

Notes for General Topology

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Chapter 1

Topological Spaces

1.1 Metric Spaces

Definition 1.1.1. Let X be any set. A mapping $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is *metric on X* if and only if it satisfies the *metric axioms*. That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ if and only if $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

In this case, the pair $M = (X, d)$ is called a *metric space*.

Definition 1.1.2. A $M = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$. An *open ε -ball*, or just ε -ball, about x is defined to be the set

$$B_\varepsilon(x; d) = \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) = \{y \in X : d(x, y) \leq \varepsilon\}.$$

Note 1.1.1. As

$$M = (X, d), \quad M' = (X, d'), \quad M'' = (X, d''), \quad \dots$$

are different although they share the same set X , for any $x \in X$ and any $\varepsilon \in \mathbb{R}_{>0}$,

$$B_\varepsilon(x; d), \quad B_\varepsilon(x; d'), \quad B_\varepsilon(x; d''), \quad \dots$$

are also different. However, if confusion is unlikely, we simply write “ $B_\varepsilon(x)$ ” for “ $B_\varepsilon(x; d)$ ”.

Example 1.1.1. The *Euclidean metric space* $M = (X, d)$ is an n -dimensional set X equipped with the *Euclidean metric* d defined as

$$d(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

Example 1.1.2. A *discrete metric space* $M = (X, d)$ is a set X equipped with the *discrete metric* d defined as

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{else.} \end{cases}$$

This is an equivalent definition of the discrete metric:

$$d(x, y) = (\text{sgn}(d'(x, y)))^2,$$

where $\text{sgn}(\cdot)$ is a [sign function](#), and d' is any metric on X .

Example 1.1.3. ¹ Denote $C[a, b]$ for the set of all continuous mapping $\mathbb{R}_{[a, b]} \rightarrow \mathbb{R}$. On $C[a, b]$, we can define a metric d as

$$d_p(f, g) = \left(\int_a^b |f(t) - g(t)|^p dt \right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(f, g) = \sup_{t \in \mathbb{R}_{[a, b]}} |f(t) - g(t)|.$$

¹ See [Minkowski inequality](#).

Example 1.1.4. ² Let $M = (X, d)$ be a metric space. The *Hausdorff metric* d_H on $2^X \setminus \{\emptyset\}$ is defined as

$$d_H = \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\},$$

where

$$d(x, Y) = \inf_{y \in Y} d(x, y), \text{ and } d(y, X) = \inf_{x \in X} d(y, x).$$

1.2 Open sets in Metric Spaces

Definition 1.2.1. Let $M = (X, d)$ be a metric space, and let $U \subseteq X$. U is said to be *open in M* if and only if for any $y \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$, such that $B_\varepsilon(y) \subseteq U$.

Lemma 1.2.1. Let $M = (X, d)$ be a metric space, let $x \in A$ and let $\varepsilon \in \mathbb{R}_{>0}$. For any $y \in B_\varepsilon(x)$, there is a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_\varepsilon(x)$.

Proof. For any $y \in B_\varepsilon(x)$, by the definition of open balls (Definition 1.1.2), we have $d(x, y) < \varepsilon$.

Let $\delta \in \mathbb{R}_{>0}$ such that $\delta + d(x, y) = \varepsilon$.

By M3 in metric axioms (Definition 1.1.1), for any $z \in A$ with $d(y, z) < \delta$, we have

$$d(x, z) \leq d(y, z) + d(x, y) < \varepsilon.$$

Thus, again, by the definition of open balls, we have $B_\delta(y) \subseteq B_\varepsilon(x)$. ■

Theorem 1.2.1. ³ Let $M = (X, d)$ be a metric space, and let $U \subseteq X$. U is open in M if and only if it is a union of open balls.

Proof. First, prove \Rightarrow .

As U is open, for any $y \in U$, there exists $\varepsilon_y \in \mathbb{R}_{>0}$ such that $B_{\varepsilon_y}(y) \subseteq U$.

Therefore,

$$U = \bigcup_{y \in U} B_{\varepsilon_y}(y).$$

□

Now, prove \Leftarrow .

Aiming for a contradiction, suppose U is a union of open balls but not open.

² See [Hausdorff distance](#).

³ Shared on [ProofWiki](#).

As U is not open, there is a $y \in U$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \not\subseteq U$.

As U is a union of open balls, there is an $x \in U$ and $r \in \mathbb{R}_{>0}$ such that $y \in B_r(x)$.

By Lemma 1.2.1, there exists a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_r(x)$.

This is a contradiction by the assumption.

Thus, U has to be open. ■

Theorem 1.2.2. Let $M = (X, d)$ be any metric space. M is *Hausdorff*. That is, For any distinct points $x, y \in X$, we can always find an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset.$$

Proof. Aiming for a contradiction, suppose there are $x, y \in X$ with $x \neq y$, such that for any $\varepsilon \in \mathbb{R}_{>0}$, we can always find a $z \in X$ such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let $r = d(x, y)/2$, and let $z \in B_r(x) \cap B_r(y)$.

As $z \in B_r(x)$, by the definition of open balls (Definition 1.1.2), $d(x, z) < r$; as $z \in B_r(y)$, similarly, $d(y, z) < r$. Then we have

$$d(x, z) + d(y, z) < 2r = d(x, y).$$

By M3 in metric axioms (Definition 1.1.1), this is impossible. ■