Notes for General Topology

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Contents

Chapter 1

Topological Spaces

1.1 Interiors

Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

Definition 1.1.1. The *interior* of A, denoted A°, is defined to be the union of all open subsets of A; i.e.,

$$A^\circ = \bigcup \mathcal{U}, \quad \mathcal{U} = \mathcal{P}(X) \cap \mathcal{T}.$$

Proposition 1.1.1. $A^\circ \subseteq A$.

Proof. Let $\mathcal{U} = \mathcal{P}(A) \cap \mathcal{T}$. Clearly

$$A^\circ = \bigcup \mathcal{U} \subseteq A.$$

$\square$

Proposition 1.1.2. $A \in \mathcal{T}$ if and only if $A = A^\circ$.

Proof. Let $\mathcal{U} = \mathcal{P}(A) \cap \mathcal{T}$.

$\mathcal{U}$ is closed under arbitrary union, so if $A = A^\circ = \bigcup \mathcal{U}$, then $A \in \mathcal{T}$.

On the other hand, suppose $A \in \mathcal{T}$ but $A \neq A^\circ$, then $A^\circ \subsetneq A$. As $A \in \mathcal{T}$, there exists $U \in \mathcal{P}(A) \cap \mathcal{T}$ with $U \ni x$. But as $x \in X \setminus A^\circ$, U could not be a subset of A°. Then we have $U \in \mathcal{T}$ but $U \not\subseteq \bigcup \mathcal{U}$, which is contradicted to the assumption. $\square$

Proposition 1.1.3. $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proof. $(A \cap B)^\circ \in \mathcal{T}$, so there exists $\mathcal{U} \subseteq \mathcal{T}$ such that

$$\bigcup \mathcal{U} = (A \cap B)^\circ.$$

Clearly, $\mathcal{U} = \mathcal{P}(A \cap B) \cap \mathcal{T}$.

Let $\mathcal{I} = \mathcal{P}(A) \cap \mathcal{T}$ and $\mathcal{J} = \mathcal{P}(B) \cap \mathcal{T}$, then $\mathcal{I} = A^\circ$ and $\mathcal{J} = B^\circ$.

$$\begin{aligned} \mathcal{I} \cap \mathcal{J} &= \mathcal{P}(A) \cap \mathcal{T} \cap \mathcal{P}(B) \cap \mathcal{T} \\ &= \mathcal{P}(A \cap B) \cap \mathcal{T} \\ &= \mathcal{U}. \end{aligned}$$

Then we have

$$(A \cap B)^\circ = \bigcup \mathcal{U} = \bigcup \mathcal{I} \cup \bigcup \mathcal{J} = A^\circ \cap B^\circ.$$

□

Proposition 1.1.4. If $A \subseteq B$, then $A^\circ \subseteq B^\circ$.

Proof. Let $\mathcal{U} = \mathcal{P}(A) \cap \mathcal{T}$, and let $\mathcal{V} = \mathcal{P}(B) \cap \mathcal{T}$

$A^\circ \subseteq A$, so $A \subseteq B$ implies $A^\circ \subseteq B$, so $\mathcal{U} \subseteq \mathcal{V}$. Thus,

$$A^\circ = \bigcup \mathcal{U} \subseteq \bigcup \mathcal{V} = B^\circ.$$

□

Note 1.1.1. $A^\circ \subseteq B^\circ$ dose not implies $A \subseteq B$. For example, let $X = \mathbb{R}^n$, let $\mathcal{T}$ be induce by Euclidean metric ρ on X, and let

$$\begin{aligned} &B = B_\rho(\vec{0}, 1), \text{ and} \\ &A = \{-\hat{e}_1 + (-\hat{e}_1 - \hat{e})t : t \in (0, 1]\}, \end{aligned}$$

where $\hat{e}_i$ denotes the i-th unit vector in X.

Clearly $A^\circ = \emptyset \subseteq B^\circ$, but $A \not\subseteq B$ for $\hat{e}_1 \in A \setminus B$.

Proposition 1.1.5. Let $\mathcal{T}$ be induced by a metric ρ on X. The interior of A° is the union of all open balls in A; i.e., there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$A^\circ = \bigcup_{x \in A} B(x, \varepsilon).$$

Proof. Let $\mathcal{U} = \mathcal{P}(A) \cap \mathcal{T}$, then any $U \in \mathcal{U}$,

$$U = \bigcup_{x \in U} B(x, \varepsilon).$$

For any open subsets $U \in \mathcal{U}$ and for any $x \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$A^\circ = \bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \bigcup_{x \in U} B(x, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon).$$

□

1.2 Closures

Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

Definition 1.2.1. The *closure* of A, denoted $\overline{A}$, is defined to be the intersection of all closed supersets of A.

Proposition 1.2.1. $X \setminus A^\circ = \overline{X \setminus A}$.

Proof. Let $\mathcal{U} = \mathcal{P}(A) \cap \mathcal{T}$, then $A^\circ = \bigcup \mathcal{U}$. By De Morgan's Law,

$$X \setminus A^\circ = \bigcap_{U \in \mathcal{U}} (X \setminus U).$$

As $\mathcal{U}$ is the family of all open subsets of A, the set of all $X \setminus U$ is the family of all closed superset of $X \setminus A$. Thus

$$\bigcap_{U \in \mathcal{U}} (X \setminus U) = \overline{X \setminus A}.$$

□

Proposition 1.2.2. $\overline{A}$ is closed.

Proof. As it is the intersection of some closed sets, $\overline{A}$ is closed. □

Proposition 1.2.3. $\overline{A}$ is closed if and only if $A = \overline{A}$.

Proof. Let $\mathcal{U} = \mathcal{P}(X \setminus A) \cap \mathcal{T}$, then we have

$$\begin{aligned} X \setminus \bigcup \mathcal{U} &= X \setminus (X \setminus A)^\circ \\ &= \overline{X \setminus (X \setminus A)} \\ &= \overline{A}. \end{aligned}$$

□

Proposition 1.2.4. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof.

$$\begin{aligned} A \subseteq B &\iff (X \setminus A) \supseteq (X \setminus B) \\ &\implies (X \setminus A)^\circ \supseteq (X \setminus B)^\circ \\ &\iff X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ \\ &\iff \overline{X \setminus (X \setminus A)} \subseteq \overline{X \setminus (X \setminus B)} \\ &\iff \overline{A} \subseteq \overline{B}. \end{aligned}$$

□

Proposition 1.2.5. If A is closed, then $A \supseteq B$ if and only if $A \supseteq \overline{B}$.

Proof. A is closed, so $X \setminus A$ is open. $A \supseteq B$ if and only if $X \setminus A \subseteq X \setminus B$. These conditions hold if and only if $X \setminus A \subseteq (X \setminus B)^\circ$. This holds if and only if

$$\begin{aligned} X \setminus A \subseteq (X \setminus B)^\circ &\iff X \setminus (X \setminus A) \supseteq X \setminus (X \setminus B)^\circ \\ &\iff A \supseteq \overline{B}. \end{aligned}$$

□

1.3 Boundary Sets

Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

Definition 1.3.1. The *boundary* of A, denoted ∂A, is defined to be the complement of the interior of A in the closure of A; i.e.,

$$\partial A = \overline{A} \setminus A^\circ.$$

Note 1.3.1. In Euclidean n-space, the volume of the boundary of any set must zero. For the topological spaces with countable elements, this property still holds, for the volume of any set in such spaces is zero. But for the topological spaces with uncountable elements, it is not necessarily the case.

For example, let $(\mathbb{R}^n, \mathcal{T})$ be a topological space where $\mathcal{T}$ is generated by $\{B_\rho(\vec{0}, 1)\}$ where ρ is the Euclidean metric. In this case, $\overline{B_\rho(\vec{0}, 1)} = \mathbb{R}^n$, and the volume of its boundary is infinite.

Proposition 1.3.1. ∂A is closed.

Proof. $\overline{A}$ is closed, and so is $X \setminus A^\circ$. Then we have

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (X \setminus A^\circ).$$

□

Proposition 1.3.2. $A^\circ \cap \partial A = \emptyset$.

Proof.

$$\partial A = \overline{A} \setminus A^\circ \iff A^\circ \cap \partial A = \overline{A} \setminus A^\circ \cap A^\circ = \overline{A} \cap \emptyset = \emptyset.$$

□

Proposition 1.3.3. $\overline{A} = A^\circ \cup \partial A$.

Proof.

$$\begin{aligned} \partial A = \overline{A} \setminus A^\circ &\iff \partial A \cup A^\circ = \overline{A} \setminus A^\circ \cup A^\circ = \overline{A} \cap (X \setminus A^\circ \cup A^\circ) \\ &\iff \partial A \cup A^\circ = \overline{A} \cap X = \overline{A}. \end{aligned}$$

□

Proposition 1.3.4. A is closed if and only if $\partial A \subseteq A$.

Proof. A is closed if and only if $A = \overline{A} = A^\circ \cap \partial A$. □

1.4 Limit Points and Isolated Points

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Also, let $u : I \to X$ with $I \subseteq \mathbb{N}$ and $|I| = \aleph_0$.

Definition 1.4.1. A point $x \in X$ is a *limit point* (or *cluster point* or *accumulation point*) of A, denoted $L(A)$, if and only if for any neighbourhood N of x, $N \setminus \{x\} \cap A \neq \emptyset$.

Definition 1.4.2. A point $x \in X$ is called a *limit* of u if and only if for any neighbourhood N of x,

$$|u[I] \setminus N| \in \mathbb{N}.$$

In words, any such N contains all but finite element of $u[I]$.

In this case, u is said to be convergent in X, and we say that x converges to x.

Note 1.4.1. In this definition, x is a limit point of $u[I]$ only if u is not constant. If u is constant, then $u[I]$ contains only one element, x. As $u[I] = \{x\}$, any set $(N \setminus \{x\}) \cap \{x\} = \emptyset$.

Note 1.4.2. Being a limit point of the image of a sequence does not means that the sequence converges itself to this point. For example, let u defined by

$$u_n = \sin(n),$$

and let $\mathcal{T}$ be an Euclidean topology on X. Any $x \in \mathbb{R}_{[0,1]}$ is a limit point of $u[I]$. But, clearly, u is not convergent in X.

Proposition 1.4.1. If $(X, \mathcal{T})$ is Hausdorff, then u converges to a unique point $x \in X$.

Proof. Suppose u converges to $x, y \in X$ with $x \neq y$, then for any neighbourhoods N_x of x and N_y of y, N_x contains a cofinite subset $A \subseteq u[I]$ and N_y contains a cofinite subset $B \subseteq u[I]$. If this were true, $N_x \cap N_y$ should be non-empty, otherwise N_x or N_y should be finite. But it is impossible as $(X, \mathcal{T})$ is a Hausdorff space. $\square$

Note 1.4.3. As all metric space are Hausdorff, so any convergent sequence in a metrizable space converges to at most one point.

Definition 1.4.3. A point $x \in A$ is an *isolated point* of A, denoted $I(A)$, if and only if there exists neighbourhood N of x such that $N \setminus \{x\} \cap A = \emptyset$.

Example 1.4.1. If $\mathcal{T}$ is a discrete topology on X, then any $x \in A$ is isolated. X is discrete if and only if for any $x \in X$, there exists neighbourhood N of x, such that $N \setminus \{x\} = \emptyset$, which means X contains only isolated points. It is easy to show that $A = I(A)$ also.

Proposition 1.4.2. The closure of A is a disjoint union of its limit points and isolated points; i.e.,

$$\overline{A} = L(A) \sqcup I(A).$$

Proof. $x \in \overline{A}$ if and only if for any neighbourhood N of x, $N \cap A \neq \emptyset$.

If $x \in \overline{A} \setminus I(A)$, then, for any neighbourhood N of x, $N \setminus \{x\} \cap A \neq \emptyset$. This is precisely the definition of limit points, so $x \in L(A)$.

On the other hand, if $x \in \overline{A} \setminus L(A)$, then there exists some neighbourhood N of x, such that $N \setminus \{x\} \cap A = \emptyset$. This is precisely the definition of isolated points, so $x \in I(A)$.

Above all, any elements in $\overline{A}$ is either an element of $I(A)$ or an element of $L(A)$, but not both. So $\overline{A} = L(A) \sqcup I(A)$. $\square$

Proposition 1.4.3. *A is closed if and only if it contains all its limit points.*

Proof. A is closed if and only if $A = \overline{A}$. As we have proved, in this case, $\overline{A} = L(A) \sqcup I(A)$. So, $L(A) \subseteq \overline{A}$.

On the other hand, let $x \in A \setminus L(A)$, then there exists neighbourhood N of x, such that $N \setminus \{x\} \cap A = \emptyset$. Thus such x must be an isolated points of A. If $A = L(A) \sqcup I(A)$, then $\overline{A} = A$, in which case A is closed. Then we have

$$\begin{aligned} X = L(X) \sqcup I(X) &\iff X = L(X) \sqcup X \\ &\iff L(X) = \emptyset. \end{aligned}$$

It is easy to prove that $L(A) \subseteq L(X)$. Thus $L(A) = \emptyset$. $\square$

1.5 Dense Sets

Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

Definition 1.5.1. A is *dense* in B if and only if for any $x \in B$, $x \in A$ or x is a limit point of A.

Example 1.5.1. Let $X = \mathbb{R}^n$, let $\mathcal{T}$ be induced by Euclidean metric ρ on X, let

$$A = B(\vec{0}, 1) \sqcup \{\vec{x} \in \mathbb{R}^n \setminus \mathbb{Q}^n\} \setminus \mathbb{Q}^n,$$

and let

$$B = A \cup \mathbb{Q}^n.$$

A is dense in B, although $x \notin L(A)$ and $x \in B$.

Example 1.5.2. For any topological space $(X, \mathcal{T})$, X is dense in itself, even if $X = \emptyset$ (vacuously true).

Proposition 1.5.1. *A is dense in B if and only if $B \subseteq \overline{A}$.*

Proof. A is dense in B, so for any $x \in B$, $x \in L(A)$. Now, we have $B \subseteq L(A)$. Thus, $B \subseteq \overline{A}$.

On the other hand, let $B \subseteq \overline{A}$. $\overline{A} = L(A) \sqcup I(A)$, so any $x \in B$ is an element of A or a limit point of A. Thus A is dense in B. $\square$

Note 1.5.1. Naturally, A is dense in X if and only if $\overline{A} = X$. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 1.5.2. A is said to be *nowhere dense* in X if and only if for any $U \in \mathcal{T}$, A is not dense in U.

Proposition 1.5.2. A is said to be *nowhere dense* in X if and only if the interior of its closure is empty; i.e.,

$$(\overline{A})^\circ = \emptyset.$$

Proof. Suppose A is nowhere dense, but $(\overline{A})^\circ \neq \emptyset$. There exists $U \in \mathcal{T}$ with $U \subseteq \overline{A}$. As $U \in \mathcal{T}$, any $x \in \overline{A}$ is also an element of $\overline{A}$; and as $\overline{A} = L(A) \sqcup I(A)$, such an x is either an element of A or a limit point of A. So A is dense in U, which is contradicted to the condition that A is nowhere dense.

On the other hand, $(\overline{A})^\circ = \emptyset$ implies that $\emptyset$ is the only open subset of $\overline{A}$. Thus, for any $U \subseteq \mathcal{T}$, $U \setminus \overline{A} \neq \emptyset$. As $U \setminus \overline{A}$ is also open, for any $x \in U \setminus \overline{A}$, there exists neighbourhood N of x, $N \cap \overline{A} = \emptyset$, in this case, naturally, $N \cap A = \emptyset$. Thus A is not dense in U. $\square$

Example 1.5.3. Let $f : (\mathbb{R}^n, \rho) \to (\mathbb{R}^m, \rho)$ be a function continuous over A, where ρ is the Euclidean metric, and $n < m$. $f[A]$ is nowhere dense in $\mathbb{R}^m$.

Proposition 1.5.3. A is nowhere dense if and only if $\mathcal{P}(\overline{A}) \cap \mathcal{T} = \emptyset$.

Proof. A is nowhere dense if and only if $(\overline{A})^\circ = \emptyset$. Naturally, there exists no non-empty subset in $\overline{A}$.

On the other hand, suppose there exists non-empty $U \in \mathcal{T}$ with $\overline{A} \supseteq U$, then we have

$$U \subseteq \overline{A}\big|_{U \in \mathcal{T}} \implies U \subseteq (\overline{A})^\circ\big|_{U \in \mathcal{T}}$$

But U is not empty and, as U is open, for any $x \in U$, there exists neighbourhood N of x such that $N \cap \overline{A} \neq \emptyset$. Clearly, such x can not be isolated in A, so $x \in L(A)$. This implies A is dense in U, which is contradicted to the condition. $\square$

1.6 Continuous Maps

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let f be any map from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$.

Definition 1.6.1. f is said to be *continuous on* $U_x \in \mathcal{T}_X \cap \mathcal{P}(X)$ if and only if for any $U_y \in \mathcal{T}_Y$, the preimage $f^{-1}[U_y] \cap U_x \in \mathcal{T}_X$.

Definition 1.6.2. f is said to be *continuous* if and only if for any $U \in \mathcal{T}_Y$, the preimage $f^{-1}[U] \in \mathcal{T}_X$.

Example 1.6.1. Even if for any $U \in \mathcal{T}_X$ such that $f[X] \in \mathcal{T}_Y$, f is not necessarily continuous. For example, let $X = \mathbb{R}$ and let $\mathcal{T}_X$ be induced by Euclidean metric, let $Y = \mathbb{R}$ and $\mathcal{T}_Y = \mathcal{P}(Y)$, and let f defined by

$$f(x) = x.$$

For any $U \subseteq X$ (even unnecessary to be open), $f[U] \in \mathcal{T}$. But f is not continuous, for there exists $U \in \mathcal{T}_Y$ such that $f^{-1}[U] \notin \mathcal{T}_X$; for example, for any $U \subseteq \mathbb{Q}$, or for any $\{y\} \in \mathcal{T}_Y$, etc.

This example also shows that being bijective dose not implies continuity.

Example 1.6.2. If $\mathcal{T}_X = \mathcal{P}(X)$, then any map from $\mathcal{T}_X$ to $\mathcal{T}_Y$ is continuous.

Definition 1.6.3. f is said to be *continuous at a point* $x \in X$ if and only if there is an open neighbourhood U_x of x such that for any open neighbourhood U_y of y, the preimage $f^{-1}[U_y] \cap U_x \in \mathcal{T}_X$.

Proposition 1.6.1. f is continuous at $x \in X$ if and only if for any neighbourhood N_y of $f(x)$, there exists neighbourhood N_x of x such that $f(N_x) \subseteq N_y$.

Proof. Let U_y be an open neighbourhood of $f(x)$.

If f is continuous at x, then there must be an open neighbourhood U_x of x such that $f^{-1}[U_y] \cap U_x \in \mathcal{T}_X$. $U_y \ni f(x)$, so $f^{-1}[U_y] \cap U_x \ni x$. As f is continuous, $U_y \in \mathcal{T}_Y$ implies $f^{-1}[U_y] \in \mathcal{T}$, then, as $U_x \in \mathcal{T}$, $f^{-1}[U_y] \cap U_x$ is an open neighbourhood of x. And we have

$$f[f^{-1}[U_y] \cap U_x] \subseteq U_y.$$

On the other hand, if there is an open neighbourhood N_x of x such that $f[N_x] \subseteq U_y$, then we have

$$\begin{aligned} f^{-1}[f[N_x]] \subseteq f^{-1}[U_y] &\implies N_x \subseteq f^{-1}[U_y] \\ &\implies N_x = f^{-1}[U_y] \cap N_x. \end{aligned}$$

As $N_x \in \mathcal{T}$, it is also true that $f^{-1}[U_y] \cap N_x \in \mathcal{T}$. Thus f is continuous on x. $\square$

1.7 Homeomorphisms

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let f be a map from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$.

Definition 1.7.1. f is called a *homeomorphism* between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ if and only if

(i) f is bijective;

(ii) f is a continuous map;

(iii) f^{-1} is a continuous map.

Note 1.7.1. A function should satisfy all of the three properties to be a homeomorphism. For example, if f is not one-to-one, then f^{-1} is not a map; if f is not surjective, then f^{-1} is not defined on whole Y; let $\mathcal{T}_X \supsetneq \mathcal{T}_Y$, and let f be defined by $f(x) = x$, then f is bijective and continuous, but f^{-1} is bijective but not continuous.

Definition 1.7.2. $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, if and only if there is an homeomorphism between them.

Proposition 1.7.1. Being homeomorphic is an equivalent relation.

Proof. Being homeomorphic is a reflexive relation, for any topological space are homeomorphic to itself.

Being homeomorphic is a symmetric relation. Let f be a homeomorphism between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. f is bijective if and only if f^{-1} is.

Being homeomorphic is a transitive relation. Let $(Z, \mathcal{T}_Z)$ be another topological space. If f is a homeomorphism between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, and g is a homeomorphism between $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$, then it is easy to show that $f \circ g$ is a homeomorphism between $(X, \mathcal{T}_X)$ and $(Z, \mathcal{T}_Z)$ by the properties of bijections and continuous maps. $\square$

Proposition 1.7.2. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic, then X and Y have the same cardinality.

Proof. Suppose they are homeomorphic but with different cardinality, say $|X| < |Y|$, then there is no surjection from $|X|$ to $|Y|$. So this is impossible. $\square$

Note 1.7.2. It is a simple and beautiful fact that $|X| = |Y|$ does not imply that they are homeomorphic.

For example, let $X = \mathbb{R}_{[0,1]}$ and $Y = \mathbb{R}_{[0,1)}$, and let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be Euclidean topologies on X and Y respectively. Clearly, $|X| = |Y| = \mathfrak{c}$, but we can find no homeomorphism these two space. Intuitively, there is not any continuous way to deform from one of them to the other.

Suppose there exists a homeomorphism $f : (X, \mathcal{T}_Y) \to (Y, \mathcal{T}_Y)$. f should be continuous everywhere, so for $0, 1 \in X$, there should be neighbourhoods N_0 of 0 and N_1 of 1 such that $f[N_0] \subseteq N_{f(0)}$ and $f[N_1] \subseteq N_{f(1)}$ for any neighbourhoods $N_{f(0)}$ of $f(0)$ and $N_{f(1)}$ of $f(1)$. In this case, f can not be bijective, for if f is bijective, there are only to possible case:

$$f(0) = \min Y \wedge f(1) = \max Y,$$
$$f(0) = \max Y \wedge f(1) = \min Y.$$

Both of the case are impossible, for $\max Y$ does not exist. Thus $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are not homeomorphic.

Note 1.7.3. In the examples below, all topological spaces are Hausdorff.

Example 1.7.1. $\mathbb{R}^n$ and $\mathbb{R}^m$ $(n \neq m)$ are not homeomorphic, although $|\mathbb{R}^n| = |\mathbb{R}^m| = \mathfrak{c}$.

Example 1.7.2. Any proper open intervals in $\mathbb{R}^n$ are homeomorphic. But for any interval $I \in \mathbb{R}^n$, if $\max I$ or $\min I$ exists, then I is not homeomorphic to $\mathbb{R}^2$.

Example 1.7.3. Let S^n be an n-dimensional sphere with center $\vec{o} \in \mathbb{R}^{n+1}$ and radius $r \in \mathbb{R}_{>0}$, i.e.,

$$S^n = \left\{ \vec{x} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} |x_i - o_i|^2 = r^2 \right\}.$$

Let

$$\varepsilon \in \left(0, \max_{\vec{x}, \vec{y} \in S^n} |\vec{x} - \vec{y}| \right),$$

and, for any $\vec{x} \in S^n$, let

$$U = \left\{ \vec{u} \in \mathbb{R}^{n+1} : 0 \leq \sum_{i=1}^{n} |u_i - x_i|^2 \leq \varepsilon \right\}.$$

Then, $S^n \setminus U \cong \mathbb{R}^n$.

Example 1.7.4. In $\mathbb{R}^3$ a piecewise smooth coffee mug and a smooth donut are homeomorphic, but they are not homeomorphic to any closed ball, for a ball does not have a hole.

Example 1.7.5. In $\mathbb{R}^3$, a 1-knot is homeomorphic to any half-closed or half-open interval in $\mathbb{R}$.

Example 1.7.6. The graph of

$$f = \left\{ \left(x, \sin \frac{1}{x} \right) \in \mathbb{R}^2 : x \in \mathbb{R}_{>0} \right\}$$

is not homeomorphic to any open intervals in $\mathbb{R}$.

Chapter 2

Creating New Spaces

2.1 Subspace

Let $(X, \mathcal{T}_X)$ and $(S, \mathcal{T}_S)$ be topological spaces.

Definition 2.1.1. $\mathcal{T}_S$ is a *subspace topology* on S if and only if for any $U_S \subseteq S$, there is a $U_X \in \mathcal{T}_X$, such that $S \cap U_X = U_S$; i.e.,

$$\mathcal{T}_S = \{S \cap U_X : U_X \in \mathcal{T}_X\}.$$

Note 2.1.1. "$S \subseteq X$" would be a redundant condition in the definition. For if $U_S = U_X \cap S$, then of course $U_S \subseteq U_X$. As for any $U_S \in \mathcal{T}_S$ including the case $U_S = S$, there exists $U_X \in \mathcal{T}_X$, such that $U_S \subseteq U_X$, then of course we have $S \subseteq X$.

Definition 2.1.2. $(S, \mathcal{T}_S)$ is an *open subspace* of $(X, \mathcal{T}_X)$ if and only if $(S, \mathcal{T}_S)$ is a subspace of $(X, \mathcal{T}_X)$ and $\mathcal{T}_S \subseteq \mathcal{T}_X$.

Below, let $(X, \mathcal{T}_X)$ and $(S, \mathcal{T}_S)$ be topological spaces, where $(S, \mathcal{T}_S)$ is a subspace of $(X, \mathcal{T}_X)$.

Proposition 2.1.1. Let ι be the inclusion map from S to X. For any topological space $(Z, \mathcal{T}_Z)$, a map $f : Z \to S$ is continuous if and only if $\iota \circ f$ is.

Proof. For any $U \subseteq X$, $\iota^{-1}[U] = U \cap S \in \mathcal{T}_S$, so ι is continuous.

If f is continuous, then for any $U_S \in \mathcal{T}_S$, $f^{-1}[U_S] \in \mathcal{T}_Z$. As ι is also continuous, for any $U_X \in \mathcal{T}_X$, $U_X \in \mathcal{T}_S$, hence $U_X \in \mathcal{T}_Z$. Thus $\iota \circ f$ is continuous.

On the other hand, if $\iota \circ f$ is continuous, then for any $U \in \mathcal{T}_X$, $(\iota \circ f)^{-1}[U] \in \mathcal{T}_Z$. Suppose f is not continuous, then there is an $U_S \in \mathcal{T}_S$ such that $f^{-1}[U_S] \notin \mathcal{T}_Z$.

For such U_S, there is an $U_X \in \mathcal{T}_X$, such that $U_X \cap S = U_S$. As ι is the inclusion map, that means for $(\iota \circ f)^{-1}[U_X] \notin \mathcal{T}_Z$. This implies $\iota \circ f$ is not continuous, which is contradicted to the condition. $\square$

Proposition 2.1.2. For any closed subset V_S of S, there is a closed subset V_X of X such that $V_S = V_X \cap S$.

Proof. As V_S is closed in S, there exists $U_S \in \mathcal{T}_S$ such that $V_X = S \setminus U_S$. And, there exists $U_X \in \mathcal{T}_X$ such that $U_S = U_X \cap S$, then we have

$$V_S = S \setminus U_S = (X \setminus U_X) \cap S,$$

where $X \setminus U_X$ is closed in X. $\square$

Proposition 2.1.3. For any subspace $(Z, \mathcal{T}_Z)$ of $(S, \mathcal{T}_S)$, $(Z, \mathcal{T}_Z)$ is also a subspace of $(X, \mathcal{T}_X)$.

Proof. As $(Z, \mathcal{T}_Z)$ is a subspace of $(S, \mathcal{T}_S)$, for any $U_Z \in \mathcal{T}_Z$, there is a $U_S \in \mathcal{T}_S$, such that $U_Z = U_S \cap Z$.

As $(S, \mathcal{T}_S)$ os a subspace of $(X, \mathcal{T}_X)$, for any $U_S \in \mathcal{T}_S$, there is a $U_X \in \mathcal{T}_X$ such that $U_S = U_X \cap S$.

Then we have

$$U_Z = (U_X \cap S) \cap Z = U_X \cap (S \cap Z) = U_X \cap Z.$$

Thus $(Z, \mathcal{T}_Z)$ is a subspace of $(X, \mathcal{T}_X)$. $\square$

Proposition 2.1.4. If $(Z, \mathcal{T}_Z)$ is an open subspace of $(S, \mathcal{T}_S)$ and $(S, \mathcal{T}_S)$ is an open subspace of $(X, \mathcal{T}_X)$, then $(Z, \mathcal{T}_Z)$ is an open subspace of $(X, \mathcal{T}_X)$.

Proof. Clearly, this relation implies that $(Z, \mathcal{T}_Z)$ is a subspace of $(X, \mathcal{T}_X)$.

If $(Z, \mathcal{T}_Z)$ is an open subspace of $(S, \mathcal{T}_S)$ then $\mathcal{T}_Z \subseteq \mathcal{T}_S$. If $(S, \mathcal{T}_S)$ is an open subspace of $(X, \mathcal{T}_X)$, then $\mathcal{T}_S \subseteq \mathcal{T}_X$. Then we have $\mathcal{T}_Z \subseteq \mathcal{T}_X$.

$(Z, \mathcal{T}_Z)$ is a subspace of $(X, \mathcal{T}_X)$ and $\mathcal{T}_Z \subseteq \mathcal{T}_X$, thus $(Z, \mathcal{T}_Z)$ is an open subspace of $(X, \mathcal{T}_X)$. $\square$

Proposition 2.1.5. If $\mathcal{B}_X$ is a base for $(X, \mathcal{T}_X)$, then

$$\mathcal{B}_S = \{B_X \cap S : B_X \in \mathcal{B}_X\}$$

is a base for $(S, \mathcal{T}_S)$.

Proof. For any $B_X \in \mathcal{B}_X$, $B_X \in \mathcal{T}_X$; and as $(S, \mathcal{T}_S)$ is a subspace of $(X, \mathcal{T}_X)$, $B_X \cap S \in \mathcal{T}_S$. The collection of all such $B_X \cap S$ is

$$\mathcal{B}_S = \{B_X \cap S : B_X \in \mathcal{B}_X\}.$$

Now, we need to prove $\mathcal{B}_S$ is a base of S.

As $\mathcal{B}_X$ is a base of X, $\bigcup \mathcal{B}_X = X$. Then we have

$$S = X \cap S = \bigcup \mathcal{B}_X \cap S = \bigcup \mathcal{B}_S.$$

Thus $\mathcal{B}_S$ is a cover of S.

For any B_X and B'_X in $\mathcal{B}_X$, there exists $\mathcal{I}_X \subseteq \mathcal{B}_X$, $B_X \cap B'_X = \bigcup \mathcal{I}_X$. Then we have

$$(B_X \cap S) \cap (B'_X \cap S) = \bigcup \mathcal{I}_X \cap S = \bigcup \mathcal{I}_S,$$

where $\mathcal{I}_S = \{I_X \cap S : I_X \in \mathcal{I}_X\}$. As any such I_X is an element of $\mathcal{B}_X$, $I_X \cap S$ is an element of $\mathcal{B}_S$, thus $\mathcal{I}_S \subseteq \mathcal{B}_S$.

Above all, $\mathcal{B}_S$ is a base of S. $\square$

2.2 Quotient Space

Definition 2.2.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let f be a continuous map from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$. f is called a *quotient map* if and only if it is surjective and $U \in \mathcal{T}_Y$ if and only if $f^{-1}[U] \in \mathcal{T}_X$.

Example 2.2.1. Let $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}_{[0,1)}, \mathcal{T}')$ be topological spaces where $\mathcal{T}$ and $\mathcal{T}'$ are Euclidean topologies on $\mathbb{R}$ and $\mathbb{R}_{[0,1)}$ respectively. let $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}_{[0,1)}, \mathcal{T}')$ be defined by

$$f(x) = x - [x],$$

where $[x]$ denotes the integer part of x. f is a quotient map from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}_{[0,1)}, \mathcal{T}')$.

Definition 2.2.2. $\mathcal{T}_{X/\sim}$ is a *quotient topology* on $X/\sim$ if and only if