Notes for Vector Calculus

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Let $f : \mathbb{R}^m \to \mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^m$.

f is said to be *differentiable* at $\mathbf{p}$ iff for any $\mathbf{t} \in \mathbb{R}^m \setminus \{\mathbf{0}_{\mathbb{R}^m}\}$,

$$\lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}} \frac{f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \mathbf{0}_{\mathbb{R}^n}.$$

Lemma 2.1.1 (Alternative Definition of Differentiable mapping)**.**

With the condition of Definition 2.1.1, f is continuous at $\mathbf{p}$, iff

$$f(\mathbf{p}+\mathbf{t}) = f(\mathbf{p}) + \phi(\mathbf{t}) + o(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}.$$

Proof. By Theorem 1.1.1, the limit in Definition 2.1.1 is zero, iff there exists a neighbourhood N of $\mathbf{p}$, and an $\alpha : \mathbb{R}^m \to \mathbb{R}^n$ with $\alpha(\mathbf{t}) \to \mathbf{0}_{\mathbb{R}^n}$ at $\mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}$ such that for any $\mathbf{t} \in N \setminus \{\mathbf{p}\} - \{\mathbf{p}\}$,

$$\frac{f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \alpha(\mathbf{t})$$

Then,

$$\lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}} \frac{\|\mathbf{t}\|_{\mathbb{R}^m} \alpha(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \mathbf{0}_{\mathbb{R}^m}$$

$$\iff \|\mathbf{t}\|_{\mathbb{R}^m} \alpha(\mathbf{t}) = o(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}$$

Thus, we have

$$f(\mathbf{p}+\mathbf{t}) = f(\mathbf{p}) + \phi(\mathbf{t}) + o(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}.$$

■

Theorem 2.1.1. In Definition 2.1.1, ϕ is unique.

Proof. The equation in Definition 2.1.1 can be considered as: there exists a neighbourhood N of $\mathbf{p}$ and an $\alpha : \mathbb{R}^m \to \mathbb{R}^n$ with $\alpha(\mathbf{t}) \to \mathbf{0}_{\mathbb{R}^n}$ as $\mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}$, such that for any $\mathbf{t} \in \mathbb{R}^m \setminus \{\mathbf{0}_{\mathbb{R}^m}\}$ with $\mathbf{p}+\mathbf{t} \in N$,

$$f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t}) = \|\mathbf{t}\|_{\mathbb{R}^m}\alpha(\mathbf{t}).$$

Suppose there exists another linear mapping $\lambda : \mathbb{R}^m \to \mathbb{R}^n$, such that there exists a neighbourhood N' of $\mathbf{p}$ and a $\beta : \mathbb{R}^m \to \mathbb{R}^n$ with $\beta(\mathbf{t}) \to \mathbf{0}_{\mathbb{R}^n}$ as $\mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}$, such that for any $\mathbf{t} \in \mathbb{R}^m \setminus \{\mathbf{0}_{\mathbb{R}^m}\}$ with $\mathbf{p}+\mathbf{t} \in N'$,

$$f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \lambda(\mathbf{t}) = \|\mathbf{t}\|_{\mathbb{R}^m}\beta(\mathbf{t}).$$

Then, we have

$$\begin{aligned} &\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^m}} \frac{\phi(\mathbf{t}) - \lambda(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^m}} (\beta(\mathbf{t}) - \alpha(\mathbf{t})) \\ \iff\ &\phi(\hat{\mathbf{t}}) - \lambda(\hat{\mathbf{t}}) = \mathbf{0}_{\mathbb{R}^m}. \end{aligned}$$

As $\mathbf{t}$ is arbitrarily taken from $N \cap N' - \mathbf{p}$, and there must be an open subset $U \subseteq N \cap N'$, thus,

$$\left\{ \hat{\mathbf{t}} = \frac{\mathbf{t}}{\|\mathbf{t}\|_{\mathbb{R}^m}} : \mathbf{t} \in N \cap N' - \mathbf{p} \right\}$$

contains all possible direction in $\mathbb{R}^m$. Thus as ϕ and λ are linear, $\phi(\hat{\mathbf{t}}) = \lambda(\hat{\mathbf{t}})$ iff $\phi = \lambda$. ■

Theorem 2.1.2. With the condition in Definition 2.1.1, if f is differentiable at $\mathbf{p}$, then f is continuous at $\mathbf{p}$.

Proof. As f is differentiable at $\mathbf{p}$, ■

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