Stability Analysis on Replicator Games with Perturbations

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ToG

1. Replicator Dynamics
2. Stability Analysis
3. Single Population Game
4. Multi-Population Game
5. Perturbations

1. Replicator Dynamics

Replicator Dynamics

Game Settings

- A game within a population.
- All have the same game matrix Π and strategy:
$$S = \{s_1, \cdots, s_n\}.$$
- Ratio of players playing strategy s_k at an instance is x_k. Let the state of game be
$$\boldsymbol{x} = [x_1, \cdots, x_n].$$
- The expected payoff playing s_k is
$$\pi_{k\sigma} = (\Pi \boldsymbol{x})_k.$$

Replicator Dynamics

Replicator Dynamics:

- At any instance, players of the population are paired up to play the game.
- If the enemy has a higher payoff, one might replicate the enemy’s strategy.
- The game dynamics:

$$\dot{x}_k = \alpha x_k[(\Pi \boldsymbol{x})_k - \boldsymbol{x}^{\mathrm{T}}\Pi \boldsymbol{x}]$$

(replicator dynamics has $n - 1$ DEs)

2. Stability Analysis

Stability Analysis

For game described as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

It can be approximated linearly as

$$\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{a}) \coloneqq A(\boldsymbol{x} - \boldsymbol{a})$$

Where $\boldsymbol{f}(\boldsymbol{a}) = \mathbf{0}$ is a stationary point.

- If $A \prec 0$, it is negative definite, then it is asymptotically stable.
- Imaginary E-val. of A creates oscillatory orbits.

Stability Analysis

- **Nash Equilibrium**

$$\boldsymbol{x}^N \text{ is Nash(pure/mixed) of } \Pi$$
$$\Updownarrow$$
$$\boldsymbol{f}(\boldsymbol{x}^N) = \boldsymbol{0} \text{ and } \frac{\partial \boldsymbol{f}(\boldsymbol{x}^N)}{\partial \boldsymbol{x}} \nsucc 0$$

This definition comes straightly from the dynamics of the phase portrait.

3. Single Population Game

Single Population RPS Game

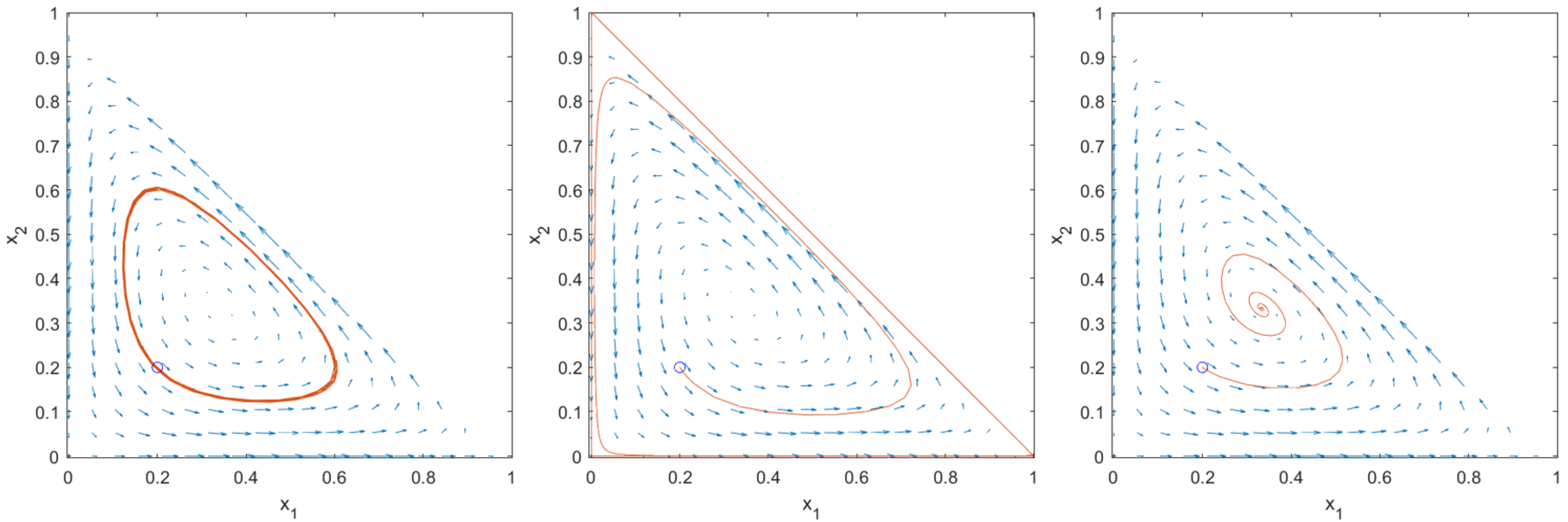
(Rock-Paper-Scissors Game)

Opponent / P	Rock x_1	Paper x_2	Scissor $1-x_1-x_2$
Rock x_1	0	-1	a
Paper x_2	a	0	-1
Scissor $1-x_1-x_2$	-1	a	0

$$\Pi = \begin{bmatrix} 0 & -1 & a \\ a & 0 & -1 \\ -1 & a & 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 - x_1 - x_2 \end{bmatrix}$$

Single Population RPS Game

(Rock-Paper-Scissors Game)



$a = 1$ $a = 0.5$ $a = 2$

4. Multi-Population Game

Extension to Multi-Population

- Population $\boldsymbol{x}$ and $\boldsymbol{y}$ compete with each other.
- Same strategy set S and payoff matrix Π.
- The Dynamics:

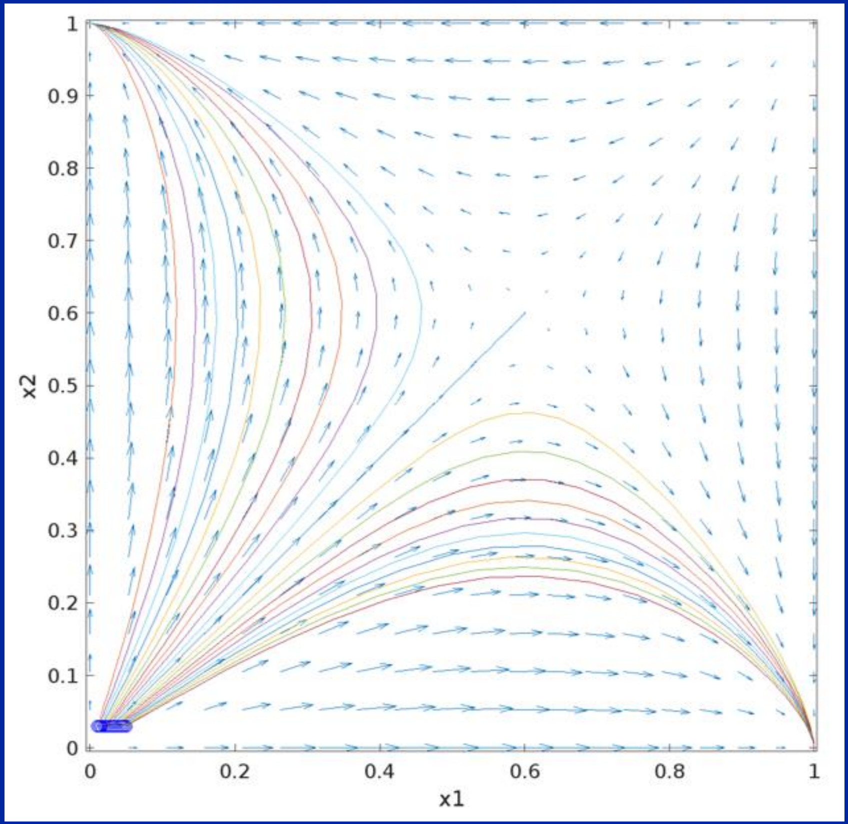
$$\dot{x}_k = \alpha x_k[(\Pi\boldsymbol{y})_k - \boldsymbol{x}^{\mathrm{T}}\Pi\boldsymbol{y}]$$
$$\dot{y}_k = \beta y_k[(\Pi\boldsymbol{x})_k - \boldsymbol{y}^{\mathrm{T}}\Pi\boldsymbol{x}]$$

- The same idea of changing strategy based on difference of payoff.

Extension to Multi-Population

Game of Chicken (Cold War)

US \ USSR	Launch x_2	Don't $1-x_2$
Launch x_1	-5,-5	3,-3
Don't $1-x_1$	-3,3	0,0



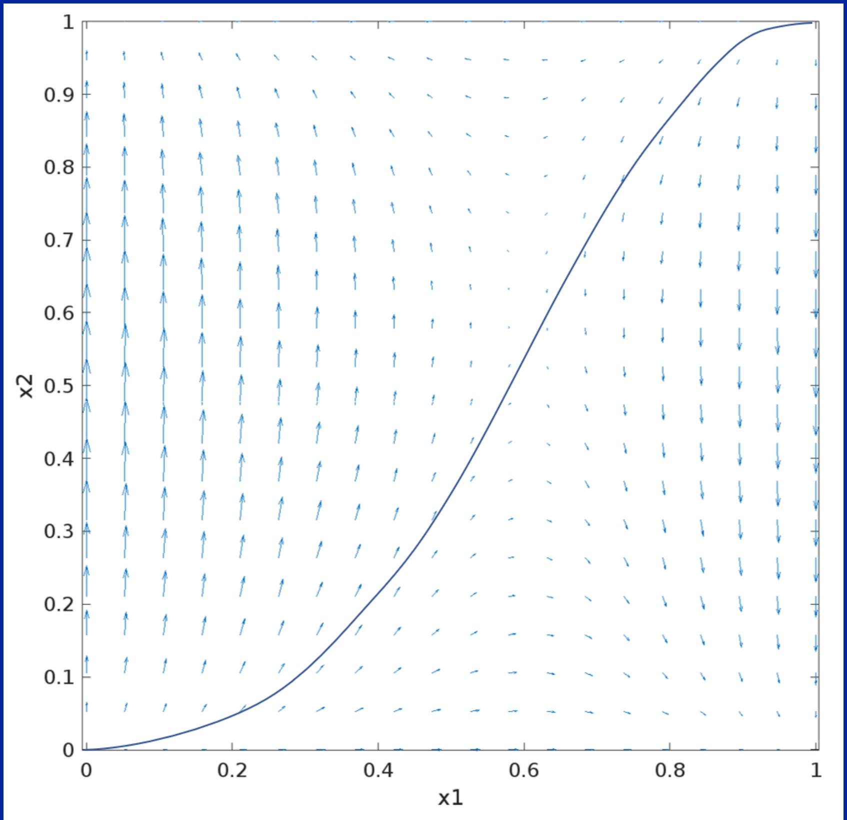
$$\dot{x}_1 = x_1(1-x_1)(3-5x_2)$$

$$\dot{x}_2 = x_2(1-x_2)(3-5x_1)$$

Extension to Multi-Population

Game of Chicken (Cold War)

US \ USSR	Launch x_2	Don't $1-x_2$
Launch x_1	-5,-5	3,-3
Don't $1-x_1$	-3,3	0,0



$$\dot{x}_1 = x_1(1-x_1)(3-5x_2)$$

$$\dot{x}_2 = 5x_2(1-x_2)(3-5x_1)$$

Extension to Multi-Population

Problems:

1. No derivation, formed from viewing the population as a whole.
2. The dynamical equations neglect the individual choices.
3. The dynamics of x_n is not compatible.

5. Perturbations

Perturbations

- **Perturbed Dynamics:**

$$\dot{x}_k = \alpha x_k \left[(\Pi \boldsymbol{x})_k - \boldsymbol{x}^{\mathrm{T}} \Pi \boldsymbol{x} - \varepsilon \cdot v(\boldsymbol{x}) \right]$$

where

$$v(\boldsymbol{x}) = -\sum_{k=1}^{n} x_k \ln x_k$$

"The Gibbs entropy is a measure of unexpectedness."

Perturbations

Surprise:

- Expectedness: $\mathcal{S}(1) = 0$
- Unexpectedness: $\mathcal{S}(0) = \infty$
- Additivity: $\mathcal{S}(x_1 \cdot x_2) = \mathcal{S}(x_1) + \mathcal{S}(x_2)$
- Continuity: $\mathcal{S} \in C$

$$\rightarrow \mathcal{S}(x) = -\ln x$$

Expected Surprise:

$$v(\boldsymbol{x}) = -\sum_k x_k \ln x_k$$

Perturbations

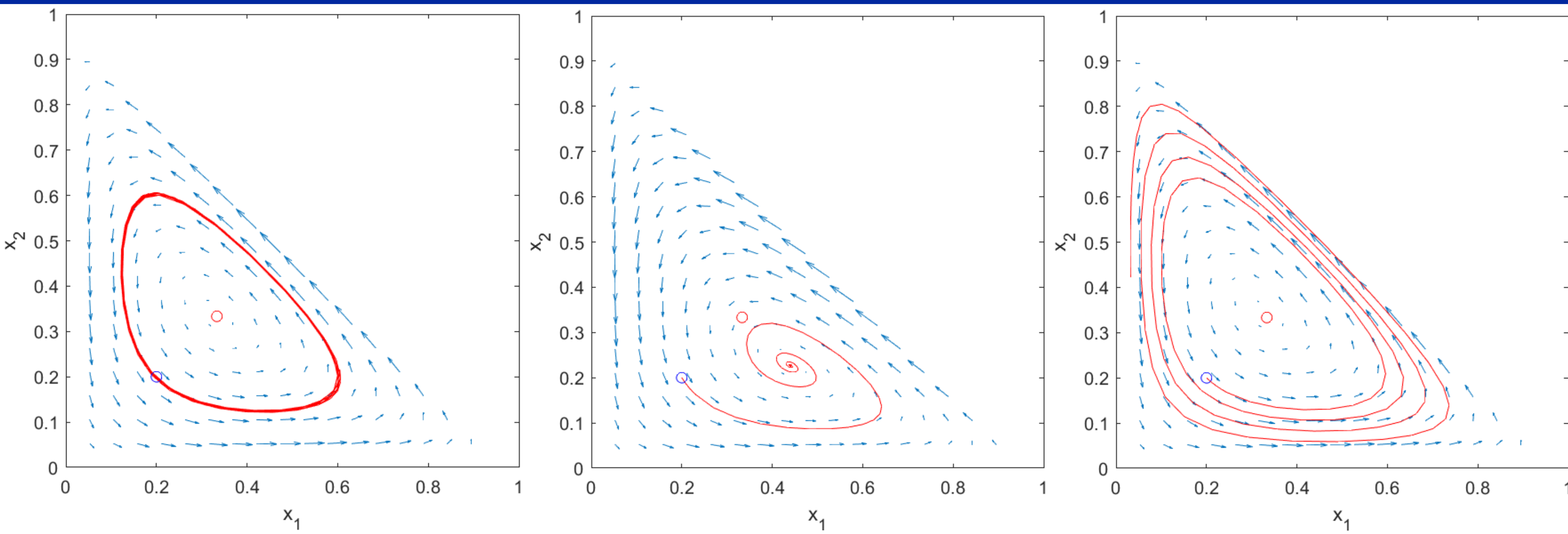
$$\dot{x}_k = \alpha x_k [(\Pi \boldsymbol{x})_k - \boldsymbol{x}^{\mathrm{T}} \Pi \boldsymbol{x} - \varepsilon \cdot v(\boldsymbol{x})]$$

The ε term measures how the population cope with unexpectedness.

- $\varepsilon > 0$: stick to ones strategy even if lost.
- $\varepsilon < 0$: change strategy even if won.

Perturbations

Perturbed Rock-Paper-Scissors Game ($a = 1$)



$\varepsilon = 0$ $\varepsilon = 0.1$ $\varepsilon = -0.01$

Perturbations

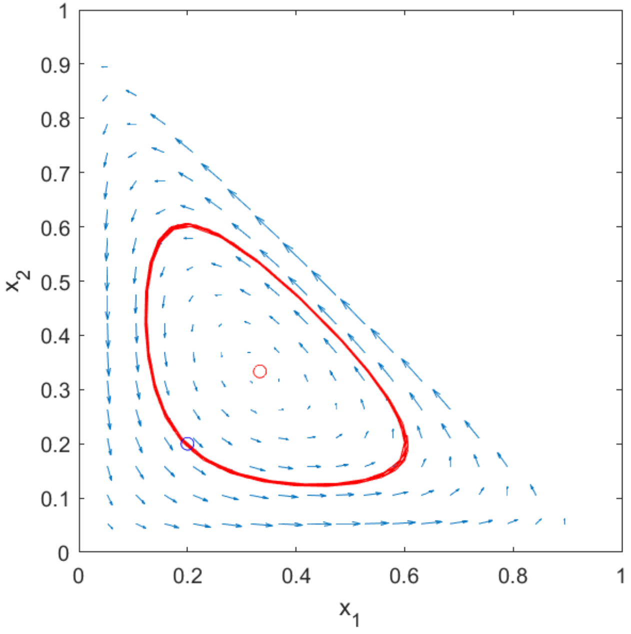
- **A better choice:**

$$\dot{x}_k = \alpha x_k[(\Pi \boldsymbol{x})_k - \boldsymbol{x}^{\mathrm{T}}\Pi \boldsymbol{x} - \varepsilon \cdot (v(\boldsymbol{x}) + \ln x_k)]$$

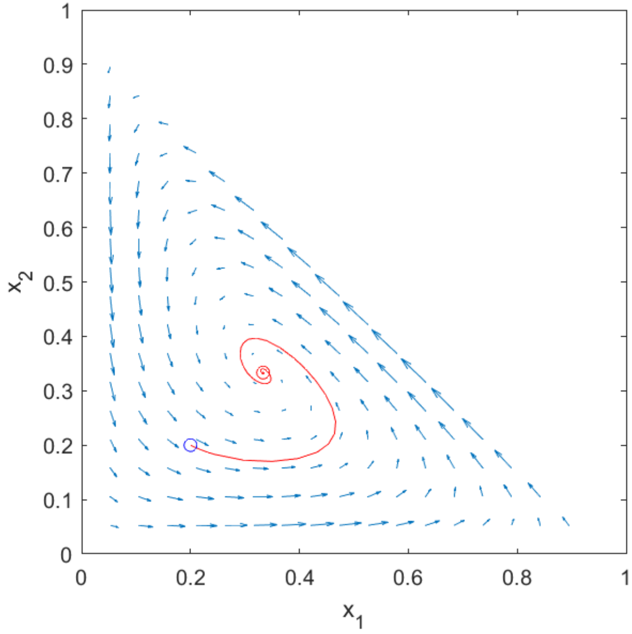
1. Preserves equilibrium of uniform distribution.
2. Compatible with dynamics of x_n ($n-1$ degrees of freedom as before).

Perturbations

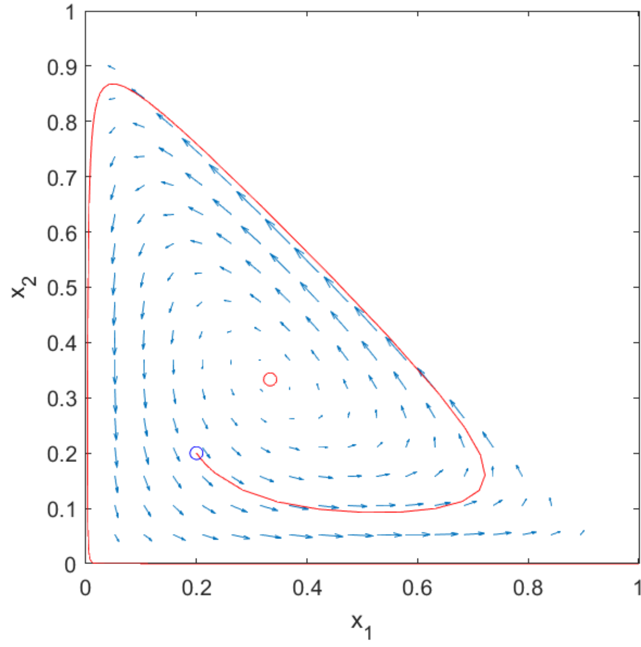
Perturbed Rock-Paper-Scissors Game (new) $(a = 1)$



$\varepsilon = 0$

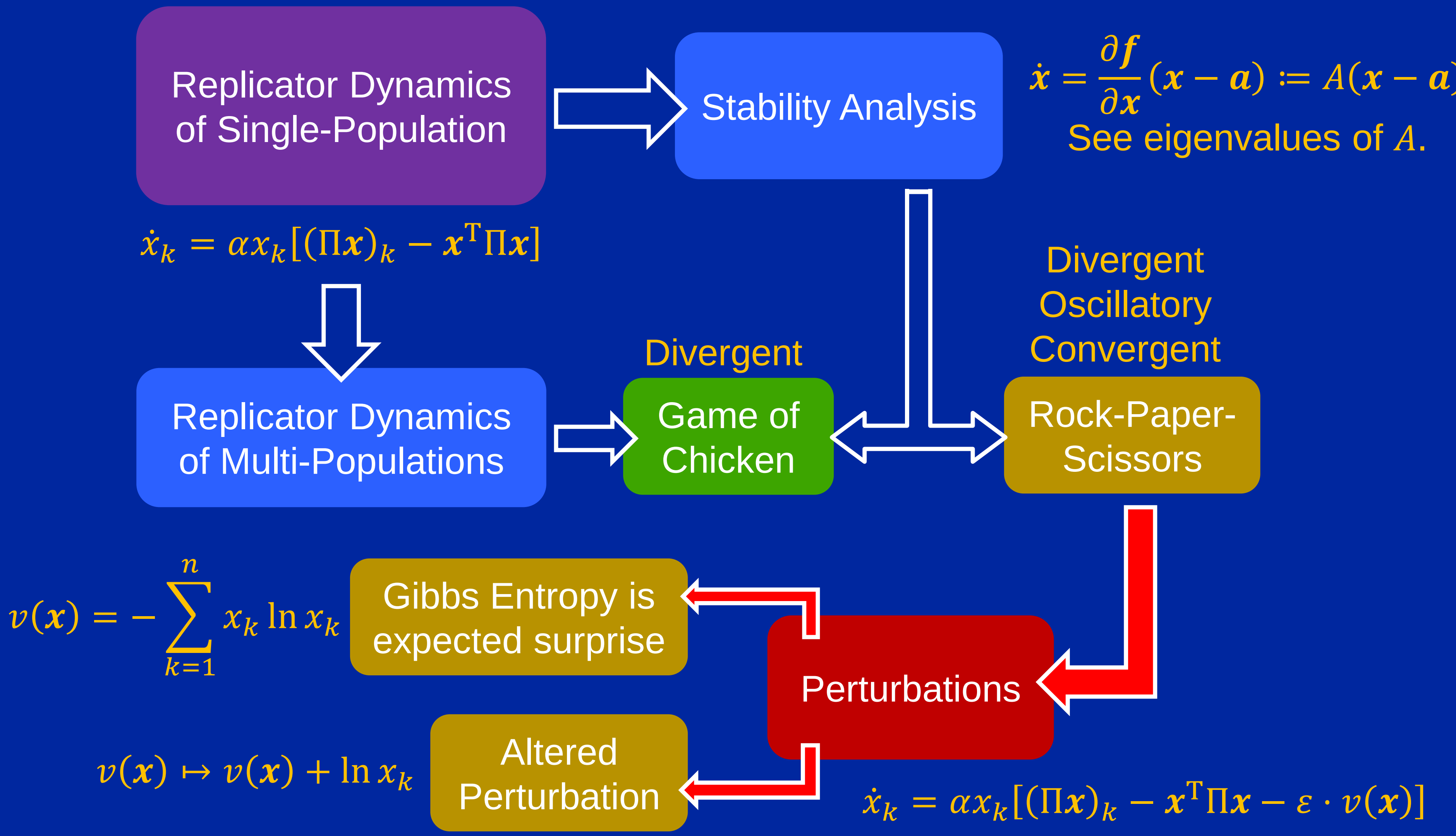


$\varepsilon = 0.1$



$\varepsilon = -0.1$

Conclusion



Replicator Dynamics of Single-Population
$\dot{x}_k = \alpha x_k[(\Pi \boldsymbol{x})_k - \boldsymbol{x}^{\mathrm{T}}\Pi\boldsymbol{x}]$
Stability Analysis
$\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{a}) \coloneqq A(\boldsymbol{x} - \boldsymbol{a})$
See eigenvalues of A.
Replicator Dynamics of Multi-Populations
Divergent
Game of Chicken
Divergent
Oscillatory
Convergent
Rock-Paper-Scissors
$v(\boldsymbol{x}) = -\sum_{k=1}^{n} x_k \ln x_k$
Gibbs Entropy is expected surprise
Perturbations
$v(\boldsymbol{x}) \mapsto v(\boldsymbol{x}) + \ln x_k$
Altered Perturbation
$\dot{x}_k = \alpha x_k[(\Pi \boldsymbol{x})_k - \boldsymbol{x}^{\mathrm{T}}\Pi\boldsymbol{x} - \varepsilon \cdot v(\boldsymbol{x})]$

QnA and Feedback