Week 5 Graph

Thursday, 30 April 2020 8:55 PM

Degree of a vertex
i.e., the number of edges attached to the vertex
Regular graph: all degrees are equal

Di = no. of vertices of degree i
D0 + D1 + ...+ Dk=
thus the sum of vertex degrees is always even.

A **path** in a graph (V; E) is a sequence of edges
length of the path is the number of edges: n

Connected graph | each pair of vertices joined by a path

- Connected component of G | a connected subgraph of G that is not contained in a larger connected subgraph of G

Cycle -- closed path, all other vi pairwise distinct
$C_n = (e_1, \ldots, e_n)$ is a cycle *iff* removing *any* single edge leaves an acyclic path. (Show that the 'any' condition is needed!)

Trees
Acyclic graph -- graph that doesn't contain any cycle
Tree -- connected acyclic graph
A graph is acyclic iff it is a forest (collection of unconnected trees)

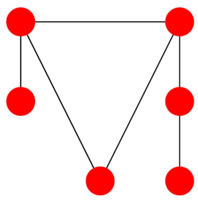
Graph G is a tree.
$\Leftrightarrow$ G is acyclic and $|V_G| = |E_G| + 1$.

Two graphs are called **isomorphic** if there exists (at least one) isomorphism between them.
$\iota : G \longrightarrow H$ is a *graph isomorphism* if
(i) $\iota : V_G \longrightarrow V_H$ is 1–1 and onto (a so-called *bijection*)
(ii) $\{x, y\} \in E_G$ iff $\{\iota(x), \iota(y)\} \in E_H$

Automorphisms and Asymmetric Graphs

An isomorphism from a graph to itself is called *automorphism*.
Every graph has at least the trivial automorphism
(trivial means: $\iota(v) = v$ for all $v \in V_G$)

Graphs with no non-trivial automorphisms are called *asymmetric*.
The smallest non-trivial asymmetric graphs have 6 vertices.



Edge Traversal

Definition

- **Euler path** — path containing every edge exactly once
- **Euler circuit** — closed Euler path

(Named after Leonhard Euler (Switzerland), 1707–1783)

Characterisations

- G (connected) has an Euler circuit iff $\deg(v)$ is even for all $v \in V$.
- G (connected) has an Euler path iff either it has an Euler circuit (above) or it has exactly two vertices of odd degree.

NB

- These characterisations apply to graphs with loops as well

- **Complete graph K_n**
  n vertices, all pairwise connected, $\frac{n(n-1)}{2}$ edges.

- **Complete bipartite graph $K_{m,n}$**

No. of edges is $m \cdot n$

- **Complete k-partite graph $K_{m_1,\ldots,m_k}$**

No. of edges is $\sum_{i<j} m_i m_j = \frac{1}{2} \sum_{i \neq j} m_i m_j$

- These graphs generalise the complete graphs $K_n = K_{\underbrace{1,\ldots,1}_{n}}$

Vertex Traversal

Definition

- **Hamiltonian path** visits every vertex of graph exactly once
- **Hamiltonian circuit** visits every vertex exactly once except the last one, which duplicates the first

(Named after Sir William Rowan Hamilton (Ireland), 1805–1865)

Examples (when the circuit exists)

- K_m for all $m \geq 3$; $K_{m,n}$ iff $m = n$ and $m, n \geq 2$

if a circuit is given, it is immediate to verify that it is a Hamiltonian circuit.

Colouring --chromatic number of a graph is denoted,

- $\chi(K_n) = n$
- If G has n vertices and $\chi(G) = n$ then $G = K_n$

- If $\chi(G) = 1$ then G is totally disconnected: it has 0 edges.
- If $\chi(G) = 2$ then G is bipartite.
- For any tree $\chi(T) = 2$.
- For any cycle C_n its chromatic number depends on the parity of n — for n even $\chi(C_n) = 2$, while for n odd $\chi(C_n) = 3$.

A **clique** in G is a *complete* subgraph of G. A clique of k nodes is called *k-clique*.
The size of the largest clique is called the *clique number* of the graph and denoted $\kappa(G)$.

Theorem

$\chi(G) \geq \kappa(G)$.

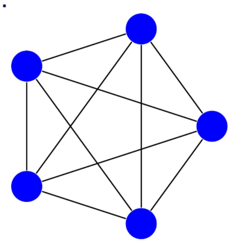
- $\kappa(K_n) = n$, $\kappa(K_{m,n}) = 2$, $\kappa(K_{m_1,\ldots,m_r}) = r$.
- If $\kappa(G) = 1$ then G is totally disconnected.
- For a tree $\kappa(T) = 2$.
- For a cycle C_n
  $\kappa(C_3) = 3, \quad \kappa(C_4) = \kappa(C_5) = \ldots = 2$

The difference between $\kappa(G)$ and $\chi(G)$ is apparent with just $\kappa(G) = 2$ — this does not imply that G is bipartite. For example, the cycle C_n for any odd n has $\chi(C_n) = 3$.

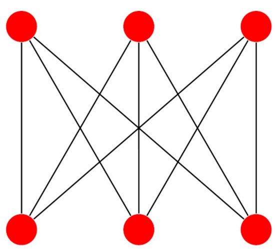
A graph is **planar** if it can be embedded in a plane without its edges intersecting.

Two minimal nonplanar graphs

K_5 :



$K_{3,3}$:



Theorem

A graph is nonplanar iff it contains K_5 or $K_{3,3}$ as a minor.

Exercises

Exercise

$\boxed{6.1.20(a)}$ Graph with $e(G) = 21$ edges has a degree sequence
$D_0 = 0, D_1 = 7, D_2 = 3, D_3 = 7, D_4 = ?$
Find $v(G)$!

$\boxed{6.1.20(b)}$ How would your answer change, if at all, when $D_0 = 6$?

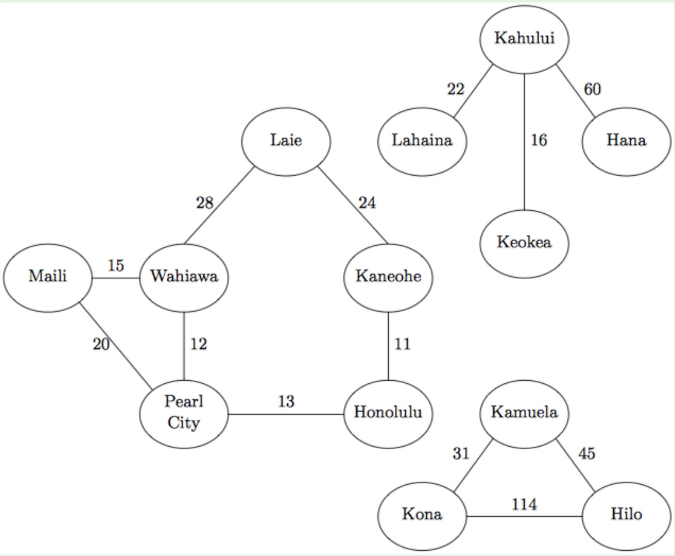
Exercise

$\boxed{6.7.3 \text{ (supp)}}$ Tree with n vertices, $n \geq 3$.
Always true, false or could be either?
(a) $e(T) \stackrel{?}{=} n$
(b) at least one vertex of deg 2
(c) at least two v_1, v_2 s.t. $\deg(v_1) = \deg(v_2)$

$\boxed{6.2.14}$ Which complete graphs K_n have an Euler circuit?
When do bipartite, 3-partite complete graphs have an Euler circuit?

$\boxed{6.5.5(a)}$ How many Hamiltonian circuits does $K_{n,n}$ have?

9.10.1 (Aho & Ullman)

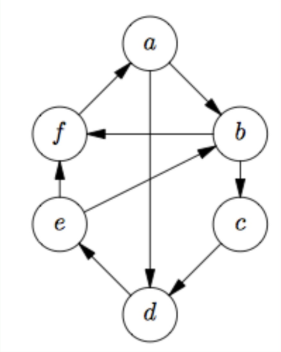


$\chi(G)$? $\kappa(G)$? A largest clique?

9.10.3 (Aho & Ullman) Let $G = (V, E)$ be an undirected graph. What inequalities must hold between

- the maximal $deg(v)$ for $v \in V$
- $\chi(G)$
- $\kappa(G)$

9.10.2 (Aho & Ullman)



Is (the undirected version of) this graph planar?

Question

Are all $K_{m,1}$ planar?

Question

Are all $K_{m,2}$ planar?