SI231B: Matrix Computations, 2024 Spring

Programming Homework – Part II

Acknowledgments:

1) Deadline: **2024-06-23 23:59:59**
2) You have 5 "free days" in total for late submission

Problem 3. (Hamiltonian-Schur Decomposition, 25 pts) (ref. Ch7.8 [Golub-van-Loan'13]) The algebraic Riccati equation (ARE) is a type of nonlinear matrix equation that often arises in control theory, particularly in the context of linear-quadratic regulator (LQR) problems and Kalman filtering. The general form of the continuous-time ARE is

$$\mathbf{G} + \mathbf{X}\mathbf{A} + \mathbf{A}^\top \mathbf{X} - \mathbf{X}\mathbf{F}\mathbf{X} = \mathbf{0},$$

where $\mathbf{G}, \mathbf{A}, \mathbf{F} \in \mathbb{R}^{n \times n}$ and $\mathbf{X} \in \mathbb{S}^{n \times n}$ is the symmetric matrix to be determined. We want to find a solution such that the eigenvalues of $\mathbf{A} - \mathbf{F}\mathbf{X}$ are in the open left half plane (i.e. the real part < 0). For solving this problem, we can construct a Hamiltonian Matrix in the form of

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{G} & -\mathbf{A}^\top \end{bmatrix} \in \mathbb{R}^{2n \times 2n}.$$

Suppose $\mathbf{M}$ has no purely imaginary eigenvalues, then there exists an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{2n \times 2n}$ satisfying

$$\mathbf{Q}^\top \mathbf{M} \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ -\mathbf{Q}_2 & \mathbf{Q}_1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{G} & -\mathbf{A}^\top \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ -\mathbf{Q}_2 & \mathbf{Q}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{0} & -\mathbf{T}^\top \end{bmatrix},$$

where $\mathbf{T}, \mathbf{R} \in \mathbb{R}^{n \times n}$. This is called the *real Hamiltonian-Schur decomposition*. If $\mathbf{Q}_1$ is invertible and $\mathbf{Q}_1^\top \mathbf{Q}_2$ is symmetric, answer the following questions:

1) Show that $\mathbf{X} = \mathbf{Q}_2 \mathbf{Q}_1^{-1}$ is a solution of the ARE (3 pts).
2) Further assume $\mathbf{I} = \mathbf{Q}_1^\top \mathbf{Q}_1 + \mathbf{Q}_2^\top \mathbf{Q}_2$. Show that $\mathbf{A} - \mathbf{F}\mathbf{X} = \mathbf{Q}_1 \mathbf{T} \mathbf{Q}_1^{-1}$ (i.e. $\mathbf{A} - \mathbf{F}\mathbf{X}$ is similar to $\mathbf{T}$), and so the eigenvalues of $\mathbf{A} - \mathbf{F}\mathbf{X}$ are the eigenvalues of $\mathbf{T}$ (5 pts).
3) The following questions are solved through programming.

   Consider the ARE problem

   $$\mathbf{C}_1^\top \mathbf{X} + \mathbf{X}\mathbf{C}_1 - \mathbf{X}\mathbf{C}_2\mathbf{C}_4^{-1}\mathbf{C}_2^\top \mathbf{X} + \mathbf{C}_3 = \mathbf{0},$$

   in which

   $$\mathbf{C}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C}_4 = 1.$$

- Find the symmetric solution $\mathbf{X}$ by computing the real Hamiltonian-Schur decomposition and ordering the eigenvalues such that $\lambda(\mathbf{T})$ is in the left half plane (15 pts). *Hint: you are recommended to use the 'schur()' and 'ordschur()' functions in **Matlab**.*

- Calculate the residual to determine whether the solution is accurate enough (2 pts).

Note: In this problem, ***Matlab*** *is recommended as the programming language. If not, please indicate the programming language you use in your solution.*

Solution 1. 1. Given the Hamiltonian-Schur decomposition:

$$\mathbf{Q}^\top \mathbf{M}\mathbf{Q} = \begin{bmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{0} & -\mathbf{T}^\top \end{bmatrix},$$

where $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{G} & -\mathbf{A}^\top \end{bmatrix}$.

We can write $\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ -\mathbf{Q}_2 & \mathbf{Q}_1 \end{bmatrix}$. From the decomposition:

$$\mathbf{Q}^\top \mathbf{M}\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ -\mathbf{Q}_2 & \mathbf{Q}_1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{G} & -\mathbf{A}^\top \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ -\mathbf{Q}_2 & \mathbf{Q}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{0} & -\mathbf{T}^\top \end{bmatrix}.$$

Expanding the left-hand side:

$$\mathbf{Q}^\top \mathbf{M}\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1^\top & -\mathbf{Q}_2^\top \\ \mathbf{Q}_2^\top & \mathbf{Q}_1^\top \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{G} & -\mathbf{A}^\top \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ -\mathbf{Q}_2 & \mathbf{Q}_1 \end{bmatrix}.$$

Carrying out the multiplication:

$$\begin{bmatrix} \mathbf{Q}_1^\top & -\mathbf{Q}_2^\top \\ \mathbf{Q}_2^\top & \mathbf{Q}_1^\top \end{bmatrix} \begin{bmatrix} \mathbf{A}\mathbf{Q}_1 + \mathbf{F}\left(-\mathbf{Q}_2\right) & \mathbf{A}\mathbf{Q}_2 + \mathbf{F}\mathbf{Q}_1 \\ \mathbf{G}\mathbf{Q}_1 + \left(-\mathbf{A}^\top\right)\left(-\mathbf{Q}_2\right) & \mathbf{G}\mathbf{Q}_2 + \left(-\mathbf{A}^\top\right)\mathbf{Q}_1 \end{bmatrix}.$$

This results in:

$$\begin{bmatrix} \mathbf{Q}_1^\top\left(\mathbf{A}\mathbf{Q}_1 - \mathbf{F}\mathbf{Q}_2\right) - \mathbf{Q}_2^\top\left(\mathbf{G}\mathbf{Q}_1 + \mathbf{A}^\top\mathbf{Q}_2\right) & \mathbf{Q}_1^\top\left(\mathbf{A}\mathbf{Q}_2 + \mathbf{F}\mathbf{Q}_1\right) - \mathbf{Q}_2^\top\left(\mathbf{G}\mathbf{Q}_2\mathbf{A}^\top\mathbf{Q}_1\right) \\ \mathbf{Q}_2^\top\left(\mathbf{A}\mathbf{Q}_1 - \mathbf{F}\mathbf{Q}_2\right) + \mathbf{Q}_1^\top\left(\mathbf{G}\mathbf{Q}_1 + \mathbf{A}^\top\mathbf{Q}_2\right) & \mathbf{Q}_2^\top\left(\mathbf{A}\mathbf{Q}_2 + \mathbf{F}\mathbf{Q}_1\right) + \mathbf{Q}_1^\top\left(\mathbf{G}\mathbf{Q}_2 - \mathbf{A}^\top\mathbf{Q}_1\right) \end{bmatrix}.$$

Comparing this with $\begin{bmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{0} & -\mathbf{T}^\top \end{bmatrix}$, we deduce:

$$\mathbf{Q}_1^\top\left(\mathbf{A}\mathbf{Q}_1 - \mathbf{F}\mathbf{Q}_2\right) - \mathbf{Q}_2^\top\left(\mathbf{G}\mathbf{Q}_1 + \mathbf{A}^\top\mathbf{Q}_2\right) = \mathbf{T},$$

$$\mathbf{Q}_2^\top\left(\mathbf{A}\mathbf{Q}_1 - \mathbf{F}\mathbf{Q}_2\right) + \mathbf{Q}_1^\top\left(\mathbf{G}\mathbf{Q}_1 + \mathbf{A}^\top\mathbf{Q}_2\right) = \mathbf{0}.$$

Plug $\mathbf{X} = \mathbf{Q}_2\mathbf{Q}_1^{-1}$ into $\mathbf{G} + \mathbf{X}\mathbf{A} + \mathbf{A}^\top\mathbf{X} - \mathbf{X}\mathbf{F}\mathbf{X} = \mathbf{0}$, and use the property taht $\mathbf{Q}_1$ is invertible and $\mathbf{Q}_1^\top\mathbf{Q}_2$ is symmetric, the result is verified for the same equation.

2. Given $\mathbf{I} = \mathbf{Q}_1^\top\mathbf{Q}_1 + \mathbf{Q}_2^\top\mathbf{Q}_2$, and konwing $\mathbf{X} = \mathbf{Q}_2\mathbf{Q}_1^{-1}$, we compute $\mathbf{A} - \mathbf{F}\mathbf{X} = \mathbf{A} - \mathbf{F}\mathbf{Q}_2\mathbf{Q}_1^{-1}$. From the Hamiltonian-Schur decomposition:

$$\mathbf{Q}^\top \mathbf{M}\mathbf{Q} = \begin{bmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{0} & -\mathbf{T}^\top \end{bmatrix}.$$

We know:

$$\mathbf{MQ} = \mathbf{Q}\begin{bmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{0} & -\mathbf{T}^\top \end{bmatrix}.$$

Substitute $\mathbf{Q}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{F} \\ \mathbf{G} & -\mathbf{A}^\top \end{bmatrix}\begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ -\mathbf{Q}_2 & \mathbf{Q}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ -\mathbf{Q}_2 & \mathbf{Q}_1 \end{bmatrix}\begin{bmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{0} & -\mathbf{T}^\top \end{bmatrix}.$$

Multiplying out:

$$\begin{bmatrix} \mathbf{AQ}_1 + \mathbf{F}\left(-\mathbf{Q}_2\right) & \mathbf{AQ}_2 + \mathbf{FQ}_1 \\ \mathbf{GQ}_1 + \left(-\mathbf{A}^\top\right)\left(-\mathbf{Q}_2\right) & \mathbf{GQ}_2 + \left(-\mathbf{A}^\top\right)\mathbf{Q}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1\mathbf{T} & \mathbf{Q}_1\mathbf{R} + \mathbf{Q}_2\left(-\mathbf{T}^\top\right) \\ -\mathbf{Q}_2\mathbf{T} & -\mathbf{Q}_2\mathbf{R} + \mathbf{Q}_1\left(-\mathbf{T}^\top\right) \end{bmatrix}.$$

Equating terms, we find:

$$\mathbf{AQ}_1 - \mathbf{FQ}_2 = \mathbf{Q}_1\mathbf{T}.$$

Thus,

$$\mathbf{A} - \mathbf{FX} = \mathbf{Q}_1\mathbf{T}\mathbf{Q}_1^{-1},$$

showing $\mathbf{A} - \mathbf{FX}$ is similar to $\mathbf{T}$ and they are the same eigenvalues.

3. The code is in the zip file. $\mathbf{X} = \begin{bmatrix} 1.3852 & -0.6148 \\ -0.6148 & 0.3852 \end{bmatrix}$. Residual $\begin{bmatrix} -1.8374 & 1.7033 \\ 1.7033 & 1.2440 \end{bmatrix}$. The residual norm is 3.2750. Thus the solution is not accurate enough.

Problem 4. (SVD decomposition, 25 pts)

In this problem you will see one of the application of SVD, what is doing handwritten digits classification on MNIST dataset. "mnist.mat" contains 60,000 training samples and 10,000 test samples. Each sample is a 28×28 digit image vectorized into a 784-dimensional digit vector. Given a $p \times n$ data matrix $\mathbf{M}_z$ corresponding to digit z, where p is the dimension of digit vector, and n is the number of samples. Our classification algorithm performs SVD decomposition on $\mathbf{M}_z$ as $\mathbf{M}_z = \mathbf{U}_z \boldsymbol{\Sigma}_z \mathbf{V}_z^T$ and uses $\mathbf{U}_z$ to classify an unknown digit vector $\mathbf{q}$. For any unknown digit vector $\mathbf{q}$, we define **residual** between $\mathbf{q}$ and the column space of the digit z given r as the distance between $\mathbf{q}$ and $\text{Proj}_{\mathbf{U}_z}(\mathbf{q})$, which is given as

$$\left\| \mathbf{q} - \sum_{i=1}^{r} \langle \mathbf{q}, \mathbf{u}_{z,i} \rangle \mathbf{u}_{z,i} \right\|_2 .$$

Next, we want to find z, such that

$$\min_{0 \le z \le 9} \quad \left\| \mathbf{q} - \sum_{i=1}^{r} \langle \mathbf{q}, \mathbf{u}_{z,i} \rangle \mathbf{u}_{z,i} \right\|_2 , \tag{1}$$

and we conclude z is the classification of the unknown digit. More details can be founded in exercise 6 of SVD lab course.

1) Load the dataset "mnist.mat" and construct a training set containing 100 data points. Perform SVD decomposition on the training set corresponding to each digits. For each digits, plot its mean image, the first left singular space $\mathbf{u}_1$, the second left singular space $\mathbf{u}_2$, and the tenth left singular space $\mathbf{u}_{10}$ in your answer. Describe what you have observed. (10 pts)
2) Construct a test set containing 200 data points. Implement the objective function (1) in your code to classify the test set, and report the accuracy rates for selecting different r in the table below. (15 pts)

Table I

SUMMARY OF ACCURACY RATES

r	2	3	4	5	6	7	8
accuracy rate							

*Note: In this problem, **Matlab** is recommended as the programming language. If not, please indicate the programming language you use in your solution. You can consider using the provided* ImageCrop2 *function to crop the* 28×28 *images to* 16×16 *to remove the margins.*

Solution 2. 1. The code is in the zip file. 2.

Table II

SUMMARY OF ACCURACY RATES

r	2	3	4	5	6	7	8
accuracy rate	0.275	0.48	0.61	0.615	0.635	0.705	0.765

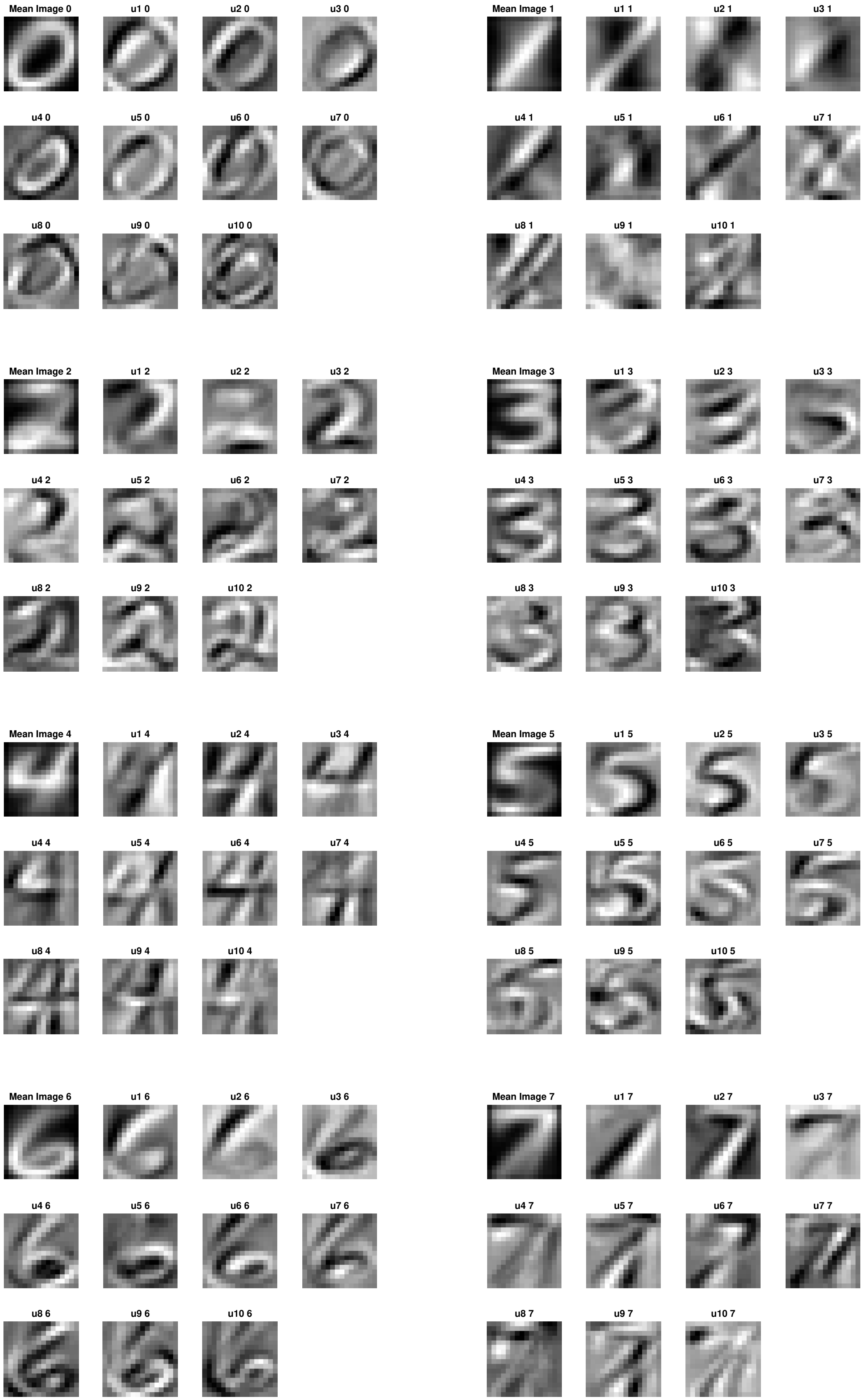

Figure 1. Fig. 1

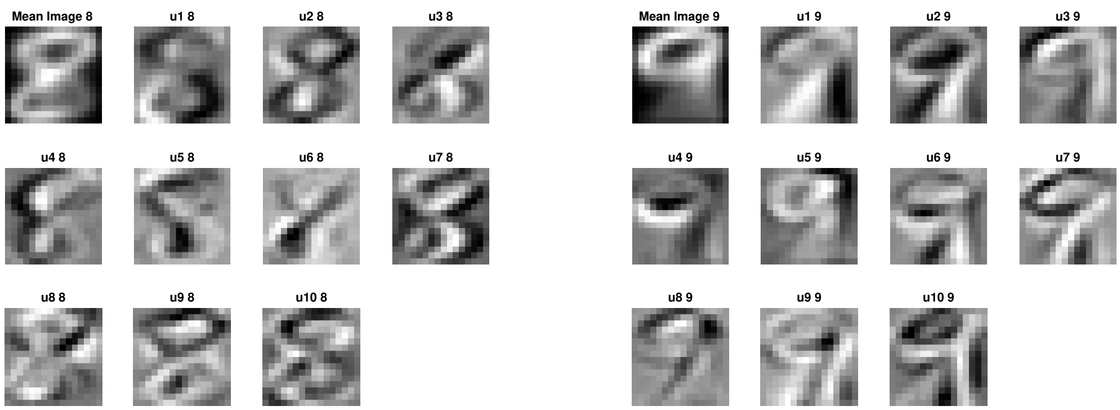

Figure 2. Fig. 2