Notes for General Topology

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June 9, 2021

Contents

Chapter 1

Topological Spaces

1.1 Interior

Definition 1.1.1. Let $(X, \mathcal{T})$ be any topological space, and let $A \subseteq X$. The *interior* of A, denoted A°, is defined to be the union of all open subsets of A; i.e.,

$$A^\circ = \bigcup \mathcal{U}, \quad \mathcal{U} = \mathcal{P}(X) \cap \mathcal{T}.$$

Proposition 1.1.1. $A^\circ \subseteq A$.

Proof. Let $\mathcal{U} = \mathcal{P}(X) \cap \mathcal{T}$. Clearly

$$A^\circ = \bigcup \mathcal{U} \subseteq A.$$

$\square$

Proposition 1.1.2. $A \in \mathcal{T}$ if and only if $A = A^\circ$.

Proof. Let $\mathcal{U} = \mathcal{P}(X) \cap \mathcal{T}$.

$\mathcal{U}$ is closed under arbitrary union, so if $A = A^\circ = \bigcup \mathcal{U}$, then $A \in \mathcal{T}$.

On the other hand, suppose $A \in \mathcal{T}$ but $A \neq A^\circ$, then $A^\circ \subsetneq A$. As $A \in \mathcal{T}$, there exists $U \in \mathcal{P}(A) \cap \mathcal{T}$ with $U \ni x$. But as $x \in X \setminus A^\circ$, U could not be a subset of A°. Then we have $U \in \mathcal{T}$ but $U \not\subseteq \bigcup \mathcal{U}$, which is contradicted to the assumption. $\square$

Proposition 1.1.3. $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proof. $(A \cap B)^\circ \in \mathcal{T}$, so there exists $\mathcal{U} \subseteq \mathcal{T}$ such that

$$\bigcup \mathcal{U} = (A \cap B)^\circ.$$

Clearly, $\mathcal{U} = \mathcal{P}(A \cap B) \cap \mathcal{T}$.

Let $\mathcal{I} = \mathcal{P}(A) \cap \mathcal{T}$ and $\mathcal{J} = \mathcal{P}(B) \cap \mathcal{T}$, then $\mathcal{I} = A^\circ$ and $\mathcal{J} = B^\circ$.

$$\begin{aligned} \mathcal{I} \cap \mathcal{J} &= \mathcal{P}(A) \cap \mathcal{T} \cap \mathcal{P}(B) \cap \mathcal{T} \\ &= \mathcal{P}(A \cap B) \cap \mathcal{T} \\ &= \mathcal{U}. \end{aligned}$$

Then we have

$$(A \cap B)^\circ = \bigcup \mathcal{U} = \bigcup \mathcal{I} \cup \bigcup \mathcal{J} = A^\circ \cap B^\circ.$$

□

Proposition 1.1.4. If $A \subseteq B$, then $A^\circ \subseteq B^\circ$.

Proof. Let $\mathcal{U} = \mathcal{P}(A) \cap \mathcal{T}$, and let $\mathcal{V} = \mathcal{P}(B) \cap \mathcal{T}$

$A^\circ \subseteq A$, so $A \subseteq B$ implies $A^\circ \subseteq B$, so $\mathcal{U} \subseteq \mathcal{V}$. Thus,

$$A^\circ = \bigcup \mathcal{U} \subseteq \bigcup \mathcal{V} = B^\circ.$$

□

Proposition 1.1.5. Let $\mathcal{T}$ be induced by a metric ρ on X. The interior of A° is the union of all open balls in A; i.e., there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$A^\circ = \bigcup_{x \in A} B(x, \varepsilon).$$

Proof. Let $\mathcal{U} = \mathcal{P}(A) \cap \mathcal{T}$, then any $U \in \mathcal{U}$,

$$U = \bigcup_{x \in U} B(x, \varepsilon).$$

??????????

For any open subsets $U \in \mathcal{U}$ and for any $x \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$A^\circ = \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \bigcup_{x \in U} B(x, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon).$$

□