

Notes for Mathematical Analysis

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Chapter 1

Set Theory

1.1 Countability of Sets

Definition 1.1.1. A set A is said to be *countable* iff there is a bijection $f : I \rightarrow A$ with $I \subseteq \mathbb{N}$.

A is said to be *uncountable* iff for any injection $f : I \rightarrow A$ with $I \supseteq \mathbb{N}$, f is not surjection.

Definition 1.1.2. The *cardinal number* of a set A , denoted $|A|$ or $\#A$, is defined to the quantity of it elements.

Customarily, we write \aleph_0 for $|\mathbb{N}|$, and \mathfrak{c} for $|\mathbb{R}|$.

Definition 1.1.3. Given A and B as sets, we define the following:

- (i) $|A| = |B|$ iff there is a bijection $f : A \rightarrow B$;
- (ii) $|A| \leq |B|$ iff there is an injection $g : A \rightarrow B$;
- (iii) $|A| < |B|$ iff for any injection $g : A \rightarrow B$, g is not surjection.

Definition 1.1.4. A set A is said to be *finite* iff $|A| < \aleph_0$; it is *infinite* iff $|A| \geq \aleph_0$.

By 1.1.3, A is finite iff for all injection $f : A \rightarrow \mathbb{N}$, f is not surjection. Respectively, A is infinite iff there exists injection $f : \mathbb{N} \rightarrow A$.

Proposition 1.1.1. For any countable set A , $|A| \leq \aleph_0$.

Proof. If A is finite, i.e., $|A| < \aleph_0$, it is clearly countable.

If A is infinite, by Definition 1.1.4, $|A| \geq \aleph_0$. As A is countable, there must be an bijective $f : I \rightarrow A$ with $I \subseteq \mathbb{N}$, then (iii) in Definition 1.1.3 fails, so $|A| = \aleph_0$. \square

Proposition 1.1.2. For any set A , A is uncountable iff $|A| > \aleph_0$.

Proof. By Definition 1.1.1, A is uncountable iff for any injection $f : I \rightarrow A$ with $I \supseteq \mathbb{N}$, f is not surjection. This holds iff for any $I \subseteq \mathbb{N}$, $|I| < |A|$. $\mathbb{N} \subseteq \mathbb{N}$, Thus $\aleph_0 < |A|$. \square

Proposition 1.1.3. The subsets of any countable sets are countable.

Proof. Clearly, by intuition or by Definition 1.1.3, for any sets A and B , $A \subseteq B$ implies $|A| \leq |B|$. By Proposition 1.1.2, B is countable iff $|B| \leq \aleph_0$. Then we have $|A| \leq \aleph_0$. This holds iff A is countable. \square

Proposition 1.1.4. The super sets of any uncountable sets are uncountable.

Proof. Let A be an uncountable set. $|A| > \aleph_0$ implies that for any $B \supseteq A$, $|B| > |A| > \aleph_0$. Thus, by Proposition 1.1.2. \square

Proposition 1.1.5. The Cartesian product of countable sets is countable.

Proof. If A or B is empty, $A \times B$ is empty. The empty set is countable.

Let A and B be both infinite countable, then there exist $f : \mathbb{N} \rightarrow A$ and $g : \mathbb{N} \rightarrow B$. Let $h : \mathbb{N} \rightarrow \mathcal{P}(A \times B)$ defined by

$$h(x) = \begin{cases} \{(f_0, g_0)\} & x = 0 \\ \{(f_0, g_1), (f_1, g_0)\} & x = 1 \\ \{(f_0, g_2), (f_1, g_1), (f_2, g_0)\} & x = 2 \\ \{(f_0, g_3), (f_1, g_2), (f_2, g_1), (f_3, g_0)\} & x = 3 \\ \vdots & \vdots \end{cases}$$

Now we have

$$A \times B = \bigcup_{x=0}^{\infty} f(x).$$

Thus,

$$|A \times B| = \left| \bigcup_{x=0}^{\infty} f(x) \right| = \sum_{x=0}^{\infty} (x+1).$$

Clearly,

$$\aleph_0 = |\{0\} \cup \{1, 2\} \cup \{3, 4, 5\} \cup \dots| = |A \times B|.$$

Thus, $A \times B$ is countable. \square

Proposition 1.1.6. The countable unions of countable sets is countable.

Proof. Similar to 1.1.5. \square

Proposition 1.1.7. If A is a countable set but B is not, then $B \setminus A$ is uncountable.

Proof. If $B \setminus A$ is countable, $B \setminus A \cup A$ must be countable. Then $B \subseteq B \setminus A \cup A$ is also countable, contradicted to the condition. \square

Proposition 1.1.8. There is no countably infinite $\mathcal{P}(X)$ for any set X .

Proof. As \aleph_0 is the smallest infinite cardinal number, let $X = \mathbb{N}$.

□

Proposition 1.1.9. \mathbb{R} is uncountable.

Proof. If the interval $[0, 1) \subseteq \mathbb{R}$ is countable, there must be a bijection $f : I \rightarrow \mathbb{R}$ with $I \subseteq \mathbb{N}$, then we have the following list:

$$\begin{aligned} f_0 &: 0.a_{0_0}a_{0_1}a_{0_2}\dots \\ f_1 &: 0.a_{1_0}a_{1_1}a_{1_2}\dots \\ f_2 &: 0.a_{2_0}a_{2_1}a_{2_2}\dots \\ &\vdots \end{aligned}$$

Construct a new decimal number as following:

$$z = 0.x_1x_2x_3\dots$$

where $x_n = 0$ if $a_{n_n} \neq 0$ and $x_n = 1$ if $a_{n_n} = 0$. Thus z is not an element of $f[I]$ for any $I \subseteq \mathbb{N}$. It implies that f is not bijective. Thus $[0, 1)$ is uncountable, thus $\mathbb{R} \supsetneq [0, 1)$ is also uncountable. □