

# Notes for Vector Calculus

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November 17, 2021

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# Chapter 1.

## *Differentiation*

### §1.1 Differentiable Mapping

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**Observation 1.1.1.** Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , and denote  $f_i$  for the  $i$ -th factor of  $f$ , i.e.,  $f = \langle f_i \rangle_i^n$ . Assume  $f_i$  is smooth at a point  $\mathbf{p} \in \mathbb{R}^m$ .

Intuitively,  $f_i$  is smooth at  $\mathbf{p}$  iff there exists a neighbourhood  $N$  of  $\mathbf{p}$  and a plane described by  $P_i : \mathbb{R}^m \rightarrow \mathbb{R}$ , such that

$$f_i[N] \approx P_i[N].$$

As  $\mathbb{R}^m$  is considered as a metric space, any open ball  $B(\mathbf{p}, \delta) \subseteq N$  is also required neighbourhood of  $\mathbf{p}$ . In this sense, the approximation can be considered as,

$$\lim_{\delta \rightarrow 0} f_i[B(\mathbf{p}, \delta)] = \lim_{\delta \rightarrow 0} P_i[B(\mathbf{p}, \delta)].$$

In the term of elements, that is, there exists  $\delta \in \mathbb{R}_{>0}$ , such that for any  $\mathbf{t} \in \mathbb{R}^m$  with  $\mathbf{p} + \mathbf{t} \in B(\mathbf{p}, \delta)$ ,

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} f_i(\mathbf{p} + \mathbf{t}) = \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} P_i(\mathbf{p} + \mathbf{t}).$$

As  $P_i$  describes a plane, it can be considered as a translated linear mapping, and as this plane must be a tangent plane of  $f_i$  at  $\mathbf{p}$ , there exists a linear mapping  $\phi_i : \mathbb{R}^m \rightarrow \mathbb{R}$  such that

$$P_i(\mathbf{p} + \mathbf{t}) = \phi_i(\mathbf{p} + \mathbf{t} - \mathbf{p}) + f_i(\mathbf{p}).$$

Thus, we have

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} f_i(\mathbf{p} + \mathbf{t}) = \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \phi_i(\mathbf{t}) + f_i(\mathbf{p}).$$

Rearrange the equation, we have

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{f_i(\mathbf{p} + \mathbf{t}) - f_i(\mathbf{p})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{\phi_i(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}}.$$

As  $\phi_i$  is linear, the right hand side of the equation is a constant in  $\mathbb{R}$ . Thus, by rearrange the equation again, we have

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{f_i(\mathbf{p} + \mathbf{t}) - f_i(\mathbf{p}) - \phi_i(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = 0$$

The last equation describes the smoothness of  $f_i$  at  $\mathbf{p}$ , in calculus,  $f_i$  is said to be differentiable at  $\mathbf{p}$ .

Now, assume for any  $i \in \{1, \dots, n\}$ ,  $f_i$  is differentiable at  $\mathbf{p}$ . That is, for any  $f_i$ , there exists a  $\phi_i : \mathbb{R}^m \rightarrow \mathbb{R}$ , such that

$$\left\langle \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{f_i(\mathbf{p} + \mathbf{t}) - f_i(\mathbf{p}) - \phi_i(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} \right\rangle_{i=1}^n = \langle 0 \rangle_{i=1}^n.$$

By vector sum, we have

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{f(\mathbf{p} + \mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \mathbf{0}_{\mathbb{R}^n},$$

where  $\phi = \langle \phi_i \rangle_{i=1}^n$ .

In this sense, Definition 1.1.1 is introduced as following.

**Definition 1.1.1** (Differentiable Mappings).

Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ .

$f$  is said to be *differentiable* at a point  $\mathbf{p} \in \mathbb{R}^m$  iff for any  $\mathbf{u} \in \mathbb{R}^m \setminus \{\mathbf{0}_{\mathbb{R}^m}\}$ , there exists a linear mapping  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{f(\mathbf{p} + \mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \mathbf{0}_{\mathbb{R}^n}.$$

**Theorem 1.1.1.** In Definition 1.1.1,  $\phi$  is unique.

*Proof.* As Definition holds for  $\phi$ , there exists an  $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$  with

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \alpha(\mathbf{t}) = \alpha(\mathbf{0}_{\mathbb{R}^m}) = \mathbf{0}_{\mathbb{R}^n},$$

and a neighbourhood  $N$  of  $\mathbf{p}$  such that for any  $\mathbf{t} \in \mathbb{R}^m$  with  $\mathbf{p} + \mathbf{t} \in N$ ,

$$\frac{f(\mathbf{p} + \mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \alpha(\mathbf{t}).$$

Suppose Definition 1.1.1 also holds for another  $\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , then there exists an  $\beta : \mathbb{R}^m \rightarrow \mathbb{R}^n$  with

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \beta(\mathbf{t}) = \beta(\mathbf{0}_{\mathbb{R}^m}) = \mathbf{0}_{\mathbb{R}^n},$$

and a neighbourhood  $N'$  of  $\mathbf{p}$  such that for any  $\mathbf{t} \in \mathbb{R}^m$  with  $\mathbf{p} + \mathbf{t} \in N'$ ,

$$\frac{f(\mathbf{p} + \mathbf{t}) - f(\mathbf{p}) - \lambda(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \beta(\mathbf{t}).$$

Let  $\gamma = \phi - \lambda$ . As  $\phi$  and  $-\lambda$  are both linear,  $\gamma$  is also linear. Then, we have

$$\begin{aligned} \frac{\gamma(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} &= \alpha(\mathbf{t}) - \beta(\mathbf{t}) \\ \iff \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \gamma(\hat{\mathbf{t}}) &= \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} (\alpha(\mathbf{t}) - \beta(\mathbf{t})) \\ \iff \gamma(\hat{\mathbf{t}}) &= \mathbf{0}_{\mathbb{R}^n}. \end{aligned}$$

As  $\mathbf{t}$  is arbitrarily picked from  $U \cap U'$ , and  $U \cap U'$  is open in  $\mathbb{R}^m$  as  $U$  and  $U'$  are open, the set  $\{\hat{\mathbf{t}} : \mathbf{t} \in U \cap U' - \mathbf{p}\}$  gives all possible directions in  $\mathbb{R}^m$ . And, as  $\gamma(s\hat{\mathbf{t}}) = \mathbf{0}_{\mathbb{R}^n}$  for any  $\mathbf{t} \in \mathbb{R}^m$  and any  $s \in \mathbb{R}$ ,  $\gamma[\mathbb{R}^m] = \{\mathbf{0}_{\mathbb{R}^n}\}$ . Thus,  $\phi = \lambda$ . ■

**Theorem 1.1.2.** With the condition in Definition 1.1.1, if  $f$  is differentiable at  $\mathbf{p}$ , then  $f$  is continuous at  $\mathbf{p}$ .

*Proof.* As  $f$  is differentiable at  $\mathbf{p}$ , there exists an  $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$  with

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \alpha(\mathbf{t}) = \alpha(\mathbf{0}_{\mathbb{R}^m}) = \mathbf{0}_{\mathbb{R}^n},$$

such that

$$\frac{f(\mathbf{p} + \mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \alpha(\mathbf{t}).$$

By rearranging the equation, we observe

$$\begin{aligned} \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} [f(\mathbf{p} + \mathbf{t}) - \phi(\mathbf{t})] &= \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} [\|\mathbf{t}\|_{\mathbb{R}^m} \alpha(\mathbf{t}) + f(\mathbf{p})] \\ \iff \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} f(\mathbf{p} + \mathbf{t}) &= f(\mathbf{p}). \end{aligned}$$

Thus,  $f$  is continuous at  $\mathbf{p}$ . ■

**Theorem 1.1.3.** With the condition in Definition 1.1.1, let  $g : \mathbb{R}^n \rightarrow \mathbb{R}^k$ .

If  $f$  is differentiable at  $\mathbf{p}$  and  $g$  is differentiable at  $f(\mathbf{p})$ , then  $g \circ f$  is differentiable at  $\mathbf{p}$ .

*Proof.* As  $f$  is differentiable at  $\mathbf{p}$ , there exists a linear mapping  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$  and a neighbourhood  $N$  of  $\mathbf{p}$  such that for any  $\mathbf{t} \in \mathbb{R}^m$  with  $\mathbf{p} + \mathbf{t} \in \mathbb{R}^m$ ,

$$f(\mathbf{p}) + \phi(\mathbf{t}) = f(\mathbf{p} + \mathbf{t}) - \|\mathbf{t}\|_{\mathbb{R}^m} \alpha(\mathbf{t}).$$

As  $g$  is differentiable at  $f(\mathbf{p})$ , there exists a linear mapping  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^k$  such that

$$\lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{g(f(\mathbf{p}) + \|\mathbf{t}\|_{\mathbb{R}^m} \phi(\hat{\mathbf{t}})) - g(f(\mathbf{p})) - \lambda(\|\mathbf{t}\|_{\mathbb{R}^m} \phi(\hat{\mathbf{t}}))}{\left\| \|\mathbf{t}\|_{\mathbb{R}^m} \phi(\hat{\mathbf{t}}) \right\|_{\mathbb{R}^n}} = \mathbf{0}_{\mathbb{R}^k}.$$

As  $\phi$  is linear, we have

$$\|\mathbf{t}\|_{\mathbb{R}^m} \phi(\hat{\mathbf{t}}) = \phi(\mathbf{t}).$$

By scalar multiplication, we have

$$\left\| \|\mathbf{t}\|_{\mathbb{R}^m} \phi(\hat{\mathbf{t}}) \right\|_{\mathbb{R}^n} = \|\mathbf{t}\|_{\mathbb{R}^m} \|\phi(\hat{\mathbf{t}})\|_{\mathbb{R}^n}.$$

Now, we have

$$\begin{aligned} \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{g(f(\mathbf{p}) + \phi(\mathbf{t})) - g(f(\mathbf{p})) - \lambda(\phi(\mathbf{t}))}{\|\mathbf{t}\|_{\mathbb{R}^m}} &= \mathbf{0}_{\mathbb{R}^k} \\ \iff \lim_{\mathbf{t} \rightarrow \mathbf{0}_{\mathbb{R}^m}} \frac{g(f(\mathbf{p} + \mathbf{t})) - g(f(\mathbf{p})) - (\lambda \circ \phi)(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} &= \mathbf{0}_{\mathbb{R}^k}. \end{aligned}$$

As  $\lambda$  and  $\phi$  are both linear,  $\lambda \circ \phi$  are also linear.

By Definition 1.1.1,  $g \circ f$  is differentiable at  $\mathbf{p}$ . ■

## §1.2 Directional Derivatives

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**Observation 1.2.1.** Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , and let  $g : \mathbb{R} \rightarrow \mathbb{R}^m$  be defined as

$$g(t) := \mathbf{p} + t\mathbf{u},$$

where  $\mathbf{p}, \mathbf{u} \in \mathbb{R}^m$  and  $\mathbf{u} \neq \mathbf{0}_{\mathbb{R}^m}$ .

Let  $h = f \circ g$  and define  $h' : D_{h'} \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$  as

$$h'(t) := \lim_{t \rightarrow t_0} \frac{h(t) - h(t_0)}{t - t_0},$$

where for any  $t \in D_{h'}$ , the this limit exists in  $\mathbb{R}^n$ . Thus,

$$h'(0) = \lim_{t \rightarrow 0} \frac{f(\mathbf{p} + t\mathbf{u}) - f(\mathbf{p})}{t}$$

describes the instantaneous rate of change of  $f$  along the straight line  $\{\mathbf{p} + t\mathbf{u} : t \in \mathbb{R}\}$  with  $\|\mathbf{u}\|_{\mathbb{R}^m}$  as the unit length.  $h'(0)$  is so-called the  $\mathbf{u}$ -directional derivative of  $f$  at  $\mathbf{p}$  (See Definition 1.2.1).

**Definition 1.2.1** (Directional Derivatives).

Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let  $\mathbf{u} \in \mathbb{R}^m \setminus \{\mathbf{0}_{\mathbb{R}^m}\}$  and let  $\mathbf{p} \in \mathbb{R}^m$ . The  $\mathbf{u}$ -directional derivative of  $f$  at  $\mathbf{p}$  is defined as

$$\nabla_{\mathbf{u}} f(\mathbf{p}) := \lim_{t \rightarrow 0} \frac{f(\mathbf{p} + t\mathbf{u}) - f(\mathbf{p})}{t},$$

if the limit exists in  $\mathbb{R}^n$ .

**Theorem 1.2.1.** With the condition in Definition 1.2.1, for any  $s \in \mathbb{R} \setminus \{0\}$ ,

$$\nabla_{s\mathbf{u}} f(\mathbf{p}) = s \nabla_{\mathbf{u}} f(\mathbf{p}).$$

*Proof.* Let  $\theta = ts^{-1}$ , then, by Definition 1.2.1, we have

$$\begin{aligned}
s\nabla_{\mathbf{u}}f(\mathbf{p}) &= s \lim_{t \rightarrow 0} \frac{f(\mathbf{p} + t\mathbf{u}) - f(\mathbf{p})}{t} \\
&= \lim_{t \rightarrow 0} \frac{f(\mathbf{p} + t\mathbf{u}) - f(\mathbf{p})}{ts^{-1}} \\
&= \lim_{t \rightarrow 0} \frac{f(\mathbf{p} + \theta(s\mathbf{u})) - f(\mathbf{p})}{\theta} \\
&= \nabla_{s\mathbf{u}}f(\mathbf{p}).
\end{aligned}$$

■

**Theorem 1.2.2.** With the condition in Definition 1.2.1, if  $\nabla_{\mathbf{u}}f(\mathbf{p})$  exists, then there exists an open subset  $U \subseteq \mathbb{R}^m$  with  $\mathbf{p} \in U$  such that  $f$  is relative continuous on the line described by  $\mathbf{p} + t\mathbf{u}$  for some  $t \in \mathbb{R}$ .

*Proof.* Let  $U$  be an open subset of  $\mathbb{R}^m$ , and let  $g : \mathbb{R} \rightarrow \mathbb{R}^m$  be defined as

$$g(t) := \mathbf{p} + t\mathbf{u}.$$

Then  $f$  is relative continuous on the line defined by  $\mathbf{p} + t\mathbf{u}$  for some  $t \in \mathbb{R}$  iff  $f \upharpoonright_{g[\mathbb{R}]}$  is continuous on  $U \cap g[\mathbb{R}]$ .

Let  $h = f \circ g$ , then

$$\nabla_{\mathbf{u}}f(\mathbf{p}) = \lim_{t \rightarrow 0} \frac{h(t) - h(0)}{t} = \mathbf{v} \in \mathbb{R}^n.$$

Then, there exists an  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^n$  with  $\alpha(t) \rightarrow \mathbf{0}_{\mathbb{R}^n}$  as  $t \rightarrow 0$ , such that there exists an open subset  $I \subseteq \mathbb{R}$  with  $0 \in I$ , such that for any  $t \in I$ ,

$$h(t) = t\mathbf{v} + t\alpha(t) + h(0).$$

Then we have

$$\begin{aligned}
\lim_{t \rightarrow 0} h(t) &= \lim_{t \rightarrow 0} [t\mathbf{v} + t\alpha(t) + h(0)] \\
&\iff \lim_{t \rightarrow 0} h(t) = h(0).
\end{aligned}$$

Thus,  $h$  is continuous at 0.



As it is easy to show  $g$  is bijective,  $g \circ g^{-1}$  is an identity mapping on  $g[\mathbb{R}] \subseteq \mathbb{R}^m$ . As composition of mappings is associative, we have

$$\begin{aligned} h = f \circ g &\iff h \circ g^{-1} = f \circ g \circ g^{-1} \\ &\iff h \circ g^{-1} = f \circ (g \circ g^{-1}) \\ &\iff h \circ g^{-1} = f \upharpoonright_{g[\mathbb{R}]} . \end{aligned}$$

It is also easy to find that  $g^{-1}$  is continuous everywhere, thus, as  $h$  is continuous at 0,  $f \upharpoonright_{g[\mathbb{R}]}$  is continuous on  $U \cap g[\mathbb{R}]$ . Thus,  $f$  is relative continuous on the line defined by  $\mathbf{p} + t\mathbf{u}$  for some  $t \in \mathbb{R}$ . ■

### §1.3 Mean Value Theorem in Vector Valued Functions

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**Theorem 1.3.1** (Mean Value Theorem on  $\mathbb{R}^m \rightarrow \mathbb{R}^n$ ).

Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^m$ , for convenience, let  $g : \mathbb{R} \rightarrow \mathbb{R}^m$  be defined as

$$g(t) := \mathbf{p} + t(\mathbf{q} - \mathbf{p}).$$

If  $f \upharpoonright_{g[\mathbb{R}]}$  is continuous on  $g[(0, 1)]$ , and differentiable on  $g[[0, 1]]$ , then

$$\|f(\mathbf{q}) - f(\mathbf{p})\|_{\mathbb{R}^n} \leq \sup_{\mathbf{x} \in g[(a, b)]} \|\nabla_{\mathbf{u}} f(\mathbf{x})\|_{\mathbb{R}^n}.$$

**Lemma 1.3.1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and let  $a, b \in \mathbb{R}$  with  $a < b$ . Suppose  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and  $0 \notin f[(a, b)]$ .

Then,  $f$  is monotonic.

**Lemma 1.3.2** (Rolle's Theorem).

Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$  with  $f(\mathbf{a}) = f(\mathbf{b})$ . Suppose  $f$  is relative continuous on  $\ell[\mathbf{a}, \mathbf{b}]$ , and relative differentiable on  $\ell(\mathbf{a}, \mathbf{b})$ .

Then, there exists  $\mathbf{c} \in \ell(\mathbf{a}, \mathbf{b})$  such that  $\nabla_{\mathbf{u}} f(\mathbf{c}) = \mathbf{0}_{\mathbb{R}^n}$ , where  $\mathbf{u} = \mathbf{b} - \mathbf{a}$ .

*Proof.* First, consider  $f = \langle f_i \rangle_{i=1}^n$ .

Suppose for any  $\mathbf{c} \in \ell(\mathbf{a}, \mathbf{b})$ ,  $\nabla_{\mathbf{u}} f(\mathbf{c}) \neq \mathbf{0}_{\mathbb{R}^n}$ , then there exists  $i \in \{1, \dots, n\}$  such that  $\nabla_{\mathbf{u}} f_i(\mathbf{c}) \neq 0$ .

Let  $g : \mathbb{R} \rightarrow \mathbb{R}^m$  be defined as

$$g(t) = \mathbf{b} - t\mathbf{a},$$

and let  $h_i = f_i \circ g$ . Then, for any  $t \in (0, 1)$ ,  $h'_i(t) \neq 0$ .

As  $f_i$  is differentiable on  $g[(0, 1)]$ , and  $g$  is differentiable on  $(0, 1)$ , by Theorem 1.1.3,  $h_i$  is differentiable on  $(0, 1)$ . In this case,  $0 \notin h'_i[(0, 1)]$  implies  $h_i$  is monotonic (Lemma 1.3.1). This implies

$$h_i(0) = f_i(\mathbf{a}) \neq f_i(\mathbf{b}) = h_i(1).$$

As  $f(\mathbf{a}) = \langle f_i(\mathbf{a}) \rangle_{i=1}^n$  and  $f(\mathbf{b}) = \langle f_i(\mathbf{b}) \rangle_{i=1}^n$ , we have  $f(\mathbf{a}) \neq f(\mathbf{b})$ . This contradicts the assumption that  $f(\mathbf{a}) = f(\mathbf{b})$ .

Thus, there has to be a  $\mathbf{c} \in \ell(\mathbf{a}, \mathbf{b})$  such that  $\nabla_{\mathbf{u}} f_i(\mathbf{c}) = 0$ . ■

**Lemma 1.3.3.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}^n$ . If  $f$  is differentiable on open subset  $(a, b)$ , and continuous on closed interval  $[a, b]$ , then there exists a  $c \in I$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

*Proof.* Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$  be defined as

$$\phi(t) := t \frac{f(b) - f(a)}{b - a}.$$

Let  $h : \mathbb{R} \rightarrow \mathbb{R}^n$  be defined as

$$h(t) := f(t) - \phi(t).$$

Then it is easy to find that

$$h(a) = h(b).$$

As  $f$  and  $\phi$  are differentiable on  $(a, b)$ , so is  $h$ . (Why?)

As  $f$  and  $\phi$  are continuous on  $[a, b]$ , so is  $h$ . (Why?)

Thus, by Lemma 1.3.2, there exists a  $c \in (a, b)$  such that we have

$$\begin{aligned} 0 &= h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \\ \iff f'(c) &= \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

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*Proof for Theorem 1.3.1.* Let  $h = f \circ g$ . As  $f$  is continuous on  $g[(0, 1)]$  and  $g$  is continuous everywhere,  $h$  is continuous on  $(0, 1)$ . By Theorem 1.1.3, as  $f$  is differentiable on  $g[[0, 1]]$  and  $g$  is differentiable on  $[0, 1]$ , then, by Theorem 1.1.3,  $h$  is differentiable on  $[0, 1]$ .

Let  $h' : D \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$  be defined as

$$h'(t) := \lim_{t \rightarrow 0} \frac{h(c + t) - h(t)}{t},$$

where  $D$  is the set of all points in  $\mathbb{R}$  such that the limit exists in  $\mathbb{R}^n$ .

By Lemma 1.3.3, there exists a  $c \in (0, 1)$  such that

$$h'(c) = \frac{h(1) - h(0)}{1 - 0}.$$

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