How do We Rotate?

Wen Perng

Electrical Engineering, NTU

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How do we rotate things?

When you are coding with vPython or MATLAB, we often do three dimensional vector rotations via rotational matrices:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and etc.}$$

But is this the only way we can do rotations?
Moreover, matrices are

1. computationally costly,
2. lacking of intuition,
3. computationally unstable.

We need better ways.

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Descriptions of rotations are crucial in many areas including:

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Many important mathematical concepts are met along the way. Let us walk through the various descriptions of rotation, and discover a new tool called quaternions.

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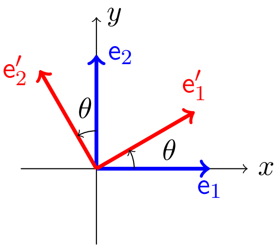
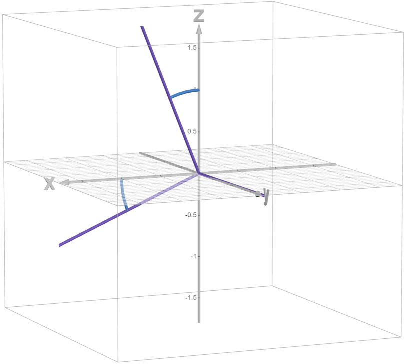
$$R = \left[\begin{array}{c:c} \mathsf{e}_1' & \mathsf{e}_2' \end{array}\right] = \left[\begin{array}{c:c} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$



3D Rotational Matrices

The same procedure is used to produce the previous rotational matrices R_x, R_y and R_z. Take R_y for example:

$$R_y(\theta) = \left[\begin{array}{c:c:c} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{array}\right]$$



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Hence, we have

$$R(\hat{\boldsymbol{n}}, \eta) = R_z(\phi)R_y(\theta)R_z(\eta)R_y(-\theta)R_z(-\phi). \tag{1}$$

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$$\text{Rotate about } \hat{\boldsymbol{x}}\text{: } R_x(\theta) = \left[\begin{array}{c:c:c} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{array}\right] \tag{2}$$

$$\text{Rotate about } \hat{\boldsymbol{y}}\text{: } R_y(\theta) = \left[\begin{array}{c:c:c} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{array}\right] \tag{3}$$

$$\text{Rotate about } \hat{\boldsymbol{z}}\text{: } R_z(\theta) = \left[\begin{array}{c:c:c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{4}$$

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For example, when we first rotate around fixed $\hat{\boldsymbol{x}}$ by θ_3, then rotate around fixed $\hat{\boldsymbol{z}}$ by θ_2, followed by a last rotation of θ_1 about $\hat{\boldsymbol{x}}$, the system changes by

$$\boldsymbol{v} \mapsto R_x(\theta_1) R_z(\theta_2) R_x(\theta_3) \boldsymbol{v}.$$

See [desmos] for demo on $R_x(\theta_1) R_z(\theta_2) R_x(\theta_3)$.

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From the demo, we know that, amazingly:

$$R_{\underbrace{R_{R_x(\theta_1)\hat{z}}(\theta_2)\ R_x(\theta_1)\ \hat{x}}_{\text{3rd axis}}}(\theta_3) \cdot R_{\underbrace{R_x(\theta_1)\ \hat{z}}_{\text{2nd axis}}}(\theta_2) \cdot R_x(\theta_1)$$

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$$= R_x(\theta_1)R_z(\theta_2)R_x(\theta_3).$$

(I know this is some terrible notation, but that's how it goes for matrices.) We will prove this near the end of the presentation.

Rodrigue's Formula

Thm. (Rodrigue's Rotation Formula)

A simple formula used to describe rotation of vector $\boldsymbol{v}$ by angle θ around axis $\hat{\boldsymbol{n}}$ ($|\hat{\boldsymbol{n}}| = 1$):

$$\boldsymbol{v}_{\mathsf{rot}} = \boldsymbol{v}\cos\theta + (\hat{\boldsymbol{n}} \times \boldsymbol{v})\sin\theta + \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{v})(1 - \cos\theta) \tag{5}$$

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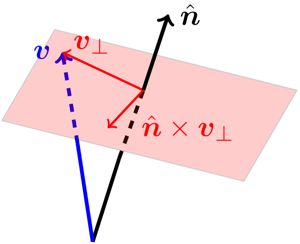
[pf.]

$$\begin{aligned}\boldsymbol{v}_{\text{rot}} &= \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}\cos\theta + \hat{\boldsymbol{n}} \times \boldsymbol{v}_{\perp}\sin\theta \\ &= \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{v}) + \big(\boldsymbol{v} - \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{v})\big)\cos\theta \\ &\quad + \hat{\boldsymbol{n}} \times \boldsymbol{v}\sin\theta\end{aligned}$$



Thus it is proven.
This equation is far easier to calculate.

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Transpose

Here we define a special operation on vectors and matrices:

Def. (Transpose)

Flip the matrix along its diagonal:

$$\left[\begin{array}{c:c:c} \color{red}{a_{11}} & a_{12} & a_{13} \\ a_{21} & \color{red}{a_{22}} & a_{23} \end{array}\right]^{\mathsf{T}} = \left[\begin{array}{cc} \color{red}{a_{11}} & a_{21} \\ \hdashline a_{12} & \color{red}{a_{22}} \\ \hdashline a_{13} & a_{23} \end{array}\right]. \tag{6}$$

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$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

Properties on Rotational Matrices

We can clearly observe that, under rotation, orthogonal vector remain orthogonal.

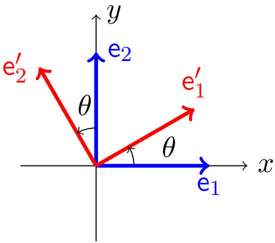
Orthogonal Matrices

A matrix is called orthogonal if:

$$A^\mathsf{T} A = A A^\mathsf{T} = \mathbb{1}.$$

A rotational matrix is orthogonal since

$$R^\mathsf{T} R = \begin{bmatrix} \mathsf{e}_1'^{\mathsf{T}} \\ \hdashline \vdots \\ \hdashline \mathsf{e}_n'^{\mathsf{T}} \end{bmatrix} \left[\begin{array}{c:c:c} \mathsf{e}_1' & \cdots & \mathsf{e}_n' \end{array} \right] = \mathbb{1}.$$



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We call rotational matrices an element of the special orthogonal group of order n:

$$\begin{aligned} \mathrm{SO}(n) &= \{n \times n \text{ rotational matrices}\} \\ &= \left\{ M \in \mathbb{R}^{n \times n} \,\middle|\, M^\mathsf{T} M = \mathbb{1}, \det(M) = 1 \right\}. \end{aligned}$$

Infinitesimal Generators of Rotation

Let's go back to the case of 2D rotation. If we only rotated a small angle $\mathrm{d}\theta$:

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$X\theta$ is coined the generator of the rotation $R(\theta)$.

Infinitesimal Generators of Rotation

Hence we have:

$$R(\theta) = \mathrm{e}^{X\theta}, X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The converse can be easily checked.
Since $X^2 = -\mathbb{1}$, we have:

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&= \cos\theta \mathbb{1} + X \sin\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.
\end{aligned}$$

Infinitesimal Generators of Rotation

For three dimensional rotations, we have:

$$L_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \; R_x(\theta) = \mathrm{e}^{L_x \theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$L_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; R_y(\theta) = \mathrm{e}^{L_y \theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{8}$$

$$L_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; R_z(\theta) = \mathrm{e}^{L_z \theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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\tag{8}
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Interestingly, all L_x, L_y and L_z are skew-symmetric matrices, they are the infinitesimal generators of special orthogonal matrices. We will also see later that these are very similar to quaternions.

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$$(\text{column } 1) \cdot (\text{column } 2) = 4.2 \times 10^{-10}$$

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4. Hard to **interpolate** rotation: for animations, find a rotation $R(t)$ s.t. $R(0) = R_0$ and $R(1) = R_1$.

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Axis: $[1, 2, 0]$, angle: $45°$.

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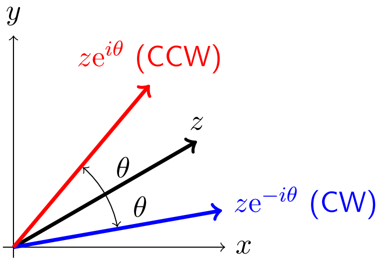
Complex Numbers for Rotation

We often associate a unit complex number as a rotation operator:

$$z \mapsto z\mathrm{e}^{i\theta} \equiv \text{rotate } z \text{ CCW by } \theta.$$

(CCW = counterclockwise; CW = clockwise)



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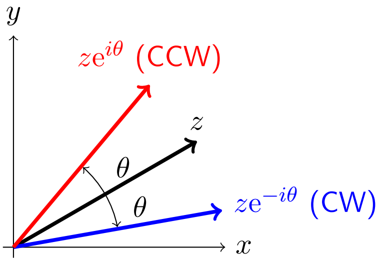

How can we extend the idea of “multiply by a unit complex number” to do rotations in **three dimensional space**?

Extension of Complex Numbers to 3D

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This fails as a good number system. We need a third complex number.

Quaternion

Def. (Quaternions)

The **quaternions**, denoted by $\mathbb{H}$, are spanned by 1, i, j and k, satisfying the multiplication relation of

$$\begin{aligned} 1^2 &= 1, \\ i^2 = j^2 = k^2 &= -1, \\ ijk &= -1. \end{aligned} \tag{9}$$

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A general quaternion is expressed as

$$q = s + v_1 i + v_2 j + v_3 k = s + \boldsymbol{v}, \tag{10}$$

$s = \mathrm{Re}\{q\}$ is called the real/scalar part, $\boldsymbol{v} = \mathrm{Im}\{q\}$ is called the imaginary/vector part.

Quaternion

From the defining relations

$$\begin{aligned} 1^2 &= 1, \\ i^2 = j^2 = k^2 &= -1, \\ ijk &= -1, \end{aligned}$$

we can deduce the following multiplication relationships:

$$ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik.$$

Quaternion (some history)

Quaternions were conceived by the Irish mathematician Sir William Rowan Hamilton in 1843. It was on a bridge in Dublin when the defining relations of quaternions

$$i^2 = j^2 = k^2 = ijk = -1$$

dawned on him.

This algebraic system successfully extended complex numbers to three dimensions, explaining rotations elegantly. Though replaced by vector algebra not long after its invention, quaternions regained traction in fields of computer graphics nowadays.

Figure: W.Hamilton (Wiki)

Quaternion

e.g. Quaternion Multiplication:

$$(1 + i)(2 - ji + j) = ?$$

Quaternion

Let us introduce some more functions and terms regarding quaternions, leading up to its description of rotation.

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- **Norm**: the norm of a quaternion is defined as

$$|q| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}. \tag{13}$$

Note that for vector components $\boldsymbol{v}^2 \leq 0$.

Quaternion

- **Inverse**: the inverse of q satisfying $qq^{-1} = q^{-1}q = 1$ is

$$q^{-1} = \frac{q^*}{qq^*} = \frac{q^*}{|q|^2} \quad (|q| \neq 0). \tag{14}$$

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- **Exponentials**:

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- **Versor**: a versor is a unit quaternion, i.e. $|q| = 1$. Any versor can be written as

$$q = e^{t\boldsymbol{r}} = \cos t + \boldsymbol{r} \sin t, \tag{16}$$

where $\boldsymbol{r}$ is a unit vector (expressed in quaternions: $\boldsymbol{r}^2 = -1$).

Quaternionic Rotation

So, how do we describe rotations with quaternions?

Quaternionic Rotation

So, how do we describe rotations with quaternions?

Thm. (Quaternion rotation)

To rotate a vector $\boldsymbol{p}$ (in the sense of quaternionic vector) by an angle θ around the unit vector $\hat{\boldsymbol{n}}$ (right hand rotation), let us define the versor:

$$q = \mathrm{e}^{\hat{\boldsymbol{n}}\theta/2} = \cos\frac{\theta}{2} + \hat{\boldsymbol{n}}\sin\frac{\theta}{2},$$

then the rotated vector will be:

$$\boldsymbol{p}' = q\boldsymbol{p}q^*.$$

Quaternionic Rotation

e.g. Rotate $\boldsymbol{p} = 1i$ by 120° around the axis $\hat{\boldsymbol{n}} = \frac{1}{\sqrt{3}}(i + j + k)$.
[sol.]

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$$\begin{aligned}\boldsymbol{p}' &= q\boldsymbol{p}q^* \\ &= \left(\frac{1}{2} + \frac{1}{2}(i+j+k)\right)(i)\left(\frac{1}{2} - \frac{1}{2}(i+j+k)\right)\end{aligned}$$

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Note that both q and $-q$ results in the same rotation.

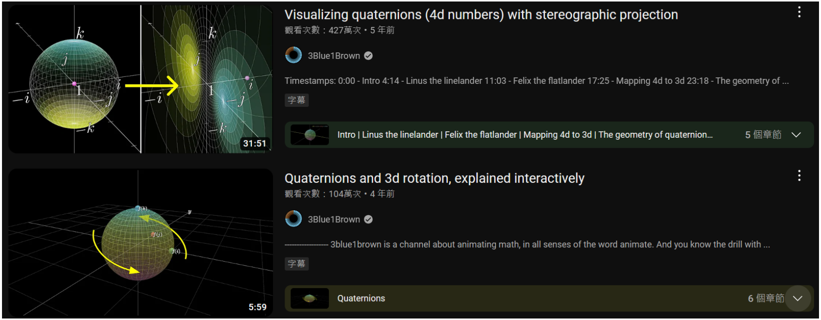
Advantages of Quaternions

When compared with matrices, quaternions have many advantages that made them more useful in fields of computer graphics and much more.

1. **Axis-angle rotation is super intuitive**!
2. Only requires 4 floats to store.
3. Computationally cheaper.
4. Invulnerable to floating point errors, since it is easy to re-normalize:
$$\text{versor} = \frac{q}{|q|}.$$
5. Easy to interpolate:
$$q(t) = \frac{(1-t)q_0 + tq_1}{\left|(1-t)q_0 + tq_1\right|}.$$

Disadvantage of Quaternions

Even though quaternions are really useful, its development were hindered by the fact that "it is not easy to understand quaternions"!
3Blue1Brown even made videos explaining it..., requiring a mapping of 4D objects into 3D space. It is not a good explanation.



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Let us try to understand quaternions... in 3D!

$$\boldsymbol{p}' = q\boldsymbol{p}q^*$$

$$q = \cos\frac{\theta}{2} + \hat{\boldsymbol{n}}\sin\frac{\theta}{2}$$

We can do this by introducing the idea of multivectors and rotors in the study of geometric algebra / Clifford algebra.

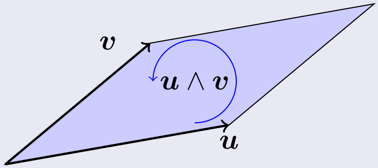
Outer Product

Between two vector, we can take their inner product and their cross product, now let us introduce a new kind of product:

Def. (Outer product)

The outer product (wedge product) of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as the **oriented area** element B formed by the two vectors:

$$B = \boldsymbol{u} \wedge \boldsymbol{v}.$$



The product is associative.

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Some properties:

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3. Moreover, we call this oriented area a **bivector**, it has a magnitude of

$$|\boldsymbol{u} \wedge \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| \sin\theta = |\boldsymbol{u}_\perp||\boldsymbol{v}| \, .$$

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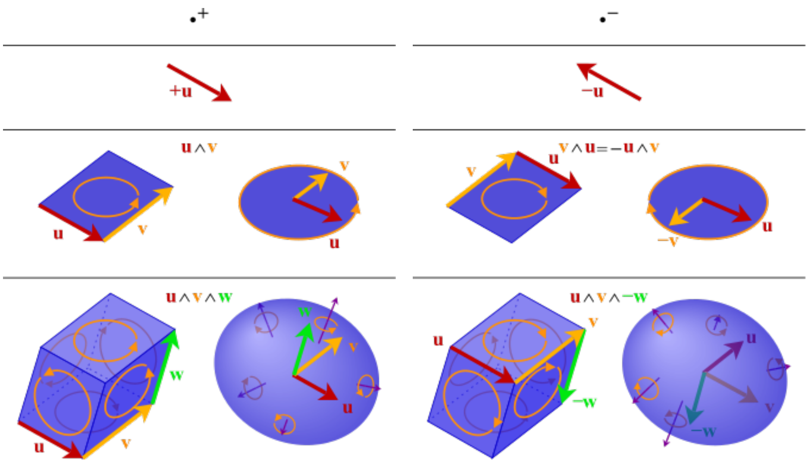
5. The element $\boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w}$ is called a **trivector**, an oriented volume.

Outer Product



(a) Positive orientation. (b) Negative orientation.

Figure: (Wiki)

Outer Product

Here we introduce a new term called **grade**.

Object	1	$\boldsymbol{u}$	$\boldsymbol{u} \wedge \boldsymbol{v}$	$\boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w}$	$\cdots$	$\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_k$
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Between two vectors, we have their inner and outer product as

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Wait! Can we add a scalar to an oriented area?

Why not? It's just like complex numbers and quaternions.
In fact, all objects of different grades can be added together, forming a general **multivector**, e.g.

$$M = 3 + 4\boldsymbol{u} \wedge \boldsymbol{v} + \boldsymbol{w}.$$

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If we have the canonical basis for $\mathbb{R}^n$ as $\{\mathsf{e}_i\}_{i=1}^n$, i.e. they satisfy

$$\begin{cases} \mathsf{e}_i^2 = \mathsf{e}_i\mathsf{e}_i = \mathsf{e}_i \cdot \mathsf{e}_i = 1 \\ \mathsf{e}_i\mathsf{e}_j = \mathsf{e}_i \wedge \mathsf{e}_j = -\mathsf{e}_j \wedge \mathsf{e}_i = -\mathsf{e}_j\mathsf{e}_i. \end{cases}$$

The relations can be concisely written as:

$$\mathsf{e}_i^2 = 1, \qquad \mathsf{e}_i\mathsf{e}_j = -\mathsf{e}_j\mathsf{e}_i. \tag{18}$$

Geometric Product

Let us calculate some examples: consider $\{\mathsf{e}_i\}_{i=1}^{n}$ the canonical basis for $\mathbb{R}^n$,
e.g.(1)

$$\boldsymbol{uv} = (u_1\mathsf{e}_1 + u_2\mathsf{e}_2)(v_1\mathsf{e}_1 + v_2\mathsf{e}_2)$$

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&= u_1v_1 1 + u_1v_2\mathsf{e}_1\mathsf{e}_2 - u_2v_1\mathsf{e}_1\mathsf{e}_2 + u_2v_2 1
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&= \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\wedge\boldsymbol{v}
\end{aligned}$$

Geometric Product

e.g.(2)

$$(\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3 + \mathsf{e}_1\mathsf{e}_2)(\mathsf{e}_2\mathsf{e}_1 + \mathsf{e}_1\mathsf{e}_3) =$$

Geometric Product

e.g.(2)

$$
\begin{aligned}
(e_1e_2e_3 + e_1e_2)(e_2e_1 + e_1e_3) = & e_1e_2e_3e_2e_1 + e_1e_2e_2e_1 \\
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\end{aligned}
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(e_1e_2e_3 + e_1e_2)(e_2e_1 + e_1e_3) =& e_1e_2e_3e_2e_1 + e_1e_2e_2e_1 \\
&+ e_1e_2e_3e_1e_3 + e_1e_2e_1e_3 \\
=& e_3 + 1 + e_2 - e_2e_3
\end{aligned}
$$

Geometric Product

e.g.(2)

$$\begin{aligned}(\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3 + \mathrm{e}_1\mathrm{e}_2)(\mathrm{e}_2\mathrm{e}_1 + \mathrm{e}_1\mathrm{e}_3) =& \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_2\mathrm{e}_1 + \mathrm{e}_1\mathrm{e}_2\mathrm{e}_2\mathrm{e}_1 \\ &+ \mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_1\mathrm{e}_3 + \mathrm{e}_1\mathrm{e}_2\mathrm{e}_1\mathrm{e}_3 \\ =& \mathrm{e}_3 + 1 + \mathrm{e}_2 - \mathrm{e}_2\mathrm{e}_3\end{aligned}$$

e.g.(3) We can define the inverse of a vector $\boldsymbol{u}$ as $\boldsymbol{u}^{-1}$, satisfying

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$$

I.e.

$$
\boldsymbol{u} := \frac{\boldsymbol{u}}{|\boldsymbol{u}|^2}. \tag{19}
$$

Geometric Product

The geometric product is super easy to compute!
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Hence,

$$\boldsymbol{u} \cdot \boldsymbol{v} = \frac{1}{2}(\boldsymbol{uv} + \boldsymbol{vu}), \tag{20}$$

$$\boldsymbol{u} \wedge \boldsymbol{v} = \frac{1}{2}(\boldsymbol{uv} - \boldsymbol{vu}). \tag{21}$$

Geometric Algebra

Given a canonical basis $\{\mathsf{e}_i\}$ of $\mathbb{R}^n$, we cam create a set of **basis multivectors** via **geometric products** between them:

$$1 \qquad \mathsf{e}_i \qquad \mathsf{e}_i\mathsf{e}_j \qquad \mathsf{e}_i\mathsf{e}_j\mathsf{e}_k \qquad \cdots \qquad \mathsf{e}_1\mathsf{e}_2\cdots\mathsf{e}_n$$

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Def. (n choose k)

The amount of choices of k objects from n objects is $\binom{n}{k} = \mathrm{C}_k^n$.

We often call the highest grade element the **pseudoscalar**, and denote it by

$$I := \mathrm{e}_1\mathrm{e}_2\cdots\mathrm{e}_n.$$

Geometric Algebra

Def. (Geometric Algebra)

For $\mathbb{R}^n$, we define the geometric algebra generated by it as a vector space spanned by the basis multivectors mentioned previously, i.e.

$$\mathcal{G}_n = \mathsf{span}\{1, \mathsf{e}_i, \mathsf{e}_i\mathsf{e}_j, \cdots, \mathsf{e}_1\mathsf{e}_2\cdots\mathsf{e}_n\}.$$

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An example:

$$\mathbb{R}^2 = \mathrm{span}\{\mathrm{e}_1, \mathrm{e}_2\},$$
$$\mathcal{G}_2 = \mathrm{span}\{1, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_1\mathrm{e}_2\}.$$

A general element (multivector) in $\mathcal{G}_2$ can be written as

$$M = \lambda + \alpha\mathrm{e}_1 + \beta\mathrm{e}_2 + \mu\mathrm{e}_1\mathrm{e}_2.$$

Geometric Algebra in 2D

$$\mathcal{G}_2 = \mathsf{span}\{1, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_1\mathsf{e}_2\}$$

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vectors: $\boldsymbol{x} = u\mathsf{e}_1 + v\mathsf{e}_2$ and **complex numbers**: $Z = u + vI$.

Immediately, we can see that

$$\mathsf{e}_1\boldsymbol{x} = Z, \mathsf{e}_1 Z = \boldsymbol{x}.$$

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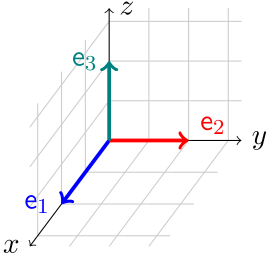
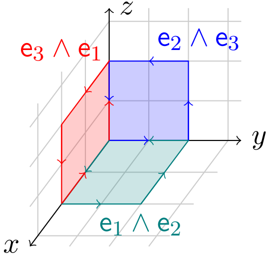
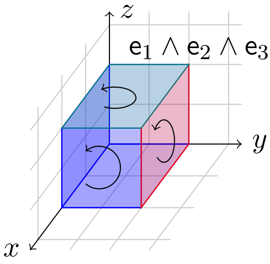
(this is just the same idea as orthogonal complements!), we can rewrite the basis multivectors as:

$$\mathcal{G}_3 = \text{span}\{1, \mathrm{e}_i, I\mathrm{e}_i, I\}$$

Geometric Algebra in 3D







Mirroring

Let's see how geometric algebra describes **reflections**, this will in turn lead us to the beautiful formulation of rotation in geometric algebra, and in turn explain the quaternion rotations.

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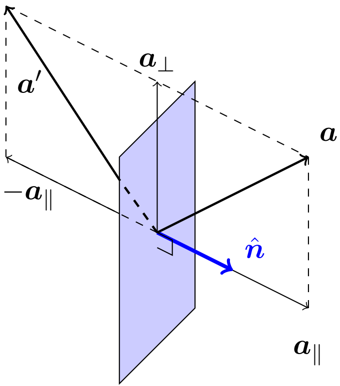

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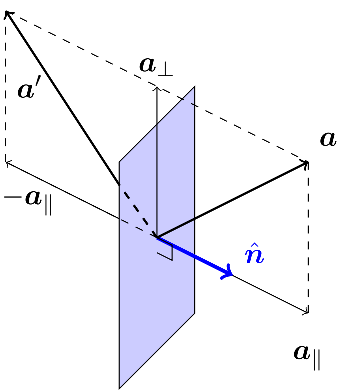

$$
\begin{aligned}
\boldsymbol{a}' &= \boldsymbol{a} - 2\boldsymbol{a}_\parallel \\
&= \boldsymbol{a} - 2(\boldsymbol{a} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \\
&= \boldsymbol{a} - (\boldsymbol{a}\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}}\boldsymbol{a})\boldsymbol{n} \\
&= \boldsymbol{a} - \boldsymbol{a}\hat{\boldsymbol{n}}\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}\boldsymbol{a}\hat{\boldsymbol{n}} \\
&= -\hat{\boldsymbol{n}}\boldsymbol{a}\hat{\boldsymbol{n}}
\end{aligned}
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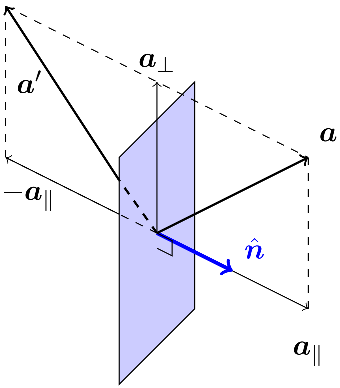

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\end{aligned}
$$

Hence, mirroring across the $\hat{\boldsymbol{n}}$ direction will be

$$\boldsymbol{a}' = -\hat{\boldsymbol{n}}\boldsymbol{a}\hat{\boldsymbol{n}}. \tag{22}$$

Rotation

What's the relation between reflections and rotations?

Two reflections = a rotation.

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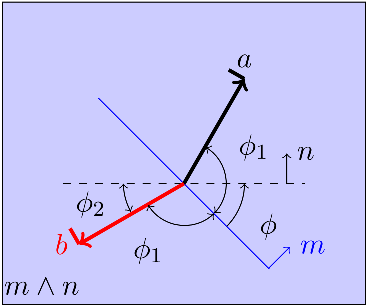

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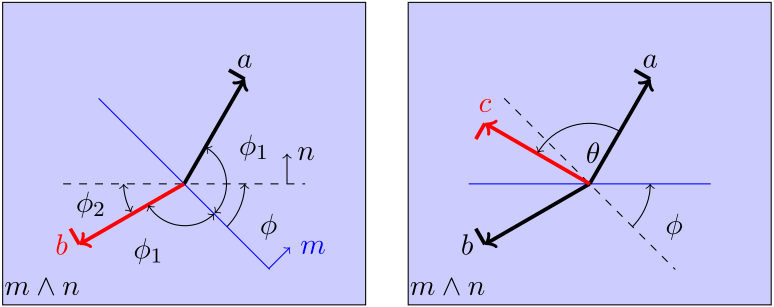

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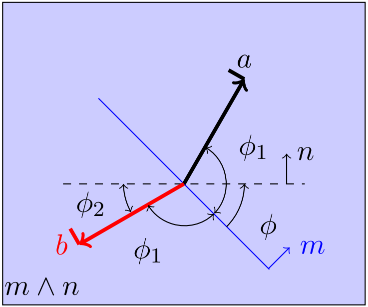
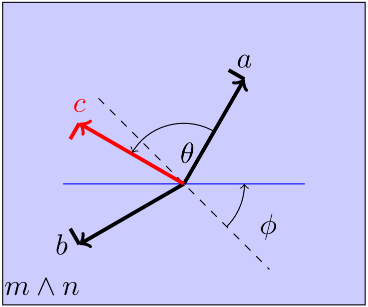
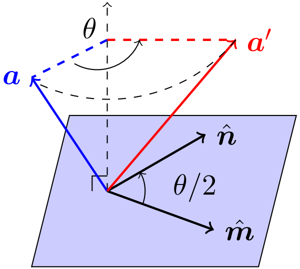




$$\theta = 2\phi$$

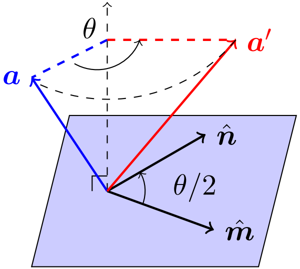
Rotation



$$\boldsymbol{a}' = -\hat{\boldsymbol{n}}(-\hat{\boldsymbol{m}}\boldsymbol{a}\hat{\boldsymbol{m}})\hat{\boldsymbol{n}}$$

Rotation

$$\begin{aligned} \boldsymbol{a}' &= -\hat{\boldsymbol{n}}(-\hat{\boldsymbol{m}}\boldsymbol{a}\hat{\boldsymbol{m}})\hat{\boldsymbol{n}} \\ &= (\hat{\boldsymbol{n}}\hat{\boldsymbol{m}})\boldsymbol{a}(\hat{\boldsymbol{m}}\hat{\boldsymbol{n}}) \end{aligned}$$



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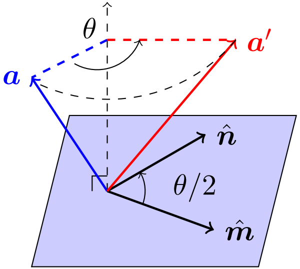
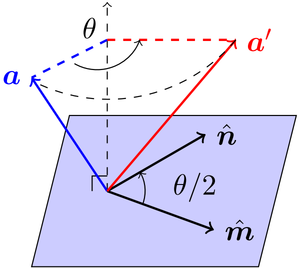


$$\begin{aligned} a' &= -\hat{n}(-\hat{m}a\hat{m})\hat{n} \\ &= (\hat{n}\hat{m})a(\hat{m}\hat{n}) \\ &= (\hat{n}\hat{m})a(\hat{n}\hat{m})^{\sim} \end{aligned}$$

Rotation



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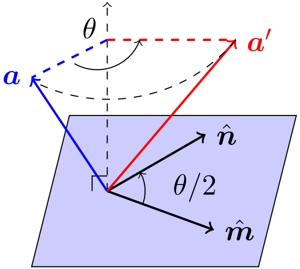

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$$

We define the object

$$R = \hat{\boldsymbol{n}}\hat{\boldsymbol{m}}$$

as a rotor, used to describe rotations.
Note: $(\cdot)^{\sim}$ is called "reversion", it reverses the order of geometric product.

Rotation

We can rewrite the rotor in a more inspiring fashion.

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$$R = \hat{\boldsymbol{n}}\hat{\boldsymbol{m}} = \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{m}} - \hat{\boldsymbol{m}} \wedge \hat{\boldsymbol{n}}.$$

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We have $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{m}} = \cos\frac{\theta}{2}$, and $\hat{\boldsymbol{m}} \wedge \hat{\boldsymbol{n}} = B \sin\frac{\theta}{2}$, where

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But since B is a bivector with unit area, $B^2 = -1$, we have

$$R = \mathrm{e}^{-B\frac{\theta}{2}}. \tag{23}$$

In the equation above, θ is the angle of rotation, and B is the plane in which objects rotate.

Quaternion Rotations

Rotor Rotation	$\boldsymbol{a}' = R\boldsymbol{a}R^{\sim}$	$R = \mathrm{e}^{-B\frac{\theta}{2}}$
Quaternionic Rotation	$\boldsymbol{p} = q\boldsymbol{p}q^{*}$	$q = \mathrm{e}^{\hat{\boldsymbol{n}}\frac{\theta}{2}}$

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The relation between quaternion and geometric algebra is now evident. Quaternions can be obtained by simply applying the following substitution:

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We've finally understand quaternions: they are the bivectors describing rotational planes living in $\mathbb{R}^3$, and the half angle is due to the fact that double-reflection equals a rotation.

Body-Axis Rotation proof

Let us proof the equation:

$$R_{\underbrace{R_{R_x(\theta_1)\hat{z}}(\theta_2)\ R_x(\theta_1)\ \hat{x}}_{\text{3rd axis}}}(\theta_3) \cdot R_{\underbrace{R_x(\theta_1)\ \hat{z}}_{\text{2nd axis}}}(\theta_2) \cdot R_x(\theta_1)$$

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$$= R_x(\theta_1) R_z(\theta_2) R_x(\theta_3).$$

Table of Contents

What more?

Rotations are important:

1. Using matrices to rotate vectors.

$$\boldsymbol{v}' = R_{\hat{\boldsymbol{n}}}(\theta)\boldsymbol{v}$$

2. Using quaternions to rotate vectors.

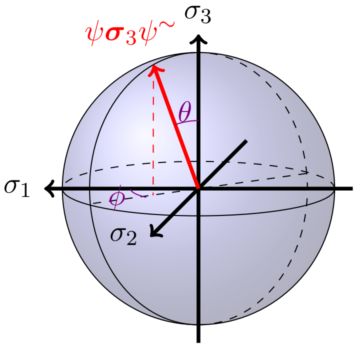
$$\boldsymbol{p}' = q\boldsymbol{p}q^*$$

3. Using geometric algebra to rotate vectors.

$$\boldsymbol{a}' = R\boldsymbol{a}R^{\sim}$$

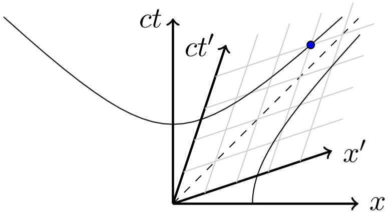
What more?

4. In quantum mechanics: a quantum state $|\psi\rangle$ can **rotate** the $|0\rangle$ state to others on the Bloch sphere. In which case we call $|\psi\rangle$ a **spinor** instead of a rotor.



What more?

5. In special relativity: the **Lorentz transformation** is a **hyperbolic rotation** in spacetime.



$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

That's all folks.