Week 3 Functions and Relations

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Functions

$f : S \longrightarrow T$ describes pairing of the sets: it means that f assigns to every element $s \in S$ a unique element $t \in T$.

S — **domain** of f, symbol: $\mathrm{Dom}(f)$
T — **codomain** of f, symbol: $\mathrm{Codom}(f)$
$\{ f(x) : x \in \mathrm{Dom}(f) \}$ — **image** of f, symbol: $\mathrm{Im}(f)$

$$\mathrm{Im}(f) \subseteq \mathrm{Codom}(f)$$

We observe that every function maps its domain **into** its codomain, but only **onto** its image.

$$f : x \mapsto y, \quad f : A \mapsto B$$

The former means that x is mapped to y; the latter means that B is the image of A under f.

NB

Observe the difference between $\longrightarrow$ and $\mapsto$

Composition of functions is described as

$$g \circ f : x \mapsto g(f(x)), \quad \text{requiring } \mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$$

Composition is associative

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \text{can write } h \circ g \circ f$$

Identity function on S

$$\mathrm{Id}_S(x) = x, x \in S; \mathrm{Dom}(i) = \mathrm{Codom}(i) = \mathrm{Im}(i) = S$$

For $g : S \longrightarrow T \quad g \circ \mathrm{Id}_S = g, \ \mathrm{Id}_T \circ g = g$

Function is called **onto** (or **surjective**) if every element of the codomain is mapped to by at least one x in the domain, i.e.

$$\mathrm{Im}(f) = T$$

Function is called **1–1** (**one-to-one**) or **injective** if different x implies different $f(x)$, i.e.

$$f(x) = f(y) \Rightarrow x = y$$

Basic Matrix Operations

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 3 & -2 & 1 \\ 4 & -2 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & -1 & 5 & 7 \\ 5 & 5 & -3 & 3 \\ 8 & -2 & 1 & 5 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad 2\mathbf{A} = \begin{bmatrix} 4 & -2 & 0 & 8 \\ 6 & 4 & -2 & 4 \\ 8 & 0 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

(all w.r.t. set S)

Identity (diagonal, equality) $\quad E = \{\, (x,x) : x \in S \,\}$

Empty $\emptyset$

Universal $U = S \times S$

Important Properties of Binary Relations $\mathcal{R} \subseteq S \times S$

(R) reflexive $(x,x) \in \mathcal{R}$ for all $x \in S$ $\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \bullet & \circ \\ \bullet & \circ & \bullet \end{bmatrix}$

(AR) antireflexive $(x,x) \notin \mathcal{R}$ $\begin{bmatrix} \circ & \bullet & \bullet \\ \circ & \circ & \circ \\ \bullet & \circ & \circ \end{bmatrix}$

(S) symmetric $(x,y) \in \mathcal{R} \Rightarrow (y,x) \in \mathcal{R}$ $\begin{bmatrix} \bullet & \circ & \bullet \\ \circ & \circ & \bullet \\ \bullet & \bullet & \circ \end{bmatrix}$

(AS) antisymmetric $(x,y), (y,x) \in \mathcal{R} \Rightarrow x = y$
$\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{bmatrix}$

(T) transitive $(x,y), (y,z) \in \mathcal{R} \Rightarrow (x,z) \in \mathcal{R}$
$\begin{bmatrix} \circ & \circ & \bullet \\ \bullet & \bullet & \bullet \\ \circ & \circ & \circ \end{bmatrix}$

Most important kinds of relations on S

- total order $\begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \bullet \end{bmatrix}$
- partial order $\begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$, $\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$
- equivalence $\begin{bmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$
- identity $\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$

Exercise

$\boxed{1.5.4}$ Σ^* as above and $g(n) \stackrel{\text{def}}{=} \{ \omega \in \Sigma^* : \text{length}(\omega) \leq n \}$, $n \in \mathbb{N}$
Here $g(n)$ is a function that has a complex object as its value for any given argument — it maps $\mathbb{N}$ into $\text{Pow}(\Sigma^*)$
(a) $g(0) = \{\lambda\}$
(b) $g(1) = \{\lambda, a, b\}$
(c) $g(2) = \{\lambda, a, b, aa, ab, ba, bb\}$
In general $g(n) = \bigcup_{i=0}^{n} \Sigma^i = \Sigma^{\leq n}$
(d) Are all $g(n)$ finite?
Yes; $|g(n)| = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$

$\boxed{1.7.5}$ f and g are 'shift' functions $\mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = n + 1$, and $g(n) = \max(0, n - 1)$

(c) Is f 1–1? onto?
(d) Is g 1-1? onto?
(e) Do f and g commute, i.e. $\forall n((f \circ g)(n) = (g \circ f)(n))$?

Exercise

$\boxed{1.7.6}$ $\Sigma = \{a, b, c\}$
(c) Is length : $\Sigma^* \longrightarrow \mathbb{N}$ onto? Yes: $\text{length}^{\leftarrow}(\{n\}) = \Sigma^n \neq \emptyset$

(d) $\text{length}^{\leftarrow}(2) = \{aa, ab, ac, bb, \ldots, cc\}$

Exercise

$\boxed{1.7.12}$ Verify that $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x + y, x - y)$ is invertible.

The inverse is $f^{-1}(a, b) = (\frac{a+b}{2}, \frac{a-b}{2})$; substituting shows that $f^{-1} \circ f = \mathrm{Id}_{\mathbb{R} \times \mathbb{R}}$

Exercise

$\boxed{1.8.16}$ $\Sigma = \{a, b\}$
(a) Is there an onto function of the form $\Sigma \longrightarrow \Sigma^*$?
(b) Is there an onto function of the form $\Sigma^* \longrightarrow \Sigma$?

$\boxed{3.1.2(e)}$ Write the following relation on $A = \{0, 1, 2\}$ as a matrix.

$(m, n) \in \mathcal{R}$ if $m \cdot n = m$

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \circ \end{bmatrix} \end{array}$$

Exercise

$\boxed{3.1.1}$ The following relations are on $S = \{1, 2, 3\}$.
Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in \mathcal{R}$ if $m + n = 3$

(AR) and (S)

(e) $(m, n) \in \mathcal{R}$ if $\max\{m, n\} = 3$

(S)

$\boxed{3.1.2(b)}$ $(m, n) \in \mathcal{R}$ if $m < n$

(AR), (AS), (T)

Exercise

$f^{\leftarrow}$ is a relation; when is it a function?

Exercise

$\boxed{3.1.9}$ Find the properties of the *empty relation* $\emptyset \subset S \times S$ and the *universal relation* $U = S \times S$. Assume that S is a nonempty domain.

$\boxed{3.1.10(a)}$ Give examples of relations with specified properties.
(AS), (T), ¬(R).

Examples over $\mathbb{N}$, $\mathsf{Pow}(\mathbb{N})$:

- strict order of numbers $x < y$
- simple (weak) order, but with some pairs (x, x) removed from $\mathcal{R}$
- being a prime divisor
  $(p, n) \in \mathcal{R}$ iff p is prime and $p|n$
  - not reflexive: $(1, 1) \notin \mathcal{R}, (4, 4) \notin \mathcal{R}, (6, 6) \notin \mathcal{R}$
  - transitivity is meaningful only for the pairs $(p, p), (p, n),\ p|n$ for p prime

$\boxed{3.1.10(b)}$ Give examples of relations with specified properties.
(S), ¬(R), ¬(T).

Easiest examples: inequality

- $\mathcal{R} = \{(x, y)|x \neq y,\ \ x, y \in \mathbb{N}\}$
- $\mathcal{R} = \{(A, B)|A \neq B,\ \ A, B \subseteq S\}$

$\boxed{3.1.14}$ Which properties carry from individual relations to their union?
(a) $\mathcal{R}_1, \mathcal{R}_2 \in$ (R) $\Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in$ (R)
(b) $\mathcal{R}_1, \mathcal{R}_2 \in$ (S) $\Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in$ (S)
(c) $\mathcal{R}_1, \mathcal{R}_2 \in$ (T) $\not\Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in$ (T)
Eg. $S = \{a, b, c\}, a\mathcal{R}_1 b, b\mathcal{R}_2 c$
and no other relationships