Notes for Vector Calculus

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Contents

Chapter 1.

Vector Spaces

§1.1 Linear Maps

Definition 1.1.1. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *linear*, iff for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $a, b \in \mathbb{R}$,

$$f(a\mathbf{u} + b\mathbf{v}) = af(\mathbf{u}) + bf(\mathbf{v}).$$

Note 1.1.1. Assume $b = 0$, then we have

$$f(a\mathbf{u}) = af(\mathbf{u}).$$

Assume $a = b = 1$, then Definition 1.1.1 gives

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}).$$

Lemma 1.1.1. With the condition above, $f(\mathbf{x}) = \mathbf{0}_{\in\mathbb{R}^m}$ iff $\mathbf{x} = \mathbf{0}_{\in\mathbb{R}^n}$.

Proof.

$$\begin{aligned} f(\mathbf{x}) = \mathbf{0}_{\in\mathbb{R}^m} &\iff f(\mathbf{x} + 0\mathbf{x}) = \mathbf{0}_{\in\mathbb{R}^m} \\ &\iff f(\mathbf{x}) + 0f(\mathbf{x}) = \mathbf{0}_{\in\mathbb{R}^m} \\ &\iff f(\mathbf{x}) = \mathbf{0}_{\in\mathbb{R}^m}. \end{aligned}$$

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Lemma 1.1.2. With the condition above, f is injective.

Proof. For any $\mathbf{y} \in \mathbb{R}^m$, there is an $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{y} = f(\mathbf{x}).$$

Assume there is an $\mathbf{x}' \in \mathbb{R}^n$ such that

$$\mathbf{y} = f(\mathbf{x}').$$

Then we have

$$f(\mathbf{x}) - f(\mathbf{x}') = \mathbf{0} \iff f(\mathbf{x} - \mathbf{x}') = \mathbf{0}.$$

By Lemma 1.1.1, we have

$$\mathbf{x} = \mathbf{x}'.$$

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Lemma 1.1.3. With the condition above, f is continuous.

Proof. Let $\varepsilon \in \mathbb{R}$, and let $\mathbf{p} \in \mathbb{R}^n$. For any $\mathbf{q} \in \mathbb{R}^n$ such that $f(\mathbf{q}) \in B(f(\mathbf{p}), \varepsilon) \setminus \{f(\mathbf{p})\}$,

$$\|f(\mathbf{p}) - f(\mathbf{q})\| = \sqrt{\sum_{i=1}^{m} (q_i - y_i)^2} < \varepsilon.$$

As $f(\mathbf{q}) \neq f(\mathbf{p})$, $\mathbf{p} \neq \mathbf{q}$, thus

$$\|\mathbf{p} - \mathbf{q}\| > 0.$$

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Lemma 1.1.4. With the condition above, f^{-1} is linear.

§1.2 Linear Spans and Combinations

Definition 1.2.1. Let $\langle \mathbf{v}_i \rangle_{i=1}^n$ be a sequence of vectors in $\mathbb{R}^m$.

The *span* of $\langle \mathbf{v}_i \rangle$ is a subset of $\mathbb{R}^m$ defined as

$$\operatorname{span}\langle \mathbf{v}_i \rangle := \left\{ \mathbf{a} \cdot \langle \mathbf{v}_i \rangle : \mathbf{a} \in \mathbb{R}^n \right\}.$$

An element $\mathbf{u} \in \mathbb{R}^m$ is a *linear combination* of $\langle \mathbf{v}_i \rangle$ iff $\mathbf{u} \in \operatorname{span}\langle \mathbf{v}_i \rangle$.

Note 1.2.1. By dot product,

$$\mathbf{u} = \sum_{i=1}^n a_i \mathbf{v}_i.$$

Note 1.2.2. For any vector $\mathbf{p} \in \mathbb{R}^m$, the linear combination form of $\mathbf{p}$ is

$$\mathbf{p} = \sum_{i=1}^n p_i \hat{e}_i,$$

where $\hat{e}_i$ denotes the i-th basis of $\mathbb{R}^n$; i.e., all terms but the i-th term of $\hat{e}_i$ are 0.

Lemma 1.2.1. With the condition above, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined as

$$f(\mathbf{x}) := \mathbf{x} \cdot \langle \mathbf{v}_i \rangle.$$

Then f is a linear map.

Proof. Let $a \in \mathbb{R}$, and let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ Then we have

$$\begin{aligned} f(\mathbf{u} + \mathbf{v}) &= (\mathbf{u} + \mathbf{v}) \cdot \langle \mathbf{v}_i \rangle \\ &= \sum_{i=1}^n (u_i + v_i) \mathbf{v}_i \\ &= \sum_{i=1}^n u_i \mathbf{v}_i + \sum_{i=1}^n v_i \mathbf{v}_i \\ &= f(\mathbf{u}) + f(\mathbf{v}). \end{aligned}$$

And

$$\begin{aligned} f(a\mathbf{u}) &= (a\mathbf{u}) \cdot \langle \mathbf{v}_i \rangle \\ &= a(\mathbf{u} \cdot \langle \mathbf{v}_i \rangle) \\ &= af(\mathbf{u}). \end{aligned}$$

By Definition 1.1.1, f is linear. ■

§1.3 Linear Dependency

Definition 1.3.1. A sequence $\langle \mathbf{v}_i \rangle_{i=1}^n$ of vectors in $\mathbb{R}^m$ is said to be *linearly independent* iff for any $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\mathbf{a} \cdot \langle \mathbf{v}_i \rangle \neq \mathbf{0}.$$

$\langle \mathbf{v}_i \rangle$ is *linearly dependent* iff it is not linearly independent.

Lemma 1.3.1. With the condition above, $\langle \mathbf{v}_i \rangle_{i=1}^n$ is linearly dependent iff there exists $k \in \{1, \ldots, n\}$ such that there exists $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ with $a_k = 0$, such that

$$\mathbf{v}_k = \mathbf{a} \cdot \langle \mathbf{v}_i \rangle.$$

Proof.

$$\begin{aligned} \mathbf{v}_k = \mathbf{a} \cdot \langle \mathbf{v}_i \rangle &\iff \mathbf{0} = \mathbf{a} \cdot \langle \mathbf{v}_i \rangle - \mathbf{v}_k \\ &\iff \mathbf{0} = (a_1\mathbf{v}_1 + \cdots + 0\mathbf{v}_k + \cdots + a_n\mathbf{v}_n) - \mathbf{v}_k \\ &\iff \mathbf{0} = (a_1\mathbf{v}_1 + \cdots + (-1)\mathbf{v}_k + \cdots + a_n\mathbf{v}_n) \\ &\iff \mathbf{0} = (a_1, \ldots, a_{k-1}, -1, a_{k+1}, \ldots, a_n) \cdot \langle \mathbf{v}_i \rangle. \end{aligned}$$

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Note 1.3.1. Lemma 1.3.1 can define Definition 1.3.1 in a more geometric way.

Lemma 1.3.2. Let $\langle \mathbf{v}_i \rangle_{i=1}^{n}$ be a sequence of vectors in $\mathbb{R}^m$, where $n < m$.

Then $\operatorname{span}\langle \mathbf{v}_i \rangle$ is homeomorphic to $\mathbb{R}^n$.

Proof. Let $f : \mathbb{R}^n \to \operatorname{span}\langle \mathbf{v}_i \rangle$ be defined as

$$f(\mathbf{x}) := \mathbf{x} \cdot \langle \mathbf{v}_i \rangle.$$

By Lemma 1.2.1, as $\operatorname{span}\langle \mathbf{v}_i \rangle \subseteq \mathbb{R}^m$, f is linear. By Lemma 1.1.2, f is injective.

By Definition 1.2.1, for any $\mathbf{u} \in \operatorname{span}\langle \mathbf{v}_i \rangle$, there is an $\mathbf{a} \in \mathbb{R}^n$, such that

$$\mathbf{u} = \mathbf{a} \cdot \langle \mathbf{v}_i \rangle,$$

Thus f is surjective.

Now, it is proved that f is bijective.

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Lemma 1.3.3. Let $\langle \mathbf{v}_i \rangle_{i=1}^{n}$ be a sequence of vectors in $\mathbb{R}^n$ (notice the n here).

Then $\langle \mathbf{v}_i \rangle$ is linearly independent iff

$$\operatorname{span}\langle \mathbf{v}_i \rangle = \mathbb{R}^n.$$

Proof. Assume $\langle \mathbf{v}_i \rangle$ is linearly independent.

By Definition 1.2.1, for any $\mathbf{a} \in \mathbb{R}^n$,

$$\mathbf{a} \cdot \langle \mathbf{v}_i \rangle \in \operatorname{span}\langle \mathbf{v}_i \rangle.$$

So $\mathbb{R}^n \subseteq \operatorname{span}\langle \mathbf{v}_i \rangle$. As $\operatorname{span}\langle \mathbf{v}_i \rangle \subseteq \mathbb{R}^n$ also, we have

$$\operatorname{span}\langle \mathbf{v}_i \rangle = \mathbb{R}^n.$$

□

Conversely, assume $\operatorname{span}\langle \mathbf{v}_i \rangle = \mathbb{R}^n$, but $\operatorname{span}\langle \mathbf{v}_i \rangle$ is not linearly independent.

Let $\langle \mathbf{v}_{i_k} \rangle_{i=1}^{n-1}$ be a subsequence of $\langle \mathbf{v}_i \rangle$, and assume $\langle \mathbf{v}_{i_k} \rangle$ is linearly independent. ■