogeneity ratio  $r$  is marked beside each data set and closely follows Eq. (8). Figure 3(b) shows  $r$  for  $n = 300$  and different values of  $p$ . The error bar represents standard deviation estimated from 1000 samples. The data confirm our prediction that  $r > 1$  for all considered simplicial complexes, and the difference in degree heterogeneity is most pronounced when the pairwise connections are sparse.

Next we turn to the case of random hypergraphs and explain why higher-order interactions promote synchronization in this case (assume  $p = p_{\Delta}$ ). For Erdős-Rényi graphs  $G(n, p)$ , the degree of each node is a random variable drawn from the binomial distribution  $B(k; n, p) = \binom{n}{k} p^k q^{n-k}$ , where  $\binom{n}{k}$  is the binomial coefficient and  $q = 1 - p$ . There are some correlations among the degrees, because if an edge connects nodes  $i$  and  $j$ , then it adds to

the degree of both nodes. However, the induced correlations are weak and the degrees can almost be treated as independent random variables for sufficiently large  $n$  (the degrees would be truly independent if the Erdős–Rényi graphs were directed). The distribution of the maximum degree for large  $n$  is given in Ref. [42]:

$$P\left(k_{\max}^{(1)} < pn + (2pqn \log n)^{1/2} f(n, y)\right) \approx e^{-e^{-y}}, \quad (9)$$

where  $f(n, y) = 1 - \frac{\log \log n}{4 \log n} - \frac{\log(\sqrt{2\pi})}{2 \log n} + \frac{y}{2 \log n}$ .

For generalized degrees  $k^{(2)}$ , the degree correlation induced by three-body couplings is stronger than the case of pairwise interactions, but it is still a weak correlation for large  $n$ . To estimate the expected value of the maximum degree, one needs to solve the following problem from order statistics: Given a binomial distribution and  $n$  independent random variables  $k_i$  drawn from it, what is the expected value of the largest random variable  $E[k_{\max}]$ ? Denoting the cumulative distribution of  $B(N, p)$  as  $F(N, p)$ , where  $N = (n-1)(n-2)/2$  is the number of possible 2-simplices attached to a node, the cumulative distribution of  $k_{\max}^{(2)}$  is simply given by  $F(N, p)^n$ . However, because  $F(N, p)$  does not have a closed-form expression, it is not easy to extract useful information from the result above.

To gain analytical insights, we turn to Eq. (9) with  $n$  replaced by  $N$ , which serves as an upper bound for the distribution of  $k_{\max}^{(2)}$ . To see why, notice that Eq. (9) gives the distribution of  $k_{\max}^{(1)}$  for  $n$  (weakly-correlated) random variables  $k_i^{(1)}$  drawn from  $B(n, p)$ . For  $k_{\max}^{(2)}$ , we are looking at  $n$  random variables  $k_i^{(2)}$  with slightly stronger correlations than  $k_i^{(1)}$ , now drawn from  $B(N, p)$ . Thus, Eq. (9) with  $n$  replaced by  $N$  gives the distribution of  $k_{\max}^{(2)}$  if one had more samples ( $N$  instead of  $n$ ) and weaker correlations. Both factors lead to an overestimation of  $E[k_{\max}]$ , but their effects are expected to be small.

To summarize, we have

$$P\left(k_{\max}^{(2)} < pN + (2pqN \log N)^{1/2} f(N, y)\right) > e^{-e^{-y}}. \quad (10)$$

Solving  $e^{-e^{-y_0}} = \frac{1}{2}$  gives  $y_0 \approx 0.52$ . Plugging  $y_0$  into the left hand side of Eqs. (9) and (10) yields an estimate of the expected values of  $k_{\max}^{(1)}$  and  $k_{\max}^{(2)}$ , respectively. Through symmetry, one can also easily obtain the expected values of  $k_{\min}^{(1)}$  and  $k_{\min}^{(2)}$ . To measure the degree heterogeneity, we can compute the heterogeneity indexes

$$h^{(1)} = (E[k_{\max}^{(1)}] - pn)/pn, \quad h^{(2)} = (E[k_{\max}^{(2)}] - pN)/pN, \quad (11)$$

which controls  $\lambda_2$  through degree-based bounds.

Now, how does the first-order and second-order degree heterogeneities compare against each other? Using

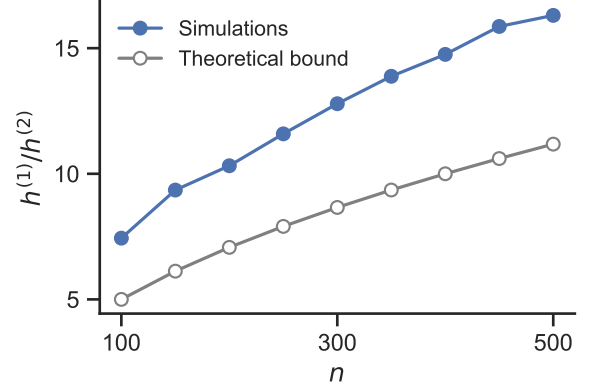


FIG. 4. Higher-order interactions decrease degree heterogeneity in random hypergraphs. The degree heterogeneity (measured by  $h^{(1)}$  and  $h^{(2)}$ ) is stronger in  $\mathcal{L}^{(1)}$  than in  $\mathcal{L}^{(2)}$ , and the difference increases with  $n$ . The theoretical lower bound of  $\frac{h^{(1)}}{h^{(2)}}$  is given by  $\frac{\sqrt{n}}{2}$  and is independent of  $p$ . The simulation results are obtained from random hypergraphs with different sizes  $n$ , using 500 samples for each  $n$ . The connection probabilities are set to  $p = p_{\Delta} = 0.1$ .

Eqs. (9) to (11), we see that

$$\frac{h^{(1)}}{h^{(2)}} > \frac{(2qn^{-1} \log n)^{1/2} f(n, y_0)}{(2qN^{-1} \log N)^{1/2} f(N, y_0)}. \quad (12)$$

For large  $n$ , we can assume  $f(n, y_0) \approx f(N, y_0) \approx 1$  and simplify Eq. (12) into

$$\frac{h^{(1)}}{h^{(2)}} > \frac{(n^{-1} \log n)^{1/2}}{(N^{-1} \log N)^{1/2}} \approx \frac{\sqrt{n}}{2}. \quad (13)$$

First, note that  $\frac{h^{(1)}}{h^{(2)}} > 1$  for almost all  $n$ , which translates into better synchronization stability in the presence of higher-order interactions. The scaling also tells us that as  $n$  is increased, the difference in degree heterogeneities becomes more pronounced. The theoretical lower bound [Eq. (13)] is compared to simulation results in Fig. 4, which show good agreement. Intuitively, the (normalized) second-order Laplacian has a much narrower spectrum compared to the first-order Laplacian with the same  $p$  because binomial distributions are more concentrated for larger  $n$  (i.e., there is much less relative fluctuation around the mean degree for  $k^{(2)}$  compared to  $k^{(1)}$ ).

To conclude, using simple phase oscillators, we have shown that higher-order interactions typically promote synchronization in random hypergraphs but impede it in simplicial complexes. We identify higher-order degree heterogeneity as the underlying mechanism driving the opposite trends. While we only considered two-body and three-body couplings, the same framework naturally extends to the case of larger group interactions. So far, most studies on higher-order interactions have treated the differences between simplicial complexes and hypergraphs as inconsequential and often make decisions about



which representation to use based on technical conveniences. Our results suggest that simplicial complexes and hypergraphs cannot always be used interchangeably and future work on higher-order interactions should consider the influence of the chosen representation when interpreting their results.

Do the lessons obtained here for phase oscillators carry over to more general oscillator dynamics? The generalized Laplacians used here have been shown to work for arbitrary oscillator dynamics and coupling functions [28]. Similarly, the spread of eigenvalues of each Laplacian carries critical information on the synchronizability of the corresponding level of interactions. Thus, once different orders of coupling functions have been properly normalized, we expect the findings here to transfer to systems beyond coupled phase oscillators. That is, for generic oscillator dynamics, higher-order interactions should promote synchronization if the hyperedges are more uniformly distributed than their pairwise counterpart. For the future, it would be interesting to generalize our results to systems whose interactions can be nonreciprocal [43–46].

We thank Alessio Lapolla, George Cantwell, Giovanni Petri, and Steve Strogatz for insightful discussions. Y.Z. acknowledges support from the Schmidt Science Fellowship. M.L. acknowledges partial support from Intesa Sanpaolo Innovation Center. The code for the analysis is available at [https://github.com/maximelucas/higherorder\\_sync\\_promoted](https://github.com/maximelucas/higherorder_sync_promoted).

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\* Y.Z. and M.L. contributed equally to this work.

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