Notes for Vector Calculus

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Contents

Chapter 1.

Differentiation

§1.1 Infintesimal

Definition 1.1.1. Let $f, g : \mathbb{R}^m \to \mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^m$.

Then f is a *little-o* of g as $\mathbf{x} \to \mathbf{p}$, i.e.,

$$f(\mathbf{x}) = o(g(\mathbf{x})) \text{ as } \mathbf{x} \to \mathbf{p},$$

iff for any $\varepsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood of U of $\mathbf{p}$ such that for any $\mathbf{x} \in U$, $\|f(\mathbf{x})\| \leq \varepsilon \|g(\mathbf{x})\|$. Equivalently, that is,

$$\lim_{\mathbf{x} \to \mathbf{p}} \frac{f(\mathbf{x})}{\|g(\mathbf{x})\|_{\mathbb{R}^n}} = \mathbf{0}_{\mathbb{R}^n}.$$

Note 1.1.1. In the case that $f(\mathbf{x}) = o(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{0}_{\mathbb{R}^m}$, I will simply write $f(\mathbf{x}) = o(g(\mathbf{x}))$.

Lemma 1.1.1.

$$o(f(\mathbf{x})) + o(g(\mathbf{x})) = o(\|f(\mathbf{x})\|_{\mathbb{R}^n} + \|g(\mathbf{x})\|_{\mathbb{R}^n}).$$

Proof. By Definition 1.1.1, for any $\varepsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood of U of $\mathbf{p}$ such that for any $\mathbf{x} \in U$,

$$\|o(f(\mathbf{x}))\|_{\mathbb{R}^n} \leq \varepsilon \|f(\mathbf{x})\|.$$

Then, there exists some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that

$$o(f(\mathbf{x})) = \varepsilon \|f(\mathbf{x})\| \mathbf{u} \text{ and } o(g(\mathbf{x})) = \varepsilon \|g(\mathbf{x})\| \mathbf{v}.$$

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By Definition 1.1.1, now we have

$$o(f(\mathbf{x})) + o(g(\mathbf{x}))$$

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§1.2 Differentiable Mapping

Definition 1.2.1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^m$.

Then, f is said to be *differentiable* at $\mathbf{p}$, iff there exists a linear map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ and an open subset $U \subseteq \mathbb{R}^m$, such that for any $\mathbf{h} \in \mathbb{R}^m \setminus \{\mathbf{0}_{\mathbb{R}^m}\}$ with $\mathbf{p} + \mathbf{h} \in U$,

$$f(\mathbf{p} + \mathbf{h}) = f(\mathbf{p}) + \phi(\mathbf{h}) + o(\phi(\mathbf{h})).$$

Lemma 1.2.1. The relation in Definition 1.2.1 holds for a unique ϕ.

Proof. Aiming for a contradiction, suppose there is another linear map $\lambda : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$f(\mathbf{p} + \mathbf{h}) = f(\mathbf{p}) + \lambda(\mathbf{h}) + o(\lambda(\mathbf{h})),$$

then we have

$$\phi(\mathbf{h}) - \lambda(\mathbf{h}) = o(\phi(\mathbf{h})) - o(\lambda(\mathbf{h})).$$

Let $g : \mathbb{R} \to \mathbb{R}^n$ be defined as $g(t) := \phi(t\mathbf{u})$, then ■

§1.3 Derivatives

Definition 1.3.1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$, let $\mathbf{u} \in \mathbb{R}^m \setminus \{\mathbf{0}_{\mathbb{R}^m}\}$, and let $\mathbf{p} \in \mathbb{R}^m$.

The *directional derivative* of f along $\mathbf{u}$ at $\mathbf{p}$ is defined as

$$\nabla_{\mathbf{u}} f(\mathbf{p}) := \lim_{h \to 0} \frac{f(\mathbf{p} + h\mathbf{u}) - f(\mathbf{p})}{h},$$

if the limit exists in $\mathbb{R}^n$.

Lemma 1.3.1. With the conditions in Definition 1.3.1, if $\nabla_{\mathbf{u}} f(\mathbf{p})$ exists at $\mathbf{p}$, then there exists open subset $U \subseteq \mathbb{R}^m$ such that f is relative continuous on $U \cap \{\mathbf{p} + h\mathbf{u} : h \in \mathbb{R}\}$ for some U.

Proof. ∎

Lemma 1.3.2. With the conditions in Definition 1.3.1, let $s \in \mathbb{R} \setminus \{0\}$, then

$$\nabla_{s\mathbf{u}} f(\mathbf{p}) = s \nabla_{\mathbf{u}} f(\mathbf{p})$$

if $\nabla_{\mathbf{u}} f(\mathbf{p})$ exists in $\mathbb{R}^n$.

Proof. Let $g : \mathbb{R} \to \mathbb{R}^n$ be defined as

$$g(h) := f(\mathbf{p} + h\mathbf{u}).$$

Then, we have

$$\begin{aligned} \nabla_{s\mathbf{u}} f(\mathbf{p}) &= \lim_{h \to 0} \frac{f(\mathbf{p} + hs\mathbf{u}) - f(\mathbf{p})}{h} \\ &= \lim_{h \to 0} \frac{g(sh) - g(0)}{h}. \end{aligned}$$

As this is a 0/0 limit, thus, by L'Hópital's rule, we have

$$\nabla_{s\mathbf{u}} f(\mathbf{p}) = \lim_{h \to 0} \frac{\mathrm{d}g(hs)}{\mathrm{d}h}.$$

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