

# Notes for General Topology

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## Chapter 1

# Metric Spaces

## Chapter 2

# Topological Spaces

### 2.1 Topological Spaces

**Definition 2.1.1** (topology). Let  $X$  be a set, and let a family  $\mathcal{T} \subseteq \mathcal{P}(X)$ .  $\mathcal{T}$  is called a topology on  $X$  iff

- (i)  $\emptyset, X \in \mathcal{T}$ ;
- (ii)  $\mathcal{T}$  is closed under arbitrary union;
- (iii)  $\mathcal{T}$  is closed under finite intersection.

The pair  $(X, \mathcal{T})$  is called a *topological space*. The elements of  $\mathcal{T}$  are called *open sets* in  $(X, \mathcal{T})$ .

**Definition 2.1.2** (topological spaces). Let  $X$  be any set, and let  $\mathcal{T}$  be a topology on  $X$ , then the pair  $(X, \mathcal{T})$  is called a *topological space*. All subsets of  $X$  in  $\mathcal{T}$  are called *open sets* of  $(X, \mathcal{T})$ .

**Definition 2.1.3** (finer and coarser topology). Let  $X$  be any set, and let  $\mathcal{T}, \mathcal{T}'$  be topologies on  $X$ .  $\mathcal{T}$  is said to be *finer* than  $\mathcal{T}'$  iff  $\mathcal{T} \supseteq \mathcal{T}'$ ; respectively,  $\mathcal{T}$  is said to be *coarser* than  $\mathcal{T}'$  iff  $\mathcal{T} \subseteq \mathcal{T}'$ .

**Definition 2.1.4** (neighbourhood). Given  $(X, \mathcal{T})$  as a topological space and a point  $x \in X$ , a subset  $N \subseteq X$  is called a *neighbourhood* iff it contains an open set  $U$  containing  $x$ .

**Proposition 2.1.1.** Given  $(X, \mathcal{T})$  as a topological space and  $U \subseteq X$ ,  $U$  is open iff for all  $x \in U$ , there is a neighbourhood  $N$  of  $x$  contained in  $U$ .

*Proof.* If  $U$  is open, then  $U$  itself is a neighbourhood of  $x$  contained in  $U$ .

Conversely, if for all  $x \in U$ , there is a neighbourhood  $N_x$  of  $x$  contained in  $U$ , then there is a open neighbourhood  $U_x \ni x$  contained in  $N_x$ . Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose  $U$  is not open, then  $U$  is a proper superset in the relation above.  $\square$

## 2.2 Untitled

**Definition 2.2.1** (cover). Let  $(X, \mathcal{T})$  be a topological space, and let  $U \subseteq X$ , then a family  $\mathcal{C} \subseteq \mathcal{P}(X)$  is called a *cover* of  $U$  iff the union of  $\mathcal{C}$  is a superset of  $U$ . That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If  $\mathcal{C} \subseteq \mathcal{T}$ , then we call  $\mathcal{C}$  an *open cover* of  $U$ .

Let  $\mathcal{C}' \subseteq \mathcal{C}$ , iff the union of  $\mathcal{C}'$  is still a superset of  $U$ , then we call  $\mathcal{C}'$  a subcover of  $\mathcal{C}$ .

**Definition 2.2.2** (basis). Let  $(X, \mathcal{T})$  be a topological space, let  $U \subseteq X$ , and let  $\mathcal{B} \subseteq \mathcal{P}(X)$  be a cover of  $U$ . We call  $\mathcal{B}$  a *base* of  $(X, \mathcal{T})$  iff  $\mathcal{B} \subseteq \mathcal{T}$  and the union of  $\mathcal{B}$  is exactly  $U$  itself. That is,

$$\mathcal{B} \subseteq \mathcal{T}, \text{ and } U = \bigcup \mathcal{B}.$$

**Definition 2.2.3** (subspace topology). Let  $(X, \mathcal{T})$  be a topological space and let  $A \subseteq X$ . The *subspace topology*  $\mathcal{T}_A$  on  $A$  is defined to be the family of the intersections of open sets in  $(X, \mathcal{T})$  and  $A$ . That is,

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}.$$

**Definition 2.2.4** (quotient topology). Let  $(X, \mathcal{T})$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . The *quotient topology* is a topology on  $\mathcal{P}(X/\sim)$ ; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

**Definition 2.2.5** (continuous functions). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A function  $f : X \rightarrow Y$  is said to be *continuous* iff for all open subset  $U$  of  $Y$ , the preimage  $f^{-1}[U]$  is open in  $X$ . That is,

$$\forall U \in \mathcal{T}_Y : f^{-1}[U] \in \mathcal{T}_X.$$

**Definition 2.2.6** (homeomorphisms). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A bijection  $f : X \rightarrow Y$  is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

**Definition 2.2.7** (homeomorphic). Two topological spaces  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are said to be *homeomorphic* or *topologically equivalent*, denoted  $X \cong Y$ , iff there is an homeomorphism between them.

**Definition 2.2.8** (compactness). A topological space  $(X, \mathcal{T})$  is said to be *compact* iff every open cover of  $X$  has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

**Definition 2.2.9** (connectedness). Let  $(X, \mathcal{T})$  be a topological space.  $(X, \mathcal{T})$  is said to be *connected* iff  $X$  is not empty and it is not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

**Definition 2.2.10** (path-connectedness). Let  $(X, \mathcal{T})$  be a topological space.

- (i) A map  $\gamma : [0, 1] \rightarrow X$  is called a *path* in  $X$  iff it is continuous. If  $\gamma(0) = x$  and  $\gamma(1) = y$ , we say that  $\gamma$  is path from  $x$  to  $y$  in  $X$ .
- (ii)  $X$  is said to be *path-connected* iff for all  $x, y \in X$  there is a path from  $x$  to  $y$  in  $X$ .

**Definition 2.2.11** (topologically indistinguishable). Let  $(X, \mathcal{T})$  be a topological space. Two points  $x, y \in X$  are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let  $\mathcal{N}_x$  be the family of all neighbourhoods of  $x$  and let  $\mathcal{N}_y$  be the family of all neighbourhoods of  $y$ , we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively,  $x, y$  are said to be *topologically distinguishable* iff they are not topologically indistinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

**Definition 2.2.12** (separated sets). Let  $(X, \mathcal{T})$  be a topological space, and let  $A, B \in \mathcal{P}(X)$ .

- (i)  $A$  and  $B$  are said to be *separated* iff each is disjoint from other's closure.

- (ii)  $A$  and  $B$  are said to be *separated by neighbourhoods* iff there are neighbourhoods  $N_A$  of  $A$  and  $N_B$  of  $B$  such that  $N_A$  and  $N_B$  are disjoint.
- (iii)  $A$  and  $B$  are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods  $\overline{N}_A$  of  $A$  and  $\overline{N}_B$  of  $B$  such that  $\overline{N}_A$  and  $\overline{N}_B$  are disjoint.
- (iv)  $A$  and  $B$  are said to be *separated by a continuous function* iff there is a continuous function  $f : X \rightarrow \mathbb{R}$ , such that  $f[A] = \{0\}$  and  $f[B] = \{1\}$ .
- (v)  $A$  and  $B$  are said to be *precisely separated by a continuous function* iff there is a continuous function  $f : X \rightarrow \mathbb{R}$ , such that  $f^{-1}[\{0\}] = A$  and  $f^{-1}[\{1\}] = B$

[See Wikipedia.org](https://en.wikipedia.org/wiki/Topological_spaces)

**Definition 2.2.13** ( $T_0$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_0$  or *Kolmogorov*, iff all distinct points  $x, y \in X$  are *topologically distinguishable*.

**Definition 2.2.14** ( $R_0$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $R_0$  iff any two topologically distinguishable points in  $X$  are separated.

**Definition 2.2.15** ( $T_1$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_1$  or *Fréchet* iff any two distinct points in  $X$  are separated.

**Proposition 2.2.1.** All singletons in a  $T_1$  space are closed, That is, if a topological space  $(X, \mathcal{T})$  is  $T_1$ , then

$$\forall x \in (X, \mathcal{T}) : \exists U \in \mathcal{T} : \{x\} = X \setminus U.$$

**Definition 2.2.16** ( $T_2$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_2$  or *Hausdorff* or *separated* iff any two distinct points in  $(X, \mathcal{T})$  are separated by neighbourhoods.

**Definition 2.2.17** ( $T_{21/2}$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_{21/2}$  or *Urysohn* iff two distinct points in  $X$  are separated by closed neighbourhoods.

**Definition 2.2.18** ( $T_3$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_3$  or *regular* iff it is  $T_0$  and given any point  $x \in (X, \mathcal{T})$  and closed set  $V \subseteq X$  with  $x \notin V$  are separated by neighbourhoods.

**Definition 2.2.19** ( $T_{31/2}$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_{31/2}$ , or *Tychonoff* or, *completely*  $T_3$ , or *completely regular*, iff it is  $T_0$  and given any point  $x$  and closed set  $V \subseteq X$  with  $x \notin V$ , they are separated by a continuous function.

**Definition 2.2.20** ( $T_4$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_4$  or *normal* iff it is Hausdorff and any two disjoint closed subsets of  $X$  are separated by neighbourhoods.

**Proposition 2.2.2** (Urysohn's lemma). A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

**Definition 2.2.21** ( $T_5$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_5$  or *completely  $T_4$*  iff it is  $T_1$  and any two separated sets are separated by neighbourhoods.

**Proposition 2.2.3.** Every subspace of a  $T_5$  space is normal.

**Definition 2.2.22** ( $T_6$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_6$ , or *perfectly  $T_4$*  or *perfectly normal* iff it is  $T_1$  and any two disjoint closed sets are precisely separated by a continuous function.

**Proposition 2.2.4** (Tietze extension theorem). Let  $(X, \mathcal{T})$  be normal topological space, and let  $f : A \rightarrow (\mathbb{R}, \mathcal{T}')$  be a continuous map where  $A$  is a closed subset of  $X$  and  $\mathcal{T}'$  is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \rightarrow (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$