



Notes for General Topology

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Contents

1	Metric Spaces	2
1.1	Metric Spaces	2
1.2	Open Sets in Metric Spaces	4
1.3	Restrictions and Metric Subspaces	7
2	Topological Spaces	9
2.1	Basic Definitions	9
2.2	Some Important Topologies	11
2.3	Comparison of Topologies	11
2.4	Interiors	12
2.5	Limit Points and Isolated Points	15
2.6	Closures	16
2.7	Density	18
2.8	Neighbourhood Systems	19
3	Sequences	21
3.1	Convergent Sequences	21
3.2	Accumulation Points of Sequences	22
4	Countable Axioms	24
4.1	Covers and Bases	24
4.2	First-Countable Spaces	25
4.3	Second-Countable Spaces	26

Chapter 1.

Metric Spaces

§1.1 Metric Spaces

Definition 1.1.1. Let X be any set.

A function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is *metric function*, or, simply, *metric on X* iff it satisfies the *metric axioms*. That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ iff $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.1.2. Let X be any set and let d be a structure on X .

The pair (X, d) is called a *metric space* iff d is a metric on X .

Definition 1.1.3. A $\mathbb{X} = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$.

An *open ε -ball*, or just ε -ball, about x is defined to be the set

$$B_\varepsilon(x; d) := \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) := \{y \in X : d(x, y) \leq \varepsilon\}.$$

Note 1.1.1. As

$$\mathbb{X}_0 = (X, d_0), \mathbb{X}_1 = (X, d_1), \mathbb{X}_2 = (X, d_2), \dots$$

are different although they share the same set X , for any $x \in X$ and any $\varepsilon \in \mathbb{R}_{>0}$,

$$B_\varepsilon(x; d_1), B_\varepsilon(x; d_2), B_\varepsilon(x; d_3), \dots$$

are also different. However, if confusion is unlikely, we simply write “ $B_\varepsilon(x)$ ” for “ $B_\varepsilon(x; d)$ ”.

Example 1.1.1. The *Euclidean metric space* $\mathbb{X} = (X, d)$ is an n -dimensional set X equipped with the *Euclidean metric* d defined as

$$d(x, y) := \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x, y) := \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

Example 1.1.2. A *discrete metric space* $\mathbb{X} = (X, d)$ is a set X equipped with the *discrete metric* d_{dis} defined as

$$d_{\text{disc}}(x, y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{else.} \end{cases}$$

This is an equivalent definition of the discrete metric:

$$d_{\text{disc}}(x, y) := (\text{sgn}(d(x, y)))^2,$$

where $\text{sgn}(\cdot)$ is a [sign function](#), and d is any metric on X .

Example 1.1.3. ¹ Let $\mathbb{I} = (C[a, b], d_p)$ be a metric space where $C[a, b]$ denotes the set of all continuous mapping $\mathbb{R}_{[a, b]} \rightarrow \mathbb{R}$, and $p > 0$, and the metric d_p is defined as

$$d_p(f, g) := \left(\int_a^b |f(t) - g(t)|^p dt \right)^{\frac{1}{p}}.$$

In particular,

$$d_\infty(f, g) := \sup_{t \in \mathbb{R}_{[a, b]}} |f(t) - g(t)|.$$

¹ See [Minkowski inequality](#).

Example 1.1.4. ² Let $\mathbb{X} = (X, d)$ be a metric space. The *Hausdorff metric* d_H on $2^X \setminus \{\emptyset\}$ is defined as

$$d_H := \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\},$$

where

$$d(x, Y) := \inf_{y \in Y} d(x, y), \text{ and } d(y, X) := \inf_{x \in X} d(y, x).$$

§1.2 Open Sets in Metric Spaces

Definition 1.2.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $U \subseteq X$.

U is said to be *open in \mathbb{X}* , iff for any $y \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$, such that $B_\varepsilon(y) \subseteq U$.

Proposition 1.2.1. Let $\mathbb{X} = (X, d)$ be a metric space, let $x \in A$ and let $\varepsilon \in \mathbb{R}_{>0}$.

For any $y \in B_\varepsilon(x)$, there is a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_\varepsilon(x)$.

Proof. For any $y \in B_\varepsilon(x)$, by the definition of open balls (Definition 1.1.3), we have $d(x, y) < \varepsilon$.

Let $\delta \in \mathbb{R}_{>0}$ such that $\delta + d(x, y) = \varepsilon$.

By M3 in metric axioms (Definition 1.1.1), for any $z \in A$ with $d(y, z) < \delta$, we have

$$d(x, z) \leq d(y, z) + d(x, y) < \varepsilon.$$

Thus, again, by the definition of open balls, we have $B_\delta(y) \subseteq B_\varepsilon(x)$. ■

Proposition 1.2.2. Let $\mathbb{X} = (X, d)$ be a metric space, and let $U \subseteq X$.

U is open in \mathbb{X} iff it is a union of open balls.

Proof. First, prove \Rightarrow .

As U is open, for any $y \in U$, there exists $\varepsilon_y \in \mathbb{R}_{>0}$ such that $B_{\varepsilon_y}(y) \subseteq U$.

Therefore,

$$U = \bigcup_{y \in U} B_{\varepsilon_y}(y).$$

□

² See [Hausdorff distance](#).

Now, prove \Leftarrow .

Aiming for a contradiction, suppose U is a union of open balls but not open.

As U is not open, there is a $y \in U$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \not\subseteq U$.

As U is a union of open balls, there is an $x \in U$ and $r \in \mathbb{R}_{>0}$ such that $y \in B_r(x)$.

By Proposition 1.2.1, there exists a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_r(x)$.

This is a contradiction by the assumption.

Thus, U has to be open. ■

Proposition 1.2.3. Let $\mathbb{X} = (X, d)$ be any metric space.

\mathbb{X} is *Hausdorff*. That is, For any distinct points $x, y \in X$, we can always find an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset.$$

Proof. Aiming for a contradiction, suppose there are $x, y \in X$ with $x \neq y$, such that for any $\varepsilon \in \mathbb{R}_{>0}$, we can always find a $z \in X$ such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let $r = d(x, y)/2$, and let $z \in B_r(x) \cap B_r(y)$.

As $z \in B_r(x)$, by the definition of open balls (Definition 1.1.3), $d(x, z) < r$; as $z \in B_r(y)$, similarly, $d(y, z) < r$. Then we have

$$d(x, z) + d(y, z) < 2r = d(x, y).$$

This contradicts the metric axioms M3 (Definition 1.1.1).

Thus \mathbb{X} is Hausdorff. ■

Definition 1.2.2. Let $\mathbb{X} = (X, d)$ be any metric space, and let $V \subseteq X$.

V is said to be *closed* in \mathbb{X} , iff there is an open set U satisfies $X \setminus U = V$.

Proposition 1.2.4. In a metric space, any singleton is closed.

Proof. Let $\mathbb{X} = (X, d)$ be a metric space, let $x \in X$, and let $y \in X \setminus \{x\}$.

As M is Hausdorff (Proposition 1.2.3), there is an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$0 < \varepsilon < d(x, y),$$

thus $X \setminus \{x\}$ is open, hence, by Definition 1.1.1, its complement $\{x\}$ is open. ■

Proposition 1.2.5. Let $\mathbb{X} = (X, d)$ be a metric space, denote \mathcal{T} for the family of open subsets of X .

Then \mathcal{T} satisfies the following conditions:

- O1.** $X, \emptyset \in \mathcal{T}$;
- O2.** For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, \mathcal{T} is closed under arbitrary union;
- O3.** For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, \mathcal{T} is closed under finite intersection.

Proof.

- O1.** As \emptyset is the subset of any set, $\emptyset \in \mathcal{T}$. $\bigcup \emptyset = \emptyset \in \mathcal{T}$.

By Definition 1.2.2, $X = X \setminus \emptyset$.

□

- O2.** Let $\mathcal{U} \subseteq \mathcal{T}$, and denote \mathcal{O} for the open balls in M .

For any $U \in \mathcal{U}$, there is an $\mathcal{O}_U \subseteq \mathcal{O}$ such that $U = \bigcup \mathcal{O}_U$.

Then we have

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \left(\bigcup \mathcal{O}_U \right) = \bigcup_{U \in \mathcal{U}} \mathcal{O}_U.$$

By Proposition 1.2.2, $\bigcup \mathcal{U}$ is open.

□

- O3.** Let \mathcal{V} be a finite subset of \mathcal{T} .

Aiming for a contradiction, suppose $\bigcap \mathcal{V}$ is not open.

By Definition 1.2.1, there exists a $y \in \bigcap \mathcal{V}$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \setminus \bigcap \mathcal{V} \neq \emptyset$.

By De Morgan's law, we have

$$\bigcup_{V \in \mathcal{V}} (B_\varepsilon(y) \setminus V) \neq \emptyset.$$

Thus, there exists $V \in \mathcal{V}$ such that $B_\varepsilon(y) \setminus V \neq \emptyset$.

As $V \in \mathcal{T}$ and ε is arbitrarily given, by Proposition 1.2.1, $y \notin V$. This is a contradiction.

Thus, $\bigcap \mathcal{V}$ is open.

□

Thus, the theorem is proved. ■

Proposition 1.2.6. Infinite intersections of open sets in some metric spaces are not necessarily open.

Proof. Consider \mathbb{R} is a Euclidean metric space, and denote \mathcal{T} .

Clearly, for any $n \in \mathbb{N}_{>0}$ and for any $x \in X$, the open interval $B_{\frac{1}{n}}(x)$ is open, but

$$\bigcap \left\{ B_{\frac{1}{n}}(x) : n \in \mathbb{N}_{>0} \right\} = \{x\}.$$

For any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(x) \setminus \{x\}$ is not empty, thus $\{x\}$ is not open. ■

§1.3 Restrictions and Metric Subspaces

Restriction of metric function is a useful tool to describe the relation between metric spaces with different sets but “same” metric function on the sets.

As a restriction of a relation R on $X \times Y$ to a subset $A \times B \subseteq X \times Y$ is defined to be

$$R \upharpoonright_{A \times B} := R \cap (X \times Y),$$

a restriction of a metric d on a set S to a subset $U \subseteq S$ is defined to be

$$d \upharpoonright_{(U \times U) \times \mathbb{R}_{>0}} := d \cap ((U \times U) \times \mathbb{R}_{>0}).$$

If $B = Y$, customarily, we simply write $R \upharpoonright_A$ for $R \upharpoonright_{A \times B}$. Similarly, as the codomain of a metric function is always $\mathbb{R}_{>0}$, so we simply write $d \upharpoonright_{U \times U}$ instead of $d \upharpoonright_{(U \times U) \times \mathbb{R}_{>0}}$.

Definition 1.3.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$.

The *metric on A induced by d* , or the *subspace metric of d with respect to A* is defined to be

$$d_A := d \upharpoonright_{A \times A}.$$

Proposition 1.3.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$ and let $d_A := d \upharpoonright_{A \times A}$.

Then $\mathbb{A} = (A, d_A)$ is a metric space.

Proof. As metric axioms (Definition 1.1.1) holds for any $x, y \in X$, and $A \subseteq X$, they also holds for any $a, b \in A$. As d_A is the subspace metric of d with respect to A , d_A is a metric on A .

Thus, \mathbb{A} is a metric space.

Definition 1.3.2. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$.

$\mathbb{A} = (A, d_A)$ is a *metric subspace* of \mathbb{X} iff d_A is a subspace metric of d with respect to A .

Chapter 2.

Topological Spaces

§2.1 Basic Definitions

Definition 2.1.1. Let X be any set, and let $\mathcal{T} \subseteq 2^X$.

\mathcal{T} is a *topology on X* iff it satisfies the *open set axioms*. That is,

- O1.** $X \in \mathcal{T}$;
- O2.** For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, \mathcal{T} is closed under arbitrary union.
- O3.** For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, \mathcal{T} is closed under finite intersection.

A subset $U \subseteq X$ is said to be *open in M* iff it is an element of \mathcal{T} .

Definition 2.1.2. Let X be any set, and let \mathcal{T} be a structure on X .

The pair (X, \mathcal{T}) is called a *topological space* iff \mathcal{T} is a topology on X .

Proposition 2.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

Then $\emptyset \in \mathcal{T}$.

Proof. As empty set is an element of any set, it also an element of \mathcal{T} .

Therefore, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

■

Definition 2.1.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

A subset $A \subseteq X$ is said to be *closed in \mathbb{X}* iff there exists a $U \in \mathcal{T}$ such that $A = X \setminus U$.

Proposition 2.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and denote \mathcal{C} for the family of all closed sets in M .

Then \mathcal{C} satisfies the following conditions:

- C1.** $X, \emptyset \in \mathcal{C}$;
- C2.** For any $\mathcal{A} \subseteq \mathcal{C}$, $\bigcap \mathcal{A} \in \mathcal{C}$;
- C3.** For any finite $\mathcal{B} \subseteq \mathcal{C}$, $\bigcup \mathcal{B} \in \mathcal{C}$.

Proof.

C1. As $\emptyset \in \mathcal{T}$ and $X = X \setminus \emptyset$, by Definition 2.1.3, X is closed.

Similarly, as $X \in \mathcal{T}$ and $\emptyset = X \setminus X$, \emptyset is closed.

□

C2. For any $\mathcal{A} \subseteq \mathcal{C}$, there exists a $\mathcal{U} \subseteq \mathcal{T}$ such that

$$\forall A \in \mathcal{A} : \exists U \in \mathcal{U} : A = X \setminus U. \quad (\text{Definition 2.1.3.})$$

Then we have

$$\begin{aligned} \mathcal{A} = \{X \setminus U : U \in \mathcal{U}\} &\iff \bigcap \mathcal{A} = \bigcap_{U \in \mathcal{U}} X \setminus U \\ &\iff \bigcap \mathcal{A} = X \setminus \bigcup \mathcal{U}. \end{aligned}$$

As $\bigcup \mathcal{U} \in \mathcal{T}$ by Definition 2.1.1 O2, its complement $\bigcap \mathcal{A} \in \mathcal{C}$ by Definition 2.1.3.

□

C3. For any finite $\mathcal{B} \subseteq \mathcal{C}$, there exists a finite $\mathcal{U} \subseteq \mathcal{T}$ such that

$$\forall B \in \mathcal{B} : \exists U \in \mathcal{U} : A = X \setminus U. \quad (\text{Definition 2.1.3.})$$

Then we have

$$\begin{aligned} \mathcal{B} = \{X \setminus U : U \in \mathcal{U}\} &\iff \bigcup \mathcal{B} = \bigcup_{U \in \mathcal{U}} X \setminus U \\ &\iff \bigcup \mathcal{B} = X \setminus \bigcap \mathcal{U}. \end{aligned}$$

As $\bigcap \mathcal{U} \in \mathcal{T}$ by Definition 2.1.1 O3, its complement $\bigcup \mathcal{B} \in \mathcal{C}$ by Definition 2.1.3.

□

Thus, the proof is done.

■

§2.2 Some Important Topologies

Definition 2.2.1. Let X be any set.

A family $\mathcal{T} \subseteq 2^X$ is a *discrete topology on X* iff $\mathcal{T} = 2^X$.

Definition 2.2.2. Let X be any set.

A family $\mathcal{T} \subseteq 2^X$ is an *indiscrete topology on X* iff $\mathcal{T} = \{X, \emptyset\}$.

Definition 2.2.3. Let $\mathbb{X} = (X, d)$ be a metric space.

A family $\mathcal{T} \subseteq 2^X$ is a *topology induced by d* iff \mathcal{T} is the set of all open sets in \mathbb{X} .

§2.3 Comparison of Topologies

Definition 2.3.1. Let X be any set and let \mathcal{T}_1 and \mathcal{T}_2 be topologies on X .

We say that \mathcal{T} is *coarser* than \mathcal{T}_1 , or \mathcal{T}_2 is *finer* than \mathcal{T}_1 , iff $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Note 2.3.1. By the definition of cardinality and inclusion mapping, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$, it is certainly true that $|\mathcal{T}_1| \leq |\mathcal{T}_2|$. But, on the contrary, $|\mathcal{T}_1| \leq |\mathcal{T}_2|$ does not implies $\mathcal{T}_1 \subseteq \mathcal{T}_2$. It is easy to find counter-example about this.

Example 2.3.1. By Definition 2.3.1, for any set X , if a family \mathcal{U} of open sets is given, then we can find the coarsest topology on X containing \mathcal{U} by

$$\mathcal{T} = \left\{ \bigcup \mathcal{I}, \bigcap \mathcal{I}, X : \mathcal{I} \subseteq \mathcal{U} \right\}.$$

For example, let $X = \{1, 2, 3, 4, 5\}$, and let

$$\mathcal{U} = \{\{1, 2\}, \{2, 3\}, \{4\}\}.$$

Then a topology on X contains at least these sets:

$$\begin{aligned} &\{1, 2, 3, 4\}, \{\}, \\ &\{1, 2\}, \{2, 3\}, \{4\}, \\ &\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \\ &\{2\}. \end{aligned}$$

Example 2.3.2. The discrete topology is the finest topology on any X , while the indiscrete topology is the coarsest.

§2.4 Interiors

Definition 2.4.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

The *interior* of A is defined as

$$\text{Int}_{\mathcal{T}}(A) := \bigcup (\mathcal{T} \cap 2^A).$$

Note 2.4.1. Let $\mathbb{X}_1 = (X, \mathcal{T}_1)$, $\mathbb{X}_2 = (X, \mathcal{T}_2)$, and $A \subseteq X$. Then $\mathcal{T}_1 \neq \mathcal{T}_2$ iff $\text{Int}_{\mathcal{T}_1}(A) \neq \text{Int}_{\mathcal{T}_2}(A)$. In this case, the subscript for “Int” is necessary.

But, if the confusion is unlikely, we can also simply write $\text{Int}(A)$ for $\text{Int}_{\mathcal{T}}A$. In this case, it is also common to write A° for $\text{Int}(A)$.

Proposition 2.4.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$A \in \mathcal{T}$ iff $A = A^\circ$.

Proof. First, prove \Rightarrow .

If $A \in \mathcal{T}$, then we have

$$\mathcal{T} \cap 2^A = \mathcal{T} \cap \{A\} \cap 2^A = \{A\} \cap 2^A = \{A\}.$$

By Definition 2.4.1,

$$A^\circ = \bigcup (\mathcal{T} \cap 2^A) = \bigcup \{A\} = A.$$

□

Now, prove \Leftarrow .

By Definition 2.4.1, we have

$$A = \bigcup (\mathcal{T} \cap 2^A).$$

As $\mathcal{T} \cap 2^A \subseteq \mathcal{T}$, thus, by open set axioms O2 (Definition 2.1.1 O2), $A \in \mathcal{T}$.

□

Thus, the proof is done.

■

Proposition 2.4.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \in \mathcal{T}$. For any $x \in A$, there is a $U \in \mathcal{T} \cap 2^A$ such that $x \in U$.

Proof.

$$x \in A \iff x \in A^\circ \quad (\text{Proposition 2.4.1})$$

$$\iff x \in \bigcup (\mathcal{T} \cap 2^A) \quad (\text{Definition 2.4.1})$$

$$\iff \exists U \in \mathcal{T} \cap 2^A : x \in U.$$

■

Proposition 2.4.3. Let X be any set, let I be an index set, and let $\mathcal{A}_i \subseteq 2^X$ for any $i \in I$.

Then we have

$$\bigcup \left(\bigcap_{i \in I} \mathcal{A}_i \right) \subseteq \bigcap_{i \in I} \left(\bigcup \mathcal{A}_i \right).$$

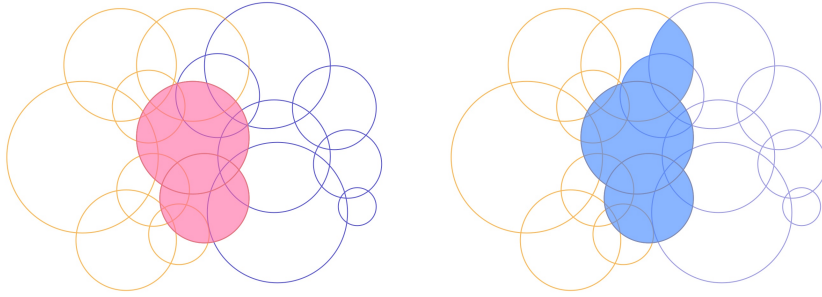


Figure 2.1: Diagram of the relation in Proposition 2.4.3.

Proposition 2.4.4. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $\mathcal{A} \subseteq 2^X$.

Then we have

$$\left(\bigcap \mathcal{A} \right)^\circ \subseteq \bigcap_{A \in \mathcal{A}} A^\circ.$$

Proof.

$$\begin{aligned}
\left(\bigcap \mathcal{A}\right)^\circ &= \bigcup \left(\mathcal{T} \cap 2^{\bigcap \mathcal{A}}\right) && \text{(Definition 2.4.1)} \\
&= \bigcup \left(\mathcal{T} \cap \bigcap_{A \in \mathcal{A}} 2^A\right) && \text{(intersection of power sets)} \\
&= \bigcup \left(\bigcap_{A \in \mathcal{A}} (\mathcal{T} \cap 2^A)\right) && \text{(intersection is idempotent} \\
&&& \text{and associative)} \\
&\subseteq \bigcap_{A \in \mathcal{A}} \left(\bigcup (\mathcal{T} \cap 2^A)\right) && \text{(Proposition 2.4.3)} \\
&= \bigcap_{A \in \mathcal{A}} A^\circ. && \text{(Definition 2.4.1)}
\end{aligned}$$

■

Example 2.4.1. The equality in Proposition 2.4.4 may not hold.

Let $\mathbb{T} = (\mathbb{R}, \mathcal{T})$ be a topological space with

$$\mathcal{T} = \{X, (0, 2), (1, 3), \emptyset\}.$$

Then we have

$$((0, 2) \cap (1, 3))^\circ = \emptyset \quad \subsetneq \quad (0, 2)^\circ \cap (1, 3) = (1, 2).$$

Proposition 2.4.5. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

If $A \subseteq B$, then $A^\circ \subseteq B^\circ$.

Proof.

$$\begin{aligned}
A \subseteq B &\implies 2^A \subseteq 2^B && \text{(power set of subset)} \\
&\implies \mathcal{T} \cap 2^A \subseteq \mathcal{T} \cap 2^B \\
&\implies \bigcup (\mathcal{T} \cap 2^A) \subseteq \bigcup (\mathcal{T} \cap 2^B) \\
&\implies A^\circ \subseteq B^\circ && \text{(Definition 2.4.1)}
\end{aligned}$$

■

Note 2.4.2. Note that, $A^\circ \subseteq B^\circ$ does not implies $A \subseteq B$. Consider \mathbb{R} as a Euclidean metric space, and let

$$A = \{0\}, \quad B \subseteq \mathbb{R} \setminus \{0\}.$$

As $A^\circ = \emptyset$, $A^\circ \subseteq B^\circ$, but $A \setminus B = \{0\}$, so $A \not\subseteq B$.

§2.5 Limit Points and Isolated Points

Definition 2.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in X$ is a *limit point of A* iff for any $U \in \mathcal{T}$ with $x \in U$

$$A \cap U \setminus \{x\} \neq \emptyset.$$

The *derived set of A* is the set of all limit points of A .

Definition 2.5.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in A$ is said to be *isolated* iff there is $U \in \mathcal{T}$ with $x \in U$, such that

$$A \cap U \setminus \{x\} = \emptyset.$$

Notations. The Derived set of A is usually denoted A' .¹ But sometime it is also necessary to know in which space (with its topology) the derived set of A is. For example, for topological spaces $\mathbb{X}_1 = (X, \mathcal{T}_1)$ and $\mathbb{X}_2 = (X, \mathcal{T}_2)$, if $\mathcal{T}_1 \neq \mathcal{T}_2$, the derived sets of a set A in \mathbb{X}_1 and \mathbb{X}_2 may be different. So, below, the notation A' is used only if the confusions are unlikely; else, we denote $L_{\mathcal{T}}A$ for A' with respect to the topology \mathcal{T} .

Sometime, the set of isolated points of A is denoted by A^i . For avoiding confusions, we denote $I_{\mathcal{T}}(A)$ for A^i with respect to the topology \mathcal{T} .

Proposition 2.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then,

$$A \subseteq L(A) \sqcup I(A).$$

Proof. By Definition 2.5.1, $x \notin L(A)$ iff there exists a $U \in \mathcal{T}$ of $x \in U$ such that $A \cap U \setminus \{x\} = \emptyset$. This precisely satisfies Definition 2.5.2. Thus

$$A \subseteq L(A) \cup I(A).$$

As Definition 2.5.1 and 2.5.2 are precisely logical complement for each other, $x \in I(A) \cap L(A)$ always fails, i.e., $I(A) \cap L(A) = \emptyset$. Thus

$$A \subseteq L(A) \sqcup I(A).$$

■

¹See [ProofWiki](#) and [Wikipedia](#).

Proposition 2.5.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff $L(A) \subseteq A$.

Proof. First, prove \Rightarrow .

Aiming for a contradiction, suppose A is closed but there exists a $y \in L(A) \setminus A$.

By Definition 2.1.3, as A is closed, then A^c is open.

As $y \in A^c$ and A^c is open, then, by Proposition 2.4.2, there exists a $U \in \mathcal{T}$ with $y \in U$, such that $U \subseteq A^c$.

As U is an open set containing y and $A \cap U \setminus \{y\} = \emptyset$, then $y \notin L(A)$. This contradicts the assumption.

Thus $L(A) \subseteq A$. ■

§2.6 Closures

Definition 2.6.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

The *closure* of A is defined as

$$\text{Cl}_{\mathcal{T}}(A) := A \cup L(A).$$

When the confusions are unlikely, we simply write $\text{Cl}(A)$, \overline{A} or A^- for $\text{Cl}_{\mathcal{T}}(A)$.

Proposition 2.6.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff $A = A^-$

Proof.

$$A \text{ is closed} \iff A \supseteq L(A) \quad (\text{Proposition 2.5.2})$$

$$\iff A = A \cup L(A)$$

$$\iff A = A^-. \quad (\text{Definition 2.6.1})$$
■

Proposition 2.6.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff

$$A = I(A) \sqcup L(A).$$

Proof. As A is closed, we have

$$\begin{aligned}
A &= \text{Cl}(A) && \text{(Proposition 2.6.1)} \\
&= A \cup \text{L}(A) && \text{(Definition 2.6.1)} \\
&= A \setminus \text{L}(A) \sqcup \text{L}(A) \\
&= \text{I}(A) \sqcup \text{L}(A). && \text{(Proposition 2.5.1)}
\end{aligned}$$

■

Proposition 2.6.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$$A^- = \bigcap \{K \supseteq A : K \text{ closed in } \mathbb{X}\}.$$

Proof. By Proposition 2.6.1, A^- is closed. Thus, by Definition 2.1.3, $X \setminus A^-$ is open. Then we have

$$\begin{aligned}
X \setminus (X \setminus A^-) &= X \setminus (X \setminus A^-)^\circ && \text{(Proposition: 2.4.1)} \\
&= X \setminus \bigcup (\mathcal{T} \cap 2^{X \setminus A^-}) && \text{(Definition: 2.4.1)} \\
&= X \setminus \bigcup \{U \subseteq A : U \text{ open in } \mathbb{X}\} \\
&= \bigcap \{X \setminus U \supseteq A : U \text{ open in } \mathbb{X}\} && \text{(De Morgan's Law)} \\
&= \bigcap \{K \supseteq A : K \text{ closed in } \mathbb{X}\}. && \text{(Definition: 2.1.3)}
\end{aligned}$$

■

Proposition 2.6.4. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then we have

$$X \setminus A^\circ = (X \setminus A)^-.$$

Proof. First, we have

$$\begin{aligned}
X \setminus A^\circ &= X \setminus \bigcup (\mathcal{T} \cap 2^A) && \text{(Definition 2.4.1)} \\
&= \bigcap_{K \in \mathcal{T} \cap 2^A} (X \setminus K) && \text{(De Morgan's Law)}
\end{aligned}$$

For any K , $X \setminus K$ is a closed superset of $X \setminus A$.

As closed sets are closed under arbitrary intersection (Proposition 2.1.2), and $X \setminus A^\circ$ is the intersection of all closed superset of $X \setminus A$, by Proposition 2.6.3, $X \setminus A^\circ = (X \setminus A)^-$.

■

Proposition 2.6.5. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.
If $A \subseteq B$, then $A^- \subseteq B^-$.

Proof.

$$\begin{aligned}
A \subseteq B &\iff X \setminus A \supseteq X \setminus B \\
&\implies (X \setminus A)^\circ \supseteq (X \setminus B)^\circ && \text{(Proposition 2.4.5)} \\
&\iff X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ \\
&\iff (X \setminus (X \setminus A))^- \subseteq (X \setminus (X \setminus B))^- && \text{(Proposition 2.6.4)} \\
&\iff A^- \subseteq B^-.
\end{aligned}$$

■

§2.7 Density

Definition 2.7.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.
Then A is said to be *everywhere dense*, or simply *dense*, in \mathbb{X} iff

$$A^- = X.$$

Proposition 2.7.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.
Then A is dense in \mathbb{X} iff for any $U \in \mathcal{T}$, $A \cap U \neq \emptyset$.

Proof. First, prove \Rightarrow .

Assume A is dense in \mathbb{X} , then, by Definition 2.7.1, $A^- = X$.

By Definition 2.5.2, for any $x \in I(A)$, $x \in A$.

By Definition 2.5.1, for any $x \in L(A)$ and for any $U \in \mathcal{T}$ with $x \in U$,
 $U \cap A \neq \emptyset$.

As $A^- = X$, then, by Proposition 2.6.2, $X = I(A) \sqcup L(A)$.

Thus for any $U \in \mathcal{T}$, $A \cap U \neq \emptyset$.

□

Now, prove \Leftarrow .

Aiming for a contradiction, suppose for any $U \in \mathcal{T}$, $A \cap U \neq \emptyset$, but A is not dense in \mathbb{X} .

As, $A \subseteq X$, by Proposition 2.6.5, $A^- \subseteq X^-$. And, as X is closed in \mathbb{X} , by Proposition 2.6.1, $X = X^-$. Therefore, $A^- \subseteq X$.

As A is not dense in X , by Definition 2.7.1, $A^- \neq X$. Therefore, $A^- \subsetneq X$. This implies that $X \setminus A^-$ is non-empty. And, by Definition 2.6.1, $X \setminus A^- \in \mathcal{T}$.

By Proposition 2.4.2, for any $x \in X \setminus A^-$, there exists a $U \in \mathcal{T}$ with $x \in U$, such that $U \in X \setminus A^-$. Then $U \cap A = \emptyset$. This contradicts the assumption we have.

Therefore, A has to be dense in \mathbb{X} .

□

Thus, the proof is done.

■

Definition 2.7.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then A is said to be *nowhere dense* in \mathbb{X} iff

$$(A^-)^\circ = \emptyset.$$

Proposition 2.7.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then A is nowhere dense in \mathbb{X} iff for any $U \in \mathcal{T} \setminus \{\emptyset\}$,

$$U \setminus A^- \neq \emptyset.$$

Proof.

A is nowhere dense in \mathbb{X}

$$\iff (A^-)^\circ = \emptyset$$

$$\iff (\forall U \in \mathcal{T} : U \subseteq A^-)$$

§2.8 Neighbourhood Systems

Definition 2.8.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A subset $N \subseteq X$ is a *neighbourhood* of A iff

$$(\exists U \in \mathcal{T}) \quad A \subseteq U \subseteq N.$$

If $A = \{x\}$, we simply call N a *neighbourhood* of x .

If $N \in \mathcal{T}$ also, then N is an *open neighbourhood* of A ; and if N is closed, then N is a *closed neighbourhood* of A .

Proposition 2.8.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$A \in \mathcal{T}$ iff for any $x \in A$, A is a neighbourhood of x .

Proof. First, prove \Rightarrow .

If $A \in \mathcal{T}$, then, by Definition 2.8.1, for any $x \in A$, we have

$$x \in A \subseteq A.$$

□

Now, prove \Leftarrow .

Aiming for a contradiction, suppose for any $x \in A$, A is a neighbourhood of x , but $A \notin \mathcal{T}$.

As $X \setminus A$ is not closed, (otherwise, by Definition 2.1.3, $A = X \setminus (X \setminus A)$ is open) by Proposition 2.5.2, there exists $x \in \text{L}(X \setminus A) \setminus (X \setminus A)$.

Then, for such an $x \in A$ (for $x \notin X \setminus A$), for any $U \in \mathcal{T}$ with $x \in U$,

$$U \cap (X \setminus A) \neq \emptyset. \quad (\text{Definition 2.5.1})$$

By Definition 2.8.1, A fails to be a neighbourhood of x . This contradicts the assumption.

Thus A has to be open.

■

Chapter 3.

Sequences

§3.1 Convergent Sequences

Definition 3.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $u : \mathbb{N} \rightarrow X$.
u converges to a limit $x \in X$ iff for any $U \in \mathcal{T}$ with $x \in U$,

$$(\exists k \in \mathbb{R}_{>0}) \quad u[\mathbb{N}_{>k}] \subseteq U.$$

Proposition 3.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $u : \mathbb{N} \rightarrow X$.
u converges to a limit $x \in X$ iff for any $U \in \mathcal{T}$ with $x \in U$,

$$\mathbb{N} \setminus u^{-1}[U] \text{ is finite.}$$

Proof. First, prove \Rightarrow .

By Definition 3.1.1, as u converges to x , let $U \in \mathcal{T}$ with $x \in U$, then there exists a $k \in \mathbb{R}_{>0}$ such that $u[\mathbb{N}_{>k}] \subseteq U$.

Then we have

$$\begin{aligned} u[\mathbb{N}_{>k}] \subseteq U &\implies u^{-1}[u[\mathbb{N}_{>k}]] \subseteq u^{-1}[U] \\ &\implies \mathbb{N}_{>k} \subseteq u^{-1}[U] && \text{(image of inverse image)} \\ &\implies \mathbb{N} \setminus \mathbb{N}_{>k} \supseteq \mathbb{N} \setminus u^{-1}[U]. \end{aligned}$$

As $\mathbb{N} \setminus \mathbb{N}_{>k}$ is finite, its subset $\mathbb{N} \setminus u^{-1}[U]$ is finite.

□

Now, prove \Leftarrow .

By [image of inverse image](#), we have

$$u[u^{-1}[U]] \subseteq U.$$

As $u^{-1}[U]$ is a cofinite subset of \mathbb{N} , there exists a $k \in \mathbb{N}$ such that $I \supseteq \mathbb{N}_{>k}$. Then we have

$$U \supseteq u[\mathbb{N}_{>k}].$$

This precisely satisfies Definition 3.1.1.

□

Therefore the proof is done.

■

§3.2 Accumulation Points of Sequences

Definition 3.2.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $u : \mathbb{N} \rightarrow X$.

A point $x \in X$ is an *accumulation point* of u iff for any $U \in \mathcal{T}$ with $x \in U$, U contains infinitely many terms of u ; i.e.,

$$\forall U \in \mathcal{T} : x \in U \implies (\exists I \subseteq \mathbb{N} : |I| = \aleph_0 \implies u[I] \subseteq U).$$

Note 3.2.1. Sometime, an accumulation point of a sequence is also a limit of the range of the sequence. But this not always holds.

Consider \mathbb{R} as a Euclidean, and let $u : \mathbb{N} \rightarrow \mathbb{R}$ be defined as

$$u(n) := \left| \sin \left(\frac{\pi n}{2} \right) \right|.$$

Then 1 is an accumulation point of $u[\mathbb{N}]$, but $u[\mathbb{N}] = (u[\mathbb{N}])^i = \{0, 1\}$, so it has no limit point at all.

Proposition 3.2.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, let $u : \mathbb{N} \rightarrow X$, and let $x \in X$ be a limit of $u[\mathbb{N}]$.

Then x is an accumulation point of u .

Proof. Let $U \in \mathcal{T}$ with $x \in U$, then we have

$$u[u^{-1}[U]] \subseteq U.$$

By Proposition 3.1.1, as u converges to x , $u^{-1}[U]$ is a cofinite subset of \mathbb{N} . Thus $u^{-1}[U]$ is infinite.

As $u^{-1}[U]$ is infinite and $x \in U \in \mathcal{T}$, by Definition 3.2.1, x is an accumulation point of u .

■

Definition 3.2.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in X$ is an ω -accumulation point of A iff for any $U \in \mathcal{T}$ with $x \in U$,

$$|U \cap A| \geq \aleph_0.$$

Proposition 3.2.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, let $u : \mathbb{N} \rightarrow X$ be an injection, and let $x \in X$ be an accumulation point of u .

Then x is an ω -accumulation point of $u[\mathbb{N}]$.

Proof. By Definition 3.2.1, as x is an accumulation point of u , let $U \in \mathcal{T}$ with $x \in U$, there exists an infinite $I \subseteq \mathbb{N}$ such that $u[I] \subseteq U$.

As u is injective and I is infinite, $u[I]$ is also infinite.

As $u[I] \subseteq U$ and $U \in \mathcal{T}$ with $x \in U$ is arbitrarily given, by Definition 3.2.2, x is an ω -accumulation point of $u[\mathbb{N}]$. ■

Chapter 4.

Countable Axioms

§4.1 Covers and Bases

Definition 4.1.1. Let X be any set, and let $A \subseteq X$.

A family $\mathcal{C} \subseteq 2^X$ is a *cover for* A iff $A \subseteq \bigcup \mathcal{C}$.

\mathcal{C} is an *open cover* iff $\mathcal{C} \subseteq \mathcal{T}$.

Definition 4.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

A family $\mathcal{B} \subseteq 2^X$ is an *analytic basis for* \mathcal{T} iff

- (i) $\mathcal{B} \subseteq \mathcal{T}$;
- (ii) For any $U \in \mathcal{T}$, there is an $\mathcal{A} \subseteq \mathcal{B}$, such that $U = \bigcup \mathcal{A}$.

Definition 4.1.3. Let X be any set.

A family $\mathcal{B} \subseteq 2^X$ is a *synthetic basis on* X iff

- (i) \mathcal{B} is a cover for X ;
- (ii) For any $U, V \in \mathcal{B}$, there exists $\mathcal{A} \subseteq \mathcal{B}$, such that $U \cap V = \bigcup \mathcal{A}$.

Definition 4.1.4. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $x \in X$.

A family $\mathcal{B} \subseteq 2^X$ is a *local basis at* x iff

- (i) $\mathcal{B} \subseteq \mathcal{T}$;
- (ii) For any $B \in \mathcal{B}$, $x \in B$;
- (iii) For any $U \in \mathcal{T}$ with $x \in U$, there exists a $B \in \mathcal{B}$ such that $B \subseteq U$.

§4.2 First-Countable Spaces

Definition 4.2.1. A topological space $\mathbb{X} = (X, \mathcal{T})$ is said to be *first-countable* iff any $x \in X$ has a countable basis.

Proposition 4.2.1. Metric spaces are first-countable.

Proof. Let $\mathbb{X} = (X, d)$ be a metric space.

For any $x \in X$, let $\mathcal{B}_x : \mathbb{N} \rightarrow \mathcal{T}$ be defined as

$$\mathcal{B}_x(n) := B_{1/n}(x).$$

Clearly, the image $\mathcal{B}_x[\mathbb{N}]$ is countable.

Let $U \in \mathcal{T}$. As U is open, and as $x \in U$, then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(x) \subseteq U$.

By Archimedean Principle, there exists an $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. Then we have

$$\mathcal{B}_x(n) = B_{1/n}(x) \subseteq B_\varepsilon(x) \subseteq U.$$

As U is arbitrarily given, for any $x \in X$, $\mathcal{B}_x[\mathbb{N}]$ is a countable local basis at x . ■

Proposition 4.2.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a first-countable topological space, let $u : \mathbb{N} \rightarrow X$, and let $x \in X$ be an accumulation point of u .

Then x is a *subsequential limit* of u . That is, there exists an infinite $I \subseteq \mathbb{N}$, such that $u \upharpoonright_I$ converges to x (as a limit).

Proof.¹ By Definition 4.2.1, as \mathbb{X} is first-countable, there exists a countable local basis \mathcal{B} at x .

Let $\mathcal{B}_x : \mathbb{N} \rightarrow \mathcal{T}$ such that $\mathcal{B}_x[\mathbb{N}]$ is a local base at x and for any $n \in \mathbb{N}$,

$$\mathcal{B}_x(n) \supseteq \bigcup \mathcal{B}_x[\mathbb{N}_{>n}].$$

Let $w : I \rightarrow u[\mathbb{N}]$ (I infinite) such that for any $i \in I$, $w(i) \in \mathcal{B}_x(i)$.

Then, for any $k \in \mathbb{N}$, we have $w[I_{>k}] \subseteq \mathcal{B}_x(k)$. Thus, by Definition 3.1.1, w is a subsequence of u converging to x . ■

¹ The detail of this proof is incomplete.

§4.3 Second-Countable Spaces

Definition 4.3.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

\mathbb{X} is said to be *second countable* iff \mathcal{T} has a countable (analytic) basis.

Proposition 4.3.1. Second-countable spaces are first-countable.

Proof. Let $\mathbb{X} = (X, \mathcal{T})$ be a second-countable space.

By Definition 4.3.1, \mathcal{T} has a countable analytic basis.

Let $x \in X$ and let $U \in \mathcal{T}$ with $x \in U$. By Definition 4.1.2 there exists a countable $\mathcal{B} \subseteq \mathcal{T}$, such that for any $U \in \mathcal{T}$, there is an $\mathcal{A} \subseteq \mathcal{B}$, such that $U = \bigcup \mathcal{A}$.

As $U \in \mathcal{T}$ and $U = \bigcup \mathcal{A}$, by Proposition 2.4.2, there exists a $A \in \mathcal{A}$ such that $x \in A \subseteq U$.

Let $\mathcal{C} \subseteq \mathcal{B}$ be the family of all such A containing x , then, by Definition 4.1.4, \mathcal{C} is a local basis at x . And as \mathcal{B} is countable, as a subset, \mathcal{C} is also countable.

Therefore \mathcal{C} is a countable local basis at x .

As x is arbitrarily given, \mathbb{X} is first-countable. ■

Proposition 4.3.2. Consider \mathbb{R} as a Euclidean metric space.

\mathbb{R} is second-countable.

Proof. For any $x \in \mathbb{Q}$, let $\mathcal{O}_x : \mathbb{N} \rightarrow \mathcal{T}$ be defined as

$$\mathcal{O}_x(n) := B_{1/n}(x).$$

For any $r \in \mathbb{R} \setminus \mathbb{Q}$ and for any open set $U \ni r$, there exists $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(r) \subseteq U$.

There exists some $q \in \mathbb{Q}$ such that $q \in B_\delta(r)$. As $B_\delta(r)$ is open, by Definition 1.2.1, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(q) \subseteq B_\delta(r)$.

By Archimedean property, there exists $k \in \mathbb{N}$ such that $k > \frac{1}{\varepsilon}$. Then we have

$$\mathcal{O}_x(k) = B_{1/k}(q) \subseteq B_\varepsilon(q) \subseteq B_\delta(r).$$