

Notes for General Topology

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Chapter 1

Topological Spaces

1.1 Topological Spaces

Definition 1.1.1 (topology). Let X be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. \mathcal{T} is called a topology on X iff

- (i) $\emptyset, X \in \mathcal{T}$;
- (ii) \mathcal{T} is closed under arbitrary union;
- (iii) \mathcal{T} is closed under finite intersection.

Definition 1.1.2 (topological spaces). Let X be any set, and let \mathcal{T} be a topology on X , then the pair (X, \mathcal{T}) is called a *topological space*. All subsets of X in \mathcal{T} are called *open sets* in (X, \mathcal{T}) .

Definition 1.1.3 (closed sets). Let (X, \mathcal{T}) be a topological space. A subset V of X is said to be *closed* iff there is an open set U in X such that

$$V = X \setminus U.$$

Proposition 1.1.1. Let (X, \mathcal{T}) be a topological space, and let \mathcal{C} be the family of all closed sets in X . Then

- (i) $\emptyset, X \in \mathcal{C}$;
- (ii) \mathcal{C} is closed under arbitrary intersection;
- (iii) \mathcal{C} is closed under finite union.

Proof.

- (i) $X \in \mathcal{T}$ implies $X \setminus X = \emptyset \in \mathcal{C}$; and $\emptyset \in \mathcal{T}$ implies $X \setminus \emptyset = X \in \mathcal{C}$;
- (ii) As \mathcal{T} is closed under arbitrary union, then by Definition 1.1.3 and De Morgan's Law, \mathcal{C} is closed under arbitrary intersection.
- (iii) As \mathcal{T} is closed under finite intersection, then by Definition 1.1.3 and De Morgan's Law, \mathcal{C} is closed under finite union.

□

Definition 1.1.4 (finer and coarser topology). Let X be any set, and let $\mathcal{T}, \mathcal{T}'$ be topologies on X . \mathcal{T} is said to be *finer* than \mathcal{T}' iff $\mathcal{T} \supseteq \mathcal{T}'$; respectively, \mathcal{T} is said to be *coarser* than \mathcal{T}' iff $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 1.1.5 (neighbourhood). Given (X, \mathcal{T}) as a topological space and a point $x \in X$, a subset $N \subseteq X$ is called a *neighbourhood* iff it contains an open set U containing x .

Proposition 1.1.2. Given (X, \mathcal{T}) as a topological space and $U \subseteq X$, U is open iff for all $x \in U$, there is a neighbourhood N of x contained in U .

Proof. If U is open, then U itself is a neighbourhood of x contained in U .

Conversely, if for all $x \in U$, there is a neighbourhood N_x of x contained in U , then there is a open neighbourhood $U_x \ni x$ contained in N_x . Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose U is not open, then U is a proper superset in the relation above. Then there exists $y \in U$ which is not in any U_x . This implies that such a y does not have any neighbourhood N_y in U , for such an N_y must contains an open $U_y \ni y$. For if it does, then there must be a U_x contains y . This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open.

□

1.2 Metrizable

Definition 1.2.1 (metric spaces). Let X be any set. A *metric* ρ on X is a function $\rho : X \times X \rightarrow \mathbb{R}$ satisfying the following conditions: for all $x, y, z \in X$

- (i) $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ iff $x = y$;
- (ii) $\rho(x, y) = \rho(y, z)$;
- (iii) $\rho(x, z) + \rho(z, y) \geq \rho(x, y)$.

Definition 1.2.2 (balls). Let (X, ρ) be a metric space, let $x \in X$, and let $\varepsilon \in \mathbb{R}_{>0}$. The *open ε -ball about x* or just *ε -ball about x* is defined to be

$$B(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}.$$

The *closed ε -ball about x* is defined to be

$$\overline{B}(x, \varepsilon) = \{y \in X : \rho(x, y) \leq \varepsilon\}.$$

Example 1.2.1. Let X be any set, and let metric ρ_p on X^n ($n \in \mathbb{Z}_{>0}$) defined by

$$\rho_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}},$$

where $p \in \mathbb{R}_{\geq 1}$. ρ_2 is so called the *standard Euclidean metric*. If $X = \mathbb{R}$, then the metric space (\mathbb{R}^n, ρ_2) is so-called *Euclidean n -space*.

For all $p, q \in \mathbb{R}_{\geq 1}$, if $p < q$, then for all $\varepsilon \in \mathbb{R}_{>0}$ and for all $x, y \in X$, $\rho_p(x, y) \geq \rho_q(x, y)$; in particular, $\rho_p = \rho_q$ iff there is a unique $k \in \{1, \dots, n\}$, such that for all $i \in \{1, \dots, n\} \setminus \{k\}$, $x_i = 0$. As $\rho_p(x, y)$ is always “overestimated” than $\rho_q(x, y)$, we have $B_{\rho_p}(x, \varepsilon) \supseteq B_{\rho_q}(x, \varepsilon)$.

Example 1.2.2. Let X be any set. The *discrete metric ρ* on X is defined to be

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Example 1.2.3. Let $a, b \in \mathbb{R}$ with $a < b$, and let metric ρ_p on $C[a, b]$ defined by

$$\rho_p(f, g) = \left(\int_a^b |f(t) - g(t)|^p dt \right)^{\frac{1}{p}},$$

where $p \geq 1$. In particular,

$$\rho_\infty(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|.$$

Proposition 1.2.1. Let (X, ρ) be a metric space, then for all $x, y \in X$ ($x \neq y$), there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$.

Proof. Suppose for all $\varepsilon > 0$, $B(x, \varepsilon) \cap B(y, \varepsilon) \neq \emptyset$, then there must be a $z \in X$ such that $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. $z \in B(x, \varepsilon)$ only if $\rho(x, z) < \varepsilon$, and $z \in B(y, \varepsilon)$ only if $\rho(y, z) < \varepsilon$. Thus

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

As the assumption holds for all $\varepsilon > 0$, we may put

$$\varepsilon = \frac{\rho(x, y)}{2}.$$

Then, we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which is impossible. \square

Definition 1.2.3 (induced topologies). Let (X, ρ) be a metric space. A topology \mathcal{T} on X is said to be *induced* by ρ iff for all $\varepsilon > 0$, any $U \in \mathcal{T}$ is the union of ball(s) in X ; i.e.,

$$\mathcal{T} = \left\{ U \subseteq X : U = \bigcup_{x \in X} B(x, \varepsilon) \right\}.$$

In this case, \mathcal{T} is called the *underlying topology* of ρ .

Definition 1.2.4 (metrizable spaces). Let (X, \mathcal{T}) be a topological space. If there is any ρ induce \mathcal{T} , then (X, \mathcal{T}) is said to be *metrizable*.

Definition 1.2.5 (Lipschitz equivalence). Let X be any set, and let ρ and ρ' be metrics on X . ρ and ρ' are said to be *Lipschitz equivalent* iff there exist $c, C > 0$, such that for all $x, y \in X$,

$$c\rho(x, y) \leq \rho'(x, y) \leq C\rho(x, y).$$

Proposition 1.2.2. Lipschitz equivalence is an equivalence relation.

Proof. Clearly, Definition 1.2.5 also holds for $\rho = \rho'$. So, Lipschitz equivalence is reflexive. In Definition 1.2.5, the relation also holds for $\frac{1}{C}\rho' \leq \rho \leq \frac{1}{c}\rho'$. So Lipschitz equivalence is symmetric.

If there is another ρ'' be Lipschitz equivalent to ρ' , then there is $r, R > 0$, such that for all $x, y \in X$,

$$r\rho''(x, y) \leq \rho'(x, y) \leq R\rho''(x, y).$$

By the conditions in Definition 1.2.5, we have

$$\frac{c}{r}\rho(x, y) \leq \rho''(x, y) \leq \frac{C}{R}\rho(x, y),$$

i.e., ρ and ρ'' are also Lipschitz equivalent. So Lipschitz equivalence is transitive.

Above all, Lipschitz equivalence is an equivalence relation. \square

Proposition 1.2.3. Let X be any set, and let ρ and ρ' be metrics on X . If ρ and ρ' are Lipschitz equivalent, then ρ and ρ' induce the same topology.

Proof. As ρ and ρ' are Lipschitz equivalent, by Definition 1.2.5, there is a $c > 0$ such that for all $x, y \in X$,

$$c\rho(x, y) \leq \rho'(x, y).$$

Given $r \in \mathbb{R}_{>0}$ and for all $x \in X$, we have

$$B_{\rho'}(x, cr) \subseteq B_{c\rho}(x, r) = B_{\rho}\left(x, \frac{1}{c}r\right).$$

For all $U \in \mathcal{T}_{\rho}$, for all $x \in U$, there is an $\varepsilon \in \mathbb{R}_{>0}$, such that

$$B_{\rho'}(x, \varepsilon) \subseteq B_{\rho}(x, \varepsilon) \subseteq U.$$

So $U \in \mathcal{T}_{\rho'}$. Then we have $\mathcal{T}_{\rho} \subseteq \mathcal{T}_{\rho'}$.

Similarly, $U \in \mathcal{T}_{\rho'}$ only if $U \in \mathcal{T}_{\rho}$. Then we have $\mathcal{T}_{\rho'} \subseteq \mathcal{T}_{\rho}$.

Above all, $\mathcal{T}_{\rho} = \mathcal{T}_{\rho'}$. \square

Note 1.2.1. In this proposition, \mathcal{T}_{ρ} and $\mathcal{T}_{\rho'}$ are said to be homeomorphic or topologically equivalent (see Definition 1.5.2). And ρ and ρ' are also said to be topologically equivalent.

Example 1.2.4. In Example 1.2.1, for all $p, q \geq 1$, all ρ_p and ρ_q induce the same topology. Let X be any subset of \mathbb{R}^n , then for all $x, y \in X$, if $p < q$, then

$$\rho_p(x, y) \geq \rho_q(x, y).$$

Thus, if ρ_1 and ρ_{∞} are Lipschitz equivalent, then any other ρ_p and ρ_q are Lipschitz equivalent. We have

$$\rho_1(x, y) = \sum_{i=1}^n |x_i - y_i| \geq \max_{i \in \{1, \dots, n\}} |x_i - y_i| = \rho_{\infty}(x, y).$$

Clearly,

$$\rho_{\infty}(x, y) \leq \rho_1(x, y) \leq n\rho_{\infty}(x, y).$$

By Definition 1.2.5, ρ_1 and ρ_{∞} are Lipschitz equivalent, hence for all $p, q \geq 1$, ρ_p and ρ_q are Lipschitz equivalent. Thus, by Proposition 1.2.3, they induce the same topology.

1.3 Separation Axioms. From Hausdorff to T_0

Definition 1.3.1 (Hausdorff Spaces). A topological space is said to be *Hausdorff* or T_1 iff all distinct points in the space are separated by neighbourhoods.

Proposition 1.3.1. All metrizable spaces are Hausdorff

Proof. Let (X, \mathcal{T}) be a metrizable space. There exists a metric ρ on X that induces \mathcal{T} . Given distinct points $x, y \in X$, suppose for all $\varepsilon \in \mathbb{R}_{>0}$, there exists $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. Then $\rho(x, z) < \varepsilon$ and $\rho(y, z) < \varepsilon$. Now we have

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

Put $\rho(x, y) > 2\varepsilon$ as x and y are arbitrarily given. Then we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which implies that ρ is not a metric on X . Hence, (X, \mathcal{T}) is not metrizable which is contradicted to the condition. \square

Proposition 1.3.2. All singletons in a Hausdorff space are closed.

Proof. Let (X, \mathcal{T}) be a Hausdorff space, and let $x \in X$. For all $y \in X$ with $x \neq y$, there is an open neighbourhood U_y of y such that $x \notin U_y$. Then, for all such U_y , we have

$$\forall y \in X, x \in X \setminus U_y = \{x\} \iff x \in \bigcap_{y \in X \setminus \{x\}} X \setminus U_y = \{x\}.$$

As all $X \setminus U_y$ are closed, their intersection $\{x\}$ is closed. \square

Definition 1.3.2 (Fréchet spaces). A topological space (X, \mathcal{T}) is said to be T_1 or *Fréchet* iff for all singletons in X are closed.

Proposition 1.3.3. In a Hausdorff space, any two topologically distinguishable points are separated.

Proof. Clearly, in a Hausdorff space, any two points are topologically distinguishable points iff they are distinct. Then, of course, they are separated. \square

Definition 1.3.3 (R_0 spaces). A topological space (X, \mathcal{T}) is said to be R_0 iff any two topologically distinguishable points in X are separated.

Example 1.3.1 (Fréchet but not Hausdorff). Let X be a nonempty set, let $p \in X$, let \mathcal{T}' be a Hausdorff topology on $X \setminus \{p\}$, and let

$$\mathcal{T} = \{X\} \cup \mathcal{T}'.$$

Then, all $x \in (X, \mathcal{T})$ are closed, thus (X, \mathcal{T}) is Fréchet. But the only neighbourhood of p is X , so its closure is X . Then, for any $x \in X \setminus \{p\}$, x and p are not separated, in which case (X, \mathcal{T}) is not R_0 . Thus, (X, \mathcal{T}) is not Hausdorff.

Note 1.3.1. A topological space (X, \mathcal{T}) is Fréchet only if it is R_0 . Because for distinct points all $x, y \in X$, $x \notin \overline{\{y\}} = \{y\}$ (being R_0), as all singletons are closed (being Fréchet). But, on the contrary, being R_0 does not imply being T_1 , for we can find topological spaces being R_0 but not Fréchet.

Example 1.3.2. Let X be any set, let $U \subsetneq X$ with $|U| > 1$, let \mathcal{T}' be a T_1 topology on $X \setminus U$.

$$\mathcal{T} = \mathcal{T}' \cup \{X\}.$$

—

Example 1.3.3 (R_0 but not T_1). Let X be any set, let $U \subsetneq X$ with $|X \setminus U| > 1$, let $V \subsetneq U$ with $|V| = 1$, and let

$$\mathcal{T} = \mathcal{P}(X) \setminus \mathcal{P}(X \setminus U) \setminus \mathcal{P}(V) \cup \{\emptyset, X\}.$$

For any two distinct points $x, y \in U$, the families of their neighbourhoods are different; and for all such x and y , $x \notin \overline{\{y\}} = \{y\}$ (i.e., they are separated; but, be caution, they are not necessarily be separated by neighbourhoods; for if $y \in V$, the smallest neighbourhood of y is X). Thus (X, \mathcal{T}) is R_0 . But X is not T_0 , because for two distinct points $x, y \in X \setminus U \cup V$, the families of their neighbourhoods are the same.

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Definition 1.3.4 (T_0 spaces). A topological space (X, \mathcal{T}) is said to be T_0 or *Kolmogorov*, iff all distinct points $x, y \in X$ are *topologically distinguishable*.

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Definition 1.3.5 (topologically indistinguishable). Let (X, \mathcal{T}) be a topological space. Two points $x, y \in X$ are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let \mathcal{N}_x be the family of all neighbourhoods of x and let \mathcal{N}_y be the family of all neighbourhoods of y , we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively, x, y are said to be *topologically distinguishable* iff they are not topologically indistinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

Definition 1.3.6 (separated sets). Let (X, \mathcal{T}) be a topological space, and let $A, B \in \mathcal{P}(X)$.

- (i) A and B are said to be *separated* iff each is disjoint from other's closure.
- (ii) A and B are said to be *separated by neighbourhoods* iff there are neighbourhoods N_A of A and N_B of B such that N_A and N_B are disjoint.
- (iii) A and B are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods \overline{N}_A of A and \overline{N}_B of B such that \overline{N}_A and \overline{N}_B are disjoint.
- (iv) A and B are said to be *separated by a continuous function* iff there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $f[A] = \{0\}$ and $f[B] = \{1\}$.
- (v) A and B are said to be *precisely separated by a continuous function* iff there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$.

Example 1.3.4 (non- T_0 sets). The a set X with the discrete topology is T_0 iff $|X| \in \{0, 1\}$ (vacuously true).

Example 1.3.5 (T_0 but not R_0). Let $(\mathbb{R}_{\geq 0}, \mathcal{T})$ be a topological space with

$$\mathcal{T} = \{U \subseteq \mathbb{R} : \forall i \in \mathbb{R}_{\geq 0}, U_i = [0, i)\},$$

Then for all $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$, if $x \neq y$, then there are $|y - x|$ neighbourhoods N_x of x do not contain y . Thus, it is T_0 .

On the other hand, it is not R_0 , because for all $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$ with $x < y$, $x \in [0, y]$

Proposition 1.3.4 (alternative definitions of R_0 spaces). Let (X, \mathcal{T}) be R_0 , then the following conditions are equivalent.

- (i) The closure of all singletons in X are not T_0 subspace.
- (ii) For any two points $x, y \in X$, $x \in \overline{\{y\}}$ iff $y \in \overline{\{x\}}$.
- (iii) Every open set is the union of closed sets.

Proof.

- (i) By Definition 1.3.3, if y and x are topologically distinguishable, by Definition 1.3.3, x and y are separated; i.e., $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$.
- (ii) By Definition 1.3.3, for all $x, y \in X$, x, y are not separated only if they are topologically indistinguishable. By Definition 1.3.5, they share all their neighbourhoods, thus they have the same closure; i.e., $\overline{\{x\}} = \overline{\{y\}}$.
- (iii) (Remained as a problem!)

□

Proposition 1.3.5 (alternative definitions of T_1 spaces). Let (X, \mathcal{T}) be T_1 , then the following conditions are equivalent.

- (i) All singletons in X are closed.
- (ii) Every subset of X is the intersection of all open sets containing it.
- (iii) Every cofinite subset of X is open.

Proof.

□

Proposition 1.3.6. All singletons in a T_1 space are closed, That is, if a topological space (X, \mathcal{T}) is T_1 , then

$$\forall x \in (X, \mathcal{T}) : \exists U \in \mathcal{T} : \{x\} = X \setminus U.$$

Definition 1.3.7 (R_1 spaces). A topological space (X, \mathcal{T}) is said to be R_1 iff any two topological distinguishable points in X are separated by neighbourhoods.

Example 1.3.6 (R_0 but not R_1). (Remained as a problem!)

Proposition 1.3.7. All metric spaces are Hausdorff.

1.4 Continuity

Definition 1.4.1 (continuous maps). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f : X \rightarrow Y$ is said to be *continuous* iff for any open set U in Y , its preimage in X under f is open.

Note 1.4.1. In Definition 1.4.1, note that even if for any open set U in X , $f[X]$ is open in Y , f is not necessarily continuous. For example, let $X = (\mathbb{R}, \mathcal{T}_X)$ with \mathcal{T}_X induced by standard Euclidean metric, let $Y = (\mathbb{R}, \mathcal{T}_Y)$ with \mathcal{T}_Y as a indiscrete topology, and define

$$f(x) = [x],$$

where $[x]$ denotes the integer part of x . Then for all $U \subseteq X$, $f[U]$ is open in Y , but by Definition 1.4.1, f is not continuous.

Note 1.4.2. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, if \mathcal{T}_X is the discrete topology on X , then any function with domain X is continuous. If \mathcal{T}_Y is the indiscrete topology on Y , then any function with codomain Y is continuous.

Note 1.4.3. A function is continuous bijection does not implies that its inverse is continuous. For example, let X be any set and let \mathcal{T} and \mathcal{T}' be its topologies. If \mathcal{T} is finer than \mathcal{T}' , then any bijection $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}')$ is continuous. In this case, however, if $\mathcal{T} \neq \mathcal{T}'$, then f^{-1} is not continuous.

Proposition 1.4.1. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f : X \rightarrow Y$ is continuous at $x \in X$ iff for any neighbourhood N_y of $f(x)$, there is a neighbourhood N_x of x , such that $f[N_x] \subseteq N_y$.

Proof. Let N_y be a neighbourhood of $f(x)$. Clearly, there exists an open set U_y contains y .

By Definition 1.4.1, f is continuous at x iff $x \in f^{-1}[U_y] \in \mathcal{T}_X$. Clearly, $f^{-1}[U_y]$ is a neighbourhood of x . We have $f[f^{-1}[U_y]] = U_y \subseteq N_y$.

By Proposition 1.1.2, there U_x must contains at least one neighbourhood N_x of x , thus, $f[N_x] \subseteq U_y$. \square

Proposition 1.4.2. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be metrizable spaces. A map $f : X \rightarrow Y$ is continuous at $p \in X$ iff for any $\varepsilon > 0$, there is a $\delta > 0$, such that for all $x \in B_X(p, \delta)$, $f(x) \in B_Y(f(p), \varepsilon)$, where B_X is defined by any metrics ρ_X induces \mathcal{T}_X , and B_Y is defined by any metrics ρ_Y induces \mathcal{T}_Y .

Proof. Clearly, for all $\varepsilon > 0$, $B_Y(f(x), \varepsilon)$ is an open neighbourhood of $f(x)$.

f is not necessarily be injective, so $f^{-1}[B_Y(f(x), \varepsilon)] = U \ni x$. By Definition 1.4.1, U is open, so for some $\delta > 0$, $B_X(x, \delta) \subseteq U$. Thus, By Proposition 1.4.1, f is continuous iff $f[B_X(x, \delta)] \subseteq B_Y(f(x), \varepsilon)$. This satisfies the conditions we have. \square

Proposition 1.4.3. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A function $f : X \rightarrow Y$ is said to be continuous iff for any closed set V in Y , its preimage in X under f is closed.

Proof. Let U_Y be any open set in Y , let U_X be the preimage of U_Y under f . By Definition 1.4.1, U_X is open in X . Let

$$V_X = f^{-1}[Y \setminus U_Y] = X \setminus U_X,$$

Then V_X is closed. \square

Definition 1.4.2 (convergence of sequences). Let (X, \mathcal{T}) be a topological space, and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be *converges* in X iff there is an $x \in X$, such that for any open neighbourhood U_x of x , it contains a cofinite subset $A \subseteq \{x_n\}$. That is, there exists N in the domain of $\{x_n\}$, for any natural numbers $n \geq N$, $x_n \in U_x$.

Example 1.4.1.

1. In a discrete topological space, a sequence $\{x_n\}$ converges iff there is an N in the domain of $\{x_n\}$, for any natural numbers $m > N$, $x_N = x_m$.
2. In a indiscrete topological space, any sequence $\{x_n\}$ in X converges in X .
And

$$\lim_{n \rightarrow \infty} \{x_n\} = X.$$

Proposition 1.4.4. In a Hausdorff space, any convergent sequence converges to a unique point in the space.

Proof. Let (X, \mathcal{T}) be a Hausdorff space, and let $\{x_n\}$ be a sequence in X . Suppose $\{x_n\}$ converges to more than one point, say to $x, y \in X$ with $x \neq y$, then, for all neighbourhoods N_x of x and N_y of y , N_x contains a cofinite subset $A \subseteq \{x_n\}$ and N_y contains a cofinite subset $B \subseteq \{x_n\}$. If this were true, $N_x \cap N_y$ should be non-empty, otherwise N_x or N_y should be finite.

Then, x and y are not separated by neighbourhoods, thus (X, \mathcal{T}) is not Hausdorff. This is a contradiction.

But, as (X, \mathcal{T}) is Hausdorff, there must be mutually disjoint N_x and N_y . Thus, the assumption cause a contradiction. \square

Note 1.4.4. As all metrizable spaces are Hausdorff, so any convergent sequence in a metrizable space converges to at most one point.

Proposition 1.4.5. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological space, let $f : X \rightarrow Y$ be a map, and let $\{x_n\}$ be a convergent sequence in X . If f is continuous, then $f[\{x_n\}]$ is a sequence convergent in Y .

Proof. Let U_y be any open neighbourhood of $f(x)$. By Definition 1.4.1, $f^{-1}[U_y]$ is also an open neighbourhood of x . By Definition 1.4.2, $f^{-1}[U_y]$ contains a cofinite subset $A \subseteq \{x_n\}$. Then $f[A]$ is a cofinite subset of $f[\{x_n\}]$. As $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$, $f[\{x_n\}]$ converges in $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$. \square

Note 1.4.5. In this proposition, even if $f[\{x_n\}]$ converges in Y , f might be discontinuous. For example, let X any set, let \mathcal{T} be the indiscrete topology on X , let U be another cofinite subset of X with $X \neq U$, and let $\mathcal{T}' = \{\emptyset, X, U\}$. Let $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}')$ be defined by

$$f(x) = x.$$

By Definition 1.4.1, f is not continuous. But, for any convergent sequence $\{x_n\}$ in (X, \mathcal{T}) , $f[\{x_n\}]$ also convergent in (X, \mathcal{T}) .

1.5 Homeomorphisms

Definition 1.5.1 (homeomorphisms). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A bijection $f : X \rightarrow Y$ is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

Definition 1.5.2 (homeomorphic). Two topological spaces (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

1.6 Cover and Basis

Definition 1.6.1 (cover). Let (X, \mathcal{T}) be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of U iff the union of all sets in \mathcal{C} is a superset of U . That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call \mathcal{C} an *open cover* of U .

Let $\mathcal{S} \subseteq \mathcal{C}$, iff the union of \mathcal{S} is still a superset of U , then we call \mathcal{S} a *subcover* of \mathcal{C} .

Definition 1.6.2 (basis). Let (X, \mathcal{T}) be a topological space, let $U \subseteq X$, and let \mathcal{B} be a open cover of X . We call \mathcal{B} a *base* of X iff the union of \mathcal{B} is precisely U itself, i.e.,

$$U = \bigcup \mathcal{B}.$$

Definition 1.6.3 (synthetic basis). Let (X, \mathcal{T}) be a topological space, and let \mathcal{B} be a base of X . \mathcal{B} is said to be *synthetic* iff for any $A, B \in \mathcal{B}$,

$$A \cap B = \bigcup_{i=1}^n B_i, \quad B_i \in \mathcal{B}.$$

Definition 1.6.4 (generated by basis). Let X be any set and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be its cover. A topology \mathcal{T} on X is said to be *generated* by the base \mathcal{B} iff

- (i) for all $U \in \mathcal{T}$, U is the union of \mathcal{B} -sets;
- (ii) for all $U \in \mathcal{T}$, U is the finite intersection of \mathcal{B} -sets.

Proposition 1.6.1. Let (X, \mathcal{T}) be a topological space be genrated by a base \mathcal{B} . For all $U \in \mathcal{T}$, there is a $B \in \mathcal{B}$ such that $U \subseteq B$.

Proof. By Definition 1.6.4, if \mathcal{T} is generated by \mathcal{B} , then for all $U \in \mathcal{T}$, there is an finite set I , such that

$$U = \bigcap_{i \in I} B_i, \quad B_i \in \mathcal{B}.$$

Thus, for at least one $k \in I$, $U \subseteq B_k$. □

Proposition 1.6.2. Let X be any set, and let \mathcal{T} and \mathcal{T}' be its topologies generated by basis \mathcal{B} and \mathcal{B}' respectively. Then \mathcal{T}' is finer than \mathcal{T} iff for any $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $B' \subseteq B$.

Proof. If \mathcal{T} is generated by \mathcal{B} , then for all $U' \in \mathcal{T}'$,

$$U' = \bigcup_{j \in J} B'_j,$$

where $B_j \in \mathcal{B}$.

As \mathcal{T} is generated by \mathcal{B} , then, certainly, $\mathcal{B} \subseteq \mathcal{T}$.

By the conditions we have, $\mathcal{T} \subseteq \mathcal{T}'$ iff for all $B \in \mathcal{B}$, there is $W' \in \mathcal{T}$ such that

$$B = W' = \bigcup_{i \in I} B'_i,$$

where $B'_i \in \mathcal{B}'$. Certainly, all such B'_i are contained in B . \square

Proposition 1.6.3. Let X be any set, and let $\mathcal{T} \subseteq \mathcal{P}(X)$. \mathcal{T} is a topology on X iff it generates itself.

Proof. If \mathcal{T} is a topology on X , then, by Definition 1.6.4, any open set generated by \mathcal{T} is still a member of \mathcal{T} . On the other hand, if \mathcal{T} generates itself, then, \emptyset and X must be members of \mathcal{T} , and, by Definition 1.6.4, \mathcal{T} is a topology on X . \square

1.7 Interiors and Closures

Definition 1.7.1 (interiors). The *interior* of a set A , denoted A° , is defined to be the union of all open subsets of A .

Definition 1.7.2 (closure). The *closure* of a set A , denoted \overline{A} , is defined to be the intersection of all closed supersets of A .

Definition 1.7.3 (dense sets). Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$. A is said to be *dense*, iff $\overline{A} = X$.

Definition 1.7.4 (nowhere dense sets). A set A is said to be *nowhere dense* iff the interior of its closure is empty.

Proposition 1.7.1 (properties of interiors). Let (X, \mathcal{T}) be any topological space and $A, B \subseteq X$.

- (i) (Intensive) $A^\circ \subseteq A$.
- (ii) A is open iff $A = A^\circ$.
- (iii) (Idempotence) $(A^\circ)^\circ = A^\circ$.

- (iv) $(A \cap B)^\circ = A^\circ \cap B^\circ$.
- (v) $A \subseteq B \implies A^\circ \subseteq B^\circ$.
- (vi) If B is open, then $B \subseteq A$ iff $B \subseteq A^\circ$.

Proof.

- (i) By Definition 1.7.1, naturally, $A^\circ \subseteq A$.
- (ii) By Definition 1.1.2, A° is the union of open sets hence it is open. A is open iff it is the union of all open subsets of A . Thus $A = A^\circ$.
- (iii) A° is open, thus $(A^\circ)^\circ = A^\circ$.
- (iv) By Definition 1.7.1, we have

$$\begin{aligned}
 (A \cap B)^\circ &= \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq A \cap B \right\} \\
 &= \left\{ \bigcup U : (U \in \mathcal{T} \wedge U \subseteq A) \wedge (U \in \mathcal{T} \wedge U \subseteq B) \right\} \\
 &= \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq A \right\} \cap \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq B \right\} \\
 &= A^\circ \cap B^\circ.
 \end{aligned}$$

- (v) Clearly, $A^\circ \subseteq A$, thus,

$$A \subseteq B \implies A^\circ \subseteq B$$

Suppose $A^\circ \not\subseteq B^\circ$, then $A^\circ \setminus B^\circ$ is not empty (\emptyset is the subset of any set, so A° is not empty).

Then there exists $x \in A^\circ$ with $x \in \partial B$ ($x \in B$ but $x \notin B^\circ$). Then there exists neighbourhood $N_x \ni x$, and $N_x \cap \partial B \neq \emptyset$. But this is impossible, for $A^\circ \subseteq B$ implies that $A^\circ \cap \partial B = \emptyset$ (This is a straight consequence of $A^\circ \cap \partial A = \emptyset$. See Proposition 1.8.1), so such N_x does not exist. Thus,

$$A^\circ \subseteq B^\circ.$$

- (vi) If B is open, then $B = B^\circ$. Then $B \subseteq A$ iff $B^\circ \subseteq A^\circ$.

□

Proposition 1.7.2 (properties of closures). Let (X, \mathcal{T}) be a topological space, and let $A, B \subseteq X$.

- (i) \overline{A} is closed.
- (ii) A is closed iff $A = \overline{A}$.
- (iii) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.
- (iv) If A is closed, then $A \supseteq B$ iff $A \supseteq \overline{B}$

Proof.

- (i) By Definition 1.7.2, \overline{A} is the intersection of closed sets. By Proposition 1.1.1, \overline{A} is closed.
- (ii) Proposition 1.1.1 implies that any closed set is the intersection of closed sets, this is precisely what Definition 1.7.2 says.
- (iii) $A \subseteq B$ iff $X \setminus A \supseteq X \setminus B$. Then we have

$$X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ$$

Clearly, $(X \setminus A)^\circ$ is the union of all open set disjoint from A , then, by De Morgan's laws, $X \setminus (X \setminus A)^\circ$ is the intersection of all closed sets containing A . By Definition 1.7.2, we have $(X \setminus A)^\circ = \overline{A}$. Thus

$$\overline{A} \subseteq \overline{B}.$$

- (iv) If A is closed, then $A = \overline{A}$. Suppose $B \subseteq A$, then we have

$$\overline{B} \subseteq \overline{A} \iff \overline{B} \subseteq A.$$

□

1.8 Boundaries

Definition 1.8.1 (boundaries). Let A be any set, the *boundary* of A , denoted ∂A , is defined to be the complement of the interior of A in the closure of A ; i.e.,

$$\partial A = \overline{A} \setminus A^\circ.$$

Proposition 1.8.1 (properties of boundaries). Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$.

- (i) ∂A is closed.

- (ii) $A^\circ \cap \partial A = \emptyset$.
- (iii) $\overline{A} = A^\circ \cup \partial A$.
- (iv) A is closed iff $\partial A \subseteq A$.
- (v) ∂A is nowhere dense.
- (vi) $\partial \overline{A} \subseteq \partial A \subseteq \partial A^\circ$.
- (vii) $\partial A = \partial(X \setminus A)$.
- (viii) A is dense iff $\partial A = X \setminus A^\circ$.

Proof.

- (i) \overline{A} is closed, and $X \setminus A^\circ$ is also closed. Thus

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (X \setminus A)$$

is closed.

- (ii) By Definition 1.8.1, we have

$$\partial A = \overline{A} \setminus A^\circ \iff \partial A \cap A^\circ = \overline{A} \setminus A^\circ \cap A^\circ = \overline{A} \cap \emptyset = \emptyset.$$

- (iii) We have

$$\begin{aligned} \partial A = \overline{A} \setminus A^\circ &\iff \partial A \cup A^\circ = \overline{A} \setminus A^\circ \cup A^\circ = \overline{A} \cap (X \setminus A^\circ \cup A^\circ) \\ &\iff \partial A \cup A^\circ = \overline{A} \cap X|_{\text{for } A^\circ \subseteq X} = \overline{A}. \end{aligned}$$

- (iv) As A is closed, $A = \overline{A}$ (this can be straightly proved by Definition 1.7.2).
By Definition 1.8.1, it is clear that $\partial A \subseteq \overline{A}$, thus $\partial A \subseteq A$.

- (v) By Definition 1.7.4, ∂A is nowhere dense iff $\overline{\partial A}^\circ$ is empty. We have

$$\begin{aligned} \overline{\partial A}^\circ &= \overline{\overline{A} \setminus A^\circ}^\circ \\ &= (\overline{A} \setminus A^\circ) \cup (\overline{A} \setminus A^\circ) \setminus (\overline{A} \setminus A^\circ) \\ &= \emptyset. \end{aligned}$$

- (vi) $\overline{A} \supseteq A^\circ$ implies $\overline{A}^\circ \supseteq (A^\circ)^\circ = A^\circ$, then we have,

$$\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^\circ \subseteq \overline{A} \setminus A^\circ = \partial A.$$

$A^\circ \subseteq A$ implies $\overline{A}^\circ \subseteq \overline{A}$, then we have,

$$\partial A^\circ = \overline{A^\circ} \setminus (A^\circ)^\circ \supseteq \overline{A} \setminus A^\circ.$$

(vii) We have

$$\begin{aligned}
\partial(X \setminus A) &= \overline{X \setminus A} \setminus (X \setminus A)^\circ \\
&= X \setminus A^\circ \setminus (X \setminus \overline{A}) \\
&= X \setminus A^\circ \cap \overline{A} \\
&= \overline{A} \setminus A^\circ \\
&= \partial A.
\end{aligned}$$

(viii) By Definition 1.7.3, A is dense in X iff $\overline{A} = X$. Then we have,

$$\begin{aligned}
\overline{A} = X &\iff \overline{A} \setminus A^\circ = X \setminus A^\circ \\
&\iff \partial A = X \setminus A^\circ.
\end{aligned}$$

□

1.9 Limit Points

Definition 1.9.1 (limit points). Let (X, \mathcal{T}_X) be a topological space, and let $A \subseteq X$. A point $x \in X$ is called a *limit point* of A iff for all neighbourhood N_x of x , $N_x \setminus \{x\}$ intersects A .

Proposition 1.9.1. Let A be any set, and let x be a limit point of A , then x is an element of the closure of A .

Proof. If A is empty, then this is vacuously true. So, suppose A is not empty.

By Definition 1.9.1, for all neighbourhood N_x of x , $N_x \setminus \{x\} \cap A$ is not empty. Naturally, $N_x \cap A$ is not empty.

Assume that $x \notin \overline{A}$, then $X \setminus \overline{A}$ is a neighbourhood of x , by Definition 1.1.5, and is disjoint from A . This is contradicted to the conditions. □

Note 1.9.1. In this proof, the proposition also holds for $N_x \cap A^\circ = \emptyset$. Because if it is true, then

$$N_x \cap \partial A \supseteq (N_x \cap A) \setminus (N_x \cap A^\circ) = N_x \cap A.$$

This implies that $A \subseteq \partial A$. In this case, $\overline{A} = \partial A$, for

Assume that $x \notin \partial A$, then we have the same conclusion.

Then $A^\circ = A \setminus \partial A = \emptyset$.

Proposition 1.9.2. A set is closed iff it contains all its limit point.

Proof. Let A be a set. By proposition [1.9.1](#), for every limit point of A , it is also an element of the closure \overline{A} . And A is closed iff $A = \overline{A}$. \square

Definition 1.9.2 (convergent sequences). Let (X, \mathcal{T}_X) be a topological space. A sequence $\{x_n\}$ in X is said to be *convergence* in X iff there is an open set U contains all but finite terms of $\{x_n\}$.

Chapter 2

Creating New Spaces

2.1 Subspaces

Definition 2.1.1 (subspace topology). Let (X, \mathcal{T}) be a topological space and let $A \subseteq X$. The *subspace topology* \mathcal{T}_A on A is defined to be the family of the intersections of open sets in (X, \mathcal{T}) and A . That is,

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}.$$

2.2 Quotient Spaces

Definition 2.2.1 (quotient topology). Let (X, \mathcal{T}) be a topological space and let \sim be an equivalence relation on X . The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

2.3 Product Spaces

Definition 2.3.1 (product topologies).

Chapter 3

Topological Properties

3.1 Cardinal Functions

3.2 More on Separation Axioms

Definition 3.2.1 ($T_{2^{1/2}}$ spaces). A topological space (X, \mathcal{T}) is said to be $T_{2^{1/2}}$ or *Urysohn* iff two distinct points in X are separated by closed neighbourhoods.

Example 3.2.1 (T_2 but not $T_{2^{1/2}}$).¹ (Remained as a problem)

Definition 3.2.2 (T_3 spaces). A topological space (X, \mathcal{T}) is said to be T_3 or *regular* iff it is T_0 and given any point $x \in (X, \mathcal{T})$ and closed set $V \subseteq X$ with $x \notin V$ are separated by neighbourhoods.

Definition 3.2.3 ($T_{3^{1/2}}$ spaces). A topological space (X, \mathcal{T}) is said to be $T_{3^{1/2}}$, or *Tychonoff* or, *completely* T_3 , or *completely regular*, iff it is T_0 and given any point x and closed set $V \subseteq X$ with $x \notin V$, they are separated by a continuous function.

Definition 3.2.4 (T_4 spaces). A topological space (X, \mathcal{T}) is said to be T_4 or *normal* iff it is Hausdorff and any two disjoint closed subsets of X are separated by neighbourhoods.

Proposition 3.2.1 (Urysohn's lemma). A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

¹ See [MathPlanet](#).

Definition 3.2.5 (T_5 spaces). A topological space (X, \mathcal{T}) is said to be T_5 or *completely* T_4 iff it is T_1 any two separated sets are separated by neighbourhoods.

Proposition 3.2.2. Every subspace of a T_5 space is normal.

Definition 3.2.6 (T_6 spaces). A topological space (X, \mathcal{T}) is said to be T_6 , or *perfectly* T_4 or *perfectly normal* iff it is T_1 and any two disjoint closed sets are precisely separated by a continuous function.

Proposition 3.2.3 (Tietze extension theorem). Let (X, \mathcal{T}) be normal topological space, and let $f : A \rightarrow (\mathbb{R}, \mathcal{T}')$ be a continuous map where A is a closed subset of X and \mathcal{T}' is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \rightarrow (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$

3.3 Countability Axioms

3.4 Compactness

Definition 3.4.1 (compactness). A topological space (X, \mathcal{T}) is said to be *compact* iff every open cover of X has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

3.5 Connectedness

Definition 3.5.1 (connectedness). Let (X, \mathcal{T}) be a topological space. (X, \mathcal{T}) is said to be *connected* iff X is not empty and it is not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

Definition 3.5.2 (path-connectedness). Let (X, \mathcal{T}) be a topological space.

- (i) A map $\gamma : [0, 1] \rightarrow X$ is called a *path* in X iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that γ is path from x to y in X .
- (ii) X is said to be *path-connected* iff for all $x, y \in X$ there is a path from x to y in X .