Week 4 Equivalence and Order Relations

Wednesday, 27 May 2020 6:06 PM

If the relation $\sim$ is an equivalence on S and $[S]$ the corresponding partition, then

$$\nu : S \longrightarrow [S], \quad \nu : s \mapsto [s] = \{ x \in S : x \sim s \}$$

is called the *natural* map. It is always onto.

Order Relations

Total order $\leq$ on S
(R) $x \leq x$ for all $x \in S$
(AS) $x \leq y, y \leq x \Rightarrow x = y$
(T) $x \leq y, y \leq z \Rightarrow x \leq z$
(L) *Linearity* — any two elements are comparable:
for all x, y either $x \leq y$ or $y \leq x$ (and both if $x = y$)

Partial Order

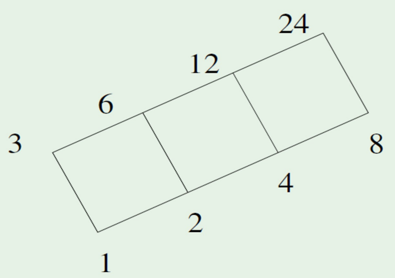
A **partial order** $\preceq$ on S satisfies (R), (AS), (T); need not be (L)
We call $(S, \preceq)$ a **poset** — partially ordered set

Hasse Diagram

Every finite poset can be represented as a **Hasse diagram**, where a line is drawn *upward* from x to y if $x \prec y$ and there is no z such that $x \prec z \prec y$

Example

$\boxed{11.1.1(a)}$ Hasse diagram for positive divisors of 24



$p \preceq q$ if, and only if, $p \mid q$

Ordering Concepts

Definition

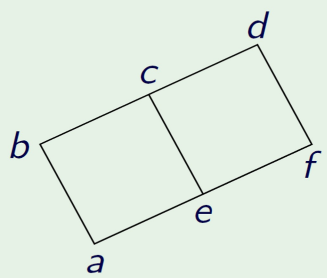
- *Minimal* and *maximal* elements (they always exist in every finite poset)
- *Minimum* and *maximum* — unique minimal and maximal element
- *lub* (least upper bound) and *glb* (greatest lower bound) of a subset $A \subseteq S$ of elements
  $\mathrm{lub}(A)$ — smallest element $x \in S$ s.t. $x \succeq a$ for all $a \in A$
  $\mathrm{glb}(A)$ — greatest element $x \in S$ s.t. $x \preceq a$ for all $a \in A$
- *Lattice* — a poset where lub and glb exist for every pair of elements
  (by induction, they then exist for every *finite* subset of elements)

Well-ordered set: every subset has a least element.

Ordering of a Poset — Topological Sort

For a poset $(S, \preceq)$ any linear order $\leq$ that is consistent with $\preceq$ is called **topological sort**. Consistency means that $a \preceq b \Rightarrow a \leq b$.

Example



The following all are topological sorts:
$a \leq b \leq e \leq c \leq f \leq d$
$a \leq e \leq b \leq f \leq c \leq d$
$\ldots\ldots$
$a \leq e \leq f \leq b \leq c \leq d$

- $\mathbb{Z}$ — neither lub nor glb;
- $\mathbb{F}(\mathbb{N})$ — all finite subsets, has no *arbitrary* lub property; glb exists, it is the intersection, hence always finite;
- $\mathbb{I}(\mathbb{N})$ — all infinite subsets, may not have an arbitrary glb; lub exists, it is the union, which is always infinite.

Practical Orderings

They are, effectively, *total* orders on the *product* of ordered sets.

- **Lexicographic order** — defined on all of Σ^*. It extends a total order already assumed to exist on Σ.
- **Lenlex** — the order on (potentially) the entire Σ^*, where the elements are ordered first by length.
  $\Sigma^{(1)} \prec \Sigma^{(2)} \prec \Sigma^{(3)} \prec \cdots$, then lexicographically within each $\Sigma^{(k)}$. In practice it is applied only to the finite subsets of Σ^*.
- **Filing order** — lexicographic order confined to the strings of the same length.
  It defines total orders on Σ^i, separately for each i.

Modular Arithmetic

Example

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

n	$-n$
0	0
1	4
2	3
3	2
4	1

$*_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

n	n^{-1}
0	—
1	1
2	3
3	2
4	4

Exercise

$\boxed{3.5.6}$ Calculate the following in $\mathbb{Z}_7$.

(b) $5 + 6 =$ 4

(c) $4 * 4 =$ 2

(d) for any $k \in \mathbb{Z}_7$, $0 + k =$ k

(e) for any $k \in \mathbb{Z}_7$, $1 * k =$ k

Solve the following for x in $\mathbb{Z}_5$.

(a) $2 + x = 1 \quad \Rightarrow x = 4$

(b) $2 * x = 1 \quad \Rightarrow x = 2^{-1} = 3$

(c) $2 * x = 3 \quad \Rightarrow x = 3 * 2^{-1} = 3 * 3 = 4$

Solve the following for x in $\mathbb{Z}_6$.

(d) $5 + x = 1 \quad \Rightarrow x = 2$

(e) $5 * x = 1 \quad \Rightarrow x = 5 \quad (\text{since } 25 \mod 6 = 1)$

(e) $2 * x = 1$ undefined $\quad (\text{since } 2 \cdot k \mod 6 \neq 1 \text{ for all } k \in \mathbb{Z}_6)$

$\boxed{3.6.6}$ Show that $m \sim n$ iff $m^2 \equiv n^2 \pmod 5$ is an equivalence on $S = \{1, \ldots, 7\}$. Find all the equivalence classes.

(a) It just means that $m \equiv n \pmod 5$ or $m \equiv -n \pmod 5$, e.g. $1 \equiv -4 \pmod 5$. This satisfies (R), (S), (T).

(b) We have

$[1] = \{1, 4, 6\}$

$[2] = \{2, 3, 7\}$

$[5] = \{5\}$

$\boxed{3.6.10}$

$\mathcal{R}$ is a binary relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^4$

$(m, n)\,\mathcal{R}\,(p, q)$ if $m \equiv p \pmod 3$ or $n \equiv q \pmod 5$.

(a) $\mathcal{R} \in$ (R)?

Yes: $(m, n) \sim (m, n)$ iff $m \equiv m \pmod 3$ or $n \equiv n \pmod 5$ iff tr
or true.

(b) $\mathcal{R} \in$ (S)?

Yes: by symmetry of $. \equiv . \pmod n$.

(c) $\mathcal{R} \in$ (T)?

No — for arbitrary two pairs (m_1, n_1) and (m_2, n_2) one can crea
a chain $(m_1, n_1)\mathcal{R}(m_2, n_1)$ and $(m_2, n_1)\mathcal{R}(m_2, n_2)$, but
$(m_1, n_1)\,\mathcal{R}(m_2, n_2)$

(b) $\mathcal{R} \in$ (S)?

(c) $\mathcal{R} \in$ (T)?

$\boxed{11.1.8}$ For $\omega_1, \omega_2 \in \Sigma^*$ define $\omega_1 \preceq \omega_2$ when $\omega_2 = \nu\omega_1\nu'$ for some ν, ν'.

Is this a partial order?

Yes.

Relation $\preceq$ means being a substring; it is a partial order:

(R) $\omega = \lambda\omega\lambda$, hence $\omega \preceq \omega$

(AS) if $\omega_1 = \nu\omega_2\nu'$ and $\omega_2 = \chi\omega_1\chi'$ for some ν, ν', χ, χ' then
$\nu = \nu' = \chi = \chi' = \lambda$, hence $\omega_1 = \omega_2$

(T) if $\omega_1 = \nu\omega_2\nu'$ and $\omega_2 = \chi\omega_3\chi'$ then $\omega_1 = \nu\chi\omega_3\chi'\nu'$

$\boxed{11.6.16}$ Properties of four relations defined on $\mathbb{P} = \{1, 2, \ldots\}$?

- $\mathcal{R}_1$ if $m|n$
- $\mathcal{R}_2$ if $|m - n| \leq 2$
- $\mathcal{R}_3$ if $2|m + n$
- $\mathcal{R}_4$ if $3|m + n$

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
(R)				
(S)				
(AS)				
(T)				
Equivalence	?	?	?	?
Partial order	?	?	?	?

Examples

- Pow($\{a, b, c\}$) with the order $\subseteq$
  $\emptyset$ is minimum; $\{a, b, c\}$ is maximum
- $\boxed{11.1.4}$
  Pow($\{a, b, c\}$) $\setminus \{\{a, b, c\}\}$ (proper subsets of $\{a, b, c\}$)
  Each two-element subset $\{a, b\}, \{a, c\}, \{b, c\}$ is maximal.
  - But there is no maximum
- $\{1, 2, 3, 4, 6, 8, 12, 24\}$ partially ordered by divisibility is a lattice
  - e.g. $\text{lub}(\{4, 6\}) = 12$; $\text{glb}(\{4, 6\}) = 2$
- $\{1, 2, 3\}$ partially ordered by divisibility is not a lattice
  - $\{2, 3\}$ has no lub
- $\{2, 3, 6\}$ partially ordered by divisibility is not a lattice
  - $\{2, 3\}$ has no glb

Examples

- $\{1, 2, 3, 12, 18, 36\}$ partially ordered by divisibility is not a lattice
  - $\{2, 3\}$ has no lub ($12, 18$ are minimal upper bounds)

$\boxed{11.1.5}$ Consider poset $(\mathbb{R}, \leq)$
(a) Is this a lattice?
(b) Give an example of a non-empty subset of $\mathbb{R}$ that has no upper bound.
(c) Find $\text{lub}(\{\ x \in \mathbb{R} : x < 73\ \})$
(d) Find $\text{lub}(\{\ x \in \mathbb{R} : x \leq 73\ \})$
(e) Find $\text{lub}(\{\ x : x^2 < 73\ \})$
(f) Find $\text{glb}(\{\ x : x^2 < 73\ \})$

$\boxed{11.1.13}$ $\mathbb{F}(\mathbb{N})$ — collection of all *finite* subsets of $\mathbb{N}$; $\subseteq$-order
(a) Does it have a maximal element?
(b) Does it have a minimal element?
(c) Given $A, B \in \mathbb{F}(\mathbb{N})$, does $\{A, B\}$ have a lub in $\mathbb{F}(\mathbb{N})$?
(d) Given $A, B \in \mathbb{F}(\mathbb{N})$, does $\{A, B\}$ have a glb in $\mathbb{F}(\mathbb{N})$?
(e) Is $(\mathbb{F}(\mathbb{N}), \subseteq)$ a lattice?

$\boxed{11.1.14}$ $\mathbb{I}(\mathbb{N}) = \text{Pow}(\mathbb{N}) \setminus \mathbb{F}(\mathbb{N})$ — all *infinite* subsets of $\mathbb{N}$
(a) Does it have a maximal element?
(b) Does it have a minimal element?
(c) Given $A, B \in \mathbb{I}(\mathbb{N})$, does $\{A, B\}$ have a lub in $\mathbb{I}(\mathbb{N})$?
(d) Given $A, B \in \mathbb{I}(\mathbb{N})$, does $\{A, B\}$ have a glb in $\mathbb{I}(\mathbb{N})$?
(e) Is $(\mathbb{I}(\mathbb{N}), \subseteq)$ a lattice?

$\boxed{11.2.1}$ Let $A = \{1, 2, 3, 4\}$ and $S = A \times A$ with the product order.
(a) A chain with seven elements?

(b) A chain with eight elements?

$\boxed{11.2.5}$ Let $\mathbb{B} = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $\mathbb{B}^*$ in the

(a) Lexicographic order

$000, 0010, 010, 10, 1000, 101, 11$

(b) Lenlex order

$10, 11, 000, 010, 101, 0010, 1000$

$\boxed{11.2.8}$ When are the lexicographic order and *lenlex* on Σ^* the same?

Only when $|\Sigma| = 1$.

$\boxed{11.6.12}$ Draw a Hasse diagram for a poset with exactly 5 members, 2 of which are maximal and 1 of which is the poset's minimum.

$\boxed{11.6.6}$ True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.

(c) Every finite partially ordered set has a Hasse diagram.

(d) Every finite partially ordered set has a topological sorting.

(e) Every finite partially ordered set has a minimum.

(f) Every finite totally ordered set has a maximum.

(g) An infinite partially ordered set cannot have a maximum.