

# How do We Rotate?

Wen Perng

Electrical Engineering, NTU

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# How do we rotate things?

When you are coding with vPython or MATLAB, we often do three dimensional vector rotations via rotational matrices:

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ and etc.}$$

But is this the only way we can do rotations?

Moreover, matrices are

- ① computationally costly,
- ② lacking of intuition,
- ③ computationally unstable.

We need better ways.

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Many important mathematical concepts are met along the way.  
Let us walk through the various descriptions of rotation, and  
discover a new tool called **quaternions**.

# Table of Contents

- 1 Descriptions of Rotation
- 2 Rotational Matrices
- 3 Quaternions
- 4 Geometric Algebra: understanding quaternions
- 5 Final Remarks

## 2D Rotational Matrices

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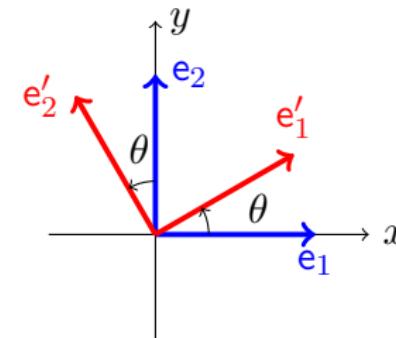
$$\mathbf{e}'_i = R\mathbf{e}_i = \begin{cases} \left[ \begin{array}{c|c} a & b \\ c & d \end{array} \right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, & i = 1 \\ \left[ \begin{array}{c|c} a & b \\ c & d \end{array} \right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}, & i = 2 \end{cases}.$$

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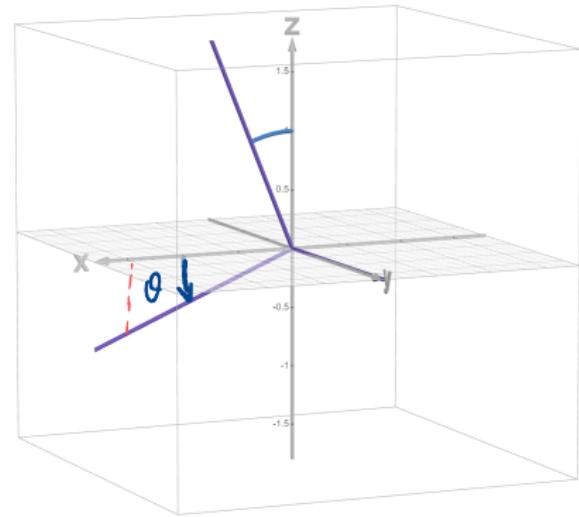
$$R = \left[ \begin{array}{c|c} \mathbf{e}'_1 & \mathbf{e}'_2 \end{array} \right] = \left[ \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right]$$



# 3D Rotational Matrices

The same procedure is used to produce the previous rotational matrices  $R_x$ ,  $R_y$  and  $R_z$ . Take  $R_y$  for example:

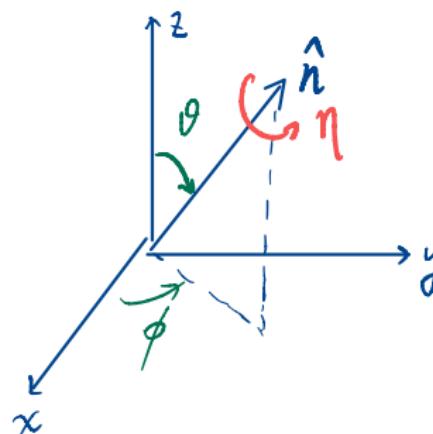
$$R_y(\theta) = \begin{bmatrix} \hat{x}' & \hat{y}' & \hat{z}' \\ \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$



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Hence, we have

$$R(\hat{n}, \eta) = R_z(\phi)R_y(\theta)R_z(\eta)R_y(-\theta)R_z(-\phi). \quad (1)$$

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$$\text{Rotate about } \hat{x}: R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad (2)$$

$$\text{Rotate about } \hat{y}: R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \quad (3)$$

$$\text{Rotate about } \hat{z}: R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4)$$

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For example, when we first rotate around fixed  $\hat{x}$  by  $\theta_3$ , then rotate around fixed  $\hat{z}$  by  $\theta_2$ , followed by a last rotation of  $\theta_1$  about  $\hat{x}$ , the system changes by

$$\mathbf{v} \mapsto R_x(\theta_1) R_z(\theta_2) R_x(\theta_3) \mathbf{v}.$$

See [[desmos](#)] for demo on  $R_x(\theta_1) R_z(\theta_2) R_x(\theta_3)$ .

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From the demo, we know that, amazingly:

$$\underbrace{R_{R_x(\theta_1)\hat{z}(\theta_2)} R_x(\theta_1)}_{\text{3rd axis}} \hat{x}(\theta_3) \cdot \underbrace{R_{R_x(\theta_1)\hat{z}(\theta_2)}}_{\text{2nd axis}} \cdot R_x(\theta_1)$$

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(I know this is some terrible notation, but that's how it goes for matrices.) We will prove this near the end of the presentation.

# Rodrigue's Formula

Thm. (Rodrigue's Rotation Formula)

A simple formula used to describe rotation of vector  $\mathbf{v}$  by angle  $\theta$  around axis  $\hat{\mathbf{n}}$  ( $|\hat{\mathbf{n}}| = 1$ ):

$$\mathbf{v}_{\text{rot}} = \mathbf{v} \cos \theta + (\hat{\mathbf{n}} \times \mathbf{v}) \sin \theta + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{v})(1 - \cos \theta) \quad (5)$$

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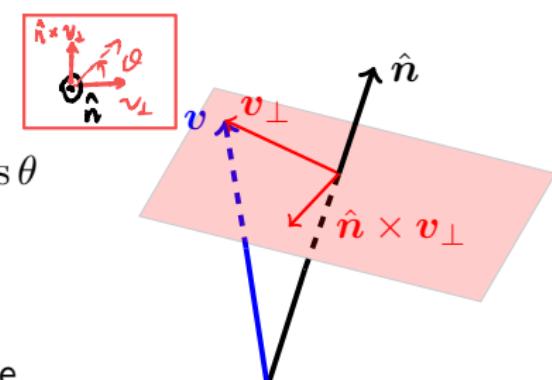
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[pf.]

$$\begin{aligned}\mathbf{v}_{\text{rot}} &= \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \cos \theta + \hat{\mathbf{n}} \times \mathbf{v}_{\perp} \sin \theta \\ &= \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{v}) + (\mathbf{v} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{v})) \cos \theta \\ &\quad + \hat{\mathbf{n}} \times \mathbf{v} \sin \theta\end{aligned}$$



Thus it is proven.

This equation is far easier to calculate.

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Here we define a special operation on vectors and matrices:

Def. (Transpose)

Flip the matrix along its diagonal:

$$\begin{bmatrix} \color{red}{a_{11}} & a_{12} & a_{13} \\ a_{21} & \color{red}{a_{22}} & a_{23} \end{bmatrix}^T = \begin{bmatrix} \color{red}{a_{11}} & a_{21} \\ a_{12} & \color{red}{a_{22}} \\ a_{13} & a_{23} \end{bmatrix}. \quad (6)$$

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$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$$

# Properties on Rotational Matrices

We can clearly observe that, under rotation, orthogonal vector remain orthogonal.

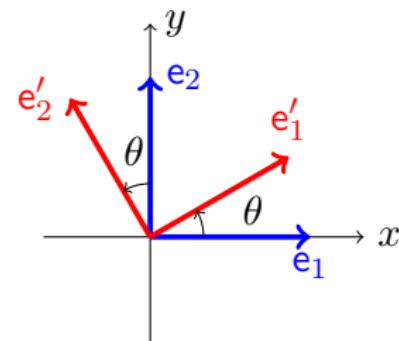
## Orthogonal Matrices

A matrix is called **orthogonal** if:

$$A^T A = A A^T = \mathbb{1}.$$

A rotational matrix is orthogonal since

$$\begin{aligned}
 R^T R &= \left[ \begin{array}{c} e_1'^T \\ \vdots \\ e_n'^T \end{array} \right] \left[ \begin{array}{c|c|c} e_1' & \cdots & e_n' \end{array} \right] = \mathbb{1}. \\
 &= \left[ \begin{array}{ccc} e_1'^T e_1' & e_1'^T e_2' & \cdots \\ e_2'^T e_1' & e_2'^T e_2' & \cdots \end{array} \right] = \left[ \begin{array}{ccc} 1 & 0 & \cdots \\ 0 & 1 & \cdots \end{array} \right]
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We call rotational matrices an element of the **special orthogonal group** of order  $n$ :

$$\begin{aligned}\text{SO}(n) &= \{n \times n \text{ rotational matrices}\} \\ &= \left\{ M \in \mathbb{R}^{n \times n} \mid M^T M = \mathbb{1}, \det(M) = 1 \right\}.\end{aligned}$$

## Infinitesimal Generators of Rotation

Let's go back to the case of 2D rotation. If we only rotated a small angle  $d\theta$ :

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$$R(\theta) = \lim_{N \rightarrow \infty} [R(\theta/N)]^N = \lim_{N \rightarrow \infty} \left( \mathbb{1} + \frac{1}{N} X \theta \right)^N = e^{X\theta}. \quad (7)$$

$X\theta$  is coined the **generator** of the rotation  $R(\theta)$ .

# Infinitesimal Generators of Rotation

Hence we have:

$$R(\theta) = e^{X\theta}, X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The converse can be easily checked.

Since  $X^2 = -\mathbb{1}$ , we have:

$$e^{X\theta} = \sum_{n=0}^{\infty} \frac{(X\theta)^n}{n!}$$

$$X^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbb{1}$$

$$X^3 = -X$$

$$X^4 = \mathbb{1}$$

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# Infinitesimal Generators of Rotation

For three dimensional rotations, we have:

$$\begin{aligned} L_x &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, R_x(\theta) = e^{L_x\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \\ L_y &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, R_y(\theta) = e^{L_y\theta} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \\ L_z &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R_z(\theta) = e^{L_z\theta} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{8}$$

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 \end{aligned} \tag{8}$$

Interestingly, all  $L_x$ ,  $L_y$  and  $L_z$  are **skew-symmetric** matrices, they are the infinitesimal generators of special orthogonal matrices. We will also see later that these are very similar to quaternions.

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↳ Strassen Alg.

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Is it orthogonal?

## Incapabilities of Rotational Matrices

Though rotational matrices are the first representation of rotation we've learnt, it isn't always the best. Here are some disadvantages of matrix:

- ① It requires 9 floats to store a matrix, 3 should be enough.
- ② Computationally expensive.
- ③ Floating point error renders it **non-orthogonal**.

Re-orthogonalization is expensive to calculate. → *Gram-Schmidt Orthogonalization*

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Though rotational matrices are the first representation of rotation we've learnt, it isn't always the best. Here are some disadvantages of matrix:

- ④ Hard to **interpolate** rotation: for animations, find a rotation  $R(t)$  s.t.  $R(0) = R_0$  and  $R(1) = R_1$ .

$$R(t) \stackrel{?}{=} (1-t)R_0 + tR_1 \text{ or } \cos(t)R_0 + \sin(t)R_1$$

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Axis:  $[1, 2, 0]$ , angle:  $45^\circ$ .

# Table of Contents

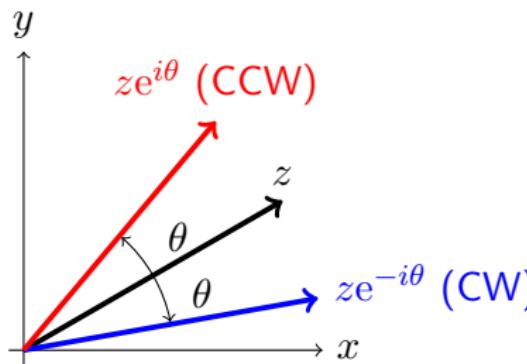
- 1 Descriptions of Rotation
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# Complex Numbers for Rotation

We often associate a unit complex number as a rotation operator:

$$z \mapsto ze^{i\theta} \equiv \text{rotate } z \text{ CCW by } \theta.$$

(CCW = counterclockwise; CW = clockwise)

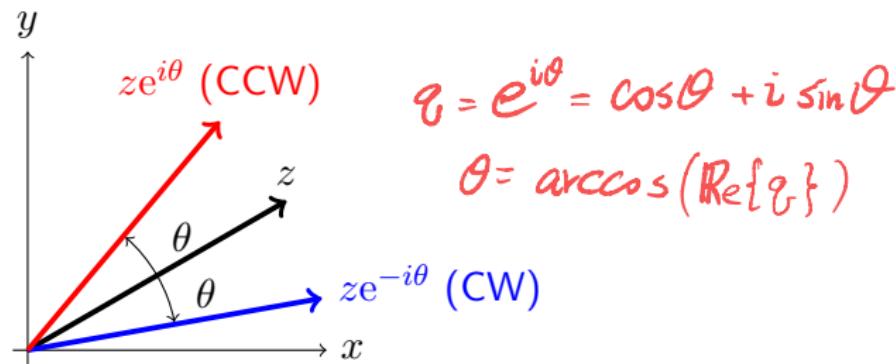


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How can we extend the idea of “multiply by a unit complex number” to do rotations in **three dimensional space**?

# Extension of Complex Numbers to 3D

Let us consider **trinions!**

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$$\cancel{ij = -1} \quad \textcolor{red}{ij = k}$$

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This fails as a good number system. We need a third complex number.

# Quaternion

## Def. (Quaternions)

The **quaternions**, denoted by  $\mathbb{H}$ , are spanned by  $1$ ,  $i$ ,  $j$  and  $k$ , satisfying the multiplication relation of

$$\begin{aligned}1^2 &= 1, \\i^2 = j^2 = k^2 &= -1, \\ijk &= -1. \quad \equiv \quad ijk\end{aligned}\tag{9}$$

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A general quaternion is expressed as

$$q = s + v_1 i + v_2 j + v_3 k = s + \mathbf{v},\tag{10}$$

$s = \text{Re}\{q\}$  is called the real/scalar part,  $\mathbf{v} = \text{Im}\{q\}$  is called the imaginary/vector part.

# Quaternion

From the defining relations

$$1^2 = 1,$$

不滿足交換律

$$i^2 = j^2 = k^2 = -1,$$
$$ijk = -1,$$

we can deduce the following multiplication relationships:

$$\underline{ij = k = -ji}, \quad jk = i = -kj, \quad ki = j = -ik.$$

$$(c_j k) k = (-1) k$$

$$-ij = -k$$

$$ij = k$$

$$ij = k$$
$$j(ij)j = j(k)j$$
$$-ji = ij = k$$

## Quaternion (some history)

*Quaternions* were conceived by the Irish mathematician Sir William Rowan Hamilton in 1843. It was on a bridge in Dublin when the defining relations of quaternions

$$i^2 = j^2 = k^2 = ijk = -1$$

dawned on him.

This algebraic system successfully extended complex numbers to three dimensions, explaining rotations elegantly. Though replaced by vector algebra not long after its invention, quaternions regained traction in fields of computer graphics nowadays.



Figure: W.Hamilton (Wiki)

# Quaternion

e.g. Quaternion Multiplication:

$$(1 + i)(2 - ji + j) = ?$$

$$= (1+i)(2+k+j)$$

$$= 2+k+j + i \cdot 2 + ik + ij$$

$$= 2+k+j + 2i - j + k$$

$$= 2+2i+2k \quad \cancel{*}$$

## Quaternion

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$$\vec{v} = i + j \quad q^* = (a + bi + cj + dk)^* = a - bi - cj - dk. \quad (12)$$

$$\vec{v}^2 = i^2 + ij + ji + j^2 = -2$$

- **Norm:** the norm of a quaternion is defined as

$$|q| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}. \quad (13)$$

Note that for vector components  $\mathbf{v}^2 \leq 0$ .

# Quaternion

- **Inverse:** the inverse of  $q$  satisfying  $qq^{-1} = q^{-1}q = 1$  is

$$q^{-1} = \frac{q^*}{qq^*} = \frac{q^*}{|q|^2} \quad (|q| \neq 0). \quad (14)$$

$$q q^{-1} = q \frac{q^*}{q q^*} = 1$$

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- **Exponentials:**

$$e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}. \quad (15)$$

- **Versor:** a versor is a unit quaternion, i.e.  $\sqrt{qq^*} = 1$ . Any versor can be written as

$$q = e^{t\mathbf{r}} = \cos t + \mathbf{r} \sin t, \quad (16)$$

where  $\mathbf{r}$  is a unit vector (expressed in quaternions:  $\mathbf{r}^2 = -1$ ).

# Quaternionic Rotation

So, how do we describe rotations with quaternions?

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Thm. (Quaternion rotation)

To rotate a vector  $p$  (in the sense of quaternionic vector) by an angle  $\theta$  around the unit vector  $\hat{n}$  (right hand rotation), let us define the versor:

$$q = e^{\hat{n}\theta/2} = \cos \frac{\theta}{2} + \hat{n} \sin \frac{\theta}{2},$$

then the rotated vector will be:

$$p' = qpq^*.$$

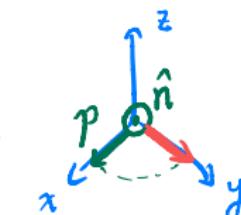
## Quaternionic Rotation

e.g. Rotate  $p = 1i$  by  $120^\circ$  around the axis  $\hat{n} = \frac{1}{\sqrt{3}}(i + j + k)$ .  
 [sol.]

$$p' = qpq^*$$

$$\hat{n}^2 = \frac{1}{3}(i+j+k)(i+j+k) = \frac{1}{3}(i^2 + j^2 + k^2) = -1$$

$$q = e^{\hat{n}\frac{\pi}{3}} = \cos \frac{\pi}{3} + \hat{n} \sin \frac{\pi}{3} = \frac{1}{2} + \frac{1}{2}(i+j+k)$$



$$p' = q(i)\left(\frac{1}{2}(1-i-j-k)\right) = \frac{1}{4}(1+i+j+k)(i+1-k+j)$$

$$\begin{aligned}
 &= \frac{1}{4} \begin{pmatrix} 1+i+j & -k \\ -1+i+j & +k \\ -1-i+j & -k \\ +1-i+j & +k \end{pmatrix} = \frac{1}{4}(4j) = j
 \end{aligned}$$

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$$\begin{aligned} p' &= qpq^* \\ &= \left( \frac{1}{2} + \frac{1}{2}(i + j + k) \right) (i) \left( \frac{1}{2} - \frac{1}{2}(i + j + k) \right) \end{aligned}$$

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$$p' = q p q^* = (-q)p(-q)^*$$

Note that both  $q$  and  $-q$  results in the same rotation.

## Advantages of Quaternions

When compared with matrices, quaternions have many advantages that made them more useful in fields of computer graphics and much more.

- ① **Axis-angle rotation is super intuitive!**
- ② Only requires 4 floats to store.
- ③ Computationally cheaper.
- ④ Invulnerable to floating point errors, since it is easy to re-normalize:

$$\text{versor} = \frac{\mathbf{q}}{|\mathbf{q}|}.$$

- ⑤ Easy to interpolate:

$$q(t) = \frac{(1-t)\mathbf{q}_0 + t\mathbf{q}_1}{|(1-t)\mathbf{q}_0 + t\mathbf{q}_1|}.$$

# Disadvantage of Quaternions

Even though quaternions are really useful, its development were hindered by the fact that “it is not easy to understand quaternions”!

3Blue1Brown even made videos explaining it..., requiring a mapping of **4D objects** into 3D space. It is not a good explanation.

The image displays two video thumbnails from the channel 3Blue1Brown, both titled "Quaternions".

The top thumbnail, titled "Visualizing quaternions (4d numbers) with stereographic projection", shows a 3D sphere with various axes and labels for  $i$ ,  $j$ ,  $k$ ,  $-i$ ,  $-j$ , and  $-k$ . A yellow arrow points from the left side of the sphere towards the right side, where a 2D stereographic projection of the sphere's surface is shown. The video duration is 31:51. Below the video are links to other related videos: "Intro", "Linus the linelander", "Felix the flatlander", "Mapping 4d to 3d", and "The geometry of quaternion...".

The bottom thumbnail, titled "Quaternions and 3d rotation, explained interactively", shows a 3D coordinate system with a red cube. A yellow arrow indicates a rotation around the  $x$ -axis. The video duration is 5:59. Below the video is a link to the channel description: "3blue1brown is a channel about animating math, in all senses of the word animate. And you know the drill with ...".

# Table of Contents

- 1 Descriptions of Rotation
- 2 Rotational Matrices
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Let us try to understand quaternions... in 3D!

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^*$$

$$\mathbf{q} = \cos \frac{\theta}{2} + \hat{\mathbf{n}} \sin \frac{\theta}{2}$$

We can do this by introducing the idea of multivectors and rotors in the study of **geometric algebra** / **Clifford algebra**.

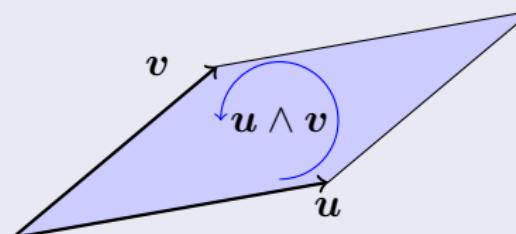
# Outer Product

Between two vector, we can take their inner product and their cross product, now let us introduce a new kind of product:

Def. (Outer product)

The outer product (wedge product) of two vectors  $u$  and  $v$  is defined as the **oriented area** element  $B$  formed by the two vectors:

$$B = u \wedge v.$$



The product is associative.

# Outer Product

Some properties:

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$$\mathbf{v} = \mathbf{u}$$

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- ⑤ The element  $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$  is called a **trivector**, an oriented volume.

# Outer Product

scalar	$\cdot^+$	$\cdot^-$
vector	$+u$	$-u$
bivector	$u \wedge v$	$v \wedge u = -u \wedge v$
trivector	$u \wedge v \wedge w$	$u \wedge v \wedge -w$

(a) Positive orientation.     
 (b) Negative orientation.

Figure: (Wiki)

# Outer Product

Here we introduce a new term called **grade**.

Object	1	$u$	$u \wedge v$	$u \wedge v \wedge w$	$\dots$	$u_1 \wedge \dots \wedge u_k$
Name	scalar	vector	bivector	trivector	$\dots$	$k$ -vector
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Between two vectors, we have their inner and outer product as

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containing the parallel and perpendicular information respectively.  
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**Wait! Can we add a scalar to an oriented area?**

Why not? It's just like complex numbers and quaternions.  
In fact, all objects of different grades can be added together,  
forming a general **multivector**, e.g.

$$M = 3 + 4\mathbf{u} \wedge \mathbf{v} + \mathbf{w}.$$

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If we have the canonical basis for  $\mathbb{R}^n$  as  $\{\mathbf{e}_i\}_{i=1}^n$ , i.e. they satisfy  
 → 相互正交·長度為 1.

$$\begin{cases} \mathbf{e}_i^2 = \mathbf{e}_i \mathbf{e}_i = \mathbf{e}_i \cdot \mathbf{e}_i = 1 \\ \mathbf{e}_i \mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i = -\mathbf{e}_j \mathbf{e}_i. \end{cases}$$

The relations can be concisely written as:

$$\mathbf{e}_i^2 = 1, \quad \mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i. \quad (18)$$

# Geometric Product

Let us calculate some examples: consider  $\{e_i\}_{i=1}^n$  the canonical basis for  $\mathbb{R}^n$ ,

e.g.(1)

$$\begin{aligned}\mathbf{uv} &= (u_1 e_1 + u_2 e_2)(v_1 e_1 + v_2 e_2) \\ &= u_1 e_1 v_1 e_1 + u_1 e_1 v_2 e_2 + u_2 e_2 v_1 e_1 + u_2 e_2 v_2 e_2\end{aligned}$$

# Geometric Product

Let us calculate some examples: consider  $\{e_i\}_{i=1}^n$  the canonical basis for  $\mathbb{R}^n$ ,  
e.g.(1)

$$\begin{aligned}\mathbf{uv} &= (u_1 e_1 + u_2 e_2)(v_1 e_1 + v_2 e_2) \\ &= u_1 v_1 \cancel{e_1 e_1} + u_1 v_2 \cancel{e_1 e_2} + u_2 v_1 \cancel{e_2 e_1} + u_2 v_2 \cancel{e_2 e_2} \\ &\quad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \\ &\quad 1 \qquad e_1 e_2 \qquad -e_1 e_2 \qquad 1\end{aligned}$$

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# Geometric Product

e.g.(2)

$$(e_1 e_2 e_3 + e_1 e_2)(e_2 e_1 + e_1 e_3) =$$

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$$(e_1 e_2 e_3 + e_1 e_2)(e_2 e_1 + e_1 e_3) = e_1 e_2 e_3 e_2 e_1 + e_1 e_2 e_2 e_1$$

$$\begin{aligned} e_i^2 &= 1 \\ e_i e_j &= -e_j e_i \end{aligned}$$

$$+ e_1 e_2 e_3 e_1 e_3 + e_1 e_2 e_1 e_3$$

$$e_1 e_2 (e_3 e_2) e_1 = -e_1 |e_2 e_2| e_3 e_1$$

$$= -(e_1 e_3) e_1$$

$$= -(-e_3 e_1) e_1$$

$$= e_3 |e_1 e_1|$$

$$= e_3$$

# Geometric Product

e.g.(2)

$$\begin{aligned} (\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 + \mathbf{e}_1 \mathbf{e}_2)(\mathbf{e}_2 \mathbf{e}_1 + \mathbf{e}_1 \mathbf{e}_3) &= \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_2 \mathbf{e}_1 + \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_2 \mathbf{e}_1 \\ &\quad + \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \mathbf{e}_1 \mathbf{e}_3 + \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_3 \\ &= \mathbf{e}_3 + 1 + \mathbf{e}_2 - \mathbf{e}_2 \mathbf{e}_3 \end{aligned}$$

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 &\quad + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3 \\
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 \end{aligned}$$

e.g.(3) We can define the inverse of a vector  $\mathbf{u}$  as  $\mathbf{u}^{-1}$ , satisfying

$$\underbrace{\mathbf{u}\mathbf{u}^{-1}}_{\text{vector}} = 1 = \vec{\mathbf{u}} \cdot \vec{\mathbf{u}}^{-1} + \vec{\mathbf{u}} \wedge \vec{\mathbf{u}}^{-1} \rightarrow \vec{\mathbf{u}}^{-1} \parallel \vec{\mathbf{u}}$$

$$\vec{\mathbf{C}} = \frac{\omega}{R} = \omega \vec{k}^{-1}$$

# Geometric Product

e.g.(2)

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 &\quad + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3 \\
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i.e.

$$\mathbf{\bar{u}} := \frac{\mathbf{u}}{|\mathbf{u}|^2}. \tag{19}$$

## Geometric Product

The geometric product is super easy to compute!

How good would it be if we can also express inner and outer product by geometric product:

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Hence,

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}), \tag{20}$$

$$\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u}). \tag{21}$$

# Geometric Algebra

Given a canonical basis  $\{e_i\}$  of  $\mathbb{R}^n$ , we can create a set of **basis multivectors** via **geometric products** between them:

$$1 \quad e_i \quad e_i e_j \quad e_i e_j e_k \quad \cdots \quad e_1 e_2 \cdots e_n$$

# Geometric Algebra

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$$\begin{array}{ccccccc} 1 & e_i & e_i e_j & e_i e_j e_k & \cdots & e_1 e_2 \cdots e_n \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \cdots & \text{grade n} \end{array}$$

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		$e_i \wedge e_j$	$e_i \wedge e_j \wedge e_k$	$\dots$	$e_1 e_2 \dots e_n$
1	$e_i$	$e_i \wedge e_j$	$e_i \wedge e_j \wedge e_k$	$\dots$	$e_1 e_2 \dots e_n$
grade 0	grade 1	grade 2	grade 3	$\dots$	grade n
$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\dots$	$\binom{n}{n}$

Def. ( $n$  choose  $k$ )

The amount of choices of  $k$  objects from  $n$  objects is  $\binom{n}{k} = C_k^n$ .

We often call the highest grade element the **pseudoscalar**, and denote it by

$$\overset{\uparrow}{e_k} \quad I := e_1 e_2 \dots e_n = e_1 \wedge e_2 \wedge \dots \wedge e_n$$

$$(e_i \wedge e_j) e_k = e_i \wedge e_j \wedge e_k$$

# Geometric Algebra

## Def. (Geometric Algebra)

For  $\mathbb{R}^n$ , we define the geometric algebra generated by it as a vector space spanned by the basis multivectors mentioned previously, i.e.

$$\mathcal{G}_n = \text{span}\{1, e_i, e_i e_j, \dots, e_1 e_2 \dots e_n\}.$$

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An example:

$$\mathbb{R}^2 = \text{span}\{e_1, e_2\},$$

$$\mathcal{G}_2 = \text{span}\{1, e_1, e_2, e_1 e_2\}.$$

A general element (multivector) in  $\mathcal{G}_2$  can be written as

$$M = \lambda + \alpha e_1 + \beta e_2 + \mu e_1 e_2.$$

# Geometric Algebra in 2D

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For  $\mathcal{G}_2$ , let us first consider its highest grade element  $e_1e_2 =: I$ ,

$$I^2 =$$

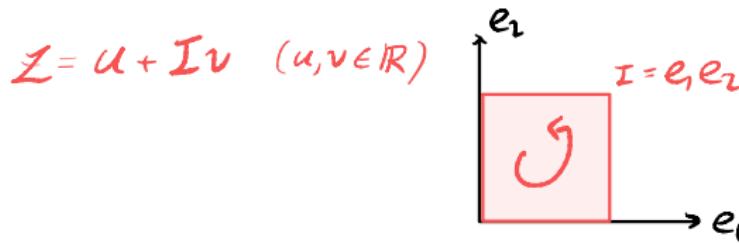
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Amazingly, the bivector  $I$  behaves just like the complex numbers!  
(It became apparent why multiplication by  $I$  rotates CCW.)



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**vectors:**  $x = ue_1 + ve_2$  and **complex numbers:**  $Z = u + vI$ .

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$$\mathcal{G}_2 = \text{span}\{1, e_1, e_2, e_1e_2\} = \mathbb{R}^2 \oplus \mathbb{C}$$

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Immediately, we can see that

$$e_1(ue_1 + ve_2) = ue_1e_1 + ve_2e_1 \stackrel{I}{=} u + Iv$$

$$e_1''x = Z, e_1Z = x.$$

$$e_1''(e_1\vec{x}) = (e_1e_1)\vec{x} = \vec{x}$$

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The **unit pseudoscalar** and **unit bivectors** all squares to negative one:

$$\begin{aligned}
 I^2 &= (e_1 e_2 e_3)^2 = -1, \quad (\overbrace{e_i e_j}^{3 \times 3})^2 = -1. \quad (i \neq j) \\
 &= e_1 e_2 e_3 e_3 e_2 e_1 \\
 &= e_1 e_1 e_2 e_3 e_3 e_2 \\
 &= e_2 e_3 (-e_3 e_2) \quad \quad \quad = -e_1 e_2 e_3 e_1 \\
 &= -1 \quad \quad \quad = -1
 \end{aligned}$$

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Moreover, since the vector and bivector components satisfy

$$e_2 e_3 = I e_1, \quad e_3 e_1 = I e_2, \quad e_1 e_2 = \underline{\underline{I e_3}}$$

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(this is just the same idea as orthogonal complements!)

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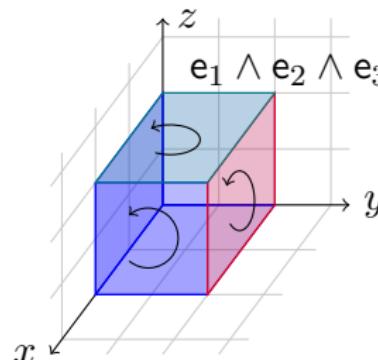
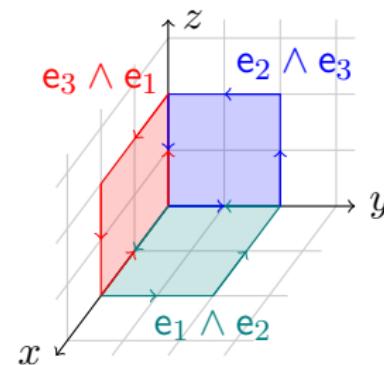
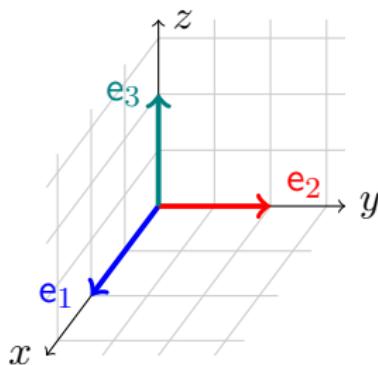
Moreover, since the vector and bivector components satisfy

$$e_2 e_3 = I e_1, \quad e_3 e_1 = I e_2, \quad e_1 e_2 = I e_3$$

(this is just the same idea as orthogonal complements!), we can rewrite the basis multivectors as:

$$\mathcal{G}_3 = \text{span}\{1, e_i, I e_i, I\}$$

# Geometric Algebra in 3D

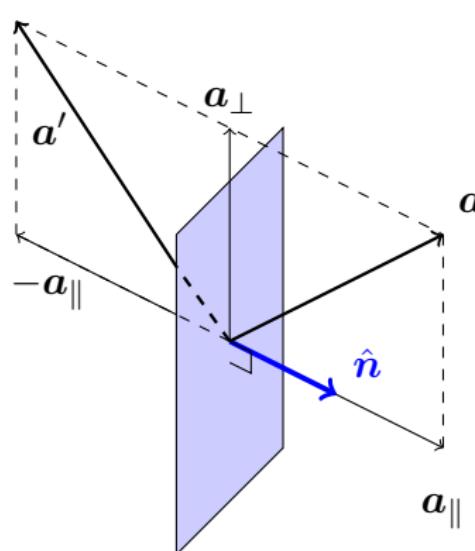


# Mirroring

Let's see how geometric algebra describes **reflections**, this will in turn lead us to the beautiful formulation of rotation in geometric algebra, and in turn explain the quaternion rotations.

# Mirroring

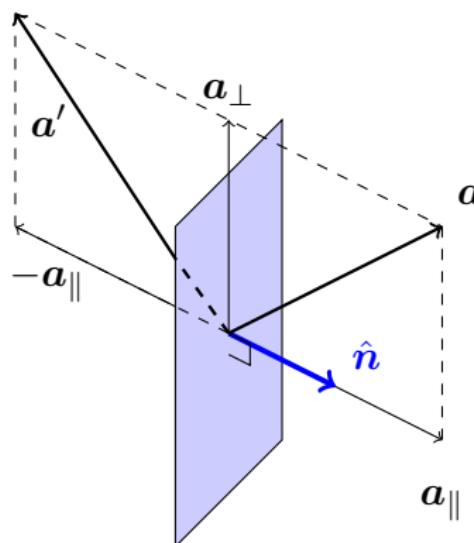
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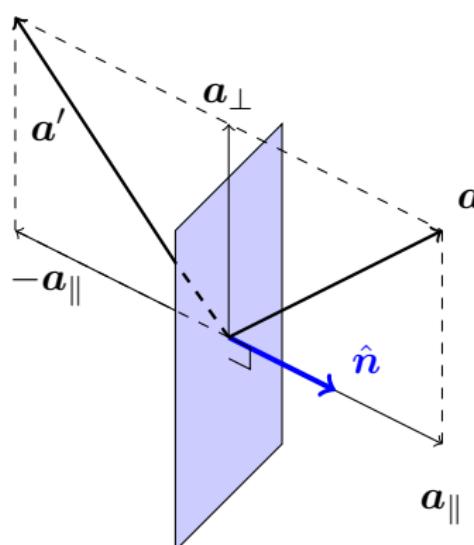


$$\begin{aligned}
 a' &= a - 2a_{\parallel} \\
 &= a - 2(a \cdot \hat{n})\hat{n} \\
 &= a - (a\hat{n} + \hat{n}a)\hat{n} \\
 &= a - a\hat{n}\hat{n} - \hat{n}a\hat{n} \\
 &= -\hat{n}a\hat{n}
 \end{aligned}$$

$$u \cdot v = \frac{1}{2}(uv + vu)$$

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 &= -\hat{n}a\hat{n}
 \end{aligned}$$

Hence, mirroring across the  $\hat{n}$  direction will be

$$a' = -\hat{n}a\hat{n}. \quad (22)$$

# Rotation

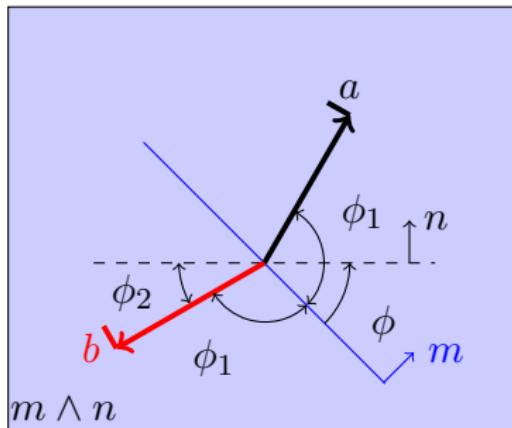
What's the relation between reflections and rotations?

**Two reflections = a rotation.**

# Rotation

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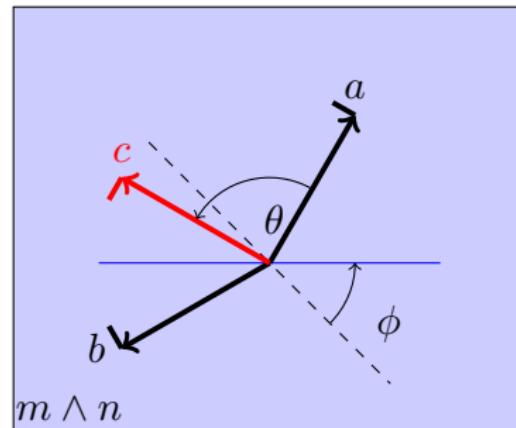
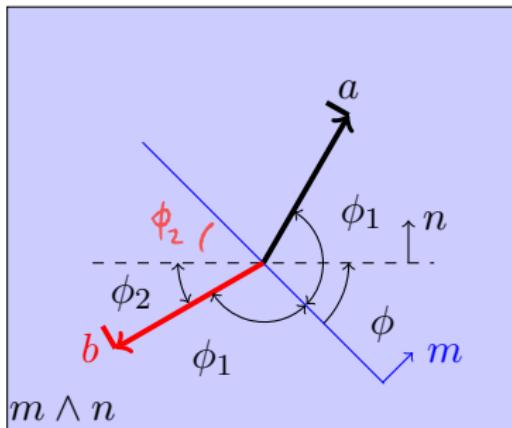
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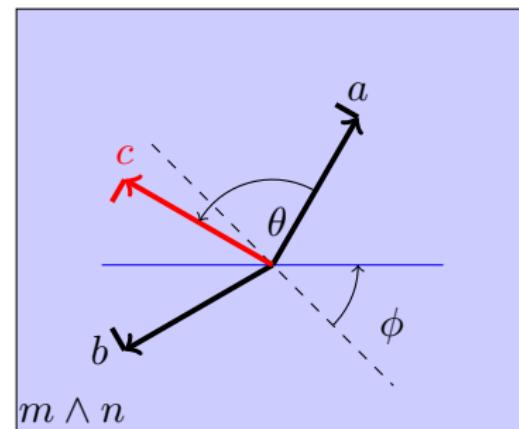
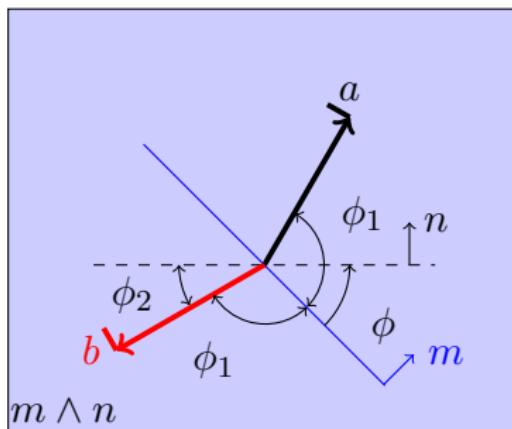


$$\begin{cases} \theta = \pi - \phi_2 - (\phi_1 - \phi) \rightarrow \theta = 2\phi \\ \pi = \phi_2 + \phi_1 + \phi \end{cases}$$

# Rotation

What's the relation between reflections and rotations?

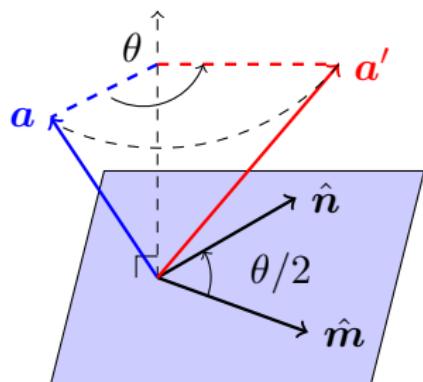
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$$\theta = 2\phi$$

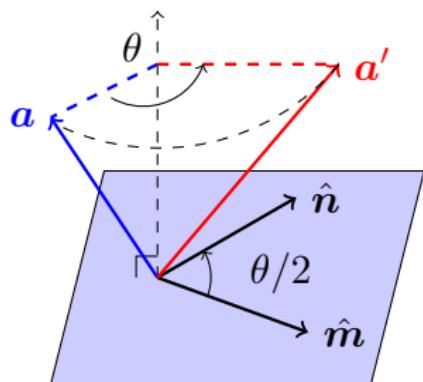
# Rotation

$$\mathbf{a}' = -\hat{\mathbf{n}}(-\hat{\mathbf{m}} \mathbf{a} \hat{\mathbf{m}})\hat{\mathbf{n}}$$



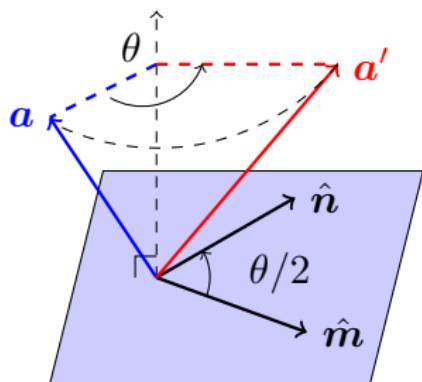
# Rotation

$$\begin{aligned} \mathbf{a}' &= -\hat{\mathbf{n}}(-\hat{\mathbf{m}}\mathbf{a}\hat{\mathbf{m}})\hat{\mathbf{n}} \\ &= (\hat{\mathbf{n}}\hat{\mathbf{m}})\mathbf{a}(\hat{\mathbf{m}}\hat{\mathbf{n}}) \end{aligned}$$

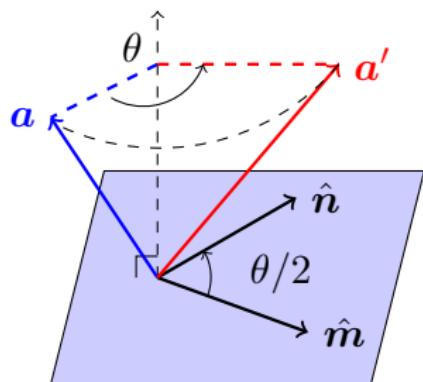


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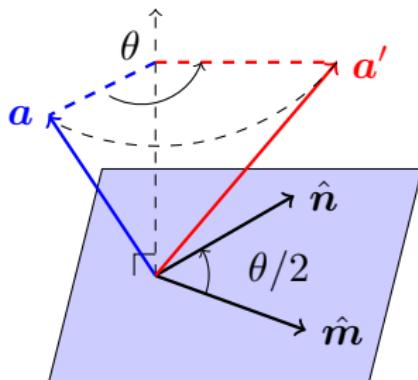


# Rotation



$$\begin{aligned} a' &= -\hat{n}(-\hat{m}a\hat{m})\hat{n} \\ &= (\hat{n}\hat{m})a(\hat{m}\hat{n}) \\ &= (\hat{n}\hat{m})a(\hat{n}\hat{m})^\sim \\ &= \mathbf{R}a\mathbf{R}^\sim \end{aligned}$$

# Rotation



$$\begin{aligned}
 a' &= -\hat{n}(-\hat{m}a\hat{m})\hat{n} \\
 &= (\hat{n}\hat{m})a(\hat{m}\hat{n}) \\
 &= (\hat{n}\hat{m})a(\hat{n}\hat{m})^\sim \\
 &= RaR^\sim
 \end{aligned}$$

We define the object

$$R = \hat{n}\hat{m}$$

as a **rotor**, used to describe rotations.

Note:  $(\cdot)^\sim$  is called “reversion”, it reverses the order of geometric product.

# Rotation

We can rewrite the rotor in a more inspiring fashion.

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$$R = e^{-B \frac{\theta}{2}} = \cos \frac{\theta}{2} - B \sin \frac{\theta}{2} \quad (23)$$

In the equation above,  $\theta$  is the angle of rotation, and  $B$  is the plane in which objects rotate.

# Quaternion Rotations

Rotor Rotation

$$\mathbf{a}' = R\mathbf{a}R^\sim$$

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Quaternionic Rotation

$$\mathbf{p} = q\mathbf{p}q^*$$

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The relation between quaternion and geometric algebra is now evident. Quaternions can be obtained by simply applying the following substitution:

$$i = -\mathbf{e}_2\mathbf{e}_3,$$

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We've finally understand quaternions: they are the bivectors describing **rotational planes living in  $\mathbb{R}^3$** , and the half angle is due to the fact that double-reflection equals a rotation.

# Body-Axis Rotation proof

Let us proof the equation:

$$\underbrace{R_{R_x(\theta_1)\hat{z}(\theta_2)} R_x(\theta_1)}_{\text{3rd axis}} \hat{x}(\theta_3) \cdot \underbrace{R_{R_x(\theta_1)\hat{z}(\theta_2)}}_{\text{2nd axis}} \cdot R_x(\theta_1)$$

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$$R_1 = e^{-e_2 e_3 \frac{\theta_1}{2}}, \quad R_2 = e^{-R_1(e_1 e_2) \tilde{R}_1 \frac{\theta_2}{2}}, \quad R_3 = e^{-R_2 R_1(e_1 e_3) \tilde{R}_1 \tilde{R}_2 \frac{\theta_3}{2}}$$

$$= R_1 e^{-e_1 e_2 \frac{\theta_1}{2} \tilde{R}_1} \quad \boxed{RR = 1 = \tilde{R}\tilde{R}} \quad R_2 R_1 e^{-e_1 e_3 \frac{\theta_3}{2} \tilde{R}_1 \tilde{R}_2}$$

$$R_2 R_1 e^{-e_1 e_3 \frac{\theta_3}{2} \tilde{R}_1 \tilde{R}_2} R_1 e^{-e_1 e_2 \frac{\theta_1}{2} \tilde{R}_1 \tilde{R}_1} = R_1 e^{-e_1 e_2 \frac{\theta_1}{2} \tilde{R}_1 \tilde{R}_1} e^{-e_1 e_3 \frac{\theta_3}{2} \tilde{R}_1 \tilde{R}_2} \cancel{R_2 R_1 e^{-e_1 e_3 \frac{\theta_3}{2} \tilde{R}_1 \tilde{R}_2}} \cancel{R_1 e^{-e_1 e_2 \frac{\theta_1}{2} \tilde{R}_1 \tilde{R}_1}}$$

$$= e^{-e_2 e_3 \frac{\theta_1}{2}} e^{-e_1 e_2 \frac{\theta_2}{2}} e^{-e_1 e_3 \frac{\theta_3}{2}} \blacksquare$$

# Table of Contents

- 1 Descriptions of Rotation
- 2 Rotational Matrices
- 3 Quaternions
- 4 Geometric Algebra: understanding quaternions
- 5 Final Remarks

# What more?

Rotations are important:

- ① Using matrices to rotate vectors.

$$\mathbf{v}' = R_{\hat{\mathbf{n}}}(\theta)\mathbf{v}$$

- ② Using quaternions to rotate vectors.

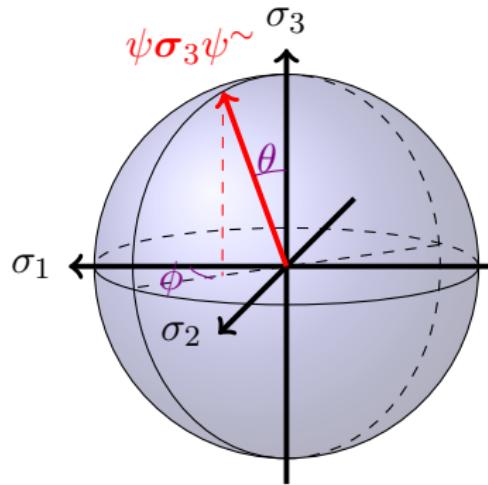
$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^*$$

- ③ Using geometric algebra to rotate vectors.

$$\mathbf{a}' = \mathbf{R}\mathbf{a}\mathbf{R}^\sim$$

# What more?

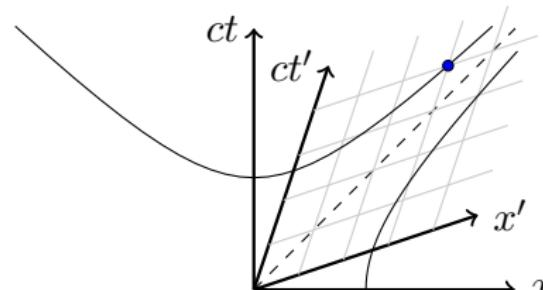
- ④ In quantum mechanics: a quantum state  $|\psi\rangle$  can **rotate** the  $|0\rangle$  state to others on the Bloch sphere. In which case we call  $|\psi\rangle$  a **spinor** instead of a rotor.



# What more?

- ⑤ In special relativity: the **Lorentz transformation** is a **hyperbolic rotation** in spacetime.

$$\vec{x}' = R \vec{x} \tilde{R}$$
$$R = e^{-B\frac{\eta}{2}}$$



$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh \eta & -\sinh \eta \\ -\sinh \eta & \cosh \eta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

That's all folks.