

Notes for General Topology

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Contents

1	Metric Spaces	2
1.1	Metric Spaces	2
1.2	Open sets in Metric Spaces	4
1.3	Restrictions and Metric Subspaces	7
2	Topological Spaces	9
2.1	Basic Definitions	9
2.2	Some Important Topologies	10
2.3	Comparison of Topologies	10

Chapter 1

Metric Spaces

1.1 Metric Spaces

Definition 1.1.1. Let X be any set. A function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is *metric function*, or, simply, *metric on X* iff it satisfies the *metric axioms*. That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ iff $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.1.2. Let X be any set and let d be a structure on X . The pair (X, d) is called a *metric space* iff d is a metric on X .

Definition 1.1.3. A $M = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$. An *open ε -ball*, or just ε -ball, about x is defined to be the set

$$B_\varepsilon(x; d) := \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) := \{y \in X : d(x, y) \leq \varepsilon\}.$$

Note 1.1.1. As

$$M = (X, d), \quad M' = (X, d'), \quad M'' = (X, d''), \quad \dots$$

are different although they share the same set X , for any $x \in X$ and any $\varepsilon \in \mathbb{R}_{>0}$,

$$B_\varepsilon(x; d), B_\varepsilon(x; d'), B(x; d''), \dots$$

are also different. However, if confusion is unlikely, we simply write “ $B_\varepsilon(x)$ ” for “ $B_\varepsilon(x; d)$ ”.

Example 1.1.1. The *Euclidean metric space* $M = (X, d)$ is an n -dimensional set X equipped with the *Euclidean metric* d defined as

$$d(x, y) := \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x, y) := \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

Example 1.1.2. A *discrete metric space* $M = (X, d)$ is a set X equipped with the *discrete metric* d defined as

$$d(x, y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{else.} \end{cases}$$

This is an equivalent definition of the discrete metric:

$$d(x, y) := (\text{sgn}(d'(x, y)))^2,$$

where $\text{sgn}(\cdot)$ is a [sign function](#), and d' is any metric on X .

Example 1.1.3. ¹ Denote $C[a, b]$ for the set of all continuous mapping $\mathbb{R}_{[a, b]} \rightarrow \mathbb{R}$. On $C[a, b]$, we can define a metric d as

$$d_p(f, g) := \left(\int_a^b |f(t) - g(t)|^p dt \right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(f, g) := \sup_{t \in \mathbb{R}_{[a, b]}} |f(t) - g(t)|.$$

¹ See [Minkowski inequality](#).

Example 1.1.4. ² Let $M = (X, d)$ be a metric space. The *Hausdorff metric* d_H on $2^X \setminus \{\emptyset\}$ is defined as

$$d_H := \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\},$$

where

$$d(x, Y) := \inf_{y \in Y} d(x, y), \text{ and } d(y, X) := \inf_{x \in X} d(y, x).$$

1.2 Open sets in Metric Spaces

Definition 1.2.1. Let $M = (X, d)$ be a metric space, and let $U \subseteq X$. U is said to be *open in M* , iff for any $y \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$, such that $B_\varepsilon(y) \subseteq U$.

Lemma 1.2.1. Let $M = (X, d)$ be a metric space, let $x \in A$ and let $\varepsilon \in \mathbb{R}_{>0}$. For any $y \in B_\varepsilon(x)$, there is a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_\varepsilon(x)$.

Proof. For any $y \in B_\varepsilon(x)$, by the definition of open balls (Definition 1.1.3), we have $d(x, y) < \varepsilon$.

Let $\delta \in \mathbb{R}_{>0}$ such that $\delta + d(x, y) = \varepsilon$.

By M3 in metric axioms (Definition 1.1.1), for any $z \in A$ with $d(y, z) < \delta$, we have

$$d(x, z) \leq d(y, z) + d(x, y) < \varepsilon.$$

Thus, again, by the definition of open balls, we have $B_\delta(y) \subseteq B_\varepsilon(x)$. □

Theorem 1.2.1. ³ Let $M = (X, d)$ be a metric space, and let $U \subseteq X$. U is open in M iff it is a union of open balls.

Proof. First, prove \Rightarrow .

As U is open, for any $y \in U$, there exists $\varepsilon_y \in \mathbb{R}_{>0}$ such that $B_{\varepsilon_y}(y) \subseteq U$.

Therefore,

$$U = \bigcup_{y \in U} B_{\varepsilon_y}(y).$$

□

Now, prove \Leftarrow .

Aiming for a contradiction, suppose U is a union of open balls but not open.

As U is not open, there is a $y \in U$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \not\subseteq U$.

² See [Hausdorff distance](#).

³ Shared on [ProofWiki](#).

As U is a union of open balls, there is an $x \in U$ and $r \in \mathbb{R}_{>0}$ such that $y \in B_r(x)$.

By Lemma 1.2.1, there exists a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_r(x)$.

This is a contradiction by the assumption.

Thus, U has to be open. ■

Theorem 1.2.2. ⁴ Let $M = (X, d)$ be any metric space. M is *Hausdorff*. That is, For any distinct points $x, y \in X$, we can always find an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset.$$

Proof. Aiming for a contradiction, suppose there are $x, y \in X$ with $x \neq y$, such that for any $\varepsilon \in \mathbb{R}_{>0}$, we can always find a $z \in X$ such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let $r = d(x, y)/2$, and let $z \in B_r(x) \cap B_r(y)$.

As $z \in B_r(x)$, by the definition of open balls (Definition 1.1.3), $d(x, z) < r$; as $z \in B_r(y)$, similarly, $d(y, z) < r$. Then we have

$$d(x, z) + d(y, z) < 2r = d(x, y).$$

By M3 in metric axioms (Definition 1.1.1), this is impossible. ■

Definition 1.2.2. Let $M = (X, d)$ be any metric space, and let $V \subseteq X$. V is said to be *closed* in M , iff there is an open set U satisfies $X \setminus U = V$.

Lemma 1.2.2. In a metric space, any singleton is closed.

Proof. Let $M = (X, d)$ be a metric space, let $x \in X$, and let $y \in X \setminus \{x\}$.

As M is Hausdorff (Theorem 1.2.2), there is an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$0 < \varepsilon < d(x, y),$$

thus $X \setminus \{x\}$ is open, hence, by Definition 1.1.1, its complement $\{x\}$ is open. □

Theorem 1.2.3. Let $M = (X, d)$ be a metric space, denote \mathcal{T} for the family of open subsets of X . Then \mathcal{T} satisfies the following conditions:

⁴ Shared on [ProofWiki](#).

O1. $X, \emptyset \in \mathcal{T}$;

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, \mathcal{T} is closed under arbitrary union;

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, \mathcal{T} is closed under finite intersection.

Proof.

O1. As \emptyset is the subset of any set, $\emptyset \in \mathcal{T}$. $\bigcup \emptyset = \emptyset \in \mathcal{T}$.

By Definition 1.2.2, $X = X \setminus \emptyset$.

□

O2. Let $\mathcal{U} \subseteq \mathcal{T}$, and denote \mathcal{O} for the open balls in M .

For any $U \in \mathcal{U}$, there is an $\mathcal{O}_U \subseteq \mathcal{O}$ such that $U = \bigcup \mathcal{O}_U$.

Then we have

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \left(\bigcup \mathcal{O}_U \right) = \bigcup_{U \in \mathcal{U}} \mathcal{O}_U.$$

By Theorem 1.2.1, $\bigcup \mathcal{U}$ is open.

□

O3. ⁵ Let \mathcal{V} be a finite subset of \mathcal{T} .

Aiming for a contradiction, suppose $\bigcap \mathcal{V}$ is not open.

By Definition 1.2.1, there exists a $y \in \bigcap \mathcal{V}$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \setminus \bigcap \mathcal{V} \neq \emptyset$.

By De Morgan's law, we have

$$\bigcup_{V \in \mathcal{V}} (B_\varepsilon(y) \setminus V) \neq \emptyset.$$

Thus, there exists $V \in \mathcal{V}$ such that $B_\varepsilon(y) \setminus V \neq \emptyset$.

As $V \in \mathcal{T}$ and ε is arbitrarily given, by Lemma 1.2.1, $y \notin V$. This is a contradiction.

Thus, $\bigcap \mathcal{V}$ is open.

□

⁵ Shared on [ProofWiki](#).

Thus, the theorem is proved. ■

Theorem 1.2.4. Infinite intersections of open sets in some metric spaces are not necessarily open.

Proof. Consider \mathbb{R} is a Euclidean metric space, and denote \mathcal{T} .

Clearly, for any $n \in \mathbb{N}_{>0}$ and for any $x \in X$, the open interval $B_{\frac{1}{n}}(x)$ is open, but

$$\bigcap \left\{ B_{\frac{1}{n}}(x) : n \in \mathbb{N}_{>0} \right\} = \{x\}.$$

For any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(x) \setminus \{x\}$ is not empty, thus $\{x\}$ is not open. ■

1.3 Restrictions and Metric Subspaces

Restriction of metric function is a useful tool to describe the relation between metric spaces with different sets but “same” metric function on the sets.

As a restriction of a relation R on $X \times Y$ to a subset $A \times B \subseteq X \times Y$ is defined to be

$$R \upharpoonright_{A \times B} := R \cap (X \times Y),$$

a restriction of a metric d on a set S to a subset $U \subseteq S$ is defined to be

$$d \upharpoonright_{(U \times U) \times \mathbb{R}_{>0}} := d \cap ((U \times U) \times \mathbb{R}_{>0}).$$

If $B = Y$, customarily, we simply write $R \upharpoonright_A$ for $R \upharpoonright_{A \times B}$. Similarly, as the codomain of a metric function is always $\mathbb{R}_{>0}$, so we simply write $d \upharpoonright_{U \times U}$ instead of $d \upharpoonright_{(U \times U) \times \mathbb{R}_{>0}}$.

Definition 1.3.1. Let $M = (X, d)$ be a metric space, and let $A \subseteq X$. The *metric on A induced by d* , or the *subspace metric of d with respect to A* is defined to be

$$d_A := d \upharpoonright_{A \times A}.$$

Theorem 1.3.1. Let $M = (X, d)$ be a metric space, and let $A \subseteq X$ and let $d_A := d \upharpoonright_{A \times A}$. Then (A, d_A) is a metric space.

Proof. As metric axioms (Definition 1.1.1) holds for any $x, y \in X$, and $A \subseteq X$, they also holds for any $a, b \in A$. As d_A is the subspace metric of d with respect to A , d_A is a metric on A .

Thus, (A, d_A) is a metric space.

Definition 1.3.2. Let $M = (X, d)$ be a metric space, and let $A \subseteq X$. (A, d_A) is a *metric subspace* of M iff d_A is a subspace metric of d with respect to A .

Chapter 2

Topological Spaces

2.1 Basic Definitions

Definition 2.1.1. Let X be any set, and let $\mathcal{T} \subseteq 2^X$. We call \mathcal{T} a *topology on X* iff it satisfies the *open set axioms*. That is,

- O1.** $X \in \mathcal{T}$;
- O2.** For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, \mathcal{T} is closed under arbitrary union.
- O3.** For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, \mathcal{T} is closed under finite intersection.

A subset $U \subseteq X$ is said to be *open in M* iff it is an element of \mathcal{T} .

Definition 2.1.2. Let X be any set, and let \mathcal{T} be a structure on X . The pair (X, \mathcal{T}) is called a *topological space* iff \mathcal{T} is a topology on X .

Theorem 2.1.1. Let $M = (X, \mathcal{T})$ be a topological space. Then $\emptyset \in \mathcal{T}$.

Proof. As empty set is an element of any set, it also an element of \mathcal{T} .

Therefore, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

■

Definition 2.1.3. Let $M = (X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is said to be *closed in M* iff there exists a $U \in \mathcal{T}$ such that $A = X \setminus U$.

2.2 Some Important Topologies

Definition 2.2.1. Let X be any set. A family $\mathcal{T} \subseteq 2^X$ is a *discrete topology* on X iff $\mathcal{T} = 2^X$.

Definition 2.2.2. Let X be any set. A family $\mathcal{T} \subseteq 2^X$ is an *indiscrete topology* on X iff $\mathcal{T} = \{X, \emptyset\}$.

Definition 2.2.3. Let $M = (X, d)$ be a metric space. A family $\mathcal{T} \subseteq 2^X$ is a *topology induced by d* iff \mathcal{T} is the set of all open sets in M .

2.3 Comparison of Topologies

Definition 2.3.1. Let X be any set and let \mathcal{T} and \mathcal{T}' be topologies on X . We say that \mathcal{T} is *coarser* than \mathcal{T}' , or \mathcal{T}' is *finer* than \mathcal{T} , iff $\mathcal{T} \subseteq \mathcal{T}'$.

Note 2.3.1. By the definition of cardinality and inclusion mapping, if $\mathcal{T} \subseteq \mathcal{T}'$, it is certainly true that $|\mathcal{T}| \leq |\mathcal{T}'|$. But, on the contrary, $|\mathcal{T}| \leq |\mathcal{T}'|$ does not implies $\mathcal{T} \subseteq \mathcal{T}'$. It is easy to find counter-example about this.

Example 2.3.1. By Definition 2.3.1, for any set X , if a family \mathcal{U} of open sets is given, then we can find the coarsest topology on X containing \mathcal{U} by

$$\mathcal{T} = \left\{ \bigcup \mathcal{I}, \bigcap \mathcal{I}, X : \mathcal{I} \subseteq \mathcal{U} \right\}.$$

For example, let $X = \{1, 2, 3, 4, 5\}$, and let

$$\mathcal{U} = \{\{1, 2\}, \{2, 3\}, \{4\}\}.$$

Then a topology on X contains at least these sets:

$$\begin{aligned} &\{1, 2, 3, 4\}, \{\}, \\ &\{1, 2\}, \{2, 3\}, \{4\}, \\ &\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \\ &\{2\}. \end{aligned}$$

Definition 2.3.2.