# Week 8 Counting

Sunday, 3 May 2020 6:42 PM

## Basic Counting Rules (1)

**Union rule** — $S$ and $T$ *disjoint*

$$|S \cup T| = |S| + |T|$$

$S_1, S_2, \ldots, S_n$ pairwise disjoint ($S_i \cap S_j = \emptyset$ for $i \neq j$)

$$|S_1 \cup \ldots \cup S_n| = \sum |S_i|$$

### Inclusion-Exclusion

This is one of the most universal counting procedures. It allows you to compute the size of

$$A_1 \cup \ldots \cup A_n$$

from the sizes of all possible intersections

$$A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}, \quad a_{i_1} < a_{i_2} < \ldots < a_{i_k}$$

**Two sets** $|A \cup B| = |A| + |B| - |A \cap B|$

**Three sets** $|A \cup B \cup C| = |A| + |B| + |C|$
$-|A \cap B| - |A \cap C| - |B \cap C|$
$+|A \cap B \cap C|$

### Corollaries

- If $|S \cup T| = |S| + |T|$ then $S$ and $T$ are disjoint
- If $|\bigcup_{i=1}^n S_i| = \sum_{i=1}^n |S_i|$ then $S_i$ are pairwise disjoint
- If $|T \setminus S| = |T| - |S|$ then $S \subseteq T$

These properties can serve to identify cases when sets are disjoint (resp. one is contained in the other).

**Proof.**
$|S| + |T| = |S \cup T|$ implies $|S \cap T| = |S| + |T| - |S \cup T| = 0$

$|T \setminus S| = |T| - |S|$ implies $|S \cap T| = |S|$, which implies $S \subset T$ $\square$

***r*-permutations**
Selecting any $r$ objects from a set $S$ of size $n$ without repetition while *recognising* the order of selection.
Their number is

$$\Pi(n, r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

## Basic Counting Rules (2)

**Product rule**

$$|S_1 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k| = \prod_{i=1}^{k} |S_i|$$

If all $S_i = S$ (the same set) and $|S| = m$ then $|S^k| = m^k$

### Interpretation

Each $A_i$ defined as the set of objects that satisfy some property $P_i$

$$A_i = \{ x \in X : P_i(x) \}$$

Union $A_1 \cup \ldots \cup A_n$ is the set of objects that satisfy **at least one** property $P_i$

$$A_1 \cup \ldots \cup A_n = \{ x \in X : P_1(x) \vee P_2(x) \vee \ldots \vee P_n(x) \}$$

Intersection $A_{i_1} \cap \ldots \cap A_{i_r}$ is the set of objects that satisfy **all** properties $P_{i_1}, \ldots, P_{i_r}$

$$A_{i_1} \cap \ldots \cap A_{i_r} = \{ x \in X : P_{i_1}(x) \wedge P_{i_2}(x) \wedge \ldots \wedge P_{i_r}(x) \}$$

Special case $r = 1$: $A_{i_1} = \{x \in X : P_{i_1}(x)\}$

***r*-selections** (or: ***r*-combinations**)
Collecting any $r$ distinct objects without repetition;
equivalently: selecting $r$ objects from a set $S$ of size $n$ and *not* recognising the order of selection.
Their number is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{1 \cdot 2 \cdots r}$$

**NB**
These numbers are usually called *binomial coefficients* due to

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + b^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i}b^i$$

**NB**
Also defined for any $\alpha \in \mathbb{R}$ as $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$

### Exercise

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words? How many with no letter repeated?

## Exercise

$S, T$ finite. How many functions $S \longrightarrow T$ are there?

$|T|^{|S|}$

## Exercise

$\boxed{5.1.19}$ Consider a *complete* graph on $n$ vertices.

(a) No. of paths of length 3

Take any vertex to start, then every next vertex differe
preceding one. Hence $n \cdot (n-1)^3$

(b) paths of length 3 with all vertices distinct

$n(n-1)(n-2)(n-3)$

(c) paths of length 3 with all edges distinct

$n(n-1)(n-2)^2$

$\boxed{5.3.1}$ 200 people. 150 swim or jog, 85 swim and 60 do both. How many jog?

$\boxed{5.6.38 \text{ (supp)}}$ There are 100 problems, 75 of which are 'easy' and 40 'important'.
What's the smallest number of easy *and* important problems?

## Exercise

$\boxed{5.3.2}$ $S = [100 \ldots 999]$, thus $|S| = 900$.

(a) How many numbers have at least one digit that is a 3 or 7?

$A_3 = \{\text{at least one '3'}\}$

$A_7 = \{\text{at least one '7'}\}$

$$(A_3 \cup A_7)^c = \{\, n \in [100, 999] : n \text{ digits} \in \{0, 1, 2, 4, 5, 6, 8, 9$$

7 choices for the first digit and 8 choices for the later digits:

$$|(A_3 \cup A_7)^c| = |\{1, 2, 4, 5, 6, 8, 9\}| \cdot |\{0, 1, 2, 4, 5, 6, 8, 9\}|^2$$

Therefore $|A_3 \cup A_7| = 900 - 448 = 452$.

(b) How many numbers have a 3 *and* a 7?

$\boxed{5.1.2}$ Give an example of a counting problem whose answer is

(a) $\Pi(26, 10)$

(b) $\binom{26}{10}$

$\boxed{5.1.6}$ From a group of 12 men and 16 women, how many committees can be chosen consisting of

(a) 7 members? $\binom{12+16}{7}$

(b) 3 men and 4 women? $\binom{12}{3}\binom{16}{4}$

(c) 7 women or 7 men? $\binom{12}{7} + \binom{16}{7}$

$\boxed{5.1.7}$ As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

$\{\text{all committees}\} - \{\text{committees with both } A \text{ and } B\}$

$= \binom{9}{4} - \binom{7}{2} = 126 - 21 = 105$

equivalently, $\{\text{A in, B out}\} + \{\text{A out, B in}\} + \{\text{none in}\}$

$$= \binom{7}{3} + \binom{7}{3} + \binom{7}{4} = 35 + 35 + 35 = 105$$

$\boxed{5.1.15}$ A poker hand consists of 5 cards drawn without replacement from a standard deck of 52 cards

$$\{A, 2\text{-}10, J, Q, K\} \times \{\text{club, spade, heart, diamond}\}$$

(a) Number of "4 of a kind" hands (e.g. 4 Jacks)

$|\text{rank of the 4-of-a-kind}| \cdot |\text{any other card}| = 13 \cdot (52 - 4)$

(b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)

$|\text{all flush}| - |\text{straight flush}|$

$= |\text{suit}| \cdot |\text{5-hand in a given suit}| -$

$\quad |\text{suit}| \cdot |\text{rank of a straight flush in a given suit}|$

$= 4 \cdot \binom{13}{5} - 4 \cdot 10$