



## Notes for General Topology

Zhao Wenchuan

September 21, 2021

# *Contents*

<b>1</b>	<b>Metric Spaces</b>	<b>2</b>
1.1	Metric Spaces . . . . .	2
1.2	Open Sets in Metric Spaces . . . . .	4
1.3	Restrictions and Metric Subspaces . . . . .	7
<b>2</b>	<b>Topological Spaces</b>	<b>9</b>
2.1	Basic Definitions . . . . .	9
2.2	Some Important Topologies . . . . .	11
2.3	Comparison of Topologies . . . . .	11
2.4	Subspaces . . . . .	12
2.5	Interiors . . . . .	13
2.6	Limit Points and Isolated Points . . . . .	16
2.7	Closures . . . . .	17
2.8	Density . . . . .	19
<b>3</b>	<b>Sequences</b>	<b>21</b>
3.1	Convergent Sequences . . . . .	21
3.2	Accumulation Points of Sequences . . . . .	22
<b>4</b>	<b>Countable Axioms</b>	<b>24</b>
4.1	Covers and Bases . . . . .	24
4.2	First-Countable Spaces . . . . .	25
4.3	Second-Countable Spaces . . . . .	26
4.4	Separable Spaces . . . . .	27
4.5	Lindelöf Space . . . . .	29
<b>5</b>	<b>Continuous Mappings</b>	<b>31</b>
5.1	Continuous Mappings . . . . .	31

5.2	Homeomorphisms . . . . .	33
<b>6</b>	<b>Separation Axioms</b>	<b>36</b>
6.1	Neighbourhood Systems . . . . .	36
6.2	$T_0$ Spaces . . . . .	37
6.3	$R_0$ Spaces . . . . .	37
6.4	$T_1$ Spaces . . . . .	38
	<b>Appendices</b>	<b>40</b>
<b>A</b>	<b>Some Examples of Topological Spaces</b>	<b>41</b>
A.1	Sorgenfrey line . . . . .	41
A.2	Niemytzki Plane . . . . .	41

# Chapter 1.

## Metric Spaces

### §1.1 Metric Spaces

---

**Definition 1.1.1.** Let  $X$  be any set.

A function  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$  is *metric function*, or, simply, *metric on  $X$*  iff it satisfies the *metric axioms*. That is, for any  $x, y, z \in X$ :

**M1.**  $d(x, y) = 0$  iff  $x = y$ ;

**M2.**  $d(x, y) = d(y, x)$ ;

**M3.**  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Definition 1.1.2.** Let  $X$  be any set and let  $d$  be a structure on  $X$ .

The pair  $(X, d)$  is called a *metric space* iff  $d$  is a metric on  $X$ .

**Definition 1.1.3.** A  $\mathbb{X} = (X, d)$  be a metric space, let  $x \in X$  and let  $\varepsilon \in \mathbb{R}_{>0}$ .

An *open  $\varepsilon$ -ball*, or just  $\varepsilon$ -ball, about  $x$  is defined to be the set

$$B_\varepsilon(x; d) := \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) := \{y \in X : d(x, y) \leq \varepsilon\}.$$

**Note 1.1.1.** As

$$\mathbb{X}_0 = (X, d_0), \mathbb{X}_1 = (X, d_1), \mathbb{X}_2 = (X, d_2), \dots$$

are different although they share the same set  $X$ , for any  $x \in X$  and any  $\varepsilon \in \mathbb{R}_{>0}$ ,

$$B_\varepsilon(x; d_1), B_\varepsilon(x; d_2), B_\varepsilon(x; d_3), \dots$$

are also different. However, if confusion is unlikely, we simply write “ $B_\varepsilon(x)$ ” for “ $B_\varepsilon(x; d)$ ”.

**Example 1.1.1.** The *Euclidean metric space*  $\mathbb{X} = (X, d)$  is an  $n$ -dimensional set  $X$  equipped with the *Euclidean metric*  $d$  defined as

$$d(x, y) := \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x, y) := \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

**Example 1.1.2.** A *discrete metric space*  $\mathbb{X} = (X, d)$  is a set  $X$  equipped with the *discrete metric*  $d_{\text{dis}}$  defined as

$$d_{\text{disc}}(x, y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{else.} \end{cases}$$

This is an equivalent definition of the discrete metric:

$$d_{\text{disc}}(x, y) := (\text{sgn}(d(x, y)))^2,$$

where  $\text{sgn}(\cdot)$  is a [sign function](#), and  $d$  is any metric on  $X$ .

**Example 1.1.3.** <sup>1</sup> Let  $\mathbb{I} = (C[a, b], d_p)$  be a metric space where  $C[a, b]$  denotes the set of all continuous mapping  $\mathbb{R}_{[a, b]} \rightarrow \mathbb{R}$ , and  $p > 0$ , and the metric  $d_p$  is defined as

$$d_p(f, g) := \left( \int_a^b |f(t) - g(t)|^p dt \right)^{\frac{1}{p}}.$$

In particular,

$$d_\infty(f, g) := \sup_{t \in \mathbb{R}_{[a, b]}} |f(t) - g(t)|.$$

---

<sup>1</sup> See [Minkowski inequality](#).

**Example 1.1.4.** <sup>2</sup> Let  $\mathbb{X} = (X, d)$  be a metric space. The *Hausdorff metric*  $d_H$  on  $2^X \setminus \{\emptyset\}$  is defined as

$$d_H := \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\},$$

where

$$d(x, Y) := \inf_{y \in Y} d(x, y), \text{ and } d(y, X) := \inf_{x \in X} d(y, x).$$

## §1.2 Open Sets in Metric Spaces

---

**Definition 1.2.1.** Let  $\mathbb{X} = (X, d)$  be a metric space, and let  $U \subseteq X$ .

$U$  is said to be *open in  $\mathbb{X}$* , iff for any  $y \in U$ , there exists  $\varepsilon \in \mathbb{R}_{>0}$ , such that  $B_\varepsilon(y) \subseteq U$ .

**Proposition 1.2.1.** Let  $\mathbb{X} = (X, d)$  be a metric space, let  $x \in A$  and let  $\varepsilon \in \mathbb{R}_{>0}$ .

For any  $y \in B_\varepsilon(x)$ , there is a  $\delta \in \mathbb{R}_{>0}$  such that  $B_\delta(y) \subseteq B_\varepsilon(x)$ .

**Proof.** For any  $y \in B_\varepsilon(x)$ , by the definition of open balls (Definition 1.1.3), we have  $d(x, y) < \varepsilon$ .

Let  $\delta \in \mathbb{R}_{>0}$  such that  $\delta + d(x, y) = \varepsilon$ .

By M3 in metric axioms (Definition 1.1.1), for any  $z \in A$  with  $d(y, z) < \delta$ , we have

$$d(x, z) \leq d(y, z) + d(x, y) < \varepsilon.$$

Thus, again, by the definition of open balls, we have  $B_\delta(y) \subseteq B_\varepsilon(x)$ . ■

**Proposition 1.2.2.** Let  $\mathbb{X} = (X, d)$  be a metric space, and let  $U \subseteq X$ .

$U$  is open in  $\mathbb{X}$  iff it is a union of open balls.

**Proof.** First, prove  $\Rightarrow$ .

As  $U$  is open, for any  $y \in U$ , there exists  $\varepsilon_y \in \mathbb{R}_{>0}$  such that  $B_{\varepsilon_y}(y) \subseteq U$ .

Therefore,

$$U = \bigcup_{y \in U} B_{\varepsilon_y}(y).$$

□

---

<sup>2</sup> See [Hausdorff distance](#).

Now, prove  $\Leftarrow$ .

Aiming for a contradiction, suppose  $U$  is a union of open balls but not open.

As  $U$  is not open, there is a  $y \in U$  such that for any  $\varepsilon \in \mathbb{R}_{>0}$ ,  $B_\varepsilon(y) \not\subseteq U$ .

As  $U$  is a union of open balls, there is an  $x \in U$  and  $r \in \mathbb{R}_{>0}$  such that  $y \in B_r(x)$ .

By Proposition 1.2.1, there exists a  $\delta \in \mathbb{R}_{>0}$  such that  $B_\delta(y) \subseteq B_r(x)$ .

This is a contradiction by the assumption.

Thus,  $U$  has to be open. ■

**Proposition 1.2.3.** Let  $\mathbb{X} = (X, d)$  be any metric space.

$\mathbb{X}$  is *Hausdorff*. That is, For any distinct points  $x, y \in X$ , we can always find an  $\varepsilon \in \mathbb{R}_{>0}$  such that

$$B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset.$$

**Proof.** Aiming for a contradiction, suppose there are  $x, y \in X$  with  $x \neq y$ , such that for any  $\varepsilon \in \mathbb{R}_{>0}$ , we can always find a  $z \in X$  such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let  $r = d(x, y)/2$ , and let  $z \in B_r(x) \cap B_r(y)$ .

As  $z \in B_r(x)$ , by the definition of open balls (Definition 1.1.3),  $d(x, z) < r$ ; as  $z \in B_r(y)$ , similarly,  $d(y, z) < r$ . Then we have

$$d(x, z) + d(y, z) < 2r = d(x, y).$$

This contradicts the metric axioms M3 (Definition 1.1.1).

Thus  $\mathbb{X}$  is Hausdorff. ■

**Definition 1.2.2.** Let  $\mathbb{X} = (X, d)$  be any metric space, and let  $V \subseteq X$ .

$V$  is said to be *closed* in  $\mathbb{X}$ , iff there is an open set  $U$  satisfies  $X \setminus U = V$ .

**Proposition 1.2.4.** In a metric space, any singleton is closed.

**Proof.** Let  $\mathbb{X} = (X, d)$  be a metric space, let  $x \in X$ , and let  $y \in X \setminus \{x\}$ .

As  $M$  is Hausdorff (Proposition 1.2.3), there is an  $\varepsilon \in \mathbb{R}_{>0}$  such that

$$0 < \varepsilon < d(x, y),$$

thus  $X \setminus \{x\}$  is open, hence, by Definition 1.1.1, its complement  $\{x\}$  is open. ■

**Proposition 1.2.5.** Let  $\mathbb{X} = (X, d)$  be a metric space, denote  $\mathcal{T}$  for the family of open subsets of  $X$ .

Then  $\mathcal{T}$  satisfies the following conditions:

- O1.**  $X, \emptyset \in \mathcal{T}$ ;
- O2.** For any  $\mathcal{U} \subseteq \mathcal{T}$ ,  $\bigcup \mathcal{U} \in \mathcal{T}$ ; in words,  $\mathcal{T}$  is closed under arbitrary union;
- O3.** For any finite  $\mathcal{V} \subseteq \mathcal{T}$ ,  $\bigcap \mathcal{V} \in \mathcal{T}$ ; in words,  $\mathcal{T}$  is closed under finite intersection.

**Proof.**

- O1.** As  $\emptyset$  is the subset of any set,  $\emptyset \in \mathcal{T}$ .  $\bigcup \emptyset = \emptyset \in \mathcal{T}$ .

By Definition 1.2.2,  $X = X \setminus \emptyset$ .

□

- O2.** Let  $\mathcal{U} \subseteq \mathcal{T}$ , and denote  $\mathcal{O}$  for the open balls in  $M$ .

For any  $U \in \mathcal{U}$ , there is an  $\mathcal{O}_U \subseteq \mathcal{O}$  such that  $U = \bigcup \mathcal{O}_U$ .

Then we have

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \left( \bigcup \mathcal{O}_U \right) = \bigcup_{U \in \mathcal{U}} \mathcal{O}_U.$$

By Proposition 1.2.2,  $\bigcup \mathcal{U}$  is open.

□

- O3.** Let  $\mathcal{V}$  be a finite subset of  $\mathcal{T}$ .

Aiming for a contradiction, suppose  $\bigcap \mathcal{V}$  is not open.

By Definition 1.2.1, there exists a  $y \in \bigcap \mathcal{V}$  such that for any  $\varepsilon \in \mathbb{R}_{>0}$ ,  $B_\varepsilon(y) \setminus \bigcap \mathcal{V} \neq \emptyset$ .

By De Morgan's law, we have

$$\bigcup_{V \in \mathcal{V}} (B_\varepsilon(y) \setminus V) \neq \emptyset.$$

Thus, there exists  $V \in \mathcal{V}$  such that  $B_\varepsilon(y) \setminus V \neq \emptyset$ .

As  $V \in \mathcal{T}$  and  $\varepsilon$  is arbitrarily given, by Proposition 1.2.1,  $y \notin V$ . This is a contradiction.

Thus,  $\bigcap \mathcal{V}$  is open.

□



Thus, the theorem is proved. ■

**Proposition 1.2.6.** Infinite intersections of open sets in some metric spaces are not necessarily open.

**Proof.** Consider  $\mathbb{R}$  is a Euclidean metric space, and denote  $\mathcal{T}$ .

Clearly, for any  $n \in \mathbb{N}_{>0}$  and for any  $x \in X$ , the open interval  $B_{\frac{1}{n}}(x)$  is open, but

$$\bigcap \left\{ B_{\frac{1}{n}}(x) : n \in \mathbb{N}_{>0} \right\} = \{x\}.$$

For any  $\varepsilon \in \mathbb{R}_{>0}$ ,  $B_\varepsilon(x) \setminus \{x\}$  is not empty, thus  $\{x\}$  is not open. ■

### §1.3 Restrictions and Metric Subspaces

---

Restriction of metric function is a useful tool to describe the relation between metric spaces with different sets but “same” metric function on the sets.

As a restriction of a relation  $R$  on  $X \times Y$  to a subset  $A \times B \subseteq X \times Y$  is defined to be

$$R \upharpoonright_{A \times B} := R \cap (X \times Y),$$

a restriction of a metric  $d$  on a set  $S$  to a subset  $U \subseteq S$  is defined to be

$$d \upharpoonright_{(U \times U) \times \mathbb{R}_{>0}} := d \cap ((U \times U) \times \mathbb{R}_{>0}).$$

If  $B = Y$ , customarily, we simply write  $R \upharpoonright_A$  for  $R \upharpoonright_{A \times B}$ . Similarly, as the codomain of a metric function is always  $\mathbb{R}_{>0}$ , so we simply write  $d \upharpoonright_{U \times U}$  instead of  $d \upharpoonright_{(U \times U) \times \mathbb{R}_{>0}}$ .

**Definition 1.3.1.** Let  $\mathbb{X} = (X, d)$  be a metric space, and let  $A \subseteq X$ .

The *metric on  $A$  induced by  $d$* , or the *subspace metric of  $d$  with respect to  $A$*  is defined to be

$$d_A := d \upharpoonright_{A \times A}.$$

**Proposition 1.3.1.** Let  $\mathbb{X} = (X, d)$  be a metric space, and let  $A \subseteq X$  and let  $d_A := d \upharpoonright_{A \times A}$ .

Then  $\mathbb{A} = (A, d_A)$  is a metric space.

**Proof.** As metric axioms (Definition 1.1.1) holds for any  $x, y \in X$ , and  $A \subseteq X$ , they also holds for any  $a, b \in A$ . As  $d_A$  is the subspace metric of  $d$  with respect to  $A$ ,  $d_A$  is a metric on  $A$ .

Thus,  $\mathbb{A}$  is a metric space.

**Definition 1.3.2.** Let  $\mathbb{X} = (X, d)$  be a metric space, and let  $A \subseteq X$ .

$\mathbb{A} = (A, d_A)$  is a *metric subspace* of  $\mathbb{X}$  iff  $d_A$  is a subspace metric of  $d$  with respect to  $A$ .

## Chapter 2.

# Topological Spaces

### §2.1 Basic Definitions

---

**Definition 2.1.1.** Let  $X$  be any set, and let  $\mathcal{T} \subseteq 2^X$ .

$\mathcal{T}$  is a *topology on  $X$*  iff it satisfies the *open set axioms*. That is,

- O1.**  $X \in \mathcal{T}$ ;
- O2.** For any  $\mathcal{U} \subseteq \mathcal{T}$ ,  $\bigcup \mathcal{U} \in \mathcal{T}$ ; in words,  $\mathcal{T}$  is closed under arbitrary union.
- O3.** For any finite  $\mathcal{V} \subseteq \mathcal{T}$ ,  $\bigcap \mathcal{V} \in \mathcal{T}$ ; in words,  $\mathcal{T}$  is closed under finite intersection.

A subset  $U \subseteq X$  is said to be *open in  $M$*  iff it is an element of  $\mathcal{T}$ .

**Definition 2.1.2.** Let  $X$  be any set, and let  $\mathcal{T}$  be a structure on  $X$ .

The pair  $(X, \mathcal{T})$  is called a *topological space* iff  $\mathcal{T}$  is a topology on  $X$ .

**Proposition 2.1.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space.

Then  $\emptyset \in \mathcal{T}$ .

**Proof.** As empty set is an element of any set, it also an element of  $\mathcal{T}$ .

Therefore, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

■

**Definition 2.1.3.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space.

A subset  $A \subseteq X$  is said to be *closed in  $\mathbb{X}$*  iff there exists a  $U \in \mathcal{T}$  such that  $A = X \setminus U$ .

**Proposition 2.1.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and denote  $\mathcal{C}$  for the family of all closed sets in  $M$ .

Then  $\mathcal{C}$  satisfies the following conditions:

- C1.**  $X, \emptyset \in \mathcal{C}$ ;
- C2.** For any  $\mathcal{A} \subseteq \mathcal{C}$ ,  $\bigcap \mathcal{A} \in \mathcal{C}$ ;
- C3.** For any finite  $\mathcal{B} \subseteq \mathcal{C}$ ,  $\bigcup \mathcal{B} \in \mathcal{C}$ .

**Proof.**

**C1.** As  $\emptyset \in \mathcal{T}$  and  $X = X \setminus \emptyset$ , by Definition 2.1.3,  $X$  is closed.

Similarly, as  $X \in \mathcal{T}$  and  $\emptyset = X \setminus X$ ,  $\emptyset$  is closed.

□

**C2.** For any  $\mathcal{A} \subseteq \mathcal{C}$ , there exists a  $\mathcal{U} \subseteq \mathcal{T}$  such that

$$\forall A \in \mathcal{A} : \exists U \in \mathcal{U} : A = X \setminus U. \quad (\text{Definition 2.1.3.})$$

Then we have

$$\begin{aligned} \mathcal{A} = \{X \setminus U : U \in \mathcal{U}\} &\iff \bigcap \mathcal{A} = \bigcap_{U \in \mathcal{U}} X \setminus U \\ &\iff \bigcap \mathcal{A} = X \setminus \bigcup \mathcal{U}. \end{aligned}$$

As  $\bigcup \mathcal{U} \in \mathcal{T}$  by Definition 2.1.1 O2, its complement  $\bigcap \mathcal{A} \in \mathcal{C}$  by Definition 2.1.3.

□

**C3.** For any finite  $\mathcal{B} \subseteq \mathcal{C}$ , there exists a finite  $\mathcal{U} \subseteq \mathcal{T}$  such that

$$\forall B \in \mathcal{B} : \exists U \in \mathcal{U} : A = X \setminus U. \quad (\text{Definition 2.1.3.})$$

Then we have

$$\begin{aligned} \mathcal{B} = \{X \setminus U : U \in \mathcal{U}\} &\iff \bigcup \mathcal{B} = \bigcup_{U \in \mathcal{U}} X \setminus U \\ &\iff \bigcup \mathcal{B} = X \setminus \bigcap \mathcal{U}. \end{aligned}$$

As  $\bigcap \mathcal{U} \in \mathcal{T}$  by Definition 2.1.1 O3, its complement  $\bigcup \mathcal{B} \in \mathcal{C}$  by Definition 2.1.3.

□

Thus, the proof is done.

■

## §2.2 Some Important Topologies

---

**Definition 2.2.1.** Let  $X$  be any set.

A family  $\mathcal{T} \subseteq 2^X$  is a *discrete topology on  $X$*  iff  $\mathcal{T} = 2^X$ .

**Definition 2.2.2.** Let  $X$  be any set.

A family  $\mathcal{T} \subseteq 2^X$  is an *indiscrete topology on  $X$*  iff  $\mathcal{T} = \{X, \emptyset\}$ .

**Definition 2.2.3.** Let  $\mathbb{X} = (X, d)$  be a metric space.

A family  $\mathcal{T} \subseteq 2^X$  is a *topology induced by  $d$*  iff  $\mathcal{T}$  is the set of all open sets in  $\mathbb{X}$ .

## §2.3 Comparison of Topologies

---

**Definition 2.3.1.** Let  $X$  be any set and let  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be topologies on  $X$ .

We say that  $\mathcal{T}$  is *coarser* than  $\mathcal{T}_1$ , or  $\mathcal{T}_2$  is *finer* than  $\mathcal{T}_1$ , iff  $\mathcal{T}_1 \subseteq \mathcal{T}_2$ .

**Note 2.3.1.** By the definition of cardinality and inclusion mapping, if  $\mathcal{T}_1 \subseteq \mathcal{T}_2$ , it is certainly true that  $|\mathcal{T}_1| \leq |\mathcal{T}_2|$ . But, on the contrary,  $|\mathcal{T}_1| \leq |\mathcal{T}_2|$  does not implies  $\mathcal{T}_1 \subseteq \mathcal{T}_2$ . It is easy to find counter-example about this.

**Example 2.3.1.** By Definition 2.3.1, for any set  $X$ , if a family  $\mathcal{U}$  of open sets is given, then we can find the coarsest topology on  $X$  containing  $\mathcal{U}$  by

$$\mathcal{T} = \left\{ \bigcup \mathcal{I}, \bigcap \mathcal{I}, X : \mathcal{I} \subseteq \mathcal{U} \right\}.$$

For example, let  $X = \{1, 2, 3, 4, 5\}$ , and let

$$\mathcal{U} = \{\{1, 2\}, \{2, 3\}, \{4\}\}.$$

Then a topology on  $X$  contains at least these sets:

$$\begin{aligned} &\{1, 2, 3, 4\}, \{\}, \\ &\{1, 2\}, \{2, 3\}, \{4\}, \\ &\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \\ &\{2\}. \end{aligned}$$

**Example 2.3.2.** The discrete topology is the finest topology on any  $X$ , while the indiscrete topology is the coarsest.

## §2.4 Subspaces

---

**Definition 2.4.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

The *subspace topology* on  $A$  is defines as

$$\mathcal{T}_A := \{A \cap U : U \in \mathcal{T}\}.$$

In this case,  $(A, \mathcal{T}_A)$  is called a *subspace* of  $\mathbb{X}$ .

**Note 2.4.1.** Note that  $(A, \mathcal{T}_A)$  is a subspace of  $\mathbb{X}$  does not implies that  $\mathcal{T}_A \subseteq \mathcal{T}$ . Consider  $(\mathbb{R}, \mathcal{T})$  as a standard topological space. Let  $\mathcal{T}'$  be a standard topological space on  $\mathbb{R}_{\geq 0}$ , then  $(\mathbb{R}_{\geq 0}, \mathcal{T}')$  is a subspace of  $(\mathbb{R}, \mathcal{T})$ . For any  $a \in \mathbb{R}_{> 0}$ , real interval  $[0, a) \in \mathcal{T}'$ , but it is not an element in  $\mathcal{T}$ .

Here is another extreme example. Let  $\mathbb{X} = (X, \mathcal{T})$  be an indiscrete topological space, and let  $A \subseteq X$ . Then, if  $(A, \mathcal{T}_A)$  is a subspace of  $\mathbb{X}$ , then  $\mathcal{T}_A \subseteq \mathcal{T}$  iff  $A \in \{\emptyset, X\}$ .

**Note 2.4.2.** As  $\emptyset$  is the subset of any set, by Definition 2.4.1, for any topological space  $(X, \mathcal{T})$ ,

$$\mathcal{T}_\emptyset = \{\emptyset \cap U : U \in \mathcal{T}\} = \{\emptyset\}$$

Thus,  $(\emptyset, \{\emptyset\})$  is the subspace of any topological space.

**Proposition 2.4.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $\mathbb{A} = (A, \mathcal{T}_A)$  be a subspace of  $\mathbb{X}$ .

Then,

$$\mathcal{T}_A \subseteq \mathcal{T} \iff A \in \mathcal{T}.$$

**Proof.** First, prove  $\Rightarrow$ .

$S \in \mathcal{T}$ . By Definition 2.1.1 O1,  $A \in \mathcal{T}_A$ . As  $\mathcal{T}_A \subseteq \mathcal{T}$ ,  $A \in \mathcal{T}$ .

□

Now, prove  $\Leftarrow$ .

As  $A \in \mathcal{T}$ , by Definition 2.4.1, for any  $S \in \mathcal{T}_A$ ,

$$S = A \cap U, \quad U \in \mathcal{T}.$$

By Definition 2.1.1 O3,  $S \in \mathcal{T}$ .

As  $S \in \mathcal{T}_A$  is arbitrarily given, all  $S \in \mathcal{T}_A$  is also an element in  $\mathcal{T}$ . Thus  $\mathcal{T}_A \subseteq \mathcal{T}$ .

□

Thus, the proof is done.

■

## §2.5 Interiors

---

**Definition 2.5.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

The *interior* of  $A$  is defined as

$$\text{Int}_{\mathcal{T}}(A) := \bigcup (\mathcal{T} \cap 2^A).$$

**Note 2.5.1.** Let  $\mathbb{X}_1 = (X, \mathcal{T}_1)$ ,  $\mathbb{X}_2 = (X, \mathcal{T}_2)$ , and  $A \subseteq X$ . Then  $\mathcal{T}_1 \neq \mathcal{T}_2$  iff  $\text{Int}_{\mathcal{T}_1}(A) \neq \text{Int}_{\mathcal{T}_2}(A)$ . In this case, the subscript for “Int” is necessary.

But, if the confusion is unlikely, we can also simply write  $\text{Int}(A)$  for  $\text{Int}_{\mathcal{T}}A$ . In this case, it is also common to write  $A^\circ$  for  $\text{Int}(A)$ .

**Proposition 2.5.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

$A \in \mathcal{T}$  iff  $A = A^\circ$ .

**Proof.** First, prove  $\Rightarrow$ .

If  $A \in \mathcal{T}$ , then we have

$$\mathcal{T} \cap 2^A = \mathcal{T} \cap \{A\} \cap 2^A = \{A\} \cap 2^A = \{A\}.$$

By Definition 2.5.1,

$$A^\circ = \bigcup (\mathcal{T} \cap 2^A) = \bigcup \{A\} = A.$$

□

Now, prove  $\Leftarrow$ .

By Definition 2.5.1, we have

$$A = \bigcup (\mathcal{T} \cap 2^A).$$

As  $\mathcal{T} \cap 2^A \subseteq \mathcal{T}$ , thus, by open set axioms O2 (Definition 2.1.1 O2),  $A \in \mathcal{T}$ .

□

Thus, the proof is done.

■

**Proposition 2.5.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \in \mathcal{T}$ . For any  $x \in A$ , there is a  $U \in \mathcal{T} \cap 2^A$  such that  $x \in U$ .

**Proof.**

$$x \in A \iff x \in A^\circ \quad (\text{Proposition 2.5.1})$$

$$\iff x \in \bigcup (\mathcal{T} \cap 2^A) \quad (\text{Definition 2.5.1})$$

$$\iff \exists U \in \mathcal{T} \cap 2^A : x \in U.$$

■

**Proposition 2.5.3.** Let  $X$  be any set, let  $I$  be an index set, and let  $\mathcal{A}_i \subseteq 2^X$  for any  $i \in I$ .

Then we have

$$\bigcup \left( \bigcap_{i \in I} \mathcal{A}_i \right) \subseteq \bigcap_{i \in I} \left( \bigcup \mathcal{A}_i \right).$$

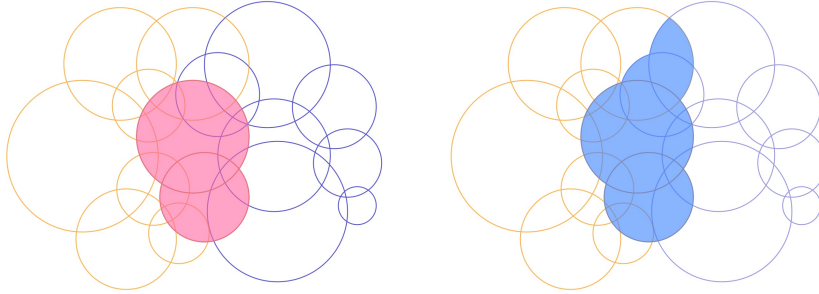


Figure 2.1: Diagram of the relation in Proposition 2.5.3.

**Proposition 2.5.4.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $\mathcal{A} \subseteq 2^X$ .

Then we have

$$\left( \bigcap \mathcal{A} \right)^\circ \subseteq \bigcap_{A \in \mathcal{A}} A^\circ.$$



**Proof.**

$$\begin{aligned}
\left(\bigcap \mathcal{A}\right)^\circ &= \bigcup \left(\mathcal{T} \cap 2^{\bigcap \mathcal{A}}\right) && \text{(Definition 2.5.1)} \\
&= \bigcup \left(\mathcal{T} \cap \bigcap_{A \in \mathcal{A}} 2^A\right) && \text{(intersection of power sets)} \\
&= \bigcup \left(\bigcap_{A \in \mathcal{A}} (\mathcal{T} \cap 2^A)\right) && \text{(intersection is idempotent} \\
&&& \text{and associative)} \\
&\subseteq \bigcap_{A \in \mathcal{A}} \left(\bigcup (\mathcal{T} \cap 2^A)\right) && \text{(Proposition 2.5.3)} \\
&= \bigcap_{A \in \mathcal{A}} A^\circ. && \text{(Definition 2.5.1)}
\end{aligned}$$

■

**Example 2.5.1.** The equality in Proposition 2.5.4 may not hold.

Let  $\mathbb{T} = (\mathbb{R}, \mathcal{T})$  be a topological space with

$$\mathcal{T} = \{X, (0, 2), (1, 3), \emptyset\}.$$

Then we have

$$((0, 2) \cap (1, 3))^\circ = \emptyset \quad \subsetneq \quad (0, 2)^\circ \cap (1, 3) = (1, 2).$$

**Proposition 2.5.5.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A, B \subseteq X$ .

If  $A \subseteq B$ , then  $A^\circ \subseteq B^\circ$ .

**Proof.**

$$\begin{aligned}
A \subseteq B &\implies 2^A \subseteq 2^B && \text{(power set of subset)} \\
&\implies \mathcal{T} \cap 2^A \subseteq \mathcal{T} \cap 2^B \\
&\implies \bigcup (\mathcal{T} \cap 2^A) \subseteq \bigcup (\mathcal{T} \cap 2^B) \\
&\implies A^\circ \subseteq B^\circ && \text{(Definition 2.5.1)}
\end{aligned}$$

■

**Note 2.5.2.** Note that,  $A^\circ \subseteq B^\circ$  does not implies  $A \subseteq B$ . Consider  $\mathbb{R}$  as a Euclidean metric space, and let

$$A = \{0\}, \quad B \subseteq \mathbb{R} \setminus \{0\}.$$

As  $A^\circ = \emptyset$ ,  $A^\circ \subseteq B^\circ$ , but  $A \setminus B = \{0\}$ , so  $A \not\subseteq B$ .

## §2.6 Limit Points and Isolated Points

---

**Definition 2.6.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

A point  $x \in X$  is a *limit point of  $A$*  iff for any  $U \in \mathcal{T}$  with  $x \in U$

$$A \cap U \setminus \{x\} \neq \emptyset.$$

The *derived set of  $A$*  is the set of all limit points of  $A$ .

**Definition 2.6.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

A point  $x \in A$  is said to be *isolated* iff there is  $U \in \mathcal{T}$  with  $x \in U$ , such that

$$A \cap U \setminus \{x\} = \emptyset.$$

**Notations.** The Derived set of  $A$  is usually denoted  $A'$ .<sup>1</sup> But sometime it is also necessary to know in which space (with its topology) the derived set of  $A$  is. For example, for topological spaces  $\mathbb{X}_1 = (X, \mathcal{T}_1)$  and  $\mathbb{X}_2 = (X, \mathcal{T}_2)$ , if  $\mathcal{T}_1 \neq \mathcal{T}_2$ , the derived sets of a set  $A$  in  $\mathbb{X}_1$  and  $\mathbb{X}_2$  may be different. So, below, the notation  $A'$  is used only if the confusions are unlikely; else, we denote  $L_{\mathcal{T}}A$  for  $A'$  with respect to the topology  $\mathcal{T}$ .

Sometime, the set of isolated points of  $A$  is denoted by  $A^i$ . For avoiding confusions, we denote  $I_{\mathcal{T}}(A)$  for  $A^i$  with respect to the topology  $\mathcal{T}$ .

**Proposition 2.6.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

Then,

$$A \subseteq L(A) \sqcup I(A).$$

**Proof.** By Definition 2.6.1,  $x \notin L(A)$  iff there exists a  $U \in \mathcal{T}$  of  $x \in U$  such that  $A \cap U \setminus \{x\} = \emptyset$ . This precisely satisfies Definition 2.6.2. Thus

$$A \subseteq L(A) \cup I(A).$$

As Definition 2.6.1 and 2.6.2 are precisely logical complement for each other,  $x \in I(A) \cap L(A)$  always fails, i.e.,  $I(A) \cap L(A) = \emptyset$ . Thus

$$A \subseteq L(A) \sqcup I(A).$$

■

---

<sup>1</sup>See [ProofWiki](#) and [Wikipedia](#).

**Proposition 2.6.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

$A$  is closed iff  $L(A) \subseteq A$ .

**Proof.** First, prove  $\Rightarrow$ .

Aiming for a contradiction, suppose  $A$  is closed but there exists a  $y \in L(A) \setminus A$ .

By Definition 2.1.3, as  $A$  is closed, then  $A^c$  is open.

As  $y \in A^c$  and  $A^c$  is open, then, by Proposition 2.5.2, there exists a  $U \in \mathcal{T}$  with  $y \in U$ , such that  $U \subseteq A^c$ .

As  $U$  is an open set containing  $y$  and  $A \cap U \setminus \{y\} = \emptyset$ , then  $y \notin L(A)$ . This contradicts the assumption.

Thus  $L(A) \subseteq A$ . ■

## §2.7 Closures

---

**Definition 2.7.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

The *closure* of  $A$  is defined as

$$\text{Cl}_{\mathcal{T}}(A) := A \cup L(A).$$

When the confusions are unlikely, we simply write  $\text{Cl}(A)$ ,  $\overline{A}$  or  $A^-$  for  $\text{Cl}_{\mathcal{T}}(A)$ .

**Proposition 2.7.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

$A$  is closed iff  $A = A^-$

**Proof.**

$$A \text{ is closed} \iff A \supseteq L(A) \quad (\text{Proposition 2.6.2})$$

$$\iff A = A \cup L(A)$$

$$\iff A = A^-. \quad (\text{Definition 2.7.1})$$
■

**Proposition 2.7.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

$A$  is closed iff

$$A = I(A) \sqcup L(A).$$

**Proof.** As  $A$  is closed, we have

$$\begin{aligned}
A &= \text{Cl}(A) && \text{(Proposition 2.7.1)} \\
&= A \cup L(A) && \text{(Definition 2.7.1)} \\
&= A \setminus L(A) \sqcup L(A) \\
&= I(A) \sqcup L(A). && \text{(Proposition 2.6.1)}
\end{aligned}$$

■

**Proposition 2.7.3.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

$$A^- = \bigcap \{K \supseteq A : K \text{ closed in } \mathbb{X}\}.$$

**Proof.** By Proposition 2.7.1,  $A^-$  is closed. Thus, by Definition 2.1.3,  $X \setminus A^-$  is open. Then we have

$$\begin{aligned}
X \setminus (X \setminus A^-) &= X \setminus (X \setminus A^-)^\circ && \text{(Proposition: 2.5.1)} \\
&= X \setminus \bigcup (\mathcal{T} \cap 2^{X \setminus A^-}) && \text{(Definition: 2.5.1)} \\
&= X \setminus \bigcup \{U \subseteq A : U \text{ open in } \mathbb{X}\} \\
&= \bigcap \{X \setminus U \supseteq A : U \text{ open in } \mathbb{X}\} && \text{(De Morgan's Law)} \\
&= \bigcap \{K \supseteq A : K \text{ closed in } \mathbb{X}\}. && \text{(Definition: 2.1.3)}
\end{aligned}$$

■

**Proposition 2.7.4.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

Then we have

$$X \setminus A^\circ = (X \setminus A)^-.$$

**Proof.** First, we have

$$\begin{aligned}
X \setminus A^\circ &= X \setminus \bigcup (\mathcal{T} \cap 2^A) && \text{(Definition 2.5.1)} \\
&= \bigcap_{K \in \mathcal{T} \cap 2^A} (X \setminus K) && \text{(De Morgan's Law)}
\end{aligned}$$

For any  $K$ ,  $X \setminus K$  is a closed superset of  $X \setminus A$ .

As closed sets are closed under arbitrary intersection (Proposition 2.1.2), and  $X \setminus A^\circ$  is the intersection of all closed superset of  $X \setminus A$ , by Proposition 2.7.3,  $X \setminus A^\circ = (X \setminus A)^-$ .

■

**Proposition 2.7.5.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A, B \subseteq X$ .  
If  $A \subseteq B$ , then  $A^- \subseteq B^-$ .

**Proof.**

$$\begin{aligned}
A \subseteq B &\iff X \setminus A \supseteq X \setminus B \\
&\implies (X \setminus A)^\circ \supseteq (X \setminus B)^\circ && \text{(Proposition 2.5.5)} \\
&\iff X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ \\
&\iff (X \setminus (X \setminus A))^- \subseteq (X \setminus (X \setminus B))^- && \text{(Proposition 2.7.4)} \\
&\iff A^- \subseteq B^-.
\end{aligned}$$

■

**Proposition 2.7.6.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \in \mathcal{T}$  such that for any  $x \in A$ , there exists a  $U \in \mathcal{T}$  such that  $x \in U \subseteq A$ .

Then  $A$  is open in  $\mathbb{X}$ .

**Proof.** Aiming for a contradiction, suppose for any  $x \in A$ , there exists a  $U \in \mathcal{T}$  such that  $x \in U \subseteq A$ , but  $A$  is not open.

By Definition 2.1.3, as  $A$  is not open,  $X \setminus A$  is not closed.

By Proposition 2.6.2, there exists  $x \in L(A) \setminus (X \setminus A)$ . Fix  $x$ .

As  $x \notin X \setminus A$ ,  $x \in A$ .

By Definition 2.6.1, for  $U \in \mathcal{T}$  with  $x \in U$ ,  $U \cap (X \setminus A) \neq \emptyset$ , i.e.,  $U \setminus A \neq \emptyset$ .

This implies that  $U \not\subseteq A$ .

This contradicts the assumption we have.

Thus  $A$  has to be open.

■

## §2.8 Density

---

**Definition 2.8.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

Then  $A$  is said to be *everywhere dense*, or simply *dense*, in  $\mathbb{X}$  iff

$$A^- = X.$$

**Proposition 2.8.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

Then  $A$  is dense in  $\mathbb{X}$  iff for any  $U \in \mathcal{T}$ ,  $A \cap U \neq \emptyset$ .

**Proof.** First, prove  $\Rightarrow$ .

Assume  $A$  is dense in  $\mathbb{X}$ , then, by Definition 2.8.1,  $A^- = X$ .

By Definition 2.6.2, for any  $x \in I(A)$ ,  $x \in A$ .

By Definition 2.6.1, for any  $x \in L(A)$  and for any  $U \in \mathcal{T}$  with  $x \in U$ ,  $U \cap A \neq \emptyset$ .

As  $A^- = X$ , then, by Proposition 2.7.2,  $X = I(A) \sqcup L(A)$ .

Thus for any  $U \in \mathcal{T}$ ,  $A \cap U \neq \emptyset$ .

□

Now, prove  $\Leftarrow$ .

Aiming for a contradiction, suppose for any  $U \in \mathcal{T}$ ,  $A \cap U \neq \emptyset$ , but  $A$  is not dense in  $\mathbb{X}$ .

As,  $A \subseteq X$ , by Proposition 2.7.5,  $A^- \subseteq X^-$ . And, as  $X$  is closed in  $\mathbb{X}$ , by Proposition 2.7.1,  $X = X^-$ . Therefore,  $A^- \subseteq X$ .

As  $A$  is not dense in  $X$ , by Definition 2.8.1,  $A^- \neq X$ . Therefore,  $A^- \subsetneq X$ . This implies that  $X \setminus A^-$  is non-empty. And, by Definition 2.7.1,  $X \setminus A^- \in \mathcal{T}$ .

By Proposition 2.5.2, for any  $x \in X \setminus A^-$ , there exists a  $U \in \mathcal{T}$  with  $x \in U$ , such that  $U \in X \setminus A^-$ . Then  $U \cap A = \emptyset$ . This contradicts the assumption we have.

Therefore,  $A$  has to be dense in  $\mathbb{X}$ .

□

Thus, the proof is done. ■

**Definition 2.8.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

Then  $A$  is said to be *nowhere dense in  $\mathbb{X}$*  iff

$$(A^-)^\circ = \emptyset.$$

**Proposition 2.8.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

Then  $A$  is nowhere dense in  $\mathbb{X}$  iff for any  $U \in \mathcal{T} \setminus \{\emptyset\}$ ,

$$U \setminus A^- \neq \emptyset.$$

**Proof.**

$A$  is nowhere dense in  $\mathbb{X}$

$$\iff (A^-)^\circ = \emptyset \quad (\text{Definition 2.8.2})$$

$$\iff (A^-)^\circ = \bigcup (\mathcal{T} \cap 2^A) = \emptyset \quad (\text{Definition 2.5.1})$$

$$\iff (\forall U \in \mathcal{T} : U \subseteq A^-) \quad U = \emptyset.$$

■

## Chapter 3.

# Sequences

### §3.1 Convergent Sequences

---

**Definition 3.1.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $u : \mathbb{N} \rightarrow X$ .  
*u converges to a limit  $x \in X$*  iff for any  $U \in \mathcal{T}$  with  $x \in U$ ,

$$(\exists k \in \mathbb{R}_{>0}) \quad u[\mathbb{N}_{>k}] \subseteq U.$$

**Proposition 3.1.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $u : \mathbb{N} \rightarrow X$ .  
*u converges to a limit  $x \in X$*  iff for any  $U \in \mathcal{T}$  with  $x \in U$ ,

$$\mathbb{N} \setminus u^{-1}[U] \text{ is finite.}$$

**Proof.** First, prove  $\Rightarrow$ .

By Definition 3.1.1, as  $u$  converges to  $x$ , let  $U \in \mathcal{T}$  with  $x \in U$ , then there exists a  $k \in \mathbb{R}_{>0}$  such that  $u[\mathbb{N}_{>k}] \subseteq U$ .

Then we have

$$\begin{aligned} u[\mathbb{N}_{>k}] \subseteq U &\implies u^{-1}[u[\mathbb{N}_{>k}]] \subseteq u^{-1}[U] \\ &\implies \mathbb{N}_{>k} \subseteq u^{-1}[U] && \text{(image of inverse image)} \\ &\implies \mathbb{N} \setminus \mathbb{N}_{>k} \supseteq \mathbb{N} \setminus u^{-1}[U]. \end{aligned}$$

As  $\mathbb{N} \setminus \mathbb{N}_{>k}$  is finite, its subset  $\mathbb{N} \setminus u^{-1}[U]$  is finite.

□

Now, prove  $\Leftarrow$ .

By [image of inverse image](#), we have

$$u[u^{-1}[U]] \subseteq U.$$

As  $u^{-1}[U]$  is a cofinite subset of  $\mathbb{N}$ , there exists a  $k \in \mathbb{N}$  such that  $I \supseteq \mathbb{N}_{>k}$ . Then we have

$$U \supseteq u[\mathbb{N}_{>k}].$$

This precisely satisfies Definition 3.1.1.

□

Therefore the proof is done.

■

### §3.2 Accumulation Points of Sequences

---

**Definition 3.2.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $u : \mathbb{N} \rightarrow X$ .

A point  $x \in X$  is an *accumulation point* of  $u$  iff for any  $U \in \mathcal{T}$  with  $x \in U$ ,  $U$  contains infinitely many terms of  $u$ ; i.e.,

$$\forall U \in \mathcal{T} : x \in U \implies (\exists I \subseteq \mathbb{N} : |I| = \aleph_0 \implies u[I] \subseteq U).$$

**Note 3.2.1.** Sometime, an accumulation point of a sequence is also a limit of the range of the sequence. But this not always holds.

Consider  $\mathbb{R}$  as a Euclidean, and let  $u : \mathbb{N} \rightarrow \mathbb{R}$  be defined as

$$u(n) := \left| \sin \left( \frac{\pi n}{2} \right) \right|.$$

Then 1 is an accumulation point of  $u[\mathbb{N}]$ , but  $u[\mathbb{N}] = (u[\mathbb{N}])^i = \{0, 1\}$ , so it has no limit point at all.

**Proposition 3.2.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, let  $u : \mathbb{N} \rightarrow X$ , and let  $x \in X$  be a limit of  $u[\mathbb{N}]$ .

Then  $x$  is an accumulation point of  $u$ .

**Proof.** Let  $U \in \mathcal{T}$  with  $x \in U$ , then we have

$$u[u^{-1}[U]] \subseteq U.$$

By Proposition 3.1.1, as  $u$  converges to  $x$ ,  $u^{-1}[U]$  is a cofinite subset of  $\mathbb{N}$ . Thus  $u^{-1}[U]$  is infinite.

As  $u^{-1}[U]$  is infinite and  $x \in U \in \mathcal{T}$ , by Definition 3.2.1,  $x$  is an accumulation point of  $u$ .

■



**Definition 3.2.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

A point  $x \in X$  is an  $\omega$ -accumulation point of  $A$  iff for any  $U \in \mathcal{T}$  with  $x \in U$ ,

$$|U \cap A| \geq \aleph_0.$$

**Proposition 3.2.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, let  $u : \mathbb{N} \rightarrow X$  be an injection, and let  $x \in X$  be an accumulation point of  $u$ .

Then  $x$  is an  $\omega$ -accumulation point of  $u[\mathbb{N}]$ .

**Proof.** By Definition 3.2.1, as  $x$  is an accumulation point of  $u$ , let  $U \in \mathcal{T}$  with  $x \in U$ , there exists an infinite  $I \subseteq \mathbb{N}$  such that  $u[I] \subseteq U$ .

As  $u$  is injective and  $I$  is infinite,  $u[I]$  is also infinite.

As  $u[I] \subseteq U$  and  $U \in \mathcal{T}$  with  $x \in U$  is arbitrarily given, by Definition 3.2.2,  $x$  is an  $\omega$ -accumulation point of  $u[\mathbb{N}]$ . ■

## Chapter 4.

# Countable Axioms

### §4.1 Covers and Bases

---

**Definition 4.1.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

Then a family  $\mathcal{C} \subseteq 2^X$  is a *cover for A* iff  $A \subseteq \bigcup \mathcal{C}$ .

$\mathcal{C}$  is an *open cover* iff  $\mathcal{C} \subseteq \mathcal{T}$ .

**Definition 4.1.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $\mathcal{C}, \mathcal{D}$  be covers for a subset  $A \subseteq X$ .

Then  $\mathcal{D}$  is a *subcover of  $\mathcal{C}$*  iff  $\mathcal{D} \subseteq \mathcal{C}$ .

**Definition 4.1.3.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space.

A family  $\mathcal{B} \subseteq 2^X$  is an *analytic basis for  $\mathcal{T}$*  iff

- (i)  $\mathcal{B} \subseteq \mathcal{T}$ ;
- (ii) For any  $U \in \mathcal{T}$ , there is an  $\mathcal{A} \subseteq \mathcal{B}$ , such that  $U = \bigcup \mathcal{A}$ .

**Proposition 4.1.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $\mathcal{B} \subseteq \mathcal{T}$ .

Then  $\mathcal{B}$  is an analytic basis for  $\mathcal{T}$  iff for any  $U \in \mathcal{T}$  and for any  $x \in U$ , there exists a  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ .

**Proof.** First, prove  $\Rightarrow$ .

By Definition 4.1.3, as  $\mathcal{B}$  is an analytic basis for  $\mathcal{T}$ , let  $U \in \mathcal{T}$ , then there is an  $\mathcal{A} \subseteq \mathcal{B}$  such that  $U = \bigcup \mathcal{A}$ .

Then, for any  $x \in U$ , there exists at least one  $A \in \mathcal{A}$  such that  $x \in A$ . As  $U = \bigcup \mathcal{A}$ ,  $A \subseteq U$ .

□

Now, prove  $\Leftarrow$ .

By Proposition 2.7.6, as for any  $U \in \mathcal{T}$  and for any  $x \in U$ , there exists a  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ , then there exists an  $\mathcal{A} \subseteq \mathcal{B}$  such that  $\bigcup \mathcal{A} = U$ .

By Definition 4.1.3,  $\mathcal{B}$  is an analytic basis for  $\mathcal{T}$ .

□

Thus, the proof is done.

■

**Definition 4.1.4.** Let  $X$  be any set.

A family  $\mathcal{B} \subseteq 2^X$  is a *synthetic basis on  $X$*  iff

- (i)  $\mathcal{B}$  is a cover for  $X$ ;
- (ii) For any  $U, V \in \mathcal{B}$ , there exists  $\mathcal{A} \subseteq \mathcal{B}$ , such that  $U \cap V = \bigcup \mathcal{A}$ .

**Definition 4.1.5.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $x \in X$ .

A family  $\mathcal{B} \subseteq 2^X$  is a *local basis at  $x$*  iff

- (i)  $\mathcal{B} \in \mathcal{T}$ ;
- (ii) For any  $B \in \mathcal{B}$ ,  $x \in B$ ;
- (iii) For any  $U \in \mathcal{T}$  with  $x \in U$ , there exists a  $B \in \mathcal{B}$  such that  $B \subseteq U$ .

## §4.2 First-Countable Spaces

---

**Definition 4.2.1.** A topological space  $\mathbb{X} = (X, \mathcal{T})$  is said to be *first-countable* iff any  $x \in X$  has a countable basis.

**Proposition 4.2.1.** Metric spaces are first-countable.

**Proof.** Let  $\mathbb{X} = (X, d)$  be a metric space.

For any  $x \in X$ , let  $\mathcal{B}_x : \mathbb{N} \rightarrow \mathcal{T}$  be defined as

$$\mathcal{B}_x(n) := B_{1/n}(x).$$

Clearly, the image  $\mathcal{B}_x[\mathbb{N}]$  is countable.

Let  $U \in \mathcal{T}$ . As  $U$  is open, and as  $x \in U$ , then there exists  $\varepsilon \in \mathbb{R}_{>0}$  such that  $B_\varepsilon(x) \subseteq U$ .

By Archimedean Principle, there exists an  $n \in \mathbb{N}$  such that  $n > \frac{1}{\varepsilon}$ . Then we have

$$\mathcal{B}_x(n) = B_{1/n}(x) \subseteq B_\varepsilon(x) \subseteq U.$$

As  $U$  is arbitrarily given, for any  $x \in X$ ,  $\mathcal{B}_x[\mathbb{N}]$  is a countable local basis at  $x$ . ■

**Proposition 4.2.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a first-countable topological space, let  $u : \mathbb{N} \rightarrow X$ , and let  $x \in X$  be an accumulation point of  $u$ .

Then  $x$  is a *subsequential limit* of  $u$ . That is, there exists an infinite  $I \subseteq \mathbb{N}$ , such that  $u \upharpoonright_I$  converges to  $x$  (as a limit).

**Proof.**<sup>1</sup> By Definition 4.2.1, as  $\mathbb{X}$  is first-countable, there exists a countable local basis  $\mathcal{B}$  at  $x$ .

Let  $\mathcal{B}_x : \mathbb{N} \rightarrow \mathcal{T}$  such that  $\mathcal{B}_x[\mathbb{N}]$  is a local base at  $x$  and for any  $n \in \mathbb{N}$ ,

$$\mathcal{B}_x(n) \supseteq \bigcup \mathcal{B}_x[\mathbb{N}_{>n}].$$

Let  $w : I \rightarrow u[\mathbb{N}]$  ( $I$  infinite) such that for any  $i \in I$ ,  $w(i) \in \mathcal{B}_x(i)$ .

Then, for any  $k \in \mathbb{N}$ , we have  $w[I_{>k}] \subseteq \mathcal{B}_x(k)$ . Thus, by Definition 3.1.1,  $w$  is a subsequence of  $u$  converging to  $x$ . ■

### §4.3 Second-Countable Spaces

---

**Definition 4.3.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space.

$\mathbb{X}$  is said to be *second countable* iff  $\mathcal{T}$  has a countable (analytic) basis.

**Proposition 4.3.1.** Second-countable spaces are first-countable.

**Proof.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a second-countable space.

By Definition 4.3.1,  $\mathcal{T}$  has a countable analytic basis.

Let  $x \in X$  and let  $U \in \mathcal{T}$  with  $x \in U$ . By Definition 4.1.3 there exists a countable  $\mathcal{B} \subseteq \mathcal{T}$ , such that for any  $U \in \mathcal{T}$ , there is an  $\mathcal{A} \subseteq \mathcal{B}$ , such that  $U = \bigcup \mathcal{A}$ .

---

<sup>1</sup> The detail of this proof is incomplete.

As  $U \in \mathcal{T}$  and  $U = \bigcup \mathcal{A}$ , by Proposition 2.5.2, there exists a  $A \in \mathcal{A}$  such that  $x \in A \subseteq U$ .

Let  $\mathcal{C} \subseteq \mathcal{B}$  be the family of all such  $A$  containing  $x$ , then, by Definition 4.1.5,  $\mathcal{C}$  is a local basis at  $x$ . And as  $\mathcal{B}$  is countable, as a subset,  $\mathcal{C}$  is also countable.

Therefore  $\mathcal{C}$  is a countable local basis at  $x$ .

As  $x$  is arbitrarily given,  $\mathbb{X}$  is first-countable. ■

**Example 4.3.1.** Consider  $\mathbb{R}$  as a Euclidean metric space.

$\mathbb{R}$  is second-countable.

**Proof.** By Proposition 4.2.1,  $\mathbb{R}$  is first-countable.

For any  $x \in \mathbb{Q}$ , let  $\mathcal{O}_x : \mathbb{N} \rightarrow \mathcal{T}$  be defined as

$$\mathcal{O}_x(n) := B_{1/n}(x).$$

For any  $r \in \mathbb{R}$  and for any open set  $U \ni r$ , there exists  $\delta \in \mathbb{R}_{>0}$  such that  $B_\delta(r) \subseteq U$ .

There exists some  $q \in \mathbb{Q}$  such that  $q \in B_\delta(r)$ . As  $B_\delta(r)$  is open, by Definition 1.2.1, there exists  $\varepsilon \in \mathbb{R}_{>0}$  such that  $B_\varepsilon(q) \subseteq B_\delta(r)$ .

By Archimedean property, there exists  $k \in \mathbb{N}$  such that  $k > \frac{1}{\varepsilon}$ . Then we have

$$\mathcal{O}_q(k) = B_{1/k}(q) \subseteq B_\varepsilon(q) \subseteq B_\delta(r).$$

[This proof is incomplete]

**Example 4.3.2.** Let  $\mathbb{X} = (\mathbb{R}, \mathcal{T})$  be a discrete topological space.

$\mathbb{X}$  is first-countable but not second-countable.

## §4.4 Separable Spaces

---

**Definition 4.4.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space.

$\mathbb{X}$  is said to be *separable* iff there exists a countable subset  $A \subseteq X$  such that  $A$  is dense in  $\mathbb{X}$ .

**Proposition 4.4.1.** Second-countable spaces are separable.

**Proof.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a second-countable topological space.

As  $\mathbb{X}$  is second-countable, by Definition 4.3.1, there is a countable base  $\mathcal{B}$  for  $\mathcal{T}$ .

Let  $f : \mathcal{B} \rightarrow X$  such that for any  $B \in \mathcal{B}$ ,

$$f(B) = \text{a random } x \in B.$$

As  $\mathcal{B}$  is countable, then  $f[\mathcal{B}]$  is countable.

Now, it suffices to show that  $f[\mathcal{B}]$  is dense in  $\mathbb{X}$ .

Aiming for a contradiction, suppose  $f[\mathcal{B}]$  is not dense in  $\mathbb{X}$ , then, there exists some  $x \in X \setminus (f[\mathcal{B}])^-$ .

By Definition 2.1.3,  $X \setminus (f[\mathcal{B}])^- \in \mathcal{T}$ ; by Definition 2.5.2, there exists  $U \in \mathcal{T}$  with  $U \ni x$  such that  $U \subseteq X \setminus (f[\mathcal{B}])^-$ . That is, for any  $B \in \mathcal{B}$ ,  $f(B) \notin U$ ; i.e.,  $f[\mathcal{B}] \cap U = \emptyset$ .

As  $U \in \mathcal{T}$  and  $\mathcal{B}$  is a base for  $\mathcal{T}$ , by Definition 4.1.3, there exists  $\mathcal{A} \subseteq \mathcal{B}$  such that  $\bigcup \mathcal{A} = U$ . Thus,  $f[\mathcal{A}]$  must be a non-empty subset of  $U$ . This contradicts  $f[\mathcal{B}] \cap U = \emptyset$ .

Thus,  $f[\mathcal{B}]$  has to be dense in  $\mathbb{X}$ . As  $f[\mathcal{B}]$  is countable, therefore,  $\mathbb{X}$  is second-countable. ■

**Example 4.4.1.** Niemytzki plane is separable but not second-countable.<sup>2</sup>

**Proposition 4.4.2.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a discrete topological space which is separable.

Then  $X$  is countable.

**Proof.** Aiming for a contradiction, suppose  $X$  is uncountable.

As  $\mathbb{X}$  is separable, by Definition 4.4.1, there exists a countable subset  $A \subseteq X$  being dense in  $\mathbb{X}$ .

By Definition 2.8.1,  $A^- = X$ .

As  $\mathbb{X}$  is discrete,  $A^- = A$ .

Now, we have  $A = X$ . As  $A$  is countable but  $X$  is not, this is impossible.

This contradiction shows that  $X$  has to be countable. ■

**Proposition 4.4.3.** Separable metric spaces are second-countable.

**Proof.** Let  $\mathbb{X} = (X, d)$  be a metric space which is separable. Denote  $\mathcal{T}$  for the topology on  $X$  induced by  $d$ .

---

<sup>2</sup> See [ProofWiki](#).

By Definition 4.4.1, let  $A \subseteq X$  be a countable set with  $A^- = X$  (by Definition 2.8.1,  $A$  dense in  $\mathbb{X}$ ).

Let  $\mathcal{B} : \mathbb{N} \times A \rightarrow \mathcal{T}$  be defined as

$$\mathcal{B}(n, a) := B_{1/n}(a).$$

Let  $\varepsilon \in \mathbb{R}_{>0}$  and let  $x \in X$ . Then  $B_\varepsilon(x)$  defines an open ball in  $\mathbb{X}$ .

As  $A^- = X$  and  $x \in X$ ,  $x \in A^-$  also. Thus, there exists an  $a \in A \cap B_\varepsilon(x)$ .

By Proposition 1.2.1, as  $a \in B_\varepsilon(x)$ , there exists a  $\delta \in \mathbb{R}_{>0}$  such that  $B_\delta(a) \subseteq B_\varepsilon(x)$ .

By Archimedean property, let  $k \in \mathbb{N}$  such that  $k > \frac{1}{\delta}$ , then we have

$$\mathcal{B}(k, a) = B_{1/k}(a) \subseteq B_\delta(a) \subseteq B_\varepsilon(x).$$

By Proposition 4.1.1,  $\mathcal{B}[\mathbb{N} \times A]$  is an analytic basis for  $\mathcal{T}$ . As  $\mathbb{N} \times A$  is countable, the image  $\mathcal{B}[\mathbb{N} \times A]$  is also countable.

Therefore,  $\mathcal{B}[\mathbb{N} \times A]$  is a countable analytic basis for  $\mathcal{T}$ . By Definition 4.3.1,  $\mathbb{X}$  is second-countable. ■

## §4.5 Lindelöf Space

---

**Definition 4.5.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space.

Then  $\mathbb{X}$  is said to be *Lindelöf* iff every open cover for  $X$  has a countable subcover.

**Proposition 4.5.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a second-countable topological space.

Then  $\mathbb{X}$  is Lindelöf.

**Proof.** As  $\mathbb{X}$  is second-countable, by Definition 4.3.1, there exists a countable basis  $\mathcal{B}$  for  $\mathcal{T}$ .

Let  $\mathcal{U}$  be an open cover of  $\mathbb{X}$ , no matter it is countable or not.

By Definition 4.1.3, for any  $U \in \mathcal{U}$ , there exists an  $\mathcal{A} \subseteq \mathcal{B}$  such that  $\bigcup \mathcal{A} = U$ .

Let  $f : \mathcal{B} \rightarrow \mathcal{U}$  be defined as

$$f(B) := \text{a random } U \in \mathcal{U} \text{ with } U \supseteq B.$$

As  $\mathcal{B}$  is an open over of  $X$  and for any  $B \in \mathcal{B}$ ,  $f(B) \supseteq B$ , thus  $f[\mathcal{B}]$  is an open cover of  $\mathcal{B}$ .

As  $\mathcal{U}$  is the codomain of  $f$ ,  $f[\mathcal{B}] \subseteq \mathcal{U}$ .

Therefore,  $f[\mathcal{B}]$  is a subcover of  $\mathcal{U}$ .

As  $\mathcal{B}$  is countable, its image  $f[\mathcal{B}]$  is countable.

Therefore,  $f[\mathcal{B}]$  is a countable subcover of  $\mathcal{U}$ .

As  $\mathcal{U}$  is arbitrarily given, by Definition 4.5.1,  $\mathbb{X}$  is Lindelöf.

**Example 4.5.1.** Sorgenfrey line is a topological space which is Lindelöf but not second-countable. (See Section A.1.)



## Chapter 5.

# Continuous Mappings

### §5.1 Continuous Mappings

---

**Definition 5.1.1.** Let  $\mathbb{X} = (X, \mathcal{T}_X)$  and  $\mathbb{Y} = (Y, \mathcal{T}_Y)$  be topological spaces, let  $f : \mathbb{X} \rightarrow \mathbb{Y}$ , and let  $A \subseteq X$  be a mapping.

Then  $f$  is said to be *continuous on  $A$*  iff there exists a  $U_X \in \mathcal{T}_X$  with  $A \subseteq U_X$ , such that for any  $U_Y \in \mathcal{T}_Y$ ,

$$f^{-1}[U_Y] \cap U_X \in \mathcal{T}_X.$$

$f$  is a *continuous mapping* iff  $A = X$ ; i.e., it is continuous on whole  $X$ .

**Note 5.1.1.** By Definition 5.1.1,  $f$  is *continuous at a point  $x \in X$* , iff it is continuous on some  $U_X \in \mathcal{T}$  with  $x \in U_X$ , as  $x$  here can be considered as a singleton  $\{x\}$ .

**Note 5.1.2.** There is a common error: if for any  $U_X \in \mathcal{T}_X$ , its image  $f[U_X] \in \mathcal{T}_Y$  also, then  $f$  is continuous. But, this condition also holds for some discontinuous mappings.

For example, let  $\mathbb{X} = (\mathbb{R}, \mathcal{T}_X)$  be a topological space where  $\mathcal{T}$  induced by Euclidean metric, and let  $\mathbb{Y} = (\mathbb{R}, \mathcal{T}_Y)$  be a discrete topological space. Let  $\iota : \mathbb{X} \rightarrow \mathbb{Y}$  be an identity mapping; i.e., it is defined as

$$\iota : \mathbb{X} \rightarrow \mathbb{Y} : x \mapsto x.$$

For any  $A \subseteq \mathbb{R}$ , clearly,  $\iota[A] \in \mathcal{T}_Y$  holds. But for some (or for all)  $B \in \mathcal{T}_Y \setminus \mathcal{T}_X$ ,  $\iota^{-1}[B] \notin \mathcal{T}$ . Thus,  $\iota$  is not a identity mapping.

Indeed, for any identity mapping  $\iota : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ ,  $\iota$  is continuous iff  $\mathcal{T}_X \supseteq \mathcal{T}_Y$ .

**Example 5.1.1.** Let  $\mathbb{X} = (X, \mathcal{T}_X)$  be a topological space, where  $\mathcal{T}_X$  is the discrete topology on  $X$ . Let  $\mathbb{Y} = (X, \mathcal{T}_Y)$  be any topological space. Then for any  $f : \mathbb{X} \rightarrow \mathbb{Y}$ ,  $f$  is continuous.

**Proposition 5.1.1.** Let  $\mathbb{X} = (X, \mathcal{T}_X)$  and  $\mathbb{Y} = (X, \mathcal{T}_Y)$  be topological spaces, let  $f : \mathbb{X} \rightarrow \mathbb{Y}$ , and let  $A \subseteq X$  be a mapping.

$f$  is continuous on  $A$  iff for any  $U_Y \in \mathcal{T}$  with  $f[A] \subseteq U_Y$ , there exists a  $U_X$  with  $A \subseteq U_X$ , such that  $f[U_X] \subseteq U_Y$ .

**Proof.** First, prove  $\Rightarrow$ .

Assume  $f$  is continuous on  $A$ , then, by Definition 5.1.1, let  $U_Y \in \mathcal{T}$  with  $f[A] \subseteq U_Y$ , then there exists  $U_X \in \mathcal{T}$  with  $A \subseteq U_X$ , such that

$$f^{-1}[U_Y] \cap U_X \in \mathcal{T}.$$

Then we have

$$\begin{aligned} U_X &\subseteq f^{-1}[U_Y] \cap U_X \\ \Rightarrow f[U_X] &\subseteq f[f^{-1}[U_Y] \cap U_X] \\ \Rightarrow f[U_X] &\subseteq f[f^{-1}[U_Y]] \cap f[U_X] \\ &\quad \text{(Image of Intersection under Mapping)} \\ \Rightarrow f[U_X] &\subseteq U_Y \cap f[U_X]. \\ &\quad \text{(Image of Inverse Image)} \\ \Rightarrow f[U_X] &\subseteq U_Y. \end{aligned}$$

■

**Proposition 5.1.2.** Let  $\mathbb{X} = (X, \mathcal{T}_X)$ ,  $\mathbb{Y} = (X, \mathcal{T}_Y)$  and  $\mathbb{Z} = (X, \mathcal{T}_Z)$  be topological spaces, let  $f : \mathbb{X} \rightarrow \mathbb{Y}$  and  $g : \mathbb{Y} \rightarrow \mathbb{Z}$  be continuous mapping.

Then  $f \circ g$  is continuous.

**Proof.** By Definition 5.1.1, as  $g$  is continuous, for any  $U_Z \in \mathcal{T}_Z$ ,  $g^{-1}[U_Z] \in \mathcal{T}_Y$ . Similarly,  $f^{-1}[g^{-1}[U_Z]] \in \mathcal{T}_X$ .

As  $U_Z \in \mathcal{T}_Z$  is arbitrarily given,  $f \circ g$  is continuous.

■

## §5.2 Homeomorphisms

---

**Definition 5.2.1.** Let  $\mathbb{X} = (X, \mathcal{T}_X)$  and  $\mathbb{Y} = (X, \mathcal{T}_Y)$  be topological spaces, and let  $f : \mathbb{X} \rightarrow \mathbb{Y}$  be a mapping.

$f$  is a *homeomorphism* iff

**H1.**  $f$  is bijective (injective and surjective);

**H2.**  $f$  is continuous;

**H3.**  $f^{-1}$  is continuous;

**Definition 5.2.2.** Let  $\mathbb{X} = (X, \mathcal{T}_X)$  and  $\mathbb{Y} = (X, \mathcal{T}_Y)$  be topological spaces.

$\mathbb{X}$  and  $\mathbb{Y}$  are said to be *homeomorphic*, denoted  $\mathbb{X} \cong \mathbb{Y}$ , iff there exists a homeomorphism between  $\mathbb{X}$  and  $\mathbb{Y}$ .

**Note 5.2.1.** Rigorously speaking, if we say that two subsets  $A, B \subseteq X$  are homeomorphic, i.e.,  $A \cong B$ ,  $A$  and  $B$  are considered as subspaces of  $\mathbb{X} = (X, \mathcal{T})$ , and these two subspaces are homeomorphic.

Indeed, being homeomorphic is a relation between topological spaces but not sets without considering their topologies.

**Proposition 5.2.1.** Being homeomorphic is an equivalent relation.

**Proof.** Let  $\mathbb{X} = (X, \mathcal{T}_X)$ ,  $\mathbb{Y} = (X, \mathcal{T}_Y)$  and  $\mathbb{Z} = (X, \mathcal{T}_Z)$  be topological spaces.

Let  $\iota : \mathbb{X} \rightarrow \mathbb{X}$  be an identity mapping.

For any  $x_1, x_2 \in X$  with  $x_1 \neq x_2$ ,  $\iota(x_1) = x_1$  and  $\iota(x_2) = x_2$ , so  $\iota(x_1) \neq \iota(x_2)$ .

Thus  $\iota$  is injective.

For any  $x \in X$ , there exists  $\iota^{-1}(x) = x \in X$ . Thus  $\iota$  is surjective.

As  $\iota$  is injective and surjective, it is bijective.

For any  $U \in \mathcal{T}_X$ ,  $\iota^{-1}[U] = U \in \mathcal{T}_X$ . Thus, by Definition 5.1.1,  $\iota$  is continuous.

Similarly,  $\iota^{-1}$  is continuous.

Therefore, by Definition 5.2.1,  $\iota$  is an homeomorphism between  $\mathbb{X}$  and  $\mathbb{X}$ . By Definition 5.2.2,  $\mathbb{X}$  is homeomorphic to itself, i.e.,  $\mathbb{X} \cong \mathbb{X}$ .

Thus, being homeomorphic is reflexive.

□

Assume  $\mathbb{X} \cong \mathbb{Y}$ .

By Definition 5.2.2, there exists a homeomorphism  $f : \mathbb{X} \rightarrow \mathbb{Y}$ .

As  $f$  is bijective, then  $f^{-1}$  is also bijective.

By Definition 5.2.1,  $f$  and  $f^{-1}$  are both continuous.

As  $f^{-1}$  is bijective, continuous, and  $(f^{-1})^{-1} = f$  is also continuous, then  $f^{-1} : \mathbb{Y} \rightarrow \mathbb{X}$  is also a homeomorphism. By Definition 5.2.2, we have  $\mathbb{Y} \cong \mathbb{X}$ .

Thus, being homeomorphic is symmetric. □

Assume  $\mathbb{X} \cong \mathbb{Y}$  and  $\mathbb{Y} \cong \mathbb{Z}$ .

By Definition 5.2.2, we have  $f : \mathbb{X} \rightarrow \mathbb{Y}$  and  $g : \mathbb{Y} \rightarrow \mathbb{Z}$  as homeomorphisms.

By Definition 5.2.1 H1,  $f$  and  $g$  are bijective. Thus,  $f \circ g$  is bijective.

By Definition 5.2.1 H2,  $f$  and  $g$  are continuous, so, by Proposition 5.1.2,  $f \circ g$  is continuous. Similarly,  $g^{-1} \circ f^{-1}$  is continuous. As  $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$  (see [inverse of composite relation](#)),  $(f \circ g)^{-1}$  is also continuous.

As  $f \circ g$  is bijective,  $f \circ g$  is continuous and  $(f \circ g)^{-1}$  is also continuous,  $f \circ g : \mathbb{X} \rightarrow \mathbb{Z}$  is a homeomorphism. By Definition 5.2.2,  $\mathbb{X} \cong \mathbb{Z}$ .

Thus, being homeomorphic is transitive. □

As being homeomorphic is reflexive, symmetric, and transitive, it is an equivalence relation. ■

**Example 5.2.1.** In Euclidean metric space  $\mathbb{R}$ , let  $a, b, c, d \in \mathbb{R}$  with  $a < b$  and  $c < d$ , then we have:

- $[a, b] \cong [c, d]$ ;
- $[a, b) \cong [c, d)$ ;
- $[a, b) \cong (c, d]$ ;
- $(a, b) \cong (c, d)$ .

**Example 5.2.2.** A donut is homeomorphic to a cup, because they both have a hole.

**Example 5.2.3.** Consider  $\mathbb{R}_{[0,1]}$  and  $\mathbb{R}^n$  as Euclidean metric spaces. Let  $A$  be an index set. For any  $\alpha \in A$ , let  $f_\alpha : I \rightarrow X$  be a continuous and piece-wise smooth injection.

Then, for any  $\alpha, \beta \in A$ ,  $f_\alpha[I] \cong f_\beta[I]$ . (See, Figure 5.1.)

**Example 5.2.4.** Consider  $\mathbb{R}^n$  as a Euclidean metric space, let  $S^{n-1} \subseteq \mathbb{R}^n$  be a  $n - 1$ -sphere, i.e., let  $o \in \mathbb{R}^n$  and  $r \in \mathbb{R}$ ,

$$S^{n-1} := \{x \in \mathbb{R}^n : d(o, x) = r\},$$

where  $d$  is the Euclidean metric on  $\mathbb{R}^n$ .

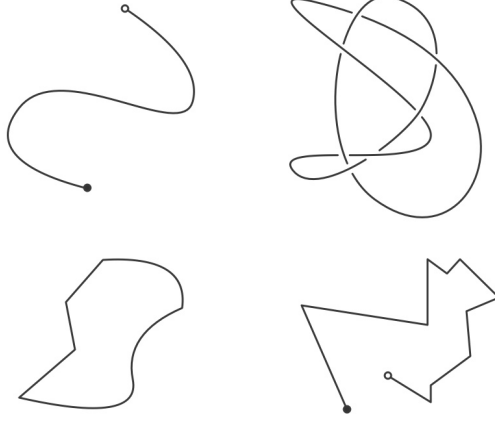


Figure 5.1: Homeomorphic curves in  $\mathbb{R}^3$ .

Let  $y \in S^{n-1}$ , and let

$$U \in \{S^{n-1} \setminus \overline{B}_\varepsilon(x), S^{n-1} \setminus \{x\}\},$$

where  $\varepsilon \in \mathbb{R}$  suffices

$$0 < \varepsilon < \max_{a,b \in S^{n-1}} d(a,b).$$

Then we have  $U \cong \mathbb{R}^{n-1}$ .

**Example 5.2.5.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space with  $\mathcal{T}$  discrete. For any  $U, V \in X$  with  $|U| = |V| = |X|$ ,  $U \cong V$ .

**Proof.** As  $\mathcal{T} = 2^X$ , for any  $U, V \in X$ ,  $(U, 2^U)$  and  $(V, 2^V)$  are subspace of  $\mathbb{X}$ .

By the definition of comparison of cardinality, if  $|U| = |V|$ , there exists a bijection  $f : U \rightarrow V$ .

For any  $A \in 2^V$ ,  $f[A] \in 2^U$ , thus, by Definition 5.1.1,  $f$  is continuous. Similarly,  $f^{-1}$  is also continuous.

As  $f$  is bijective, and bi-continuous, by Definition 5.2.1,  $f$  is a homeomorphism between  $(U, 2^U)$  and  $(V, 2^V)$ . By Definition 5.2.2,  $U \cong V$ . ■

## Chapter 6.

# Separation Axioms

### §6.1 Neighbourhood Systems

---

**Definition 6.1.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

A subset  $N \subseteq X$  is a *neighbourhood of  $A$*  iff

$$(\exists U \in \mathcal{T}) \quad A \subseteq U \subseteq N.$$

If  $A = \{x\}$ , we simply call  $N$  a *neighbourhood of  $x$* .

If  $N \in \mathcal{T}$  also, then  $N$  is an *open neighbourhood of  $A$* ; and if  $N$  is closed, then  $N$  is a *closed neighbourhood of  $A$* .

**Proposition 6.1.1.** Let  $\mathbb{X} = (X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

$A \in \mathcal{T}$  iff for any  $x \in A$ ,  $A$  is a neighbourhood of  $x$ .

**Proof.** First, prove  $\Rightarrow$ .

If  $A \in \mathcal{T}$ , then, by Definition 6.1.1, for any  $x \in A$ , we have

$$x \in A \subseteq A.$$

□

Now, prove  $\Leftarrow$ .

Aiming for a contradiction, suppose for any  $x \in A$ ,  $A$  is a neighbourhood of  $x$ , but  $A \notin \mathcal{T}$ .

As  $X \setminus A$  is not closed, (otherwise, by Definition 2.1.3,  $A = X \setminus (X \setminus A)$  is open) by Proposition 2.6.2, there exists  $x \in L(X \setminus A) \setminus (X \setminus A)$ .

Then, for such an  $x \in A$  (for  $x \notin X \setminus A$ ), for any  $U \in \mathcal{T}$  with  $x \in U$ ,

$$U \cap (X \setminus A) \neq \emptyset. \quad (\text{Definition 2.6.1})$$

By Definition 6.1.1,  $A$  fails to be a neighbourhood of  $x$ . This contradicts the assumption.

Thus  $A$  has to be open. ■

## §6.2 $T_0$ Spaces

---

**Definition 6.2.1.** A topological space  $\mathbb{X} = (X, \mathcal{T})$  is said to be  $T_0$ , or *Kolmogorov*, iff for any distinct  $x, y \in X$  are *topologically distinguishable*. That is, if we let  $\mathcal{N}_x$  be the set of all neighbourhoods of  $x$  and let  $\mathcal{N}_y$  be the set of all neighbourhoods of  $y$ , we have

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

**Example 6.2.1.** Not all topological spaces are  $T_0$ . For example, if  $\mathcal{T}$  is a indiscrete topology on  $X$ , then  $\mathbb{X}$  is a  $T_0$  space iff  $|X| \in \{1, 0\}$ .

**Proof.** First, prove  $\Rightarrow$ . Aiming for a contradiction, suppose  $\mathbb{X}$  is  $T_0$  but  $|X| > 1$ .

As  $|X| > 1$  and  $\mathcal{T}$  is an indiscrete topology on  $X$ , for any  $x, y \in X$  with  $x \neq y$ , they share their neighbourhoods. Thus, by Definition ??, they are not topological distinguishable, hence  $\mathbb{X}$  is not  $T_0$ , contradicting to the condition we have.

Now, prove  $\Leftarrow$ . If  $X \in \{1, 0\}$ ,  $X$  is a monotone or empty set. In this case “for any distinct  $x, y \in X$ ” is always false, thus the whole statement is vacuously true.

## §6.3 $R_0$ Spaces

---

**Definition 6.3.1.** A topological space  $\mathbb{X} = (X, \mathcal{T})$  is said to be  $R_0$  or *symmetric*, iff any topological distinguishable  $x, y \in X$  are *separated*. That is, there

exists closed sets  $V_x \ni x$  and  $V_y \ni y$  such that  $y \notin V_x$  and  $x \notin V_y$ .

**Example 6.3.1.** Let  $\mathbb{X} = (\mathbb{R}, \mathcal{T})$  be a topological space where  $\mathcal{T}$  is generated by base

$$\mathcal{B} = \{[n, n+1) : n \in \mathbb{Z}\}.$$

Then  $\mathbb{X}$  is not  $T_0$ .

## §6.4 $T_1$ Spaces

---

**Definition 6.4.1.** A topological space  $\mathbb{X} = (X, \mathcal{T})$  is said to be  $T_1$ , or *Fréchet space* iff it is  $T_0$  and  $R_0$ . By Definition 6.2.1 and 6.3.1, that is, for any  $x, y \in X$  with  $x \neq y$ , there exists closed sets  $V_x \ni x$  and  $V_y \ni y$  such that  $y \notin V_x$  and  $x \notin V_y$ .

**Proposition 6.4.1.** A topological space  $\mathbb{X} = (X, \mathcal{T})$  is  $T_1$  iff any monotone in  $X$  is closed.

**Proof.** First, prove  $\Rightarrow$ .

By, Definition 6.4.1, as  $\mathbb{X}$  is  $T_1$ , let  $x \in X$ , then for any  $y \in X$  with  $x \neq y$ , we can find closed  $V_x \ni x$  such that  $y \notin V_x$ .

By De Morgan's law, that is, there exists an open  $U = X \setminus V_x$  such that  $y \in U$  and  $x \notin U$ .

By Proposition 2.5.2, we have  $X \setminus \{x\}$  open.

By Definition 2.1.3,  $\{x\}$  is closed.

□

Now, prove  $\Leftarrow$ .

Assume any monotone in  $X$  is closed. Thus, let  $x, y \in X$  with  $x \neq y$ . Then  $\{x\}$  and  $\{y\}$  are closed.

Clearly,  $x \notin \{y\}$  and  $y \notin \{x\}$ , thus, by Definition 6.4.1,  $\mathbb{X}$  is  $T_1$ .

□

Thus the proposition is proved.

■

**Proposition 6.4.2.** A topological space  $\mathbb{X} = (X, \mathcal{T})$  is  $T_1$  iff any cofinite subset of  $X$  is open.

**Proof.** By Proposition 6.4.1,  $\mathbb{X}$  is  $T_1$  iff any monotone in  $X$  is closed.



First, prove  $\Rightarrow$ .

By Proposition 2.1.2, the family of closed sets of  $\mathbb{X}$  is closed under finite union, thus any finite subset  $F \subseteq X$  is closed.

By Definition 2.1.3, any cofinite subset  $X \setminus F \subseteq X$  is open.

□

Now, prove  $\Leftarrow$ .

As any cofinite subset  $S \subseteq X$  is open, by Definition 2.1.3, any finite subset  $X \setminus S \subseteq X$  is closed. Naturally, any monotone is finite, thus any monotone in  $X$  is closed. By Proposition 6.4.1,  $\mathbb{X}$  is  $T_1$ .

□

Thus the proof is done.

■

# Appendices

*Chapter A.*

***Some Examples of Topological  
Spaces***

**§A.1 Sorgenfrey line**

---

1. [Definition.](#)
2. [Sorgenfrey line is Lindelöf.](#)
3. [Sorgenfrey line is separable.](#)
4. [Sorgenfrey line is not second-countable.](#)

**§A.2 Niemytzki Plane**

---