

# Notes for General Topology

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# Chapter 1

## Topological Spaces

### 1.1 Review of Metric Spaces

**Definition 1.1.1.** Let  $X$  be a set. A *metric* on  $X$  is a function  $\rho : X \times X \rightarrow \mathbb{R}_{\geq 0}$ , such that  $\forall x, y, z \in X$ , the following (metric axioms) holds:

M1.  $\rho(x, y) = 0 \iff x = y$  (identity of indiscernibles);

M2.  $\rho(x, y) = \rho(y, x)$  (symmetry).

M3.  $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$  (triangle inequality);

A *metric space* is a set together with a metric on it, or more formally, a pair  $(X, \rho)$  where  $X$  is a set and  $\rho$  is a metric on  $X$ .

**Example 1.1.1.**

1. The function  $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  defined by  $\forall p \in \overline{\mathbb{R}}_{\geq 1}, \forall x, y \in \mathbb{R}^n$ ,

$$\rho_p(x, y) = \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}},$$

is a metric on  $\mathbb{R}^n$ . Clearly it satisfies identity of indiscernibles and symmetry. For triangle inequality, it is suggested by Minkowski inequality.

Given  $x \in \mathbb{R}^3$ ,  $r \in \mathbb{R}_{\geq 0}$ , and

$$B_\rho = \{y \in \mathbb{R}^3 \mid \rho(x, y) \leq r\}.$$

$\forall p, q \in \overline{\mathbb{R}}_{\geq 1}$ , it is true that,  $\forall x, y \in \mathbb{R}^n$ ,

$$p \leq q \implies \rho_p(x, y) \geq \rho_q(x, y).$$

Thus,  $B_p \subseteq B_q$ .

Geometrically, as  $p = 1$ ,  $B$  is a octahedron in  $\mathbb{R}^3$  with center  $x$  and radius  $r$ ; as  $p = 2$ ,  $B$  is a sphere in  $\mathbb{R}^3$  with center  $x$  and radius  $r$ . It is easy to observe that as  $p \rightarrow \infty$ ,  $B$  tends to a cube in  $\mathbb{R}^3$  with center  $x$  and edge length  $2r$ ; i.e.,

$$\rho_\infty(x, y) = \lim_{p \rightarrow \infty} \rho_p(x, y) = \sup_{i \in \{1, \dots, n\}} |x_i - y_i|.$$

2. Let  $f : (X, \rho) \rightarrow \mathbb{R}^n$  with  $X \subseteq \mathbb{R}^m$  be a continuous map on  $X$ . Let  $x, y \in X$ , then  $\rho' : f[X] \times f[X] \rightarrow \mathbb{R}_{\geq 0}$  defined by

$$\rho'_p(x, y) = \int_0^1 f(\ell(t)) d_p s(t)$$

where

$$\ell(t) = x + t(y - x)$$

and

$$d_p s(t) = \left( \sum_{i=1}^m \left| \frac{dg_i}{dt}(t) \right|^p \right)^{\frac{1}{p}} dt.$$

with  $p \geq \overline{\mathbb{R}}_{\geq 1}$  is a metric on  $f[X]$ .

Fix  $x$  and given  $r \in \mathbb{R}_{\geq 0}$ , the set

$$B_p = \{y \in \mathbb{R}^m : \rho'_p(x, y) \leq r\}$$

describes a set “attached” on  $f[X]$  with center  $x$ . If  $p = 2$ ,  $m = 2$  and  $n = 3$ , and  $f : [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}^3$  is defined by

$$f(\lambda, \phi) = \begin{cases} r \cos \lambda \sin \phi, \\ r \sin \lambda \sin \phi, \\ r \cos \phi, \end{cases}$$

then  $\rho'_2$  here is a *great circle metric* defined by

$$\rho'_2(x, y) = r \arccos(\sin x_\phi \sin y_\phi + \cos x_\phi \cos y_\phi \cos(x_\lambda - y_\lambda)).$$

3. Let  $a, b \in \mathbb{R}$  with  $a \leq b$ , and  $p \in \overline{\mathbb{R}}_{\geq 1}$ , and  $C[a, b]$  denote the set of continuous function  $[a, b] \rightarrow \mathbb{R}$ .

Then  $d_p$  defined by  $\forall f, g \in C[a, b]$ ,

$$\rho_p(f, g) = \left( \int_a^b |f - g|^p \right)^{\frac{1}{p}}$$

is a metric on  $C[a, b]$ .

Similar to  $\rho_p$  on  $\mathbb{R}^n$ ,

$$B_p = \{g \mid \rho(f, g) \leq r\}$$

defines a set with “center”  $f$  and “radius”  $r \in \mathbb{R}_{\geq 0}$ .

It also implies that, on  $C[a, b]$ ,  $\forall p, q \in \overline{\mathbb{R}}_{\geq 1}$ ,  $\forall x, y \in \mathbb{R}^n$

$$p \leq q \implies d_p(f, g) \geq d_q(f, g),$$

and, naturally,  $B_p \subseteq B_q$ . This is a straight corollary from the same case of  $d_p$  on  $\mathbb{R}^n$ .

4. Let  $A$  be a set. The *Hamming metric*  $\rho$  on a set  $A^n$  is given by  $\forall x, y \in A^n$

$$\rho(x, y) = \# \{i \in \{1, \dots, n\} : x_i \neq y_i\}.$$

An example from Wikipedia. The word “karolin” and “kathrin” can be considered as tuples

$$x = (\text{k}, \text{a}, \text{r}, \text{o}, \text{l}, \text{i}, \text{n}), \quad y = (\text{k}, \text{a}, \text{t}, \text{h}, \text{r}, \text{i}, \text{n}).$$

For all  $i \in \{0, \dots, 6\} \setminus \{0, 1, 4, 6\}$ ,  $x_i \neq y_i$ , and  $\#(\{0, \dots, 6\} \setminus \{0, 1, 4, 6\}) = 3$ , thus

$$\rho(x, y) = 3.$$

5. Let  $(M, \rho)$  be a metric space (for example,  $\rho = \rho_2$  on  $\mathbb{R}^n$ ), and  $X, Y \in \mathcal{P}(M)$ . The Hausdorff metric  $\rho_H$  on  $\mathcal{P}(M)$  is defined by

$$\rho_H(X, Y) = \max \left\{ \sup_{x \in X} \rho(x, Y), \sup_{y \in Y} \rho(X, y) \right\},$$

where  $\rho(a, B) = \inf_{b \in B} \rho(a, b)$  for all  $B \in \mathcal{P}(M)$  and  $a \in M$ .

This metric can be used to measure how close two figures (as sets of points) are.

**Definition 1.1.2.** Let  $X$  be a metric space, let  $x \in X$ , and  $\varepsilon > 0$ . The *open ball with center  $x$  and radius  $\varepsilon$* , or more briefly the *open  $\varepsilon$ -ball about  $x$*  is the subset

$$B(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\} \subseteq X.$$

Similarly, the *closed  $\varepsilon$ -ball around  $x$*  is

$$\overline{B}(x, \varepsilon) = \{y \in X : \rho(x, y) \leq \varepsilon\} \subseteq X.$$

**Note 1.1.1.** Clearly, the word “ball” does not mean it should look like a ball. Clearly, for all  $x \in \mathbb{R}^3$ , the ball  $\{y \in \mathbb{R}^3 : \rho_\infty(x, y) < 1\}$  is a cube without its surface.

And it is interesting to think that on  $C[a, b]$  with conditions above,

$$\{g \in C[a, b] : \rho_p(f, g) < 1\}$$

defines a open ball in  $C[a, b]$ .

**Note 1.1.2.** For hamming metric  $\rho$  with conditions above, for  $\varepsilon \in \mathbb{R}_{(0,1)}$ , the ball

$$\{y \in A^n : \rho(x, y) < 1\} = \{x\}.$$

is a singleton.

**Definition 1.1.3.** Let  $X$  be a metric space.

(i) A subset  $U$  of  $X$  is *open in  $X$*  (or an *open subset of  $X$* ) iff

$$\forall u \in U, \exists \varepsilon \in \mathbb{R}_{>0}, B(u, \varepsilon) \subseteq U.$$

(ii) A subset  $V$  is *closed in  $X$*  iff  $X \setminus V$  is open in  $X$ .

**Note 1.1.3.** Equivalently,  $U$  is open in  $X$  iff  $\exists \varepsilon \in \mathbb{R}_{>0}$ ,

$$U = \bigcup_{x \in U} B(x, \varepsilon);$$

and  $V$  is closed in  $X$  iff

$$V = X \setminus \bigcup_{x \in U} B(x, \varepsilon) = \bigcap_{x \in U} (X \setminus B(x, \varepsilon)),$$

by De Morgan’s Law.

**Definition 1.1.4.** Let  $X$  be a metric space, let  $\{x_n\}_{n=1}^\infty$  be a sequence in  $X$  and let  $x \in X$ . Then  $\{x_n\}$  *converges* in  $X$  iff

$$\exists x \in X, \lim_{n \rightarrow \infty} d(x_n, x) = 0.$$

Explicitly, then,  $\{x_n\}$  converges to  $x$  iff

$$\forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}_{\geq 1}, \forall n \in \mathbb{N}_{\geq N}, \quad d(x_n, x) < \varepsilon.$$

**Note 1.1.4.**

1. Equivalently,  $\{x_n\}$  converges in  $X$  iff

$$\exists x \in X, \forall \varepsilon \in \mathbb{R}_{>0}, \#(\{x_n\} \cap B(x, \varepsilon)) = \aleph_0 \wedge \#(\{x_n\} \setminus B(x, \varepsilon)) < \aleph_0.$$

In other words,  $B(x, \varepsilon)$  contains all but finitely many  $x_n$ .

2. Let  $X \subseteq S$ .  $\{x_n\}$  converges to  $x \in S$  does not mean it need to converge in  $X$ . For example  $\mathbb{Q} \subseteq \mathbb{R}$ , the sequence

$$\left\{ x_n = \frac{1}{x} + r : r^2 = 2 \right\}_{n \in \mathbb{N}}$$

does converge to  $\sqrt{2} \in \mathbb{R}$ , but  $\sqrt{2} \notin \mathbb{Q}$ , so  $\{x_n\}$  converges in  $\mathbb{R}$ , but does not converge in  $\mathbb{Q}$ .

**Lemma 1.1.1.** Let  $X$  be a metric space and  $V \subseteq X$ . Then  $V$  is closed in  $X$  iff

$$\forall \{x_n\}_{n=1}^\infty \subseteq V, \forall x \in X, \quad \lim_{n \rightarrow \infty} d(x_n, x) = 0 \implies x \in V.$$

*Proof.* Suppose  $V$  is closed in  $X$ , then  $X \setminus V$  is open in  $X$ . Suppose  $\exists x \in X \setminus V$ , such that  $\exists \{x_n\}_{n=1}^\infty \subseteq V$ ,  $\{x_n\}$  converges to  $x$ , then  $\forall \varepsilon \in \mathbb{R}_{>0}$ ,  $B(x, \varepsilon) \cap \{x_n\} \neq \emptyset$ .  $\{x_n\} \subseteq V$ , so  $B(x, \varepsilon) \cap V \neq \emptyset$ . This implies that  $X \setminus V$  is not open, then  $V$  is not closed (for if  $V$  is closed, then  $X \setminus V$  is open). It is contradicted to the assumption.

Now, suppose  $V$  is not closed in  $X$ , then  $X \setminus V$  is not open. Then,  $\exists p \in X \setminus V$ , such that  $\forall \varepsilon \in \mathbb{R}_{>0}$ ,  $B(p, \varepsilon) \cap V \neq \emptyset$ . This implies there are some  $\{x_n\}_{n=1}^\infty \subseteq V$ , such that  $B(p, \varepsilon)$  contains all but finite elements in  $\{x_n\}$ . Thus,  $\{x_n\}$  converges to  $p \in X \setminus V$ , contradicting to the conditions.  $\square$

**Lemma 1.1.2.** Let  $X$  be a metric space, and  $\mathcal{T}$  be the family of open subsets of  $X$ . Then,

- (i)  $\mathcal{T}$  is closed under arbitrary union.

(ii)  $\mathcal{T}$  is closed under finite intersection.

(iii)  $\emptyset, X \in \mathcal{T}$ .

*Proof.*

1. Let  $I$  be an index set. For all  $i \in I$ , let  $U_i \in \mathcal{T}$ . Then for some  $\varepsilon \in \mathbb{R}_{>0}$ ,

$$U_i = \bigcup_{x \in U_i} B(x, \varepsilon).$$

Let  $U = \bigcup_{i \in I} U_i$ , then we have,

$$U = \bigcup_{i \in I} \bigcup_{x \in U_i} B(x, \varepsilon) = \bigcup_{x \in U} B(x, \varepsilon)$$

for some  $\varepsilon \in \mathbb{R}_{>0}$ .

2. Let  $\mathcal{C}$  be the family of closed subsets of  $X$ , and let  $U, V \in \mathcal{C}$ . Then for all  $\{u_n\}_{n=1}^\infty \subseteq U$ ,  $\forall u \in X$ ,  $\{u_n\}$  converges to  $u$  implies that  $u \in U$ . It also holds for  $U \cup V \supseteq U$ . Similarly, for all  $\{v_m\}_{m=1}^\infty$ ,  $\forall v \in X$ ,  $\{v_m\}$  converges to  $v$  implies  $v \in V$ . It also holds for  $U \cup V \supseteq V$ . Thus  $U \cup V$  is closed.

Then,  $X \setminus (U \cup V) = (X \setminus U) \cap (X \setminus V)$  is open, where  $X \setminus U$  and  $X \setminus V$  are open for  $U$  and  $V$  are closed.

3.  $\emptyset = \bigcup_{i \in \emptyset} U_i$  for all  $U_i \in \mathcal{T}$ , so  $\emptyset$  is open.  $\emptyset = U \cap V$  for all mutually disjoint closed subsets  $U, V \subseteq X$ , so  $\emptyset$  is closed, so  $X = X \setminus \emptyset$  is open.

□

**Lemma 1.1.3.** Let  $X$  be a metric space, and  $\mathcal{C}$  be the family of all closed subsets of  $X$ . Then,

(i)  $\mathcal{C}$  is closed under arbitrary intersection.

(ii)  $\mathcal{C}$  is closed under finite union.

(iii)  $\emptyset, X \in \mathcal{C}$ .

*Proof.* Let  $\mathcal{T}$  be the family of all open subset of  $X$ , and let  $I$  be any index set.

1. It has been proved that  $\mathcal{T}$  is closed under arbitrary union, so by De Morgan's law, for any  $i \in I$ , if  $U_i \in \mathcal{T}$ , then

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i) \in \mathcal{C}.$$

2. It has been proved in Lemma 1.1.2.
3. It has been proved that  $\emptyset$  is open in  $X$ . So  $X = X \setminus \emptyset$  is closed in  $X$ .

□

**Definition 1.1.5.** Let  $(X, \rho)$  and  $(Y, \rho')$  be metric spaces. A function  $f : (X, \rho) \rightarrow (Y, \rho')$  is *continuous* on a point  $p \in X$  iff

$$\forall \varepsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}, \forall x \in B(p, \delta), \quad f(x) \in B(f(p), \varepsilon).$$

**Note 1.1.5.**

1. If  $\rho$  is a discrete metric on  $X$ , then  $B(p, \delta) = \{p\}$  for all  $\delta$ . Then, by definition, for all  $\varepsilon$ ,  $f(x) \in B(f(p), \varepsilon)$ . So  $f$  is continuous everywhere.
2. On the contrary, if  $\rho'$  is a discrete metric on  $Y$ , but for all  $p \in X$ ,  $\rho$  suffices for all  $\delta \in \mathbb{R}_{>0}$ ,  $\#B(p, \delta) \geq \aleph_0$ , then for some  $\varepsilon \in \mathbb{R}_{>0}$ , for all  $\delta \in \mathbb{R}_{>0}$ , there exists  $x \in B(p, \delta)$ , such that  $f(x) \notin B(f(p), \varepsilon)$ . Thus  $f$  is not continuous on such  $p$ .

**Lemma 1.1.4.** Let  $(X, \rho)$  and  $(Y, \rho')$  be metric spaces and let  $f : (X, \rho) \rightarrow (Y, \rho')$  be a function. The following are equivalent:

- (i)  $f$  is continuous on  $X$ ;
- (ii) for all open  $U \subseteq Y$ , the preimage  $f^{-1}[U] \subseteq X$  is open;
- (iii) for all closed  $V \subseteq Y$ , the preimage  $f^{-1}[V] \subseteq X$  is closed.

## 1.2 The Definition of Topological Space

**Definition 1.2.1.** Let  $X$  be a set. A *topological* on  $X$  is a collection  $\mathcal{T} \in \mathcal{P}(X)$  with the following properties.

- T1.  $\mathcal{T}$  is closed under arbitrary union;
- T2.  $\mathcal{T}$  is closed under finite intersection;
- T3.  $X \in \mathcal{T}$ .

The *Topological Space*  $(X, \mathcal{T})$  is a set  $X$  with a topology  $\mathcal{T}$  on  $X$ . All  $\mathcal{T}$ -sets are said to be *open* in  $(X, \mathcal{T})$ .



**Lemma 1.2.1.**  $\emptyset \in \mathcal{T}$ .

*Proof.* By T1, given  $I$  as any index set, if for all  $i \in I$ ,  $U_i \in \mathcal{T}$ , then

$$U = \bigcup_{i \in I} U_i \in \mathcal{T}.$$

If  $I = \emptyset$ , then  $U = \emptyset$ . □

**Note 1.2.1.** Let  $X = \{1, 2, 3\}$  with topology

$$\mathcal{T} = \{\{1, 2\}, \{3\}\}.$$

$\{1, 2\} \in \mathcal{T}$  implies  $\{3\} = X \setminus \{1, 2\}$  is closed;  $\{3\} \in \mathcal{T}$  implies that  $\{1, 2\} = X \setminus \{3\}$  is closed.  $\{2\} \in \mathcal{P}(X)$ , but  $\{2\} \notin \mathcal{T}$ , so  $\{2\}$  is not open in  $(X, \mathcal{T})$ ,  $\{1, 3\} = X \setminus \{2\}$  is not closed. For any  $U \in \mathcal{T}$ ,  $\{2\} \neq X \setminus U$ , so  $\{2\}$  is not open.

**Definition 1.2.2.** Given  $(X, \rho)$  as a metric space, the topology

$$\mathcal{T}_\rho = \left\{ U \in \mathcal{P}(X) : U = \bigcup_{x \in U} B(x, \delta) \right\},$$

then we call  $\mathcal{T}_\rho$  the topology *induced* by  $\rho$ , and  $(X, \mathcal{T}_\rho)$  the *underlying topological space* of metric space  $(X, \rho)$ .

**Note 1.2.2.** These topology is induced by metric.

1. In this case,  $U$  is open in  $(X, \rho)$  iff  $U \in \mathcal{T}_\rho$ .
2. The metric  $\rho_p : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  (surjective) induces  $\mathcal{T}_{\rho_p} \subseteq \mathcal{P}(X)$ . And we'll see that for all  $p, q \geq 1$ ,  $\mathcal{T}_{\rho_p} = \mathcal{T}_{\rho_q}$ .
3. The discrete topology  $\rho_{\text{disc}} : X \times X \rightarrow \mathbb{R}_{>0}$  (non-surjective) induces  $\mathcal{T}_{\rho_{\text{disc}}} = \mathcal{P}(X)$ . It is the largest topology on  $X$ , and  $\rho[X \times X] \subseteq \{0, 1\}$ .
4. The metric  $\rho_p : C[a, b] \times C[a, b] \rightarrow \mathbb{R}_{>0}$  (surjective) induces  $\mathcal{T}_{\rho_p} \subseteq \mathcal{P}(X)$ . And we'll see that  $\mathcal{T}_{\rho_1} \neq \mathcal{T}_{\rho_\infty}$ .
5. Given  $X$  as a space, the Hausdorff metric  $\rho_H : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_{>0}$  (surjective) induces  $\mathcal{T}_{\rho_H} \subseteq \mathcal{P}(\mathcal{P}(X))$ .
6. Given  $A$  as a set, the hamming metric  $\rho : A^n \times A^n \rightarrow \mathbb{R}_{>0}$  (non-surjective) with  $n \in \mathbb{N}$  induces  $\mathcal{T}_\rho \subseteq \mathcal{P}(X)$ .  $\rho[A^n \times A^n] = \mathbb{N}_{\leq n}$ .

These topology is not induced by any metric.

1. The indiscrete topology  $\mathcal{T} = \{\emptyset, X\}$  on  $X$  is not induced by any metric space. Suppose it was, then there would be a metric  $\rho$  such that for all  $x \in X$ , for all  $\varepsilon > 0$ ,  $B(x, \varepsilon) \in \mathcal{T}$ . But, clearly, for those  $\varepsilon \in (0, \phi X)$ ,  $B(x, \varepsilon) \notin \mathcal{T}$ .
2. Let  $X = \{1, 2, 3\}$  with topology

$$\mathcal{T} = \{\{1, 2\}, \{3\}\}.$$

There is no such metric  $\rho$  induces  $\mathcal{T}$  for same reason.

**Definition 1.2.3.** Let  $X$  be a set and  $\mathcal{T}, \mathcal{T}'$  be topologies on  $X$ . If  $\mathcal{T} \subseteq \mathcal{T}'$ , then we say that  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$ , or  $\mathcal{T}$  is *coarser* than  $\mathcal{T}'$ .

**Note 1.2.3.**

1. Given  $X$  as a set, for all topology  $\mathcal{T}$  on  $X$ ,  $\mathcal{T} \subseteq \mathcal{T}_{\text{disc}}$  and  $\mathcal{T} \supseteq \mathcal{T}_{\text{indisc}}$ . Thus,  $\mathcal{T}_{\text{disc}}$  is the finest topology on  $X$ , and  $\mathcal{T}_{\text{indisc}}$  is the coarsest.
2.  $\rho_p$  and  $\rho_{\text{disc}}$  induced same topology on  $\mathbb{Z}$ . But on  $\mathbb{Q}$ ,  $\mathcal{T}_{\rho_p}$  is coarser than  $\mathcal{T}_{\rho_{\text{disc}}}$ .

**Definition 1.2.4.** Given  $(X, \mathcal{T})$  as a topological space, a set  $V \subseteq X$  is said to be *closed* in  $(X, \mathcal{T})$  iff  $X \setminus V \in \mathcal{T}$ .

**Definition 1.2.5.**

1. In the discrete topology on  $X$ , all subsets are closed. Because for all  $U \in \mathcal{T}_{\text{disc}}$ ,  $X \setminus U \in \mathcal{T}_{\text{disc}}$ .
2. In the indiscrete topology on  $X$ , only  $\emptyset$  and  $X$  is closed.

**Lemma 1.2.2.** Let  $X = (X, \mathcal{T})$  be a topological space, and let

$$\mathcal{C} = \{V \subseteq X : V = X \setminus U, U \in \mathcal{T}\}.$$

- (i)  $\mathcal{C}$  is closed under arbitrary intersection;
- (ii)  $\mathcal{C}$  is closed under finite intersection;
- (iii)  $\emptyset, X \in \mathcal{C}$ .

*Proof.*

(i). By De Morgan's laws,

$$V = X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i).$$

So, if  $U_i \in \mathcal{T}$ , then  $V \in \mathcal{C}$ .

(ii). By De Morgan's laws,

$$V = X \setminus \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X \setminus U_i).$$

(iii).

$$\emptyset = X \setminus X, \quad X = X \setminus \emptyset.$$

□

**Definition 1.2.6.** Let  $(X, \mathcal{T})$  be a topological space, and let  $x \in X$ . An *open neighbourhood* of  $x$  is a set  $N_x \in \mathcal{T}$  with  $x \in N_x$ . A *neighbourhood* of  $x$  is any  $N'_x \supseteq N_x$ .

**Note 1.2.4.** Given  $(X, \mathcal{T})$  as a topological space. If  $A \in \mathcal{T}$ ,

$$A = \bigcup_{x \in A} B, \quad B \ni x, \text{ and } B \in \mathcal{T}.$$

If  $\mathcal{T} = \mathcal{T}_\rho$  for some metric  $\rho$  on  $X$ , then  $A \in \mathcal{T}$  implies

$$A = \bigcup_{x \in A} B(x, \varepsilon)$$

for some  $\varepsilon > 0$ .

**Lemma 1.2.3.** Let  $(X, \mathcal{T})$  be a topological space and  $U \subseteq X$ . Then  $U \in \mathcal{T}$  iff for all  $x \in U$ , there is a neighbourhood  $N'_x \subseteq U$ .

*Proof.* If  $U \ni x$  and  $U \in \mathcal{T}$ , then  $U$  is an open neighbourhood of  $x$ , naturally, it is a neighbourhood of  $x$ .

For only if, clearly, if for all  $x \in U$ , there is a neighbourhood  $N'_x \subseteq U$ , then, by definition, there is  $N_x \subseteq N'_x$  with  $N_x \in \mathcal{T}$ . Now we have  $x \in N_x \subseteq N'_x \subseteq U$ , then,

$$U = \bigcup_{x \in U} N_x.$$

By definition,  $\mathcal{T}$  is closed under arbitrary union, thus  $U$  is open. □

### 1.3 Metrics versus Topologies

**Definition 1.3.1.** Let  $X$  be a set, and let  $\rho$  and  $\rho'$  be metrics on  $X$ . We say that  $\rho$  and  $\rho'$  are *topologically equivalent* if they induce the same topology on  $X$ .

**Definition 1.3.2.**  $\rho$  and  $\rho'$  are *Lipschitz equivalent* iff there exist  $c, C \in \mathbb{R}_{>0}$  such that for all  $x, y \in X$ ,

$$c\rho(x, y) \leq \rho'(x, y) \leq C\rho(x, y).$$

**Lemma 1.3.1.** Lipschitz equivalence implies topological equivalence.

*Proof.* As  $\rho$  and  $\rho'$  are Lipschitz equivalent, by definition, there exist  $c \in \mathbb{R}_{>0}$  such that for all  $x, y \in X$ ,

$$c\rho(x, y) \leq \rho'(x, y).$$

Given  $r > 0$  and  $x \in X$ ,

$$B_{c\rho}(x, r) = \{y \in X : c\rho(x, y) < r\}$$

and

$$B_{\rho'}(x, r) = \{y \in X : \rho'(x, y) < r\}.$$

As  $r$  is non-underestimated compared to  $\rho'$ , then

$$B_{\rho'}(x, r) \supseteq B_{c\rho}(x, r) = B_{\rho}\left(x, \frac{1}{c}r\right)$$

is an open neighbourhood of  $x$  in  $(X, \rho')$  and is a subset

Let  $U \in \mathcal{T}_{\rho'}$ , then for some  $\varepsilon > 0$ ,

$$U \supseteq B_{\rho'}(x, \varepsilon) \supseteq B_{\rho}\left(x, \frac{1}{c}\varepsilon\right).$$

Thus  $U$  is open with respect to  $\rho$ , i.e.,  $U \in \mathcal{T}_{\rho}$ .

It is not necessary to prove converse for there always exists  $C \in \mathbb{R}_{>0}$  such that  $c = \frac{1}{C}$ .  $\square$

**Note 1.3.1.**

1. For all  $p \geq 0$ ,  $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  are topologically equivalent.

2. On  $C[a, b]$ ,  $\rho_1$  and  $\rho_\infty$  induce different topologies, hence they are not topologically equivalent, and in particular, they are not Lipschitz equivalent. As Lipschitz equivalence implies topological equivalence, but not vice versa. So Lipschitz in-equivalence do nothing to the proof the topological in-equivalence between  $\rho_1$  and  $\rho_\infty$ .
3.  $\rho_p$  and  $\rho_{\text{disc}}$  on  $\mathbb{Z}$  are topologically equivalent. Firstly, topology  $\mathcal{T}_{\rho_p} = \mathcal{P}(\mathbb{Z})$ , because for all  $B_{\rho_p}(x, \varepsilon)$  for all  $x \in \mathbb{Z}$  and  $\varepsilon \in \mathbb{R}_{(0,1)}$ ,  $B_{\rho_p}(x, \varepsilon) = \{x\}$ . Thus, for all,  $U \in \mathcal{P}(\mathbb{Z})$ ,

$$U = \bigcup_{x \in U} B_{\rho_p}(x, \varepsilon) = \bigcup_{x \in U} \{x\} \in \mathcal{T}_{\rho_p}.$$

Thus  $\mathcal{P}(\mathbb{Z}) \subseteq \mathcal{T}_{\rho_p}$ , but  $\mathcal{T}_{\rho_p} \subseteq \mathcal{P}(\mathbb{Z})$ , so  $\mathcal{P}(\mathbb{Z}) = \mathcal{T}_{\rho_p}$ . Thus  $\mathcal{T}_{\rho_p} = \mathcal{T}_{\text{disc}}$ .

**Definition 1.3.3.** A topological space  $(X, \mathcal{T})$  is *metrizable* iff  $\mathcal{T}$  is induced by some metric on  $X$ .

**Note 1.3.2.**

1. Let  $(\mathbb{Z}, \mathcal{T})$  with

$$\mathcal{T} = \{U \in \mathcal{P}(\mathbb{Z}) : |U| \leq 1\},$$

Then  $\mathcal{T}$  is not induced by any metric. Suppose it were, then all open set  $U \in \mathcal{T}$  should be monotone, and for all  $\varepsilon > 0$ , and for all  $x \in \mathbb{Z}$ ,  $B(x, \varepsilon)$  should be monotone. But if  $\mathcal{T}$  is induced by some metric, then for all  $I \in \mathcal{P}(X)$  with  $|I| > 1$ , a set

$$W = \bigcup_{x \in I} B(x, \varepsilon) \in \mathcal{T},$$

then  $|W| > 1$ , which is contradicted to the conditions.

**Definition 1.3.4.**

- (i) A topological space  $(X, \mathcal{T})$  is said to be  $T_1$  iff every monotone in  $\mathcal{P}(X)$  is closed.
- (ii) A topological space  $(X, \mathcal{T})$  is said to be  $T_2$  or *Hausdorff* iff

$$\forall x, y \in X (x \neq y), \exists U, W \in \mathcal{T} (U \cap W = \emptyset), \quad x \in U \wedge y \in W.$$

**Note 1.3.3.**

1.  $(X, \mathcal{T}_{\rho_{\text{disc}}})$  is  $T_1$ , for as any set  $U \subseteq X$  is closed for  $X \setminus U \in \mathcal{T}_{\rho_{\text{disc}}}$  as well. It is also Hausdorff, because for all  $x, y \in X$ ,  $\{x\}, \{y\} \in \mathcal{T}_{\rho_{\text{disc}}}$  and  $\{x\} \cap \{y\} = \emptyset$  if  $x \neq y$ .
2. On the other hand,  $(X, \{\emptyset, X\})$  is  $T_1$  iff  $|X| = 1$ . And  $(X, \{\emptyset, X\})$  is not Hausdorff, because there exist  $x, y \in X$  with  $x \neq y$ , the only open set contains  $x$  is  $X$ , and the only open set contains  $y$  is  $X$ . Clearly,  $X \cap X$

**Lemma 1.3.2.**

- (i) Every metrizable space is Hausdorff.
- (ii) Every Hausdorff topological space is  $T_1$ .

*Proof.*

- (i) Let  $(X, \rho)$  be metric space, then for all  $x, y \in X$ , let  $r = \frac{\rho(x, y)}{2}$ . Suppose  $(X, \rho)$  is not Hausdorff, i.e., there is  $z \in B(x, r) \cap B(y, r)$ . By metric axioms, we have

$$\rho(x, z) + \rho(y, z) \geq \rho(x, y) = 2r.$$

But  $z \in B(x, r)$  implies that  $\rho(x, z) < r$ , and  $z \in B(y, r)$  implies that  $\rho(y, z) < r$ , then we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which is contradicted to the metric axioms.

- (ii) (Just an outline...) Let  $(X, \mathcal{T})$  be Hausdorff. Suppose  $X$  is not  $T_1$ , then there is  $\{x\} \subseteq X$  which is not closed. Then there must be a smallest  $V \supsetneq \{x\}$  which is closed (Why?). Then there must be a smallest  $U \in \mathcal{T}$  with  $U \supseteq V$  (Why?). Then for all  $x, y \in U$ , there is no disjoint  $U_x, U_y$  such that  $U_x \ni x$  and  $U_y \ni y$ .

□

**Definition 1.3.5.** Let  $(X, \mathcal{T})$  be a topological space, let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$ , and let  $x \in X$ . Then  $\{x_n\}$  converges in  $X$  iff there is an  $x \in X$ , for all  $U \in \mathcal{T}$  with  $x \in U$ ,  $U$  contains all but finite elements in  $\{x_n\}$ .

**Note 1.3.4.**

1. If  $(X, \mathcal{T})$  is metrizable, i.e., there is a metric  $\rho$  can induce  $\mathcal{T}$ . If  $\{x_n\} \subseteq X$  converges in  $X$ , then there exists  $x \in X$ , for all  $\varepsilon > 0$ ,  $B(x, \varepsilon)$  contains all but finite elements in  $\{x_n\}$ .

2. If  $\mathcal{T}$  is a discrete topology, a sequence  $\{x_n\}$  converges in  $(X, \mathcal{T})$  iff there is an  $N$  such that for all  $n \geq N$ ,  $x_n = x_{n+1}$ .
3. If  $\mathcal{T}$  is an indiscrete topology, then any  $\{x_n\} \subseteq X$  converges to any point in  $X$ , for there is only one non-empty open set which is  $X$  itself.

**Lemma 1.3.3.** In Hausdorff topological space, any convergent sequence converges to at most one point.

*Proof.* Let  $(X, \mathcal{T})$  be a Hausdorff topological space. Suppose there is a sequence  $\{x_n\}$  converges to  $x, y \in X$  with  $x \neq y$ . By the definition of topological convergence, there are  $U_x, U_y \in \mathcal{T}$  both contains all but finite elements in  $\{x_n\}$ .  $U_x \cap U_y$  must be non-empty (Explain!).  $x, y \in U_x \cap U_y$ , for if they were not, by Hausdorff property, there must be open  $V_x \subseteq U_x$  and  $V_y \subseteq U_x$  with  $V_x \ni x$  and  $V_y \ni y$ , and they both contains all but finite elements in  $\{x_n\}$ , which is not possible. Thus, there is no such open sets  $V_x \ni x$  and  $V_y \ni y$  with  $V_x \cap V_y = \emptyset$ , which implies  $(X, \mathcal{T})$  is not Hausdorff. This is a contradiction.  $\square$

**Definition 1.3.6.**

- (i) A topological space  $(X, \mathcal{T})$  is *regular* iff for all closed sets  $V \subseteq X$  and  $x \in X$  with  $x \notin V$ , there exist disjoint open sets  $U, W \subseteq X$  such that  $V \subseteq U$  and  $x \in W$ .
- (ii)  $(X, \mathcal{T})$  is *normal* iff for all disjoint closed sets  $V, Z \subseteq X$ , there exist disjoint open sets  $U, W \subseteq X$  such that  $V \subseteq U$  and  $Z \subseteq W$ .

**Note 1.3.5** (To do).

1. Can I find a regular space which is not normal?
2. Can I find a normal space which is not regular?
3. Does regular implies normal or normal implies regular?

## 1.4 Continuous Maps

**Definition 1.4.1.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous* iff

$$\forall U \in \mathcal{T}_Y, \quad f^{-1}[U] \in \mathcal{T}_X.$$

**Note 1.4.1.**

1. Let  $(X, \mathcal{T})$  and  $(X, \mathcal{T}')$  be topological spaces, let identity map  $i : (X, \rho) \rightarrow (X, \rho')$  be surjective. By definition,  $i$  is continuous over  $X$  iff  $\mathcal{T} \supseteq \mathcal{T}'$ .
2. Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces if  $\mathcal{T}_X$  is the discrete topology, then any function  $f : X \rightarrow Y$  is continuous over  $X$ , because any sets in  $X$  is open. Naturally, for any  $U \in \mathcal{T}_Y$ ,  $f^{-1}[U]$ .
3. By the previous conditions, if  $\mathcal{T}_X$  is indiscrete topology but  $\mathcal{T}_Y$  is not, then no function  $f : X \rightarrow Y$  is continuous. (Check again!)
4. TO DO: Find valued example by point sets, for example let  $\mathcal{T} = \{a, b, c, d, e\}$ .

**Lemma 1.4.1.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological space. A function:  $f : X \rightarrow Y$  is continuous iff for any closed subset  $V$  of  $Y$ , the preimage  $f^{-1}[V]$  is closed in  $X$ .

*Proof.* If  $V$  is closed in  $Y$ , then there is a  $U \in \mathcal{T}_Y$ , such that  $V = Y \setminus U$ .

$$f^{-1}[V] = f^{-1}[Y \setminus U] = X \setminus f^{-1}[U] \in \mathcal{T}_X.$$

□

**Lemma 1.4.2.** Continuous maps preserve convergence of sequence..

**Note 1.4.2.** This is not vise versa, i.e., there exists discontinuous map preserve convergence of sequence either.

For example, let  $(X, \mathcal{T})$  and  $(X, \mathcal{T}')$  be topological space. Let  $\mathcal{T}'$  be discrete topology, and let

$$\mathcal{T} = \{\text{monotones of } X\} \cup \{\emptyset, X\}.$$

Then any  $\{x_n\} \subseteq X$  converges iff it is in the form of

$$x, x, x, \dots$$

Clearly, for all  $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}')$ , for any  $\{x_n\}$  converges to  $x \in (X, \mathcal{T})$ ,  $\{f(x_n)\}$  also converges to  $f(x) \in (X, \mathcal{T}')$ . It is in the form of

$$f(x), f(x), f(x), \dots$$

But,  $f$  is not continuous over  $(X, \mathcal{T})$ . (For any  $U \subsetneq X$  with  $|U| > 1$ ,  $f^{-1}[U] \notin \mathcal{T}$ ).