Notes for General Topology

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Contents

Chapter 1

Topological Spaces

1.1 Topological Spaces

Definition 1.1.1 (topology)**.** Let X be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is called a topology on X iff

(i) $\emptyset, X \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under arbitrary union;

(iii) $\mathcal{T}$ is closed under finite intersection.

Definition 1.1.2 (topological spaces)**.** Let X be any set, and let $\mathcal{T}$ be a topology on X, then the pair $(X, \mathcal{T})$ is called a *topological space*. All subsets of X in $\mathcal{T}$ are called *open sets* in $(X, \mathcal{T})$.

Definition 1.1.3 (closed sets)**.** Let $(X, \mathcal{T})$ be a topological space. A subset V of X is said to be *closed* iff there is an open set U in X such that

$$V = X \setminus V.$$

Proposition 1.1.1. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C}$ be the family of all closed sets in X. Then

(i) $\emptyset, X \in \mathcal{C}$;

(ii) $\mathcal{C}$ is closed under arbitrary intersection;

(iii) $\mathcal{C}$ is closed under finite union.

Proof.

(i) $X \in \mathcal{T}$ implies $X \setminus X = \emptyset \in \mathcal{C}$; and $\emptyset \in \mathcal{T}$ implies $X \setminus \emptyset = X \in \mathcal{C}$;

(ii) As $\mathcal{T}$ is closed under arbitrary union, then by Definition 1.1.3 and De Morgan's Law, $\mathcal{C}$ is closed under arbitrary intersection.

(iii) As $\mathcal{T}$ is closed under finite intersection, then by Definition 1.1.3 and De Morgan's Law, $\mathcal{C}$ is closed under finite union.

□

Definition 1.1.4 (finer and coarser topology)**.** Let X be any set, and let $\mathcal{T}, \mathcal{T}'$ be topologies on X. $\mathcal{T}$ is said to be *finer* than $\mathcal{T}'$ iff $\mathcal{T} \supseteq \mathcal{T}'$; respectively, $\mathcal{T}$ is said to be *coarser* than $\mathcal{T}'$ iff $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 1.1.5 (neighbourhood)**.** Given $(X, \mathcal{T})$ as a topological space and a point $x \in X$, a subset $N \subseteq X$ is called a *neighbourhood* iff it contains an open set U containing x.

Proposition 1.1.2. Given $(X, \mathcal{T})$ as a topological space and $U \subseteq X$, U is open iff for all $x \in U$, there is a neighbourhood N of x contained in U.

Proof. If U is open, then U itself is a neighbourhood of x contained in U.

Conversely, if for all $x \in U$, there is a neighbourhood N_x of x contained in U, then there is a open neighbourhood $U_x \ni x$ contained in N_x. Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose U is not open, then U is a proper superset in the relation above. Then there exists $y \in U$ which is not in any U_x. This implies that such a y does not have any neighbourhood N_y in U, for such an N_y must contains an open $U_y \ni y$. For if it does, then there must be a U_x contains y. This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open. □

1.2 Metrizability

Definition 1.2.1 (metric spaces)**.** Let X be any set. A *metric* ρ on X is a function $\rho : X \times X \to \mathbb{R}$ satisfying the following conditions: for all $x, y, z \in X$

(i) $\rho(x,y) \geq 0$, and $\rho(x,y) = 0$ iff $x = y$;

(ii) $\rho(x,y) = \rho(y,z)$;

(iii) $\rho(x,z) + \rho(z,y) \geq \rho(x,y)$.

Definition 1.2.2 (balls)**.** Let (X,ρ) be a metric space, let $x \in X$, and let $\varepsilon \in \mathbb{R}_{>0}$. The *open ε-ball about x* or just *ε-ball about x* is defined to be

$$B(x,\varepsilon) = \{y \in X : \rho(x,y) < \varepsilon\}.$$

The *closed ε-ball about x* is defined to be

$$\overline{B}(x,\varepsilon) = \{y \in X : \rho(x,y) \leq \varepsilon\}.$$

Example 1.2.1. Let X be any set, and let metric ρ_p on X^n ($n \in \mathbb{Z}_{>0}$) defined by

$$\rho_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}},$$

where $p \in \mathbb{R}_{\geq 1}$. ρ_2 is so called the *standard Euclidean metric.* If $X = \mathbb{R}$, then the metric space $(\mathbb{R}^n, \rho_2)$ is so-called *Euclidean n-space.*

For all $p, q \in \mathbb{R}_{\geq 1}$, if $p < q$, then for all $\varepsilon \in \mathbb{R}_{>0}$ and for all $x, y \in X$, $\rho_p(x,y) \geq \rho_q(x,y)$; in particular, $\rho_p = \rho_q$ iff there is a unique $k \in \{1, \ldots, n\}$, such that for all $i \in \{1, \ldots, n\} \setminus \{k\}$, $x_i = 0$. As $\rho_p(x,y)$ is always "overestimated" than $\rho_q(x,y)$, we have $B_{\rho_p}(x,\varepsilon) \supseteq B_{\rho_q}(x,\varepsilon)$.

Example 1.2.2. Let X be any set. The *discrete metric* ρ on X is defined to be

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Example 1.2.3. Let $a, b \in \mathbb{R}$ with $a < b$, and let metric ρ_p on $C[a,b]$ defined by

$$\rho_p(f,g) = \left(\int_a^b |f(t) - g(t)|^p dt\right)^{\frac{1}{p}},$$

where $p \geq 1$. In particular,

$$\rho_\infty(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|.$$

Proposition 1.2.1. Let (X,ρ) be a metric space, then for all $x, y \in X$ ($x \neq y$), there is an $\varepsilon > 0$ such that $B(x,\varepsilon) \cap B(y,\varepsilon) = \emptyset$.

Proof. Suppose for all $\varepsilon > 0$, $B(x,\varepsilon) \cap B(y,\varepsilon) \neq \emptyset$, then there must be a $z \in X$ such that $z \in B(x,\varepsilon) \cap B(y,\varepsilon)$. $z \in B(x,\varepsilon)$ only if $\rho(x,z) < \varepsilon$, and $z \in B(y,\varepsilon)$ only if $\rho(z,y) < \varepsilon$. Thus

$$\rho(x,z) + \rho(y,z) < 2\varepsilon.$$

As the assumption holds for all $\varepsilon > 0$, we may put

$$\varepsilon = \frac{\rho(x,y)}{2}.$$

Then, we have

$$\rho(x,z) + \rho(y,z) < \rho(x,y),$$

which is impossible. □

Definition 1.2.3 (induced topologies)**.** Let (X,ρ) be a metric space. A topology $\mathcal{T}$ on X is said to be *induced* by ρ iff for all $\varepsilon > 0$, any $U \in \mathcal{T}$ is the union of ball(s) in X; i.e.,

$$\mathcal{T} = \left\{ U \subseteq X : U = \bigcup_{x \in X} B(x,\varepsilon) \right\}.$$

In this case, $\mathcal{T}$ is called the *underlying topology* of ρ.

Definition 1.2.4 (metrizable spaces)**.** Let $(X,\mathcal{T})$ be a topological space. If there is any ρ induce $\mathcal{T}$, then $(X,\mathcal{T})$ is said to be *metrizable*.

Definition 1.2.5 (Lipschitz equivalence)**.** Let X be any set, and let ρ and ρ' be metrics on X. ρ and ρ' are said to be *Lipschitz equivalent* iff there exist $c, C > 0$, such that for all $x, y \in X$,

$$c\rho(x,y) \leq \rho'(x,y) \leq C\rho(x,y).$$

Proposition 1.2.2. Lipschitz equivalence is an equivalence relation.

Proof. Clearly, Definition 1.2.5 also holds for $\rho = \rho'$. So, Lipschitz equivalence is reflexive. In Definition 1.2.5, the relation also holds for $\frac{1}{C}\rho' \leq \rho \leq \frac{1}{c}\rho'$. So Lipschitz equivalence is symmetric.

If there is another ρ'' be Lipschitz equivalent to ρ', then there is $r, R > 0$, such that for all $x, y \in X$,

$$r\rho''(x,y) \leq \rho'(x,y) \leq R\rho''(x,y).$$

By the conditions in Definition 1.2.5, we have

$$\frac{c}{r}\rho(x,y) \le \rho''(x,y) \le \frac{C}{R}\rho(x,y),$$

i.e., ρ and ρ'' are also Lipschitz equivalent. So Lipschitz equivalence is transitive.

Above all, Lipschitz equivalence is an equivalence relation. □

Proposition 1.2.3. Let X be any set, and let ρ and ρ' be metrics on X. If ρ and ρ' are Lipschitz equivalent, then ρ and ρ' induce the same topology.

Proof. As ρ and ρ' are Lipschitz equivalent, by Definition 1.2.5, there is a $c > 0$ such that for all $x, y \in X$,

$$c\rho(x,y) \le \rho'(x,y).$$

Given $r \in \mathbb{R}_{>0}$ and for all $x \in X$, we have

$$B_{\rho'}(x, cr) \subseteq B_{c\rho}(x, r) = B_{\rho}\left(x, \frac{1}{c}r\right).$$

For all $U \in \mathcal{T}_\rho$, for all $x \in U$, there is an $\varepsilon \in \mathbb{R}_{>0}$, such that

$$B_{\rho'}(x,\varepsilon) \subseteq B_\rho(x,\varepsilon) \subseteq U.$$

So $U \in \mathcal{T}'_\rho$. Then we have $\mathcal{T}_\rho \subseteq \mathcal{T}_{\rho'}$.

Similarly, $U \in \mathcal{T}_{\rho'}$ only if $U \in \mathcal{T}_\rho$. Then we have $\mathcal{T}_{\rho'} \subseteq \mathcal{T}_\rho$.

Above all, $\mathcal{T}_\rho = \mathcal{T}_{\rho'}$. □

Note 1.2.1. In this proposition, $\mathcal{T}_\rho$ and $\mathcal{T}_{\rho'}$ are said to be homeomorphic or topologically equivalent (see Definition 1.5.2). And ρ and ρ' are also said to be topologically equivalent.

Example 1.2.4. In Example 1.2.1, for all $p, q \ge 1$, all ρ_p and ρ_q induce the same topology. Let X be any subset of $\mathbb{R}^n$, then for all $x, y \in X$, if $p < q$, then

$$\rho_p(x,y) \ge \rho_q(x,y).$$

Thus, if ρ_1 and ρ_∞ are Lipschitz equivalent, then any other ρ_p and ρ_q are Lipschitz equivalent. We have

$$\rho_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| \ge \max_{i\in\{1,\ldots,n\}} |x_i - y_i| = \rho_\infty(x,y).$$

Clearly,

$$\rho_\infty(x,y) \le \rho_1(x,y) \le n\rho_\infty(x,y).$$

By Definition 1.2.5, ρ_1 and ρ_∞ are Lipschitz equivalent, hence for all $p, q \ge 1$, ρ_p and ρ_q are Lipschitz equivalent. Thus, by Proposition 1.2.3, they induce the same topology.

1.3 Separation Axioms. From T_0 to Hausdorff

Definition 1.3.1 (saperated)**.** In a topological space, two sets are said to be *separated* iff each is disjoint from other's closure.

Definition 1.3.2 (separated by neighbourhoods)**.** In a topological space $(X, \mathcal{T})$, two sets A and B are said to be *separated by neighbourhood* iff there are neighbourhoods N_A of A and N_B of B such that N_A and N_B are disjoint.

Definition 1.3.3 (topologically indistinguishable)**.** Let $(X, \mathcal{T})$ be a topological space. Two points $x, y \in X$ are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let $\mathcal{N}_x$ be the family of all neighbourhoods of x and let $\mathcal{N}_y$ be the family of all neighbourhoods of y, we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively, x, y are said to be *topologically distinguishable* iff they are not topologically distinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

Example 1.3.1. In an indiscrete topological space, all distinct points are topologically indistinguishable.

T_0 Spaces

Definition 1.3.4 (T_0 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_0 or *Kolmogorov*, iff all distinct points $x, y \in X$ are topologically distinguishable.

Example 1.3.2. Let X be any set and let $\mathcal{T}$ be the indiscrete topology on X. $(X, \mathcal{T})$ is T_0 iff $|X| \in \{0, 1\}$.

T_1 Spaces

Definition 1.3.5 (R_0 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be R_0 iff any two topologically distinguishable points in X are separated.

Definition 1.3.6 (T_1 Spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_1 or *Fréchet* iff it is T_0 and R_0; i.e., all distinct pionts $x, y \in X$ are separated.

Example 1.3.3 (R_0 but not T_0)**.** Let $\mathcal{T}$ be a countable family of disjoint proper intervals on $\mathbb{R}^n$, and $\bigcup \mathcal{T} = \mathbb{R}^n$. $(X, \mathcal{T})$ is R_0, but not T_0.

Example 1.3.4 (T_0 but not R_0)**.** Let $(\mathbb{R}_{\geq 0}, \mathcal{T})$ be a topological space with

$$\mathcal{T} = \{U \subseteq \mathbb{R} : \forall i \in \mathbb{R}_{\geq 0},\ U_i = [0, i)\},$$

Then for all $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$, if $x \neq y$, then there are $|y - x|$ neighbourhoods N_x of x do not contain y. Thus, it is T_0.
On the other hand, it is not R_0, because for all $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$ with $x < y$, $x \in \overline{\{y\}} = [0, y]$.

Example 1.3.5 (R_0 but not T_1)**.** Let X be any set with $|X| \geq 3$, let $U \subsetneq X$ with $|U| \geq 2$, let $\mathcal{T}_{X \setminus U}$ be a T_1 topology on $X \setminus U$, and let $\mathcal{T}$

$$\mathcal{T} = \mathcal{T}_{X \setminus U} \cup \{X, U\}.$$

For all $x, y \in X$, if $x \neq y$, then they are separated. Thus, the space is R_0.
But $(X, \mathcal{T})$ is not T_1, because all $\{u\} \in U$ share the same closure which is U itself.

Proposition 1.3.1 (alternative definitions of T_1 spaces)**.** Let $(X, \mathcal{T})$ be T_1, then the following conditions are equivalent.

(i) All singletons in X are closed.

(ii) Every subset of X is the intersection of all open sets containing it.

(iii) Every cofinite subset of X is open.

Proof.

(i) Suppose there exists $\{x\} \subseteq X$ with $\overline{\{x\}} \neq \{x\}$, then there exists $y \in \overline{\{x\}}$ with $x \neq y$. By Definition 1.3.6, this is impossible.

(ii) Let $A \subseteq X$.

□

Proposition 1.3.2 (alternative definitions of R_0 spaces)**.** Let $(X, \mathcal{T})$ be R_0, then the following conditions are equivalent.

(i) The closure of all singletons in X are not T_0 subspace.

(ii) For any two points $x, y \in X$, $x \in \overline{\{y\}}$ iff $y \in \overline{\{x\}}$.

(iii) Every open set is the union of closed sets.

Proof.

(i) By Definition 1.3.5, if y and x are topologically distinguishable, by Definition 1.3.5, x and y are separated; i.e., $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$.

(ii) By Definition 1.3.5, for all $x, y \in X$, x, y are not separated only if they are topologically indistinguishable. By Definition 1.3.3, they share all their neighbourhoods, thus they have the same closure; i.e., $\overline{\{x\}} = \overline{\{y\}}$.

(iii) For any $U \in \mathcal{T}$,

$$U = \bigcup_{x \in U} \{x\}.$$

If $(X, \mathcal{T})$ is T_1, then we are done. Suppose $(X, \mathcal{T})$ is not T_1, then there exists $A \in \mathcal{T}$ with $|A| > 1$, and for all $B \subsetneq A$, $B \notin \mathcal{T}$ (proof omitted). For such A, $X \setminus A$ is open, for $X \setminus A = \bigcup(\mathcal{T} \setminus \{A\})$, thus A is also closed.

Suppose for any such A with $A \cap U \neq \emptyset$, $A \subseteq U$. Suppose it fails, i.e., $A \cap U \neq A$, then we have $A \cap U \subsetneq A$ and $A \cap U \in \mathcal{T}$, which is contradicted to the condition of A. Now we have

$$U = \bigcup \mathcal{A} \cup \bigcup_{x \in I} \{x\}$$

where $\mathcal{A}$ is the family of such A, and I is the union of all closed singletons in U. Thus U is open.

□

Hausdorff Spaces

Definition 1.3.7 (R_1 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be R_1 iff any two topological distinguishable points in X are separated by neighbourhoods.

Definition 1.3.8 (Hausdorff Spaces)**.** A topological space $(X, \mathcal{T})$ is said to be *Hausdorff* or T_2 iff it is T_0 and R_1; i.e., all distinct points $x, y \in X$ are separated by neighbourhoods.

Proposition 1.3.3. All metrizable spaces are Hausdorff

Proof. Let $(X, \mathcal{T})$ be a metrizable space. There exists a metric ρ on X that induces $\mathcal{T}$. Given distinct points $x, y \in X$, suppose for all $\varepsilon \in \mathbb{R}_{>0}$, there exists $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. Then $\rho(x, z) < \varepsilon$ and $\rho(y, z) < \varepsilon$. Now we have

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

Put $\rho(x,y) > 2\varepsilon$ as x and y are arbitrarily given. Then we have

$$\rho(x,z) + \rho(y,z) < \rho(x,y),$$

which implies that ρ is not a metric on X. Hence, $(X, \mathcal{T})$ is not metrizable which is contradicted to the condition. $\square$

Proposition 1.3.4. All singletons in a Hausdorff space are closed.

Proof. Let $(X, \mathcal{T})$ be a Hausdorff space, and let $x \in X$. For all $y \in X$ with $x \neq y$, there is a open neighbourhood U_y of y such that $x \notin U_y$. Then, for all such U_y, we have

$$\forall y \in X,\ x \in X \setminus U_y = \{x\} \iff x \in \bigcap_{y \in X \setminus \{x\}} X \setminus U_y = \{x\}.$$

As all $X \setminus U_y$ are closed, their intersection $\{x\}$ is closed. $\square$

Example 1.3.6 (T_1 but not Hausdorff)**.** Let X be a nonempty set, let $p \in X$, let $\mathcal{T}'$ be a Hausdorff topology on $X \setminus \{p\}$, and let

$$\mathcal{T} = \{X\} \cup \mathcal{T}'.$$

Then, all $x \in (X, \mathcal{T})$ are closed, thus $(X, \mathcal{T})$ is Fréchet. But the only neighbourhood of p is X, so its closure is X. Then, for any $x \in X \setminus \{p\}$, x and p are not separated, in which case $(X, \mathcal{T})$ is not R_0. Thus, $(X, \mathcal{T})$ is not Hausdorff.

1.4 Continuity

Definition 1.4.1 (continuous maps)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is said to be *continuous* iff for any open set U in Y, its preimage in X under f is open.

Note 1.4.1. In Definition 1.4.1, note that even if for any open set U in X, $f[X]$ is open in Y, f is not necessarily continuous. For example, let $X = (\mathbb{R}, \mathcal{T}_X)$ with $\mathcal{T}_X$ induced by standard Euclidean metric, let $Y = (\mathbb{R}, \mathcal{T}_Y)$ with $\mathcal{T}_Y$ as a indiscrete topology, and define

$$f(x) = [x],$$

where $[x]$ denotes the integer part of x. Then for all $U \subseteq X$, $f[U]$ is open in Y, but by Definition 1.4.1, f is not continuous.

Note 1.4.2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, if $\mathcal{T}_X$ is the discrete topology on X, then any function with domain X is continuous. If $\mathcal{T}_Y$ is the indiscrete topology on Y, then any function with codomain Y is continuous.

Note 1.4.3. A function is continuous bijection does not implies that its inverse is continuous. For example, let X be any set and let $\mathcal{T}$ and $\mathcal{T}'$ be its topologies. If $\mathcal{T}$ is finer than $\mathcal{T}'$, then any bijection $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous. In this case, however, if $\mathcal{T} \neq \mathcal{T}'$, then f^{-1} is not continuous.

Proposition 1.4.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is continuous at $x \in X$ iff for any neighbourhood N_y of $f(x)$, there is a neighbourhood N_x of x, such that $f[N_x] \subseteq N_y$.

Proof. Let N_y be a neighbourhood of $f(x)$. Clearly, there exists an open set U_y contains y.
By Definition 1.4.1, f is continuous at x iff $x \in f^{-1}[U_y] \in \mathcal{T}_X$. Clearly, $f^{-1}[U_y]$ is a neighbourhood of x. We have $f[f^{-1}[U_y]] = U_y \subseteq N_y$.
By Proposition 1.1.2, there U_x must contains at least one neighbourhood N_x of x, thus, $f[N_x] \subseteq U_y$. □

Proposition 1.4.2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be metrizable spaces. A map $f : X \to Y$ is continuous at $p \in X$ iff for any $\varepsilon > 0$, there is a $\delta > 0$, such that for all $x \in B_X(p, \delta)$, $f(x) \in B_Y(f(p), \varepsilon)$, where B_X is defined by any metrics ρ_X induces $\mathcal{T}_X$, and B_Y is defined by any metrics ρ_Y induces $\mathcal{T}_Y$.

Proof. Clearly, for all $\varepsilon > 0$, $B_Y(f(x,), \varepsilon)$ is an open neighbourhood of $f(x)$.
f is not necessarily be injective, so $f^{-1}[B_Y(f(x), \varepsilon)] = U \in x$. By Definition 1.4.1, U is open, so for some $\delta > 0$, $B_X(x, \delta) \subseteq U$. Thus, By Proposition 1.4.1, f is continuous iff $f[B_X(x, \delta)] \subseteq B_Y(f(x), \varepsilon)$. This satisfies the conditions we have. □

Proposition 1.4.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is said to be continuous iff for any closed set V in Y, its preimage in X under f is closed.

Proof. Let U_Y be any open set in Y, let U_X be the preimage of U_Y under f. By Definition 1.4.1, U_X is open in X. Let

$$V_X = f^{-1}[Y \setminus U_Y] = X \setminus U_X,$$

Then V_X is closed. □

Definition 1.4.2 (convergence of sequences)**.** Let $(X, \mathcal{T})$ be a topological space, and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *converges* in X iff there is an $x \in X$, such that for any open neighbourhood U_x of x, it contains a cofinite subset $A \subseteq \{x_n\}$. That is, there exists N in the domain of $\{x_n\}$, for any natural numbers $n \geq N$, $x_n \in U_x$.

Example 1.4.1.

1. In a discrete topological space, a sequence $\{x_n\}$ converges iff there is an N in the domain of $\{x_n\}$, for any natural numbers $m > N$, $x_N = x_m$.

2. In a indiscrete topological space, any sequence $\{x_n\}$ in X converges in X. And
$$\lim_{n \to \infty} \{x_n\} = X.$$

Proposition 1.4.4. In a Hausdorff space, any convergent sequence converges to a unique point in the space.

Proof. Let $(X, \mathcal{T})$ be a Hausdorff space, and let $\{x_n\}$ be a sequence in X. Suppose $\{x_n\}$ converges to more than one point, say to $x, y \in X$ with $x \neq y$, then, for all neighbourhoods N_x of x and N_y of y, N_x contains a cofinite subset $A \subseteq \{x_n\}$ and N_y contains a cofinite subset $B \subseteq \{x_n\}$. If this were true, $N_x \cap N_y$ should be non-empty, otherwise N_x or N_y should be finite.
Then, x and y are not separated by neighbourhoods, thus $(X, \mathcal{T})$ is not Hausdorff. This is a contradiction.
But, as $(X, \mathcal{T})$ is Hausdorff, there must be mutually disjoint N_x and N_y. Thus, the assumption cause a contradiction. $\square$

Note 1.4.4. As all metrizable spaces are Hausdorff, so any convergent sequence in a metrizable space converges to at most one point.

Proposition 1.4.5. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological space, let $f : X \to Y$ be a map, and let $\{x_n\}$ be a convergent sequence in X. If f is continuous, then $f[\{x_n\}]$ is a sequence convergent in Y.

Proof. Let U_y be any open neighbourhood of $f(x)$. By Definition 1.4.1, $f^{-1}[U_y]$ is also an open neighbourhood of x. By Definition 1.4.2, $f^{-1}[U_y]$ contains a cofinite subset $A \subseteq \{x_n\}$. Then $f[A]$ is a cofinite subset of $f[\{x_n\}]$. As $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$, $f[\{x_n\}]$ converges in $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$. $\square$

Note 1.4.5. In this proposition, even if $f[\{x_n\}]$ converges in Y, f might be discontinuous. For example, let X any set, let $\mathcal{T}$ be the indiscrete topology on X, let U be another cofinite subset of X with $X \neq U$, and let $\mathcal{T}' = \{\emptyset, X, U\}$. Let $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ be defined by

$$f(x) = x.$$

By Definition 1.4.1, f is not continuous. But, for any convergent sequence $\{x_n\}$ in $(X, \mathcal{T})$, $f[\{x_n\}]$ also convergent in $(X, \mathcal{T})$.

1.5 Homeomorphisms

Definition 1.5.1 (homeomorphisms)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

Definition 1.5.2 (homeomorphic)**.** Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

1.6 Cover and Basis

Definition 1.6.1 (cover)**.** Let $(X, \mathcal{T})$ be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of U iff the union of all sets in $\mathcal{C}$ is a superset of U. That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call $\mathcal{C}$ an *open cover* of U.
Let $\mathcal{S} \subseteq \mathcal{C}$, iff the union of $\mathcal{S}$ is still a superset of U, then we call $\mathcal{S}$ a *subcover* of $\mathcal{C}$.

Definition 1.6.2 (basis)**.** Let $(X, \mathcal{T})$ be a topological space, let $U \subseteq X$, and let $\mathcal{B}$ be a open cover of X. We call $\mathcal{B}$ a *base* of X iff the union of $\mathcal{B}$ is precisely U itself, i.e.,

$$U = \bigcup \mathcal{B}.$$

Definition 1.6.3 (synthetic basis)**.** Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a base of X. $\mathcal{B}$ is said to be *synthetic* iff for any $A, B \in \mathcal{B}$,

$$A \cap B = \bigcup_{i=1}^{n} B_i, \quad B_i \in \mathcal{B}.$$

Definition 1.6.4 (generated by basis)**.** Let X be any set and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be its cover. A topology $\mathcal{T}$ on X is said to be *generated* by the base $\mathcal{B}$ iff

(i) for all $U \in \mathcal{T}$, U is the union of $\mathcal{B}$-sets;

(ii) for all $U \in \mathcal{T}$, U is the finite intersection of $\mathcal{B}$-sets.

Proposition 1.6.1. Let $(X, \mathcal{T})$ be a topological space be genrated by a base $\mathcal{B}$. For all $U \in \mathcal{T}$, there is a $B \in \mathcal{B}$ such that $U \subseteq \mathcal{B}$.

Proof. By Definition 1.6.4, if $\mathcal{T}$ is generated by $\mathcal{B}$, then for all $U \in \mathcal{T}$, there is an finite set I, such that

$$U = \bigcap_{i \in I} B_i, \quad B_i \in \mathcal{B}.$$

Thus, for at least one $k \in I$, $U \subseteq B_k$. □

Proposition 1.6.2. Let X be any set, and let $\mathcal{T}$ and $\mathcal{T}'$ be its topologies generated by basis $\mathcal{B}$ and $\mathcal{B}'$ respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for any $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $B' \subseteq B$.

Proof. If $\mathcal{T}$ is generated by $\mathcal{B}$, then for all $U' \in \mathcal{T}'$,

$$U' = \bigcup_{j \in J} B'_j,$$

where $B_j \in \mathcal{B}$.
As $\mathcal{T}$ is generated by $\mathcal{B}$, then, certainly, $\mathcal{B} \subseteq \mathcal{T}$.
By the conditions we have, $\mathcal{T} \subseteq \mathcal{T}'$ iff for all $B \in \mathcal{B}$, there is $W' \in \mathcal{T}$ such that

$$B = W' = \bigcup_{i \in I} B'_i,$$

where $B'_i \in \mathcal{B}'$. Certainly, all such B'_i are contained in B. □

Proposition 1.6.3. Let X be any set, and let $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is a topology on X iff it generates itself.

Proof. If $\mathcal{T}$ is a topology on X, then, by Definition 1.6.4, any open set generated by $\mathcal{T}$ is still a member of $\mathcal{T}$. On the other hand, if $\mathcal{T}$ generates itself, then, $\emptyset$ and X must be members of $\mathcal{T}$, and, by Definition 1.6.4, $\mathcal{T}$ is a topology on X. □

1.7 Interiors and Closures

Definition 1.7.1 (interiors)**.** The *interior* of a set A, denoted A°, is defined to be the union of all open subsets of A.

Definition 1.7.2 (closure)**.** The *closure* of a set A, denoted $\overline{A}$, is defined to be the intersection of all closed supersets of A.

Definition 1.7.3 (dense sets)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. A is said to be dense, iff $\overline{A} = X$.

Definition 1.7.4 (nowhere dense sets)**.** A set A is said to be *nowhere dense* iff the interior of its closure is empty.

Proposition 1.7.1 (properties of interiors)**.** Let $(X, \mathcal{T})$ be any topological space and $A, B \subseteq X$.

(i) (Intensive) $A^\circ \subseteq A$.

(ii) A is open iff $A = A^\circ$.

(iii) (Idempotence) $(A^\circ)^\circ = A^\circ$.

(iv) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(v) $A \subseteq B \implies A^\circ \subseteq B^\circ$.

(vi) If B is open, then $B \subseteq A$ iff $B \subseteq A^\circ$.

Proof.

(i) By Definition 1.7.1, naturally, $A^\circ \subseteq A$.

(ii) By Definition 1.1.2, A° is the union of open sets hence it is open. A is open iff it is the union of all open subsets of A. Thus $A = A^\circ$.

(iii) A° is open, thus $(A^\circ)^\circ = A^\circ$.

(iv) By Definition 1.7.1, we have

$$\begin{aligned}
(A \cap B)^\circ &= \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq A \cap B\right\} \\
&= \left\{\bigcup U : (U \in \mathcal{T} \wedge U \subseteq A) \wedge (U \in \mathcal{T} \wedge U \subseteq B)\right\} \\
&= \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq A\right\} \cap \left\{\bigcup U : U \in \mathcal{T} \wedge U \subseteq B\right\} \\
&= A^\circ \cap B^\circ.
\end{aligned}$$

(v) Clearly, $A^\circ \subseteq A$, thus,

$$A \subseteq B \implies A^\circ \subseteq B$$

Suppose $A^\circ \not\subseteq B^\circ$, then $A^\circ \setminus B^\circ$ is not empty ($\emptyset$ is the subset of any set, so A° is not empty).

Then there exists $x \in A^\circ$ with $x \in \partial B$ ($x \in B$ but $x \notin B^\circ$). Then there exists neighbourhood $N_x \ni x$, and $N_x \cap \partial B \neq \emptyset$. But this is impossible, for $A^\circ \subseteq B$ implies that $A^\circ \cap \partial B = \emptyset$ (This is a straight consequence of $A^\circ \cap \partial A = \emptyset$. See Proposition 1.8.1), so such N_x does not exist. Thus,

$$A^\circ \subseteq B^\circ.$$

(vi) If B is open, then $B = B^\circ$. Then $B \subseteq A$ iff $B^\circ \subseteq A^\circ$.

□

Proposition 1.7.2 (properties of closures)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

(i) $\overline{A}$ is closed.

(ii) A is closed iff $A = \overline{A}$.

(iii) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

(iv) If A is closed, then $A \supseteq B$ iff $A \supseteq \overline{B}$

Proof.

(i) By Definition 1.7.2, $\overline{A}$ is the intersection of closed sets. By Proposition 1.1.1, $\overline{A}$ is closed.

(ii) Proposition 1.1.1 implies that any closed set is the intersection of closed sets, this is precisely what Definition 1.7.2 says.

(iii) $A \subseteq B$ iff $X \setminus A \supseteq X \setminus B$. Then we have

$$X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ$$

Clearly, $(X \setminus A)^\circ$ is the union of all open set disjoint from A, then, by De Morgan's laws, $X \setminus (X \setminus A)^\circ$ is the intersection of all closed sets containing A. By Definition 1.7.2, we have $(X \setminus A)^\circ = \overline{A}$. Thus

$$\overline{A} \subseteq \overline{B}.$$

(iv) If A is closed, then $A = \overline{A}$. Suppose $B \subseteq A$, then we have

$$\overline{B} \subseteq \overline{A} \iff \overline{B} \subseteq A.$$

$\square$

1.8 Boundaries

Definition 1.8.1 (boundaries)**.** Let A be any set, the *boundary* of A, denoted ∂A, is defined to be the complement of the interior of A in the closure of A; i.e.,

$$\partial A = \overline{A} \setminus A^{\circ}.$$

Proposition 1.8.1 (properties of boundaries)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

(i) ∂A is closed.

(ii) $A^{\circ} \cap \partial A = \emptyset$.

(iii) $\overline{A} = A^{\circ} \cup \partial A$.

(iv) A is closed iff $\partial A \subseteq A$.

(v) ∂A is nowhere dense.

(vi) $\partial \overline{A} \subseteq \partial A \subseteq \partial A^{\circ}$.

(vii) $\partial A = \partial (X \setminus A)$.

(viii) A is dense iff $\partial A = X \setminus A^{\circ}$.

Proof.

(i) $\overline{A}$ is closed, and $X \setminus A^{\circ}$ is also closed. Thus

$$\partial A = \overline{A} \setminus A^{\circ} = \overline{A} \cap (X \setminus A)$$

is closed.

(ii) By Definition 1.8.1, we have

$$\partial A = \overline{A} \setminus A^{\circ} \iff \partial A \cap A^{\circ} = \overline{A} \setminus A^{\circ} \cap A^{\circ} = \overline{A} \cap \emptyset = \emptyset.$$

(iii) We have

$$\partial A = \overline{A} \setminus A^\circ \iff \partial A \cup A^\circ = \overline{A} \setminus A^\circ \cup A^\circ = \overline{A} \cap (X \setminus A^\circ \cup A^\circ)$$
$$\iff \partial A \cup A^\circ = \overline{A} \cap X|_{\text{for } A^\circ \subseteq X} = \overline{A}.$$

(iv) As A is closed, $A = \overline{A}$ (this can be straightly proved by Definition 1.7.2). By Definition 1.8.1, it is clear that $\partial A \subseteq \overline{A}$, thus $\partial A \subseteq A$.

(v) By Definition 1.7.4, ∂A is nowhere dense iff $\overline{\partial A}^\circ$ is empty. We have

$$\begin{aligned}\overline{\partial A}^\circ &= \overline{\overline{A} \setminus A^\circ}^\circ \\ &= (\overline{A} \setminus A^\circ) \cup (\overline{A} \setminus A^\circ) \setminus (\overline{A} \setminus A^\circ) \\ &= \emptyset.\end{aligned}$$

(vi) $\overline{A} \supseteq A^\circ$ implies $\overline{A}^\circ \supseteq (A^\circ)^\circ = A^\circ$, then we have,

$$\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^\circ \subseteq \overline{A} \setminus A^\circ = \partial A.$$

$A^\circ \subseteq A$ implies $\overline{A^\circ} \subseteq \overline{A}$, then we have,

$$\partial A^\circ = \overline{A^\circ} \setminus (A^\circ)^\circ \supseteq \overline{A} \setminus A^\circ.$$

(vii) We have

$$\begin{aligned}\partial (X \setminus A) &= \overline{X \setminus A} \setminus (X \setminus A)^\circ \\ &= X \setminus A^\circ \setminus (X \setminus \overline{A}) \\ &= X \setminus A^\circ \cap \overline{A} \\ &= \overline{A} \setminus A^\circ \\ &= \partial A.\end{aligned}$$

(viii) By Definition 1.7.3, A is dense in X iff $\overline{A} = X$. Then we have,

$$\overline{A} = X \iff \overline{A} \setminus A^\circ = X \setminus A^\circ$$
$$\iff \partial A = X \setminus A^\circ.$$

$\square$

1.9 Limit Points

Definition 1.9.1 (limit points)**.** Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$. A point $x \in X$ is called a *limit point* of A iff for all neighbourhood N_x of x, $N_x \setminus \{x\}$ intersects A.

Proposition 1.9.1. Let A be any set, and let x be a limit point of A, then x is an element of the closure of A.

Proof. If A is empty, then this is vacuously true. So, suppose A is not empty. By Definition 1.9.1, for all neighbourhood N_x of x, $N_x \setminus \{x\} \cap A$ is not empty. Naturally, $N_x \cap A$ is not empty.
Assume that $x \notin \overline{A}$, then $X \setminus \overline{A}$ is a neighbourhood of x, by Definition 1.1.5, and is disjoint from A. This is contradicted to the conditions. $\square$

Note 1.9.1. In this proof, the proposition also holds for $N_x \cap A^\circ = \emptyset$. Because if it is true, then

$$N_x \cap \partial A \supseteq (N_x \cap A) \setminus (N_x \cap A^\circ) = N_x \cap A.$$

This implies that $A \subseteq \partial A$. In this case, $\overline{A} = \partial A$, for
Assume that $x \notin \partial A$, then we have the same conclusion.
Then $A^\circ = A \setminus \partial A = \emptyset$.

Proposition 1.9.2. A set is closed iff it contains all its limit point.

Proof. Let A be a set. By proposition 1.9.1, for every limit point of A, it is also an element of the closure $\overline{A}$. And A is closed iff $A = \overline{A}$. $\square$

Definition 1.9.2 (convergent sequences)**.** Let $(X, \mathcal{T}_X)$ be a topological space. A sequence $\{x_n\}$ in X is said to be *convergence* in X iff there is an open set U contains all but finite terms of $\{x_n\}$.

Chapter 2

Creating New Spaces

2.1 Subspaces

Definition 2.1.1 (subspace topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The *subspace topology* $\mathcal{T}_A$ on A is defined to be the family of the intersections of open sets in $(X, \mathcal{T})$ and A. That is,

$$\mathcal{T}_A = \{U \cap A : \ U \in \mathcal{T}\} .$$

2.2 Quotient Spaces

Definition 2.2.1 (quotient topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\} .$$

2.3 Product Spaces

Definition 2.3.1 (product topologies)**.**

Chapter 3

Topological Properties

3.1 Cardinal Functions

3.2 More on Separation Axioms

Definition 3.2.1 (saperated sets)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \in \mathcal{P}(X)$.

(i) A and B are said to be *separated* iff each is disjoint from other's closure.

(ii) A and B are said to be *separated by neighbourhoods* iff there are neighbourhoods N_A of A and N_B of B such that N_A and N_B are disjoint.

(iii) A and B are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods $\overline{N}_A$ of A and $\overline{N}_B$ of B such that $\overline{N}_A$ and $\overline{N}_B$ are disjoint.

(iv) A and B are said to be *separated by a continuous function* iff there is a continuous function $f : X \to \mathbb{R}$, such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

(v) A and B are said to be *precisely separated by a continuous function* iff there is a continuous function $f : X \to \mathbb{R}$, such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$

Definition 3.2.2 ($T_{2^1/2}$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_{2^1/2}$ or *Urysohn* iff two distinct points in X are separated by closed neighbourhoods.

Example 3.2.1 (T_2 but not $T_{2^1/2}$)**.** [1] (Remained as a problem)

[1] See MathPlanet.

Definition 3.2.3 (T_3 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_3 or *regular* iff it is T_0 and given any point $x \in (X, \mathcal{T})$ and closed set $V \subseteq X$ with $x \notin V$ are separated by neighbourhoods.

Definition 3.2.4 ($T_{3^1/_2}$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_{3^1/_2}$, or *Tychonoff* or, *completely* T_3, or *completely regular*, iff it is T_0 and given any point x and closed set $V \subseteq X$ with $x \notin V$, they are separated by a continuous function.

Definition 3.2.5 (T_4 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_4 or *normal* iff it is Hausdorff and any tow disjoint closed subsets of X are separated by neighbourhoods.

Proposition 3.2.1 (Urysohn's lemma)**.** A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

Definition 3.2.6 (T_5 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_5 or *completely* T_4 iff it is T_1 any two separated sets are separated by neighbourhoods.

Proposition 3.2.2. Every subspace of a T_5 space is normal.

Definition 3.2.7 (T_6 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_6, or *perfectly* T_4 or *perfectly normal* iff it is T_1 and any two disjoint closed sets are precisely separated by a continuous function.

Proposition 3.2.3 (Tietze extension theorem)**.** Let $(X, \mathcal{T})$ be normal topological space, and let $f : A \to (\mathbb{R}, \mathcal{T}')$ be a continuous map where A is a closed subset of X and $\mathcal{T}'$ is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$

3.3 Countability Axioms

3.4 Compactness

Definition 3.4.1 (compactness)**.** A topological space $(X, \mathcal{T})$ is said to be *compact* iff every open cover of X has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

3.5 Connectedness

Definition 3.5.1 (connectedness)**.** Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is said to be *connected* iff X is not empty and it it not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

Definition 3.5.2 (path-connectedness)**.** Let $(X, \mathcal{T})$ be a topological space.

(i) A map $\gamma : [0,1] \to X$ is called a *path* in X iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that γ is path from x to y in X.

(ii) X is said to be *path-connected* iff for all $x, y \in X$ there is a path from x to y in X.