For garage j suppose its six inputs are a_{1j} to a_{6j} and its three outputs are a_{7j} to a_{9j}. If we choose a specific garage k, we will seek to find a mixture of the 28 garages whose combined inputs do not exceed those of k but whose combined output does. Let us choose x_j units of each of the garages. Our constraints are then

$$\sum_j a_{ij} x_j \le a_{ik} \quad i = 1, 2, \ldots, 6;$$

$$\sum_j a_{ij} x_j \ge a_{ik} w \quad i = 7, 8, 9;$$

$$x_j, w \ge 0 \quad j = 1, 2, \ldots, 28.$$

If we choose to maximize w then, so long as we can make w larger than 1, we will have chosen a mixture of garages whose combined inputs do not exceed that of the garage under consideration but whose combined outputs do exceed some of the outputs, demonstrating its *inefficiency* compared with this mixture. $1/w$ is usually referred to as the *efficiency number*.

Should it not be possible to find such a mixture, then the result will be to use one unit of the garage itself, resulting in a value of 1 for w.

In order to solve this problem, it is necessary to solve the model 28 times for each value of k. With some modelling/optimization systems, it is possible to do this automatically.

13.23 Milk collection

This problem is an extension of the travelling salesman problem whose formulation is discussed in Section 9.5. We extend the formulation given there.

13.23.1 Variables

$x_{ijk} = 1$ if the tour on day k goes directly between farms i and j (in either direction) for $i < j, k = 1, 2,$

$= 0$ otherwise;

$y_{ik} = 1$ if farm i is visited on the tour on day k for $i = 11$ to $21, k = 1, 2,$

$= 0$ otherwise.

13.23.2 Constraints

The limited tanker capacity gives

$$\sum_i K_i y_{ik} \le C \quad \text{for } k = 1, 2,$$

where K_i is the daily pickup requirement from farm i and C is the tanker capacity.

The limit on visiting some farms only every other day gives

$$y_{i1} + y_{i2} = 1 \quad \text{for } i = 11 \text{ to } 21.$$

The need to visit the 'every day' farms on each tour gives

$$\sum_{j:j>i} x_{ijk} + \sum_{j:j<i} x_{jik} = 2 \quad \text{for } i = 1 \text{ to } 10,\ k = 1, 2.$$

The need to visit the 'every other day' farms on the chosen day gives

$$\sum_{j:j>i} x_{ijk} + \sum_{j:j<i} x_{jik} - 2y_{ik} = 0 \quad \text{for } i = 11 \text{ to } 21,\ k = 1, 2.$$

Taking into account the considerations discussed in Chapter 10, these constraints imply those below for which the associated linear programming relaxation is more constrained, making the model much easier to solve:

$$x_{ijk} - y_{ik} \leq 0 \quad \text{for } i = 11 \text{ to } 21, j = 11 \text{ to } 21, \quad j > i, k = 1, 2;$$

$$x_{jik} - y_{ik} \leq 0 \quad \text{for } i = 11 \text{ to } 21, j = 1 \text{ to } 21, \quad i > j, k = 1, 2.$$

In order to prevent unnecessary (and computationally more costly) symmetric alternative solutions (switching days of visiting farms), it is convenient to set $y_{11,1}$ to 1, forcing farm 11 to be visited on the first day.

13.23.3 Objective

$$\text{Minimize} \quad \sum_{\substack{i,\ j \\ i<j}} c_{ij} x_{ij}$$

where c_{ij} is the distance between farm i and farm j.

This model has 65 constraints and 442 variables (all 0–1 integer).

As it is only a *relaxation* of the true model (the subtour elimination constraints have been ignored), it will almost certainly be necessary to add these on an 'as needed' basis during the course of optimization in a similar manner as for the travelling salesman problem, as described in Section 9.5.

13.24 Yield management

In order to solve this problem, a *stochastic program* (as mentioned in Section 1.2) will be built. This will be a three-period model. Solving the model for the first time will give recommended price levels and sales three weeks from to produce outputs of:

1.518 (1000s)	Alpha sales
0.568 (1000s)	Beta sales
1.568 (million pounds)	Profit

Clearly, this uses no more than the inputs used by Petworth but produces outputs at least 1.0119 times as great.

There is also a useful interpretation of the dual values on the input and output constraints. These can be regarded as *weightings* that the particular garage would like to place on its inputs and outputs so as to maximize this weighted ratio of outputs to inputs but keep the corresponding ratios for the other garages above 1 (i.e. not allow them to appear inefficient). In the case of Petworth, the dual values from solving the model turn out to be 0, 0.1557, 0.0618, 0.0158, 0 and 0 for the inputs and 0.1551, 0 and 0.495 for the outputs.

This gives a weighted ratio of

$$\frac{0.1551 \times 1.5 + 0.495 \times 1.55}{0.1557 \times 5.5 + 0.0618 \times 2 + 0.0158 \times 2} = 0.988.$$

This is clearly the efficiency number. Petworth weighs most heavily high outputs that bring it out in the best light and least heavily high inputs.

The *dual* formulation referred to in Sections 3.2 and 13.22 chooses the weights so as to maximize this ratio while not allowing the ratios for other garages (with these weights) to fall below 1.

14.23 Milk collection

The optimal solution is given in Figure 14.9 with dashes representing the first day's routes and dotted lines those of the second day. The total distance covered is 1229 miles.

In the formulation given in Section 13.23, this first solution produced subtours on day 1 around farms 2, 5 and 18; around 3, 16, 13, 4 and 19 and around 1, 8, 21, 9, 11, 7, 6 and 10 and on day 2, around farms 6, 7 and 20 and around all the other farms. This infeasible solution covers 1214 miles (there are alternative, equally good solutions). Subtour elimination constraints were then introduced to prevent the subtours (on both days to prevent them re-arising on the other day), resulting in another solution with no subtours on day 1 but subtours on day 2 around farms 1, 2, 17, 6, 7 and 10; around 4, 15, 3, 5, 14, 16 and 13 and around 8, 9 and 21. Subtour elimination constraints were introduced to prevent these subtours (on both days). This resulted in the optimal solution (no subtours) given in Figure 14.9.

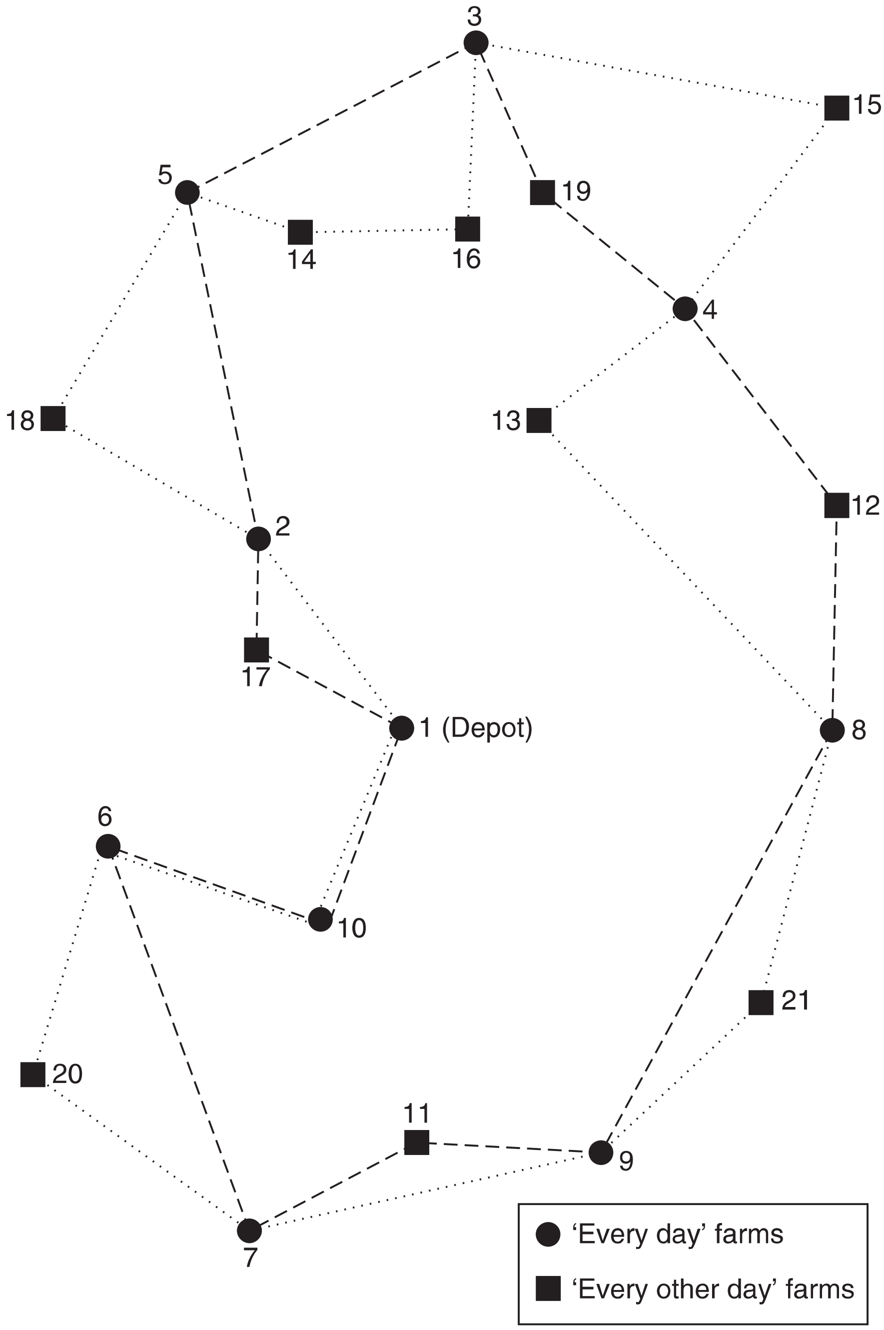

Figure 14.9

These stages required 34, 13 and 272 nodes, respectively. Out of interest, not introducing the 'disaggregated' constraints described in Section 13.23 resulted in the first stage taking 1707 nodes.

This problem is based on a larger one described by Butler, Williams and Yarrow (1997). That larger problem required a more sophisticated solution approach using generalizations of known results concerning the structure of the travelling salesman polytope.