Notes for General Topology

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Chapter 1.

Metric Spaces

§1.1 Metric Spaces

Definition 1.1.1. Let X be any set. A function $d : X \times X \to \mathbb{R}_{\geq 0}$ is *metric function*, or, simply, *metric on* X iff it satisfies the *metric axioms*. That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ iff $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.1.2. Let X be any set and let d be a structure on X. The pair (X, d) is called a *metric space* iff d is a metric on X.

Definition 1.1.3. A $M = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$. An *open ε-ball*, or just ε-ball, about x is defined to be the set

$$B_\varepsilon(x; d) := \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) := \{y \in X : d(x, y) \leq \varepsilon\}.$$

Note 1.1.1. As

$$M = (X, d),\ M' = (X, d'),\ M'' = (X, d''),\ \ldots$$

are different although they share the same set X, for any $x \in X$ and any $\varepsilon \in \mathbb{R}_{>0}$,

$$B_\varepsilon(x; d),\ B_\varepsilon(x; d'),\ B(x; d''),\ \ldots$$

are also different. However, if confusion is unlikely, we simply write "$B_\varepsilon(x)$" for "$B_\varepsilon(x; d)$".

Example 1.1.1. The *Euclidean metric space* $M = (X, d)$ is an n-dimensional set X equipped with the *Euclidean metric* d defined as

$$d(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

Example 1.1.2. A *discrete metric space* $M = (X, d)$ is a set X equiped with the *discrete metric* d defined as

$$d(x, y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{else.} \end{cases}$$

This is an equivalent definition of the discrete metric:

$$d(x, y) := (\operatorname{sgn}(d'(x, y)))^2,$$

where $\operatorname{sgn}(\cdot)$ is a sign function, and d' is any metric on X.

Example 1.1.3. [1] Denote $C[a, b]$ for the set of all continuous mapping $\mathbb{R}_{[a,b]} \to \mathbb{R}$. On $C[a, b]$, we can define a metric d as

$$d_p(f, g) := \left(\int_a^b |f(t) - g(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}, \quad p \geq 1.$$

[1] See Minkowski inequality.

In particular,

$$d_\infty(f,g) := \sup_{t\in\mathbb{R}_{[a,b]}} |f(t)-g(t)|.$$

Example 1.1.4. [2] Let $M=(X,d)$ be a metric space. The *Hausdorff metric* d_H on $2^X\setminus\{\emptyset\}$ is defined as

$$d_H := \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\},$$

where

$$d(x,Y) := \inf_{y\in Y}(x,y), \text{ and } d(y,X) := \inf_{x\in X}(y,x).$$

§1.2 Open sets in Metric Spaces

Definition 1.2.1. Let $M=(X,d)$ be a metric space, and let $U\subseteq X$. U is said to be *open in* M, iff for any $y\in U$, there exists $\varepsilon\in\mathbb{R}_{>0}$, such that $B_\varepsilon(y)\subseteq U$.

Lemma 1.2.1. Let $M=(X,d)$ be a metric space, let $x\in A$ and let $\varepsilon\in\mathbb{R}_{>0}$. For any $y\in B_\varepsilon(x)$, there is a $\delta\in\mathbb{R}_{>0}$ such that $B_\delta(y)\subseteq B_\varepsilon(x)$.

Proof. For any $y\in B_\varepsilon(x)$, by the definition of open balls (Definition 1.1.3), we have $d(x,y)<\varepsilon$.

Let $\delta\in\mathbb{R}_{>0}$ such that $\delta+d(x,y)=\varepsilon$.

By M3 in metric axioms (Definition 1.1.1), for any $z\in A$ with $d(y,z)<\delta$, we have

$$d(x,z)\le d(y,z)+d(x,y)<\varepsilon.$$

Thus, again, by the definition of open balls, we have $B_\delta(y)\subseteq B_\varepsilon(x)$.

□

Theorem 1.2.1. Let $M=(X,d)$ be a metric space, and let $U\subseteq X$. U is open in M iff it is a union of open balls.

Proof. First, prove $\Rightarrow$.

As U is open, for any $y\in U$, there exists $\varepsilon_y\in\mathbb{R}_{>0}$ such that $B_{\varepsilon_y}(y)\subseteq U$.

Therefore,

$$U=\bigcup_{y\in U} B_{\varepsilon_y}(y).$$

[2] See Hausdorff distance.

□

Now, prove $\Leftarrow$.

Aiming for a contradiction, suppose U is a union of open balls but not open.

As U is not open, there is a $y \in U$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \nsubseteq U$.

As U is a union of open balls, there is an $x \in U$ and $r \in \mathbb{R}_{>0}$ such that $y \in B_r(x)$.

By Lemma 1.2.1, there exists a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_r(x)$.

This is a contradiction by the assumption.

Thus, U has to be open.

■

Theorem 1.2.2. Let $M = (X, d)$ be any metric space. M is *Hausdorff*. That is, For any distinct points $x, y \in X$, we can always find an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset.$$

Proof. Aiming for a contradiction, suppose there are $x, y \in X$ with $x \neq y$, such that for any $\varepsilon \in \mathbb{R}_{>0}$, we can always find a $z \in X$ such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let $r = d(x, y)/2$, and let $z \in B_r(x) \cap B_r(y)$.

As $z \in B_r(x)$, by the definition of open balls (Definition 1.1.3), $d(x, z) < r$; as $z \in B_r(y)$, similarly, $d(y, z) < r$. Then we have

$$d(x, z) + d(y, z) < 2r = d(x, y).$$

This contradicts the metric axioms M3 (Definition 1.1.1).

■

Definition 1.2.2. Let $M = (X, d)$ be any metric space, and let $V \subseteq X$. V is said to be *closed* in M, iff there is an open set U satisfies $X \setminus U = V$.

Lemma 1.2.2. In a metric space, any singleton is closed.

Proof. Let $M = (X, d)$ be a metric space, let $x \in X$, and let $y \in X \setminus \{x\}$.

As M is Hausdorff (Theorem 1.2.2), there is an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$0 < \varepsilon < d(x, y),$$

thus $X \setminus \{x\}$ is open, hence, by Definition 1.1.1, its complement $\{x\}$ is open.

□

Theorem 1.2.3. Let $M = (X, d)$ be a metric space, denote $\mathcal{T}$ for the family of open subsets of X. Then $\mathcal{T}$ satisfies the following conditions:

O1. $X, \emptyset \in \mathcal{T}$;

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under arbitrary union;

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under finite intersection.

Proof.

O1. As $\emptyset$ is the subset of any set, $\emptyset \in \mathcal{T}$. $\bigcup \emptyset = \emptyset \in \mathcal{T}$.

By Definition 1.2.2, $X = X \setminus \emptyset$.

$\square$

O2. Let $\mathcal{U} \subseteq \mathcal{T}$, and denote $\mathcal{O}$ for the open balls in M.

For any $U \in \mathcal{U}$, there is an $\mathcal{O}_U \subseteq \mathcal{O}$ such that $U = \bigcup \mathcal{O}_U$.

Then we have

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \left(\bigcup \mathcal{O}_U \right) = \bigcup_{U \in \mathcal{U}} \mathcal{O}_U.$$

By Theorem 1.2.1, $\bigcup \mathcal{U}$ is open.

$\square$

O3. Let $\mathcal{V}$ be a finite subset of $\mathcal{T}$.

Aiming for a contradiction, suppose $\bigcap \mathcal{V}$ is not open.

By Definition 1.2.1, there exists a $y \in \bigcap \mathcal{V}$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \setminus \bigcap \mathcal{V} \neq \emptyset$.

By De Morgan's law, we have

$$\bigcup_{V \in \mathcal{V}} (B_\varepsilon(y) \setminus V) \neq \emptyset.$$

Thus, there exists $V \in \mathcal{V}$ such that $B_\varepsilon(y) \setminus V \neq \emptyset$.

As $V \in \mathcal{T}$ and ε is arbitrarily given, by Lemma 1.2.1, $y \notin V$. This is a contradiction.

Thus, $\bigcap \mathcal{V}$ is open.

$\square$

Thus, the theorem is proved.

■

Theorem 1.2.4. Infinite intersections of open sets in some metric spaces are not necessarily open.

Proof. Consider $\mathbb{R}$ is a Euclidean metric space, and denote $\mathcal{T}$.

Clearly, for any $n \in \mathbb{N}_{>0}$ and for any $x \in X$, the open interval $B_{\frac{1}{n}}(x)$ is open, but

$$\bigcap \left\{ B_{\frac{1}{n}}(x) : n \in \mathbb{N}_{>0} \right\} = \{x\}.$$

For any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(x) \setminus \{x\}$ is not empty, thus $\{x\}$ is not open.

■

§1.3 Restrictions and Metric Subspaces

Restriction of metric function is a useful tool to describe the relation between metric spaces with different sets but "same" metric function on the sets.

As a restriction of a relation R on $X \times Y$ to a subset $A \times B \subseteq X \times Y$ is defined to be

$$R \restriction_{A \times B} := R \cap (X \times Y),$$

a restriction of a metric d on a set S to a subset $U \subseteq S$ is defined to be

$$d \restriction_{(U \times U) \times \mathbb{R}_{>0}} := d \cap ((U \times U) \times \mathbb{R}_{>0}).$$

If $B = Y$, customarily, we simply write $R \restriction_A$ for $R \restriction_{A \times B}$. Similarly, as the codomain of a metric function is alway $\mathbb{R}_{>0}$, so we simply write $d \restriction_{U \times U}$ instead of $d \restriction_{(U \times U) \times \mathbb{R}_{>0}}$.

Definition 1.3.1. Let $M = (X, d)$ be a metric space, and let $A \subseteq X$. The *metric on A induced by d*, or the *subspace metric of d with respect to A* is defined to be

$$d_A := d \restriction_{A \times A} .$$

Theorem 1.3.1. Let $M = (X, d)$ be a metric space, and let $A \subseteq X$ and let $d_A := d \restriction_{A \times A}$. Then (A, d_A) is a metric space.

Proof. As metric axioms (Definition 1.1.1) holds for any $x, y \in X$, and $A \subseteq X$, they also holds for any $a, b \in A$. As d_A is the subspace metric of d with respect to A, d_A is a metric on A.

Thus, (A, d_A) is a metric space.

Definition 1.3.2. Let $M = (X, d)$ be a metric space, and let $A \subseteq X$. (A, d_A) is a *metric subspace* of M iff d_A is a subspace metric of d with respect to A.

Chapter 2.

Topological Spaces

§2.1 Basic Definitions

Definition 2.1.1. Let X be any set, and let $\mathcal{T} \subseteq 2^X$. We call $\mathcal{T}$ a *topology on* X iff it satisfies the *open set axioms.* That is,

O1. $X \in \mathcal{T}$;

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under arbitrary union.

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under finite intersection.

A subset $U \subseteq X$ is said to be *open in* M iff it is an element of $\mathcal{T}$.

Definition 2.1.2. Let X be any set, and let $\mathcal{T}$ be a structure on X. The pair $(X, \mathcal{T})$ is called a *topological space* iff $\mathcal{T}$ is a topology on X.

Theorem 2.1.1. Let $M = (X, \mathcal{T})$ be a topological space. Then $\emptyset \in \mathcal{T}$.

Proof. As empty set is an element of any set, it also an element of $\mathcal{T}$. Therefore, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

■

Definition 2.1.3. Let $M = (X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is said to be *closed in* M iff there exists a $U \in \mathcal{T}$ such that $A = X \setminus U$.

Theorem 2.1.2. Let $M = (X, \mathcal{T})$ be a topological space, and denote $\mathcal{C}$ for the family of all closed sets in M. Then $\mathcal{C}$ satisfies the following condition

C1. $X, \emptyset \in \mathcal{C}$;

C2. For any $\mathcal{A} \subseteq \mathcal{C}$, $\bigcap \mathcal{A} \in \mathcal{C}$;

C3. For any finite $\mathcal{B} \subseteq \mathcal{C}$, $\bigcup \mathcal{B} \in \mathcal{C}$.

Proof.

C1. As $\emptyset \in \mathcal{T}$ and $X = X \setminus \emptyset$, by Definition 2.1.3, X is closed.

Similarly, as $X \in \mathcal{T}$ and $\emptyset = X \setminus X$, $\emptyset$ is closed.

□

§2.2 Some Important Topologies

Definition 2.2.1. Let X be any set. A family $\mathcal{T} \subseteq 2^X$ is a *discrete topology on* X iff $\mathcal{T} = 2^X$.

Definition 2.2.2. Let X be any set. A family $\mathcal{T} \subseteq 2^X$ is an *indiscrete topology on* X iff $\mathcal{T} = \{X, \emptyset\}$.

Definition 2.2.3. Let $M = (X, d)$ be a metric space. A family $\mathcal{T} \subseteq 2^X$ is a *topology induced by* d iff $\mathcal{T}$ is the set of all open sets in M.

§2.3 Comparison of Topologies

Definition 2.3.1. Let X be any set and let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. We say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or $\mathcal{T}'$ is *finer* than $\mathcal{T}$, iff $\mathcal{T} \subseteq \mathcal{T}'$.

Note 2.3.1. By the definition of cardinality and inclusion mapping, if $\mathcal{T} \subseteq \mathcal{T}'$, it is certainly true that $|\mathcal{T}| \leq |\mathcal{T}'|$. But, on the contrary, $|\mathcal{T}| \leq |\mathcal{T}'|$ does not implies $\mathcal{T} \subseteq \mathcal{T}'$. It is easy to find counter-example about this.

Example 2.3.1. By Definition 2.3.1, for any set X, if a family $\mathcal{U}$ of open sets is given, then we can find the coarsest topology on X containing $\mathcal{U}$ by

$$\mathcal{T} = \left\{ \bigcup \mathcal{I}, \bigcap \mathcal{I}, X : \mathcal{I} \subseteq \mathcal{U} \right\}.$$

For example, let $X = \{1, 2, 3, 4, 5\}$, and let

$$\mathcal{U} = \{\{1, 2\}, \{2, 3\}, \{4\}\}.$$

Then a topology on X contains at least these sets:

$$\begin{gathered} \{1, 2, 3, 4\}, \{\}, \\ \{1, 2\}, \{2, 3\}, \{4\}, \\ \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \\ \{2\}. \end{gathered}$$

Example 2.3.2. The discrete topology is the finest topology on any X, while the indiscrete topology is the coarsest.