Week 9 Probability

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Elementary Probability

Sample space:

$$\Omega = \{\omega_1, \ldots, \omega_n\}$$

Each point represents an outcome, each outcome ω_i equally likely:

$$P(\omega_1) = P(\omega_2) = \ldots = P(\omega_n) = \frac{1}{n}$$

This a called a **uniform probability distribution** over Ω

Non-uniform Probability

Slight modification is needed to define an arbitrary (in general non-uniform) probability distribution:

$$\Omega = \{\omega_1, \ldots, \omega_n\}$$

Let

$$P(\omega_1) = p_1, P(\omega_2) = p_2, \ldots, P(\omega_n) = p_n$$

Then

$$\sum_{i=1}^{n} p_i = 1$$

Computing Probabilities by Counting

Computing probabilities with respect to a *uniform* distribution comes down to counting the size of the event.

If $E = \{e_1, \ldots, e_k\}$ then

$$P(E) = \sum_{i=1}^{k} P(e_i) = \sum_{i=1}^{k} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

Inclusion-Exclusion and Probability

Inclusion-Exclusion is a very common method for deriving probabilities from other probabilities.

Two sets

Inclusion-Exclusion and Probability

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Two sets

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Three sets

$$\begin{aligned}
P(A \cup B \cup C) &= P(A \cup B) + P(C) - P((A \cup B) \cap C) \\
&= P(A) + P(B) - P(A \cap B) + P(C) \\
&\quad - P((A \cap C) \cup (B \cap C)) \\
&= P(A) + P(B) - P(A \cap B) + P(C) \\
&\quad - (P(A \cap C) + P(B \cap C) - P(A \cap C \cap B \cap C)) \\
&= P(A) + P(B) + P(C) \\
&\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\
&\quad + P(A \cap B \cap C)
\end{aligned}$$

Conditional Probability

Definition

Conditional probability of E **given** S:

$$P(E|S) = \frac{P(E \cap S)}{P(S)}, \quad E, S \subseteq \Omega$$

It is defined only when $P(S) \neq 0$

If P is the uniform distribution over a finite set Ω, then

$$P(E|S) = \frac{\frac{|E \cap S|}{|\Omega|}}{\frac{|S|}{|\Omega|}} = \frac{|E \cap S|}{|S|}$$

Some General Rules

Fact

- $A \subseteq B \Rightarrow P(A|B) \geq P(A)$
- $A \subseteq B \Rightarrow P(B|A) = 1$
- $P(A \cap B|B) = P(A|B)$
- $P(\emptyset|A) = 0$ *for* $A \neq \emptyset$
- $P(A|\Omega) = P(A)$
- $P(A^c|B) = 1 - P(A|B)$

NB

- $P(A|B)$ and $P(A|B^c)$ are not related
- $P(A|B), P(B|A), P(A^c|B^c), P(B^c|A^c)$ are not related

NB

Bayes' Formula: $P(S|B) \cdot P(B) = P(B|S) \cdot P(S)$

Stochastic Independence

Definition

A and B are **stochastically independent** (notation: $A \perp B$) if $P(A \cap B) = P(A) \cdot P(B)$

If $P(A) \neq 0$ and $P(B) \neq 0$, all of the following are *equivalent*:

- $P(A \cap B) = P(A)P(B)$
- $P(A|B) = P(A)$ (i.e. B does not affect the probability of A)
- $P(B|A) = P(B)$ (i.e. A does not affect the probability of B)
- $P(A^c|B) = P(A^c)$ or $P(A|B^c) = P(A)$ or $P(A^c|B^c) = P(A^c)$

The last one means that

$$A \perp B \Leftrightarrow A^c \perp B \Leftrightarrow A \perp B^c \Leftrightarrow A^c \perp B^c$$

Example

$\boxed{\text{5.6.38 (supp)}}$ Of 100 problems, 75 are 'easy' and 40 'important'.
(b) n problems chosen randomly. What is the probability that all n are important?

$$p = \frac{\binom{40}{n}}{\binom{100}{n}} = \frac{40 \cdot 39 \cdots (41-n)}{100 \cdot 99 \cdots (101-n)}$$

Exercise

$\boxed{5.2.3}$ A 4-letter word is selected at random from Σ^4, where $\Sigma = \{a, b, c, d, e\}$. What is the probability that

(a) the letters in the word are all distinct?
(b) there are no vowels ("a", "e") in the word?
(c) the word begins with a vowel?

Exercise

$\boxed{5.2.5}$ An urn contains 3 red and 4 black balls. 3 balls are removed without replacement. What are the probabilities that

(a) all 3 are red
(b) all 3 are black
(c) one is red, two are black

$\boxed{5.2.11}$ Two dice, a red die and a black die, are rolled.
What is the probability that
(a) the sum of the values is even?
$P(R + B \in \{2, 4, \ldots, 12\}) = \frac{18}{36} = \frac{1}{2}$

(b) the number on the red die is bigger than on the black die?
$P(R > B) = P(R < B)$; also $P(R = B) = \frac{1}{6}$
Therefore $P(R < B) = \frac{1}{2}(1 - P(R = B)) = \frac{5}{12}$

(c) the number on the black die is twice the one on the red die?
$P(R = 2 \cdot B) = P(\{(2,1),(4,2),(6,3)\}) = \frac{3}{36} = \frac{1}{12}$

$\boxed{5.2.12}$ (a) the maximum of the numbers is 4? $P(E_1) = \frac{7}{36}$
(b) their minimum is 4? $P(E_2) = \frac{5}{36}$

Check:

$P(E_1 \cup E_2) = \frac{7}{36} + \frac{5}{36} - P(E_1 \cap E_2) = \frac{7+5-1}{36} = \frac{11}{36}$
$P(\text{at least one '4'}) = 1 - P(\text{no '4'}) = 1 - \frac{5}{6} \cdot \frac{5}{6} = \frac{11}{36}$

A four-digit number n is selected at random (i.e. randomly from $[1000 \ldots 9999]$). Find the probability p that n has each of 0, 1, 2 among its digits.

Probability of Sequential Outcomes

Example

Team A has probability $p = 0.5$ of winning a game against B. What is the probability P_p of A winning a best-of-seven match if
(a) A already won the first game?
(b) A already won the first two games?
(c) A already won two out of the first three games?

(a) Sample space S — 6-sequences, formed from wins (W) and losses (L)

$$|S| = 2^6 = 64$$

Favourable sequences F — those with three to six W

$$|F| = \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 20 + 15 + 6 + 1 = 42$$

Therefore $P_{0.5} = \frac{42}{64} \approx 66\%$

Use of Recursion in Probability Computations

Question

Given n tosses of a coin, what is the probability of two HEADS *in a row? Compute for $n = 5, 10, 20, \ldots$*

Question

Given n tosses, what is the probability q_n of at least one HHH?

Question

A coin is tossed 'indefinitely'. Which pattern is more likely (and by how much) to appear first, HTH or HHT?

Question

Two dice are rolled repeatedly. What is the probability that '6–6' will occur before two consecutive (back-to-back) 'totals seven'?

NB

The probability of either occurring at a given roll is the same: $\frac{1}{36}$.

$\boxed{9.1.6}$ A coin is tossed four times. What is the probability of
(a) two consecutive HEADS
(b) two consecutive HEADS *given* that ≥ 2 tosses are HEADS

$\boxed{9.1.12}$ What is the probability of a flush given that all five cards in a Poker hand are red?

Two dice are rolled and the outcomes recorded as b for the black die, r for the red die and $s = b + r$ for their total.
Define the events $B = \{b \geq 3\},\ R = \{r \geq 3\},\ S = \{s \geq 6\}$.

$\boxed{9.1.9}$ Consider three red and eight black marbles; draw two without replacement. We write b_1 — Black on the first draw, b_2 — Black on the second draw, r_1 — Red on first draw, r_2 — Red on second draw
Using conditional probabilities, find the probabilities
(a) both Red:

$$P(r_1 \wedge r_2) = P(r_1)P(r_2|r_1) = \frac{3}{11} \cdot \frac{2}{10} = \frac{3}{55}$$

Equivalently:
$|\text{two-samples}| = \binom{11}{2} = 55$; $|\text{Red two-samples}| = \binom{3}{2} = 3$
$P(\cdot) = \frac{\binom{3}{2}}{\binom{11}{2}} = \frac{3}{55}$

(b) both Black:

$$P(b_1 \wedge b_2) = P(b_1)P(b_2|b_1) = \frac{8}{11} \cdot \frac{7}{10} = \frac{28}{55} \quad (= \frac{\binom{8}{2}}{\binom{11}{2}})$$

(c) one Red, one Black:

$\boxed{9.1.22}$ Prove the following:
If $P(A|B) > P(A)$ ("positive correlation") then $P(B|A) > P(B)$

$P(A|B) > P(A)$

$\Rightarrow P(A \cap B) > P(A) \cdot P(B)$

$\Rightarrow \frac{P(A \cap B)}{P(A)} > P(B)$

$\Rightarrow P(B|A) > P(B)$

$\boxed{9.1.7}$ Suppose that an experiment leads to events A, B and C with $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.1$

(a) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4}$

(b) $P(A^c) = 1 - P(A) = 0.7$

(c) Is $A \perp B$? No. $P(A) \cdot P(B) = 0.12 \neq P(A \cap B)$

(d) Is $A^c \perp B$? No, as can be seen from (c).

Note: $P(A^c \cap B) = P(B) - P(A \cap B) = 0.4 - 0.1 = 0.3$
$P(A^c) \cdot P(B) = 0.7 \cdot 0.4 = 0.28$

$\boxed{9.1.8}$ Given $A \perp B$, $P(A) = 0.4$, $P(B) = 0.6$

$P(A|B) = P(A) = 0.4$

$P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.76$

$P(A^c \cap B) = P(A^c)P(B) = 0.36$

$\boxed{\text{9.5.5 (supp)}}$ We are given two events with $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$. True, false or could be either?

(a) $P(A \cap B) = \frac{1}{12}$ — possible; it holds when $A \perp B$

(b) $P(A \cup B) = \frac{7}{12}$ — possible; it holds when A, B are disjoint

(c) $P(B|A) = \frac{P(B)}{P(A)}$ — false; correct is: $P(B|A) = \frac{P(B \cap A)}{P(A)}$

(d) $P(A|B) \geq P(A)$ — possible (it means that B "supports" A)

(e) $P(A^c) = \frac{3}{4}$ — true, since $P(A^c) = 1 - P(A)$

(f) $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$ — true