# Notes for Mathematical Analysis

Zhao Wenchuan

June 24, 2021

# Chapter 1

# Set Theory

## 1.1 Countability of Sets

**Definition 1.1.1.** A set $A$ is said to be *countable* iff there is a bijection $f : I \to A$ with $I \subseteq \mathbb{N}$.

$A$ is said to be *uncountable* iff for any injection $f : I \to A$ with $I \supseteq \mathbb{N}$, $f$ is not surjection.

**Definition 1.1.2.** The *cardinal number* of a set $A$, denoted $|A|$ or $\#A$, is defined to the quantity of it elements.

Customarily, we write $\aleph_0$ for $|\mathbb{N}|$, and $\mathfrak{c}$ for $|\mathbb{R}|$.

**Definition 1.1.3.** Given $A$ and $B$ as sets, we define the following:

(i) $|A| = |B|$ iff there is a bijection $f : A \to B$;

(ii) $|A| \leq |B|$ iff there is an injection $g : A \to B$;

(iii) $|A| < |B|$ iff for any injection $g : A \to B$, $g$ is not surjection.

**Definition 1.1.4.** A set $A$ is said to be *finite* iff $|A| < \aleph_0$; it is *infinite* iff $|A| \geq \aleph_0$.

By 1.1.3, $A$ is finite iff for all injection $f : A \to \mathbb{N}$, $f$ is not surjection. Respectively, $A$ is infinite iff there exists injection $f : \mathbb{N} \to A$.

**Proposition 1.1.1.** For any countable set $A$, $|A| \leq \aleph_0$.

*Proof.* If $A$ is finite, i.e., $|A| < \aleph_0$, it is clearly countable.

If $A$ is infinite, by Definition 1.1.4, $|A| \geq \aleph_0$. As $A$ is countable, there must be an bijective $f : I \to A$ with $I \subseteq \mathbb{N}$, then (iii) in Definition 1.1.3 fails, so $|A| = \aleph_0$. □

**Proposition 1.1.2.** For any set $A$, $A$ is uncountable iff $|A| > \aleph_0$.

*Proof.* By Definition 1.1.1, $A$ is uncountable iff for any injection $f : I \to A$ with $I \supseteq \mathbb{N}$, $f$ is not surjection. This holds iff for any $I \subseteq \mathbb{N}$, $|I| < |A|$. $\mathbb{N} \subseteq \mathbb{N}$, Thus $\aleph_0 < |A|$. □

**Proposition 1.1.3.** The subsets of any countable sets are countable.

*Proof.* Clearly, by intuition or by Definition 1.1.3, for any sets $A$ and $B$, $A \subseteq B$ implies $|A| \leq |B|$. By Proposition 1.1.2, $B$ is countable iff $|B| \leq \aleph_0$. Then we have $|A| \leq \aleph_0$. This holds iff $A$ is countable. □

**Proposition 1.1.4.** The super sets of any uncountable sets are uncountable.

*Proof.* Let $A$ be an uncountable set. $|A| > \aleph_0$ implies that for any $B \supseteq A$, $|B| > |A| > \aleph_0$. Thus, by Proposition 1.1.2. □

**Proposition 1.1.5.** The Cartesian product of countable sets is countable.

*Proof.* If $A$ or $B$ is empty, $A \times B$ is empty. The empty set is countable.

Let $A$ and $B$ be both infinite countable, then there exist $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Let $h : \mathbb{N} \to \mathcal{P}(A \times B)$ defined by

$$h(x) = \begin{cases} \{(f_0, g_0)\} & x = 0 \\ \{(f_0, g_1), (f_1, g_0)\} & x = 1 \\ \{(f_0, g_2), (f_1, g_1), (f_2, g_0)\} & x = 2 \\ \{(f_0, g_3), (f_1, g_2), (f_2, g_1), (f_3, g_0)\} & x = 3 \\ \vdots & \vdots \end{cases}$$

Now we have

$$A \times B = \bigcup_{x=0}^{\infty} f(x).$$

Thus,

$$|A \times B| = \left| \bigcup_{x=0}^{\infty} f(x) \right| = \sum_{x=0}^{\infty} (x + 1).$$

Clearly,

$$\aleph_0 = |\{0\} \cup \{1,2\} \cup \{3,4,5\} \cup \dots| = |A \times B|.$$

Thus, $A \times B$ is countable. $\square$

**Proposition 1.1.6.** The countable unions of countable sets is countable.

*Proof.* Similar to Proposition 1.1.5. $\square$

**Proposition 1.1.7.** If $A$ is a countable set but $B$ is not, then $B \setminus A$ is uncountable.

*Proof.* If $B \setminus A$ is countable, $B \setminus A \cup A$ must be countable. Then $B \subseteq B \setminus A \cup A$ is also countable, contradicted to the condition. $\square$

**Proposition 1.1.8.** There is no countably infinite $\mathcal{P}(X)$ for any set $X$.

*Proof.* As $\aleph_0$ is the smallest infinite cardinal number, let $X = \mathbb{N}$. Let

$$\mathcal{U} = \left\{ U = \bigcup_{i=0}^{\infty} \{x_i \in \{2i, 2i+1\}\} \right\} \subseteq \mathcal{P}(X).$$

Suppose $\mathcal{U}$ is countable, then there is a list

$$\mathcal{W} = \bigcup_{k=0}^{\infty} \{U_k = \{x_{k,i}\}\} \supseteq \mathcal{U}.$$

Now construct a new set

$$W = \{w_i\}_{i=0}^{\infty}$$

where for all $i \in \mathbb{N}$, $w_i = 2i$ if $x_{i,i} = 2i+1$, and $w_i = 2i+1$ if $x_{i,i} = 2i$. Now we have $W \in \mathcal{U}$ but $W \notin \mathcal{W}$, which is contradicted to the condition. Thus $\mathcal{U}$ is not countable, thus $\mathcal{P}(X) \supseteq \mathcal{U}$ is not either. $\square$

**Proposition 1.1.9.** $\mathbb{R}$ is uncountable.

*Proof.* Similar to Proposition 1.1.8. $\square$