

Notes for Undergraduate Algebra by Serge Lang

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Chapter 1

The Integers

1.1 Terminology of Sets

1.2 Basic Properties

Theorem 1.2.1 (Induction: First Form). Suppose that for each integer $n \geq 1$ we are given an assertion $A(n)$, and that we can prove the following two properties:

- (1) The assertion $A(1)$ is true.
- (2) For each integer $n \geq 1$, if $A(n)$ is true, then $A(n + 1)$ is true.

Then for all integers $n \geq 1$, the assertion $A(n)$ is true.

Theorem 1.2.2 (Induction: Second Form). Suppose that for each integer $n \geq 0$ we are given an assertion $A(n)$, and that we can prove the following two properties:

- (i') The assertion $A(0)$ is true;
- (ii') For each integer $n > 0$, if $A(k)$ is true for every integer k with $0 \leq k < n$, then $A(n)$ is true.

Then the assertion $A(n)$ is true for all integers $n \geq 0$.

Theorem 1.2.3 (Euclidean Algorithm). Let m, n be integers and $m > 0$. Then there exists integers q, r with $0 \leq r < m$ such that

$$n = qm + r.$$

The integers q, r are uniquely determined by these conditions.

Proof. For $m = n$, then $q = 1$ and $r = 0$ are unique.

For $m < n$, there is a greatest integer q such that

$$0 \leq n - qm < m.$$

Because if q is not the greatest, then there must be $q+1$ such that the inequality holds. But

$$0 \leq n - (q+1)m \iff m \leq n - qm,$$

which is impossible. Thus q must be the greatest one.

Secondly, there is a smallest integer q such that

$$0 \leq n - qm < m.$$

Because if it is not, then $q-1$ makes the inequality holds. But

$$n - (q-1)m < m \iff n - qm < 0,$$

which is impossible. Thus q must be the smallest one.

As q is the greatest as well as the smallest one, then q is unique.

Suppose r is not unique, then there must be $s \in \mathbb{Z}_{[0,m)}$ with $s \neq r$ such that

$$n = qm + r, \text{ and}$$

$$n = qm + s.$$

then, we have

$$0 = r - s \neq 0,$$

a contradiction. So r is unique. □

Exercises

1. If m, n are integers ≥ 1 and $n \geq m$, define the **binomial coefficient**

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

As usual, $n! = n \cdot (n-1) \cdots 1$ is the product of the first n integers. We define $0! = 1$ and $\binom{n}{0} = 1$. Prove that

$$\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m}.$$

Proof. This one can be straightly proved by the definition of binomial coef-

ficient as following.

$$\begin{aligned}
\binom{n}{m-1} + \binom{n}{m} &= \frac{n!}{(m-1)!(n-m+1)!} + \frac{n!}{m!(n-m)!} \\
&= \frac{n!m}{m!(n-m+1)!} + \frac{n!(n-m+1)}{m!(n-m+1)!} \\
&= \frac{n!}{m!(n-m+1)!} (m + n - m + 1) \\
&= \frac{(n+1)!}{m!(n+1-m)!} \\
&= \binom{n+1}{m}.
\end{aligned}$$

□

2. Prove by induction that for any integers x, y we have

$$(x+y)^n = \sum_{i=1}^n \binom{n}{i} x^i y^{n-i} = y^n + \binom{n}{1} xy^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + x^n.$$

Proof. The equation holds for $n = 1$, because

$$(x+y)^1 = x+y.$$

Assume the equation holds for any integer $n \geq 1$, then

$$\begin{aligned}
(x+y)^{n+1} &= (x+y) \sum_{i=0}^n \binom{n}{i} x^i y^{n-i} \\
&= \sum_{i=0}^n \left[\binom{n}{i} x^i y^{n+i} + \binom{n}{i} x^{i+1} y^{n-i-1} \right].
\end{aligned}$$

By Exercise 1, it is easy to prove that

$$\binom{n}{k} = \binom{n+1}{k+1} - \binom{n}{k+1}.$$

Then the equation is

$$\begin{aligned}
&\sum_{i=0}^{n+1} \left[\binom{n+1}{i} x^i y^{n+1-i} - \binom{n}{i} x^i y^{n+1-i} + \binom{n}{i} x^i y^{n+1-i} \right] \\
&= \sum_{i=0}^{n+1} \binom{n+1}{i} x^i y^{n+1-i} \Big|_{\text{let } k = n+1} \\
&= \sum_{i=0}^k \binom{k}{i} x^i y^{k-i}.
\end{aligned}$$

□

3. Prove the following statements for all positive integers:

- (a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$;
- (b) $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$;
- (c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = [n(n + 1)/2]^2$.

Proof. (a) Clearly the equation holds for $n = 1$. Suppose it holds for all integer $n \geq 1$, then we have

$$\sum_{i=1}^{n+1} (2i - 1) = n^2 + 2n + 1 = (n + 1)^2$$

(b) Clearly the equation holds for $n = 1$. Suppose it holds for all integer $n \geq 1$, then we have

$$\begin{aligned} \sum_{i=1}^{n+1} i^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \Big|_{\text{let } k = n+1} \\ &= \frac{k(k+1)(2k+1)}{6}. \end{aligned}$$

(c) Clearly the equation holds for $n = 1$. Suppose it holds for all integer $n \geq 1$, then we have

$$\begin{aligned} \sum_{i=1}^{n+1} i^3 &= \left(\frac{n(n+1)}{2} \right)^2 + (n+1)^3 \\ &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} \\ &= \left(\frac{(n+1)(n+2)}{2} \right)^2 \Big|_{\text{let } k = n+1} \\ &= \left(\frac{k(k+1)}{2} \right)^2 \end{aligned}$$

□

4. Prove that

$$\left(1 + \frac{1}{1}\right)^1 \left(1 + \frac{1}{2}\right)^2 \cdots \left(1 + \frac{1}{n-1}\right)^{n-1} = \frac{n^{n-1}}{(n+1)!}$$

Proof. The equation holds for $n = 2$, because

$$\left(1 + \frac{1}{1}\right)^1 = 2 = \frac{2}{1!}.$$

Assume the equation holds for any integer $n \geq 2$, then

$$\begin{aligned} \prod_{i=1}^n \left(1 + \frac{1}{i}\right)^i &= \frac{n^{n-1}}{(n-1)!} \left(1 + \frac{1}{n}\right)^n \\ &= \frac{n^{n-1}}{(n-1)!} \frac{(n+1)^n}{n^n} \\ &= \frac{n^{n-1}(n+1)^n}{n!n^{n-1}} \\ &= \frac{(n+1)^n}{n!} \Big|_{\text{let } k = n+1} \\ &= \frac{k^{k-1}}{(k-1)!}. \end{aligned}$$

□

5. Let x be a real number. Prove that there exists an integer q and a real number s with $0 \leq s < 1$ such that $x = q + s$, and that q, s are uniquely determined. Can you deduce the Euclidean algorithm from this result without using induction?

Proof. This is just a straight corollary of Euclidean algorithm. □

1.3 Greatest Common Divisor

Definition 1.3.1. Given $n, d \in \mathbb{Z} \setminus \{0\}$, we shall say that d divides n , or d is a divisor of n , denoted $d|n$, iff

$$\exists q \in \mathbb{Z}, \quad n = dq.$$

The divisors of n is a set

$$\text{div}(n) = \{d \in \mathbb{Z} \setminus \{0\} : d|n\}.$$

For example,

$$\begin{aligned} \text{div}(8) &= \{\pm 1, \pm 2, \pm 4, \pm 8\}, \\ \text{div}(-24) &= \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12\}, \\ \text{div}(35) &= \{\pm 1, \pm 5, \pm 7, \pm 35\}. \end{aligned}$$

Clearly, for all $n \in \mathbb{Z} \setminus \{0\}$, for all $x \in \text{div}(n) \setminus \{\pm n\}$

$$|x| \leq \frac{|n|}{2}.$$

Definition 1.3.2. Given $m, n \in \mathbb{Z} \setminus \{0\}$, the *common divisor* is defined to be the set

$$\text{cd}(m, n) = \{d \in \mathbb{Z}_{>0} : d|m \wedge d|n\}.$$

Thus,

$$\text{cd}(m, n) = \text{div}(m)_{>0} \cap \text{div}(n)_{>0}$$

For example,

$$\begin{aligned}\text{cd}(18, 12) &= \{2, 3, 6\}, \\ \text{cd}(-18, 12) &= \{2, 3, 6\}, \\ \text{cd}(24, -20) &= \{2, 4\},\end{aligned}$$