UM-SJTU JOINT INSTITUTE
VE477 Introduction to Algorithms

Homework 2

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Ex.1 Basic complexity

1. a) If

$$n^3 - 3n^2 - n + 1 = \Theta(n^3)$$

Then there exists c_1, c_2, n_0 such that $\forall n \geq n_0$

$$c_1 n^3 \leq n^3 - 3n^2 - n + 1 \leq c_2 n^2$$

$$c_1 \leq \frac{1}{n^3} - \frac{1}{n^2} - \frac{3}{n} + 1 \leq c_2$$

When $n_0 = 4$, $c_1 = 0.05$, $c_2 = 1$, it satisfies.

b) If

$$n^2 = O(2^n)$$

Then there exists c_1, n_0 such that $\forall n \geq n_0$

$$n^2 \leq c_1 2^n$$

When $n_0 = 4$, $c_1 = 1$, it satisfies.

c) If

$$\forall a \in \mathbb{R}, b \in \mathbb{R}^+, (n+a)^b = \Theta(n^b)$$

Then there exists c_1, c_2, n_0 such that $\forall n \geq n_0$

$$c_1 n^b \leq (n+a)^b \leq c_2 n^b$$

$$c_1 \leq (\frac{n+a}{n})^b \leq c_2$$

When $a \geq 0$, $n_0 = 2a$, $c_1 = 1$, $c_2 = 1.5^b$, it satisfies.
When $a < 0$, $n_0 = -2a$, $c_1 = 0.5^b$, $c_2 = 1$, it satisfies.

2. a) $f(n) = O(g(n))$

b) $f(n) = \Omega(g(n))$

3. (a) None.

(b) $f(n) = 2^n - n^2$, $g(n) = n^4 + n^2$

4. Considering f_2 and f_3, which are easy to compare

$$\frac{f_3(n)}{f_2(n)} = \sqrt{\frac{n}{\log n}}$$

Apparently, $f_2(n) < f_3(n)$.

$$f_1(n) = \sum_{i=1}^{n} \sqrt{i} > \frac{n(1+\sqrt{n})}{2}$$

$$(1+\sqrt{n})^2 - (2\sqrt{\log n})^2 = n + 2\sqrt{n} + 1 - 4\log n > n + 1 - 2\log n > 0$$

Hence,

$$f_1(n) > \frac{n(1+\sqrt{n})}{2} > n\sqrt{\log n} = f_3(n)$$

$$f_4(n) = \Theta(n\sqrt{n})$$

Hence,

$$f_4 > f_1 > f_3 > f_2$$

Ex.2 Master Theorem

1. a)

$$f(n)$$

$$f(n/b) \quad \cdots \quad b \text{ childs in total} \quad \cdots \quad f(n/b)$$

$$f(n/b^2) \quad b \text{ childs in total} \quad f(n/b^2) \qquad f(n/b^2) \quad b \text{ childs in total} \quad f(n/b^2)$$

$$\cdots$$

   b) i - $\log_b n + 1$

   ii - $a^{\log_b n}$

   iii - $a^k f(n/b^k)$

   iv -

$$\begin{aligned} T(n) &= \sum_{j=0}^{log_b n} a^j f(\frac{n}{b^j}) \\ &= a^{\log_b n} f(1) + \sum_{j=0}^{log_b n-1} a^j f(\frac{n}{b^j}) \\ &= n^{\log_b a} f(1) + \sum_{j=0}^{log_b n-1} a^j f(\frac{n}{b^j}) \\ &= \Theta(n^{\log_b a}) + \sum_{j=0}^{log_b n-1} a^j f(\frac{n}{b^j}) \end{aligned}$$

2. a) i - If

$$g(n) = \Theta\left(\sum_{j=0}^{log_b n-1} a^j (\frac{n}{b^j})^{\log_b a}\right)$$

Then there exists c_1, c_2, n_0 such that $\forall n \geq n_0$

$$c_1 \sum_{j=0}^{log_b n-1} a^j (\frac{n}{b^j})^{\log_b a} \leq g(n) \leq c_2 \sum_{j=0}^{log_b n-1} a^j (\frac{n}{b^j})^{\log_b a}$$

According to the previous question, $f(n) = \Theta(n^{\log_b a})$. Then there exists n'_0, c'_1, c'_2 such that $\forall n \geq n'_0$

$$c'_1 n^{\log_b a} \leq f(n) \leq c'_2 n^{\log_b a}$$

$$c'_1 \left(\frac{n}{b^j}\right)^{\log_b a} \leq a^j f\left(\frac{n}{b^j}\right) \leq c'_2 \left(\frac{n}{b^j}\right)^{\log_b a}$$

Hence $\forall n'_0 = n_0/b^j$, there exist c_1, c_2 such that

$$c_1 \sum_{j=0}^{log_b n-1} a^j (\frac{n}{b^j})^{\log_b a} \leq \sum_{j=0}^{log_b n-1} a^j f(\frac{n}{b^j}) \leq c_2 \sum_{j=0}^{log_b n-1} a^j (\frac{n}{b^j})^{\log_b a},$$

which is exactly the formula we obtained at first.

ii -

$$\sum_{j=0}^{log_b n-1} a^j (\frac{n}{b^j})^{\log_b a} = n^{\log_b a} \sum_{j=0}^{log_b n-1} \frac{a^j}{(b^{\log_b a})^j}$$
$$= n^{\log_b a} \log_b n$$

iii -

$$g(n) = \Theta\left(\sum_{j=0}^{log_b n-1} a^j (\frac{n}{b^j})^{\log_b a}\right) = \Theta(n^{\log_b a} \log_b n)$$

b) i - Skipped, similar with 2-ii

ii -

$$\sum_{j=0}^{log_b n-1} a^j (\frac{n}{b^j})^{\log_b a-\epsilon} = n^{\log_b a-\epsilon} \sum_{j=0}^{log_b n-1} \frac{a^j}{(b^{\log_b a-\epsilon})^j}$$
$$= n^{\log_b a-\epsilon} \sum_{j=0}^{log_b n-1} b^{\epsilon j}$$
$$= n^{\log_b a-\epsilon} \frac{(b^{\log_b n})^{\epsilon} - 1}{b^{\epsilon} - 1}$$
$$= \frac{n^{\epsilon} - 1}{b^{\epsilon} - 1} n^{\log_b a-\epsilon}$$

iii -

$$g(n) = O\left(\sum_{j=0}^{\log_b n-1} a^j (\frac{n}{b^j})^{\log_b a-\epsilon}\right) = O\left(\frac{n^{\epsilon} - 1}{b^{\epsilon} - 1} n^{\log_b a-\epsilon}\right)$$

c) i -

$$\sum_{j=0}^{log_b n} a^j f(\frac{n}{b^j}) = f(n) + \sum_{j=1}^{log_b n} a^j f(\frac{n}{b^j}) > f(n)$$

Hence, $g(n) = \Omega(f(n))$.

ii - Consider $j = 1$, then $af(a/b) \leq cf(n)$ is true.
We assume that when $j = k$, the formula is true. Then as for $j = k + 1$

$$a^{j+1} f(n/b^{j+1}) = a^{j+1} f((n/b^j)/j) \leq a^j cf(n/b^j) \leq c^{j+1} f(n)$$

iii -

$$a^j f(n/b^j) \leq c^j f(n)$$
$$\sum_{j=0}^{\log_b n-1} a^j f(n/b^j) \leq \sum_{j=0}^{\log_b n-1} c^j f(n)$$
$$g(n) \leq \sum_{j=0}^{\log_b n-1} c^j f(n)$$
$$g(n) = O(f(n))$$

iv - $g(n) = \Omega(f(n)), g(n) = O(f(n)) \Rightarrow g(n) = \Theta(f(n))$

3. As for the Master Theorem in the case where n is a power of $b > 1$.

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & f(n) = \Theta(n^{\log_b a}) \\ \Theta(n^{\log_b a}) & f(n) = O(n^{\log_b a - \epsilon}) \\ \Theta(f(n)) & f(n) \geq \dfrac{a}{c} f(n/b) \end{cases}$$

Ex.3 Ramanujam numbers

Algorithm 1 Ramanujam numbers

Input: Integer n
Output: All the Ramanujam numbers smaller or equal to n.

Initalize an array *Buckets* with n zeros.
$Ramanujam \leftarrow \emptyset$
$i \leftarrow 0$
for all $i < n^{1/3}$ **do**
    for all $j < i$ **do**
        $sum = i^3 + j^3$
        if $sum < n$ **then**
            $Buckets[sum] += 1$
        end if
    end for
end for
for each in *Buckets* **do**
    Append the index of the element with value of 2 in *Ramanujam*
end for
return *Ramanujam*

$$1 + 2 + 3 + 4 + \cdots + n^{1/3} = O(n^{2/3})$$
$$O(n^{2/3}) + O(n) = O(n)$$

Ex. 4 Critical thinking

Two pirates left: No.5 and No.6 are sharing the gold. No.5 will keep all 300 pieces and will be support by himself.

Three pirates left: No.5 must be aganist to No.4 to get all of the gold. No.4 needs to give No.6 1 piece of gold so that No.6 will vote for him because if No.4 died, No.6 would get nothing. No.4 keeps 299 pieces.

Four pirates left: No.4 must be aganist to No.3 to 299 pieces of gold. No.3 needs to give No.5 1 piece of gold so that No.5 will vote for him because if No.3 died, No.5 would get nothing. Then he gets enough support. And then he keeps 299 pieces.

Five pirates left: No.3 must be aganist to No.2 to get 299 pieces of gold. No.2 needs to give No.4 and No.6 each one a piece of gold so that they will vote for him because of if No.2 died, they would get nothing. And then he keeps 298 pieces.

Six pirates left: Similarly, No.1 298, No.3 1, No.5 1.