Notes for General Topology

Zhao Wenchuan

September 6, 2021

Contents

Chapter 1.

Metric Spaces

§1.1 Metric Spaces

Definition 1.1.1. Let X be any set.

A function $d : X \times X \to \mathbb{R}_{\geq 0}$ is *metric function*, or, simply, *metric on* X iff it satisfies the *metric axioms*. That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ iff $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.1.2. Let X be any set and let d be a structure on X.

The pair (X, d) is called a *metric space* iff d is a metric on X.

Definition 1.1.3. A $\mathbb{X} = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$.

An *open ε-ball*, or just ε-ball, about x is defined to be the set

$$B_\varepsilon(x; d) := \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) := \{y \in X : d(x, y) \leq \varepsilon\}.$$

Note 1.1.1. As

$$\mathbb{X}_0 = (X, d_0),\ \mathbb{X}_1 = (X, d_1),\ \mathbb{X}_2 = (X, d_2),\ \ldots$$

are different although they share the same set X, for any $x \in X$ and any $\varepsilon \in \mathbb{R}_{>0}$,

$$B_\varepsilon(x; d_1),\ B_\varepsilon(x; d_2),\ B(x; d_3),\ \ldots$$

are also different. However, if confusion is unlikely, we simply write "$B_\varepsilon(x)$" for "$B_\varepsilon(x; d)$".

Example 1.1.1. The *Euclidean metric space* $\mathbb{X} = (X, d)$ is an n-dimensional set X equipped with the *Euclidean metric* d defined as

$$d(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

Example 1.1.2. A *discrete metric space* $\mathbb{X} = (X, d)$ is a set X equiped with the *discrete metric* d_{dsic} defined as

$$d_{\text{disc}}(x, y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{else.} \end{cases}$$

This is an equivalent definition of the discrete metric:

$$d_{\text{disc}}(x, y) := (\operatorname{sgn}(d(x, y)))^2,$$

where $\operatorname{sgn}(\cdot)$ is a sign function, and d is any metric on X.

Example 1.1.3. [1] Let $\mathbb{I} = (C[a, b], d_p)$ be a metric space where $C[a, b]$ denotes the set of all continuous mapping $\mathbb{R}_{[a,b]} \to \mathbb{R}$, and $p > 0$, and the metric d_p is defined as

$$d_p(f, g) := \left(\int_a^b |f(t) - g(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

In particular,

$$d_\infty(f, g) := \sup_{t \in \mathbb{R}_{[a,b]}} |f(t) - g(t)|.$$

[1] See Minkowski inequality.

Example 1.1.4. [2] Let $\mathbb{X} = (X, d)$ be a metric space. The *Hausdorff metric* d_H on $2^X \setminus \{\emptyset\}$ is defined as

$$d_H := \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\},$$

where

$$d(x, Y) := \inf_{y \in Y} (x, y), \text{ and } d(y, X) := \inf_{x \in X} (y, x).$$

§1.2 Open Sets in Metric Spaces

Definition 1.2.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $U \subseteq X$.

U is said to be *open in* $\mathbb{X}$, iff for any $y \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$, such that $B_\varepsilon(y) \subseteq U$.

Proposition 1.2.1. Let $\mathbb{X} = (X, d)$ be a metric space, let $x \in A$ and let $\varepsilon \in \mathbb{R}_{>0}$.

For any $y \in B_\varepsilon(x)$, there is a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_\varepsilon(x)$.

Proof. For any $y \in B_\varepsilon(x)$, by the definition of open balls (Definition 1.1.3), we have $d(x, y) < \varepsilon$.

Let $\delta \in \mathbb{R}_{>0}$ such that $\delta + d(x, y) = \varepsilon$.

By M3 in metric axioms (Definition 1.1.1), for any $z \in A$ with $d(y, z) < \delta$, we have

$$d(x, z) \leq d(y, z) + d(x, y) < \varepsilon.$$

Thus, again, by the definition of open balls, we have $B_\delta(y) \subseteq B_\varepsilon(x)$.

■

Proposition 1.2.2. Let $\mathbb{X} = (X, d)$ be a metric space, and let $U \subseteq X$.

U is open in $\mathbb{X}$ iff it is a union of open balls.

Proof. First, prove $\Rightarrow$.

As U is open, for any $y \in U$, there exists $\varepsilon_y \in \mathbb{R}_{>0}$ such that $B_{\varepsilon_y}(y) \subseteq U$.

Therefore,

$$U = \bigcup_{y \in U} B_{\varepsilon_y}(y).$$

□

[2] See Hausdorff distance.

Now, prove $\Leftarrow$.

Aiming for a contradiction, suppose U is a union of open balls but not open.

As U is not open, there is a $y \in U$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \not\subseteq U$.

As U is a union of open balls, there is an $x \in U$ and $r \in \mathbb{R}_{>0}$ such that $y \in B_r(x)$.

By Proposition 1.2.1, there exists a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_r(x)$.

This is a contradiction by the assumption.

Thus, U has to be open.

■

Proposition 1.2.3. Let $\mathbb{X} = (X, d)$ be any metric space.

$\mathbb{X}$ is *Hausdorff*. That is, For any distinct points $x, y \in X$, we can always find an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset.$$

Proof. Aiming for a contradiction, suppose there are $x, y \in X$ with $x \neq y$, such that for any $\varepsilon \in \mathbb{R}_{>0}$, we can always find a $z \in X$ such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let $r = d(x, y)/2$, and let $z \in B_r(x) \cap B_r(y)$.

As $z \in B_r(x)$, by the definition of open balls (Definition 1.1.3), $d(x, z) < r$; as $z \in B_r(y)$, similarly, $d(y, z) < r$. Then we have

$$d(x, z) + d(y, z) < 2r = d(x, y).$$

This contradicts the metric axioms M3 (Definition 1.1.1).

Thus $\mathbb{X}$ is Hausdorff.

■

Definition 1.2.2. Let $\mathbb{X} = (X, d)$ be any metric space, and let $V \subseteq X$.

V is said to be *closed* in $\mathbb{X}$, iff there is an open set U satisfies $X \setminus U = V$.

Proposition 1.2.4. In a metric space, any singleton is closed.

Proof. Let $\mathbb{X} = (X, d)$ be a metric space, let $x \in X$, and let $y \in X \setminus \{x\}$.

As M is Hausdorff (Proposition 1.2.3), there is an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$0 < \varepsilon < d(x, y),$$

thus $X \setminus \{x\}$ is open, hence, by Definition 1.1.1, its complement $\{x\}$ is open.

■

Proposition 1.2.5. Let $\mathbb{X} = (X, d)$ be a metric space, denote $\mathcal{T}$ for the family of open subsets of X.

Then $\mathcal{T}$ satisfies the following conditions:

O1. $X, \emptyset \in \mathcal{T}$;

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under arbitrary union;

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under finite intersection.

Proof.

O1. As $\emptyset$ is the subset of any set, $\emptyset \in \mathcal{T}$. $\bigcup \emptyset = \emptyset \in \mathcal{T}$.

By Definition 1.2.2, $X = X \setminus \emptyset$.

□

O2. Let $\mathcal{U} \subseteq \mathcal{T}$, and denote $\mathcal{O}$ for the open balls in M.

For any $U \in \mathcal{U}$, there is an $\mathcal{O}_U \subseteq \mathcal{O}$ such that $U = \bigcup \mathcal{O}_U$.

Then we have

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \left(\bigcup \mathcal{O}_U \right) = \bigcup_{U \in \mathcal{U}} \mathcal{O}_U.$$

By Proposition 1.2.2, $\bigcup \mathcal{U}$ is open.

□

O3. Let $\mathcal{V}$ be a finite subset of $\mathcal{T}$.

Aiming for a contradiction, suppose $\bigcap \mathcal{V}$ is not open.

By Definition 1.2.1, there exists a $y \in \bigcap \mathcal{V}$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \setminus \bigcap \mathcal{V} \neq \emptyset$.

By De Morgan's law, we have

$$\bigcup_{V \in \mathcal{V}} (B_\varepsilon(y) \setminus V) \neq \emptyset.$$

Thus, there exists $V \in \mathcal{V}$ such that $B_\varepsilon(y) \setminus V \neq \emptyset$.

As $V \in \mathcal{T}$ and ε is arbitrarily given, by Proposition 1.2.1, $y \notin V$. This is a contradiction.

Thus, $\bigcap \mathcal{V}$ is open.

□

Thus, the theorem is proved.

■

Proposition 1.2.6. Infinite intersections of open sets in some metric spaces are not necessarily open.

Proof. Consider $\mathbb{R}$ is a Euclidean metric space, and denote $\mathcal{T}$.

Clearly, for any $n \in \mathbb{N}_{>0}$ and for any $x \in X$, the open interval $B_{\frac{1}{n}}(x)$ is open, but

$$\bigcap \left\{ B_{\frac{1}{n}}(x) : n \in \mathbb{N}_{>0} \right\} = \{x\}.$$

For any $\varepsilon \in \mathbb{R}_{>0}$, $B_{\varepsilon}(x) \setminus \{x\}$ is not empty, thus $\{x\}$ is not open.

■

§1.3 Restrictions and Metric Subspaces

Restriction of metric function is a useful tool to describe the relation between metric spaces with different sets but "same" metric function on the sets.

As a restriction of a relation R on $X \times Y$ to a subset $A \times B \subseteq X \times Y$ is defined to be

$$R \upharpoonright_{A \times B} := R \cap (X \times Y),$$

a restriction of a metric d on a set S to a subset $U \subseteq S$ is defined to be

$$d \upharpoonright_{(U \times U) \times \mathbb{R}_{>0}} := d \cap ((U \times U) \times \mathbb{R}_{>0}).$$

If $B = Y$, customarily, we simply write $R \upharpoonright_A$ for $R \upharpoonright_{A \times B}$. Similarly, as the codomain of a metric function is alway $\mathbb{R}_{>0}$, so we simply write $d \upharpoonright_{U \times U}$ instead of $d \upharpoonright_{(U \times U) \times \mathbb{R}_{>0}}$.

Definition 1.3.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$.

The *metric on A induced by d*, or the *subspace metric of d with respect to A* is defined to be

$$d_A := d \upharpoonright_{A \times A} .$$

Proposition 1.3.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$ and let $d_A := d \upharpoonright_{A \times A}$.

Then $\mathbb{A} = (A, d_A)$ is a metric space.

Proof. As metric axioms (Definition 1.1.1) holds for any $x, y \in X$, and $A \subseteq X$, they also holds for any $a, b \in A$. As d_A is the subspace metric of d with respect to A, d_A is a metric on A.

Thus, $\mathbb{A}$ is a metric space.

Definition 1.3.2. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$.

$\mathbb{A} = (A, d_A)$ is a *metric subspace* of $\mathbb{X}$ iff d_A is a subspace metric of d with respect to A.

Chapter 2.

Topological Spaces

§2.1 Basic Definitions

Definition 2.1.1. Let X be any set, and let $\mathcal{T} \subseteq 2^X$.

$\mathcal{T}$ is a *topology on* X iff it satisfies the *open set axioms.* That is,

O1. $X \in \mathcal{T}$;

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under arbitrary union.

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under finite intersection.

A subset $U \subseteq X$ is said to be *open in* M iff it is an element of $\mathcal{T}$.

Definition 2.1.2. Let X be any set, and let $\mathcal{T}$ be a structure on X.

The pair $(X, \mathcal{T})$ is called a *topological space* iff $\mathcal{T}$ is a topology on X.

Proposition 2.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

Then $\emptyset \in \mathcal{T}$.

Proof. As empty set is an element of any set, it also an element of $\mathcal{T}$.

Therefore, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

■

Definition 2.1.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

A subset $A \subseteq X$ is said to be *closed in* $\mathbb{X}$ iff there exists a $U \in \mathcal{T}$ such that $A = X \setminus U$.

Proposition 2.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and denote $\mathcal{C}$ for the family of all closed sets in M.

Then $\mathcal{C}$ satisfies the following conditions:

C1. $X, \emptyset \in \mathcal{C}$;

C2. For any $\mathcal{A} \subseteq \mathcal{C}$, $\bigcap \mathcal{A} \in \mathcal{C}$;

C3. For any finite $\mathcal{B} \subseteq \mathcal{C}$, $\bigcup \mathcal{B} \in \mathcal{C}$.

Proof.

C1. As $\emptyset \in \mathcal{T}$ and $X = X \setminus \emptyset$, by Definition 2.1.3, X is closed.

Similarly, as $X \in \mathcal{T}$ and $\emptyset = X \setminus X$, $\emptyset$ is closed.

□

C2. For any $\mathcal{A} \subseteq \mathcal{C}$, there exists a $\mathcal{U} \subseteq \mathcal{T}$ such that

$$\forall A \in \mathcal{A} : \exists U \in \mathcal{U} : A = X \setminus U. \quad \text{(Definition 2.1.3.)}$$

Then we have

$$\begin{aligned} \mathcal{A} = \{X \setminus U : U \in \mathcal{U}\} &\iff \bigcap \mathcal{A} = \bigcap_{U \in \mathcal{U}} X \setminus U \\ &\iff \bigcap \mathcal{A} = X \setminus \bigcup \mathcal{U}. \end{aligned}$$

As $\bigcup \mathcal{U} \in \mathcal{T}$ by Definition 2.1.1 O2, its complement $\bigcap \mathcal{A} \in \mathcal{C}$ by Definition 2.1.3.

□

C3. For any finite $\mathcal{B} \subseteq \mathcal{C}$, there exists a finite $\mathcal{U} \subseteq \mathcal{T}$ such that

$$\forall B \in \mathcal{B} : \exists U \in \mathcal{U} : A = X \setminus U. \quad \text{(Definition 2.1.3.)}$$

Then we have

$$\begin{aligned} \mathcal{B} = \{X \setminus U : U \in \mathcal{U}\} &\iff \bigcup \mathcal{B} = \bigcup_{U \in \mathcal{U}} X \setminus U \\ &\iff \bigcup \mathcal{B} = X \setminus \bigcap \mathcal{U}. \end{aligned}$$

As $\bigcap \mathcal{U} \in \mathcal{T}$ by Definition 2.1.1 O3, its complement $\bigcup \mathcal{A} \in \mathcal{C}$ by Definition 2.1.3.

□

Thus, the proof is done.

■

§2.2 Some Important Topologies

Definition 2.2.1. Let X be any set.

A family $\mathcal{T} \subseteq 2^X$ is a *discrete topology on* X iff $\mathcal{T} = 2^X$.

Definition 2.2.2. Let X be any set.

A family $\mathcal{T} \subseteq 2^X$ is an *indiscrete topology on* X iff $\mathcal{T} = \{X, \emptyset\}$.

Definition 2.2.3. Let $\mathbb{X} = (X, d)$ be a metric space.

A family $\mathcal{T} \subseteq 2^X$ is a *topology induced by* d iff $\mathcal{T}$ is the set of all open sets in $\mathbb{X}$.

§2.3 Comparison of Topologies

Definition 2.3.1. Let X be any set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X.

We say that $\mathcal{T}$ is *coarser* than $\mathcal{T}_1$, or $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$, iff $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Note 2.3.1. By the definition of cardinality and inclusion mapping, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$, it is certainly true that $|\mathcal{T}_1| \leq |\mathcal{T}_2|$. But, on the contrary, $|\mathcal{T}_1| \leq |\mathcal{T}_2|$ does not implies $\mathcal{T}_1 \subseteq \mathcal{T}_2$. It is easy to find counter-example about this.

Example 2.3.1. By Definition 2.3.1, for any set X, if a family $\mathcal{U}$ of open sets is given, then we can find the coarsest topology on X containing $\mathcal{U}$ by

$$\mathcal{T} = \left\{ \bigcup \mathcal{I}, \bigcap \mathcal{I}, X : \mathcal{I} \subseteq \mathcal{U} \right\}.$$

For example, let $X = \{1, 2, 3, 4, 5\}$, and let

$$\mathcal{U} = \{\{1, 2\}, \{2, 3\}, \{4\}\}.$$

Then a topology on X contains at least these sets:

$$\begin{gathered} \{1, 2, 3, 4\}, \{\}, \\ \{1, 2\}, \{2, 3\}, \{4\}, \\ \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \\ \{2\}. \end{gathered}$$

Example 2.3.2. The discrete topology is the finest topology on any X, while the indiscrete topology is the coarsest.

§2.4 Interiors

Definition 2.4.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

The *interior* of A is defined as

$$\operatorname{Int}_{\mathcal{T}}(A) := \bigcup \left(\mathcal{T} \cap 2^A\right).$$

Note 2.4.1. Let $\mathbb{X}_1 = (X, \mathcal{T}_1)$, $\mathbb{X}_2 = (X, \mathcal{T}_2)$, and $A \subseteq X$. Then $\mathcal{T}_1 \neq \mathcal{T}_2$ iff $\operatorname{Int}_{\mathcal{T}_1}(A) \neq \operatorname{Int}_{\mathcal{T}_2}(A)$. In this case, the subscript for "Int" is necessary.

But, if the confusion is unlikely, we can also simply write $\operatorname{Int}(A)$ for $\operatorname{Int}_{\mathcal{T}} A$. In this case, it is also common to write A° for $\operatorname{Int}(A)$.

Proposition 2.4.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$A \in \mathcal{T}$ iff $A = A^\circ$.

Proof. First, prove $\Rightarrow$.

If $A \in \mathcal{T}$, then we have

$$\mathcal{T} \cap 2^A = \mathcal{T} \cap \{A\} \cap 2^A = \{A\} \cap 2^A = \{A\}.$$

By Definition 2.4.1,

$$A^\circ = \bigcup(\mathcal{T} \cap 2^A) = \bigcup\{A\} = A.$$

□

Now, prove $\Leftarrow$.

By Definition 2.4.1, we have

$$A = \bigcup(\mathcal{T} \cap 2^A).$$

As $\mathcal{T} \cap 2^A \subseteq \mathcal{T}$, thus, by open set axioms O2 (Definition 2.1.1 O2), $A \in \mathcal{T}$.

□

Thus, the proof is done.

■

Proposition 2.4.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \in \mathcal{T}$. For any $x \in A$, there is a $U \in \mathcal{T} \cap 2^A$ such that $x \in U$.

Proof.

$$\begin{aligned} x \in A &\iff x \in A^\circ && \text{(Proposition 2.4.1)} \\ &\iff x \in \bigcup \left(\mathcal{T} \cap 2^A\right) && \text{(Definition 2.4.1)} \\ &\iff \exists U \in \mathcal{T} \cap 2^A : x \in U. \end{aligned}$$

■

Proposition 2.4.3. Let X be any set, let I be an index set, and let $\mathcal{A}_i \subseteq 2^X$ for any $i \in I$.

Then we have

$$\bigcup \left(\bigcap_{i \in I} \mathcal{A}_i \right) \subseteq \bigcap_{i \in I} \left(\bigcup \mathcal{A}_i \right).$$

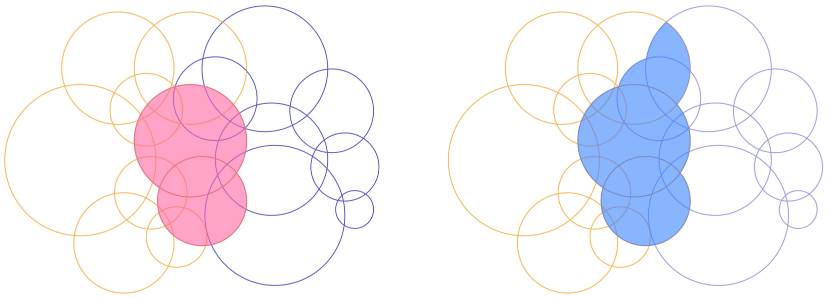

Figure 2.1: Diagram of the relation in Proposition 2.4.3.

Proposition 2.4.4. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $\mathcal{A} \subseteq 2^X$.

Then we have

$$\left(\bigcap \mathcal{A} \right)^\circ \subseteq \bigcap_{A \in \mathcal{A}} A^\circ.$$

Proof.

$$\begin{aligned}
\left(\bigcap \mathcal{A}\right)^{\circ} &= \bigcup \left(\mathcal{T} \cap 2^{\bigcap \mathcal{A}}\right) && \text{(Definition 2.4.1)} \\
&= \bigcup \left(\mathcal{T} \cap \bigcap_{A \in \mathcal{A}} 2^{A}\right) && \text{(intersection of power sets)} \\
&= \bigcup \left(\bigcap_{A \in \mathcal{A}} (\mathcal{T} \cap 2^{A})\right) && \text{(intersection is idempotent} \\
& && \text{and associative)} \\
&\subseteq \bigcap_{A \in \mathcal{A}} \left(\bigcup (\mathcal{T} \cap 2^{A})\right) && \text{(Proposition 2.4.3)} \\
&= \bigcap_{A \in \mathcal{A}} A^{\circ}. && \text{(Definition 2.4.1)}
\end{aligned}$$

■

Example 2.4.1. The equality in Proposition 2.4.4 may not hold.

Let $\mathbb{T} = (\mathbb{R}, \mathcal{T})$ be a topological space with

$$\mathcal{T} = \{X, (0,2), (1,3), \emptyset\}.$$

Then we have

$$((0,2) \cap (1,3))^{\circ} = \emptyset \quad \subsetneq \quad (0,2)^{\circ} \cap (1,3) = (1,2).$$

Proposition 2.4.5. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. If $A \subseteq B$, then $A^{\circ} \subseteq B^{\circ}$.

Proof.

$$\begin{aligned}
A \subseteq B &\implies 2^{A} \subseteq 2^{B} && \text{(power set of subset)} \\
&\implies \mathcal{T} \cap 2^{A} \subseteq \mathcal{T} \cap 2^{B} \\
&\implies \bigcup (\mathcal{T} \cap 2^{A}) \subseteq \bigcup (\mathcal{T} \cap 2^{B}) \\
&\implies A^{\circ} \subseteq B^{\circ} && \text{(Definition 2.4.1)}
\end{aligned}$$

■

Note 2.4.2. Note that, $A^{\circ} \subseteq B^{\circ}$ does not implies $A \subseteq B$. Consider $\mathbb{R}$ as a Euclidean metric space, and let

$$A = \{0\}, \quad B \subseteq \mathbb{R} \setminus \{0\}.$$

As $A^{\circ} = \emptyset$, $A^{\circ} \subseteq B^{\circ}$, but $A \setminus B = \{0\}$, so $A \nsubseteq B$.

§2.5 Limit Points and Isolated Points

Definition 2.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in X$ is a *limit point of* A iff for any $U \in \mathcal{T}$ with $x \in U$

$$A \cap U \setminus \{x\} \neq \emptyset.$$

The *derived set of* A is the set of all limit points of X.

Definition 2.5.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in A$ is said to be *isolated* iff there is $U \in \mathcal{T}$ with $x \in U$, such that

$$A \cap U \setminus \{x\} = \emptyset.$$

Notations. The Derived set of A is usually denoted A'.[1] But sometime it is also necessary to know in which space (with its topology) the derived set of A is. For example, for topological spaces $\mathbb{X}_1 = (X, \mathcal{T}_1)$ and $\mathbb{X}_2 = (X, \mathcal{T}_2)$, if $\mathcal{T}_1 \neq \mathcal{T}_2$, the derived sets of a set A in $\mathbb{X}_1$ and $\mathbb{X}_2$ may be different. So, below, the notation A' is used only if the confusions are unlikely; else, we denote $\mathrm{L}_{\mathcal{T}} A$ for A' with respect to the topology $\mathcal{T}$.

Sometime, the set of isolated points of A is denoted by A^i. For avoiding confusions, we denote $\mathrm{I}_{\mathcal{T}}(A)$ for A^i with respect to the topology $\mathcal{T}$.

Proposition 2.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then,

$$A \subseteq \mathrm{L}(A) \sqcup \mathrm{I}(A).$$

Proof. By Definition 2.5.1, $x \notin \mathrm{L}(A)$ iff there exists a $U \in \mathcal{T}$ of $x \in U$ such that $A \cap N \setminus \{x\} = \emptyset$. This precisely satisfies Definition 2.5.2. Thus

$$A \subseteq \mathrm{L}(A) \cup \mathrm{I}(A).$$

As Definition 2.5.1 and 2.5.2 are precisely logical complement for each other, $x \in \mathrm{I}(A) \cap \mathrm{L}(A)$ always fails, i.e., $\mathrm{I}(A) \cap \mathrm{L}(A) = \emptyset$. Thus

$$A \subseteq \mathrm{L}(A) \sqcup \mathrm{I}(A).$$

∎

[1]See ProofWiki and Wikipedia.

Proposition 2.5.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff $\mathrm{L}(A) \subseteq A$.

Proof. First, prove $\Rightarrow$.

Aiming for a contradiction, suppose A is closed but there exists a $y \in \mathrm{L}(A) \setminus A$.

By Definition 2.1.3, as A is closed, then $A^{\complement}$ is open.

As $y \in A^{\complement}$ and $A^{\complement}$ is open, then, by Proposition 2.4.2, there exists a $U \in \mathcal{T}$ with $y \in U$, such that $U \subseteq A^{\complement}$.

As U is an open set containing y and $A \cap U \setminus \{y\} = \emptyset$, then $y \notin \mathrm{L}(A)$. This contradicts the assumption.

Thus $\mathrm{L}(A) \subseteq A$.

■

§2.6 Closures

Definition 2.6.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

The *closure of* A is defined as

$$\mathrm{Cl}_{\mathcal{T}}(A) := A \cup \mathrm{L}(A).$$

When the confusions are unlikely, we simply write $\mathrm{Cl}(A)$, $\overline{A}$ or A^{-} for $\mathrm{Cl}_{\mathcal{T}}(A)$.

Proposition 2.6.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff $A = A^{-}$

Proof.

$$\begin{aligned} A \text{ is closed} &\iff A \supseteq \mathrm{L}(A) && \text{(Proposition 2.5.2)} \\ &\iff A = A \cup \mathrm{L}(A) \\ &\iff A = A^{-}. && \text{(Definition 2.6.1)} \end{aligned}$$

■

Proposition 2.6.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff

$$A = \mathrm{I}(A) \sqcup \mathrm{L}(A).$$

Proof. As A is closed, we have

$$\begin{aligned} A &= \mathrm{Cl}(A) && \text{(Proposition 2.6.1)} \\ &= A \cup \mathrm{L}(A) && \text{(Definition 2.6.1)} \\ &= A \setminus \mathrm{L}(A) \sqcup \mathrm{L}(A) \\ &= \mathrm{I}(A) \sqcup \mathrm{L}(A). && \text{(Proposition 2.5.1)} \end{aligned}$$

■

Proposition 2.6.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$$A^- = \bigcap \{K \supseteq A : K \text{ closed in } \mathbb{X}\}.$$

Proof. By Proposition 2.6.1, A^- is closed. Thus, by Definition 2.1.3, $X \setminus A^-$ is open. Then we ahve

$$\begin{aligned} X \setminus (X \setminus A^-) &= X \setminus (X \setminus A^-)^\circ && \text{(Proposition: 2.4.1)} \\ &= X \setminus \bigcup \left(\mathcal{T} \cap 2^{X \setminus A^-}\right) && \text{(Definition: 2.4.1)} \\ &= X \setminus \bigcup \{U \subseteq A : U \text{ open in } \mathbb{X}\} \\ &= \bigcap \{X \setminus U \supseteq A : U \text{ open in } \mathbb{X}\} && \text{(De Morgan's Law)} \\ &= \bigcap \{K \supseteq A : K \text{ closed in } \mathbb{X}\}. && \text{(Definition: 2.1.3)} \end{aligned}$$

■

Proposition 2.6.4. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then we have

$$X \setminus A^\circ = (X \setminus A)^-.$$

Proof. First, we have

$$\begin{aligned} X \setminus A^\circ &= X \setminus \bigcup (\mathcal{T} \cap 2^A) && \text{(Definition 2.4.1)} \\ &= \bigcap_{K \in \mathcal{T} \cap 2^A} (X \setminus K) && \text{(De Morgan's Law)} \end{aligned}$$

For any K, $X \setminus K$ is a closed superset of $X \setminus A$.

As closed sets are closed under arbitrary intersection (Proposition 2.1.2), and $X \setminus A^\circ$ is the intersection of all closed superset of $X \setminus A$, by Proposition 2.6.3, $X \setminus A^\circ = (X \setminus A)^-$.

■

Proposition 2.6.5. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

If $A \subseteq B$, then $A^- \subseteq B^-$.

Proof.

$$\begin{aligned}
A \subseteq B &\iff X \setminus A \supseteq X \setminus B \\
&\implies (X \setminus A)^\circ \supseteq (X \setminus B)^\circ && \text{(Proposition 2.4.5)} \\
&\iff X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ \\
&\iff (X \setminus (X \setminus A))^- \subseteq (X \setminus (X \setminus B))^-. && \text{(Proposition 2.6.4)} \\
&\iff A^- \subseteq B^-.
\end{aligned}$$

■

Proposition 2.6.6. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \in \mathcal{T}$ such that for any $x \in A$, there exists a $U \in \mathcal{T}$ such that $x \in U \subseteq A$.

Then A is open in $\mathbb{X}$.

Proof. Aiming for a contradiction, suppose for any $x \in A$, there exists a $U \in \mathcal{T}$ such that $x \in U \subseteq A$, but A is not open.

By Definition 2.1.3, as A is not open, $X \setminus A$ is not closed.

By Proposition 2.5.2, there exists $x \in \mathrm{L}(A) \setminus (X \setminus A)$. Fix x.

As $x \notin X \setminus A$, $x \in A$.

By Definition 2.5.1, for $U \in \mathcal{T}$ with $x \in U$, $U \cap (X \setminus A) \neq \emptyset$, i.e., $U \setminus A \neq \emptyset$. This implies that $U \not\subseteq A$.

This contradicts the assumption we have.

Thus A has to be open.

■

§2.7 Density

Definition 2.7.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then A is said to be *everywhere dense*, or simply *dense*, in $\mathbb{X}$ iff

$$A^- = X.$$

Proposition 2.7.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then A is dense in $\mathbb{X}$ iff for any $U \in \mathcal{T}$, $A \cap U \neq \emptyset$.

Proof. First, prove $\Rightarrow$.

Assume A is dense in $\mathbb{X}$, then, by Definition 2.7.1, $A^{-} = X$.

By Definition 2.5.2, for any $x \in \mathrm{I}(A)$, $x \in A$.

By Definition 2.5.1, for any $x \in \mathrm{L}(A)$ and for any $U \in \mathcal{T}$ with $x \in U$, $U \cap A \neq \emptyset$.

As $A^{-} = X$, then, by Proposition 2.6.2, $X = \mathrm{I}(A) \sqcup \mathrm{L}(A)$.

Thus for any $U \in \mathcal{T}$, $A \cap U \neq \emptyset$.

□

Now, prove $\Leftarrow$.

Aiming for a contradiction, suppose for any $U \in \mathcal{T}$, $A \cap U \neq \emptyset$, but A is not dense in $\mathbb{X}$.

As, $A \subseteq X$, by Proposition 2.6.5, $A^{-} \subseteq X^{-}$. And, as X is closed in $\mathbb{X}$, by Proposition 2.6.1, $X = X^{-}$. Therefore, $A^{-} \subseteq X$.

As A is not dense in X, by Definition 2.7.1, $A^{-} \neq X$. Therefore, $A^{-} \subsetneq X$. This implies that $X \setminus A^{-}$ is non-empty. And, by Definition 2.6.1, $X \setminus A^{-} \in \mathcal{T}$.

By Proposition 2.4.2, for any $x \in X \setminus A^{-}$, there exists a $U \in \mathcal{T}$ with $x \in U$, such that $U \in X \setminus A^{-}$. Then $U \cap A = \emptyset$. This contradicts the assumption we have.

Therefore, A has to be dense in $\mathbb{X}$.

□

Thus, the proof is done.

■

Definition 2.7.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then A is said to be *nowhere dense in* $\mathbb{X}$ iff

$$\left(A^{-}\right)^{\circ} = \emptyset.$$

Proposition 2.7.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then A is nowhere dense in $\mathbb{X}$ iff for any $U \in \mathcal{T} \setminus \{\emptyset\}$,

$$U \setminus A^{-} \neq \emptyset.$$

Proof.

$$\begin{aligned}
&A \text{ is nowhere dense in } \mathbb{X} \\
\iff &(A^{-})^{\circ} = \emptyset && \text{(Definition 2.7.2)} \\
\iff &(A^{-})^{\circ} = \bigcup \left(\mathcal{T} \cap 2^{A}\right) = \emptyset && \text{(Definition 2.4.1)} \\
\iff &(\forall U \in \mathcal{T} : U \subseteq A^{-}) \quad U = \emptyset.
\end{aligned}$$

■

§2.8 Neighbourhood Systems

Definition 2.8.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A subset $N \subseteq X$ is a *neighbourhood of* A iff

$$(\exists U \in \mathcal{T}) \quad A \subseteq U \subseteq N.$$

If $A = \{x\}$, we simply call N a *neighbourhood of* x.

If $N \in \mathcal{T}$ also, then N is an *open neighbourhood of* A; and if N is closed, then N is a *closed neighbourhood of* A.

Proposition 2.8.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$A \in \mathcal{T}$ iff for any $x \in A$, A is a neighbourhood of x.

Proof. First, prove $\Rightarrow$.

If $A \in \mathcal{T}$, then, by Definition 2.8.1, for any $x \in A$, we have

$$x \in A \subseteq A.$$

$\square$

Now, prove $\Leftarrow$.

Aiming for a contradiction, suppose for any $x \in A$, A is a neighbourhood of x, but $A \notin \mathcal{T}$.

As $X \setminus A$ is not closed, (otherwise, by Definition 2.1.3, $A = X \setminus (X \setminus A)$ is open) by Proposition 2.5.2, there exists $x \in \mathrm{L}(X \setminus A) \setminus (X \setminus A)$.

Then, for such an $x \in A$ (for $x \notin X \setminus A$), for any $U \in \mathcal{T}$ with $x \in U$,

$$U \cap (X \setminus A) \neq \emptyset. \qquad \text{(Definition 2.5.1)}$$

By Definition 2.8.1, A fails to be a neighbourhood of x. This contradicts the assumption.

Thus A has to be open.

$\blacksquare$

Chapter 3.

Sequences

§3.1 Convergent Sequences

Definition 3.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $u : \mathbb{N} \to X$.

u converges to a limit $x \in X$ iff for any $U \in \mathcal{T}$ with $x \in U$,

$$(\exists k \in \mathbb{R}_{>0}) \quad u[\mathbb{N}_{>k}] \subseteq U.$$

Proposition 3.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $u : \mathbb{N} \to X$.

u converges to a limit $x \in X$ iff for any $U \in \mathcal{T}$ with $x \in U$,

$$\mathbb{N} \setminus u^{-1}[U] \text{ is finite.}$$

Proof. First, prove $\Rightarrow$.

By Definition 3.1.1, as u converges to x, let $U \in \mathcal{T}$ with $x \in U$, then there exists a $k \in \mathbb{R}_{>0}$ such that $u[\mathbb{N}_{>k}] \subseteq U$.

Then we have

$$\begin{aligned} u[\mathbb{N}_{>k}] \subseteq U &\implies u^{-1}[u[\mathbb{N}_{>k}]] \subseteq u^{-1}[U] \\ &\implies \mathbb{N}_{>k} \subseteq u^{-1}[U] \qquad \text{(image of inverse image)} \\ &\implies \mathbb{N} \setminus \mathbb{N}_{>k} \supseteq \mathbb{N} \setminus u^{-1}[U]. \end{aligned}$$

As $\mathbb{N} \setminus \mathbb{N}_{>k}$ is finite, its subset $\mathbb{N} \setminus u^{-1}[U]$ is finite.

□

Now, prove $\Leftarrow$.

By image of inverse image, we have

$$u[u^{-1}[U]] \subseteq U.$$

As $u^{-1}[U]$ is a cofinite subset of $\mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $I \supseteq \mathbb{N}_{>k}$. Then we have

$$U \supseteq u[\mathbb{N}_{>k}].$$

This precisely satisfies Definition 3.1.1.

$\square$

Therefore the proof is done.

$\blacksquare$

§3.2 Accumulation Points of Sequences

Definition 3.2.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $u : \mathbb{N} \to X$.

A point $x \in X$ is an *accumulation point* of u iff for any $U \in \mathcal{T}$ with $x \in U$, U contains infinitely many terms of u; i.e.,

$$\forall U \in \mathcal{T} : x \in U \implies (\exists I \subseteq \mathbb{N} : |I| = \aleph_0 \implies u[I] \subseteq U).$$

Note 3.2.1. Sometime, an accumulation point of a sequence is also a limit of the range of the sequence. But this not always holds.

Consider $\mathbb{R}$ as a Euclidean, and let $u : \mathbb{N} \to \mathbb{R}$ be defined as

$$u(n) := \left|\sin\left(\frac{\pi n}{2}\right)\right|.$$

Then 1 is an accumulation point of $u[\mathbb{N}]$, but $u[\mathbb{N}] = (u[\mathbb{N}])^i = \{0, 1\}$, so it has no limit point at all.

Proposition 3.2.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, let $u : \mathbb{N} \to X$, and let $x \in X$ be a limit of $u[\mathbb{N}]$.

Then x is an accumulation point of u.

Proof. Let $U \in \mathcal{T}$ with $x \in U$, then we have

$$u[u^{-1}[U]] \subseteq U.$$

By Proposition 3.1.1, as u converges to x, $u^{-1}[U]$ is a cofinite subset of $\mathbb{N}$. Thus $u^{-1}[U]$ is infinite.

As $u^{-1}[U]$ is infinite and $x \in U \in \mathcal{T}$, by Definition 3.2.1, x is an accumulation point of u.

$\blacksquare$

Definition 3.2.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in X$ is an ω-*accumulation point* of A iff for any $U \in \mathcal{T}$ with $x \in U$,

$$|U \cap A| \geq \aleph_0.$$

Proposition 3.2.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, let $u : \mathbb{N} \to X$ be an injection, and let $x \in X$ be an accumulation point of u.

Then x is an ω-accumulation point of $u[\mathbb{N}]$.

Proof. By Definition 3.2.1, as x is an accumulation point of u, let $U \in \mathcal{T}$ with $x \in U$, there exists an infinite $I \subseteq \mathbb{N}$ such that $u[I] \subseteq U$.

As u is injective and I is infinite, $u[I]$ is also infinite.

As $u[I] \subseteq U$ and $U \in \mathcal{T}$ with $x \in U$ is arbitrarily given, by Definition 3.2.2, x is an ω-accumulation point of $u[\mathbb{N}]$.

■

Chapter 4.

Countable Axioms

§4.1 Covers and Bases

Definition 4.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then a family $\mathcal{C} \subseteq 2^X$ is a *cover for* A iff $A \subseteq \bigcup \mathcal{C}$.

$\mathcal{C}$ is an *open cover* iff $\mathcal{C} \subseteq \mathcal{T}$.

Definition 4.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $\mathcal{C}, \mathcal{D}$ be covers for a subset $A \subseteq X$.

Then $\mathcal{D}$ is a *subcover of* $\mathcal{C}$ iff $\mathcal{D} \subseteq \mathcal{C}$.

Definition 4.1.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

A family $\mathcal{B} \subseteq 2^X$ is an *analytic basis for* $\mathcal{T}$ iff

(i) $\mathcal{B} \subseteq \mathcal{T}$;

(ii) For any $U \in \mathcal{T}$, there is an $\mathcal{A} \subseteq \mathcal{B}$, such that $U = \bigcup \mathcal{A}$.

Proposition 4.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $\mathcal{B} \subseteq \mathcal{T}$.

Then $\mathcal{B}$ is an analytic basis for $\mathcal{T}$ iff for any $U \in \mathcal{T}$ and for any $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proof. First, prove $\Rightarrow$.

By Definition 4.1.3, as $\mathcal{B}$ is an analytic basis for $\mathcal{T}$, let $U \in \mathcal{T}$, then there is an $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}$.

Then, for any $x \in U$, there exists at least one $A \in \mathcal{A}$ such that $x \in A$. As $U = \bigcup \mathcal{A}$, $A \subseteq U$.

□

Now, prove $\Leftarrow$.

By Proposition 2.6.6, as for any $U \in \mathcal{T}$ and for any $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then there exists an $\mathcal{A} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{A} = U$.

By Definition 4.1.3, $\mathcal{B}$ is an analytic basis for $\mathcal{T}$.

□

Thus, the proof is done.

■

Definition 4.1.4. Let X be any set.

A family $\mathcal{B} \subseteq 2^X$ is a *synthetic basis on* X iff

(i) $\mathcal{B}$ is a cover fir X;

(ii) For any $U, V \in \mathcal{B}$, there exists $\mathcal{A} \subseteq \mathcal{B}$, such that $U \cap V = \bigcup \mathcal{A}$.

Definition 4.1.5. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $x \in X$.

A family $\mathcal{B} \subseteq 2^X$ is a *local basis at* x iff

(i) $\mathcal{B} \in \mathcal{T}$;

(ii) For any $B \in \mathcal{B}$, $x \in B$;

(iii) For any $U \in \mathcal{T}$ with $x \in U$, there exists a $B \in \mathcal{B}$ such that $B \subseteq U$.

§4.2 First-Countable Spaces

Definition 4.2.1. A topological space $\mathbb{X} = (X, \mathcal{T})$ is said to be *first-countable* iff any $x \in X$ has a countable basis.

Proposition 4.2.1. Metric spaces are first-countable.

Proof. Let $\mathbb{X} = (X, d)$ be a metric space.

For any $x \in X$, let $\mathcal{B}_x : \mathbb{N} \to \mathcal{T}$ be defined as

$$\mathcal{B}_x(n) := B_{1/n}(x).$$

Clearly, the image $\mathcal{B}_x[\mathbb{N}]$ is countable.

Let $U \in \mathcal{T}$. As U is open, and as $x \in U$, then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(x) \subseteq U$.

By Archimedean Principle, there exists an $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. Then we have

$$\mathcal{B}_x(n) = B_{1/n}(x) \subseteq B_\varepsilon(x) \subseteq U.$$

As U is arbitrarily given, for any $x \in X$, $\mathcal{B}_x[\mathbb{N}]$ is a countable local basis at x.

■

Proposition 4.2.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a first-countable topological space, let $u : \mathbb{N} \to X$, and let $x \in X$ be an accumulation point of u.

Then x is a *subsequential limit* of u. That is, there exists an infinite $I \subseteq \mathbb{N}$, such that $u \restriction_I$ converges to x (as a limit).

Proof.[1] By Definition 4.2.1, as $\mathbb{X}$ is first-countable, there exists a countable local basis $\mathcal{B}$ at x.

Let $\mathcal{B}_x : \mathbb{N} \to \mathcal{T}$ such that $\mathcal{B}_x[\mathbb{N}]$ is a local base at x and for any $n \in \mathbb{N}$,

$$\mathcal{B}_x(n) \supseteq \bigcup \mathcal{B}_x[\mathbb{N}_{>n}].$$

Let $w : I \to u[\mathbb{N}]$ (I infinite) such that for any $i \in I$, $w(i) \in \mathcal{B}_x(i)$.

Then, for any $k \in \mathbb{N}$, we have $w[I_{>k}] \subseteq \mathcal{B}_x(k)$. Thus, by Definition 3.1.1, w is a subsequence of u converging to x.

■

§4.3 Second-Countable Spaces

Definition 4.3.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

$\mathbb{X}$ is said to be *second countable* iff $\mathcal{T}$ has a countable (analytic) basis.

Proposition 4.3.1. Second-countable spaces are first-countable.

Proof. Let $\mathbb{X} = (X, \mathcal{T})$ be a second-countable space.

By Definition 4.3.1, $\mathcal{T}$ has a countable analytic basis.

Let $x \in X$ and let $U \in \mathcal{T}$ with $x \in U$. By Definition 4.1.3 there exists a countable $\mathcal{B} \subseteq \mathcal{T}$, such that for any $U \in \mathcal{T}$, there is an $\mathcal{A} \subseteq \mathcal{B}$, such that $U = \bigcup \mathcal{A}$.

---

[1] The detail of this proof is incomplete.

As $U \in \mathcal{T}$ and $U = \bigcup \mathcal{A}$, by Proposition 2.4.2, there exists a $A \in \mathcal{A}$ such that $x \in A \subseteq U$.

Let $\mathcal{C} \subseteq \mathcal{B}$ be the family of all such A containing x, then, by Definition 4.1.5, $\mathcal{C}$ is a local basis at x. And as $\mathcal{B}$ is countable, as a subset, $\mathcal{C}$ is also countable.

Therefore $\mathcal{C}$ is a countable local basis at x.

As x is arbitrarily given, $\mathbb{X}$ is first-countable.

■

Example 4.3.1. Consider $\mathbb{R}$ as a Euclidean metric space.

$\mathbb{R}$ is second-countable.

Proof. By Proposition 4.2.1, $\mathbb{R}$ is first-countable.

For any $x \in \mathbb{Q}$, let $\mathcal{O}_x : \mathbb{N} \to \mathcal{T}$ be defined as

$$\mathcal{O}_x(n) := B_{1/n}(x).$$

For any $r \in \mathbb{R}$ and for any open set $U \ni r$, there exists $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(r) \subseteq U$.

There exists some $q \in \mathbb{Q}$ such that $q \in B_\delta(r)$. As $B_\delta(r)$ is open, by Definition 1.2.1, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(q) \subseteq B_\delta(r)$.

By Archimedean property, there exists $k \in \mathbb{N}$ such that $k > \frac{1}{\varepsilon}$. Then we have

$$\mathcal{O}_q(k) = B_{1/k}(q) \subseteq B_\varepsilon(q) \subseteq B_\delta(r).$$

[This proof is incomplete]

Example 4.3.2. Let $\mathbb{X} = (\mathbb{R}, \mathcal{T})$ be a discrete topological space.

$\mathbb{X}$ is first-countable but not second-countable.

§4.4 Separable Spaces

Definition 4.4.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

$\mathbb{X}$ is said to be *separable* iff there exists a countable subset $A \subseteq X$ such that A is dense in $\mathbb{X}$.

Proposition 4.4.1. Second-countable spaces are separable.

Proof. Let $\mathbb{X} = (X, \mathcal{T})$ be a second-countable topological space.

As $\mathbb{X}$ is second-countable, by Definition 4.3.1, there is a countable base $\mathcal{B}$ for $\mathcal{T}$.

Let $f : \mathcal{B} \to X$ such that for any $B \in \mathcal{B}$,

$$f(B) = \text{a random } x \in B.$$

As $\mathcal{B}$ is countable, then $f[\mathcal{B}]$ is countable.

Now, it suffices to show that $f[\mathcal{B}]$ is dense in $\mathbb{X}$.

Aiming for a contradiction, suppose $f[\mathcal{B}]$ is not dense in $\mathbb{X}$, then, there exists some $x \in X \setminus (f[\mathcal{B}])^-$.

By Definition 2.1.3, $X \setminus (f[\mathcal{B}])^- \in \mathcal{T}$; by Definition 2.4.2, there exists $U \in \mathcal{T}$ with $U \ni x$ such that $U \subseteq X \setminus (f[\mathcal{B}])^-$. That is, for any $B \in \mathcal{B}$, $f(B) \notin U$; i.e., $f[\mathcal{B}] \cap U = \emptyset$.

As $U \in \mathcal{T}$ and $\mathcal{B}$ is a base for $\mathcal{T}$, by Definition 4.1.3, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{A} = U$. Thus, $f[\mathcal{A}]$ must be a non-empty subset of U. This contradicts $f[\mathcal{B}] \cap U = \emptyset$.

Thus, $f[\mathcal{B}]$ has to be dense in $\mathbb{X}$. As $f[\mathcal{B}]$ is countable, therefore, $\mathbb{X}$ is second-countable.

■

Example 4.4.1. Niemytzki plane is separable but not second-countable.[2]

Proposition 4.4.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a discrete topological space which is separable.

Then X is countable.

Proof. Aiming for a contradiction, suppose X is uncountable.

As $\mathbb{X}$ is separable, by Definition 4.4.1, there exists a countable subset $A \subseteq X$ being dense in $\mathbb{X}$.

By Definition 2.7.1, $A^- = X$.

As $\mathbb{X}$ is discrete, $A^- = A$.

Now, we have $A = X$. As A is countable but X is not, this is impossible.

This contradiction shows that X has to be countable.

■

Proposition 4.4.3. Separable metric spaces are second-countable.

Proof. Let $\mathbb{X} = (X, d)$ be a metric space which is separable. Denote $\mathcal{T}$ for the topology on X induced by d.

[2] See ProofWiki.

By Definition 4.4.1, let $A \subseteq X$ be a countable set with $A^- = X$ (by Definition 2.7.1, A dense in $\mathbb{X}$).

Let $\mathcal{B} : \mathbb{N} \times A \to \mathcal{T}$ be defined as

$$\mathcal{B}(n, a) := B_{1/n}(a).$$

Let $\varepsilon \in \mathbb{R}_{>0}$ and let $x \in X$. Then $B_\varepsilon(x)$ defines an open ball in $\mathbb{X}$.

As $A^- = X$ and $x \in X$, $x \in A^-$ also. Thus, there exists an $a \in A \cap B_\varepsilon(x)$.

By Proposition 1.2.1, as $a \in B_\varepsilon(x)$, there exists a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(a) \subseteq B_\varepsilon(x)$.

By Archimedean property, let $k \in \mathbb{N}$ such that $k > \frac{1}{\delta}$, then we have

$$\mathcal{B}(k, a) = B_{1/k}(a) \subseteq B_\delta(a) \subseteq B_\varepsilon(x).$$

By Proposition 4.1.1, $\mathcal{B}[\mathbb{N} \times A]$ is an analytic basis for $\mathcal{T}$. As $\mathbb{N} \times A$ is countable, the image $\mathcal{B}[\mathbb{N} \times A]$ is also countable.

Therefore, $\mathcal{B}[\mathbb{N} \times A]$ is a countable analytic basis for $\mathcal{T}$. By Definition 4.3.1, $\mathbb{X}$ is second-countable.

■

§4.5 Lindelöf Space

Definition 4.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

Then $\mathbb{X}$ is said to be *Lindelöf* iff every open cover for X has a countable subcover.

Proposition 4.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a second-countable topological space.

Then $\mathbb{X}$ is Lindelöf.

Proof. As $\mathbb{X}$ is second-countable, by Definition 4.3.1, there exists a countable basis $\mathcal{B}$ for $\mathcal{T}$.

Let $\mathcal{U}$ be an open cover of $\mathbb{X}$, no matter it is countable or not.

By Definition 4.1.3, for any $U \in \mathcal{U}$, there exists an $\mathcal{A} \subseteq \mathcal{B}$ such that $\bigcup \mathcal{A} = U$.

Let $f : \mathcal{B} \to \mathcal{U}$ be defined as

$$f(B) := \text{a random } U \in \mathcal{B} \text{ with } U \supseteq B.$$

As $\mathcal{B}$ is an open over of X and for any $B \in \mathcal{B}$, $f(B) \supseteq B$, thus $f[\mathcal{B}]$ is an open cover of $\mathcal{B}$.

As $\mathcal{U}$ is the codomain of f, $f[\mathcal{B}] \subseteq \mathcal{U}$.
Therefore, $f[\mathcal{B}]$ is a subcover of $\mathcal{U}$.
As $\mathcal{B}$ is countable, it image $f[\mathcal{B}]$ is countable.
Therefore, $f[\mathcal{B}]$ is a countable subcover of $\mathcal{U}$.
As $\mathcal{U}$ is arbitrarily given, by Definition 4.5.1, $\mathbb{X}$ is Lindelöf.

Example 4.5.1. Sorgenfrey line is a topological space which is Lindelöf but not second-countable. (See Section A.1.)

Chapter 5.

Continuous Mappings

§5.1 Continuous Mappings

Definition 5.1.1. Let $\mathbb{X} = (X, \mathcal{T}_X)$ and $\mathbb{Y} = (X, \mathcal{T}_Y)$ be topological spaces, let $f : \mathbb{X} \to \mathbb{Y}$, and let $A \subseteq X$ be a mapping.

Then f is said to be *continuous on* A iff there exists a $U_X \in \mathcal{T}_X$ with $A \subseteq U_X$, such that for any $U_Y \in \mathcal{T}_Y$,

$$f^{-1}[U_Y] \cap U_X \in \mathcal{T}_X.$$

f is a *continuous mapping* iff $A = X$; i.e., it is continuous on whole X.

Note 5.1.1. By Definition 5.1.1, f is *continuous at a point* $x \in X$, iff it is continuous on some $U_X \in \mathcal{T}$ with $x \in U_X$, as x here can be considered as a singleton $\{x\}$.

Note 5.1.2. There is a common error: if for any $U_X \in \mathcal{T}_X$, its image $f[U_X] \in \mathcal{T}_Y$ also, then f is continuous. But, this condition also holds for some discontinuous mappings.

For example, let $\mathbb{X} = (\mathbb{R}, \mathcal{T}_X)$ be a topological space where $\mathcal{T}$ induced by Euclidean metric, and let $\mathbb{Y} = (\mathbb{R}, \mathcal{T}_Y)$ be a discrete topological space. Let $\imath : \mathbb{X} \to \mathbb{Y}$ be an identity mapping; i.e., it is defined as

$$\imath : \mathbb{X} \to \mathbb{Y} : x \mapsto x.$$

For any $A \subseteq \mathbb{R}$, clearly, $\imath[A] \in \mathcal{T}_Y$ holds. But for some (or for all) $B \in \mathcal{T}_Y \backslash \mathcal{T}_X$, $\imath^{-1}[B] \notin \mathcal{T}$. Thus, $\imath$ is not a identity mapping.

Indeed, for any identity mapping $\imath : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, $\imath$ is continuous iff $\mathcal{T}_X \supseteq \mathcal{T}_Y$.

Example 5.1.1. Let $\mathbb{X} = (X, \mathcal{T}_X)$ be a topological space, where $\mathcal{T}_X$ is the discrete topology on X. Let $\mathbb{Y} = (X, \mathcal{T}_Y)$ be any topological space. Then for any $f : \mathbb{X} \to \mathbb{Y}$, f is continuous.

Proposition 5.1.1. Let $\mathbb{X} = (X, \mathcal{T}_X)$ and $\mathbb{Y} = (X, \mathcal{T}_Y)$ be topological spaces, let $f : \mathbb{X} \to \mathbb{Y}$, and let $A \subseteq X$ be a mapping.

f is continuous on A iff for any $U_Y \in \mathcal{T}$ with $f[A] \subseteq U_Y$, there exists a U_X with $A \subseteq U_X$, such that $f[U_X] \subseteq U_Y$.

Proof. First, prove $\Rightarrow$.

Assume f is continuous on A, then, by Definition 5.1.1, let $U_Y \in \mathcal{T}$ with $f[A] \subseteq U_Y$, then there exists $U_X \in \mathcal{T}$ with $A \subseteq U_X$, such that

$$f^{-1}[U_Y] \cap U_X \in \mathcal{T}.$$

Then we have

$$\begin{aligned}
& U_X \subseteq f^{-1}[U_Y] \cap U_X \\
\Longrightarrow\ & f[U_X] \subseteq f[f^{-1}[U_Y] \cap U_X] \\
\Longrightarrow\ & f[U_X] \subseteq f[f^{-1}[U_Y]] \cap f[U_X] \\
& (\text{Image of Intersection under Mapping}) \\
\Longrightarrow\ & f[U_X] \subseteq U_Y \cap f[U_X]. \\
& (\text{Image of Inverse Image}) \\
\Longrightarrow\ & f[U_X] \subseteq U_Y.
\end{aligned}$$

■

Proposition 5.1.2. Let $\mathbb{X} = (X, \mathcal{T}_Y)$, $\mathbb{Y} = (X, \mathcal{T}_Y)$ and $\mathbb{Z} = (X, \mathcal{T}_Z)$ be topological spaces, let $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{Z}$ be continuous mapping.

Then $f \circ g$ is continuous.

Proof. By Definition 5.1.1, as g is continuous, for any $U_Z \in \mathcal{T}_Z$, $g^{-1}[U_Z] \in \mathcal{T}_Y$. Similarly, $f^{-1}[g^{-1}[U_Z]] \in \mathcal{T}_X$.

As $U_Z \in \mathcal{T}_Z$ is arbitrarily given, $f \circ g$ is continuous.

■

§5.2 Homeomorphisms

Definition 5.2.1. Let $\mathbb{X} = (X, \mathcal{T}_X)$ and $\mathbb{Y} = (X, \mathcal{T}_Y)$ be topological spaces, and let $f : \mathbb{X} \to \mathbb{Y}$ be a mapping.

f is a *homeomorphism* iff

H1. f is bijective (injective and surjective);

H2. f is continuous;

H3. f^{-1} is continuous;

Definition 5.2.2. Let $\mathbb{X} = (X, \mathcal{T}_X)$ and $\mathbb{Y} = (X, \mathcal{T}_Y)$ be topological spaces.

$\mathbb{X}$ and $\mathbb{Y}$ are said to be *homeomorphic*, denoted $\mathbb{X} \cong \mathbb{Y}$, iff there exists a homeomorphism between $\mathbb{X}$ and $\mathbb{Y}$.

Proposition 5.2.1. Being homeomorphic is an equivalent relation.

Proof. Let $\mathbb{X} = (X, \mathcal{T}_Y)$, $\mathbb{Y} = (X, \mathcal{T}_Y)$ and $\mathbb{Z} = (X, \mathcal{T}_Z)$ be topological spaces.

Let $\imath : \mathbb{X} \to \mathbb{X}$ be an identity mapping.

For any $x_1, x_2 \in X$ with $x_1 \neq x_2$, $\imath(x_1) = x_2$ and $\imath(x_2) = x_2$, so $\imath(x_1) \neq \imath(x_2)$. Thus $\imath$ is injective.

For any $x \in X$, there exists $\imath^{-1}(x) = x \in X$. Thus $\imath$ is surjective.

As $\imath$ is injective and surjective, it is bijective.

For any $U \in \mathcal{T}_X$, $\imath^{-1}[U] = U \in \mathcal{T}_X$. Thus, by Definition 5.1.1, $\imath$ is continuous. Similarly, $\imath^{-1}$ is continuous.

Therefore, by Definition 5.2.1, $\imath$ is an homeomorphism between $\mathbb{X}$ and $\mathbb{X}$. By Definition 5.2.2, $\mathbb{X}$ is homeomorphic to itself, i.e., $\mathbb{X} \cong \mathbb{X}$.

Thus, being homeomorphic is reflexive.

□

Assume $\mathbb{X} \cong \mathbb{Y}$.

By Definition 5.2.2, there exists a homeomorphism $f : \mathbb{X} \to \mathbb{Y}$.

As f is bijective, then f^{-1} is also bijective.

By Definition 5.2.1, f and f^{-1} are both continuous.

As f^{-1} is bijective, continuous, and $(f^{-1})^{-1} = f$ is also continuous, then $f^{-1} : \mathbb{Y} \to \mathbb{X}$ is also a homeomorphism. By Definition 5.2.2, we have $\mathbb{Y} \cong \mathbb{X}$.

Thus, being homeomorphic is symmetric.

□

Assume $\mathbb{X} \cong \mathbb{Y}$ and $\mathbb{Y} \cong \mathbb{Z}$.

By Definition 5.2.2, we have $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{Z}$ as homeomorphisms.

By Definition 5.2.1 H1, f and g are bijective. Thus, $f \circ g$ is bijective.

By Definition 5.2.1 H2, f and g are continuous, so, by Proposition 5.1.2, $f \circ g$ is continuous. Similarly, $g^{-1} \circ f^{-1}$ is continuous. As $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$ (see inverse of composite relation), $(f \circ g)^{-1}$ is also continuous.

As $f \circ g$ is bijective, $f \circ g$ is continuous and $(f \circ g)^{-1}$ is also continuous, $f \circ g : \mathbb{X} \to \mathbb{Z}$ is a homeomorphism. By Definition 5.2.2, $\mathbb{X} \cong \mathbb{Z}$.

Thus, being homeomorphic is transitive.

□

As being homeomorphic is reflexive, symmetric, and transitive, it is an equivalence relation.

■

Example 5.2.1. In Euclidean metric space $\mathbb{R}$, let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, then we have:

- $[a, b] \cong [c, d]$;
- $[a, b) \cong [c, d)$;
- $[a, b) \cong (c, d]$;
- $(a, b) \cong (c, d)$.

Example 5.2.2. A donut is homeomorphic to a cup, because they both have a hole.

Example 5.2.3. Consider $\mathbb{R}_{[0,1]}$ and $\mathbb{R}^n$ as Euclidean metric spaces. For any α, let $f_\alpha : I \to X$ be a continuous and piece-wise smooth injection.

Then, for any α, β, $f_\alpha[I] \cong f_\beta[I]$.

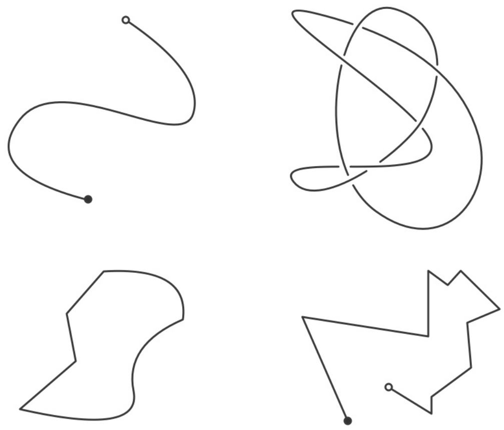

Figure 5.1: Homeomorphic curves in $\mathbb{R}^3$.

Appendices

Chapter A.

Some Examples of Topological Spaces

§A.1 Sorgenfrey line

1. Definition.
2. Sorgenfrey line is Lindelöf.
3. Sorgenfrey line is separable.
4. Sorgenfrey line is not second-countable.

§A.2 Niemytzki Plane