Notes for General Topology

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Contents

Chapter 1

Metric Spaces

1.1 Metric Spaces

Definition 1.1.1. Let X be any set. A mapping $d : X \times X \to \mathbb{R}_{\geq 0}$ is *metric on* X iff it satisfies the *metric axioms*. That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ iff $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

In this case, the pair $M = (X, d)$ is called a *metric space*.

Definition 1.1.2. A $M = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$. An *open* ε*-ball*, or just ε-ball, about x is defined to be the set

$$B_\varepsilon(x; d) = \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) = \{y \in X : d(x, y) \leq \varepsilon\}.$$

Note 1.1.1. As

$$M = (X, d),\ M' = (X, d'),\ M'' = (X, d''),\ \ldots$$

are different although they share the same set X, for any $x \in X$ and any $\varepsilon \in \mathbb{R}_{>0}$,

$$B_\varepsilon(x; d),\ B_\varepsilon(x; d'),\ B(x; d''),\ \ldots$$

are also different. However, if confusion is unlikely, we simply write "$B_\varepsilon(x)$" for "$B_\varepsilon(x; d)$".

Example 1.1.1. The *Euclidean metric space* $M = (X, d)$ is an n-dimensional set X equipped with the *Euclidean metric* d defined as

$$d(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(x,y) = \max_{1\leq i\leq n} |x_i - y_i|.$$

Example 1.1.2. A *discrete metric space* $M = (X, d)$ is a set X equiped with the *discrete metric* d defined as

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{else.} \end{cases}$$

This is an equivalent definition of the discrete metric:

$$d(x,y) = (\operatorname{sgn}(d'(x,y)))^2,$$

where $\operatorname{sgn}(\cdot)$ is a sign function, and d' is any metric on X.

Example 1.1.3. [1] Denote $C[a,b]$ for the set of all continuous mapping $\mathbb{R}_{[a,b]} \to \mathbb{R}$. On $C[a,b]$, we can define a metric d as

$$d_p(f,g) = \left(\int_a^b |f(t) - g(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(f,g) = \sup_{t\in\mathbb{R}_{[a,b]}} |f(t) - g(t)|.$$

[1] See Minkowski inequality.

Example 1.1.4. [2] Let $M = (X, d)$ be a metric space. The *Hausdorff metric* d_H on $2^X \setminus \{\emptyset\}$ is defined as

$$d_H = \max\left\{\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\right\},$$

where

$$d(x, Y) = \inf_{y\in Y}(x, y), \text{ and } d(y, X) = \inf_{x\in X}(y, x).$$

1.2 Open sets in Metric Spaces

Definition 1.2.1. Let $M = (X, d)$ be a metric space, and let $U \subseteq X$. U is said to be *open in* M, iff for any $y \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$, such that $B_\varepsilon(y) \subseteq U$.

Lemma 1.2.1. Let $M = (X, d)$ be a metric space, let $x \in A$ and let $\varepsilon \in \mathbb{R}_{>0}$. For any $y \in B_\varepsilon(x)$, there is a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_\varepsilon(x)$.

Proof. For any $y \in B_\varepsilon(x)$, by the definition of open balls (Definition 1.1.2), we have $d(x, y) < \varepsilon$.

Let $\delta \in \mathbb{R}_{>0}$ such that $\delta + d(x, y) = \varepsilon$.

By M3 in metric axioms (Definition 1.1.1), for any $z \in A$ with $d(y, z) < \delta$, we have

$$d(x, z) \leq d(y, z) + d(x, y) < \varepsilon.$$

Thus, again, by the definition of open balls, we have $B_\delta(y) \subseteq B_\varepsilon(x)$.

□

Theorem 1.2.1. [3] Let $M = (X, d)$ be a metric space, and let $U \subseteq X$. U is open in M iff it is a union of open balls.

Proof. First, prove $\Rightarrow$.

As U is open, for any $y \in U$, there exists $\varepsilon_y \in \mathbb{R}_{>0}$ such that $B_{\varepsilon_y}(y) \subseteq U$.

Therefore,

$$U = \bigcup_{y\in U} B_{\varepsilon_y}(y).$$

□

Now, prove $\Leftarrow$.

Aiming for a contradiction, suppose U is a union of open balls but not open.

As U is not open, there is a $y \in U$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \not\subseteq U$.

[2] See Hausdorff distance.

[3] Shared on ProofWiki.

As U is a union of open balls, there is an $x \in U$ and $r \in \mathbb{R}_{>0}$ such that $y \in B_r(x)$.

By Lemma 1.2.1, there exists a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_r(x)$.

This is a contradiction by the assumption.

Thus, U has to be open.

■

Theorem 1.2.2. Let $M = (X, d)$ be any metric space. M is *Hausdorff*. That is, For any distinct points $x, y \in X$, we can always find an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset.$$

Proof. Aiming for a contradiction, suppose there are $x, y \in X$ with $x \neq y$, such that for any $\varepsilon \in \mathbb{R}_{>0}$, we can always find a $z \in X$ such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let $r = d(x, y)/2$, and let $z \in B_r(x) \cap B_r(y)$.

As $z \in B_r(x)$, by the definition of open balls (Definition 1.1.2), $d(x, z) < r$; as $z \in B_r(y)$, similarly, $d(y, z) < r$. Then we have

$$d(x, z) + d(y, z) < 2r = d(x, y).$$

By M3 in metric axioms (Definition 1.1.1), this is impossible.

■

Definition 1.2.2. Let $M = (X, d)$ be any metric space, and let $V \subseteq X$. V is said to be *closed* in M, iff there is an open set U satisfies $X \setminus U = V$.

Lemma 1.2.2. In a metric space, any singleton is closed.

Proof. Let $M = (X, d)$ be a metric space, let $x \in X$, and let $y \in X \setminus \{x\}$.

As M is Hausdorff (Theorem 1.2.2), there is an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$0 < \varepsilon < d(x, y),$$

thus $X \setminus \{x\}$ is open, hence, by Definition 1.1.1, its complement $\{x\}$ is open.

□

Theorem 1.2.3. Let $M = (X, d)$ be a metric space, denote $\mathcal{T}$ for the family of open subsets of X. Then $\mathcal{T}$ satisfies the following conditions:

O1. $X, \emptyset \in \mathcal{T}$;

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under arbitrary union;

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under finite intersection.

Proof. First, prove O1.

As $\emptyset$ is the subset of any set, $\emptyset \in \mathcal{T}$. $\bigcup \emptyset = \emptyset \in \mathcal{T}$.

By Definition 1.2.2, $X = X \setminus \emptyset$.

□

Then, prove O2.

Let $\mathcal{U} \subseteq \mathcal{T}$, and denote $\mathcal{O}$ for the open balls in M.

For any $U \in \mathcal{U}$, there is an $\mathcal{O}_U \subseteq \mathcal{O}$ such that $U = \bigcup \mathcal{O}_U$.

Let $\mathcal{O} : \mathcal{T} \to$

Then we have

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \left(\bigcup \mathcal{O}(U) \right).$$

Thus $\mathcal{T}$ is closed under arbitrary union.