Notes for General Topology

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Chapter 1

Metric Spaces

Chapter 2

Topological Spaces

2.1 Topological Spaces

Definition 2.1.1 (topology)**.** Let X be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is called a topology on X iff

(i) $\emptyset, X \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under arbitrary union;

(iii) $\mathcal{T}$ is closed under finite intersection.

Definition 2.1.2 (topological spaces)**.** Let X be any set, and let $\mathcal{T}$ be a topology on X, then the pair $(X, \mathcal{T})$ is called a *topological space*. All subsets of X in $\mathcal{T}$ are called *open sets* in $(X, \mathcal{T})$.

Definition 2.1.3 (closed sets)**.** Let $(X, \mathcal{T})$ be a topological space. A subset V of X is said to be *closed* iff there is an open set U in X such that

$$V = X \setminus V.$$

Proposition 2.1.1. Let X be a set, and let $\mathcal{C}$ be the family of all closed sets in X. Then

(i) $\emptyset, X \in \mathcal{C}$;

(ii) $\mathcal{C}$ is closed under arbitrary intersection;

(iii) $\mathcal{C}$ is closed under finite union.

Definition 2.1.4 (finer and coarser topology)**.** Let X be any set, and let $\mathcal{T}, \mathcal{T}'$ be topologies on X. $\mathcal{T}$ is said to be *finer* than $\mathcal{T}'$ iff $\mathcal{T} \supseteq \mathcal{T}'$; respectively, $\mathcal{T}$ is said to be *coarser* than $\mathcal{T}'$ iff $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 2.1.5 (neighbourhood)**.** Given $(X, \mathcal{T})$ as a topological space and a point $x \in X$, a subset $N \subseteq X$ is called a *neighbourhood* iff it contains an open set U containing x.

Proposition 2.1.2. Given $(X, \mathcal{T})$ as a topological space and $U \subseteq X$, U is open iff for all $x \in U$, there is a neighbourhood N of x contained in U.

Proof. If U is open, then U itself is a neighbourhood of x contained in U.

Conversely, if for all $x \in U$, there is a neighbourhood N_x of x contained in U, then there is a open neighbourhood $U_x \ni x$ contained in N_x. Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose U is not open, then U is a proper superset in the relation above. Then there exists $y \in U$ which is not in any U_x. This implies that such a y does not have any neighbourhood N_y in U, for such an N_y must contains an open $U_y \ni y$. For if it does, then there must be a U_x contains y. This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open. □

2.2 Metrizable Spaces

2.3 Continuity

Definition 2.3.1 (continuous maps)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is said to be *continuous* iff for any open set U in Y, its preimage in X under f is open.

Proposition 2.3.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is continuous at $x \in X$ iff for any neighbourhood N_y of $f(x)$, there is a neighbourhood N_x of x, such that $f[N_x] \subseteq N_y$.

Proof. Let N_y be a neighbourhood of $f(x)$. Clearly, there exists an open set U_y contains y.

By Definition 2.3.1, f is continuous at x iff $x \in f^{-1}[U_y] \in \mathcal{T}_X$. Clearly, $f^{-1}[U_y]$ is a neighbourhood of x. We have $f[f^{-1}[U_y]] = U_y \subseteq N_y$.

By Proposition 2.1.2, there U_x must contains at least one neighbourhood N_x of x, thus, $f[N_x] \subseteq U_y$. □

Proposition 2.3.2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be metrizable spaces. A map $f : X \to Y$ is continuous at $p \in X$ iff for any $\varepsilon > 0$, there is a $\delta > 0$, such that for all $x \in B_X(p, \delta)$, $f(x) \in B_Y(f(p), \varepsilon)$, where B_X is defined by any metrics ρ_X induces $\mathcal{T}_X$, and B_Y is defined by any metrics ρ_Y induces $\mathcal{T}_Y$.

Proof. Clearly, for all $\varepsilon > 0$, $B_Y(f(x,), \varepsilon)$ is an open neighbourhood of $f(x)$.

f is not necessarily be injective, so $f^{-1}[B_Y(f(x), \varepsilon)] = U \in x$. By Definition 2.3.1, U is open, so for some $\delta > 0$, $B_X(x, \delta) \subseteq U$. Thus, By Proposition 2.3.1, f is continuous iff $f[B_X(x, \delta)] \subseteq B_Y(f(x), \varepsilon)$. This satisfies the conditions we have. □

Proposition 2.3.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is said to be continuous iff for any closed set V in Y, its preimage in X under f is closed.

Proof. Let U_Y be any open set in Y, let U_X be the preimage of U_Y under f. By Definition 2.3.1, U_X is open in X. Let

$$V_X = f^{-1}[Y \setminus U_Y] = X \setminus U_X,$$

Then V_X is closed. □

2.4 Cover

Definition 2.4.1 (cover)**.** Let $(X, \mathcal{T})$ be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of U iff the union of all sets in $\mathcal{C}$ is a superset of U. That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call $\mathcal{C}$ an *open cover* of U.

Let $\mathcal{S} \subseteq \mathcal{C}$, iff the union of $\mathcal{S}$ is still a superset of U, then we call $\mathcal{S}$ a *subcover* of $\mathcal{C}$.

Definition 2.4.2 (basis)**.** Let $(X, \mathcal{T})$ be a topological space, let $U \subseteq X$, and let $\mathcal{B}$ be a open cover of X. We call $\mathcal{B}$ a *base* of X iff the union of $\mathcal{B}$ is precisely U itself, i.e.,

$$U = \bigcup \mathcal{B}.$$

Definition 2.4.3 (synthetic basis)**.** Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a base of X. $\mathcal{B}$ is said to be *synthetic* iff for any $A, B \in \mathcal{B}$,

$$A \cap B = \bigcup_{i=1}^{n} B_i, \quad B_i \in \mathcal{B}.$$

2.5 Untitled

Definition 2.5.1 (subspace topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The *subspace topology* $\mathcal{T}_A$ on A is defined to be the family of the intersections of open sets in $(X, \mathcal{T})$ and A. That is,

$$\mathcal{T}_A = \{U \cap A : \ U \in \mathcal{T}\}.$$

Definition 2.5.2 (quotient topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

Definition 2.5.3 (homeomorphisms)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

Definition 2.5.4 (homeomorphic)**.** Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

Definition 2.5.5 (compactness)**.** A topological space $(X, \mathcal{T})$ is said to be *compact* iff every open cover of X has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

Definition 2.5.6 (connectedness)**.** Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is said to be *connected* iff X is not empty and it it not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

Definition 2.5.7 (path-connectedness)**.** Let $(X, \mathcal{T})$ be a topological space.

(i) A map $\gamma : [0, 1] \to X$ is called a *path* in X iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that γ is path from x to y in X.

(ii) X is said to be *path-connected* iff for all $x, y \in X$ there is a path from x to y in X.

Definition 2.5.8 (topologically indistinguishable)**.** Let $(X, \mathcal{T})$ be a topological space. Two points $x, y \in X$ are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let $\mathcal{N}_x$ be the family of all neighbourhoods of x and let $\mathcal{N}_y$ be the family of all neibourhoods of y, we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively, x, y are said to be *topologically distinguishable* iff they are not topologically distinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

Definition 2.5.9 (saperated sets)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \in \mathcal{P}(X)$.

(i) A and B are said to be *separated* iff each is disjoint from other's closure.

(ii) A and B are said to be *separated by neighbourhoods* iff there are neighbourhoods N_A of A and N_B of B such that N_A and N_B are disjoint.

(iii) A and B are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods $\overline{N}_A$ of A and $\overline{N}_B$ of B such that $\overline{N}_A$ and $\overline{N}_B$ are disjoint.

(iv) A and B are said to be *separated by a continuous function* iff there is a continuous function $f : X \to \mathbb{R}$, such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

(v) A and B are said to be *precisely separated by a continuous function* iff there is a continuous function $f : X \to \mathbb{R}$, such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$

See Wikipedia.org

Definition 2.5.10 (T_0 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_0 or *Kolmogorov*, iff all distinct points $x, y \in X$ are *topologically distinguishable.*

Definition 2.5.11 (R_0 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be R_0 iff any two topologically distinguishable points in X are separated.

Definition 2.5.12 (T_1 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_1 or *Fréchet* iff any two distinct points in X are separated.

Proposition 2.5.1. All singletons in a T_1 space are closed, That is, if a topological space $(X, \mathcal{T})$ is T_1, then

$$\forall x \in (X, \mathcal{T}) : \exists U \in \mathcal{T} : \{x\} = X \setminus U.$$

Definition 2.5.13 (T_2 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_2 or *Hausdorff* or *separated* iff any two distinct points in $(X, \mathcal{T})$ are separated by neighbourhoods.

Definition 2.5.14 ($T_{2^1/_2}$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_{2^1/_2}$ or *Urysohn* iff two distinct points in X are separated by closed neighbourhoods.

Definition 2.5.15 (T_3 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_3 or *regular* iff it is T_0 and given any point $x \in (X, \mathcal{T})$ and closed set $V \subseteq X$ with $x \notin V$ are separated by neighbourhoods.

Definition 2.5.16 ($T_{3^1/_2}$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_{3^1/_2}$, or *Tychonoff* or, *completely* T_3, or *completely regular*, iff it is T_0 and given any point x and closed set $V \subseteq X$ with $x \notin V$, they are separated by a continuous function.

Definition 2.5.17 (T_4 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_4 or *normal* iff it is Hausdorff and any tow disjoint closed subsets of X are separated by neighbourhoods.

Proposition 2.5.2 (Urysohn's lemma)**.** A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

Definition 2.5.18 (T_5 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_5 or *completely* T_4 iff it is T_1 any two separated sets are separated by neighbourhoods.

Proposition 2.5.3. Every subspace of a T_5 space is normal.

Definition 2.5.19 (T_6 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_6, or *perfectly* T_4 or *perfectly normal* iff it is T_1 and any two disjoint closed sets are precisely separated by a continuous function.

Proposition 2.5.4 (Tietze extension theorem)**.** Let $(X, \mathcal{T})$ be normal topological space, and let $f : A \to (\mathbb{R}, \mathcal{T}')$ be a continuous map where A is a closed subset of X and $\mathcal{T}'$ is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$

2.6 Boundaries

Definition 2.6.1 (interiors)**.** The *interior* of a set A, denoted A°, is defined to be the union of all open subsets of A.

Definition 2.6.2 (closure)**.** The *closure* of a set A, denoted $\overline{A}$, is defined to be the intersection of all closed supersets of A.

Definition 2.6.3 (dense sets)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. A is said to be dense, iff $\overline{A} = X$.

Definition 2.6.4 (nowhere dense sets)**.** A set A is said to be *nowhere dense* iff the interior of its closure is empty.

Definition 2.6.5 (boundaries)**.** Let A be any set, the *boundary* of A, denoted ∂A, is defined to be the complement of the interior of A in the closure of A; i.e.,

$$\partial A = \overline{A} \setminus A^\circ.$$

Proposition 2.6.1 (properties of interiors)**.** Let $(X, \mathcal{T})$ be any topological space and $A, B \subseteq X$.

(i) (Intensive) $A^\circ \subseteq A$.

(ii) A is open iff $A = A^\circ$.

(iii) (Idempotence) $(A^\circ)^\circ = A^\circ$.

(iv) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(v) $A \subseteq B \implies A^\circ \subseteq B^\circ$.

(vi) If B is open, then $B \subseteq A$ iff $B \subseteq A^\circ$.

Proof.

(i) By Definition 2.6.1, naturally, $A^\circ \subseteq A$.

(ii) By Definition 2.1.2, A° is the union of open sets hence it is open. A is open iff it is the union of all open subsets of A. Thus $A = A^\circ$.

(iii) A° is open, thus $(A^\circ)^\circ = A^\circ$.

(iv) By Definition 2.6.1, we have

$$\begin{aligned}(A \cap B)^\circ &= \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq A \cap B\right\} \\ &= \left\{\bigcup U : (U \in \mathcal{T} \wedge U \subseteq A) \wedge (U \in \mathcal{T} \wedge U \subseteq B)\right\} \\ &= \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq A\right\} \cap \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq B\right\} \\ &= A^\circ \cap B^\circ.\end{aligned}$$

(v) Clearly, $A^\circ \subseteq A$, thus,

$$A \subseteq B \implies A^\circ \subseteq B$$

Suppose $A^\circ \not\subseteq B^\circ$, then $A^\circ \setminus B^\circ$ is not empty ($\emptyset$ is the subset of any set, so A° is not empty).

Then there exists $x \in A^\circ$ with $x \in \partial B$ ($x \in B$ but $x \notin B^\circ$). Then there exists neighbourhood $N_x \ni x$, and $N_x \cap \partial B \neq \emptyset$. But this is impossible, for $A^\circ \subseteq B$ implies that $A^\circ \cap \partial B = \emptyset$ (This is a straight consequence of $A^\circ \cap \partial A = \emptyset$. See Proposition 2.6.3), so such N_x does not exist. Thus,

$$A^\circ \subseteq B^\circ.$$

(vi) If B is open, then $B = B^\circ$. Then $B \subseteq A$ iff $B^\circ \subseteq A^\circ$.

□

Proposition 2.6.2 (properties of closures)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

(i) $\overline{A}$ is closed.

(ii) A is closed iff $A = \overline{A}$.

(iii) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

(iv) If A is closed, then $A \supseteq B$ iff $A \supseteq \overline{B}$

Proof.

(i) By Definition 2.6.2, $\overline{A}$ is the intersection of closed sets. By Proposition 2.1.1, $\overline{A}$ is closed.

(ii) Proposition 2.1.1 implies that any closed set is the intersection of closed sets, this is precisely what Definition 2.6.2 says.

(iii) $A \subseteq B$ iff $X \setminus A \supseteq X \setminus B$. Then we have

$$X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ$$

Clearly, $(X \setminus A)^\circ$ is the union of all open set disjoint from A, then, by De Morgan's laws, $X \setminus (X \setminus A)^\circ$ is the intersection of all closed sets containing A. By Definition 2.6.2, we have $(X \setminus A)^\circ = \overline{A}$. Thus

$$\overline{A} \subseteq \overline{B}.$$

(iv) If A is closed, then $A = \overline{A}$. Suppose $B \subseteq A$, then we have

$$\overline{B} \subseteq \overline{A} \iff \overline{B} \subseteq A.$$

$\square$

Proposition 2.6.3 (properties of boundaries)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

(i) ∂A is closed.

(ii) $A^\circ \cap \partial A = \emptyset$.

(iii) $\overline{A} = A^\circ \cup \partial A$.

(iv) A is closed iff $\partial A \subseteq A$.

(v) ∂A is nowhere dense.

(vi) $\partial \overline{A} \subseteq \partial A \subseteq \partial A^\circ$.

(vii) $\partial A = \partial(X \setminus A)$.

(viii) A is dense iff $\partial A = X \setminus A^\circ$.

Proof.

(i) $\overline{A}$ is closed, and $X \setminus A^\circ$ is also closed. Thus

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (X \setminus A)$$

is closed.

(ii) By Definition 2.6.5, we have

$$\partial A = \overline{A} \setminus A^\circ \iff \partial A \cap A^\circ = \overline{A} \setminus A^\circ \cap A^\circ = \overline{A} \cap \emptyset = \emptyset.$$

(iii) We have

$$\begin{aligned} \partial A = \overline{A} \setminus A^\circ &\iff \partial A \cup A^\circ = \overline{A} \setminus A^\circ \cup A^\circ = \overline{A} \cap (X \setminus A^\circ \cup A^\circ) \\ &\iff \partial A \cup A^\circ = \overline{A} \cap X|_{\text{for } A^\circ \subseteq X} = \overline{A}. \end{aligned}$$

(iv) As A is closed, $A = \overline{A}$ (this can be straightly proved by Definition 2.6.2). By Definition 2.6.5, it is clear that $\partial A \subseteq \overline{A}$, thus $\partial A \subseteq A$.

(v) By Definition 2.6.4, ∂A is nowhere dense iff $\overline{\partial A}^\circ$ is empty. We have

$$\begin{aligned} \overline{\partial A}^\circ &= \overline{\overline{A} \setminus A^\circ}^\circ \\ &= (\overline{A} \setminus A^\circ) \cup (\overline{A} \setminus A^\circ) \setminus (\overline{A} \setminus A^\circ) \\ &= \emptyset. \end{aligned}$$

(vi) $\overline{A} \supseteq A^\circ$ implies $\overline{A}^\circ \supseteq (A^\circ)^\circ = A^\circ$, then we have,

$$\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^\circ \subseteq \overline{A} \setminus A^\circ = \partial A.$$

$A^\circ \subseteq A$ implies $\overline{A^\circ} \subseteq \overline{A}$, then we have,

$$\partial A^\circ = \overline{A^\circ} \setminus (A^\circ)^\circ \supseteq \overline{A} \setminus A^\circ.$$

(vii) We have

$$\begin{aligned} \partial(X \setminus A) &= \overline{X \setminus A} \setminus (X \setminus A)^\circ \\ &= X \setminus A^\circ \setminus (X \setminus \overline{A}) \\ &= X \setminus A^\circ \cap \overline{A} \\ &= \overline{A} \setminus A^\circ \\ &= \partial A. \end{aligned}$$

(viii) By Definition 2.6.3, A is dense in X iff $\overline{A} = X$. Then we have,

$$\begin{aligned}\overline{A} = X &\iff \overline{A} \setminus A^\circ = X \setminus A^\circ \\ &\iff \partial A = X \setminus A^\circ.\end{aligned}$$

□

2.7 Limit Points

Definition 2.7.1 (limit points)**.** Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$. A point $x \in X$ is called a *limit point* of A iff for all neighbourhood N_x of x, $N_x \setminus \{x\}$ intersects A.

Proposition 2.7.1. Let A be any set, and let x be a limit point of A, then x is an element of the closure of A.

Proof. If A is empty, then this is vacuously true. So, suppose A is not empty.

By Definition 2.7.1, for all neighbourhood N_x of x, $N_x \setminus \{x\} \cap A$ is not empty. Naturally, $N_x \cap A$ is not empty.

Assume that $x \notin \overline{A}$, then $X \setminus \overline{A}$ is a neighbourhood of x, by Definition 2.1.5, and is disjoint from A. This is contradicted to the conditions. □

Note 2.7.1. In this proof, the proposition also holds for $N_x \cap A^\circ = \emptyset$. Because if it is true, then

$$N_x \cap \partial A \supseteq (N_x \cap A) \setminus (N_x \cap A^\circ) = N_x \cap A.$$

This implies that $A \subseteq \partial A$. In this case, $\overline{A} = \partial A$, for

Assume that $x \notin \partial A$, then we have the same conclusion.

Then $A^\circ = A \setminus \partial A = \emptyset$.

Proposition 2.7.2. A set is closed iff it contains all its limit point.

Proof. Let A be a set. By proposition 2.7.1, for every limit point of A, it is also an element of the closure $\overline{A}$. And A is closed iff $A = \overline{A}$. □

Definition 2.7.2 (convergent sequences)**.** Let $(X, \mathcal{T}_X)$ be a topological space. A sequence $\{x_n\}$ in X is said to be *convergence* in X iff there is an open set U contains all but finite terms of $\{x_n\}$.