

Notes for General Topology

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Chapter 1.

Topological Spaces

§1.1 Basic Definitions

Definition 1.1.1. Let X be any set, and let $\mathcal{T} \subseteq 2^X$, where 2^X denotes the power set of X .

Then \mathcal{T} is called a **topology on X** iff it satisfies the **open set axioms**. That is,

O1. $\emptyset, X \in \mathcal{T}$

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; i.e., \mathcal{T} is closed under arbitrary union.

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; i.e., \mathcal{T} is closed under finite intersection.

The ordered pair $\mathbb{X} = (X, \mathcal{T})$ is called a **topological space**.

A subset $U \subseteq X$ is said to be **open** iff it is an element of \mathcal{T} .

Note 1.1.1. Rigorously, $\emptyset \in \mathcal{T}$ is not necessary for O1 in Definition 1.1.1, because it can be proved in a simple way.

As empty set is an element of any set, it is also an element of \mathcal{T} . Therefore,

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

The most intuitive example of topological space is no doubt the **Euclidean topological space**, it is a topological space $\mathbb{X} = (X, \mathcal{T})$ with X is the cartesian product of a sequence of sets $(X_i)_{i=1}^n$ and the **Euclidean topology** \mathcal{T} on X . That is, for any U open in \mathbb{X} (i.e., $U \in \mathcal{T}$), for any $A \subseteq U$ and for any $x \in A$, there exists $\varepsilon_x \in \mathbb{R}_{>0}$, such that U can be represented as the union of all ε_x -balls around x ; i.e.,

$$U = \bigcup_{x \in A} B_d(x, \varepsilon_x),$$

where d is the **Euclidean metric** on X ; i.e., $d : X \times X \rightarrow \mathbb{R}_{>0}$ is a function defined by the Pythagoras theorem,

$$d(x, y) := \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}},$$

where a_i denotes the i -th projection of $a \in X$ on X .

In this case, the topology \mathcal{T} is said to be induce by metric d . As a general consequence, any metric space can induce a unique topology. This will be proved later.

In Euclidean spaces, the idea of “open” represents intuitively, but it doesn’t mean that every topological space should be induce in such a natural way. Here is an easy example.

Example 1.1.1. Let $X = \{1, 2, 3, 4\}$ and let \mathcal{T} be the smallest topology on X containing $\{1, 2\}$ and $\{2, 3\}$, i.e., for topology \mathcal{T}' on X containing these two sets is a superset of \mathcal{T} .

By Open Set Axiom O1, $\emptyset, X \in \mathcal{T}$.

By Open Set Axiom O2, $\{1, 2, 3\} = \{1, 2\} \cup \{2, 3\} \in \mathcal{T}$.

By Open Set Axiom O3, $\{2\} = \{1, 2\} \cap \{2, 3\} \in \mathcal{T}$.

Therefore

$$\mathcal{T} = \{\emptyset, X, \{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{2\}\}.$$

Definition 1.1.2. Let X be any set and let \mathcal{T}_1 and \mathcal{T}_2 be topologies on X .

Then \mathcal{T}_1 is said to be **coarser** than \mathcal{T}_2 , or \mathcal{T}_2 is said to be **finer** than \mathcal{T}_1 , iff $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Example 1.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

Then \mathbb{X} is a **discrete topological space**, namely, \mathcal{T} is a **discrete topology** on X iff $\mathcal{T} = 2^X$.

It is the finest topology on X .

Example 1.1.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

Then \mathbb{X} is a **indiscrete topological space**, namely, \mathcal{T} is a **indiscrete topology** on X iff $\mathcal{T} = \{\emptyset, X\}$.

It is the coarsest topology on X .

§1.2 Interiors

Definition 1.2.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in A$ is said to be **interior to** A iff there is a $U \in \mathcal{T}$ with $x \in U$, such that $U \subseteq A$.

The **interior of** A , denoted $\text{Int}_{\mathcal{T}}(A)$, is defined as the set of all interior points of A .

Sometime, we write A° for $\text{Int}_{\mathcal{T}}(A)$, if the confusion of topology is unlikely in the context.

Proposition 1.2.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then we have

$$A^\circ = \bigcup (\mathcal{T} \cap 2^A).$$

Proof. Let $\mathcal{U} = \mathcal{T} \cap 2^A$.

$$\begin{aligned} x \in \bigcup (\mathcal{T} \cap 2^A) &\iff x \in \bigcup_{U \in \mathcal{U}} U \\ &\iff (\exists U \in \mathcal{U}) x \in U. \end{aligned}$$

By assumption, $\mathcal{U} \subseteq \mathcal{T}$, thus for any $U \in \mathcal{U}$, $U \in \mathcal{T}$. Also, $\mathcal{U} \subseteq 2^A$ implies that for any $U \in \mathcal{U}$, $U \subseteq A$.

Now, we have $x \in \bigcup(\mathcal{T} \cap 2^A)$ iff

$$(\exists U \in \mathcal{T} \mid U \subseteq A) \quad x \in U.$$

By the definition of existential quantifier and the associativity of logical conjunction, we have

$$\begin{aligned} (U \in \mathcal{T} \wedge U \subseteq A) \wedge x \in U &\iff (U \in \mathcal{T} \wedge x \in U) \wedge U \subseteq A \\ &\iff (\exists U \in \mathcal{T} \mid x \in U) \wedge U \subseteq A. \end{aligned}$$

This is precisely the statement of Definition 1.2.1 ■

Proposition 1.2.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

Then $A^\circ \subseteq B^\circ$ if $A \subseteq B$.

Proof. As $A \subseteq B$, we have

$$\begin{aligned} 2^A \subseteq 2^B &\implies \mathcal{T} \cap 2^A \subseteq \mathcal{T} \cap 2^B \\ &\implies \bigcup(\mathcal{T} \cap 2^A) \subseteq \bigcup(\mathcal{T} \cap 2^B) \end{aligned}$$

By Definition 1.2.1, $A^\circ \subseteq B^\circ$. ■

Note that $A^\circ \subseteq B^\circ$ does not imply $A \subseteq B$. For example, let $\mathbb{X} = (X, \mathcal{T})$ with $X = \{1, 2\}$ and $\mathcal{T} = \{\emptyset, X, \{2\}\}$, and let

$$A = \{1\}, \quad B = \{2\}.$$

Then, $A^\circ = \emptyset$ and $B^\circ = \{2\}$. In this case, $A^\circ \subseteq B^\circ$, but $A \not\subseteq B$.

Proposition 1.2.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then $A \in \mathcal{T}$ iff $A = A^\circ$.

Proof. Assume $A \in \mathcal{T}$.

As $A \in 2^A$ also, then $A \in \mathcal{T} \cap 2^A$. In the term of family, we have

$$\begin{aligned} \{A\} \subseteq \mathcal{T} \cap 2^A &\implies \bigcup \{A\} \subseteq \bigcup (\mathcal{T} \cap 2^A) \\ &\implies A \subseteq A^\circ. \end{aligned}$$

By Definition 1.2.1, $A^\circ \subseteq A$ is clear, therefore $A = A^\circ$. □

Conversely, Assume $A = A^\circ$.

By Proposition 1.2.1 that is

$$A = \bigcup (\mathcal{T} \cap 2^A).$$

Clearly, $\mathcal{T} \cap 2^A \subseteq \mathcal{T}$, thus, by O2, Definition 1.1.1, $A \in \mathcal{T}$. ■

Proposition 1.2.4. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $\mathcal{A} \subseteq 2^X$.

Then we have

$$\left(\bigcap \mathcal{A} \right)^\circ \subseteq \bigcap_{A \in \mathcal{A}} A^\circ.$$

Proof. By Definition 1.2.1, for any $x \in \left(\bigcap \mathcal{A} \right)^\circ$, there exists $U \in \mathcal{T}$ with $x \in U$ such that $U \subseteq \bigcap \mathcal{A}$. Thus, for any $A \in \mathcal{A}$, $U \subseteq A$.

As $U \in \mathcal{T}$ and $U \subseteq A$, $U \subseteq A^\circ$.

As $x \in U$ and $U \subseteq A^\circ$ for any $A \in \mathcal{A}$, we have

$$x \in \bigcap_{A \in \mathcal{A}} A^\circ.$$

■

Note that the relation \subseteq in Proposition 1.2.4 cannot be reversed.