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Abstract

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1. Introduction

2. Process description of steelmaking-continuous casting

SCC process is of significant importance to the iron and steel operations, it's a bottleneck in the steel production, this is mainly because its production capacity is usually lower than that of hot rolling and cold rolling production[**?**]. The principal production of SCC process usually needs to pass through three stages: steelmaking, refining and continuous casting, which is shown in Fig. 1.

In the steelmaking stage, steelmaking process is one of the most energy-intensive processes for producing molten steel by smelting iron ore and scrap on electric furnace or basic oxygen furnace. Meanwhile, impurities such as nitrogen, silicon, phosphorus, sulfur and excess carbon are removed from the raw iron, and alloying elements such as manganese, nickel, chromium and vanadium are added to produce different grades of steel. At this stage, a group of molten iron processed on the same electric furnace or basic oxygen furnace is called as a charge, the basic unit of steelmaking process[**?**].

In the refining stage, the temperature of molten steel and the ingredients of alloy materials are further adjusted and the impurities are further eliminated to improve the quality of the molten steel after the refining process.

In the continuous casting stage, continuous casting is a main process of steelmaking-continuous casting whereby molten metal is solidified into qualified steel billet. Furthermore, a sequence of charges must be continuously cast on the same continuous caster. At this stage, a sequence of charges processed on the same continuous caster is described as a cast, a basic unit in the SCC production[**?**].
The critical characteristics of SCC process can be summarized as follows:

1. in each stage, there are several identical parallel machines which are chosen to process the jobs;
2. all the jobs must be processed in the same processing sequence without skipping;
3. there exist transportation times between any two adjacent stages;
4. a setup time is considered to change equipments in the last stage;
5. each job must exactly be processed on one and only one machine of each stage;
6. on the same cast, all the jobs must be processed continuously, which should not be interrupted;
7. a position can process one and only one job;
8. a job can be processed at one and only position.

3. Mathematical formulation of the SCC scheduling problem

The following notations are needed to describe the mathematical formulation for scheduling problem of the SCC process.
Indices:
j: stage index;
$i, \hat{i}$: charge index;
r: position index;
k: machine index;
l: cast index;
Parameters:
$P_{i,j}$: processing time of the job i at the stage j;
$T_{j,j+1}$: transportation time between the stage j and stage $j+1$;
d_l: due time of the cast l;
Su: setup time between the adjacent casts on the same continuous cast machine in the last stage;
M_j: number of the identical parallel machines in stage j;

W_k: the total number of position processed on the kth machine;
$\Phi(r,k)$: index of the rth position on the kth machine, $\Phi(r,k) = \Phi(W_{k-1}, k-1) + r, \Phi(W_0, 0) = 0$.
Ω: the set of all charges, $|\Omega|$ is the total number of charges, $|\Omega| = \Phi(W_{M_j}, M_j)$ for each stage j;
Ω_l: the set of all charges in the lth cast, $l = \{1, 2, \cdots, N\}$, where N is the total number of casts, $\Omega_{l_1} \bigcap \Omega_{l_2} = \emptyset, \Omega_{l_1} \cup \Omega_{l_2} \cup \cdots \cup \Omega_{l_N} = \Omega$, for all $l_1 \neq l_2 \in \{1, 2, \cdots, N\}$;
B_k: the set of indices of all casts on the kth machine at the last stage;
$s(l)$: index of the last job in the lth cast, $\Omega_l = \{s(l-1)+1, \cdots, s(l)\}, s(l) = s(l-1) + |\Omega_l|, s(0) = 0, s(N) = |\Omega|$;
$b(k)$: index of the last cast on the machine k at the last stage, $b(k) = b(k-1)+|B_k|, b(0) = 0, b(M_S) = N, 1 \leq k \leq M_S$, $B_k = \{b(k-1)+1, \cdots, b(k)\}$, where S is the total number of stages.
C_j: penalty coefficient for the waiting time of any job from the stage j to stage $j+1$;
D_1: penalty coefficient for the total of starting time before its predefined time in each cast;
D_2: penalty coefficient for the total of starting time after its predefined time in each cast.
Decision variables
$x_{i,j,\Phi(r,k)}$: 0/1 variable, equal to 1 if and only if the job i is processed on the rth position of the kth machine at the jth stage;
$t_{i,j}$: starting time of the job i in the stage j.
Objective function:

$$\min F = F_1 + F_2 + F_3, \tag{1}$$

where

$$F_1 = \sum_{i=1}^{|\Omega|} \sum_{j=1}^{S-1} C_j (t_{i,j+1} - t_{i,j} - P_{i,j}), \tag{2}$$

$$F_2 = D_1 \sum_{l=1}^{N} \max(0, d_l - t_{s(l-1)+1,S}), \tag{3}$$

$$F_3 = D_2 \sum_{l=1}^{N} \max(0, t_{s(l-1)+1,S} - d_l). \tag{4}$$

Constraints:
(1) For the same job, its next operation can be started after finishing the preceding operation and transferring to the corresponding machine.

$$t_{i,j+1} - t_{i,j} - P_{i,j} \geq T_{j,j+1}, \forall i \in \Omega, 1 \leq j < S. \tag{5}$$

(2) Any job must be processed at one and only one position at each stage.

$$\sum_{k=1}^{M_j} \sum_{r=1}^{W_k} x_{i,j,\Phi(r,k)} = 1, \forall i \in \Omega, 1 \leq j < S. \tag{6}$$

(3) Any position of each stage can process one and only one job.

$$\sum_{i=1}^{|\Omega|} x_{i,j,\Phi(r,k)} = 1, 1 \leq r \leq W_k, 1 \leq k \leq M_j, 1 \leq j < S. \tag{7}$$

(4) There exists the machine capacity constraint, which means that a machine can process at most one job at a time.

$$x_{i,j,\Phi(r,k)} x_{\hat{i},j,\Phi(r+1,k)} (t_{\hat{i},j} - t_{i,j} - P_{i,j}) \geq 0, \tag{8}$$

where $\forall i \neq \hat{i} \in \Omega, 1 \leq r < W_k, 1 \leq k \leq M_j, 1 \leq j < S$.
(5) In the same cast, the adjacent jobs must be processed continuously without any waiting time.

$$t_{i+1,S} = t_{i,S} + P_{i,S}, \forall i, i+1 \in \Omega_l, \forall l \in \{1, 2, \cdots, N\}. \tag{9}$$

(6) There is a setup time between two adjacent casts to change equipment on the same machine in the last stage.

$$t_{i+1,S} - t_{i,S} - P_{i,S} \geq Su, \tag{10}$$

where $i = s(b(k-1)+l), \forall k \in \{1, 2, \cdots, M_s\}, \forall l \in \{1, 2, \cdots, |B_k| - 1\}$.
(7) The range of the decision variable values is given as follows.

$$x_{i,j,\Phi(r,k)} \in \{0, 1\}, \forall i \in \Omega, 1 \leq r \leq W_k, 1 \leq k \leq M_j, 1 \leq j < S. \tag{11}$$

$$t_{i,j} \geq 0, \forall i \in \Omega, 1 \leq j \leq S. \tag{12}$$

It is obvious to find that the objective functions F_2 and F_3 are nonlinearity in the above model. In order to overcome such difficulties, let $t^l_{s(n-1)+1,S} = \max(0, d_n - t_{s(n-1)+1,S}), t^u_{s(n-1)+1,S} = \max(0, t_{s(n-1)+1,S} - d_n)$, then the objective functions F_2 and F_3 can be reexpressed as follows[**?**]

$$F_2 = D_1 \sum_{n=1}^{N} t^l_{s(n-1)+1,S}, F_3 = D_2 \sum_{n=1}^{N} t^u_{s(n-1)+1,S}. \tag{13}$$

Furthermore, the new variables $t^l_{s(n-1)+1,S}$ and $t^u_{s(n-1)+1,S}$ should satisfy the additional constraints,

$$t_{s(n-1)+1,S} = t^u_{s(n-1)+1,S} - t^l_{s(n-1)+1,S} + d_n, 1 \leq n \leq N. \tag{14}$$

$$t^u_{s(n-1)+1,S} \geq 0, t^l_{s(n-1)+1,S} \geq 0, 1 \leq n \leq N. \tag{15}$$

4. Solution methodology

4.1. Lagrangian relaxation

In general, most scheduling problems are NP-hard especially for the large-scale scheduling problem in practical applications, which means that it is extremely difficult to obtain a high-quality schedule within an acceptable computation time[]. As a result, Lagrangian relaxation approach is adopted to solve the above mathematical model that can obtain a high-quality schedule in a reasonable computational time. By relaxing the formula (8), we can receive its relaxed problem, given as follows:

$$L(\lambda) = \min(F_1 + F_2 + F_3 - F_4), \tag{16}$$

with

$$F_4 = \sum_{i=1}^{|\Omega|} \sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|} \sum_{r=1}^{W_k-1} \sum_{k=1}^{M_j} \sum_{j=1}^{S-1} \lambda_{i\hat{i}jrk} x_{i,j,\Phi(r,k)} x_{\hat{i},j,\Phi(r+1,k)} (t_{\hat{i},j} - t_{i,j} - P_{i,j}), \tag{17}$$

subject to (5)-(7), (9)-(12), (14)-(15) and

$$\lambda_{i\hat{i}jrk} \geq 0. \tag{18}$$

Due to the non-separability of the Lagrangian function F_4, it is still a tough task to solve this relaxed problem. In order to overcome such drawback, a series of measures are explored to tackle this non-separability. As a result, the relaxed problem can be decomposed into two tractable subproblems by using the concave-convex procedure.

Owing to $x_{i,\Phi(r,k)} \in \{0,1\}, x_{\hat{i},\Phi(r+1,k)} \in \{0,1\}$, it follows that $x^2_{i,\Phi(r,k)} = x_{i,\Phi(r,k)}$, $x^2_{i',\Phi(r+1,k)} = x_{\hat{i},\Phi(r+1,k)}$ and $(x_{i,\Phi(r,k)}x_{\hat{i},\Phi(r+1,k)})^2 = x_{i,\Phi(r,k)}x_{\hat{i},\Phi(r+1,k)}$. Thus, we have

$$\begin{aligned} & (x_{i,j,\Phi(r,k)}x_{\hat{i},j,\Phi(r+1,k)})^2 = x_{i,j,\Phi(r,k)}x_{\hat{i},j,\Phi(r+1,k)} \\ = & \frac{1}{2}((x_{i,j,\Phi(r,k)} + x_{\hat{i},j,\Phi(r+1,k)})^2 - x^2_{i,j,\Phi(r,k)} - x^2_{\hat{i},j,\Phi(r+1,k)}) \\ = & \frac{1}{2}((x_{i,j,\Phi(r,k)} + x_{\hat{i},j,\Phi(r+1,k)})^2 - x_{i,j,\Phi(r,k)} - x_{\hat{i},j,\Phi(r+1,k)}), \end{aligned} \tag{19}$$

or

$$\begin{aligned} & (x_{i,j,\Phi(r,k)}x_{\hat{i},j,\Phi(r+1,k)})^2 = x_{i,j,\Phi(r,k)}x_{\hat{i},j,\Phi(r+1,k)} \\ = & \frac{1}{2}(x^2_{i,j,\Phi(r,k)} + x^2_{\hat{i},j,\Phi(r+1,k)} - (x_{i,j,\Phi(r,k)} - x_{\hat{i},j,\Phi(r+1,k)})^2) \\ = & \frac{1}{2}(x_{i,j,\Phi(r,k)} + x_{\hat{i},j,\Phi(r+1,k)} - (x_{i,j,\Phi(r,k)} - x_{\hat{i},j,\Phi(r+1,k)})^2). \end{aligned} \tag{20}$$

Let $z_{\hat{i}ij} = t_{\hat{i},j} - t_{i,j}$, from the formulas (19) and (20), Lagrangian function F_4

is translated into a treatable function, listed as follows

$$\begin{aligned}
F_4 &= \sum_{i=1}^{|\Omega|}\sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|}\sum_{r=1}^{W_k-1}\sum_{k=1}^{M_j}\sum_{j=1}^{S-1}\lambda_{i\hat{i}jrk}x_{i,j,\Phi(r,k)}x_{\hat{i},j,\Phi(r+1,k)}(z_{\hat{i}ij}-P_{i,j})\\
&= \sum_{i=1}^{|\Omega|}\sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|}\sum_{r=1}^{W_k-1}\sum_{k=1}^{M_j}\sum_{j=1}^{S-1}\frac{\lambda_{i\hat{i}jrk}}{2}((x_{i,j,\Phi(r,k)}x_{\hat{i},j,\Phi(r+1,k)}+z_{\hat{i}ij})^2\\
&\quad -x_{i,j,\Phi(r,k)}x_{\hat{i},j,\Phi(r+1,k)}-z_{\hat{i}ij}^2+P_{i,j}(x_{i,j,\Phi(r,k)}-x_{\hat{i},j,\Phi(r+1,k)})^2\\
&\quad -P_{i,j}x_{i,j,\Phi(r,k)}-P_{i,j}x_{\hat{i},j,\Phi(r+1,k)}) \qquad (21)\\
&= \sum_{i=1}^{|\Omega|}\sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|}\sum_{r=1}^{W_k-1}\sum_{k=1}^{M_j}\sum_{j=1}^{S-1}\lambda_{i\hat{i}jrk}(\frac{1}{2}(\frac{1}{2}((x_{i,j,\Phi(r,k)}+x_{\hat{i},j,\Phi(r+1,k)})^2\\
&\quad -x_{i,j,\Phi(r,k)}-x_{\hat{i},j,\Phi(r+1,k)})+z_{\hat{i}ij})^2-\frac{1}{4}(x_{i,j,\Phi(r,k)}+x_{\hat{i},j,\Phi(r+1,k)})\\
&\quad +\frac{1}{4}(x_{i,j,\Phi(r,k)}-x_{\hat{i},j,\Phi(r+1,k)})^2-\frac{1}{2}z_{\hat{i}ij}^2+\frac{P_{i,j}}{2}(x_{i,j,\Phi(r,k)}-x_{\hat{i},j,\Phi(r+1,k)})^2\\
&\quad -\frac{P_{i,j}}{2}x_{i,j,\Phi(r,k)}-\frac{P_{i,j}}{2}x_{\hat{i},j,\Phi(r+1,k)}).
\end{aligned}$$

Thus, the relaxed problem can be rewritten as follows

$$L(\lambda)=\min(F_1+F_2+F_3+F_5-F_6), \qquad (22)$$

with

$$\begin{aligned}
F_5 &= \frac{1}{4}\sum_{i=1}^{|\Omega|}\sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|}\sum_{r=1}^{W_k-1}\sum_{k=1}^{M_j}\sum_{j=1}^{S-1}\lambda_{i\hat{i}jrk}(2P_{i,j}+1)(x_{i,j,\Phi(r,k)}+x_{\hat{i},j,\Phi(r+1,k)})\\
&\quad +\frac{1}{2}\sum_{i=1}^{|\Omega|}\sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|}\sum_{r=1}^{W_k-1}\sum_{k=1}^{M_j}\sum_{j=1}^{S-1}\lambda_{i\hat{i}jrk}z_{\hat{i}ij}^2, \qquad (23)
\end{aligned}$$

and

$$\begin{aligned}
F_6 &= \sum_{i=1}^{|\Omega|}\sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|}\sum_{r=1}^{W_k-1}\sum_{k=1}^{M_j}\sum_{j=1}^{S-1}\lambda_{i\hat{i}jrk}(\frac{1}{2}(\frac{1}{2}((x_{i,j,\Phi(r,k)}+x_{\hat{i},j,\Phi(r+1,k)})^2-x_{i,j,\Phi(r,k)}\\
&\quad -x_{\hat{i},j,\Phi(r+1,k)})+z_{\hat{i}ij})^2+\frac{2P_{i,j}+1}{4}(x_{i,j,\Phi(r,k)}-x_{\hat{i},j,\Phi(r+1,k)})^2), \qquad (24)
\end{aligned}$$

subject to (5)-(7), (9)-(12), (14)-(15) and (18).

In the above relaxed problem, it is obvious to see that the functions F_1, F_2, F_3, F_5 and F_6 are convex if the decision variables $x_{i,\Phi(r,k)}$ and $x_{\hat{i},\Phi(r+1,k)}$ are continuous. Consequently, we consider to replace the constraint (11) with the following constraint

$$x_{i,j,\Phi(r,k)} \in [0,1], \forall i \in \Omega, 1 \leq r \leq W_k, 1 \leq k \leq M_j, 1 \leq j < S. \tag{25}$$

In the subsection 4.3, we will prove that the problem (22) is equivalent to its relaxed problem, given as follows

$$L^{'}(\lambda) = \min(F_1 + F_2 + F_3 + F_5 - F_6), \tag{26}$$

subject to (5)-(7), (9)-(10), (12), (14)-(15), (18) and (25).

4.2. Solution of the subproblems

In view of the objective functions and constraints, we know that the constraints are linear equalities or inequalities, the functions F_1, F_2, F_3, F_4 and F_6 are convex and the function F_6 is differentiable. As a result, the relaxed problem (26) is a DC optimization problem[**?**] that can be solved by the concave-convex procedure[**?**]. In the concave-convex procedure, the relaxed problem (26) is equivalent to the following convex programming problem[**?**]

$$(t_{i,j}^{l+1}, x_{i,j,\Phi(r,k)}^{l+1}) = \arg\min(F_1 + F_2 + F_3 + F_5 - f_1 - f_2 - R), \tag{27}$$

where

$$f_1 = \sum_{i=1}^{|\Omega|} \sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|} \sum_{r=1}^{W_k-1} \sum_{k=1}^{M_j} \sum_{j=1}^{S-1} \lambda_{i\hat{i}jrk} \nabla F_6(t_{i,j}^l)(t_{i,j} - t_{i,j}^l), \tag{28}$$

$$f_2 = \sum_{i=1}^{|\Omega|} \sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|} \sum_{r=1}^{W_k-1} \sum_{k=1}^{M_j} \sum_{j=1}^{S-1} \lambda_{i\hat{i}jrk} \nabla F_6(x_{i,j,\Phi(r,k)}^l)(x_{i,j,\Phi(r,k)} - x_{i,j,\Phi(r,k)}^l), \tag{29}$$

and

$$R = F_6(t_{i,j}^l, t_{\hat{i},j}^l, x_{i,j,\Phi(r,k)}^l, x_{\hat{i},j,\Phi(r+1,k)}^l), \tag{30}$$

subject to (5)-(7), (9)-(10), (12), (14)-(15), (18) and (25).

Based on the formulas (28) and (29), we can decompose the relaxed problem (27) into two tractable subproblems, one of the subproblems is given as follows

$$t_{i,j}^{l+1} = \arg\min(F_1 + F_2 + F_3 + f_3 - f_1 - R), \tag{31}$$

where

$$f_3 = \sum_{i=1}^{|\Omega|} \sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|} \sum_{r=1}^{W_k-1} \sum_{k=1}^{M_j} \sum_{j=1}^{S-1} \frac{\lambda_{i\hat{i}jrk}}{2} (t_{\hat{i},j} - t_{i,j} - P_{i,j})^2, \tag{32}$$

subject to (5), (9)-(10), (12), (14)-(15) and (18).
The other subproblem is listed as follows

$$x_{i,j,\Phi(r,k)}^{l+1} = \arg\min(f_4 - f_2), \tag{33}$$

where

$$f_4 = \sum_{i=1}^{|\Omega|} \sum_{\hat{i}=1,\hat{i}\neq i}^{|\Omega|} \sum_{r=1}^{W_k-1} \sum_{k=1}^{M_j} \sum_{j=1}^{S-1} \frac{\lambda_{i\hat{i}jrk}}{4} (x_{i,j,\Phi(r,k)} + x_{\hat{i},j,\Phi(r+1,k)}), \tag{34}$$

subject to (6)-(7), (18) and (25).

Obviously, the subproblems (31) and (33) are easily solved problems that can be solved by the standard software package directly, such as, cplex.

4.3. Convergence analysis of the concave-convex procedure

In this subsection, our aim is to provide theoretical convergence guarantees that the sequence generated by the concave-convex procedure converges to a stationary point of the relaxed problem (16). For all $(\lambda, x), (\lambda, y) \in X$, X is the feasible set of the problem (26), let $g(\lambda, x) = F_1(x) + F_2(x) + F_3(x) + F_5(\lambda, x) - F_6(\lambda, x)$ and $f(\lambda, x, y) = F_1(x) + F_2(x) + F_3(x) + F_5(\lambda, x) - \nabla F_6(\lambda, y)(x - y) - F_6(\lambda, y)$.

Theorem 4.1. For given multiplier $\hat{\lambda}$, if $f(\hat{\lambda}, x, y)$ satisfy the following assumptions:
(A1)$f(\hat{\lambda}, y, y) = g(\hat{\lambda}, y), \forall(\hat{\lambda}, y) \in X$,
(A2)$f(\hat{\lambda}, x, y) \geq g(\hat{\lambda}, x), \forall(\hat{\lambda}, x), (\hat{\lambda}, y) \in X$,
(A3)$\nabla f(\hat{\lambda}, x, y; d)|_{x=y} = \nabla g(\hat{\lambda}, y; d), \forall d$ with $(\hat{\lambda}, \mathrm{y} + \mathrm{d}) \in \mathrm{X}$,
(A4)$f(\hat{\lambda}, x, y)$ is continuous in (x, y),

then every limit point of the sequence $\{t_{i,j}^l, x_{i,r,k}^l\}$ generated by (31) and (33) is a stationary point of $L^{'}(\hat{\lambda})$.
Proof. Due to $f(\hat{\lambda}, x, y) = g(\hat{\lambda}, x) + F_6(\hat{\lambda}, x) - \nabla F_6(\hat{\lambda}, y)(x-y) - F_6(\hat{\lambda}, y)$, then it follows that

$$f(\hat{\lambda}, y, y) = g(\hat{\lambda}, y). \tag{35}$$

Since the function F_6 is convex, then we have

$$F_6(\hat{\lambda}, x) \geq F_6(\hat{\lambda}, y) + \nabla F_2(\hat{\lambda}, y)(x-y), \forall(\hat{\lambda}, x), (\hat{\lambda}, y) \in X. \tag{36}$$

Hence, we have

$$f(\hat{\lambda}, x, y) \geq g(\hat{\lambda}, x). \tag{37}$$

Based on the definition of the function $f(\hat{\lambda}, x, y)$, it is obvious to see that the function $f(\hat{\lambda}, x, y)$ is continuously differentiable, which implies that the assumption A4 holds. For any fixed $(\hat{\lambda}, y) \in X$, the directional derivative of f and g with respect to the variable x are determined by

$$\nabla g(\hat{\lambda}, x) = \nabla F_1(x) + \nabla F_2(x) + \nabla F_3(x) + \nabla F_5(\hat{\lambda}, x) - \nabla F_6(\hat{\lambda}, x), \tag{38}$$

and

$$\nabla f_x(\hat{\lambda}, x, y) = \nabla g(\hat{\lambda}, x) + \nabla F_6(\hat{\lambda}, x) - \nabla F_6(\hat{\lambda}, y). \tag{39}$$

Thus, it follows that

$$\nabla f(\hat{\lambda}, x, y; d)|_{x=y} = \nabla g(\hat{\lambda}, y; d), \tag{40}$$

then the assumptions A1, A2 and A3 hold, due to the formulas (35), (37) and (40), respectively. Thus, based on the Theorem 1 in [**?**], we know that the sequence $\{t_{i,j}^l, x_{i,j,\Phi(r,k)}^l\}$ will converge to a stationary point of $L^{'}(\hat{\lambda})$.
Theorem 4.2. For given multiplier $\hat{\lambda}$, let $(t_{i,j}^*, x_{i,j,\Phi(r,k)}^*)$ be the limit point of the sequence $\{t_{i,j}^l, x_{i,j,\Phi(r,k)}^l\}$ generated by (31) and (33), then $(t_{i,j}^*, x_{i,j,\Phi(r,k)}^*)$ is a stationary point of $L(\hat{\lambda})$.
Proof. Because the sequence $\{x_{i,j,\Phi(r,k)}^{l+1}\}$ is generated by (33), then the solution $(x_{1,1,1,1}^{l+1}, \cdots, x_{i,j,\Phi(r,k)}^{l+1}, \cdots, x_{|\Omega|,j,\Phi(W_{M_j}, M_j)}^{l+1}))$ is a vertex of the set composed by the constraints (6), (7) and (25). Since the constraints of (6) and (7) form

a totally unimodular matrix and its objective function of (33) is linear, then all vertexes of the set composed by the constraints (6), (7) and (25) are integral points, which means that the point $(t_{i,j}^{l+1}, x_{i,j,\Phi(r,k)}^{l+1})$ is a feasible solution to the problem (16). Hence, we know that the limit point $(t_{i,j}^{*}, x_{i,j,\Phi(r,k)}^{*})$ is also a feasible solution to the problem (16), then we have

$$L^{'}(\hat{\lambda}) \leq L(\hat{\lambda}) \leq L(\hat{\lambda}, t_{i,j}^{*}, x_{i,j,\Phi(r,k)}^{*}) = L^{'}(\hat{\lambda}). \tag{41}$$

Thus, we have $L(\hat{\lambda}) = L^{'}(\hat{\lambda})$. In view of the Theorem 4.1, we know that $(t_{i,j}^{*}, x_{i,j,\Phi(r,k)}^{*})$ is a stationary point of $L(\hat{\lambda})$.

4.4. Surrogate subgradient method

A major issue in Lagrangian relaxation approach is to effectively solve the Lagrangian dual problem, whose objective function is concave, piecewise linear, and not differentiable everywhere[**?**]. The Lagrangian dual problem is deemed as follows

$$\max_{\lambda \geq 0} L^{'}(\lambda). \tag{42}$$

The specific iterative procedures of the surrogate subgradient method are summarized below:

Step 1: initialization

Choose the parameters $\varepsilon > 0, M > 1, 0 < r < 1, c^0$, given an initial multiplier λ^0 and an initial feasible solution $x^0 = (t_{i,j}^0, x_{i,\Phi(r,k)}^0)$ of the relaxed problem, set $m = 0, l = 0$.

Step 2: surrogate subgradient direction

step 2.1: Calculate the feasible solution $x^{l+1} = (t_{i,j}^{l+1}, x_{i,\Phi(r,k)}^{l+1})$ of the relaxed problem by solving the problems (31) and (33).

step 2.2: Set $l = l + 1$, if $\tilde{L}(\lambda_m, x^l) < \tilde{L}(\lambda_m, x^{l-1})$ or $\|x^l - x^{l-1}\| < \varepsilon$, then go to the step 2.3; otherwise, go to the step 2.1.

step 2.3: the direction g_m is calculated as follows

$$g_m = [\nu_{1,2,1,1,1}, \nu_{1,3,1,1,1}, \cdots, \nu_{|\Omega|,|\Omega-1|,S-1,W_{M_{S-1}},M_{S-1}}], \tag{43}$$

where

$$\nu_{i,\hat{i},r,j,k} = -x_{i,\Phi(r,k)} x_{\hat{i},\Phi(r+1,k)} (t_{\hat{i},j} - t_{i,j} - P_{i,j}). \tag{44}$$

Step 3: step size
The step size c^m is computed by

$$c^m = \prod_{i=1}^{m} \alpha_i \frac{c^0 \|g_0\|}{\|g_m\|}, \tag{45}$$

where

$$\alpha_i = 1 - \frac{1}{M i^{p_i}}, p_i = 1 - \frac{1}{i^r}. \tag{46}$$

Step 4: update the multiplier
The multiplier λ_{m+1} is updated by

$$\hat{\lambda}_{m+1} = \lambda_m + c^m g_m, \tag{47}$$

$$\lambda_{m+1} = [\hat{\lambda}_{m+1}]^+, \tag{48}$$

set $m = m + 1$.
Step 5: termination conditions
If $c^m < \varepsilon$ or $\|g_m\| < \varepsilon$ or $\|\lambda_{m+1} - \lambda_m\| < \varepsilon$, then stop; otherwise go to the Step 2.

5. Numerical experiments

6. Conclusions

Acknowledgements

References