Notes for Mathematical Anaysis

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Chapter 1.

Limit and Continuity

In the sections below, the capitals with blackboard bold font, such as $\mathbb{A}$, denote the normed vector spaces.

§1.1 O Notations

Definition 1.1.1. Let $f : \mathbb{X} \to \mathbb{Y} : \mathbf{x} \mapsto f(\mathbf{x})$ and $g : \mathbb{X} \to \mathbb{S} : \mathbf{x} \mapsto g(\mathbf{x})$.

f is a *little-o of g as* $\mathbf{x} \to \mathbf{p}$, denoted

$$f(\mathbf{x}) = o(g(\mathbf{x})) \quad \text{as } \mathbf{x} \to \mathbf{p},$$

iff for any $\varepsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood N of $\mathbf{p}$, such that for any $\mathbf{x} \in N$, $\|f(\mathbf{x})\|_{\mathbb{Y}} \leq \varepsilon \|g(\mathbf{x})\|_{\mathbb{S}}$; equivalently, that is,

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{\|f(\mathbf{x})\|_{\mathbb{Y}}}{\|g(\mathbf{x})\|_{\mathbb{S}}} = 0 \text{ or, equivalently, } \lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{\|g(\mathbf{x})\|_{\mathbb{S}}} = \mathbf{0}_{\mathbb{Y}}$$

Lemma 1.1.1. With the condition in Definition 1.1.1, suppose

$$\lim_{\mathbf{x}\to\mathbf{p}} \|g(\mathbf{x})\| \in \mathbb{R},$$

Then

$$f(\mathbf{x}) = o(g(\mathbf{x})) \quad \text{as } \mathbf{x} \to \mathbf{p},$$

implies

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}_{\mathbb{Y}}.$$

Proof. Aiming for a contradiction, suppose there exists $\mathbf{r} \in \mathbb{Y} \setminus \{\mathbf{0}_\mathbb{Y}\}$, such that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{r},$$

then, we have

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{\|f(\mathbf{x})\|_\mathbb{Y}}{\|g(\mathbf{x})\|_\mathbb{S}} = \lim_{\mathbf{x}\to\mathbf{p}} \frac{\|\mathbf{r}\|_\mathbb{Y}}{\|g(\mathbf{x})\|_\mathbb{S}} > 0.$$

This contradicts the assumption. ■

Lemma 1.1.2. With the condition in Definition 1.1.1, f is a little-o of g iff $-f$ is a little-o of g.

Proof.

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{\|-f(\mathbf{x})\|_\mathbb{Y}}{\|g(\mathbf{x})\|_\mathbb{S}} = \lim_{\mathbf{x}\to\mathbf{p}} \frac{\|f(\mathbf{x})\|_\mathbb{Y}}{\|g(\mathbf{x})\|_\mathbb{S}} = 0.$$

■

Lemma 1.1.3. Let $f_1, f_2 : \mathbb{X} \to \mathbb{Y} : \mathbf{x} \mapsto f_1(\mathbf{x}), f_2(\mathbf{x})$, and let $g : \mathbb{X} \to \mathbb{S} : \mathbf{x} \mapsto g(\mathbf{x})$.

If f_1 and f_2 are both little-o of g as $\mathbf{x} \to \mathbf{p}$, i.e.,

$$f_1(\mathbf{x}) = o_1(g(\mathbf{x})) \text{ and } f_2(\mathbf{x}) = o_2(g(\mathbf{x})) \text{ as } \mathbf{x} \to \mathbf{p},$$

then $f_1 + f_2$ is also a little-o of g as $\mathbf{x} \to \mathbf{p}$, i.e.,

$$f_1(\mathbf{x}) + f_2(\mathbf{x}) = o_3(g(\mathbf{x}))$$

Proof. By triangle inequality, we have

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} \frac{\|f_1(\mathbf{x}) + f_2(\mathbf{x})\|_\mathbb{Y}}{\|g(\mathbf{x})\|_\mathbb{S}} &\leq \lim_{\mathbf{x}\to\mathbf{p}} \frac{\|f_1(\mathbf{x})\|_\mathbb{Y} + \|f_2(\mathbf{x})\|_\mathbb{Y}}{\|g(\mathbf{x})\|_\mathbb{S}} \\
&\leq 2 \lim_{\mathbf{x}\to\mathbf{p}} \frac{\max\{\|f_1(\mathbf{x})\|_\mathbb{Y} + \|f_2(\mathbf{x})\|_\mathbb{Y}\}}{\|g(\mathbf{x})\|_\mathbb{S}} \\
&= 0.
\end{aligned}$$

By Definition 1.1.1, $f_1 + f_2$ is a little-o of g as $\mathbf{x} \to \mathbf{p}$. ■

Note 1.1.1. In Lemma 1.1.3, consider A be the set of all mappings being little-o of g as $\mathbf{x} \to \mathbf{p}$, then Lemma 1.1.3 tells that A is finitely additive. That is, for any finite $B \subseteq A$,

$$\sum_{o \in B} o(g(\mathbf{p})) \in A.$$

Chapter 2.

Differentiation

§2.1 Differentiable Mappings

Definition 2.1.1. Let $f : \mathbb{X} \to \mathbb{Y}$.

f is said to be *differentiable at* $\mathbf{p} \in \mathbb{X}$ iff there exists a linear mapping $\phi : \mathbb{X} \to \mathbb{Y}$ such that for any $\mathbf{t} \in \mathbb{X}$,

$$f(\mathbf{p} + \mathbf{t}) = f(\mathbf{p}) + \phi(\mathbf{t}) + o(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}}.$$

Lemma 2.1.1. In Definition 2.1.1, the linear mapping ϕ is unique.

Proof. Suppose there is another linear mapping $\lambda : \mathbb{X} \to \mathbb{Y}$, such that for any $\mathbf{t} \in \mathbb{X}$,

$$f(\mathbf{p} + \mathbf{t}) = f(\mathbf{p}) + \lambda(\mathbf{t}) + o_\lambda(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}},$$

then we have

$$\phi(\hat{\mathbf{t}}) - \lambda(\hat{\mathbf{t}}) = \lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{X}}} \frac{\phi(\mathbf{t}) - \lambda(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{X}}} = \lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{X}}} \frac{o(\mathbf{t}) - o_\lambda(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{X}}}$$

By Lemma 1.1.2, $-o_\lambda(\mathbf{t})$ is also a little-o of $\mathbf{t}$ as $\mathbf{t} \to \mathbf{0}_{\mathbb{X}}$, thus, by Lemma 1.1.3, $o(\mathbf{t}) - o_\lambda(\mathbf{t})$ is a little-o of $\mathbf{t}$ as $\mathbf{t} \to \mathbf{0}_{\mathbb{X}}$. By Definition 1.1.1,

$$\phi(\hat{\mathbf{t}}) - \lambda(\hat{\mathbf{t}}) = \mathbf{0}_{\mathbb{X}}.$$

As $\mathbf{t}$ is arbitrarily given, $\hat{\mathbf{t}}$ defines all possible directions in $\mathbb{X}$. Thus,

$$\phi = \lambda.$$

■

Lemma 2.1.2. With the condition in Definition 2.1.1, f is differentiable at $\mathbf{p} \in \mathbb{X}$ iff there exists a linear mapping $\phi : \mathbb{X} \to \mathbb{Y}$ such that for any $\mathbf{t} \in \mathbb{X}$,

$$\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} \frac{\|f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})\|_{\mathbb{Y}}}{\|\mathbf{t}\|_{\mathbb{X}}} = 0. \tag{i}$$

Equivalently, that is,

$$\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} \frac{f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{X}}} = \mathbf{0}_{\mathbb{Y}}. \tag{i'}$$

Proof. This can be proved from both sides. Consider the equations in this proposition and in Definition 2.1.1. We observe that the equation in Definition 2.1.1 holds iff

$$\begin{aligned}
&\frac{f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{X}}} = \frac{o(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{X}}} \\
\iff &\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} \frac{f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{X}}} = \mathbf{0}_{\mathbb{X}} \qquad \text{((i') is proved)} \\
\iff &\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} \frac{\|f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})\|_{\mathbb{Y}}}{\|\mathbf{t}\|_{\mathbb{X}}} = 0. \quad \text{((i) is proved)}
\end{aligned}$$

■

Lemma 2.1.3. With the condition in Definition 2.1.1, if f is differentiable at $\mathbf{p}$, then f is continuous at $\mathbf{p}$.

Proof. As f is differentiable at $\mathbf{p}$, there exists a linear mapping $\phi : \mathbb{X} \to \mathbb{Y}$, such that for any $\mathbf{t} \in \mathbb{X}$,

$$f(\mathbf{p}+\mathbf{t}) = f(\mathbf{p}) + \phi(\mathbf{t}) + o(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}.$$

As

$$\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} \phi(\mathbf{t}) = \mathbf{0}_{\mathbb{Y}}$$

and, by Lemma 1.1.1,

$$\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} o(\mathbf{t}) = \mathbf{0}_{\mathbb{Y}},$$

we have

$$\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} f(\mathbf{p}+\mathbf{t}) = f(\mathbf{p}),$$

which implies that f is continuous at $\mathbf{p}$. ■

Lemma 2.1.4. Let $f : \mathbb{X} \to \mathbb{Y}$ and let $g : \mathbb{Y} \to \mathbb{S}$. If f is differentiable at a point $\mathbf{p} \in \mathbb{X}$, and g is differentiable at $f(\mathbf{x})$, then $g \circ f$ is differentiable at $\mathbf{p}$.

Proof. As g is differentiable at $f(\mathbf{p})$, there exists $\lambda : \mathbb{Y} \to \mathbb{S}$ such that for any $\mathbf{s} \in \mathbb{Y}$ with $f(\mathbf{p}) + \mathbf{s} \in f[\mathbb{X}]$,

$$g(f(\mathbf{p}) + \mathbf{s}) = g(f(\mathbf{p})) + \lambda(\mathbf{s}) + o(\mathbf{s}) \quad \text{as } \mathbf{s} \to \mathbf{0}_{\mathbb{Y}}.$$

As f is differentiable at $\mathbf{p}$, f is continuous at $\mathbf{p}$, thus, there exists $\mathbf{t} \in \mathbb{X}$, such that $\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} f(\mathbf{p}+\mathbf{t}) = f(\mathbf{p}) + \mathbf{s}$. Since f is differentiable at $\mathbf{p}$, there exists a linear mapping $\phi : \mathbb{X} \to \mathbb{Y}$, such that

$$f(\mathbf{p}+\mathbf{t}) = f(\mathbf{p}) + \phi(\mathbf{t}) + o_1(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}}.$$

Then we have

$$g(f(\mathbf{p}+\mathbf{t})) = g(f(\mathbf{t})) + \lambda(\Delta f) + o(\Delta f) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}},$$

where

$$\Delta f = f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) = \phi(\mathbf{t}) + o_1(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}}.$$

First, find $\lambda(\Delta f)$. As λ is linear,

$$\lambda(\Delta f) = \lambda(\phi(\mathbf{t}) + o_1(\mathbf{t})) = \lambda(\phi(\mathbf{t})) + \lambda(o_1(\mathbf{t})) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}}.$$

As λ is linear, $\lambda \circ \phi$ is also linear, and $\lambda(o_1(\mathbf{t}))$ is a little-o of $\mathbf{t}$, i.e., $o_2(\mathbf{t}) = \lambda(o_1(\mathbf{t}))$ as $\mathbf{t} \to \mathbf{0}_{\mathbb{X}}$, for

$$\begin{aligned}\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} \frac{\lambda(o_1(\mathbf{t}))}{\|\mathbf{t}\|_{\mathbb{X}}} &= \lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{X}}} \lambda\left(\frac{o_1(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{X}}}\right) \\ &= \lambda(\mathbf{0}_{\mathbb{Y}}) \\ &= \mathbf{0}_{\mathbb{S}}.\end{aligned}$$

Let $\gamma = \lambda \circ \phi$ for convenience.

Then, find $o(\Delta f)$.

$$\begin{aligned}
\mathbf{0}_{\mathbb{S}} &= \lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{X}}} \frac{o(\phi(\mathbf{t}) + o_1(\mathbf{t}))}{\|\phi(\mathbf{t}) + o_1(\mathbf{t})\|_{\mathbb{Y}}} && \text{(Definition 1.1.1)} \\
&= \lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{X}}} \frac{o(\phi(\mathbf{t}) + o_1(\mathbf{t}))\|\mathbf{t}\|_{\mathbb{X}}^{-1}}{\|\phi(\mathbf{t}) + o_1(\mathbf{t})\|_{\mathbb{Y}}\|\mathbf{t}\|_{\mathbb{X}}^{-1}} \\
&= \lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{X}}} \frac{o(\phi(\mathbf{t}) + o_1(\mathbf{t}))}{\|\mathbf{t}\|_{\mathbb{X}}\|\phi(\hat{\mathbf{t}})\|_{\mathbb{Y}}} && \text{(as } \phi \text{ is linear)} \\
&= \|\phi(\hat{\mathbf{t}})\|_{\mathbb{Y}}^{-1} \lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{X}}} \frac{o(\phi(\mathbf{t}) - o_1(\mathbf{t}))}{\|\mathbf{t}\|_{\mathbb{X}}}.
\end{aligned}$$

Thus, $o(\phi(\mathbf{t}) - o_1(\mathbf{t})) = o_3(\mathbf{t})$ as $\mathbf{t} \to \mathbf{0}_{\mathbb{X}}$.

Now, we have

$$g(f(\mathbf{p} + \mathbf{t})) = g(f(\mathbf{t})) + \gamma(\mathbf{t}) + o_2(\mathbf{t}) + o_3(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}}.$$

By Lemma 1.1.3,

$$o_2(\mathbf{t}) + o_3(\mathbf{t}) = o_4(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}}.$$

Finally, we have

$$g(f(\mathbf{p} + \mathbf{t})) = g(f(\mathbf{t})) + \gamma(\mathbf{t}) + o_4(\mathbf{t}) \quad \text{as } \mathbf{t} \to \mathbf{0}_{\mathbb{X}},$$

which implies $g \circ f$ is differentiable at $\mathbf{p}$. ■

§2.2 Directional Derivatives

Definition 2.2.1. Let $f : \mathbb{X} \to \mathbb{Y}$, and let $\mathbf{u} \in \mathbb{X} \setminus \{\mathbf{0}_{\mathbb{X}}\}$.

The $\mathbf{u}$-*directional derived mapping*

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