Notes for General Topology

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Chapter 1.

Topological Spaces

§1.1 Basic Definitions

Definition 1.1.1. Let X be any set.

A family $\mathcal{T} \subseteq \mathcal{P}(X)$ is a *topology on* X iff it satisfies the *Open Set Axioms*. That is,

1. $X \in \mathcal{T}$;
2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$;
3. For any finite $\mathcal{F} \subseteq \mathcal{T}$, $\bigcap \mathcal{U} \in \mathcal{T}$;

The ordered pair $(X, \mathcal{T})$ is called a *topological space*.

Theorem 1.1.1. $\emptyset \in \mathcal{T}$.

Proof. As $\emptyset$ is the subset of any set, $\emptyset \subseteq \mathcal{T}$. Then, we have

$$\bigcup \emptyset = \emptyset.$$

By Open Set Axiom 2, $\emptyset \in \mathcal{T}$. ■

Definition 1.1.2. Let $\mathcal{T}$ and $\mathcal{T}'$ be topological spaces on X. $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or $\mathcal{T}'$ is *finer* than $\mathcal{T}$, iff $\mathcal{T} \subseteq \mathcal{T}'$.

Example 1.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, $\mathcal{T}$ is a *indiscrete topology on* X, iff $\mathcal{T} = \{\emptyset, X\}$. It is the coarsest topology on X.

Example 1.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, $\mathcal{T}$ is a *discrete topology on* X, iff $\mathcal{T} = \mathcal{P}(X)$. It is the finest topology on X.

§1.2 Subspaces

Definition 1.2.1. Let $\mathbb{X} = (X, \mathcal{T})$ be any topological space, and let $A \subseteq X$.

The *subspace topology of* A *induced by* $\mathcal{T}$ is defined as

$$\mathcal{T}_A := \{U \cap A : U \in \mathcal{T}\}.$$

The ordered pair $(A, \mathcal{T}_A)$ is called a *topological subspace*, or just *subspace*, of $\mathbb{X}$.

Theorem 1.2.1. Let $\mathbb{A} = (A, \mathcal{T}_A)$ be a subspace of $\mathbb{X}$.

$\mathcal{T}_A$ is a topology on A; i.e., $\mathbb{A}$ is a topological space.

Proof. As $X \in \mathcal{T}$, by Definition 1.2.1, $A = X \cap A \in \mathcal{T}$. Thus, $\mathcal{T}_A$ satisfies Open Set Axiom 1. □

Let I be an indexed set and let $\{U_i\}_{i \in I} \subseteq \mathcal{T}_A$. For any $i \in I$, there exists a $U_i' \in \mathcal{T}$, such that $U_i = U_i' \cap A$. Thus, we have

$$\begin{aligned} \bigcup_{i \in I} U_i &= \bigcup_{i \in I} (U_i' \cap A) \\ &= A \cap \bigcup_{i \in I} U_i'. \end{aligned}$$

As $\mathcal{T}$ is closed under arbitrary union, $\bigcup_{i \in I} U_i \in \mathcal{T}$. By Definition 1.2.1,

$$A \cap \bigcup_{i \in I} U_i' \in \mathcal{T}_A.$$

Thus, $\mathcal{T}_A$ satisfies the Open Set Axioms 2. □

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