Notes for General Topology

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Contents

Chapter 1.
Topological Spaces

§1.1 Open Sets

Definition 1.1.1. Let X be any set, and let $\mathcal{T} \subseteq 2^X$, where 2^X denotes the power set of X.

Then $\mathcal{T}$ is called a **topology on** X iff it satisfies the **open set axioms**. That is,

O1. $\emptyset, X \in \mathcal{T}$

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; i.e., $\mathcal{T}$ is closed under arbitrary union.

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; i.e., $\mathcal{T}$ is closed under finite intersection.

The ordered pair $\mathbb{X} = (X, \mathcal{T})$ is called a **topological space**.

A subset $U \subseteq X$ is said to be **open** iff it is an element of $\mathcal{T}$.

Note 1.1.1. Rigorously, $\emptyset \in \mathcal{T}$ is not necessary for O1 in Definition 1.1.1, because it can be proved in a simple way.

As empty set is an element of any set, it is also an element of $\mathcal{T}$. Therefore,

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

The most intuitive example of topological space is no doubt the **Euclidean topological space**, it is a topological space $\mathbb{X} = (X, \mathcal{T})$ with X is the cartesian product of a sequence of sets $(X_i)_{i=1}^n$ and the **Euclidean topology** $\mathcal{T}$ on X. That is, for any U open in $\mathbb{X}$ (i.e., $U \in \mathcal{T}$), for any $A \subseteq U$ and for any $x \in A$, there exists $\varepsilon_x \in \mathbb{R}_{>0}$, such that U can be represented as the union of all ε_x-balls around x; i.e.,

$$U = \bigcup_{x \in A} B_d(x, \varepsilon_x),$$

where d is the **Euclidean metric** on X; i.e., $d : X \times X \to \mathbb{R}_{>0}$ is a function defined by the Pythagoras theorem,

$$d(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}},$$

where a_i denotes the i-th projection of $a \in X$ on X.

In this case, the topology $\mathcal{T}$ is said to be induce by metric d. As a general consequence, any metric space can induce a unique topology. This will be proved later.

In Euclidean spaces, the idea of “open” represents intuitively, but it doesn’t mean that every topological space should be induce in such a natural way. Here is an easy example.

Example 1.1.1. Let $X = \{1, 2, 3, 4\}$ and let $\mathcal{T}$ be the smallest topology on X containing $\{1, 2\}$ and $\{2, 3\}$, i.e., for topology $\mathcal{T}'$ on X containing these two sets is a superset of $\mathcal{T}$.

By Open Set Axiom O1, $\emptyset, X \in \mathcal{T}$.

By Open Set Axiom O2, $\{1, 2, 3\} = \{1, 2\} \cup \{2, 3\} \in \mathcal{T}$.

By Open Set Axiom O3, $\{2\} = \{1, 2\} \cap \{2, 3\} \in \mathcal{T}$.

Therefore

$$\mathcal{T} = \{\emptyset, X, \{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{2\}\}.$$

Definition 1.1.2. Let X be any set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X.

Then $\mathcal{T}_1$ is said to be **coarser** than $\mathcal{T}_2$, or $\mathcal{T}_2$ is said to be **finer** thatn $\mathcal{T}_1$, iff $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Example 1.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

Then $\mathbb{X}$ is a **discrete topological space**, namely, $\mathcal{T}$ is a **discrete topology** on X iff $\mathcal{T} = 2^X$.

It is the finest topology on X.

Example 1.1.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

Then $\mathbb{X}$ is a **indiscrete topological space**, namely, $\mathcal{T}$ is a **indiscrete topology** on X iff $\mathcal{T} = \{\emptyset, X\}$.

It is the coarsest topology on X.

§1.2 Closed Sets

Definition 1.2.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then A is said to be **closed in** $\mathbb{X}$ iff there is a $U \in \mathcal{T}$ such that

$$A = X \setminus U.$$

Proposition 1.2.1. Closed set axioms...

§1.3 Interiors

Definition 1.3.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in A$ is said to be **interior to** A iff there is a $U \in \mathcal{T}$ with $x \in U$, such that $U \subseteq A$.

The **interior of** A, denoted $\mathrm{Int}_{\mathcal{T}}(A)$, is defined as the set of all interior points of A.

Sometime, we write A° for $\operatorname{Int}_{\mathcal{T}}(A)$, if the confusion of topology is unlikely in the context.

Proposition 1.3.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then we have

$$A^\circ = \bigcup(\mathcal{T} \cap 2^A).$$

Proof. Let $\mathcal{U} = \mathcal{T} \cap 2^A$.

$$\begin{aligned} x \in \bigcup(\mathcal{T} \cap 2^A) &\iff x \in \bigcup_{U \in \mathcal{U}} U \\ &\iff (\exists U \in \mathcal{U})\ x \in U. \end{aligned}$$

By assumption, $\mathcal{U} \subseteq \mathcal{T}$, thus for any $U \in \mathcal{U}$, $U \in \mathcal{T}$. Also, $\mathcal{U} \subseteq 2^A$ implies that for any $U \in \mathcal{U}$, $U \subseteq A$.

Now, we have $x \in \bigcup(\mathcal{T} \cap 2^A)$ iff

$$(\exists U \in \mathcal{T} \mid U \subseteq A) \quad x \in U.$$

By the definition of existential quantifier and the associativity of logical conjunction, we have

$$\begin{aligned} (U \in \mathcal{T} \wedge U \subseteq A) \wedge x \in U &\iff (U \in \mathcal{T} \wedge x \in U)\ U \subseteq A \\ &\iff (\exists U \in \mathcal{T} \mid x \in U)\ U \subseteq A. \end{aligned}$$

This is precisely the statement of Definition 1.3.1 ∎

Proposition 1.3.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

Then $A^\circ \subseteq B^\circ$ if $A \subseteq B$.

Proof. As $A \subseteq B$, we have

$$\begin{aligned} 2^A \subseteq 2^B &\implies \mathcal{T} \cap 2^A \subseteq \mathcal{T} \cap 2^B \\ &\implies \bigcup(\mathcal{T} \cap 2^A) \subseteq \bigcup(\mathcal{T} \cap 2^B) \end{aligned}$$

By Definition 1.3.1, $A^\circ \subseteq B^\circ$. ∎

Note that $A^\circ \subseteq B^\circ$ does not imply $A \subseteq B$. For example, let $\mathbb{X} = (X, \mathcal{T})$ with $X = \{1, 2\}$ and $\mathcal{T} = \{\emptyset, X, \{2\}\}$, and let

$$A = \{1\},\ B = \{2\}.$$

Then, $A^\circ = \emptyset$ and $B^\circ = \{2\}$. In this case, $A^\circ \subseteq B^\circ$, but $A \not\subseteq B$.

Proposition 1.3.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then $A \in \mathcal{T}$ iff $A = A^\circ$.

Proof. Assume $A \in \mathcal{T}$.

As $A \in 2^A$ also, then $A \in \mathcal{T} \cap 2^A$. In the term of family, we have

$$\begin{aligned}\{A\} \subseteq \mathcal{T} \cap 2^A &\implies \bigcup\{A\} \subseteq \bigcup(\mathcal{T} \cap 2^A)\\ &\implies A \subseteq A^\circ.\end{aligned}$$

By Definition 1.3.1, $A^\circ \subseteq A$ is clear, therefore $A = A^\circ$. □

Conversely, Assume $A = A^\circ$.

By Proposition 1.3.1 that is

$$A = \bigcup(\mathcal{T} \cap 2^A).$$

Clearly, $\mathcal{T} \cap 2^A \subseteq \mathcal{T}$, thus, by O2, Definition 1.1.1, $A \in \mathcal{T}$. ■

Proposition 1.3.4. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $\mathcal{A} \subseteq 2^X$.

Then we have

$$\left(\bigcap \mathcal{A}\right)^\circ \subseteq \bigcap_{A \in \mathcal{A}} A^\circ.$$

Proof. By Definition 1.3.1, for any $x \in (\bigcap \mathcal{A})^\circ$, there exists $U \in \mathcal{T}$ with $x \in U$ such that $U \subseteq \bigcap \mathcal{A}$. Thus, for any $A \in \mathcal{A}$, $U \subseteq A$.

As $U \in \mathcal{T}$ and $U \subseteq A$, $U \subseteq A^\circ$.

As $x \in U$ and $U \subseteq A^\circ$ for any $A \in \mathcal{A}$, we have

$$x \in \bigcap_{A \in \mathcal{A}} A^\circ.$$

■

Note that the relation $\subseteq$ in Proposition 1.3.4 cannot be reversed. Consider $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\emptyset, X, \{1, 2\}, \{2, 3\}\}$, then

$$(\{1, 2\} \cap \{2, 3\})^\circ = \emptyset,$$

but,

$$\{1, 2\}^\circ \cap \{2, 3\}^\circ = \{2\}.$$

§1.4 Limit Points and Isolated Points

Definition 1.4.1. Limit point and derived set...

Definition 1.4.2. Isolated points and the set of isolated points

Proposition 1.4.1.

$$A \subseteq \mathrm{L}(A) \sqcup \mathrm{I}(A)$$

Proposition 1.4.2. A is closed iff $\mathrm{L}(A) \subseteq A$.

Chapter 2.

Metric Spaces

§2.1 Review of the Metric Spaces

Definition 2.1.1. Let X be any set, and let $d : X \times X \to \mathbb{R}_{>0}$.

Then d is a **metric on** X iff it satisfies the **metric axioms**. That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ iff $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

The ordered pair $\mathbb{X} = (X, d)$ is called **metric space**.

Definition 2.1.2. Let $\mathbb{X} = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$.

An **open** ε**-ball**, or just ε**-ball**, about x is defined to be the set

$$B_d(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}.$$

A **closed ball** is defined to be the set

$$\overline{B}_d(x, \varepsilon) := \{y \in X : d(x, y) \leq \varepsilon\}.$$

Proposition 2.1.1. Let $\mathbb{X} = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$.

Then, for any $y \in B_d(x, \varepsilon)$, there exists $\delta \in \mathbb{R}_{>0}$, such that

$$B_d(y, \delta) \subseteq B_d(x, \varepsilon).$$

Proof. Aiming for a contradiction, suppose there exists a $y \in B_d(x, \varepsilon)$, for any $\delta \in \mathbb{R}_{>0}$,

$$\exists z \in B_d(y, \delta) \setminus B_d(x, \varepsilon).$$

By Definition 2.1.2, we have

$$\begin{aligned} z \notin B_d(x, \varepsilon) &\implies d(x, z) > \varepsilon, \\ y \in B_d(z, \varepsilon) &\implies d(x, y) < \varepsilon, \\ z \in B_d(y, \delta) &\implies d(y, z) < \delta. \end{aligned}$$

By metric axioms O3, we have

$$\delta > d(y, z) \geq d(x, z) - d(x, y).$$

This implies that there exists an $r = d(x, y) \in \mathbb{R}_{(0,\varepsilon)}$, such that for any $\delta \in \mathbb{R}_{>0}$,

$$\delta > \varepsilon - r,$$

which is impossible. Thus, such a y can not exist. ■

Proposition 2.1.2. Let $\mathbb{X} = (X, d)$ be a metric space, and let $\mathcal{O}$ be a family of open balls in $\mathbb{X}$.

Then, for any $y \in \bigcup \mathcal{O}$, there is a $\delta \in \mathbb{R}_{>0}$, such that

$$B_d(y, \delta) \subseteq \bigcup \mathcal{O}.$$

Proof. As $y \in \bigcup \mathcal{O}$, there is an $O \in \mathcal{O}$ such that $y \in O$. As $\mathcal{O}$ is a family of open balls, that is, there is an $x \in \bigcup \mathcal{O}$ an $\varepsilon \in \mathbb{R}_{>0}$, such that $y \in B_d(x, \varepsilon)$. ■

§2.2 Metrizability

Proposition 2.2.1. Let $\mathbb{X} = (X, d)$, let $x \in X$, and let $\varepsilon \in \mathbb{R}_{>0}$.

Then for any $y \in B_d(x, \varepsilon)$, there is a $\delta \in \mathbb{R}_{>0}$, such that

$$B_d(y, \delta) \subseteq B_d(x, \varepsilon).$$

Proof. ■

Proposition 2.2.2. Let $\mathbb{X} = (X, d)$ be a metric space, and let $\mathcal{T} \subseteq 2^X$ such that for any $U \in \mathcal{T}$ and for any $x \in U$, there exists an $\varepsilon_x \in \mathbb{R}_{>0}$ such that

$$B_d(x, \varepsilon_x) \subseteq U.$$

Then $\mathcal{T}$ is a topology on X.

Proof. $\mathcal{T}$ is a topology on X iff it satisfies the open set axioms (Definition 1.1.1.

Proof for O1. By the definition of $\mathcal{T}$ here, for any $x \in X$, there exists an $\varepsilon \in \mathbb{R}$ such that $B_d(x, \varepsilon) \subseteq U$.

For any $x \in \emptyset$, the statement is vacuously true. □

Proof for O2. Let $\mathcal{U} \subseteq \mathcal{T}$, then for any $U \in \mathcal{U}$ and for any $x \in U$, there exists an $\varepsilon_x \in \mathbb{R}_{>0}$ such that $B_d(x, \varepsilon_x) \subseteq U$. Thus, for any $x \in U$, there exists $\varepsilon_x \in \mathbb{R}_{>0}$, such that

$$U = \bigcup_{x \in U} B_d(x, \varepsilon_x).$$

Now, we need to show that $\bigcup \mathcal{U} \in \mathcal{T}$.

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