Notes for General Topology

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1 Basic Definitions

Definition 1.1 (topological space)**.** Let X be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is called a topology on X iff

(i) $\emptyset, X \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under arbitrary union;

(iii) $\mathcal{T}$ is closed under finite intersection.

The pair $(X, \mathcal{T})$ is called a *topological space*. The elements of $\mathcal{T}$ are called *open sets* in $(X, \mathcal{T})$.

Definition 1.2 (metrizable topology)**.** Let $(X, \mathcal{T})$ be a topological space.

2 Untitled

Definition 2.1 (cover)**.** Let $(X, \mathcal{T})$ be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of U iff the union of $\mathcal{C}$ is a superset of U. That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call $\mathcal{C}$ an *open cover* of U.

Let $\mathcal{C}' \subseteq \mathcal{C}$, iff the union of $\mathcal{C}'$ is still a superset of U, then we call $\mathcal{C}'$ a subcover of $\mathcal{C}$.

Definition 2.2 (basis)**.** Let $(X, \mathcal{T})$ be a topological space, let $U \subseteq X$, and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a cover of X. We call $\mathcal{B}$ a *base* of $(X, \mathcal{T})$ iff $\mathcal{B} \subseteq \mathcal{T}$ and the union of $\mathcal{B}$ is exactly U itself. That is,

$$\mathcal{B} \subseteq \mathcal{T}, \text{ and } U = \bigcup \mathcal{B}.$$

Definition 2.3 (subspace topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The *subspace topology* $\mathcal{T}_A$ on A is defined to be the family of the intersections of open sets in $(X, \mathcal{T})$ and A. That is,

$$\mathcal{T}_A = \{U \cap A : \ U \in \mathcal{T}\}.$$

Definition 2.4 (quotient topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

Definition 2.5 (continuous functions)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is said to be *continuous* iff for all open subset U of Y, the preimage $f^{-1}[U]$ is open in X. That is,

$$\forall U \in \mathcal{T}_Y, \quad f^{-1}[U] \in \mathcal{T}_X.$$

Definition 2.6 (homeomorphisms)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

Definition 2.7 (homeomorphic)**.** Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

Definition 2.8 (compactness)**.** A topological space $(X, \mathcal{T})$ is said to be *compact* iff every open cover of X has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X, \ \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X, \quad |\mathcal{S}| < \aleph_0.$$

Definition 2.9 (connectedness)**.** Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is said to be *connected* iff X is not empty and it it not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V, \quad U \cap V \neq \emptyset.$$

Definition 2.10 (path-connectedness)**.** Let $(X, \mathcal{T})$ be a topological space.

(i) A map $\gamma : [0, 1] \to X$ is called a *path* in X iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that γ is path from x to y in X.

(ii) X is said to be *path-connected* iff for all $x, y \in X$ there is a path from x to y in X.

Definition 2.11 (topologically indistinguishable)**.** Let $(X, \mathcal{T})$ be a topological space. Two points $x, y \in X$ are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let $\mathcal{N}_x$ be the family of all neighbourhoods of x and let $\mathcal{N}_y$ be the family of all neibourhoods of y, we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively, x, y are said to be *topologically distinguishable* iff they are not topologically distinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

Definition 2.12 (T_0 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_0 or *Kolmogorov*, iff all distinct points $x, y \in X$ are *topologically distinguishable.*

Definition 2.13 (R_0 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be R_0 iff any two topologically distinguishable points in X are separated. That is, for any topologically distinguishable points $x, y \in X$, there is $U_x, U_y \in \mathcal{T}$ with $U_x \ni x$ and $U_y \ni y$, $U_x \cap U_y = \emptyset$.