Notes for Vector Calculus

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Chapter 1.

Vector Spaces

§1.1 Linear Maps

Definition 1.1.1. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *linear*, iff for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $a, b \in \mathbb{R}$,

$$f(a\mathbf{u} + b\mathbf{v}) = af(\mathbf{u}) + bf(\mathbf{v}).$$

Note 1.1.1. Assume $b = 0$, then we have

$$f(a\mathbf{u}) = af(\mathbf{u}).$$

Assume $a = b = 1$, then Definition 1.1.1 gives

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}).$$

§1.2 Linear Spans and Combinations

Definition 1.2.1. Let $(\mathbf{v}_i)_{i=1}^n$ be a tuple of vectors in $\mathbb{R}^m$.

The *span* of $(\mathbf{v}_i)$ is a subset of $\mathbb{R}^m$ defined as

$$\operatorname{span}((\mathbf{v}_i)) := \{\mathbf{a} \cdot (\mathbf{v}_i) : \mathbf{a} \in \mathbb{R}^n\}.$$

An element $\mathbf{u} \in \mathbb{R}^m$ is a *linear combination* of $(\mathbf{v}_i)$ iff $\mathbf{u} \in \operatorname{span}((\mathbf{v}_i))$.

Note 1.2.1. By dot product,

$$\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i.$$

Note 1.2.2. For any vector $\mathbf{p} \in \mathbb{R}^m$, the linear combination form of $\mathbf{p}$ is

$$\mathbf{p} = \sum_{i=1}^{n} p_i \hat{e}_i,$$

where $\hat{e}_i$ denotes the i-th basis of $\mathbb{R}^n$; i.e., all terms but the i-th term of $\hat{e}_i$ are 0.

§1.3 Linear Dependency

Definition 1.3.1. A tuple $(\mathbf{v}_i)_{i=1}^n$ of vectors in $\mathbb{R}^m$ is said to be *linearly independent* iff for any $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\mathbf{a} \cdot (\mathbf{v}_i) \neq \mathbf{0}.$$

$(\mathbf{v}_i)$ is *linearly dependent* iff it is not linearly independent.

Lemma 1.3.1. Let $(\mathbf{v}_i)_{i=1}^n$ be a tuple of vectors in $\mathbb{R}^n$ (notice the n here).

Then $(\mathbf{v}_i)$ is linearly independent iff

$$\operatorname{span}((\mathbf{v}_i)) = \mathbb{R}^n.$$