Notes for General Topology

Zhao Wenchuan

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Chapter 1

Topological Spaces

1.1 Review of Metric Spaces

Definition 1.1.1. Let X be a set. A *metric* on X is a function $\rho : X \times X \to \mathbb{R}_{\geq 0}$, such that $\forall x, y, z \in X$, the following (metric axioms) holds:

M1. $\rho(x, y) = 0 \iff x = y$ (identity of indiscernibles);

M2. $\rho(x, y) = \rho(y, x)$ (symmetry).

M3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (triangle inequality);

A *metric space* is a set together with a metric on it, or more formally, a pair (X, ρ) where X is a set and ρ is a metric on X.

Example 1.1.1.

1. The function $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $\forall p \in \overline{\mathbb{R}}_{\geq 1}$, $\forall x, y \in \mathbb{R}^n$,

$$\rho_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}},$$

is a metric on $\mathbb{R}^n$. Clearly it satisfies identity of indiscernibles and symmetry. For triangle inequality, it is suggested by Minkowski inequality.

Given $x \in \mathbb{R}^3$, $r \in \mathbb{R}_{\geq 0}$, and

$$B_\rho = \left\{ y \in \mathbb{R}^3 \mid \rho(x, y) \leq r \right\}.$$

$\forall p, q \in \overline{\mathbb{R}}_{\geq 1}$, it is true that, $\forall x, y \in \mathbb{R}^n$,

$$p \leq q \implies \rho_p(x,y) \geq \rho_q(x,y).$$

Thus, $B_p \subseteq B_q$.

Geometrically, as $p = 1$, B is a octahedron in $\mathbb{R}^3$ with center x and radius r; as $p = 2$, B is a sphere in $\mathbb{R}^3$ with center x and radius r. It is easy to observe that as $p \to \infty$, B tends to a cube in $\mathbb{R}^3$ with center x and edge length $2r$; i.e.,

$$\rho_\infty(x,y) = \lim_{p\to\infty} \rho_p(x,y) = \sup_{i\in\{1,\ldots,n\}} |x_i - y_i|.$$

2. Let $f : (X, \rho) \to \mathbb{R}^n$ with $X \subseteq \mathbb{R}^m$ be a continuous map on X. Let $x, y \in X$, then $\rho' : f[X] \times f[X] \to \mathbb{R}_{\geq 0}$ defined by

$$\rho'_p(x,y) = \int_0^1 f(\ell(t)) d_p s(t)$$

where

$$\ell(t) = x + t(y - a)$$

and

$$d_p s(t) = \left(\sum_{i=1}^{m} \left| \frac{dg_i}{dt}(t) \right|^p \right)^{\frac{1}{p}} dt.$$

with $p \geq \overline{\mathbb{R}}_{\geq 1}$ is a metric on $f[X]$.

Fix x and given $r \in \mathbb{R}_{\geq 0}$, the set

$$B_p = \left\{ y \in \mathbb{R}^m : \rho'_p(x,y) \leq r \right\}$$

describes a set "attached" on $f[X]$ with center x. If $p = 2$, $m = 2$ and $n = 3$, and $f : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$ is defined by

$$f(\lambda, \phi) = \begin{cases} r\cos\lambda\sin\phi, \\ r\sin\lambda\sin\phi, \\ r\cos\phi, \end{cases}$$

then ρ'_2 here is a *great circle metric* defined by

$$\rho'_2(x,y) = r\arccos(\sin x_\phi \sin y_\phi + \cos x_\phi \cos y_\phi \cos(x_\lambda - y_\lambda)).$$

3. Let $a, b \in \mathbb{R}$ with $a \leq b$, and $p \in \overline{\mathbb{R}}_{\geq 1}$, and $C[a, b]$ denote the set of continuous function $[a, b] \to \mathbb{R}$.

   Then d_p defined by $\forall f, g \in C[a, b]$,

   $$\rho_p(f, g) = \left(\int_a^b |f - g|^p \right)^{\frac{1}{p}}$$

   is a metric on $C[a, b]$.

   Similar to ρ_p on $\mathbb{R}^n$,

   $$B_p = \{g \mid \rho(f, g) \leq r\}$$

   defines a set with "center" f and "radius" $r \in \mathbb{R}_{\geq 0}$.

   It also implies that, on $C[a, b]$, $\forall p, q \in \overline{\mathbb{R}}_{\geq 1}$, $\forall x, y \in \mathbb{R}^n$

   $$p \leq q \implies d_p(f, g) \geq d_q(f, g),$$

   and, naturally, $B_p \subseteq B_q$. This is a straight corollary from the same case of d_p on $\mathbb{R}^n$.

4. Let A be a set. The *Hamming metric* ρ on a set A^n is given by $\forall x, y \in A^n$

   $$\rho(x, y) = \# \{i \in \{1, \ldots, n\} : x_i \neq y_i\}.$$

   An example from Wikipedia. The word "karolin" and "kathrin" can be considered as tuples

   $$x = (\mathrm{k}, \mathrm{a}, \mathrm{r}, \mathrm{o}, \mathrm{l}, \mathrm{i}, \mathrm{n}), \ y = (\mathrm{k}, \mathrm{a}, \mathrm{t}, \mathrm{h}, \mathrm{r}, \mathrm{i}, \mathrm{n}).$$

   For all $i \in \{0, \ldots, 6\} \setminus \{0, 1, 4, 6\}$, $x_i \neq y_i$, and $\#(\{0, \ldots, 6\} \setminus \{0, 1, 4, 6\}) = 3$, thus

   $$\rho(x, y) = 3.$$

5. Let (M, ρ) be a metric space (for example, $\rho = \rho_2$ on $\mathbb{R}^n$), and $X, Y \in \mathcal{P}(M)$. The Hausdorff metric ρ_{H} on $\mathcal{P}(M)$ is defined by

   $$\rho_{\mathrm{H}}(X, Y) = \max \left\{ \sup_{x \in X} \rho(x, Y), \sup_{y \in Y} \rho(X, y) \right\},$$

   where $\rho(a, B) = \inf_{b \in B} \rho(a, b)$ for all $B \in \mathcal{P}(M)$ and $a \in M$.

   This metric can be used to measure how close two figures (as sets of points) are.

Definition 1.1.2. Let X be a metric space, let $x \in X$, and $\varepsilon > 0$. The *open ball with center x and radius ε*, or more briefly the *open ε-ball about x* is the subset

$$B(x, \varepsilon) = \{y \in X : \rho(x, y) \leq \varepsilon\} \subseteq X.$$

Similarly, the *closed ε-ball around x* is

$$\overline{B}(x, \varepsilon) = \{y \in X : \rho(x, y) \leq \varepsilon\} \subseteq X.$$

Note 1.1.1. Clearly, the word "ball" does not mean it should look like a ball. Clearly, for all $x \in \mathbb{R}^3$, the ball $\{y \in \mathbb{R}^3 : \rho_\infty(x, y) < 1\}$ is a cube without its surface.

And it is interesting to think that on $C[a, b]$ with conditions above,

$$\{g \in C[a, b] : \rho_p(f, g) < 1\}$$

defines a open ball in $C[a, b]$.

Note 1.1.2. For hamming metric ρ with conditions above, for $\varepsilon \in \mathbb{R}_{(0,1)}$, the ball

$$\{y \in A^n : \rho(x, y) < 1\} = \{x\}.$$

is a singleton.

Definition 1.1.3. Let X be a metric space.

(i) A subset U of X is *open in X* (or an *open subset of X*) iff

$$\forall u \in U, \ \exists \varepsilon \in \mathbb{R}_{>0}, \ \ B(u, \varepsilon) \subseteq U.$$

(ii) A subset V is *closed in X* iff $X \setminus V$ is open in X.

Note 1.1.3. Equivalently, U is open in X iff $\exists \varepsilon \in \mathbb{R}_{>0}$,

$$U = \bigcup_{x \in U} B(x, \varepsilon);$$

and V is closed in X iff

$$V = X \setminus \bigcup_{x \in U} B(x, \varepsilon) = \bigcap_{x \in U} (X \setminus B(x, \varepsilon)),$$

by De Morgan's Law.

Definition 1.1.4. Let X be a metric space, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and let $x \in X$. Then $\{x_n\}$ *converges* in X iff

$$\exists x \in X, \lim_{n\to\infty} d(x_n, x) = 0.$$

Explicitly, then, $\{x_n\}$ converges to x iff

$$\forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}_{\geq 1}, \forall n \in \mathbb{N}_{\geq N}, \quad d(x_n, x) < \varepsilon.$$

Note 1.1.4.

1. Equivalently, $\{x_n\}$ converges in X iff

   $$\exists x \in X, \ \forall \varepsilon \in \mathbb{R}_{>0}, \#(\{x_n\} \cap B(x, \varepsilon)) = \aleph_0 \wedge \#(\{x_n\} \setminus B(x, \varepsilon)) < \aleph_0.$$

   In other words, $B(x, \varepsilon)$ contains all but finitely many x_n.

2. Let $X \subseteq S$. $\{x_n\}$ converges to $x \in S$ does not means it need to converge in X. For example $\mathbb{Q} \subseteq \mathbb{R}$, the sequence

   $$\left\{ x_n = \frac{1}{x} + r : r^2 = 2 \right\}_{n \in \mathbb{N}}$$

   does converge to $\sqrt{2} \in \mathbb{R}$, but $\sqrt{2} \notin \mathbb{Q}$, so $\{x_n\}$ converges in $\mathbb{R}$, but does not converge in $\mathbb{Q}$.

Lemma 1.1.1. Let X be a metric space and $V \subseteq X$. Then V is closed in X iff

$$\forall \{x_n\}_{n=1}^{\infty} \subseteq V, \ \forall x \in X, \quad \lim_{n\to\infty} d(x_n, x) = 0 \implies x \in V.$$

Proof. Suppose V is closed in X, then $X \setminus V$ is open in X. Suppose $\exists x \in X \setminus V$, such that $\exists \{x_n\}_{n=1}^{\infty} \subseteq V$, $\{x_n\}$ converges to x, then $\forall \varepsilon \in \mathbb{R}_{>0}$, $B(x, \varepsilon) \cap \{x_n\} \neq \emptyset$. $\{x_n\} \subseteq V$, so $B(x, \varepsilon) \cap V \neq \emptyset$. This implies that $X \setminus V$ is not open, then V is not closed (for if V is closed, then $X \setminus V$ is open). It is contradicted to the assumption.

Now, suppose V is not closed in X, then $X \setminus V$ is not open. Then, $\exists p \in X \setminus V$, such that $\forall \varepsilon \in \mathbb{R}_{>0}$, $B(p, \varepsilon) \cap V \neq \emptyset$. This implies there are some $\{x_n\}_{n=1}^{\infty} \subseteq V$, such that $B(p, \varepsilon)$ contains all but finite elements in $\{x_n\}$. Thus, $\{x_n\}$ converges to $p \in X \setminus V$, contradicting to the conditions. $\square$

Lemma 1.1.2. Let X be a metric space, and $\mathcal{T}$ be the family of open subsets of X. Then,

(i) $\mathcal{T}$ is closed under arbitrary union.

(ii) $\mathcal{T}$ is closed under finite intersection.

(iii) $\emptyset, X \in \mathcal{T}$.

Proof.

1. Let I be an index set. For all $i \in I$, let $U_i \in \mathcal{T}$. Then for some $\varepsilon \in \mathbb{R}_{>0}$,

$$U_i = \bigcup_{x \in U_i} B(x, \varepsilon).$$

Let $U = \bigcup_{i \in I} U_i$, then we have,

$$U = \bigcup_{i \in I} \bigcup_{x \in U_i} B(x, \varepsilon) = \bigcup_{x \in U} B(x, \varepsilon)$$

for some $\varepsilon \in \mathbb{R}_{>0}$.

2. Let $\mathcal{C}$ be the family of closed subsets of X, and let $U, V \in \mathcal{C}$. Then for all $\{u_n\}_{n=1}^{\infty} \subseteq U$, $\forall u \in X$, $\{u_n\}$ converges to u implies that $u \in U$. It also holds for $U \cup V \supseteq U$. Similarly, for all $\{v_m\}_{m=1}^{\infty}$, $\forall v \in X$, $\{v_m\}$ converges to v implies $v \in V$. It also holds for $U \cup V \supseteq V$. Thus $U \cup V$ is closed.

   Then, $X \setminus (U \cup V) = (X \setminus U) \cap (X \setminus V)$ is open, where $X \setminus U$ and $X \setminus V$ are open for U and V are closed.

3. $\emptyset = \bigcup_{i \in \emptyset} U_i$ for all $U_i \in \mathcal{T}$, so $\emptyset$ is open. $\emptyset = U \cap V$ for all mutually disjoint closed subsets $U, V \subseteq X$, so $\emptyset$ is closed, so $X = X \setminus \emptyset$ is open.

□

Lemma 1.1.3. Let X be a metric space, and $\mathcal{C}$ be the family of all closed subsets of X. Then,

(i) $\mathcal{C}$ is closed under arbitrary intersection.

(ii) $\mathcal{C}$ is closed under finite union.

(iii) $\emptyset, X \in \mathcal{C}$.

Proof. Let $\mathcal{T}$ be the family of all open subset of X, and let I be any index set.

1. It has been proved that $\mathcal{T}$ is closed under arbitrary union, so by De Morgan's law, for any $i \in I$, if $U_i \in \mathcal{T}$, then
$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i) \in \mathcal{C}.$$
2. It has been proved in Lemma 1.1.2.
3. It has been proved that $\emptyset$ is open in X. So $X = X \setminus \emptyset$ is closed in X.

$\square$

Definition 1.1.5. Let (X, ρ) and (Y, ρ') be metric spaces. A function $f : (X, \rho) \to (Y, \rho)$ is *continuous* on a point $p \in X$ iff
$$\forall \varepsilon \in \mathbb{R}_{>0},\ \exists \delta \in \mathbb{R}_{>0},\ \forall x \in B(p, \delta), \quad f(x) \in B(f(p), \varepsilon).$$

Note 1.1.5.

1. If ρ is a discrete metric on X, then $B(p, \delta) = \{p\}$ for all δ. Then, by definition, for all ε, $f(x) \in B(f(p), \varepsilon)$. So f is continuous everywhere.
2. On the contrary, if ρ' is a discrete metric on Y, but for all $p \in X$, ρ suffices for all $\delta \in \mathbb{R}_{>0}$, $\#B(p, \delta) \geq \aleph_0$, then for some $\varepsilon \in \mathbb{R}_{>0}$, for all $\delta \in \mathbb{R}_{>0}$, there exists $x \in B(p, \delta)$, such that $f(x) \notin B(f(p), \varepsilon)$. Thus f is not continuous on such p.

Lemma 1.1.4. Let (X, ρ) and (Y, ρ') be metric spaces and let $f : (X, \rho) \to (Y, \rho)$ be a function. The following are equivalent:

(i) f is continuous on X;

(ii) for all open $U \subseteq Y$, the preimage $f^{-1}[U] \subseteq X$ is open;

(iii) for all closed $V \subseteq Y$, the preimage $f^{-1}[V] \subseteq X$ is closed.

1.2 The Definition of Topological Space

Definition 1.2.1. Let X be a set. A *topological* on X is a collection $\mathcal{T} \in \mathcal{P}(X)$ with the following properties.

T1. $\mathcal{T}$ is closed under arbitrary union;

T2. $\mathcal{T}$ is closed under finite intersection;

T3. $X \in \mathcal{T}$.

The *Topological Space* $(X, \mathcal{T})$ is a set X with a topology $\mathcal{T}$ on X. All $\mathcal{T}$-sets are said to be *open* in $(X, \mathcal{T})$.

Lemma 1.2.1. $\emptyset \in \mathcal{T}$.

Proof. By T1, given I as any index set, if for all $i \in I$, $U_i \in \mathcal{T}$, then

$$U = \bigcup_{i \in I} U_i \in \mathcal{T}.$$

If $I = \emptyset$, then $U = \emptyset$. □

Note 1.2.1. Let $X = \{1, 2, 3\}$ with topology

$$\mathcal{T} = \big\{\{1, 2\}, \{3\}\big\}.$$

$\{1, 2\} \in \mathcal{T}$ implies $\{3\} = X \setminus \{1, 2\}$ is closed; $\{3\} \in \mathcal{T}$ implies that $\{1, 2\} = X \setminus \{3\}$ is closed. $\{2\} \in \mathcal{P}(X)$, but $\{2\} \notin \mathcal{T}$, so $\{2\}$ is not open in $(X, \mathcal{T})$, $\{1, 3\} = X \setminus \{2\}$ is not closed. For any $U \in \mathcal{T}$, $\{2\} \neq X \setminus U$, so $\{2\}$ is not open.

Definition 1.2.2. Given (X, ρ) as a metric space, the topology

$$\mathcal{T}_\rho = \left\{ U \in \mathcal{P}(X) : U = \bigcup_{x \in U} B(x, \delta) \right\},$$

then we call $\mathcal{T}_\rho$ the topology *induced* by ρ, and $(X, \mathcal{T}_\rho)$ the *underlying topological space* of metric space (X, ρ).

Note 1.2.2. These topology is induced by metric.

1. In this case, U is open in (X, ρ) iff $U \in \mathcal{T}_\rho$.
2. The metric $\rho_p : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{>0}$ (surjective) induces $\mathcal{T}_{\rho_p} \subseteq \mathcal{P}(X)$. And we'll see that for all $p, q \geq 1$, $\mathcal{T}_{\rho_p} = \mathcal{T}_{\rho_q}$.
3. The discrete topology $\rho_{\text{disc}} : X \times X \to \mathbb{R}_{>0}$ (non-sujective) induces $\mathcal{T}_{\rho_{\text{disc}}} = \mathcal{P}(X)$. It is the largest topology on X, and $\rho[X \times X] \subseteq \{0, 1\}$.
4. The metric $\rho_p : C[a, b] \times C[a, b] \to \mathbb{R}_{>0}$ (surjective) induces $\mathcal{T}_{\rho_p} \subseteq \mathcal{P}(X)$. And we'll see that $\mathcal{T}_{\rho_1} \neq \mathcal{T}_{\rho_\infty}$.

5. Given X as a space, the Hausdorff metric $\rho_H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_{>0}$ (surjective) induces $\mathcal{T}_{\rho_H} \subseteq \mathcal{P}(\mathcal{P}(X))$.

6. Given A as a set, the hamming metric $\rho : A^n \times A^n \to \mathbb{R}_{>0}$ (non-surjective) with $n \in \mathbb{N}$ induces $\mathcal{T}_\rho \subseteq \mathcal{P}(X)$. $\rho[A^n \times A^n] = \mathbb{N}_{\leq n}$.

These topology is not induced by any metric.

1. The indiscrete topology $\mathcal{T} = \{\emptyset, X\}$ on X is not induced by any metric space. Suppose it was, then there would be a metric ρ such that for all $x \in X$, for all $\varepsilon > 0$, $B(x, \varepsilon) \in \mathcal{T}$. But, clearly, for those $\varepsilon \in (0, \phi X)$, $B(x, \varepsilon) \notin \mathcal{T}$.

2. Let $X = \{1, 2, 3\}$ with topology

$$\mathcal{T} = \big\{\{1, 2\}, \{3\}\big\}.$$

   These is no such metic ρ induces $\mathcal{T}$ for same reason.

Definition 1.2.3. Let X be a set and $\mathcal{T}, \mathcal{T}'$ be topologies on X. If $\mathcal{T} \subseteq \mathcal{T}'$, then we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$, or $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

Note 1.2.3.

1. Given X as a set, for all topology $\mathcal{T}$ on X, $\mathcal{T} \subseteq \mathcal{T}_{\text{disc}}$ and $\mathcal{T} \supseteq \mathcal{T}_{\text{indisc}}$, Thus, $\mathcal{T}_{\text{disc}}$ is the finest topology on X, and $\mathcal{T}_{\text{indisc}}$ is the coarsest.

2. ρ_p and ρ_{disc} induced same topology on $\mathbb{Z}$. But on $\mathbb{Q}$, $\mathcal{T}_{\rho_p}$ is coarser than $\mathcal{T}_{\rho_{\text{disc}}}$.

Definition 1.2.4. Given $(X, \mathcal{T})$ as a topological space, a set $V \subseteq X$ is said to be *closed* in $(X, \mathcal{T})$ iff $X \setminus V \in \mathcal{T}$.

Definition 1.2.5.

1. In the discrete topology on X, all subsets are closed. Because for all $U \in \mathcal{T}_{\text{disc}}$, $X \setminus U \in \mathcal{T}_{\text{disc}}$.

2. In the indiscrete topology on X, only $\emptyset$ and X is closed.

Lemma 1.2.2. Let $X = (X, \mathcal{T})$ be a topological space, and let

$$\mathcal{C} = \{V \subseteq X : V = X \setminus U,\ U \in \mathcal{T}\}.$$

(i) $\mathcal{C}$ is closed under arbitrary intersection;

(ii) $\mathcal{C}$ is closed under finite intersection;

(iii) $\emptyset, X \in \mathcal{C}$.

Proof.

(i). By De Morgan's laws,

$$V = X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i).$$

So, if $U_i \in \mathcal{T}$, then $V \in \mathcal{C}$.

(ii). By De Morgan's laws,

$$V = X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i).$$

(iii).

$$\emptyset = X \setminus X, \ X = X \setminus \emptyset.$$

□

Definition 1.2.6. Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. An *open neighbourhood* of x is a set $N_x \in \mathcal{T}$ with $x \in N_x$. A *neighbourhood* of x is any $N'_x \supseteq N_x$.

Note 1.2.4. Given $(X, \mathcal{T})$ as a topological space. If $A \in \mathcal{T}$,

$$A = \bigcup_{x \in A} B, \quad B \in x, \text{ and } B \in \mathcal{T}.$$

If $\mathcal{T} = \mathcal{T}_\rho$ for some metric ρ on X, then $A \in \mathcal{T}$ implies

$$A = \bigcup_{x \in A} B(x, \varepsilon)$$

for some $\varepsilon > 0$.

Lemma 1.2.3. Let $(X, \mathcal{T})$ be a topological space and $U \subseteq X$. Then $U \in \mathcal{T}$ iff for all $x \in U$, there is a neighbourhood $N'_x \subseteq U$.

Proof. If $U \ni x$ and $U \in \mathcal{T}$, then U is an open neighbourhood of x, naturally, it is a neighbourhood of x.

For only if, clearly, if for all $x \in U$, there is a neighbourhood $N'_x \subseteq U$, then, by definition, there is $N_x \subseteq N'_x$ with $N_x \in \mathcal{T}$. Now we have $x \in N_x \subseteq N'_x \subseteq U$, then,

$$U = \bigcup_{x \in U} N_x.$$

By definition, $\mathcal{T}$ is closed under arbitrary union, thus U is open. $\square$

1.3 Metrics versus Topologies

Definition 1.3.1. Let X be a set, and let ρ and ρ' be metrics on X. We say that ρ and ρ' are *topologically equivalent* if they induce the same topology on X.

Definition 1.3.2. ρ and ρ' are *Lipschitz equivalent* iff there exist $c, C \in \mathbb{R}_{>0}$ such that for all $x, y \in X$,

$$c\rho(x,y) \leq \rho'(x,y) \leq C\rho(x,y).$$

Lemma 1.3.1. Lipschitz equivalence implies topological equivalence.

Proof. As ρ and ρ' are Lipschitz equivalent, by definition, there exist $c \in \mathbb{R}_{>0}$ such that for all $x, y \in X$,

$$c\rho(x,y) \leq \rho'(x,y).$$

Given $r > 0$ and $x \in X$,

$$B_{c\rho}(x,r) = \{y \in X : c\rho(x,r) < r\}$$

and

$$B_{\rho'}(x,r) = \{y \in X : \rho'(x,r) < r\}.$$

As r is non-underestimated compared to ρ', then

$$B_{\rho'}(x,r) \supseteq B_{c\rho}(x,r) = B_{\rho}\left(x, \frac{1}{c}r\right)$$

is an open neighbourhood of x in (X, ρ') and is a subset

Let $U \in \mathcal{T}_{\rho'}$, then for some $\varepsilon > 0$,

$$U \supseteq B_{\rho'}(x, \varepsilon) \supseteq B_\rho \left(x, \frac{1}{c} r \right).$$

Thus U is open with respect to ρ, i.e., $U \in \mathcal{T}_\rho$.

It is not necessary to prove converse for there always exists $C \in \mathbb{R}_{>0}$ such that $c = \frac{1}{C}$. □

Note 1.3.1.

1. For all $p \geq 0$, $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are topologically equivalent.
2. On $C[a, b]$, ρ_1 and ρ_∞ induce different topologies, hence they are not topologically equivalent, and in particular, they are not Lipschitz equivalent. As Lipschitz equivalence implies topological equivalence, but not vice versa. So Lipschitz in-equivalence do nothing to the proof the topological in-equivalence between ρ_1 and ρ_∞.
3. ρ_p and ρ_{disc} on $\mathbb{Z}$ are topologically equivalent. Firstly, topology $\mathcal{T}_{\rho_p} = \mathcal{P}(\mathbb{Z})$, because for all $B_{\rho_p}(x, \varepsilon)$ for all $x \in \mathbb{Z}$ and $\varepsilon \in \mathbb{R}_{(0,1)}$, $B_{\rho_p}(x, \varepsilon) = \{x\}$. Thus, for all, $U \in \mathcal{P}(\mathbb{Z})$,

$$U = \bigcup_{x \in U} B_{\rho_p}(x, \varepsilon) = \bigcup_{x \in U} \{x\} \in \mathcal{T}_{\rho_p}.$$

Thus $\mathcal{P}(\mathbb{Z}) \subseteq \mathcal{T}_{\rho_p}$, but $\mathcal{T}_{\rho_p} \subseteq \mathcal{P}(\mathbb{Z})$, so $\mathcal{P}(\mathbb{Z}) = \mathcal{T}_{\rho_p}$. Thus $\mathcal{T}_{\rho_p} = \mathcal{T}_{\text{disc}}$.

Definition 1.3.3. A topological space $(X, \mathcal{T})$ is *metrizable* iff $\mathcal{T}$ is induced by some metric on X.

Note 1.3.2.

1. Let $(\mathbb{Z}, \mathcal{T})$ with

$$\mathcal{T} = \{U \in \mathcal{P}(\mathbb{Z}) : |U| \leq 1\},$$

Then $\mathcal{T}$ is not induced by any metric. Suppose it were, then all open set $U \in \mathcal{T}$ should be monotone, and for all $\varepsilon > 0$, and for all $x \in \mathbb{Z}$, $B(x, \varepsilon)$ should be monotone. But if $\mathcal{T}$ is induced by some metric, then for all $I \in \mathcal{P}(X)$ with $|I| > 1$, a set

$$W = \bigcup_{x \in I} B(x, \varepsilon) \in \mathcal{T},$$

then $|W| > 1$, which is contradicted to the conditions.

Definition 1.3.4.

(i) A topological space $(X, \mathcal{T})$ is said to be T_1 iff every monotone in $\mathcal{P}(X)$ is closed.

(ii) A topological space $(X, \mathcal{T})$ is said to be T_2 or *Hausdorff* iff

$$\forall x, y \in X \ (x \neq y),\ \exists U, W \in \mathcal{T} \ (U \cap W = \emptyset), \quad x \in U \wedge y \in W.$$

Note 1.3.3.

1. $(X, \mathcal{T}_{\rho_{\text{disc}}})$ is T_1, for as any set $U \subseteq X$ is closed for $X \setminus U \in \mathcal{T}_{\rho_{\text{disc}}}$ as well. It is also Hausdorff, because for all $x, y \in X$, $\{x\}, \{y\} \in \mathcal{T}_{\rho_{\text{disc}}}$ and $\{x\} \cap \{y\} = \emptyset$ if $x \neq y$.

2. On the other hand, $(X, \{\emptyset, X\})$ is T_1 iff $|X| = 1$. And $(X, \{\emptyset, X\})$ is not Hausdorff, because there exist $x, y \in X$ with $x \neq y$, the only open set contains x is X, and the only open set contains y is X. Clearly, $X \cap X$

Lemma 1.3.2.

(i) Every metrizable space is Hausdorff.

(ii) Every Hausdorff topological space is T_1.

Proof.

(i) Let (X, ρ) be metric space, then for all $x, y \in X$, let $r = \frac{\rho(x,y)}{2}$. Suppose (X, ρ) is not Hausdorff, i.e., there is $z \in B(x, r) \cap B(y, r)$. By metric axioms, we have

$$\rho(x, z) + \rho(y, z) \geq \rho(x, y) = 2r.$$

But $z \in B(x, r)$ implies that $\rho(x, z) < r$, and $z \in B(y, r)$ implies that $\rho(y, z) < r$, then we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which is contradicted to the metric axioms.

(ii) (Just an outline...) Let $(X, \mathcal{T})$ be Hausdorff. Suppose X is not T_1, then there is $\{x\} \subseteq X$ which is not closed. Then there must be a smallest $V \supsetneq \{x\}$ which is closed (Why?). Then there must be a smallest $U \in \mathcal{T}$ with $U \supseteq V$ (Why?). Then for all $x, y \in U$, there is no disjoint U_x, U_y such that $U_x \ni x$ and $U_y \ni y$.

$\square$

Definition 1.3.5. Let $(X, \mathcal{T})$ be a topological space, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X, and let $x \in X$. Then $\{x_n\}$ *converges* in X iff there is an $x \in X$, for all $U \in \mathcal{T}$ with $x \in U$, U contains all but finite elements in $\{x_n\}$.

Note 1.3.4.

1. If $(X, \mathcal{T})$ is metrizable, i.e., there is a metric ρ can induce $\mathcal{T}$. If $\{x_n\} \subseteq X$ converges in X, then there exists $x \in X$, for all $\varepsilon > 0$, $B(x, \varepsilon)$ contains all but finite elements in $\{x_n\}$.
2. If $\mathcal{T}$ is a discrete topology, a sequence $\{x_n\}$ converges in $(X, \mathcal{T})$ iff there is an N such that for all $n \geq N$, $x_n = x_{n+1}$.
3. If $\mathcal{T}$ is an indiscrete topology, then any $\{x_n\} \subseteq X$ converges to any point in X, for there is only one non-empty open set which is X itself.

Lemma 1.3.3. In Hausdorff topological space, any convergent sequence converges to at most one point.

Proof. Let $(X, \mathcal{T})$ be a Hausdorff topological space. Suppose there is a sequence $\{x_n\}$ converges to $x, y \in X$ with $x \neq y$. By the definition of topological convergence, there are $U_x, U_y \in \mathcal{T}$ both contains all but finite elements in $\{x_n\}$. $U_x \cap U_y$ must be non-empty (Explain!). $x, y \in U_x \cap U_y$, for if they were not, by Hausdorff property, there must be open $V_x \subseteq U_x$ and $V_y \subseteq U_x$ with $V_x \ni x$ and $V_y \ni y$, and they both contains all but finite elements in $\{x_n\}$, which is not possible. Thus, there is no such open sets $V_x \ni x$ and $V_y \ni y$ with $V_x \cap V_y = \emptyset$, which implies $(X, \mathcal{T})$ is not Hausdorff. This is a contradition. $\square$

Definition 1.3.6.

(i) A topological space $(X, \mathcal{T})$ is *regular* iff for all closed sets $V \subseteq X$ and $x \in X$ with $x \notin V$, there exist disjoint open sets $U, W \subseteq X$ such that $V \subseteq U$ and $x \in W$.

(ii) $(X, \mathcal{T})$ is *normal* iff for all disjoint closed sets $V, Z \subseteq X$, there exist disjoint open sets $U, W \subseteq X$ such that $V \subseteq U$ and $Z \subseteq W$.

Note 1.3.5 (To do)**.**

1. Can I find a regular space which is not normal?

2. Can I find a normal space which is not regular?

3. Does regular implies normal or normal implies regular?

1.4 Continuous Maps

Definition 1.4.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is *continuous* iff

$$\forall U \in \mathcal{T}_Y, \quad f^{-1}[U] \in \mathcal{T}_X.$$

Note 1.4.1.

1. Find an "anti-intuitive" example.