Notes for Vector Calculus

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Chapter 1.

Directional and Partial Derivatives

§1.1 Directional Derivatives

Definition 1.1.1. Let U be an open set of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$. Let $\vec{u} \in U \setminus \{\vec{0}\}$ and $\vec{x} \in U$.

Then, the *$\vec{u}$-directional derivative of f at $\vec{x}$* is defined as

$$\nabla_{\vec{u}} f(\vec{x}) := \lim_{t \to 0} \frac{f(\vec{x} + t\vec{u}) - f(\vec{x})}{t},$$

if the limit exists in $\mathbb{R}^m$.

Note 1.1.1. By Definition 1.1.1, if we consider $\nabla_{\vec{u}} f$ as a function, the mapping between elements is

$$\vec{x} \in U \mapsto \vec{y} \in \mathbb{R}^m.$$

Thus, $\nabla_{\vec{u}} f : U \to \mathbb{R}^m$, and it can considered as the *$\vec{u}$-directional derived function of f* defined as

$$\nabla_{\vec{u}} f(\vec{x}) := \begin{cases} \lim\limits_{t \to 0} \dfrac{f(\vec{x} + t\vec{u}) - f(\vec{x})}{t} & \text{, if the limit exists in } \mathbb{R}^m; \\ 0 & \text{, otherwise.} \end{cases}$$

Lemma 1.1.1. With the condition above, let $g : \mathbb{R}^1 \to \mathbb{R}^m$ be defined as

$$g(t) := f(\vec{x} + t\vec{u}),$$

then,

$$\nabla f_{\vec{u}} = \frac{\mathrm{d}g}{\mathrm{d}t}.$$

Proof.

$$\begin{aligned}
\nabla_{\vec{u}} f(\vec{x}) &= \lim_{t \to 0} \frac{f(\vec{x} + t\vec{u}) - f(\vec{x})}{t} && \text{(by Definition 1.1.1)} \\
&= \lim_{t \to 0} \frac{g(t) - g(0)}{t} && \text{(by assumption)} \\
&= \frac{\mathrm{d}g(t)}{\mathrm{d}t}.
\end{aligned}$$

Generalized it, we have

$$\nabla_{\vec{u}} f = \frac{\mathrm{d}g}{\mathrm{d}t}.$$

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Note 1.1.2. As $\vec{x} + t\vec{u}$ defines a subset of $\mathbb{R}^n$, $g(t)$ can be considered as a function defined on a new one-dimensional axis with $\vec{x}$ as the new origin $0'$ and $\vec{u}$ as the new unit $1'$.

Let

$$L := \{\vec{x} + t\vec{u} \in \mathbb{R}^n : t \in \mathbb{R}\},$$

then we have

$$g[\mathbb{R}] = f[L] \in \mathbb{R}^m.$$

Lemma 1.1.2. With the condition above, let $s \in \mathbb{R}^1 \setminus \{0\}$, we have

$$s\nabla_{\vec{u}} f = \nabla_{s\vec{u}} f.$$

Proof. Let $g : \mathbb{R}^1 \to \mathbb{R}^m$ be defined as

$$g(t) := f(\vec{x} + t\vec{u}),$$

then we have

$$
\begin{aligned}
s\nabla_{\vec{u}} f(\vec{x}) &= s\frac{\mathrm{d}g(t)}{\mathrm{d}t} && \text{(by lemma 1.1.1)} \\
&= \frac{\mathrm{d}g(t)}{\mathrm{d}t} \cdot \frac{\mathrm{d}st}{\mathrm{d}t} \\
&= \frac{\mathrm{d}g(st)}{\mathrm{d}(t)} && \text{(by chain rule)} \\
&= \lim_{t\to 0} \frac{g(st) - g(t)}{t} \\
&= \lim_{t\to 0} \frac{f(\vec{x} + ts\vec{u}) - f(\vec{x})}{t} && \text{(by assumption)} \\
&= \nabla_{s\vec{u}} f(\vec{x}). && \text{(by Definition 1.1.1)}
\end{aligned}
$$

Generalized it, we have

$$s\nabla_{\vec{u}} f = \nabla_{s\vec{u}} f.$$

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§1.2 Partial Derivatives

Definition 1.2.1. Let U be an open set of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$. Let $\vec{x} \in U$.

The *i-th partial derivative of f at $\vec{x}$* is defined to be the $\hat{e}_i$-directional derivative of f at $\vec{x}$.

Note 1.2.1. Explicitly, by Definition 1.1.1, that is,

$$\nabla_i f(\vec{x}) = \lim_{t\to 0} \frac{f(\vec{x} + t\hat{e}_i) - f(\vec{x})}{t},$$

if the limit exists in $\mathbb{R}^m$. Here, we write ∇_i for $\nabla_{\hat{e}_i}$ for convince.

As $\hat{e}_i$ is the i-th basis of $\mathbb{R}^n$, we can let $\delta = t\hat{e}_i \in \mathbb{R}_i$, then we have

$$\nabla_i f(\vec{x}) = \lim_{\delta\to 0\in\mathbb{R}_i} \frac{f(x_1, \ldots, x_i + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{\delta}.$$

Now, let $g : \mathbb{R}_i \to \mathbb{R}^m$ be defined as

$$g(x_i) := f(x_1, \ldots, x_i, \ldots, x_n),$$

then we have

$$\nabla_i f(\vec{x}) = \lim_{\delta \to 0} \frac{g(x_i + \delta) - g(x_i)}{\delta} = \frac{\mathrm{d}g(x_i)}{\mathrm{d}x_i}.$$

In classical notation, we write

$$\frac{\partial f(\vec{x})}{\partial x_i} \text{ for } \frac{\mathrm{d}g(x_i)}{\mathrm{d}x_i}, \text{ and } \frac{\partial f}{\partial x_i} \text{ for } \frac{\mathrm{d}g}{\mathrm{d}x_i}.$$

§1.3 Gradient

Definition 1.3.1. Let U be an open set of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$. Let $\vec{x} \in U$.

The *gradient of* f *at* $\vec{x}$ is defined as

$$\nabla f(\vec{x}) := (\nabla_1 f(\vec{x}), \ldots, \nabla_n f(\vec{x})).$$

Note 1.3.1.

$$\nabla f : U \to \mathbb{R}^m \times \cdots \times \mathbb{R}^m \ (n \text{ times})$$

(Note the m and n here.)

Lemma 1.3.1. Following the conditions in Definition 1.3.1, we have

$$\nabla f = \frac{\partial f}{\partial \vec{x}},$$

where, in classical notation, $\frac{\partial f}{\partial \vec{x}} = \frac{\mathrm{d}f}{\mathrm{d}\vec{x}}$.

Proof. For any $i \in \{1, \ldots, n\}$, let $g_i : U_i \to \mathbb{R}^m$ be defined as

$$g_i(x_i) := f(x_1, \ldots, x_i, \ldots, x_n).$$

Then we have

$$\begin{aligned}\nabla f(\vec{x}) &= \left(\frac{\mathrm{d}g_1(x_1)}{\mathrm{d}x_1} \dots, \frac{\mathrm{d}g_n(x_n)}{\mathrm{d}x_n}\right) \\ &= (\mathrm{d}g_1(x_1), \dots, \mathrm{d}g_n(x_n)) \cdot \left(\frac{1}{\mathrm{d}x_1}, \dots, \frac{1}{\mathrm{d}x_n}\right) \\ &= \frac{\mathrm{d}f(\vec{x})}{\mathrm{d}\vec{x}}\end{aligned}$$

(Missing Details.) ■

Definition 1.3.2.

$$\mathrm{d}f := \nabla f \cdot \mathrm{d}\vec{x}.$$

Lemma 1.3.2. Following the condition in Definition 1.3.1, let $g : T \to U$, where T is an open subset of $\mathbb{R}$. Then we have

$$\frac{\mathrm{d}f \circ g}{\mathrm{d}t} = \nabla f \cdot \frac{\mathrm{d}g}{\mathrm{d}t}.$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}f(r(t))}{\mathrm{d}t} &= \frac{\mathrm{d}f(r(t))}{\mathrm{d}r(t)} \cdot \frac{\mathrm{d}r(t)}{\mathrm{d}t} \\ &= \nabla f(\vec{x}) \cdot \frac{\mathrm{d}r(t)}{\mathrm{d}t}.\end{aligned}$$

Generalize it, we have

$$\frac{\mathrm{d}f \circ g}{\mathrm{d}t} = \nabla f \cdot \frac{\mathrm{d}g}{\mathrm{d}t}.$$

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Lemma 1.3.3.

$$\nabla_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

Proof. By Definition 1.1.1,

$$\nabla_{\vec{u}} f(\vec{x}) = \lim_{t \to 0} \frac{f(\vec{x} + t\vec{u}) - f(\vec{x})}{t}.$$

Let

$$g(t) := \vec{x} + t\vec{u},$$

then we have

$$\begin{aligned}
\nabla_{\vec{u}} f(\vec{x}) &= \lim_{t \to 0} \frac{f(g(t)) - f(g(0))}{t} \\
&= \left. \frac{\mathrm{d} f(g(t))}{\mathrm{d} t} \right|_{t=0} \\
&= \nabla f(\vec{x}) \cdot \vec{u}.
\end{aligned}$$

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