Notes for General Topology

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Contents

Chapter 1

Topological Spaces

1.1 Metric Spaces

Definition 1.1.1. Let X be any set. A mapping $d : X \times X \to \mathbb{R}_{\geq 0}$ is *metric on* X if and only if it satisfies the *metric axioms.* That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ if and only if $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

In this case, the pair $M = (X, d)$ is called a *metric space.*

Definition 1.1.2. A $M = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$. An *open* ε*-ball*, or just ε-ball, about x is defined to be the set

$$B_\varepsilon(x; d) = \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) = \{y \in X : d(x, y) \leq \varepsilon\}.$$

Note 1.1.1. As

$$M = (X, d),\ M' = (X, d'),\ M'' = (X, d''),\ \ldots$$

are different although they share the same set X, for any $x \in X$ and any $\varepsilon \in \mathbb{R}_{>0}$,

$$B_\varepsilon(x; d),\ B_\varepsilon(x; d'),\ B(x; d''),\ \ldots$$

are also different. However, if confusion is unlikely, we simply write "$B_\varepsilon(x)$" for "$B_\varepsilon(x;d)$".

Example 1.1.1. The *Euclidean metric space* $M=(X,d)$ is an n-dimensional set X equipped with the *Euclidean metric* d defined as

$$d(x,y)=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x,y)=\left(\sum_{i=1}^{n}|x_i-y_i|^p\right)^{\frac{1}{p}},\quad p\geq 1.$$

In particular,

$$d_\infty(x,y)=\max_{1\leq i\leq n}|x_i-y_i|.$$

Example 1.1.2. A *discrete metric space* $M=(X,d)$ is a set X equiped with the *discrete metric* d defined as

$$d(x,y)=\begin{cases}0, & \text{if } x=y;\\ 1, & \text{else.}\end{cases}$$

This is an equivalent definition of the discrete metric:

$$d(x,y)=(\operatorname{sgn}(d'(x,y)))^2,$$

where $\operatorname{sgn}(\cdot)$ is a sign function, and d' is any metric on X.

Example 1.1.3. [1] Denote $C[a,b]$ for the set of all continuous mapping $\mathbb{R}_{[a,b]}\to\mathbb{R}$. On $C[a,b]$, we can define a metric d as

$$d_p(f,g)=\left(\int_a^b|f(t)-g(t)|^p\mathrm{d}t\right)^{\frac{1}{p}},\quad p\geq 1.$$

In particular,

$$d_\infty(f,g)=\sup_{t\in\mathbb{R}_{[a,b]}}|f(t)-g(t)|.$$

[1] See Minkowski inequality.

Example 1.1.4. [2] Let $M = (X, d)$ be a metric space. The *Hausdorff metric* d_H on $2^X \setminus \{\emptyset\}$ is defined as

$$d_H = \max\left\{\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\right\},$$

where

$$d(x, Y) = \inf_{y\in Y}(x, y), \text{ and } d(y, X) = \inf_{x\in X}(y, x).$$

1.2 Open sets in Metric Spaces

Definition 1.2.1. Let $M = (X, d)$ be a metric space, and let $U \subseteq X$. U is said to be *open in M* if and only if for any $y \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$, such that $B_\varepsilon(y) \subseteq U$.

Lemma 1.2.1. Let $M = (X, d)$ be a metric space, let $x \in A$ and let $\varepsilon \in \mathbb{R}_{>0}$. For any $y \in B_\varepsilon(x)$, there is a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y)$

[2] See Hausdorff distance.