# Notes for General Topology

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# Chapter 1

# Topological Spaces

## 1.1 Topological Spaces

**Definition 1.1.1** (topology)**.** Let $X$ be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is called a *topology* on $X$ iff it satisfies the *open set axioms*:

**O1.** $X \in \mathcal{T}$;

**O2.** $\mathcal{T}$ is closed under arbitrary union;

**O3.** $\mathcal{T}$ is closed under finite intersection.

**Theorem 1.1.1.** $\emptyset \in \mathcal{T}$.

*Proof.* By O2 in Definition 1.1.1, for all $\mathcal{A} \subseteq \mathcal{T}$, $\bigcup \mathcal{A} \in \mathcal{T}$. Clearly, $\emptyset \in \mathcal{T}$, then we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

□

**Definition 1.1.2** (topological spaces)**.** With the condition in Definition 1.1.1, the pair $(X, \mathcal{T})$ is called a *topological space*. All subsets of $X$ in $\mathcal{T}$ are said to be *open* in $(X, \mathcal{T})$.

**Theorem 1.1.2.** Let $(X, \rho)$ be a metric space, and let

$$\mathcal{T}_\rho = \left\{ \bigcup_{x \in U} B_\rho(x, \varepsilon) : U \subseteq X \wedge \varepsilon \in \mathbb{R}_{>0} \right\}.$$

$(X, \mathcal{T}_\rho)$ is a topological space; i.e., $\mathcal{T}_\rho$ satisfies the open set axioms.

*Proof.* For O1. Apparently, there exists (for any) $\varepsilon \in \mathbb{R}_{>0}$,

$$X = \bigcup_{x \in X} B(x, \varepsilon).$$

For O2. Let $I$ be an index set, and let $U_i \in T$ for all $i \in I$. There exists $\varepsilon \in \mathbb{R}_{>0}$ we can define

$$U = \bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{x \in U_i} B(x, \varepsilon) = \bigcup_{x \in U} B(x, \varepsilon).$$

Then, for all $x \in U$, there exists $B(x, \varepsilon) \ni x$. Thus $U \in \mathcal{T}$.

For O3. Let $U, V \in \mathcal{T}$. If $U \cap V = \emptyset$, then the proof is done. Suppose $U \cap V \neq \emptyset$, and let $x \in U \cap V$. $U$ is open and $x \in U$, so there exists $r_1 \in \mathbb{R}_{>0}$ such that $B(x, r_1) \subseteq U$; $V$ is open and $x \in V$, so there exists $r_2 \in \mathbb{R}_{>0}$ such that $B(x, r_2) \subseteq V$.

If $r_1 = r_2$, then $B(x, r_1) = B(x, r_2)$. If $r_1 \neq r_2$, say $r_1 < r_2$, then $B(x, r_1) \subseteq B(x, r_2) \subseteq V$. Above all, for any $x \in U \cap V$, there exits $\varepsilon \in \mathbb{R}_{>0}$ such that $B(x, \varepsilon) \subseteq U \cap V$. Thus, $U \cap V \in \mathcal{T}_\rho$. □

**Definition 1.1.3** (closed sets)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $V$ of $X$ is said to be *closed* iff there is an open set $U$ in $X$ such that

$$V = X \setminus V.$$

**Proposition 1.1.1.** Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C}$ be the family of all closed sets in $X$. Then

(i) $\emptyset, X \in \mathcal{C}$;

(ii) $\mathcal{C}$ is closed under arbitrary intersection;

(iii) $\mathcal{C}$ is closed under finite union.

*Proof.*

(i) $X \in \mathcal{T}$ implies $X \setminus X = \emptyset \in \mathcal{C}$; and $\emptyset \in \mathcal{T}$ implies $X \setminus \emptyset = X \in \mathcal{C}$;

(ii) As $\mathcal{T}$ is closed under arbitrary union, then by Definition 1.1.3 and De Morgan's Law, $\mathcal{C}$ is closed under arbitrary intersection.

(iii) As $\mathcal{T}$ is closed under finite intersection, then by Definition 1.1.3 and De Morgan's Law, $\mathcal{C}$ is closed under finite union.

□

**Definition 1.1.4** (finer and coarser topology)**.** Let $X$ be any set, and let $\mathcal{T}, \mathcal{T}'$ be topologies on $X$. $\mathcal{T}$ is said to be *finer* than $\mathcal{T}'$ iff $\mathcal{T} \supseteq \mathcal{T}'$; respectively, $\mathcal{T}$ is said to be *coarser* than $\mathcal{T}'$ iff $\mathcal{T} \subseteq \mathcal{T}'$.

**Definition 1.1.5** (neighbourhood)**.** Given $(X, \mathcal{T})$ as a topological space and a point $x \in X$, a subset $N \subseteq X$ is called a *neighbourhood* iff it contains an open set $U$ containing $x$.

**Proposition 1.1.2.** Given $(X, \mathcal{T})$ as a topological space and $U \subseteq X$, $U$ is open iff for all $x \in U$, there is a neighbourhood $N$ of $x$ contained in $U$.

*Proof.* If $U$ is open, then $U$ itself is a neighbourhood of $x$ contained in $U$.

Conversely, if for all $x \in U$, there is a neighbourhood $N_x$ of $x$ contained in $U$, then there is a open neighbourhood $U_x \ni x$ contained in $N_x$. Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose $U$ is not open, then $U$ is a proper superset in the relation above. Then there exists $y \in U$ which is not in any $U_x$. This implies that such a $y$ does not have any neighbourhood $N_y$ in $U$, for such an $N_y$ must contains an open $U_y \ni y$. For if it does, then there must be a $U_x$ contains $y$. This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open. $\square$

## 1.2 Metrizability

**Definition 1.2.1** (metric spaces)**.** Let $X$ be any set. A *metric* $\rho$ on $X$ is a function $\rho : X \times X \to \mathbb{R}$ satisfying the following conditions: for all $x, y, z \in X$

(i) $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ iff $x = y$;

(ii) $\rho(x, y) = \rho(y, z)$;

(iii) $\rho(x, z) + \rho(z, y) \geq \rho(x, y)$.

**Definition 1.2.2** (balls)**.** Let $(X, \rho)$ be a metric space, let $x \in X$, and let $\varepsilon \in \mathbb{R}_{>0}$. The *open $\varepsilon$-ball about $x$* or just *$\varepsilon$-ball about $x$* is defined to be

$$B(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}.$$

The *closed $\varepsilon$-ball about $x$* is defined to be

$$\overline{B}(x, \varepsilon) = \{y \in X : \rho(x, y) \leq \varepsilon\}.$$

**Example 1.2.1.** Let $X$ be any set, and let metric $\rho_p$ on $X^n$ ($n \in \mathbb{Z}_{>0}$) defined by

$$\rho_p(x, y) = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}},$$

where $p \in \mathbb{R}_{\geq 1}$. $\rho_2$ is so called the *standard Euclidean metric*. If $X = \mathbb{R}$, then the metric space $(\mathbb{R}^n, \rho_2)$ is so-called *Euclidean $n$-space*.

For all $p, q \in \mathbb{R}_{\geq 1}$, if $p < q$, then for all $\varepsilon \in \mathbb{R}_{>0}$ and for all $x, y \in X$, $\rho_p(x, y) \geq \rho_q(x, y)$; in particular, $\rho_p = \rho_q$ iff there is a unique $k \in \{1, \ldots, n\}$, such that for all $i \in \{1, \ldots, n\} \setminus \{k\}$, $x_i = 0$. As $\rho_p(x, y)$ is always "overestimated" than $\rho_q(x, y)$, we have $B_{\rho_p}(x, \varepsilon) \supseteq B_{\rho_q}(x, \varepsilon)$.

**Example 1.2.2.** Let $X$ be any set. The *discrete metric* $\rho$ on $X$ is defined to be

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

**Example 1.2.3.** Let $a, b \in \mathbb{R}$ with $a < b$, and let metric $\rho_p$ on $C[a, b]$ defined by

$$\rho_p(f, g) = \left( \int_a^b |f(t) - g(t)|^p dt \right)^{\frac{1}{p}},$$

where $p \geq 1$. In particular,

$$\rho_\infty(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|.$$

**Proposition 1.2.1.** Let $(X, \rho)$ be a metric space, then for all $x, y \in X$ ($x \neq y$), there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$.

*Proof.* Suppose for all $\varepsilon > 0$, $B(x, \varepsilon) \cap B(y, \varepsilon) \neq \emptyset$, then there must be a $z \in X$ such that $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. $z \in B(x, \varepsilon)$ only if $\rho(x, z) < \varepsilon$, and $z \in B(y, \varepsilon)$ only if $\rho(z, y) < \varepsilon$. Thus

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

As the assumption holds for all $\varepsilon > 0$, we may put

$$\varepsilon = \frac{\rho(x, y)}{2}.$$

Then, we have

$$\rho(x,z)+\rho(y,z)<\rho(x,y),$$

which is impossible. $\square$

**Definition 1.2.3** (induced topologies)**.** Let $(X,\rho)$ be a metric space. A topology $\mathcal{T}$ on $X$ is said to be *induced* by $\rho$ iff for all $\varepsilon>0$, any $U\in\mathcal{T}$ is the union of ball(s) in $X$; i.e.,

$$\mathcal{T}=\left\{U\subseteq X : U=\bigcup_{x\in X} B(x,\varepsilon)\right\}.$$

In this case, $\mathcal{T}$ is called the *underlying topology* of $\rho$.

**Definition 1.2.4** (metrizable spaces)**.** Let $(X,\mathcal{T})$ be a topological space. If there is any $\rho$ induce $\mathcal{T}$, then $(X,\mathcal{T})$ is said to be *metrizable*.

**Definition 1.2.5** (Lipschitz equivalence)**.** Let $X$ be any set, and let $\rho$ and $\rho'$ be metrics on $X$. $\rho$ and $\rho'$ are said to be *Lipschitz equivalent* iff there exist $c,C>0$, such that for all $x,y\in X$,

$$c\rho(x,y)\le\rho'(x,y)\le C\rho(x,y).$$

**Proposition 1.2.2.** Lipschitz equivalence is an equivalence relation.

*Proof.* Clearly, Definition 1.2.5 also holds for $\rho=\rho'$. So, Lipschitz equivalence is reflexive. In Definition 1.2.5, the relation also holds for $\frac{1}{C}\rho'\le\rho\le\frac{1}{c}\rho'$. So Lipschitz equivalence is symmetric.

If there is another $\rho''$ be Lipschitz equivalent to $\rho'$, then there is $r,R>0$, such that for all $x,y\in X$,

$$r\rho''(x,y)\le\rho'(x,y)\le R\rho''(x,y).$$

By the conditions in Definition 1.2.5, we have

$$\frac{c}{r}\rho(x,y)\le\rho''(x,y)\le\frac{C}{R}\rho(x,y),$$

i.e., $\rho$ and $\rho''$ are also Lipschitz equivalent. So Lipschitz equivalence is transitive.

Above all, Lipschitz equivalence is an equivalence relation. $\square$

**Proposition 1.2.3.** Let $X$ be any set, and let $\rho$ and $\rho'$ be metrics on $X$. If $\rho$ and $\rho'$ are Lipschitz equivalent, then $\rho$ and $\rho'$ induce the same topology.

*Proof.* As $\rho$ and $\rho'$ are Lipschitz equivalent, by Definition 1.2.5, there is a $c > 0$ such that for all $x, y \in X$,

$$c\rho(x, y) \leq \rho'(x, y).$$

Given $r \in \mathbb{R}_{>0}$ and for all $x \in X$, we have

$$B_{\rho'}(x, cr) \subseteq B_{c\rho}(x, r) = B_\rho\left(x, \frac{1}{c}r\right).$$

For all $U \in \mathcal{T}_\rho$, for all $x \in U$, there is an $\varepsilon \in \mathbb{R}_{>0}$, such that

$$B_{\rho'}(x, \varepsilon) \subseteq B_\rho(x, \varepsilon) \subseteq U.$$

So $U \in \mathcal{T}'_\rho$. Then we have $\mathcal{T}_\rho \subseteq \mathcal{T}_{\rho'}$.

Similarly, $U \in \mathcal{T}_{\rho'}$ only if $U \in \mathcal{T}_\rho$. Then we have $\mathcal{T}_{\rho'} \subseteq \mathcal{T}_\rho$.

Above all, $\mathcal{T}_\rho = \mathcal{T}_{\rho'}$. ☐

**Note 1.2.1.** In this proposition, $\mathcal{T}_\rho$ and $\mathcal{T}_{\rho'}$ are said to be homeomorphic or topologically equivalent (see Definition 1.5.2). And $\rho$ and $\rho'$ are also said to be topologically equivalent.

**Example 1.2.4.** In Example 1.2.1, for all $p, q \geq 1$, all $\rho_p$ and $\rho_q$ induce the same topology. Let $X$ be any subset of $\mathbb{R}^n$, then for all $x, y \in X$, if $p < q$, then

$$\rho_p(x, y) \geq \rho_q(x, y).$$

Thus, if $\rho_1$ and $\rho_\infty$ are Lipschitz equivalent, then any other $\rho_p$ and $\rho_q$ are Lipschitz equivalent. We have

$$\rho_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \geq \max_{i \in \{1,\dots,n\}} |x_i - y_i| = \rho_\infty(x, y).$$

Clearly,

$$\rho_\infty(x, y) \leq \rho_1(x, y) \leq n\rho_\infty(x, y).$$

By Definition 1.2.5, $\rho_1$ and $\rho_\infty$ are Lipschitz equivalent, hence for all $p, q \geq 1$, $\rho_p$ and $\rho_q$ are Lipschitz equivalent. Thus, by Proposition 1.2.3, they induce the same topology.

## 1.3 Separation Axioms. From $T_0$ to Hausdorff

**Definition 1.3.1** (saperated)**.** In a topological space, two sets are said to be *separated* iff each is disjoint from other's closure.

**Definition 1.3.2** (separated by neighbourhoods)**.** In a topological space $(X, \mathcal{T})$, two sets $A$ and $B$ are said to be *separated by neighbourhood* iff there are neighbourhoods $N_A$ of $A$ and $N_B$ of $B$ such that $N_A$ and $N_B$ are disjoint.

**Definition 1.3.3** (topologically indistinguishable)**.** Let $(X, \mathcal{T})$ be a topological space. Two points $x, y \in X$ are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let $\mathcal{N}_x$ be the family of all neighbourhoods of $x$ and let $\mathcal{N}_y$ be the family of all neighbourhoods of $y$, we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively, $x, y$ are said to be *topologically distinguishable* iff they are not topologically distinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

**Example 1.3.1.** In an indiscrete topological space, all distinct points are topologically indistinguishable.

## $T_0$ Spaces

**Definition 1.3.4** ($T_0$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_0$ or *Kolmogorov*, iff all distinct points $x, y \in X$ are topologically distinguishable.

**Example 1.3.2.** Let $X$ be any set and let $\mathcal{T}$ be the indiscrete topology on $X$. $(X, \mathcal{T})$ is $T_0$ iff $|X| \in \{0, 1\}$.

## $T_1$ Spaces

**Definition 1.3.5** ($R_0$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $R_0$ iff any two topologically distinguishable points in $X$ are separated.

**Definition 1.3.6** ($T_1$ Spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_1$ or *Fréchet* iff it is $T_0$ and $R_0$; i.e., all distinct pionts $x, y \in X$ are separated.

**Example 1.3.3** ($R_0$ but not $T_0$)**.** Let $\mathcal{T}$ be a countable family of disjoint proper intervals on $\mathbb{R}^n$, and $\bigcup \mathcal{T} = \mathbb{R}^n$. $(X, \mathcal{T})$ is $R_0$, but not $T_0$.

**Example 1.3.4** ($T_0$ but not $R_0$)**.** Let $(\mathbb{R}_{\geq 0}, \mathcal{T})$ be a topological space with

$$\mathcal{T} = \{U \subseteq \mathbb{R} : \forall i \in \mathbb{R}_{\geq 0},\ U_i = [0, i)\},$$

Then for all $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$, if $x \neq y$, then there are $|y - x|$ neighbourhoods $N_x$ of $x$ do not contain $y$. Thus, it is $T_0$.
On the other hand, it is not $R_0$, because for all $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$ with $x < y$, $x \in \overline{\{y\}} = [0, y]$.

**Example 1.3.5** ($R_0$ but not $T_1$)**.** Let $X$ be any set with $|X| \geq 3$, let $U \subsetneq X$ with $|U| \geq 2$, let $\mathcal{T}_{X \setminus U}$ be a $T_1$ topology on $X \setminus U$, and let $\mathcal{T}$

$$\mathcal{T} = \mathcal{T}_{X \setminus U} \cup \{X, U\}.$$

For all $x, y \in X$, if $x \neq y$, then they are separated. Thus, the space is $R_0$.
But $(X, \mathcal{T})$ is not $T_1$, because all $\{u\} \in U$ share the same closure which is $U$ itself.

**Proposition 1.3.1** (alternative definitions of $R_0$ spaces)**.** Let $(X, \mathcal{T})$ be $R_0$, then the following conditions are equivalent.

(i) The closure of all singletons in $X$ are not $T_0$ subspace.

(ii) For any two points $x, y \in X$, $x \in \overline{\{y\}}$ iff $y \in \overline{\{x\}}$.

(iii) Every open set is the union of closed sets.

*Proof.*

(i) By Definition 1.3.5, if $y$ and $x$ are topologically distinguishable, by Definition 1.3.5, $x$ and $y$ are separated; i.e., $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$.

(ii) By Definition 1.3.5, for all $x, y \in X$, $x, y$ are not separated only if they are topologically indistinguishable. By Definition 1.3.3, they share all their neighbourhoods, thus they have the same closure; i.e., $\overline{\{x\}} = \overline{\{y\}}$.

(iii) For any $U \in \mathcal{T}$,

$$U = \bigcup_{x \in U} \{x\}.$$

If $(X, \mathcal{T})$ is $T_1$, then we are done. Suppose $(X, \mathcal{T})$ is not $T_1$, then there exists $A \in \mathcal{T}$ with $|A| > 1$, and for all $B \subsetneq A$, $B \notin \mathcal{T}$ (proof omitted). For such $A$, $X \setminus A$ is open, for $X \setminus A = \bigcup(\mathcal{T} \setminus \{A\})$, thus $A$ is also closed.

Suppose for any such $A$ with $A \cap U \neq \emptyset$, $A \subseteq U$. Suppose it fails, i.e., $A \cap U \neq A$, then we have $A \cap U \subsetneq A$ and $A \cap U \in \mathcal{T}$, which is contradicted to the condition of $A$. Now we have

$$U = \bigcup \mathcal{A} \cup \bigcup_{x \in I} \{x\}$$

where $\mathcal{A}$ is the family of such $A$, and $I$ is the union of all closed singletons in $U$. Thus $U$ is open.

$\square$

**Proposition 1.3.2** (alternative definitions of $T_1$ spaces)**.** Let $(X, \mathcal{T})$ be $T_1$, then the following conditions are equivalent.

(i) All singletons in $X$ are closed.

(ii) Every subset of $X$ is the intersection of all open sets containing it.

(iii) Every cofinite subset of $X$ is open.

*Proof.*

(i) Suppose there exists $\{x\} \subseteq X$ with $\overline{\{x\}} \neq \{x\}$, then there exists $y \in \overline{\{x\}}$ with $x \neq y$. By Definition 1.3.6, this is impossible.

(ii) For any $A \subseteq X$,
$$A = \bigcup_{x \in A} \{x\}.$$
Let $B = X \setminus A$. By De Morgan's law,
$$B = \bigcap_{x \in A} X \setminus \{x\}.$$
$(X, \mathcal{T})$ is $T_1$ iff all $\{x\}$ are closed, in which case, $B$ is the intersection of all open sets $X \setminus \{x\} \supseteq B$.

(iii) Let $A$ be a cofinite subset of $X$. $X \setminus A$ is a finite union of singletons. As $(X, \mathcal{T})$ is $T_1$, any singletons in $X$ is closed. By Proposition 1.1.1, $X \setminus A$ is closed. By Definition 1.1.3, $A$ is open.

$\square$

## Hausdorff Spaces

**Definition 1.3.7** ($R_1$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $R_1$ iff any two topological distinguishable points in $X$ are separated by neighbourhoods.

**Definition 1.3.8** (Hausdorff Spaces)**.** A topological space $(X, \mathcal{T})$ is said to be *Hausdorff* or $T_2$ iff it is $T_0$ and $R_1$; i.e., all distinct points $x, y \in X$ are separated by neighbourhoods.

**Proposition 1.3.3.** All metrizable spaces are Hausdorff

*Proof.* Let $(X, \mathcal{T})$ be a metrizable space. There exists a metric $\rho$ on $X$ that induces $\mathcal{T}$. Given distinct points $x, y \in X$, suppose for all $\varepsilon \in \mathbb{R}_{>0}$, there exists $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. Then $\rho(x, z) < \varepsilon$ and $\rho(y, z) < \varepsilon$. Now we have

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

Put $\rho(x, y) > 2\varepsilon$ as $x$ and $y$ are arbitrarily given. Then we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which implies that $\rho$ is not a metric on $X$. Hence, $(X, \mathcal{T})$ is not metrizable which is contradicted to the condition. $\square$

**Proposition 1.3.4.** All singletons in a Hausdorff space are closed.

*Proof.* Let $(X, \mathcal{T})$ be a Hausdorff space, and let $x \in X$. For all $y \in X$ with $x \neq y$, there is a open neighbourhood $U_y$ of $y$ such that $x \notin U_y$. Then, for all such $U_y$, we have

$$\forall y \in X,\ x \in X \setminus U_y = \{x\} \iff x \in \bigcap_{y \in X \setminus \{x\}} X \setminus U_y = \{x\}.$$

As all $X \setminus U_y$ are closed, their intersection $\{x\}$ is closed. $\square$

**Example 1.3.6** ($T_1$ but not Hausdorff)**.** Let $X$ be a nonempty set, let $p \in X$, let $\mathcal{T}'$ be a Hausdorff topology on $X \setminus \{p\}$, and let

$$\mathcal{T} = \{X\} \cup \mathcal{T}'.$$

Then, all $x \in (X, \mathcal{T})$ are closed, thus $(X, \mathcal{T})$ is Fréchet. But the only neighbourhood of $p$ is $X$, so its closure is $X$. Then, for any $x \in X \setminus \{p\}$, $x$ and $p$ are not separated, in which case $(X, \mathcal{T})$ is not $R_0$. Thus, $(X, \mathcal{T})$ is not Hausdorff.

## 1.4 Continuity

**Definition 1.4.1** (continuous maps)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is said to be *continuous* iff for any open set $U$ in $Y$, its preimage in $X$ under $f$ is open.

**Note 1.4.1.** In Definition 1.4.1, note that even if for any open set $U$ in $X$, $f[X]$ is open in $Y$, $f$ is not necessarily continuous. For example, let $X = (\mathbb{R}, \mathcal{T}_X)$ with $\mathcal{T}_X$ induced by standard Euclidean metric, let $Y = (\mathbb{R}, \mathcal{T}_Y)$ with $\mathcal{T}_Y$ as a indiscrete topology, and define

$$f(x) = [x],$$

where $[x]$ denotes the integer part of $x$. Then for all $U \subseteq X$, $f[U]$ is open in $Y$, but by Definition 1.4.1, $f$ is not continuous.

**Note 1.4.2.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, if $\mathcal{T}_X$ is the discrete topology on $X$, then any function with domain $X$ is continuous. If $\mathcal{T}_Y$ is the indiscrete topology on $Y$, then any function with codomain $Y$ is continuous.

**Note 1.4.3.** A function is continuous bijection does not implies that its inverse is continuous. For example, let $X$ be any set and let $\mathcal{T}$ and $\mathcal{T}'$ be its topologies. If $\mathcal{T}$ is finer than $\mathcal{T}'$, then any bijection $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous. In this case, however, if $\mathcal{T} \neq \mathcal{T}'$, then $f^{-1}$ is not continuous.

**Proposition 1.4.1.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is continuous at $x \in X$ iff for any neighbourhood $N_y$ of $f(x)$, there is a neighbourhood $N_x$ of $x$, such that $f[N_x] \subseteq N_y$.

*Proof.* Let $N_y$ be a neighbourhood of $f(x)$. Clearly, there exists an open set $U_y$ contains $y$.
By Definition 1.4.1, $f$ is continuous at $x$ iff $x \in f^{-1}[U_y] \in \mathcal{T}_X$. Clearly, $f^{-1}[U_y]$ is a neighbourhood of $x$. We have $f[f^{-1}[U_y]] = U_y \subseteq N_y$.
By Proposition 1.1.2, there $U_x$ must contains at least one neighbourhood $N_x$ of $x$, thus, $f[N_x] \subseteq U_y$. □

**Proposition 1.4.2.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be metrizable spaces. A map $f : X \to Y$ is continuous at $p \in X$ iff for any $\varepsilon > 0$, there is a $\delta > 0$, such that for all $x \in B_X(p, \delta)$, $f(x) \in B_Y(f(p), \varepsilon)$, where $B_X$ is defined by any metrics $\rho_X$ induces $\mathcal{T}_X$, and $B_Y$ is defined by any metrics $\rho_Y$ induces $\mathcal{T}_Y$.

*Proof.* Clearly, for all $\varepsilon > 0$, $B_Y(f(x,), \varepsilon)$ is an open neighbourhood of $f(x)$.
$f$ is not necessarily be injective, so $f^{-1}[B_Y(f(x), \varepsilon)] = U \in x$. By Definition 1.4.1, $U$ is open, so for some $\delta > 0$, $B_X(x, \delta) \subseteq U$. Thus, By Proposition 1.4.1, $f$ is continuous iff $f[B_X(x, \delta)] \subseteq B_Y(f(x), \varepsilon)$. This satisfies the conditions we have. □

**Proposition 1.4.3.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is said to be continuous iff for any closed set $V$ in $Y$, its preimage in $X$ under $f$ is closed.

*Proof.* Let $U_Y$ be any open set in $Y$, let $U_X$ be the preimage of $U_Y$ under $f$. By Definition 1.4.1, $U_X$ is open in $X$. Let

$$V_X = f^{-1}[Y \setminus U_Y] = X \setminus U_X,$$

Then $V_X$ is closed. □

**Definition 1.4.2** (convergence of sequences)**.** Let $(X, \mathcal{T})$ be a topological space, and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is said to be *converges* in $X$ iff there is an $x \in X$, such that for any open neighbourhood $U_x$ of $x$, it contains a cofinite subset $A \subseteq \{x_n\}$. That is, there exists $N$ in the domain of $\{x_n\}$, for any natural numbers $n \geq N$, $x_n \in U_x$.

**Example 1.4.1.**

1. In a discrete topological space, a sequence $\{x_n\}$ converges iff there is an $N$ in the domain of $\{x_n\}$, for any natural numbers $m > N$, $x_N = x_m$.

2. In a indiscrete topological space, any sequence $\{x_n\}$ in $X$ converges in $X$. And
$$\lim_{n\to\infty} \{x_n\} = X.$$

**Proposition 1.4.4.** In a Hausdorff space, any convergent sequence converges to a unique point in the space.

*Proof.* Let $(X, \mathcal{T})$ be a Hausdorff space, and let $\{x_n\}$ be a sequence in $X$. Suppose $\{x_n\}$ converges to more than one point, say to $x, y \in X$ with $x \neq y$, then, for all neighbourhoods $N_x$ of $x$ and $N_y$ of $y$, $N_x$ contains a cofinite subset $A \subseteq \{x_n\}$ and $N_y$ contains a cofinite subset $B \subseteq \{x_n\}$. If this were true, $N_x \cap N_y$ should be non-empty, otherwise $N_x$ or $N_y$ should be finite.
Then, $x$ and $y$ are not separated by neighbourhoods, thus $(X, \mathcal{T})$ is not Hausdorff. This is a contradiction.
But, as $(X, \mathcal{T})$ is Hausdorff, there must be mutually disjoint $N_x$ and $N_y$. Thus, the assumption cause a contradiction. □

**Note 1.4.4.** As all metrizable spaces are Hausdorff, so any convergent sequence in a metrizable space converges to at most one point.

**Proposition 1.4.5.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological space, let $f : X \to Y$ be a map, and let $\{x_n\}$ be a convergent sequence in $X$. If $f$ is continuous, then $f[\{x_n\}]$ is a sequence convergent in $Y$.

*Proof.* Let $U_y$ be any open neighbourhood of $f(x)$. By Definition 1.4.1, $f^{-1}[U_y]$ is also an open neighbourhood of $x$. By Definition 1.4.2, $f^{-1}[U_y]$ contains a cofinite subset $A \subseteq \{x_n\}$. Then $f[A]$ is a cofinite subset of $f[\{x_n\}]$. As $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$, $f[\{x_n\}]$ converges in $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$. □

**Note 1.4.5.** In this proposition, even if $f[\{x_n\}]$ converges in $Y$, $f$ might be discontinuous. For example, let $X$ any set, let $\mathcal{T}$ be the indiscrete topology on $X$, let $U$ be another cofinite subset of $X$ with $X \neq U$, and let $\mathcal{T}' = \{\emptyset, X, U\}$. Let $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ be defined by

$$f(x) = x.$$

By Definition 1.4.1, $f$ is not continuous. But, for any convergent sequence $\{x_n\}$ in $(X, \mathcal{T})$, $f[\{x_n\}]$ also convergent in $(X, \mathcal{T})$.

## 1.5 Homeomorphisms

**Definition 1.5.1** (homeomorphisms)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* iff

(i) $f$ is a bijection;

(ii) $f$ is continuous;

(iii) $f^{-1}$ is continuous.

**Definition 1.5.2** (homeomorphic)**.** Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

**Proposition 1.5.1.** Two topological spaces are homeomorphic only if they have the same cardinality.

*Proof.* Let $X$ and $Y$ be two sets with $|X| < |Y|$. There is no surjection from $A$ to $B$. □

**Example 1.5.1.** $|X| = |Y|$ does not imply $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic, even if they are finite. For example, let $X = Y = \{1, \ldots, n\}$, and let $\mathcal{T}_X$ be indiscrete topology on $X$ and $\mathcal{T}_Y = \mathcal{P}(X)$. There is no homeomorphism between $X$ and $Y$.
On the other hand, even if $|X| = |Y| \geq \aleph_0$ and $\mathcal{T}_X$ and $\mathcal{T}_Y$ are induced by same metric, $X$ and $Y$ might not be homeomorphic. For example, if $\mathcal{T}_X$ and $\mathcal{T}_Y$ are both induced by standard Euclidean metric, and $X = [a, b] \subseteq \mathbb{R}$ and $Y = [c, d) \subseteq \mathbb{R}$ where $a < b$ and $c < d$. No doubt, $|X| = |Y| = \mathfrak{c}$, but $X$ and $Y$ are not homeomorphic.

**Example 1.5.2.** $\mathbb{R}^n$ and $\mathbb{R}^m$ $(n < m)$ are not homeomorphic, although $|\mathbb{R}^n| = |\mathbb{R}^m|$.

**Example 1.5.3.** Let $I$ be a proper interval in $\mathbb{R}^n$, let $\mathcal{T}$ be standard Euclidean topology on $\mathbb{R}^n$ and let $\mathcal{T}_I$ be a subspace topology on $I$. $I \cong \mathbb{R}^n$ iff $I$ is an open interval.
But if $\mathcal{T} = \mathcal{P}(\mathbb{R}^n)$, then there exists bijection $f : I \to \mathbb{R}^n$, for $|I| = |\mathbb{R}^n|$, and such $f$ can be bicontinuous, for any subset $A \subseteq I$ is also open in $\mathbb{R}^n$, vise versa. In this case, $I \cong \mathbb{R}^n$ whenever $I$ is a closed, half-close, half-open, or open interval respect to standard Euclidean metric.

**Example 1.5.4.** Let $S^n$ be an $n$-dimensional sphere with center $o \in \mathbb{R}^{n+1}$ and radius $r \in \mathbb{R}$, i.e.,

$$S^n = \left\{x \in \mathbb{R}^{n+1} : \rho(o, x) = r\right\},$$

where $\rho$ is the standard Euclidean metric on $\mathbb{R}^{n+1}$. For any $x \in S^n$, let $U = B(x, \varepsilon) \cap S^n$ where $0 \leq \varepsilon < \max_{x,y \in S^n} \rho(x, y)$ (here $B(x, \varepsilon) = \{x\}$ if $\varepsilon = 0$), then $S^n \setminus U \cong \mathbb{R}^n$.

**Example 1.5.5.** Indeed, $S^1 \setminus \{x\} \cong \mathbb{R}$ where $x \in S^1$. But for any interval $I \in \mathbb{R}$, $S^1 \ncong I$.

## 1.6 Cover and Basis

**Definition 1.6.1** (cover)**.** Let $(X, \mathcal{T})$ be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of $U$ iff the union of all sets in $\mathcal{C}$ is a superset of $U$. That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call $\mathcal{C}$ an *open cover* of $U$.

Let $\mathcal{S} \subseteq \mathcal{C}$, iff the union of $\mathcal{S}$ is still a superset of $U$, then we call $\mathcal{S}$ a *subcover* of $\mathcal{C}$.

**Definition 1.6.2** (basis)**.** Let $(X, \mathcal{T})$ be a topological space, let $U \subseteq X$, and let $\mathcal{B}$ be a open cover of $X$. We call $\mathcal{B}$ a *base* of $X$ iff the union of $\mathcal{B}$ is precisely $U$ itself, i.e.,

$$U = \bigcup \mathcal{B}.$$

**Definition 1.6.3** (synthetic basis)**.** Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a base of $X$. $\mathcal{B}$ is said to be *synthetic* iff for any $A, B \in \mathcal{B}$,

$$A \cap B = \bigcup_{i=1}^{n} B_i, \quad B_i \in \mathcal{B}.$$

**Definition 1.6.4** (generated by basis)**.** Let $X$ be any set and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be its cover. A topology $\mathcal{T}$ on $X$ is said to be *generated* by the base $\mathcal{B}$ iff

(i) for all $U \in \mathcal{T}$, $U$ is the union of $\mathcal{B}$-sets;

(ii) for all $U \in \mathcal{T}$, $U$ is the finite intersection of $\mathcal{B}$-sets.

**Proposition 1.6.1.** Let $(X, \mathcal{T})$ be a topological space be genrated by a base $\mathcal{B}$. For all $U \in \mathcal{T}$, there is a $B \in \mathcal{B}$ such that $U \subseteq \mathcal{B}$.

*Proof.* By Definition 1.6.4, if $\mathcal{T}$ is generated by $\mathcal{B}$, then for all $U \in \mathcal{T}$, there is an finite set $I$, such that

$$U = \bigcap_{i \in I} B_i, \quad B_i \in \mathcal{B}.$$

Thus, for at least one $k \in I$, $U \subseteq B_k$. □

**Proposition 1.6.2.** Let $X$ be any set, and let $\mathcal{T}$ and $\mathcal{T}'$ be its topologies generated by basis $\mathcal{B}$ and $\mathcal{B}'$ respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for any $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $B' \subseteq B$.

*Proof.* If $\mathcal{T}$ is generated by $\mathcal{B}$, then for all $U' \in \mathcal{T}'$,

$$U' = \bigcup_{j \in J} B'_j,$$

where $B_j \in \mathcal{B}$.
As $\mathcal{T}$ is generated by $\mathcal{B}$, then, certainly, $\mathcal{B} \subseteq \mathcal{T}$.

By the conditions we have, $\mathcal{T} \subseteq \mathcal{T}'$ iff for all $B \in \mathcal{B}$, there is $W' \in \mathcal{T}$ such that

$$B = W' = \bigcup_{i \in I} B_i',$$

where $B_i' \in \mathcal{B}'$. Certainly, all such $B_i'$ are contained in $B$. $\square$

**Proposition 1.6.3.** Let $X$ be any set, and let $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is a topology on $X$ iff it generates itself.

*Proof.* If $\mathcal{T}$ is a topology on $X$, then, by Definition 1.6.4, any open set generated by $\mathcal{T}$ is still a member of $\mathcal{T}$. On the other hand, if $\mathcal{T}$ generates itself, then, $\emptyset$ and $X$ must be members of $\mathcal{T}$, and, by Definition 1.6.4, $\mathcal{T}$ is a topology on $X$. $\square$

## 1.7 Interiors and Closures

**Definition 1.7.1** (interiors)**.** The *interior* of a set $A$, denoted $A^\circ$, is defined to be the union of all open subsets of $A$.

**Definition 1.7.2** (closure)**.** The *closure* of a set $A$, denoted $\overline{A}$, is defined to be the intersection of all closed supersets of $A$.

**Definition 1.7.3** (dense sets)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. $A$ is said to be dense, iff $\overline{A} = X$.

**Definition 1.7.4** (nowhere dense sets)**.** A set $A$ is said to be *nowhere dense* iff the interior of its closure is empty.

**Proposition 1.7.1** (properties of interiors)**.** Let $(X, \mathcal{T})$ be any topological space and $A, B \subseteq X$.

(i) (Intensive) $A^\circ \subseteq A$.

(ii) $A$ is open iff $A = A^\circ$.

(iii) (Idempotence) $(A^\circ)^\circ = A^\circ$.

(iv) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(v) $A \subseteq B \implies A^\circ \subseteq B^\circ$.

(vi) If $B$ is open, then $B \subseteq A$ iff $B \subseteq A^\circ$.

*Proof.*

(i) By Definition 1.7.1, naturally, $A^\circ \subseteq A$.

(ii) By Definition 1.1.2, $A^\circ$ is the union of open sets hence it is open. $A$ is open iff it is the union of all open subsets of $A$. Thus $A = A^\circ$.

(iii) $A^\circ$ is open, thus $(A^\circ)^\circ = A^\circ$.

(iv) By Definition 1.7.1, we have

$$\begin{aligned}(A \cap B)^\circ &= \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq A \cap B\right\}\\ &= \left\{\bigcup U : (U \in \mathcal{T} \wedge U \subseteq A) \wedge (U \in \mathcal{T} \wedge U \subseteq B)\right\}\\ &= \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq A\right\} \cap \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq B\right\}\\ &= A^\circ \cap B^\circ.\end{aligned}$$

(v) Clearly, $A^\circ \subseteq A$, thus,

$$A \subseteq B \implies A^\circ \subseteq B$$

Suppose $A^\circ \nsubseteq B^\circ$, then $A^\circ \setminus B^\circ$ is not empty ($\emptyset$ is the subset of any set, so $A^\circ$ is not empty).

Then there exists $x \in A^\circ$ with $x \in \partial B$ ($x \in B$ but $x \notin B^\circ$). Then there exists neighbourhood $N_x \ni x$, and $N_x \cap \partial B \neq \emptyset$. But this is impossible, for $A^\circ \subseteq B$ implies that $A^\circ \cap \partial B = \emptyset$ (This is a straight consequence of $A^\circ \cap \partial A = \emptyset$. See Proposition 1.8.1), so such $N_x$ does not exist. Thus,

$$A^\circ \subseteq B^\circ.$$

(vi) If $B$ is open, then $B = B^\circ$. Then $B \subseteq A$ iff $B^\circ \subseteq A^\circ$.

□

**Proposition 1.7.2** (properties of closures)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

(i) $\overline{A}$ is closed.

(ii) $A$ is closed iff $A = \overline{A}$.

(iii) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

(iv) If $A$ is closed, then $A \supseteq B$ iff $A \supseteq \overline{B}$

*Proof.*

(i) By Definition 1.7.2, $\overline{A}$ is the intersection of closed sets. By Proposition 1.1.1, $\overline{A}$ is closed.

(ii) Proposition 1.1.1 implies that any closed set is the intersection of closed sets, this is precisely what Definition 1.7.2 says.

(iii) $A \subseteq B$ iff $X \setminus A \supseteq X \setminus B$. Then we have

$$X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ$$

Clearly, $(X \setminus A)^\circ$ is the union of all open set disjoint from $A$, then, by De Morgan's laws, $X \setminus (X \setminus A)^\circ$ is the intersection of all closed sets containing $A$. By Definition 1.7.2, we have $(X \setminus A)^\circ = \overline{A}$. Thus

$$\overline{A} \subseteq \overline{B}.$$

(iv) If $A$ is closed, then $A = \overline{A}$. Suppose $B \subseteq A$, then we have

$$\overline{B} \subseteq \overline{A} \iff \overline{B} \subseteq A.$$

□

## 1.8 Boundaries

**Definition 1.8.1** (boundaries)**.** Let $A$ be any set, the *boundary* of $A$, denoted $\partial A$, is defined to be the complement of the interior of $A$ in the closure of $A$; i.e.,

$$\partial A = \overline{A} \setminus A^\circ.$$

**Proposition 1.8.1** (properties of boundaries)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

(i) $\partial A$ is closed.

(ii) $A^\circ \cap \partial A = \emptyset$.

(iii) $\overline{A} = A^\circ \cup \partial A$.

(iv) $A$ is closed iff $\partial A \subseteq A$.

(v) $\partial A$ is nowhere dense.

(vi) $\partial \overline{A} \subseteq \partial A \subseteq \partial A^\circ$.

(vii) $\partial A = \partial(X \setminus A)$.

(viii) $A$ is dense iff $\partial A = X \setminus A^\circ$.

*Proof.*

(i) $\overline{A}$ is closed, and $X \setminus A^\circ$ is also closed. Thus

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (X \setminus A)$$

is closed.

(ii) By Definition 1.8.1, we have

$$\partial A = \overline{A} \setminus A^\circ \iff \partial A \cap A^\circ = \overline{A} \setminus A^\circ \cap A^\circ = \overline{A} \cap \emptyset = \emptyset.$$

(iii) We have

$$\begin{aligned} \partial A = \overline{A} \setminus A^\circ &\iff \partial A \cup A^\circ = \overline{A} \setminus A^\circ \cup A^\circ = \overline{A} \cap (X \setminus A^\circ \cup A^\circ) \\ &\iff \partial A \cup A^\circ = \overline{A} \cap X|_{\text{for } A^\circ \subseteq X} = \overline{A}. \end{aligned}$$

(iv) As $A$ is closed, $A = \overline{A}$ (this can be straightly proved by Definition 1.7.2). By Definition 1.8.1, it is clear that $\partial A \subseteq \overline{A}$, thus $\partial A \subseteq A$.

(v) By Definition 1.7.4, $\partial A$ is nowhere dense iff $\overline{\partial A}^\circ$ is empty. We have

$$\begin{aligned} \overline{\partial A}^\circ &= \overline{\overline{A} \setminus A^\circ}^\circ \\ &= (\overline{A} \setminus A^\circ) \cup (\overline{A} \setminus A^\circ) \setminus (\overline{A} \setminus A^\circ) \\ &= \emptyset. \end{aligned}$$

(vi) $\overline{A} \supseteq A^\circ$ implies $\overline{A}^\circ \supseteq (A^\circ)^\circ = A^\circ$, then we have,

$$\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^\circ \subseteq \overline{A} \setminus A^\circ = \partial A.$$

$A^\circ \subseteq A$ implies $\overline{A^\circ} \subseteq \overline{A}$, then we have,

$$\partial A^\circ = \overline{A^\circ} \setminus (A^\circ)^\circ \supseteq \overline{A} \setminus A^\circ.$$

(vii) We have

$$\begin{aligned}\partial(X \setminus A) &= \overline{X \setminus A} \setminus (X \setminus A)^\circ \\ &= X \setminus A^\circ \setminus (X \setminus \overline{A}) \\ &= X \setminus A^\circ \cap \overline{A} \\ &= \overline{A} \setminus A^\circ \\ &= \partial A.\end{aligned}$$

(viii) By Definition 1.7.3, $A$ is dense in $X$ iff $\overline{A} = X$. Then we have,

$$\begin{aligned}\overline{A} = X &\iff \overline{A} \setminus A^\circ = X \setminus A^\circ \\ &\iff \partial A = X \setminus A^\circ.\end{aligned}$$

□

## 1.9 Limit Points

**Definition 1.9.1** (limit points)**.** Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$. A point $x \in X$ is called a *limit point* of $A$ iff for all neighbourhood $N_x$ of $x$, $N_x \setminus \{x\}$ intersects $A$.

**Proposition 1.9.1.** Let $A$ be any set, and let $x$ be a limit point of $A$, then $x$ is an element of the closure of $A$.

*Proof.* If $A$ is empty, then this is vacuously true. So, suppose $A$ is not empty. By Definition 1.9.1, for all neighbourhood $N_x$ of $x$, $N_x \setminus \{x\} \cap A$ is not empty. Naturally, $N_x \cap A$ is not empty.
Assume that $x \notin \overline{A}$, then $X \setminus \overline{A}$ is a neighbourhood of $x$, by Definition 1.1.5, and is disjoint from $A$. This is contradicted to the conditions. □

**Note 1.9.1.** In this proof, the proposition also holds for $N_x \cap A^\circ = \emptyset$. Because if it is true, then

$$N_x \cap \partial A \supseteq (N_x \cap A) \setminus (N_x \cap A^\circ) = N_x \cap A.$$

This implies that $A \subseteq \partial A$. In this case, $\overline{A} = \partial A$, for
Assume that $x \notin \partial A$, then we have the same conclusion.
Then $A^\circ = A \setminus \partial A = \emptyset$.

**Proposition 1.9.2.** A set is closed iff it contains all its limit point.

*Proof.* Let $A$ be a set. By proposition 1.9.1, for every limit point of $A$, it is also an element of the closure $\overline{A}$. And $A$ is closed iff $A = \overline{A}$. $\square$

**Definition 1.9.2** (convergent sequences)**.** Let $(X, \mathcal{T}_X)$ be a topological space. A sequence $\{x_n\}$ in $X$ is said to be *convergence* in $X$ iff there is an open set $U$ contains all but finite terms of $\{x_n\}$.

# Chapter 2

# Creating New Spaces

## 2.1 Subspaces

**Definition 2.1.1** (subspace topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The *subspace topology* $\mathcal{T}_A$ on $A$ is defined to be the family of the intersections of open sets in $(X, \mathcal{T})$ and $A$. That is,

$$\mathcal{T}_A = \{U \cap A : \; U \in \mathcal{T}\} .$$

## 2.2 Quotient Spaces

**Definition 2.2.1** (quotient topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on $X$. The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\} .$$

## 2.3 Product Spaces

**Definition 2.3.1** (product topologies)**.**

# Chapter 3

# Topological Properties

## 3.1 Cardinal Functions

## 3.2 More on Separation Axioms

**Definition 3.2.1** (saperated sets)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \in \mathcal{P}(X)$.

(i) $A$ and $B$ are said to be *separated* iff each is disjoint from other's closure.

(ii) $A$ and $B$ are said to be *separated by neighbourhoods* iff there are neighbourhoods $N_A$ of $A$ and $N_B$ of $B$ such that $N_A$ and $N_B$ are disjoint.

(iii) $A$ and $B$ are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods $\overline{N}_A$ of $A$ and $\overline{N}_B$ of $B$ such that $\overline{N}_A$ and $\overline{N}_B$ are disjoint.

(iv) $A$ and $B$ are said to be *separated by a continuous function* iff there is a continuous function $f : X \to \mathbb{R}$, such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

(v) $A$ and $B$ are said to be *precisely separated by a continuous function* iff there is a continuous function $f : X \to \mathbb{R}$, such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$

**Definition 3.2.2** ($T_{2^1/_2}$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_{2^1/_2}$ or *Urysohn* iff two distinct points in $X$ are separated by closed neighbourhoods.

**Example 3.2.1** ($T_2$ but not $T_{2^1/_2}$)**.** [1] (Remained as a problem)

[1] See MathPlanet.

**Definition 3.2.3** ($T_3$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_3$ or *regular* iff it is $T_0$ and given any point $x \in (X, \mathcal{T})$ and closed set $V \subseteq X$ with $x \notin V$ are separated by neighbourhoods.

**Definition 3.2.4** ($T_{3^1/_2}$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_{3^1/_2}$, or *Tychonoff* or, *completely* $T_3$, or *completely regular*, iff it is $T_0$ and given any point $x$ and closed set $V \subseteq X$ with $x \notin V$, they are separated by a continuous function.

**Definition 3.2.5** ($T_4$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_4$ or *normal* iff it is Hausdorff and any tow disjoint closed subsets of $X$ are separated by neighbourhoods.

**Proposition 3.2.1** (Urysohn's lemma)**.** A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

**Definition 3.2.6** ($T_5$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_5$ or *completely* $T_4$ iff it is $T_1$ any two separated sets are separated by neighbourhoods.

**Proposition 3.2.2.** Every subspace of a $T_5$ space is normal.

**Definition 3.2.7** ($T_6$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_6$, or *perfectly* $T_4$ or *perfectly normal* iff it is $T_1$ and any two disjoint closed sets are precisely separated by a continuous function.

**Proposition 3.2.3** (Tietze extension theorem)**.** Let $(X, \mathcal{T})$ be normal topological space, and let $f : A \to (\mathbb{R}, \mathcal{T}')$ be a continuous map where $A$ is a closed subset of $X$ and $\mathcal{T}'$ is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$

## 3.3 Countability Axioms

## 3.4 Compactness

**Definition 3.4.1** (compactness)**.** A topological space $(X, \mathcal{T})$ is said to be *compact* iff every open cover of $X$ has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

## 3.5 Connectedness

**Definition 3.5.1** (connectedness)**.** Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is said to be *connected* iff $X$ is not empty and it it not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

**Definition 3.5.2** (path-connectedness)**.** Let $(X, \mathcal{T})$ be a topological space.

(i) A map $\gamma : [0, 1] \to X$ is called a *path* in $X$ iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that $\gamma$ is path from $x$ to $y$ in $X$.

(ii) $X$ is said to be *path-connected* iff for all $x, y \in X$ there is a path from $x$ to $y$ in $X$.