

Notes for General Topology

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June 4, 2021

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Chapter 1

Topological Spaces

1.1 Topological Spaces

Definition 1.1.1 (topology). Let X be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. \mathcal{T} is called a *topology* on X iff it satisfies the *open set axioms*:

- O1.** $X \in \mathcal{T}$;
- O2.** \mathcal{T} is closed under arbitrary union;
- O3.** \mathcal{T} is closed under finite intersection.

Theorem 1.1.1. $\emptyset \in \mathcal{T}$.

Proof. By O2 in Definition 1.1.1, for all $\mathcal{A} \subseteq \mathcal{T}$, $\bigcup \mathcal{A} \in \mathcal{T}$. Clearly, $\emptyset \in \mathcal{T}$, then we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

□

Definition 1.1.2 (topological spaces). With the condition in Definition 1.1.1, the pair (X, \mathcal{T}) is called a *topological space*. All subsets of X in \mathcal{T} are said to be *open* in (X, \mathcal{T}) .

Definition 1.1.3 (closed sets). Let (X, \mathcal{T}) be a topological space. A subset V of X is said to be *closed* iff there is an open set U in X such that

$$V = X \setminus U.$$

Corollary 1.1.1. Let (X, \mathcal{T}) be a topological space, and let \mathcal{C} be the family of all closed sets in X . Then

- (i) $\emptyset, X \in \mathcal{C}$;
- (ii) \mathcal{C} is closed under arbitrary intersection;
- (iii) \mathcal{C} is closed under finite union.

Proof.

- (i) $X \in \mathcal{T}$ implies $X \setminus X = \emptyset \in \mathcal{C}$; and $\emptyset \in \mathcal{T}$ implies $X \setminus \emptyset = X \in \mathcal{C}$;
- (ii) As \mathcal{T} is closed under arbitrary union, then by Definition 1.1.3 and De Morgan's Law, \mathcal{C} is closed under arbitrary intersection.
- (iii) As \mathcal{T} is closed under finite intersection, then by Definition 1.1.3 and De Morgan's Law, \mathcal{C} is closed under finite union.

□

Definition 1.1.4 (finer and coarser topology). Let X be any set, and let $\mathcal{T}, \mathcal{T}'$ be topologies on X . \mathcal{T} is said to be *finer* than \mathcal{T}' iff $\mathcal{T} \supseteq \mathcal{T}'$; respectively, \mathcal{T} is said to be *coarser* than \mathcal{T}' iff $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 1.1.5 (neighbourhood). Given (X, \mathcal{T}) as a topological space and a point $x \in X$, a subset $N \subseteq X$ is called a *neighbourhood* iff it contains an open set U containing x .

Theorem 1.1.2. Given (X, \mathcal{T}) as a topological space and $U \subseteq X$, U is open iff for all $x \in U$, there is a neighbourhood N of x such that $N \subseteq U$.

Proof. If U is open, then U itself is a neighbourhood of x contained in U .

Conversely, if for all $x \in U$, there is a neighbourhood N_x of x contained in U , then there is a open neighbourhood $U_x \ni x$ contained in N_x . Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose U is not open, then U is a proper superset in the relation above. Then there exists $y \in U$ which is not in any U_x . This implies that such a y does not have any neighbourhood N_y in U , for such an N_y must contains an open $U_y \ni y$. For if it does, then there must be a U_x contains y . This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open.

□

1.2 Cover and Basis

Definition 1.2.1 (cover). Let (X, \mathcal{T}) be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of U iff the union of all sets in \mathcal{C} is a superset of U . That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call \mathcal{C} an *open cover* of U .

Let $\mathcal{S} \subseteq \mathcal{C}$, iff the union of \mathcal{S} is still a superset of U , then we call \mathcal{S} a *subcover* of \mathcal{C} .

Definition 1.2.2 (basis). Let (X, \mathcal{T}) be a topological space, and let $\mathcal{B} \subseteq \mathcal{T}$. \mathcal{B} is a *base* for \mathcal{T} iff it satisfies the following properties.

- (i) For each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, i.e., $\bigcup \mathcal{B} = X$.
- (ii) For each $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \ni x$ such that $B_3 \subseteq B_1 \cap B_2$.

Note 1.2.1. A base for a topology is not necessarily closed under finite intersections. For example, let

$$\mathcal{I} = \{(a, b) : 0 < b - a < 1\} \in \mathcal{P}(\mathbb{R}),$$

and let

$$\mathcal{J} = \{(-\infty, 2), (-2, \infty)\} \in \mathcal{P}(\mathbb{R}).$$

$\mathcal{I} \cup \mathcal{J}$ is a basis for the Euclidean topology on \mathbb{R} , but $\mathcal{I} \cup \mathcal{J}$ is not closed under finite intersection, for $\bigcap \mathcal{J} = (-2, 2)$ is not a member of $\mathcal{I} \cup \mathcal{J}$.

Corollary 1.2.1. Let (X, \mathcal{T}) be a topological space and let \mathcal{B} be a basis for \mathcal{T} . For all $U \in \mathcal{T}$, there exists $\mathcal{I}, \mathcal{J} \subseteq \mathcal{B}$ (\mathcal{J} finite) such that

$$U = \bigcup \mathcal{I} = \bigcap \mathcal{J}.$$

Example 1.2.1. Let $(\mathbb{R}, \mathcal{T})$ be a topological space where \mathcal{T} is induce from the standard Euclidean metric, and let \mathcal{B} be the family of all open proper intervals in \mathbb{R} . \mathcal{B} is a base for \mathcal{T} .

However, if \mathcal{T} is a discrete topology on \mathbb{R} , \mathcal{B} is not a base for \mathcal{T} , for some open subsets of \mathbb{R} , such as $\{0\}$, is neither any arbitrary union nor any finite intersection of \mathcal{B} -sets.

Theorem 1.2.1. Let (X, \mathcal{T}) and (X, \mathcal{T}') be topological spaces, and let \mathcal{B} and \mathcal{B}' be the basis for \mathcal{T} and \mathcal{T}' respectively. \mathcal{T}' is finer than \mathcal{T} iff for any $x \in X$ and for any $B \in \mathcal{B}'$ with $B \ni x$, there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq B$.

Proof. For \implies . Let $B \in \mathcal{B}$. Clearly, $B \in \mathcal{T}$. Also, $B \in \mathcal{T}'$ for $\mathcal{T} \subseteq \mathcal{T}'$. By Corollary 1.2.1, there exists $\mathcal{I}' \subseteq \mathcal{B}'$ such that $B = \bigcup \mathcal{I}'$. Clearly, all \mathcal{I}' -sets are subsets of B .

For \impliedby . For all $B \in \mathcal{B}$ and for all $x \in B$, there exists $B' \in \mathcal{B}'$ with $x \in B' \subseteq B$. Let \mathcal{J}' denote all such B' , then we have $B = \bigcup \mathcal{J}'$. By Definition 1.2.2, we have $B \in \mathcal{B} \subseteq \mathcal{T}'$. Above all, $\mathcal{T} \subseteq \mathcal{T}'$. \square

Note 1.2.2. In this theorem, it is not necessary that $\mathcal{B} \subseteq \mathcal{B}'$. For example, let

$$\begin{aligned}\mathcal{B} &= \{(a, b) \subseteq \mathbb{R} : b - a = 1\}, \\ \mathcal{B}' &= \{(a, b) \subseteq \mathbb{R} : b - a = 2\}.\end{aligned}$$

Obviously, $\mathcal{B} \not\subseteq \mathcal{B}'$ and $\mathcal{B}' \not\subseteq \mathcal{B}$, but they generate exactly the same topologies on \mathbb{R} .

For another example, let

$$\begin{aligned}\mathcal{B} &= \{\text{all open intervals in } \mathbb{R}\}, \text{ and} \\ \mathcal{B}' &= \{\text{all singletons in } \mathbb{R}\}.\end{aligned}$$

In this case, also, $\mathcal{B} \not\subseteq \mathcal{B}'$ and $\mathcal{B}' \not\subseteq \mathcal{B}$, but \mathcal{B}' generates the discrete topology which is the finest topology on \mathbb{R} .

1.3 Metrizable

Definition 1.3.1 (metric spaces). Let X be any set. A *metric* ρ on X is a function $\rho : X \times X \rightarrow \mathbb{R}$ satisfying the following conditions: for all $x, y, z \in X$

- (i) $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ iff $x = y$;
- (ii) $\rho(x, y) = \rho(y, z)$;
- (iii) $\rho(x, z) + \rho(z, y) \geq \rho(x, y)$.

Definition 1.3.2 (balls). Let (X, ρ) be a metric space, let $x \in X$, and let $\varepsilon \in \mathbb{R}_{>0}$. The *open ε -ball about x* or just *ε -ball about x* is defined to be

$$B(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}.$$

The *closed ε -ball* about x is defined to be

$$\overline{B}(x, \varepsilon) = \{y \in X : \rho(x, y) \leq \varepsilon\}.$$

Example 1.3.1. Let X be any set, and let metric ρ_p on X^n ($n \in \mathbb{Z}_{>0}$) defined by

$$\rho_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}},$$

where $p \in \mathbb{R}_{\geq 1}$. ρ_2 is so called the *standard Euclidean metric*. If $X = \mathbb{R}$, then the metric space (\mathbb{R}^n, ρ_2) is so-called *Euclidean n -space*.

For all $p, q \in \mathbb{R}_{\geq 1}$, if $p < q$, then for all $\varepsilon \in \mathbb{R}_{>0}$ and for all $x, y \in X$, $\rho_p(x, y) \geq \rho_q(x, y)$; in particular, $\rho_p = \rho_q$ iff there is a unique $k \in \{1, \dots, n\}$, such that for all $i \in \{1, \dots, n\} \setminus \{k\}$, $x_i = 0$. As $\rho_p(x, y)$ is always “overestimated” than $\rho_q(x, y)$, we have $B_{\rho_p}(x, \varepsilon) \supseteq B_{\rho_q}(x, \varepsilon)$.

Example 1.3.2. Let X be any set. The *discrete metric* ρ on X is defined to be

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Example 1.3.3. Let $a, b \in \mathbb{R}$ with $a < b$, and let metric ρ_p on $C[a, b]$ defined by

$$\rho_p(f, g) = \left(\int_a^b |f(t) - g(t)|^p dt \right)^{\frac{1}{p}},$$

where $p \geq 1$. In particular,

$$\rho_\infty(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|.$$

Theorem 1.3.1. Let (X, ρ) be a metric space, and let

$$\mathcal{T}_\rho = \left\{ \bigcup_{x \in U} B_\rho(x, \varepsilon) : U \subseteq X \wedge \varepsilon \in \mathbb{R}_{>0} \right\}.$$

(X, \mathcal{T}_ρ) is a topological space; i.e., \mathcal{T}_ρ satisfies the open set axioms (Definition 1.1.1).

Proof. For O1. Apparently, there exists (for any) $\varepsilon \in \mathbb{R}_{>0}$,

$$X = \bigcup_{x \in X} B(x, \varepsilon).$$

For O2. Let I be an index set, and let $U_i \in \mathcal{T}$ for all $i \in I$. There exists $\varepsilon \in \mathbb{R}_{>0}$ we can define

$$U = \bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{x \in U_i} B(x, \varepsilon) = \bigcup_{x \in U} B(x, \varepsilon).$$

Then, for all $x \in U$, there exists $B(x, \varepsilon) \ni x$. Thus $U \in \mathcal{T}$.

For O3. Let $U, V \in \mathcal{T}$. If $U \cap V = \emptyset$, then the proof is done. Suppose $U \cap V \neq \emptyset$, and let $x \in U \cap V$. U is open and $x \in U$, so there exists $r_1 \in \mathbb{R}_{>0}$ such that $B(x, r_1) \subseteq U$; V is open and $x \in V$, so there exists $r_2 \in \mathbb{R}_{>0}$ such that $B(x, r_2) \subseteq V$.

If $r_1 = r_2$, then $B(x, r_1) = B(x, r_2)$. If $r_1 \neq r_2$, say $r_1 < r_2$, then $B(x, r_1) \subseteq B(x, r_2) \subseteq V$. Above all, for any $x \in U \cap V$, there exists $r \in \mathbb{R}_{>0}$ such that $B(x, r) \subseteq U \cap V$. Thus, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$U \cap V = \bigcup_{x \in U \cap V} B(x, \varepsilon),$$

i.e., $U \cap V \in \mathcal{T}_\rho$. □

Theorem 1.3.2. Let (X, ρ) be a metric space, then for all $x, y \in X$ ($x \neq y$), there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$.

Proof. Suppose for all $\varepsilon > 0$, $B(x, \varepsilon) \cap B(y, \varepsilon) \neq \emptyset$, then there must be a $z \in X$ such that $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. $z \in B(x, \varepsilon)$ only if $\rho(x, z) < \varepsilon$, and $z \in B(y, \varepsilon)$ only if $\rho(y, z) < \varepsilon$. Thus

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

As the assumption holds for all $\varepsilon > 0$, we may put

$$\varepsilon = \frac{\rho(x, y)}{2}.$$

Then, we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which is impossible. □

Definition 1.3.3 (induced topologies). Let (X, ρ) be a metric space. A topology \mathcal{T} on X is said to be *induced* by ρ iff for all $\varepsilon > 0$, any $U \in \mathcal{T}$ is the union of ball(s) in X ; i.e.,

$$\mathcal{T} = \left\{ U \subseteq X : U = \bigcup_{x \in X} B(x, \varepsilon) \right\}.$$

In this case, \mathcal{T} is called the *underlying topology* of ρ .

Definition 1.3.4 (metrizable spaces). Let (X, \mathcal{T}) be a topological space. If there is any ρ induce \mathcal{T} , then (X, \mathcal{T}) is said to be *metrizable*.

Definition 1.3.5 (Lipschitz equivalence). Let X be any set, and let ρ and ρ' be metrics on X . ρ and ρ' are said to be *Lipschitz equivalent* iff there exist $c, C > 0$, such that for all $x, y \in X$,

$$c\rho(x, y) \leq \rho'(x, y) \leq C\rho(x, y).$$

Theorem 1.3.3. Lipschitz equivalence is an equivalence relation.

Proof. Clearly, Definition 1.3.5 also holds for $\rho = \rho'$. So, Lipschitz equivalence is reflexive. In Definition 1.3.5, the relation also holds for $\frac{1}{C}\rho' \leq \rho \leq \frac{1}{c}\rho'$. So Lipschitz equivalence is symmetric.

If there is another ρ'' be Lipschitz equivalent to ρ' , then there is $r, R > 0$, such that for all $x, y \in X$,

$$r\rho''(x, y) \leq \rho'(x, y) \leq R\rho''(x, y).$$

By the conditions in Definition 1.3.5, we have

$$\frac{c}{r}\rho(x, y) \leq \rho''(x, y) \leq \frac{C}{R}\rho(x, y),$$

i.e., ρ and ρ'' are also Lipschitz equivalent. So Lipschitz equivalence is transitive.

Above all, Lipschitz equivalence is an equivalence relation. \square

Theorem 1.3.4. Let X be any set, and let ρ and ρ' be metrics on X . If ρ and ρ' are Lipschitz equivalent, then ρ and ρ' induce the same topology.

Proof. As ρ and ρ' are Lipschitz equivalent, by Definition 1.3.5, there is a $c > 0$ such that for all $x, y \in X$,

$$c\rho(x, y) \leq \rho'(x, y).$$

Given $r \in \mathbb{R}_{>0}$ and for all $x \in X$, we have

$$B_{\rho'}(x, cr) \subseteq B_{c\rho}(x, r) = B_{\rho}\left(x, \frac{1}{c}r\right).$$

For all $U \in \mathcal{T}_{\rho}$, for all $x \in U$, there is an $\varepsilon \in \mathbb{R}_{>0}$, such that

$$B_{\rho'}(x, \varepsilon) \subseteq B_{\rho}(x, \varepsilon) \subseteq U.$$

So $U \in \mathcal{T}_{\rho'}$. Then we have $\mathcal{T}_{\rho} \subseteq \mathcal{T}_{\rho'}$.

Similarly, $U \in \mathcal{T}_{\rho'}$ only if $U \in \mathcal{T}_{\rho}$. Then we have $\mathcal{T}_{\rho'} \subseteq \mathcal{T}_{\rho}$.

Above all, $\mathcal{T}_{\rho} = \mathcal{T}_{\rho'}$. \square

Note 1.3.1. In this proposition, \mathcal{T}_ρ and $\mathcal{T}_{\rho'}$ are said to be homeomorphic or topologically equivalent (see Definition 1.6.2). And ρ and ρ' are also said to be topologically equivalent.

Example 1.3.4. In Example 1.3.1, for all $p, q \geq 1$, all ρ_p and ρ_q induce the same topology. Let X be any subset of \mathbb{R}^n , then for all $x, y \in X$, if $p < q$, then

$$\rho_p(x, y) \geq \rho_q(x, y).$$

Thus, if ρ_1 and ρ_∞ are Lipschitz equivalent, then any other ρ_p and ρ_q are Lipschitz equivalent. We have

$$\rho_1(x, y) = \sum_{i=1}^n |x_i - y_i| \geq \max_{i \in \{1, \dots, n\}} |x_i - y_i| = \rho_\infty(x, y).$$

Clearly,

$$\rho_\infty(x, y) \leq \rho_1(x, y) \leq n\rho_\infty(x, y).$$

By Definition 1.3.5, ρ_1 and ρ_∞ are Lipschitz equivalent, hence for all $p, q \geq 1$, ρ_p and ρ_q are Lipschitz equivalent. Thus, by Proposition 1.3.4, they induce the same topology.

1.4 Separation Axioms. From T_0 to Hausdorff

Definition 1.4.1 (separated). In a topological space, two sets are said to be *separated* iff each is disjoint from other's closure.

Definition 1.4.2 (separated by neighbourhoods). In a topological space (X, \mathcal{T}) , two sets A and B are said to be *separated by neighbourhood* iff there are neighbourhoods N_A of A and N_B of B such that N_A and N_B are disjoint.

Definition 1.4.3 (topologically indistinguishable). Let (X, \mathcal{T}) be a topological space. Two points $x, y \in X$ are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let \mathcal{N}_x be the family of all neighbourhoods of x and let \mathcal{N}_y be the family of all neighbourhoods of y , we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively, x, y are said to be *topologically distinguishable* iff they are not topologically distinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

Example 1.4.1. In an indiscrete topological space, all distinct points are topologically indistinguishable.

T_0 Spaces

Definition 1.4.4 (T_0 spaces). A topological space (X, \mathcal{T}) is said to be T_0 or *Kolmogorov*, iff all distinct points $x, y \in X$ are topologically distinguishable.

Example 1.4.2. Let X be any set and let \mathcal{T} be the indiscrete topology on X . (X, \mathcal{T}) is T_0 iff $|X| \in \{0, 1\}$.

T_1 Spaces

Definition 1.4.5 (R_0 spaces). A topological space (X, \mathcal{T}) is said to be R_0 iff any two topologically distinguishable points in X are separated.

Definition 1.4.6 (T_1 Spaces). A topological space (X, \mathcal{T}) is said to be T_1 or *Fréchet* iff it is T_0 and R_0 ; i.e., all distinct points $x, y \in X$ are separated.

Example 1.4.3 (R_0 but not T_0). Let \mathcal{T} be a countable family of disjoint proper intervals on \mathbb{R}^n , and $\bigcup \mathcal{T} = \mathbb{R}^n$. (X, \mathcal{T}) is R_0 , but not T_0 .

Example 1.4.4 (T_0 but not R_0). Let $(\mathbb{R}_{\geq 0}, \mathcal{T})$ be a topological space with

$$\mathcal{T} = \{U \subseteq \mathbb{R} : \forall i \in \mathbb{R}_{\geq 0}, U_i = [0, i)\},$$

Then for all $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$, if $x \neq y$, then there are $|y - x|$ neighbourhoods N_x of x do not contain y . Thus, it is T_0 .

On the other hand, it is not R_0 , because for all $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$ with $x < y$, $x \in \overline{\{y\}} = [0, y]$.

Example 1.4.5 (R_0 but not T_1). Let X be any set with $|X| \geq 3$, let $U \subsetneq X$ with $|U| \geq 2$, let $\mathcal{T}_{X \setminus U}$ be a T_1 topology on $X \setminus U$, and let \mathcal{T}

$$\mathcal{T} = \mathcal{T}_{X \setminus U} \cup \{X, U\}.$$

For all $x, y \in X$, if $x \neq y$, then they are separated. Thus, the space is R_0 .

But (X, \mathcal{T}) is not T_1 , because all $\{u\} \in U$ share the same closure which is U itself.

Corollary 1.4.1 (alternative definitions of R_0 spaces). Let (X, \mathcal{T}) be R_0 , then the following conditions are equivalent.

- (i) The closure of all singletons in X are not T_0 subspace.
- (ii) For any two points $x, y \in X$, $x \in \overline{\{y\}}$ iff $y \in \overline{\{x\}}$.

(iii) Every open set is the union of closed sets.

Proof.

- (i) By Definition 1.4.5, if y and x are topologically distinguishable, by Definition 1.4.5, x and y are separated; i.e., $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$.
- (ii) By Definition 1.4.5, for all $x, y \in X$, x, y are not separated only if they are topologically indistinguishable. By Definition 1.4.3, they share all their neighbourhoods, thus they have the same closure; i.e., $\overline{\{x\}} = \overline{\{y\}}$.
- (iii) For any $U \in \mathcal{T}$,

$$U = \bigcup_{x \in U} \{x\}.$$

If (X, \mathcal{T}) is T_1 , then we are done. Suppose (X, \mathcal{T}) is not T_1 , then there exists $A \in \mathcal{T}$ with $|A| > 1$, and for all $B \subsetneq A$, $B \notin \mathcal{T}$ (proof omitted). For such A , $X \setminus A$ is open, for $X \setminus A = \bigcup (\mathcal{T} \setminus \{A\})$, thus A is also closed.

Suppose for any such A with $A \cap U \neq \emptyset$, $A \subseteq U$. Suppose it fails, i.e., $A \cap U \neq A$, then we have $A \cap U \subsetneq A$ and $A \cap U \in \mathcal{T}$, which is contradicted to the condition of A . Now we have

$$U = \bigcup \mathcal{A} \cup \bigcup_{x \in I} \{x\}$$

where \mathcal{A} is the family of such A , and I is the union of all closed singletons in U . Thus U is open.

□

Corollary 1.4.2 (alternative definitions of T_1 spaces). Let (X, \mathcal{T}) be T_1 , then the following conditions are equivalent.

- (i) All singletons in X are closed.
- (ii) Every subset of X is the intersection of all open sets containing it.
- (iii) Every cofinite subset of X is open.

Proof.

- (i) Suppose there exists $\{x\} \subseteq X$ with $\overline{\{x\}} \neq \{x\}$, then there exists $y \in \overline{\{x\}}$ with $x \neq y$. By Definition 1.4.6, this is impossible.

(ii) For any $A \subseteq X$,

$$A = \bigcup_{x \in A} \{x\}.$$

Let $B = X \setminus A$. By De Morgan's law,

$$B = \bigcap_{x \in A} X \setminus \{x\}.$$

(X, \mathcal{T}) is T_1 iff all $\{x\}$ are closed, in which case, B is the intersection of all open sets $X \setminus \{x\} \supseteq B$.

(iii) Let A be a cofinite subset of X . $X \setminus A$ is a finite union of singletons. As (X, \mathcal{T}) is T_1 , any singleton in X is closed. By Proposition 1.1.1, $X \setminus A$ is closed. By Definition 1.1.3, A is open.

□

Hausdorff Spaces

Definition 1.4.7 (R_1 spaces). A topological space (X, \mathcal{T}) is said to be R_1 iff any two topological distinguishable points in X are separated by neighbourhoods.

Definition 1.4.8 (Hausdorff Spaces). A topological space (X, \mathcal{T}) is said to be *Hausdorff* or T_2 iff it is T_0 and R_1 ; i.e., all distinct points $x, y \in X$ are separated by neighbourhoods.

Theorem 1.4.1. All metrizable spaces are Hausdorff

Proof. Let (X, \mathcal{T}) be a metrizable space. There exists a metric ρ on X that induces \mathcal{T} . Given distinct points $x, y \in X$, suppose for all $\varepsilon \in \mathbb{R}_{>0}$, there exists $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. Then $\rho(x, z) < \varepsilon$ and $\rho(y, z) < \varepsilon$. Now we have

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

Put $\rho(x, y) > 2\varepsilon$ as x and y are arbitrarily given. Then we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which implies that ρ is not a metric on X . Hence, (X, \mathcal{T}) is not metrizable which is contradicted to the condition. □

Theorem 1.4.2. All singletons in a Hausdorff space are closed.

Proof. Let (X, \mathcal{T}) be a Hausdorff space, and let $x \in X$. For all $y \in X$ with $x \neq y$, there is a open neighbourhood U_y of y such that $x \notin U_y$. Then, for all such U_y , we have

$$\forall y \in X, x \in X \setminus U_y = \{x\} \iff x \in \bigcap_{y \in X \setminus \{x\}} X \setminus U_y = \{x\}.$$

As all $X \setminus U_y$ are closed, their intersection $\{x\}$ is closed. \square

Example 1.4.6 (T_1 but not Hausdorff). Let X be a nonempty set, let $p \in X$, let \mathcal{T}' be a Hausdorff topology on $X \setminus \{p\}$, and let

$$\mathcal{T} = \{X\} \cup \mathcal{T}'.$$

Then, all $x \in (X, \mathcal{T})$ are closed, thus (X, \mathcal{T}) is Fréchet. But the only neighbourhood of p is X , so its closure is X . Then, for any $x \in X \setminus \{p\}$, x and p are not separated, in which case (X, \mathcal{T}) is not R_0 . Thus, (X, \mathcal{T}) is not Hausdorff.

1.5 Continuity

Definition 1.5.1 (continuous maps). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f : X \rightarrow Y$ is said to be *continuous* iff for any open set U in Y , its preimage in X under f is open.

Note 1.5.1. In Definition 1.5.1, note that even if for any open set U in X , $f[X]$ is open in Y , f is not necessarily continuous. For example, let $X = (\mathbb{R}, \mathcal{T}_X)$ with \mathcal{T}_X induced by standard Euclidean metric, let $Y = (\mathbb{R}, \mathcal{T}_Y)$ with \mathcal{T}_Y as a indiscrete topology, and define

$$f(x) = [x],$$

where $[x]$ denotes the integer part of x . Then for all $U \subseteq X$, $f[U]$ is open in Y , but by Definition 1.5.1, f is not continuous.

Note 1.5.2. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, if \mathcal{T}_X is the discrete topology on X , then any function with domain X is continuous. If \mathcal{T}_Y is the indiscrete topology on Y , then any function with codomain Y is continuous.

Note 1.5.3. A function is continuous bijection does not implies that its inverse is continuous. For example, let X be any set and let \mathcal{T} and \mathcal{T}' be its topologies. If \mathcal{T} is finer than \mathcal{T}' , then any bijection $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}')$ is continuous. In this case, however, if $\mathcal{T} \neq \mathcal{T}'$, then f^{-1} is not continuous.

Theorem 1.5.1. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f : X \rightarrow Y$ is continuous at $x \in X$ iff for any neighbourhood N_y of $f(x)$, there is a neighbourhood N_x of x , such that $f[N_x] \subseteq N_y$.

Proof. Let N_y be a neighbourhood of $f(x)$. Clearly, there exists an open set U_y contains y .

By Definition 1.5.1, f is continuous at x iff $x \in f^{-1}[U_y] \in \mathcal{T}_X$. Clearly, $f^{-1}[U_y]$ is a neighbourhood of x . We have $f[f^{-1}[U_y]] = U_y \subseteq N_y$.

By Proposition 1.1.2, there U_x must contains at least one neighbourhood N_x of x , thus, $f[N_x] \subseteq U_y$. \square

Proposition 1.5.1. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be metrizable spaces. A map $f : X \rightarrow Y$ is continuous at $p \in X$ iff for any $\varepsilon > 0$, there is a $\delta > 0$, such that for all $x \in B_X(p, \delta)$, $f(x) \in B_Y(f(p), \varepsilon)$, where B_X is defined by any metrics ρ_X induces \mathcal{T}_X , and B_Y is defined by any metrics ρ_Y induces \mathcal{T}_Y .

Proof. Clearly, for all $\varepsilon > 0$, $B_Y(f(x), \varepsilon)$ is an open neighbourhood of $f(x)$. f is not necessarily be injective, so $f^{-1}[B_Y(f(x), \varepsilon)] = U \in \mathcal{T}_X$. By Definition 1.5.1, U is open, so for some $\delta > 0$, $B_X(x, \delta) \subseteq U$. Thus, By Proposition 1.5.1, f is continuous iff $f[B_X(x, \delta)] \subseteq B_Y(f(x), \varepsilon)$. This satisfies the conditions we have. \square

Proposition 1.5.2. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A function $f : X \rightarrow Y$ is said to be continuous iff for any closed set V in Y , its preimage in X under f is closed.

Proof. Let U_Y be any open set in Y , let U_X be the preimage of U_Y under f . By Definition 1.5.1, U_X is open in X . Let

$$V_X = f^{-1}[Y \setminus U_Y] = X \setminus U_X,$$

Then V_X is closed. \square

Definition 1.5.2 (convergence of sequences). Let (X, \mathcal{T}) be a topological space, and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be *converges* in X iff there is an $x \in X$, such that for any open neighbourhood U_x of x , it contains a cofinite subset $A \subseteq \{x_n\}$. That is, there exists N in the domain of $\{x_n\}$, for any natural numbers $n \geq N$, $x_n \in U_x$.

Example 1.5.1.

1. In a discrete topological space, a sequence $\{x_n\}$ converges iff there is an N in the domain of $\{x_n\}$, for any natural numbers $m > N$, $x_N = x_m$.
 2. In an indiscrete topological space, any sequence $\{x_n\}$ in X converges in X .
- And

$$\lim_{n \rightarrow \infty} \{x_n\} = X.$$

Proposition 1.5.3. In a Hausdorff space, any convergent sequence converges to a unique point in the space.

Proof. Let (X, \mathcal{T}) be a Hausdorff space, and let $\{x_n\}$ be a sequence in X . Suppose $\{x_n\}$ converges to more than one point, say to $x, y \in X$ with $x \neq y$, then, for all neighbourhoods N_x of x and N_y of y , N_x contains a cofinite subset $A \subseteq \{x_n\}$ and N_y contains a cofinite subset $B \subseteq \{x_n\}$. If this were true, $N_x \cap N_y$ should be non-empty, otherwise N_x or N_y should be finite.

Then, x and y are not separated by neighbourhoods, thus (X, \mathcal{T}) is not Hausdorff. This is a contradiction.

But, as (X, \mathcal{T}) is Hausdorff, there must be mutually disjoint N_x and N_y . Thus, the assumption cause a contradiction. \square

Note 1.5.4. As all metrizable spaces are Hausdorff, so any convergent sequence in a metrizable space converges to at most one point.

Proposition 1.5.4. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological space, let $f : X \rightarrow Y$ be a map, and let $\{x_n\}$ be a convergent sequence in X . If f is continuous, then $f[\{x_n\}]$ is a sequence convergent in Y .

Proof. Let U_y be any open neighbourhood of $f(x)$. By Definition 1.5.1, $f^{-1}[U_y]$ is also an open neighbourhood of x . By Definition 1.5.2, $f^{-1}[U_y]$ contains a cofinite subset $A \subseteq \{x_n\}$. Then $f[A]$ is a cofinite subset of $f[\{x_n\}]$. As $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$, $f[\{x_n\}]$ converges in $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$. \square

Note 1.5.5. In this proposition, even if $f[\{x_n\}]$ converges in Y , f might be discontinuous. For example, let X any set, let \mathcal{T} be the indiscrete topology on X , let U be another cofinite subset of X with $X \neq U$, and let $\mathcal{T}' = \{\emptyset, X, U\}$. Let $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}')$ be defined by

$$f(x) = x.$$

By Definition 1.5.1, f is not continuous. But, for any convergent sequence $\{x_n\}$ in (X, \mathcal{T}) , $f[\{x_n\}]$ also convergent in (X, \mathcal{T}) .

1.6 Homeomorphisms

Definition 1.6.1 (homeomorphisms). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A bijection $f : X \rightarrow Y$ is called a *homeomorphism* iff

- (i) f is a bijection;
- (ii) f is continuous;
- (iii) f^{-1} is continuous.

Definition 1.6.2 (homeomorphic). Two topological spaces (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

Proposition 1.6.1. Two topological spaces are homeomorphic only if they have the same cardinality.

Proof. Let X and Y be two sets with $|X| < |Y|$. There is no surjection from A to B . \square

Example 1.6.1. $|X| = |Y|$ does not imply (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are homeomorphic, even if they are finite. For example, let $X = Y = \{1, \dots, n\}$, and let \mathcal{T}_X be indiscrete topology on X and $\mathcal{T}_Y = \mathcal{P}(X)$. There is no homeomorphism between X and Y .

On the other hand, even if $|X| = |Y| \geq \aleph_0$ and \mathcal{T}_X and \mathcal{T}_Y are induced by same metric, X and Y might not be homeomorphic. For example, if \mathcal{T}_X and \mathcal{T}_Y are both induced by standard Euclidean metric, and $X = [a, b] \subseteq \mathbb{R}$ and $Y = [c, d] \subseteq \mathbb{R}$ where $a < b$ and $c < d$. No doubt, $|X| = |Y| = \mathfrak{c}$, but X and Y are not homeomorphic.

Example 1.6.2. \mathbb{R}^n and \mathbb{R}^m ($n < m$) are not homeomorphic, although $|\mathbb{R}^n| = |\mathbb{R}^m|$.

Example 1.6.3. Let I be a proper interval in \mathbb{R}^n , let \mathcal{T} be standard Euclidean topology on \mathbb{R}^n and let \mathcal{T}_I be a subspace topology on I . $I \cong \mathbb{R}^n$ iff I is an open interval.

But if $\mathcal{T} = \mathcal{P}(\mathbb{R}^n)$, then there exists bijection $f : I \rightarrow \mathbb{R}^n$, for $|I| = |\mathbb{R}^n|$, and such f can be bicontinuous, for any subset $A \subseteq I$ is also open in \mathbb{R}^n , vise versa. In this case, $I \cong \mathbb{R}^n$ whenever I is a closed, half-close, half-open, or open interval respect to standard Euclidean metric.

Example 1.6.4. Let S^n be an n -dimensional sphere with center $o \in \mathbb{R}^{n+1}$ and radius $r \in \mathbb{R}$, i.e.,

$$S^n = \{x \in \mathbb{R}^{n+1} : \rho(o, x) = r\},$$

where ρ is the standard Euclidean metric on \mathbb{R}^{n+1} . For any $x \in S^n$, let $U = B(x, \varepsilon) \cap S^n$ where $0 \leq \varepsilon < \max_{x, y \in S^n} \rho(x, y)$ (here $B(x, \varepsilon) = \{x\}$ if $\varepsilon = 0$), then $S^n \setminus U \cong \mathbb{R}^n$.

Example 1.6.5. Indeed, $S^1 \setminus \{x\} \cong \mathbb{R}$ where $x \in S^1$. But for any interval $I \in \mathbb{R}$, $S^1 \not\cong I$.

1.7 Interiors and Closures

Definition 1.7.1 (interiors). The *interior* of a set A , denoted A° , is defined to be the union of all open subsets of A .

Definition 1.7.2 (closure). The *closure* of a set A , denoted \overline{A} , is defined to be the intersection of all closed supersets of A .

Definition 1.7.3 (dense sets). Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$. A is said to be *dense*, iff $\overline{A} = X$.

Definition 1.7.4 (nowhere dense sets). A set A is said to be *nowhere dense* iff the interior of its closure is empty.

Proposition 1.7.1 (properties of interiors). Let (X, \mathcal{T}) be any topological space and $A, B \subseteq X$.

- (i) (Intensive) $A^\circ \subseteq A$.
- (ii) A is open iff $A = A^\circ$.
- (iii) (Idempotence) $(A^\circ)^\circ = A^\circ$.
- (iv) $(A \cap B)^\circ = A^\circ \cap B^\circ$.
- (v) $A \subseteq B \implies A^\circ \subseteq B^\circ$.
- (vi) If B is open, then $B \subseteq A$ iff $B \subseteq A^\circ$.

Proof.

- (i) By Definition 1.7.1, naturally, $A^\circ \subseteq A$.

- (ii) By Definition 1.1.2, A° is the union of open sets hence it is open. A is open iff it is the union of all open subsets of A . Thus $A = A^\circ$.
- (iii) A° is open, thus $(A^\circ)^\circ = A^\circ$.
- (iv) By Definition 1.7.1, we have

$$\begin{aligned}
(A \cap B)^\circ &= \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq A \cap B \right\} \\
&= \left\{ \bigcup U : (U \in \mathcal{T} \wedge U \subseteq A) \wedge (U \in \mathcal{T} \wedge U \subseteq B) \right\} \\
&= \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq A \right\} \cap \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq B \right\} \\
&= A^\circ \cap B^\circ.
\end{aligned}$$

- (v) Clearly, $A^\circ \subseteq A$, thus,

$$A \subseteq B \implies A^\circ \subseteq B$$

Suppose $A^\circ \not\subseteq B^\circ$, then $A^\circ \setminus B^\circ$ is not empty (\emptyset is the subset of any set, so A° is not empty).

Then there exists $x \in A^\circ$ with $x \in \partial B$ ($x \in B$ but $x \notin B^\circ$). Then there exists neighbourhood $N_x \ni x$, and $N_x \cap \partial B \neq \emptyset$. But this is impossible, for $A^\circ \subseteq B$ implies that $A^\circ \cap \partial B = \emptyset$ (This is a straight consequence of $A^\circ \cap \partial A = \emptyset$. See Proposition 1.8.1), so such N_x does not exist. Thus,

$$A^\circ \subseteq B^\circ.$$

- (vi) If B is open, then $B = B^\circ$. Then $B \subseteq A$ iff $B^\circ \subseteq A^\circ$.

□

Proposition 1.7.2 (properties of closures). Let (X, \mathcal{T}) be a topological space, and let $A, B \subseteq X$.

- (i) \overline{A} is closed.
- (ii) A is closed iff $A = \overline{A}$.
- (iii) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.
- (iv) If A is closed, then $A \supseteq B$ iff $A \supseteq \overline{B}$

Proof.

- (i) By Definition 1.7.2, \overline{A} is the intersection of closed sets. By Proposition 1.1.1, \overline{A} is closed.
- (ii) Proposition 1.1.1 implies that any closed set is the intersection of closed sets, this is precisely what Definition 1.7.2 says.
- (iii) $A \subseteq B$ iff $X \setminus A \supseteq X \setminus B$. Then we have

$$X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ$$

Clearly, $(X \setminus A)^\circ$ is the union of all open set disjoint from A , then, by De Morgan's laws, $X \setminus (X \setminus A)^\circ$ is the intersection of all closed sets containing A . By Definition 1.7.2, we have $(X \setminus A)^\circ = \overline{A}$. Thus

$$\overline{A} \subseteq \overline{B}.$$

- (iv) If A is closed, then $A = \overline{A}$. Suppose $B \subseteq A$, then we have

$$\overline{B} \subseteq \overline{A} \iff \overline{B} \subseteq A.$$

□

1.8 Boundaries

Definition 1.8.1 (boundaries). Let A be any set, the *boundary* of A , denoted ∂A , is defined to be the complement of the interior of A in the closure of A ; i.e.,

$$\partial A = \overline{A} \setminus A^\circ.$$

Proposition 1.8.1 (properties of boundaries). Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$.

- (i) ∂A is closed.
- (ii) $A^\circ \cap \partial A = \emptyset$.
- (iii) $\overline{A} = A^\circ \cup \partial A$.
- (iv) A is closed iff $\partial A \subseteq A$.
- (v) ∂A is nowhere dense.
- (vi) $\partial \overline{A} \subseteq \partial A \subseteq \partial A^\circ$.

(vii) $\partial A = \partial(X \setminus A)$.

(viii) A is dense iff $\partial A = X \setminus A^\circ$.

Proof.

(i) \overline{A} is closed, and $X \setminus A^\circ$ is also closed. Thus

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (X \setminus A)$$

is closed.

(ii) By Definition 1.8.1, we have

$$\partial A = \overline{A} \setminus A^\circ \iff \partial A \cap A^\circ = \overline{A} \setminus A^\circ \cap A^\circ = \overline{A} \cap \emptyset = \emptyset.$$

(iii) We have

$$\begin{aligned} \partial A = \overline{A} \setminus A^\circ &\iff \partial A \cup A^\circ = \overline{A} \setminus A^\circ \cup A^\circ = \overline{A} \cap (X \setminus A^\circ \cup A^\circ) \\ &\iff \partial A \cup A^\circ = \overline{A} \cap X|_{\text{for } A^\circ \subseteq X} = \overline{A}. \end{aligned}$$

(iv) As A is closed, $A = \overline{A}$ (this can be straightly proved by Definition 1.7.2).

By Definition 1.8.1, it is clear that $\partial A \subseteq \overline{A}$, thus $\partial A \subseteq A$.

(v) By Definition 1.7.4, ∂A is nowhere dense iff $\overline{\partial A}^\circ$ is empty. We have

$$\begin{aligned} \overline{\partial A}^\circ &= \overline{\overline{A} \setminus A^\circ}^\circ \\ &= (\overline{A} \setminus A^\circ) \cup (\overline{A} \setminus A^\circ) \setminus (\overline{A} \setminus A^\circ) \\ &= \emptyset. \end{aligned}$$

(vi) $\overline{A} \supseteq A^\circ$ implies $\overline{A}^\circ \supseteq (A^\circ)^\circ = A^\circ$, then we have,

$$\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^\circ \subseteq \overline{A} \setminus A^\circ = \partial A.$$

$A^\circ \subseteq A$ implies $\overline{A}^\circ \subseteq \overline{A}$, then we have,

$$\partial A^\circ = \overline{A^\circ} \setminus (A^\circ)^\circ \supseteq \overline{A} \setminus A^\circ.$$

(vii) We have

$$\begin{aligned} \partial(X \setminus A) &= \overline{X \setminus A} \setminus (X \setminus A)^\circ \\ &= X \setminus A^\circ \setminus (X \setminus \overline{A}) \\ &= X \setminus A^\circ \cap \overline{A} \\ &= \overline{A} \setminus A^\circ \\ &= \partial A. \end{aligned}$$

(viii) By Definition 1.7.3, A is dense in X iff $\overline{A} = X$. Then we have,

$$\begin{aligned}\overline{A} = X &\iff \overline{A} \setminus A^\circ = X \setminus A^\circ \\ &\iff \partial A = X \setminus A^\circ.\end{aligned}$$

□

1.9 Limit Points

Definition 1.9.1 (limit points). Let (X, \mathcal{T}_X) be a topological space, and let $A \subseteq X$. A point $x \in X$ is called a *limit point* of A iff for all neighbourhood N_x of x , $N_x \setminus \{x\}$ intersects A .

Proposition 1.9.1. Let A be any set, and let x be a limit point of A , then x is an element of the closure of A .

Proof. If A is empty, then this is vacuously true. So, suppose A is not empty. By Definition 1.9.1, for all neighbourhood N_x of x , $N_x \setminus \{x\} \cap A$ is not empty. Naturally, $N_x \cap A$ is not empty.

Assume that $x \notin \overline{A}$, then $X \setminus \overline{A}$ is a neighbourhood of x , by Definition 1.1.5, and is disjoint from A . This is contradicted to the conditions. □

Note 1.9.1. In this proof, the proposition also holds for $N_x \cap A^\circ = \emptyset$. Because if it is true, then

$$N_x \cap \partial A \supseteq (N_x \cap A) \setminus (N_x \cap A^\circ) = N_x \cap A.$$

This implies that $A \subseteq \partial A$. In this case, $\overline{A} = \partial A$, for

Assume that $x \notin \partial A$, then we have the same conclusion.

Then $A^\circ = A \setminus \partial A = \emptyset$.

Proposition 1.9.2. A set is closed iff it contains all its limit point.

Proof. Let A be a set. By proposition 1.9.1, for every limit point of A , it is also an element of the closure \overline{A} . And A is closed iff $A = \overline{A}$. □

Definition 1.9.2 (convergent sequences). Let (X, \mathcal{T}_X) be a topological space. A sequence $\{x_n\}$ in X is said to be *convergence* in X iff there is an open set U contains all but finite terms of $\{x_n\}$.

Chapter 2

Creating New Spaces

2.1 Subspaces

Definition 2.1.1 (subspace topology). Let (X, \mathcal{T}) be a topological space and let $A \subseteq X$. The *subspace topology* \mathcal{T}_A on A is defined to be the family of the intersections of open sets in (X, \mathcal{T}) and A . That is,

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}.$$

2.2 Quotient Spaces

Definition 2.2.1 (quotient topology). Let (X, \mathcal{T}) be a topological space and let \sim be an equivalence relation on X . The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

2.3 Product Spaces

Definition 2.3.1 (product topologies).

Chapter 3

Topological Properties

3.1 Cardinal Functions

3.2 More on Separation Axioms

Definition 3.2.1 (separated sets). Let (X, \mathcal{T}) be a topological space, and let $A, B \in \mathcal{P}(X)$.

- (i) A and B are said to be *separated* iff each is disjoint from other's closure.
- (ii) A and B are said to be *separated by neighbourhoods* iff there are neighbourhoods N_A of A and N_B of B such that N_A and N_B are disjoint.
- (iii) A and B are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods \overline{N}_A of A and \overline{N}_B of B such that \overline{N}_A and \overline{N}_B are disjoint.
- (iv) A and B are said to be *separated by a continuous function* iff there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $f[A] = \{0\}$ and $f[B] = \{1\}$.
- (v) A and B are said to be *precisely separated by a continuous function* iff there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$

Definition 3.2.2 ($T_{2^{1/2}}$ spaces). A topological space (X, \mathcal{T}) is said to be $T_{2^{1/2}}$ or *Urysohn* iff two distinct points in X are separated by closed neighbourhoods.

Example 3.2.1 (T_2 but not $T_{2^{1/2}}$).¹ (Remained as a problem)

¹ See [MathPlanet](#).

Definition 3.2.3 (T_3 spaces). A topological space (X, \mathcal{T}) is said to be T_3 or *regular* iff it is T_0 and given any point $x \in (X, \mathcal{T})$ and closed set $V \subseteq X$ with $x \notin V$ are separated by neighbourhoods.

Definition 3.2.4 ($T_{3\frac{1}{2}}$ spaces). A topological space (X, \mathcal{T}) is said to be $T_{3\frac{1}{2}}$, or *Tychonoff* or, *completely T_3* , or *completely regular*, iff it is T_0 and given any point x and closed set $V \subseteq X$ with $x \notin V$, they are separated by a continuous function.

Definition 3.2.5 (T_4 spaces). A topological space (X, \mathcal{T}) is said to be T_4 or *normal* iff it is Hausdorff and any two disjoint closed subsets of X are separated by neighbourhoods.

Proposition 3.2.1 (Urysohn's lemma). A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

Definition 3.2.6 (T_5 spaces). A topological space (X, \mathcal{T}) is said to be T_5 or *completely T_4* iff it is T_1 any two separated sets are separated by neighbourhoods.

Proposition 3.2.2. Every subspace of a T_5 space is normal.

Definition 3.2.7 (T_6 spaces). A topological space (X, \mathcal{T}) is said to be T_6 , or *perfectly T_4* or *perfectly normal* iff it is T_1 and any two disjoint closed sets are precisely separated by a continuous function.

Proposition 3.2.3 (Tietze extension theorem). Let (X, \mathcal{T}) be normal topological space, and let $f : A \rightarrow (\mathbb{R}, \mathcal{T}')$ be a continuous map where A is a closed subset of X and \mathcal{T}' is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \rightarrow (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$

3.3 Countability Axioms

3.4 Compactness

Definition 3.4.1 (compactness). A topological space (X, \mathcal{T}) is said to be *compact* iff every open cover of X has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

3.5 Connectedness

Definition 3.5.1 (connectedness). Let (X, \mathcal{T}) be a topological space. (X, \mathcal{T}) is said to be *connected* iff X is not empty and it is not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

Definition 3.5.2 (path-connectedness). Let (X, \mathcal{T}) be a topological space.

- (i) A map $\gamma : [0, 1] \rightarrow X$ is called a *path* in X iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that γ is path from x to y in X .
- (ii) X is said to be *path-connected* iff for all $x, y \in X$ there is a path from x to y in X .