Notes for General Topology by Tom Leinster

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Chapter 1

Topological Spaces

1.1 Review of Metric Spaces

Definition 1.1.1. Let X be a set. A *metric* on X is a function $\rho : X \times X \to \mathbb{R}_{\geq 0}$, such that $\forall x, y, z \in X$, the following (metric axioms) holds:

M1. $\rho(x, y) = 0 \iff x = y$ (identity of indiscernibles);

M2. $\rho(x, y) = \rho(y, x)$ (symmetry).

M3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (triangle inequality);

A *metric space* is a set together with a metric on it, or more formally, a pair (X, ρ) where X is a set and ρ is a metric on X.

Example 1.1.1.

(i) The function $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $\forall p \in \mathbb{R}_{\geq 1}$, $\forall x, y \in \mathbb{R}^n$,

$$\rho_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}},$$

is a metric on $\mathbb{R}^n$. Clearly it satisfies identity of indiscernibles and symmetry. For triangle inequality, it is suggested by Minkowski inequality.

Given $x \in \mathbb{R}^3$, $r \in \mathbb{R}_{\geq 0}$, and

$$B_\rho = \left\{ y \in \mathbb{R}^3 \mid \rho(x, y) \leq r \right\}.$$

$\forall p, q \in \mathbb{R}_{\geq 1}$, it is true that, $\forall x, y \in \mathbb{R}^n$,

$$p \leq q \implies \rho_p(x, y) \geq \rho_q(x, y).$$

Thus, $B_p \subseteq B_q$.

Geometrically, as $p = 1$, B is a octahedron in $\mathbb{R}^3$ with center x and radius r; as $p = 2$, B is a sphere in $\mathbb{R}^3$ with center x and radius r. It is easy to observe that as $p \to \infty$, B tends to the cube in $\mathbb{R}^3$ with center x and edge length $2r$; i.e.,

$$\rho_\infty(x, y) = \lim_{p\to\infty} \rho_p(x, y) = \sup_{i\in\{1,\ldots,n\}} |x_i - y_i|.$$

(ii) Let $f : (X, \rho) \to \mathbb{R}^n$ with $X \subseteq \mathbb{R}^m$ be a continuous map on X. Let $x, y \in X$, then $\rho' : f[X] \times f[X] \to \mathbb{R}_{\geq 0}$ defined by

$$\rho'_p(x, y) = \int_0^1 f(\ell(t))d_p s(t)$$

where

$$\ell(t) = x + t(y - a)$$

and

$$d_p s(t) = \left(\sum_{i=1}^{m} \left|\frac{dg_i}{dt}(t)\right|^p\right)^{\frac{1}{p}} dt.$$

with $p \geq \mathbb{R}_{\geq 1}$ is a metric on $f[X]$.

Fix x and given $r \in \mathbb{R}_{\geq 0}$, the set

$$B_p = \left\{y \in \mathbb{R}^m : \rho'_p(x, y) \leq r\right\}$$

describes a set "attached" on $f[X]$ with center x. If $p = 2$, $m = 2$ and $n = 3$, and $f : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$ is defined by

$$f(\lambda, \phi) = \begin{cases} r\cos\lambda \sin\phi, \\ r\sin\lambda \sin\phi, \\ r\cos\phi, \end{cases}$$

then ρ'_2 here is a *great circle metric* defined by

$$\rho'_2(x, y) = r \arccos(\sin x_\phi \sin y_\phi + \cos x_\phi \cos y_\phi \cos(x_\lambda - y_\lambda)).$$

(iii) Let $a, b \in \mathbb{R}$ with $a \leq b$, and $p \in \mathbb{R}_{\geq 1}$, and $C[a, b]$ denote the set of continuous function $[a, b] \to \mathbb{R}$.

Then d_p defined by $\forall f, g \in C[a, b]$,

$$\rho_p(f, g) = \left(\int_a^b |f - g|^p\right)^{\frac{1}{p}}$$

is a metric on $C[a, b]$.

Similar to ρ_p on $\mathbb{R}^n$,

$$B_p = \{g \mid \rho(f, g) \leq r\}$$

defines a set with "center" f and "radius" $r \in \mathbb{R}_{\geq 0}$.

It also implies that, on $C[a,b]$, $\forall p,q \in \mathbb{R}_{\geq 1}$, $\forall x,y \in \mathbb{R}^n$

$$p \leq q \implies d_p(f,g) \geq d_q(f,g),$$

and, naturally, $B_p \subseteq B_q$. This is a straight corollary from the same case of d_p on $\mathbb{R}^n$.

(iv) Let A be a set. The *Hamming metric* ρ on a set A^n is given by $\forall x,y \in A^n$

$$\rho(x,y) = \#\left\{i \in \{1,\ldots,n\} : x_i \neq y_i\right\}.$$

An example from Wikipedia. The word "karolin" and "kathrin" can be considered as tuples

$$x = (\mathrm{k},\mathrm{a},\mathrm{r},\mathrm{o},\mathrm{l},\mathrm{i},\mathrm{n}),\ y = (\mathrm{k},\mathrm{a},\mathrm{t},\mathrm{h},\mathrm{r},\mathrm{i},\mathrm{n}).$$

For all $i \in \{0,\ldots,6\} \setminus \{0,1,4,6\}$, $x_i \neq y_i$, and $\#(\{0,\ldots,6\} \setminus \{0,1,4,6\}) = 3$, thus

$$\rho(x,y) = 3.$$

(v) Let (M,ρ) be a metric space (for example, $\rho = \rho_2$ on $\mathbb{R}^n$), and $X, Y \in \mathcal{P}(M)$. The Hausdorff metric ρ_{H} on $\mathcal{P}(M)$ is defined by

$$\rho_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x\in X} \rho(x,Y), \sup_{y\in Y} \rho(X,y)\right\},$$

where $\rho(a,B) = \inf_{b\in B} \rho(a,b)$ for all $B \in \mathcal{P}(M)$ and $a \in M$.

This metric can be used to measure how close two figures (as sets of points) are.

Definition 1.1.2. Let X be a metric space, let $x \in X$, and $\varepsilon > 0$. The *open ball with center x and radius ε*, or more briefly the *open ε-ball about x* is the subset

$$B(x,\varepsilon) = \{y \in X : \rho(x,y) \leq \varepsilon\} \subseteq X.$$

Similarly, the *closed ε-ball around x* is

$$\overline{B}(x,\varepsilon) = \{y \in X : \rho(x,y) \leq \varepsilon\} \subseteq X.$$

Note 1.1.1. Clearly, the word "ball" does not mean it should look like a ball. Clearly, for all $x \in \mathbb{R}^3$, the ball $\{y \in \mathbb{R}^3 : \rho_\infty(x,y) < 1\}$ is a cube without its surface.

And it is interesting to think that on $C[a,b]$ with conditions above,

$$\{g \in C[a,b] : \rho_p(f,g) < 1\}$$

defines a open ball in $C[a,b]$.

Note 1.1.2. For hamming metric ρ with conditions above, for $\varepsilon \in \mathbb{R}_{(0,1)}$, the ball

$$\{y \in A^n : \rho(x,y) < 1\} = \{x\}.$$

is a singleton.

Definition 1.1.3. Let X be a metric space.

(i) A subset U of X is *open in* X (or an *open subset of* X) iff

$$\forall u \in U,\ \exists \varepsilon \in \mathbb{R}_{>0},\ \ B(u,\varepsilon) \subseteq U.$$

(ii) A subset V is *closed in* X iff $X \setminus V$ is open in X.

Note 1.1.3. Equivalently, U is open in X iff $\exists \varepsilon \in \mathbb{R}_{>0}$,

$$U = \bigcup_{x \in U} B(x,\varepsilon);$$

and V is closed in X iff

$$V = \bigcap_{x \in U} (X \setminus B(x,\varepsilon)),$$

by De Morgan's Law.

Definition 1.1.4. Let X be a metric space, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$. Then $\{x_n\}$ *converges* to x iff

$$d(x_n, x) \to 0 \text{ as } n \to \infty.$$

Explicitly, then, $\{x_n\}$ converges to x iff

$$\forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}_{\geq 1}, \forall n \in \mathbb{N}_{\geq N}, \quad d(x_n, x) < \varepsilon.$$

Note 1.1.4. Equivalently, $\{x_n\}$ converges to x iff

$$\forall \varepsilon \in \mathbb{R}_{>0}, \#(\{x_n\} \cap B(x,\varepsilon)) = \aleph_0 \wedge \#(\{x_n\} \setminus B(x,\varepsilon)) < \aleph_0.$$

In other words, $B(x,\varepsilon)$ contains all but finitely many x_n.