Stability Analysis on Replicator Games with Perturbations

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1 Motivation and Objectives of Study

1.1 Needs and Significance

Over the course, we were taught of many different kinds of strategies and equilibria of a game. Of all the different equilibria discussed, pure or mixed Nash equilibria, we have yet to identify what the ”meaning” of them are to the game. I believe we are able to gain further insights into the different equilibria via replicator dynamics.

Furthermore, during my presentation on ”Learning in Repeated Games”, the idea–”convergence of strategy by introduction of perturbations to payoffs” was introduced. Throughout my survey on multiple literature ([1],[2],[5]), the perturbation were introduced merely as a product of mathematical manipulation/abstraction. But game theory should not be confined to only mathematical ambiguity, it should also represent reality. Thus, I plan on to come up with a physical meaning of the perturbation via replicator dynamics, too.

1.2 Objectives

1. Introduction to replicator dynamics and stability analysis on the phase portrait.
2. Replicator dynamics on a single player game.
3. Replicator dynamics on a two player game. For example, game of chicken, and the game of rock-paper-scissors.
4. Demonstration of convergence in strategy under perturbations.
5. Physical interpretation of perturbation on payoffs.

2 Literature Survey

What are replicator dynamics? The papers [3] and [4] provide detailed information on the properties of replicator dynamics. Within it, ideas of stability of the equilibria were also established. A small thing to include within is to analyze the game via direct observation of the phase portrait created from the equations of replicator dynamics. And from it, derive some physical interpretations of games.

What are the effects of perturbations? The paper [2] provides examples of it for one to learn more. From it, one can see how the perturbation affects a game.

For one to learn more, more detailed proofs in information were provided in [5]. However, it doesn't provide any physical interpretation of it, nor can I find one on the internet.

3 Approach of Study

3.1 Replicator Dynamics

Replicator dynamics is a game in which a population (or more) of decision players randomly (uniformly) competes with each other. If ones payoff is lower than his opponent, he will change to other's strategy proportionally. Such simple learning strategy creates rich interpretations and many non-trivial results.

Suppose the players have the same strategy set, with mixed strategy $\boldsymbol{x} = [x_1 \ x_2 \ \cdots \ x_n]^T$, for x_k being the probability of doing action k. And the game matrix is Π (for payoff), then the replicator dynamics state that

$$\dot{x}_i(t) = \alpha x_i(t) \left[(\Pi \boldsymbol{x})_i - \boldsymbol{x}^T \Pi \boldsymbol{x} \right]$$

where α is a positive constant.

3.2 Stability

Say the mixed strategy (the state) evolves over time subject to the following nonlinear dynamics:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) = \begin{bmatrix} f_1(\boldsymbol{x}(t)) \\ f_2(\boldsymbol{x}(t)) \\ \cdots \\ f_n(\boldsymbol{x}(t)) \end{bmatrix}$$

we can take first order approximation of the system at a *stationary point* $\boldsymbol{a}$ ($\boldsymbol{f}(\boldsymbol{a}) = 0$):

$$\dot{\boldsymbol{x}} \approx \cancel{\boldsymbol{f}(\boldsymbol{a})} + \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}} (\boldsymbol{x} - \boldsymbol{a})$$

We define a translation of coordinates: $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{a}$. And the term

$$A := \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_1(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\boldsymbol{x})}{\partial x_n} \\ \frac{\partial f_2(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_2(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\boldsymbol{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_n(\boldsymbol{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\boldsymbol{x})}{\partial x_n} \end{bmatrix}$$

is known as the *Jacobian* of the system, or it can also be viewed as the system matrix of the linearized dynamics in the neighborhood of a stationary point.

We can define terms to characterize the stability of the origin (a stationary point):

1. If A is positive definite (positive eigenvalues), it is unstable.
2. If A is negative definite (negative eigenvalues), it is asymptotically stable.
3. If A is negative semi-definite (non-positive eigenvalues), it is (Lyapunov) stable.
4. If A has eigenvalues of both positive and negative, it is a stable point.

Such definitions are straightforward, it describes how the magnitude of $\boldsymbol{x}$ evolves as t goes to infinity.

3.3 Perturbations

We introduce the *Gibbs entropy* for a state $\boldsymbol{x} = [x_1\ x_2\ \cdots\ x_n]^T$ as:

$$v(\boldsymbol{x}) = \sum_i x_i \log x_i$$

It is demonstrated in [2] and [5] that the perturbed dynamics of

$$\dot{x}_i(t) = \alpha x_i \left[(\Pi \boldsymbol{x})_i - \boldsymbol{x}^T \Pi \boldsymbol{x} - \varepsilon(t) \cdot v(\boldsymbol{x}) \right]$$

converges to the mixed strategy, given that $\varepsilon(t)$ is a suitable positive decreasing function.

4 Challenges and Planned Approach

One of the hardest part of the project is to analyse the effects of the perturbation. One is how to select the suitable $\varepsilon(t)$, another is what its physical meaning is.

[5] includes some detailed derivations of the effects of perturbations, but I need to dig deeper into it before drawing any conclusions. And perhaps, by conducting more simulations of replicator dynamics with perturbation on MATLAB, I can come up with a suitable interpretation.

References

[1] Jun Wang, *Learning in Repeated Games*, Computer Science, UCL, Oct 15 2018, SJTU.

[2] Justin Kang, *A Review of Smooth Fictitious Play and Fictitious Play in Repeated Games*, Department of Electrical and Computer Engineering, University of Toronto. https://justinkang221.github.io/files/paper5.pdf, June 10, 2021.

[3] Tim Rees, *An Introduction to Evolutionary Game Theory*, UBC Department of Computer Science.

[4] *The Replicator Dynamic (Draft)*, https://www.ma.imperial.ac.uk/ svanstri/GamesAndDynamics/The%20Replicator%20Dynamic%20(Draft).pdf.

[5] Lucas Baudin, and Rida Laraki, *Smooth Fictitious Play in Stochastic Games with Perturbed Payoffs and Unknown Transitions*, Université Paris-Dauphine - PSL, arXiv:2207.03109v1 [cs.GT], July 7, 2022.