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Searching for Optimal Network Topology with Best Possible Synchronizability

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Abstract

A large number of real-world complex networks or their sub-networks possess excellent dynamical properties such as high dynamic synchronizability, optimal controllability, strong resistance to attacks, fast information transmission capability, and natural emergence of cooperation in evolutionary games, etc., but existing network models are unable to well represent these intrinsic features and ubiquitous phenomena. This paper examines an optimal homogeneous network model which can well describe at least one of such optimal dynamical behaviors—the best possible synchronizability.

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1. Introduction

Regular (e.g., lattice and ring-shaped) networks, in which every node is connected to its nearest neighbors, have some common topological features such as homogeneous node-degree distribution, large clustering coefficient, and long average path-length. One significant progress in the study of complex networks is the introduction of the small-world network model [1] through rewiring links on a regular network. This model preserves a large clustering feature while considerably shortening the average path-length, however destroys the node-degree homogeneity. Meanwhile, to reproduce the typical social phenomenon of "rich gets richer" in network description, a scale-free network model was proposed [2], which has a fat-tailed (power-law) degree-distribution distinguishable from the bell-shaped (Poisson) distribution of random and small-world networks. These pioneering works have subsequently triggered a wide-spreading research interest and endeavor on complex networks in various fields of science and technology, revealing the theoretical significance and practical importance of network science and engineering, regarding particularly intentional attacks [3], virus spreading dynamics [4], [5], synchronization [6], [7], [8], [9], control [10], and evolutionary games [11], to name just a few.

Barabási asked [12], "Is there a chance that, despite their diversity, these dynamical processes share some common characteristics?" He said that "I suspect that such commonalities do exist; we just have not yet found the framework to unveil their universality." And he referred this to "the next frontier." Recently, Kitsak et al. reported [5] that "the best spreaders do not correspond to the most highly connected or the most central people, but are those located within the core of the network as identified by the *k*-shell decomposition analysis." A recent comment of Liu et al. [10] is that "sparse inhomogeneous networks are the most difficult to control, but that dense and homogeneous networks can be controlled using a few driver nodes."

It has been noticed that a large number of real-world complex networks or their sub-networks possess excellent dynamical properties such as high dynamic synchronizability, good controllability, strong resistance to attacks, fast information transmission capability, and natural emergence of cooperation in evolutionary games, etc., but existing network models are unable to well represent these intrinsic features and ubiquitous phenomena. Motivated by these studies and comments, this paper aims to establish an optimal homogeneous network model which can well describe at least one of such optimal dynamical behaviors—the best possible synchronizability.

Throughout this paper, a network always refers to a "simple and connected network," which means a connected, unweighted and undirected graph containing no self-loops nor multi-edges. More precisely, consider a dynamic network of N nodes interconnected in a certain topology described by

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}Hx_j, \quad i = 1, \cdots, N, \tag{1}$$

where $x_i \in R^n$ is the state vector of node i, $f(\cdot)$ is a (usually nonlinear) function satisfying the classic Lipschitz condition therefore the global dynamics of the above equation can be dominated by the summation term via appropriately choosing the constant coupling coefficient $c > 0$ and constant inner coupling matrix H, and the outer coupling matrix (known as the adjacency matrix) $A = [a_{ij}]$, in which $a_{ii} = 0$ and $a_{ij} = a_{ji}$, with $a_{ij} = 1$ if nodes i and j are connected but $= 0$ otherwise, $i,j = 1, \cdots, N$.

For the adjacency matrix $A = [a_{ij}]$ in the above dynamic network model, let $d_i = \sum_{j=1}^{N} a_{ij}$, $i = 1, \cdots, N$, and call $\{d_1, \cdots, d_N\}$ the *degree sequence*, with $D = \mathrm{diag}\{d_1, \cdots, d_N\}$. Then, the Laplacian matrix is defined by $L = D - A$, and its eigenvalues can be arranged as [13] $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. Here, the smallest nonzero eigenvalue λ_2 is called the *spectral gap* (or *algebraic connectivity*) and the *eigen-ratio* λ_2/λ_N to be discussed below is the reciprocal of the so-called condition number of the Laplacian matrix.

The mathematical notion of (*complete*) *synchronization* of the node states, denoted by $x_i(t)$, $i = 1, \cdots, N$, refers to the following asymptotic dynamical behavior:

$$\lim_{t\to\infty} \| x_i(t) - x_j(t) \| = 0, \quad i,j = 1, \cdots, N, \tag{2}$$

where $\| \cdot \|$ is the Euclidean norm. Physically, this means that, as time evolves, the dynamics of all node states approach each other asymptotically. This paper is concerned only with this type of (complete) synchronization. Obviously, the synchronizability of a dynamic network depends on the node dynamics, the coupling strength and the network topology.

It is now well known that a large class of dynamic networks have an unbounded synchronized region specified by $c\lambda_2 > \alpha_1 > 0$, where constant α_1 depends only on the node dynamics, and for a given α_1, a bigger spectral gap λ_2 implies a better network synchronizability, namely a smaller coupling strength is needed [6], [7], [8]; while another class of dynamic networks have a bounded synchronized region specified by $c\lambda_2/\lambda_N \in (\alpha_2, \alpha_3) \subset (0, \infty)$, where constants α_2, α_3 depend only on the node dynamics as well, and for two given α_2, α_3, a bigger eigen-ratio λ_2/λ_N implies a better network synchronizability which likewise means a smaller coupling strength is needed [9]. Thus, a natural question is what kind of network topologies have the best possible synchronizability when both the node dynamics and coupling strength are given and fixed.

This paper reports our new finding that an optimal solution likely lies in a special group of networks that have homogeneous node-degree sequence, node-girth

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and path-sum, which will be precisely defined later. Similarly to the small-world network model, this new type of homogeneous networks is constructed based on a regular network. However, differing from the small-world network, the new model preserves the same node-degree homogeneity while optimizing both node-girth and path-sum, thereby generating networks of different sizes all having the best possible synchronizability. This homogeneous network model is in sharp contrast to the heterogeneous scale-free network in that it follows a principle of reversely preferential attachment, which yields a fairly symmetrical and uniform topology in which "rich and poor are balanced."

In retrospect, in the pursuit of finding an appropriate network topology that has the best possible synchronizability, some sensible efforts have already been made in the past.

Noticeably, Nishikawa et al. [14] were the first to ask whether a small-world network is easier to synchronize? They found that networks with a homogeneous distribution of connectivity are more synchronizable than the heterogeneous ones, although the average distances of the former are generally large therefore not typically of small-world. Consequently, to reach synchronization more effectively, a balance between a small communication distance and a uniform load distribution is essential. Here, the load on a node is defined as the number of shortest paths that pass through this node. Yet, we found that a network with a uniform load distribution is not necessarily optimal (see Example 3 below).

Donetti et al. [15] proposed to seek an optimal solution by dynamically modifying the network topology while keeping the number of edges unchanged, whereby they introduced the concept of *entangled network* for minimizing the condition number of the Laplacian matrix. They also investigated the model properties and possible applications, and designed a modified simulated annealing algorithm for searching such entangled networks. However, using their objective function generally does not yield accurate optimal solutions. Moreover, they believed that a large spectral gap implies a small condition number, which however is not necessarily true (see Example 1 below). Despite the fact that some networks could be generated with a near-largest spectral gap [16], the problem of finding desirable entangled networks remains unsolved today.

Thereafter, Xuan et al. [17] continued the pursuit of finding entangled networks by further incorporating short average path-length and symmetry (node-degree homogeneity) into the original algorithm, and proposed a growing network model with homogeneous node degrees to approximate large-scale networks. They believed that if a symmetric network has a shortest average path-length then it would be an entangled network. Likewise, this sometimes may not be the case (see Example 2 below).

There are several other approximation schemes for searching entangled networks [18], [19], [20], but the found solutions are generally not entangled as has been commonly noticed before.

Motivated by the aforementioned ideas and works, our present study aims to search for a solution to the problem of finding a suitable network topology that likely has the best possible synchronizability. In the following section, some useful concepts and definitions are given, with three examples for illustration. The next section describes the problem under investigation and develops the main algorithm, with some theoretical results reported. The last section provides some concluding remarks with detailed discussions, and posts three conjectures for future research.

II. Definitions and Illustrative Examples

For network (1), the following concepts are important and will be useful:

(i) *node degree*—number of edges of node i, denoted by d_i; the node-degree sequence is $\{d_1, \cdots, d_N\}$, and the number $\bar{k} = \sum_{i=1}^{N} d_i/N$ is called the average degree of the network.

(ii) *node girth*—number of edges within a shortest loop at node i, denoted by g_i; the minimum of the node-girth sequence, denoted $g = \min\{g_1, \cdots, g_N\}$, is called the girth of the network.

(iii) *node path-sum*—total number of edges from other nodes to node i through shortest paths, denoted by ℓ_i; the number $\bar{\ell} = \sum_{i=1}^{N} \ell_i/[N(N-1)]$ is called the average path-length of the network.

If a network satisfies $d_1 = \cdots = d_N$, $g_1 = \cdots = g_N$ and $\ell_1 = \cdots = \ell_N$ simultaneously, then it is called a (k, g, ℓ) *-homogeneous network*. Furthermore, if such a homogeneous network has a maximum g^* and a minimum ℓ^*, then it is referred to as a *possible optimal network*. It is noticed that some non-homogenous networks may satisfy both $g = \min\{g_1, \cdots, g_N\} = g^*$ and $\bar{\ell} < \ell^*/(N-1)$; if so, they are referred to as *near homogeneous networks*. For these two types of networks, a network with maximum eigen-ratio is called an *optimal homogeneous network* (see Fig. 1). It is our *conjecture* that optimal homogeneous networks have the best possible synchronizability among all networks of the same size (with same number of nodes and same number of edges).

The following are three illustrative examples of homogeneous networks with average degree $\bar{k} = 3$, where a network is denoted by $C_m \cup \{i-j, \cdots\}$, in which C_m is a cycle subgraph of m nodes and $\{i-j, \cdots\}$ are the rest edges in the network.

Example 1. Consider two networks with 8 nodes (see Fig. 2).

Network 1: $G_1 = C_8 \cup \{1-4, 2-7, 3-6, 5-8\}$; node-degree $=3$, girth $=4$, path-sum $=12$, $\lambda_2 = 2.000$, $\lambda_2/\lambda_8 = 0.333$.

Network 2: $G_2 = C_8 \cup \{1-5, 2-6, 3-7, 4-8\}$; node-degree $=3$, girth $=4$, path-sum $=11$, $\lambda_2 = 2.000$, $\lambda_2/\lambda_8 = 0.369$.

Although both Network 1 and Network 2 are (k, g, ℓ)-homogeneous with same girth and same spectral gap, Network 2 has a smaller path-sum therefore according to [9] it is better than Network 1 in the sense of having a better synchronizability.

Example 2. Consider two networks with 18 nodes (see Fig. 3).

Network 1: $G_1 = C_{18} \cup \{1-9, 2-13, 3-16, 4-10, 5-14, 6-18, 7-12, 8-15, 11-17\}$; node-degree $=3$, girth $=6$, average path-length $\bar{\ell} = 2.300$, $\lambda_2 = 1.134$, $\lambda_2/\lambda_{18} = 0.200$.

Network 2: $G_2 = C_{18} \cup \{1-6, 2-9, 3-14, 4-11, 5-16\ 7-12, 8-15, 10-17, 13-18\}$; node-degree $=3$, girth $=6$, path-sum $=41$, $\lambda_2 = 1.268$, $\lambda_2/\lambda_{18} = 0.211$.

Although $\bar{\ell}_1 = 2.300 < \bar{\ell}_2 = 41/17 = 2.412$, the path-sum of Network 2 is homogeneous, and according to both [6], [7], [8] and [9] it is better than Network 1 in the sense of having a better synchronizability.

Example 3. Consider two networks with 26 nodes (see Fig. 4).

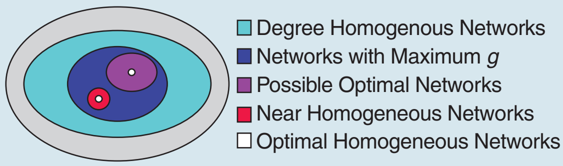

Figure 1. Diagram of the searching algorithm. Gray: feasible solutions; green: degree homogenous networks; blue: networks with maximum g; pink: possible optimal networks; red: near homogeneous networks; and white: optimal homogeneous networks.

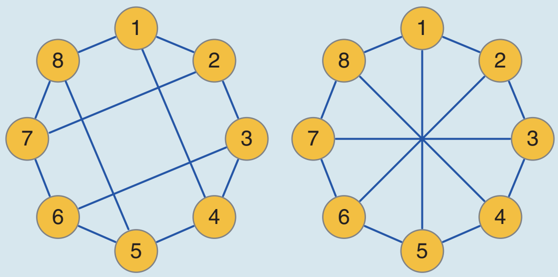

Figure 2. Two networks with 8 nodes.

Network 1: $G_1 = C_{25} \cup \{1-20, 2-9, 4-16, 5-11, 6-25, 7-14, 8-18, 10-22, 12-19, 15-21, 17-24\} \cup \{3-26, 13-26, 23-26\}$; node-degree $=3$, girth $=7$, path-sum $=67$, $\lambda_2 = 0.947$, $\lambda_2/\lambda_{18} = 0.172$.

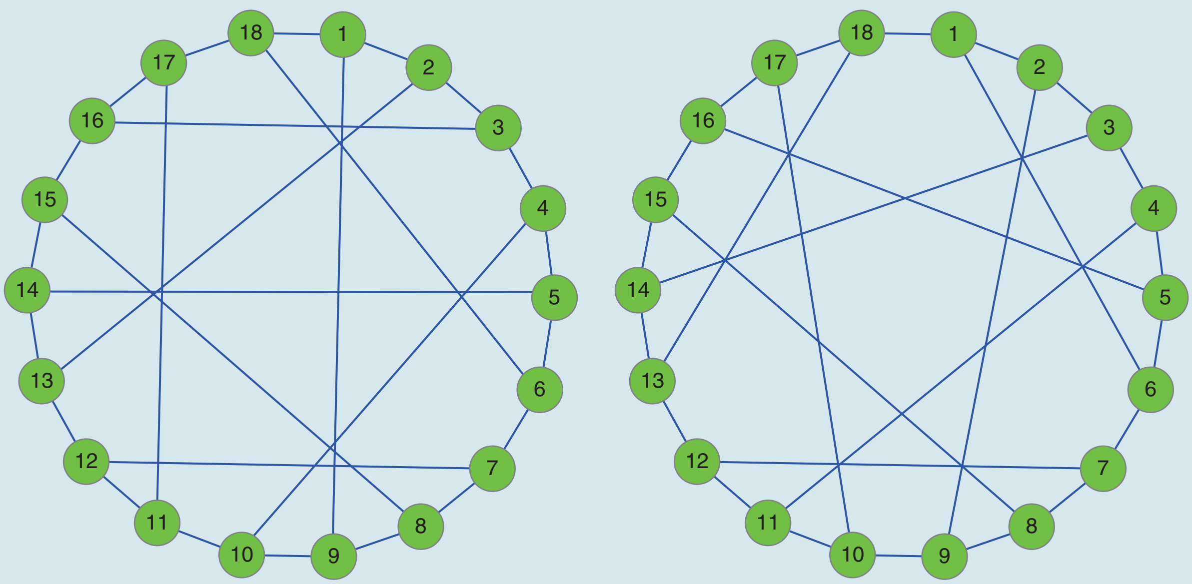

Figure 3. Two networks with 18 nodes.

Network 2: $G_2 = C_{24} \cup \{1-13, 2-9, 3-20, 5-12, 6-23, 7-19, 8-15, 11-18, 14-21, 17-24\} \cup \{4-25, 10-26, 16-25, 22-26, 25-26\}$; node-degree $= 3$, girth $= 7$, path-sum $= 67$, $\lambda_2 = 1.00$, $\lambda_2/\lambda_{18} = 0.174$.

Both Network 1 and Network 2 are the same (k, g, ℓ)-homogeneous networks, although the node-load distribution of Network 1 is uniform but that of Network 2 is not (see Table 1), so according to [6], [7], [8] and [9] Network 2 is better than Network 1 regarding their synchronizability.

It is noted that such homogeneous networks exist in real brain neuronal systems. A particular case reported is one consisting of 16 neurons [21], which is essentially homogeneous [see Fig. 5(a)].

Another example is the 3-core sub-network [see Fig. 5(b)], which can be obtained by k-shell decomposition. An advantage of such homogeneous cores is that they have more significant influence than the hub nodes or the high-betweenness nodes on epidemic spreading dynamics [5].

III. The Problem, Algorithms and Results

A. The Problem: Maximizing the Eigen-Ratio

Note that a bigger spectral gap does not necessarily imply a bigger eigen-ratio, which will be discussed elsewhere later, in the following study of optimal synchronizability, only the eigen-ratios namely only those dynamic networks with bounded synchronized regions are discussed, under the condition of keeping the number of nodes and the number of edges both unchanged.

Let A^* be the family of $N \times N$ adjacency matrices, and $x = (x_1, \cdots, x_N)$, $e = (1, \cdots, 1)^T \in R^N$. Based on matrix theory [13], the problem under investigation is formulated as the following constrained optimization:

$$\max_{A \in A^*} \lambda_2/\lambda_N \quad \text{such that} \quad \sum_{i=1}^{N} d_i = N\bar{k} \quad \text{and} \quad \lambda_2 > 0.$$

Here, notice that $\lambda_2 = 0$ if and only if the network is not connected therefore the constraint of $\lambda_2 > 0$ guarantees the network's connectivity, $\sum_{i=1}^{N} d_i = N\bar{k}$ guarantees the matrices be Laplacian under the present framework of investigation, and recall that

$$\lambda_2/\lambda_N = \min_{x^T e = 0, x \neq 0} \frac{x^T [D - A] x}{x^T x} \Bigg/ \max_{x^T e = 0, x \neq 0} \frac{x^T [D - A] x}{x^T x}.$$

This constrained optimization problem is known to be computationally intractable. To solve this difficult problem, a heuristic and constructive algorithm is proposed in the following subsections (see Fig. 1).

B. Building Networks with Node-Degree Homogeneity

According to the comparisons among regular, random and small-world networks given in [22], and in view of the simulation results on node-degree sequences and spectra of various networks reported in [23], networks with node-degree homogeneity typically have bigger eigen-ratios among all networks of the same sizes. Based on this observation, a random network model for growing node-degree homogeneous networks is proposed as follows.

First, it is assumed that the average degree $\bar{k}$ is an integer; otherwise, node-degree homogeneous networks do not exist. The network-generation algorithm consists of four steps.

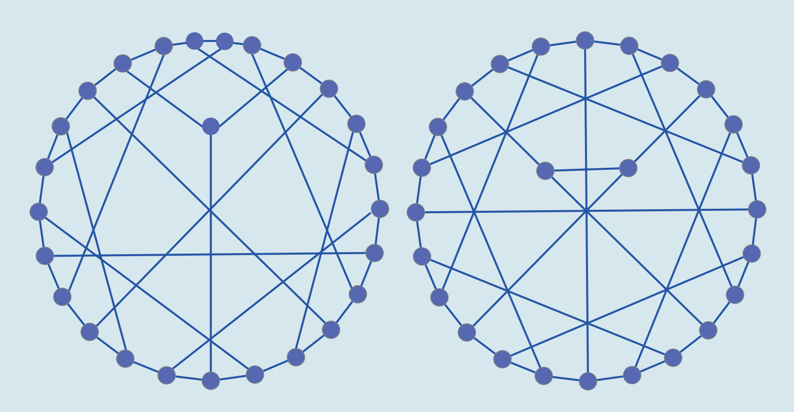

Figure 4. Two networks with 26 nodes.

Table 1.
The node-lord distribution of Network 2.

Node	1	2	3	4	5	6	7	8	9	10	11	12	13
Load	54	66	66	66	66	66	54	66	66	66	66	66	54
Node	14	15	16	17	18	19	20	21	22	23	24	25	26
Load	66	66	66	66	66	54	66	66	66	66	66	54	54

Step A1. Start from a complete graph of $\bar{k}+1$ nodes.

Step A2. If $\bar{k}$ is odd, then in each step add 2 connected nodes into the existing network, where each node brings $(\bar{k}-1)/2$ new edges into the network; if $\bar{k}$ is even, then in each step add 1 node which brings in $\bar{k}/2$ new edges.

Step A3. All newly introduced edges are connected to the existing network following a principle of reversely preferential attachment, namely choosing a node of smallest degree to connect with. If all existing nodes have the same degree, then simply perform random attachment.

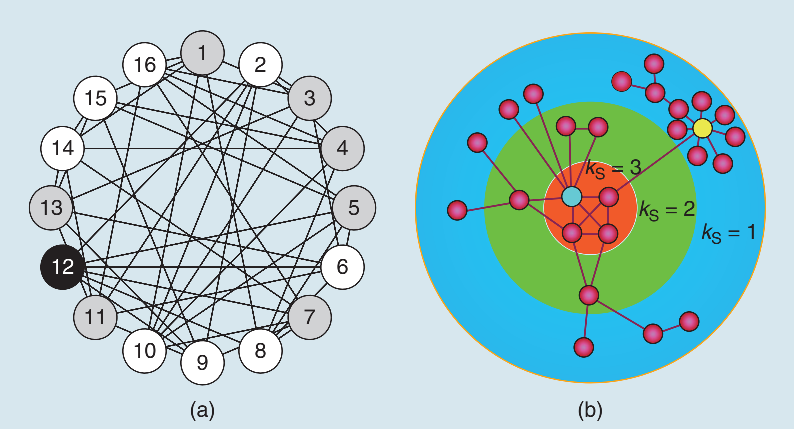

Figure 5. (a) Fig. 62 in [21]. (b) Fig. 1(a) in [5].

Step A4. From among all nodes with degrees larger than $\bar{k}$, rewire every of their edges (except the newly added ones) in the following manner: compare the degrees of two end-nodes of the chosen edge, disconnect the edge from the larger node, and then re-direct this edge-end to connect to a node with smallest degree in the network. Keep all graphs so generated if the results are not unique.

This model can generate random networks with node-degree homogeneity.

At this point, it is interesting to note that the "optimality" so obtained is consistent with the report [10] that the number of driver nodes tends to zero for such random networks that have optimal controllability.

C. Generating Networks with Longest Girths

For notational and computational simplicity, only networks with average degree $\bar{k}=3$ are discussed and presented herein.

Through extensive numerical calculations, we found that among 22 networks with 12 nodes for instance, 20 networks with girth 4 have maximum $\lambda_2 = 1.268$ and maximum $\lambda_2/\lambda_{12} = 0.234$, 2 networks with girth 5 have maximum $\lambda_2 = 1.468$ and maximum $\lambda_2/\lambda_{12} = 0.277$ (see Table 2), and so on.

Based on the above numeral simulation results, a new search algorithm for generating networks with longest girths is proposed as follows:

Step B1 (Ordering). List all random node-degree homogeneous networks generated in *Steps A1–A4* according to their girths in decreasing order; if girths are equal, then list their average path-lengths in increasing order; if path-lengths are also equal, then list their automorphisms [24] in decreasing order.

Step B2 (Reducing). If two networks have same girth sequences, path-sum sequences and automorphism numbers, then they are considered to be isomorphic, thereafter only one is kept for the following iterations.

Step B3 (Iterating). Starting from the first network in the above listed network candidates, go back to *Steps A2–A4*, until reaching the end of the list or meeting a pre-set stopping rule (e.g., 999 networks in the list that have been visited), so as to obtain all networks with the longest girths. A new list of networks is now available, with all network sizes increased. Iteration stops when a prescribed network size is reached.

The above search algorithm finds all networks with the longest girths.

Some examples of small-sized networks are summarized in Table 3 (see also [25]).

Table 2.
Spectral gaps and eigen-ratios for 22 networks with 12 nodes.

No.	1	2	3	4	5	6	7	8	9	10	11
g^*	4	4	4	4	4	4	4	4	4	4	4
λ_2	0.438	0.677	0.764	0.776	1.000	1.000	1.033	1.000	1.000	1.000	1.000
λ_2/λ_N	0.073	0.119	0.127	0.135	0.180	0.174	0.182	0.167	0.174	0.174	0.167
No.	12	13	14	15	16	17	18	19	20	21	22
g^*	4	4	4	4	4	4	*4*	4	4	5	**5**
λ_2	1.035	1.000	1.066	1.097	1.184	1.186	*1.268*	1.268	1.268	1.438	**1.468**
λ_2/λ_N	0.181	0.180	0.188	0.192	0.209	0.213	*0.234*	0.223	0.211	0.259	**0.277**

Table 3.
Number (Σ) of networks with the longest girth.

N	4	6	8	10	12	14	16	18	20	22	24	26	28	30
g	3	4	4	5	5	6	6	6	6	6	7	7	7	8
Σ	1	1	**2**	1	**2**	1	1	**5**	**32**	**385**	1	**3**	**21**	1

Table 4.
Number (Σ) of networks with minimum ℓ^* and $\bar{\ell} < \ell^*/(N-1)$.

N	8	12	18	20	22	26	28
(ℓ^*, Σ)	1	2	3	1	1	2	9
$(\bar{\ell} < \ell^*/(N-1), \Sigma)$	0	0	1	1	1	0	0

D. Finding Optimal Homogeneous Networks

The last routine is now designed, as follows.

Step C1. For each network with the longest girth obtained above, calculate the path-sum of every node, the network diameter d, and the average path-length of the whole network.

Step C2. Check the homogeneities of girth and path-sum for every network obtained above. Keep those homogeneous networks with minimum ℓ^* and those nonhomogeneous with $\bar{\ell} < \ell^*/(N-1)$ as candidates for the next step. (Results of the example are given in Table 4.)

Step C3. Calculate the eigen-ratios of all the candidates and find the biggest one among them; this yields an optimal homogeneous network.

E. Results of Small-Sized Examples

The above iterative scheme finds optimal homogenous networks of 4, 6, $\cdots$, 30 nodes with average degree $\bar{k} = 3$. (see Fig. 6). Related data are summarized in Table 5.

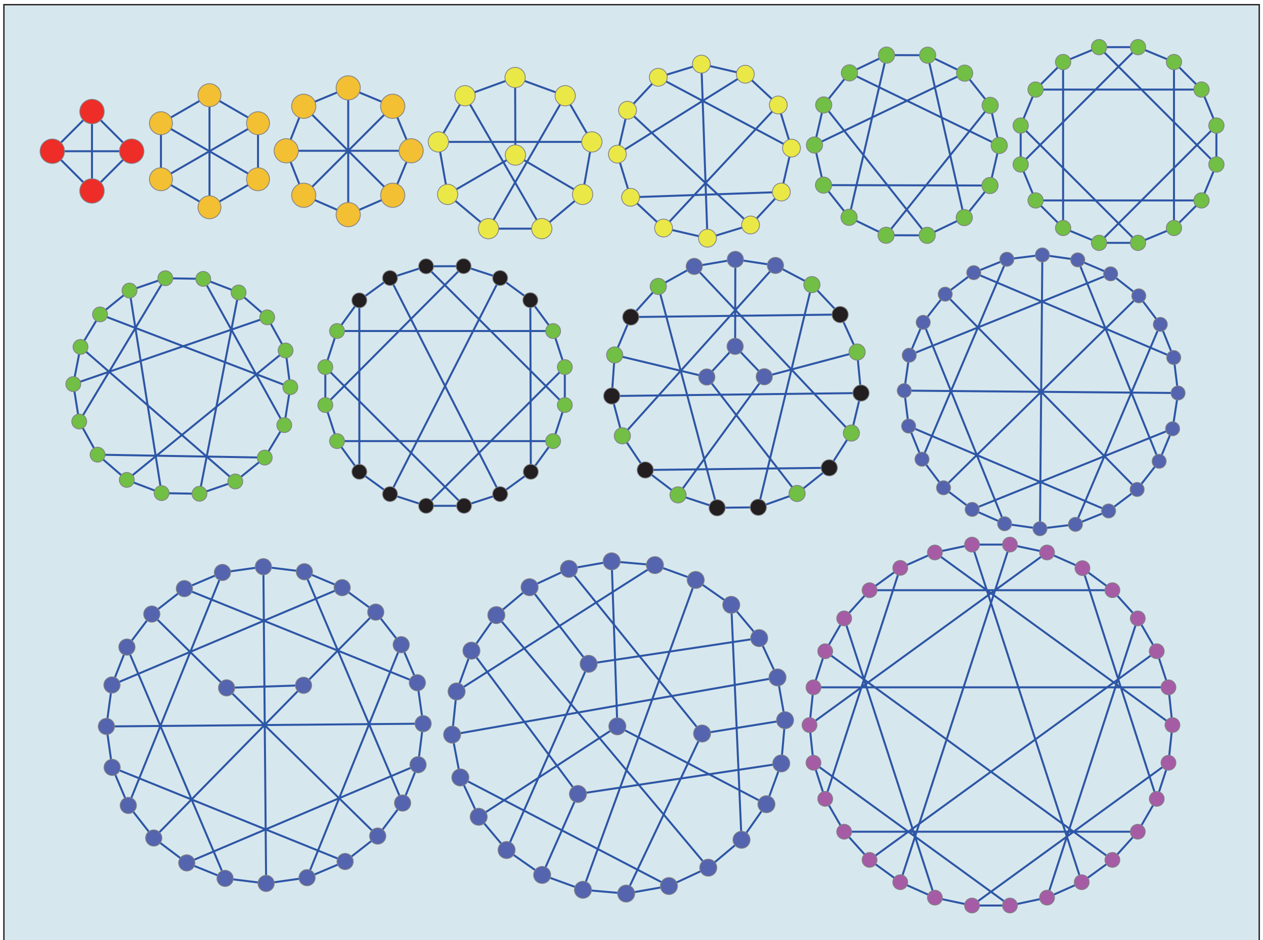

Figure 6. Figures of optimal homogenous networks of 4, 6, $\cdots$, 30 nodes with average degree $\bar{k} = 3$.

Table 5.
Data of optimal homogeneous networks.

N, d	4, 1	6, 2	8, 2	10, 2	12, 3	14, 3	16, 4
(g^*, ℓ^*)	(3,3)	(4,7)	(4,11)	(5,15)	(5,21)	(6,27)	(6,34)
λ_2	4.000	3.000	2.000	2.000	1.468	1.586	1.268
λ_2/λ_N	1.000	0.500	0.369	0.400	0.264	0.277	0.211
N, d	18, 4	20, 4	22, 4	24, 4	26, 4	28, 4	30, 4
(g^*, ℓ^*)	(6,41)	(6,*)	(*,*)	(7,59)	(7,67)	(7,75)	(8,83)
λ_2	1.268	1.065	1.000	1.000	1.000	1.000	1.000
λ_2/λ_N	0.211	0.189	0.180	0.180	0.174	0.185	0.167

Obviously, the path-sum satisfies $\ell^* = 1\times 3 + 2\times 6 + \cdots + d\times m$, where d is the network diameter and m is a smallest possible integer that satisfies $3 + 6 + \cdots + m = N - 1$. Note, however, that there are not networks with path-sums $\ell^* = 1\times 3 + 2\times 6 + 3\times 7 + 4\times 1 = 40$, $\ell^* = 1\times 3 + 2\times 6 + 3\times 9 + 4\times 1 = 46$, $\ell^* = 1\times 3 + 2\times 6 + 3\times 11 + 4\times 1 = 52$ in networks with 18, 20, 22 nodes, respectively. Thus, networks with $\bar{\ell} < \ell^*/(N-1)$ are included in networks with 18, 20, 22 nodes, respectively.

It can be observed from Table 5 that there are (only) two near homogeneous networks, where networks marked with $*$ means that their node-girth or path-sum is not homogeneous, but all the other networks are truly homogeneous. Specifically, the network with 20 nodes has homogeneous girth but not path-sum, in which 12 black nodes have path-sum 45 and the other 8 nodes have path-sum 46 with average path-length 2.389; the network with 22 nodes do not have homogenous girth and path-sum, in which 6 nodes have girth 7 with path-sum 51, and 16 nodes have girth 6, where one half have path-sum 52 and another half have 53, while it has average path-length 2.481.

IV. Conclusions and Discussions

The Laplacian matrix represents the topological structure of a network therefore determines the network synchronizability. In the pursuit of finding an optimal topology that likely yields networks with the best possible synchronizability, the earlier attempt introduces a good mathematical definition of "entangled network" which yet remains conceptually unclear today, and "symmetrical network" usually can represent approximately but not exactly an optimal solution. In contrast, the concept of "optimal homogeneous network" introduced in this paper has clearer physical meanings and we conjectured that optimal homogeneous networks are networks with the best possible synchronizability. The 14 networks shown in Fig. 6 have literally confirmed this conjecture for small-sized networks.

Reportedly several topological characteristics may significantly enhance the network synchronizability, especially the homogeneous node-degree distribution, short average path-length, small average clustering coefficient, and low betweenness centrality. In this paper, we have presented a new finding that among all networks with node-degree homogeneity, the ones with the longest girths are the best candidates. By precisely verifying small-sized networks with average degree $\bar{k} = 3$, we have shown that optimal homogeneous networks have the best synchronizability as compared to all other types of networks of the same size (with same number of nodes and same number of edges).

The optimal homogeneous networks described and obtained in this paper have many good properties, as have been verified by our numerical analysis with extensive simulations:

- Compared with regular networks, random networks, small-world networks and scale-free networks, optimal homogeneous networks have the best possible synchronizability.
- Compared with scale-free networks which are fragile to intentional attacks [3], optimal homogeneous networks are robust.
- Compared with scale-free networks with a power-law exponent $\gamma \leq 3$, which have near-zero phase-transition thresholds therefore are easy to facilitate epidemic spreading [4], optimal homogeneous networks have relatively large threshold values due to their node-degree homogeneity thereby providing better structures for preventing epidemics from rapid spreading.
- Comparing to regular, random-graph, small-world and scale-free networks, optimal homogeneous networks can easily emerge cooperation in prisoners game.
- Since the first-arrival time of a random walk on a network is proportional to the sum of the

Table 6.
Number Σ of networks with the biggest spectral gaps.

N	8	12	18	20	22	26	28
$\max\{\lambda_2(D)\}$	2.000	1.468	1.268	1.065	1.000	1.000	1.000
Σ	**2**	1	1	**3**	**4**	1	1

Table 7.
Number Σ of networks with the largest number of spanning trees versus number T of spanning trees.

N	8	20	22
Σ	1	1	1
T	3136	128440000	719634432

reciprocals of the Laplacian eigenvalues of the network [16], optimal homogeneous networks typically have very short first-arrival time due to their large spectral gap $\lambda_2 > 0$.

- Very recently, it was found that the controllability of complex networks have close relation to the node-degree homogeneity of the networks [10], which is consistent with our finding reported in this paper.

All in all, optimal homogeneous networks, or their sub-networks, appear to be an ideal structure for both theoretical and practical considerations regarding synchronization and some other desirable properties.

It is noticeable that the concept of optimal homogeneous network introduced in this paper has intrinsic relations with many important mathematical problems. Regarding graph theory, in particular, it is closely related to Hamiltonian graphs, (k, g)-cage graphs, Caley graphs, Ramanujan graphs and graph automorphism, to name a few. All these graph-theoretic issues are important but also challenging in mathematical theory as well as technological applications, leaving many interesting and important problems for future studies, such as the following:

Conjecture 1. In all networks with the same size $N\bar{k}$, where $\bar{k}$ is an integer, let $\{H\}$ be all networks with homogeneous node-degree distributions, and $\{G\}$ be the rest of networks. Then, $\max\lambda_2(\{G\}) \le \max\lambda_2(\{H\})$ and $\max\{\lambda_2(\{G\})/\lambda_N(\{G\})\} \le \max\{\lambda_2(\{H\})/\lambda_N(\{H\})\}$.

Conjecture 2. Within $\{H\}$, defined above, let $\{D\}$ be all networks with the longest girths, and $\{G\}$ be the rest of networks in $\{H\}$. Then, $\max\lambda_2(\{G\}) \le \max\lambda_2(\{D\})$ and $\max\{\lambda_2(\{G\})/\lambda_N(\{G\})\} \le \max\{\lambda_2(\{D\})/\lambda_N(\{D\})\}$.

Conjecture 3. Within $\{D\}$, defined above, let $\{S\}$ be its subclass with the biggest spectral gaps, and let $\{T\}$ be all networks with the largest numbers of spanning trees in $\{S\}$. Moreover, let $\{G\}$ be the rest of networks in $\{S\}$. Then, $\max\{\lambda_2(\{G\})/\lambda_N(\{G\})\} \le \max\{\lambda_2(\{T\})/\lambda_N(\{T\})\}$.

For small-sized examples, $\{S\}$ and $\{T\}$ can be easily obtained, as summarized in Table 6 and Table 7.

As an example, among networks with 22 nodes, there are 7,319,447 networks in $\{H\}$, with only 385 networks in $\{D\}$, 4 networks in $\{S\}$, and 1 network in $\{T\}$, respectively.

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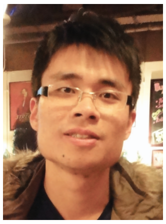

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