Notes for Vector Calculus

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Chapter 1.

Differentiation

§1.1 Infintesimal

Definition 1.1.1. Let $f, g : \mathbb{R}^m \to \mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^m$.

Then f is a *little-o* of g as $\mathbf{x} \to \mathbf{p}$, i.e.,

$$f(\mathbf{x}) = o(g(\mathbf{x})) \text{ as } \mathbf{x} \to \mathbf{p},$$

iff for any $\varepsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood of U of $\mathbf{p}$ such that for any $\mathbf{x} \in U$, $\|f(\mathbf{x})\| \leq \varepsilon \|g(\mathbf{x})\|$. Equivalently, that is,

$$\lim_{\mathbf{x} \to \mathbf{p}} \frac{f(\mathbf{x})}{\|g(\mathbf{x})\|_{\mathbb{R}^n}} = \mathbf{0}_{\mathbb{R}^n}.$$

Note 1.1.1. In the case that $f(\mathbf{x}) = o(g(\mathbf{x}))$ as $\mathbf{x} \to \mathbf{0}_{\mathbb{R}^m}$, I will simply write $f(\mathbf{x}) = o(g(\mathbf{x}))$.

Lemma 1.1.1.

$$o(f(\mathbf{x})) + o(g(\mathbf{x})) = o(\|f(\mathbf{x})\|_{\mathbb{R}^n} + \|g(\mathbf{x})\|_{\mathbb{R}^n}).$$

Proof. By Definition 1.1.1, for any $\varepsilon \in \mathbb{R}_{>0}$, there exists a neighbourhood of U of $\mathbf{p}$ such that for any $\mathbf{x} \in U$,

$$\|o(f(\mathbf{x}))\|_{\mathbb{R}^n} \leq \varepsilon \|f(\mathbf{x})\|.$$

Then, there exists some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that

$$o(f(\mathbf{x})) = \varepsilon \|f(\mathbf{x})\| \mathbf{u} \text{ and } o(g(\mathbf{x})) = \varepsilon \|g(\mathbf{x})\| \mathbf{v}.$$

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By Definition 1.1.1, now we have

$$o(f(\mathbf{x})) + o(g(\mathbf{x}))$$

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§1.2 Differentiable Mapping

Definition 1.2.1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^m$.

f is *differentiable* at $\mathbf{p}$ iff there exists a unique linear map $\phi : \mathbb{R}^m \to \mathbb{R}^n$, such that for any $\mathbf{t} \in \mathbb{R}^m$,

$$\lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{R}^n}} \frac{f(\mathbf{p} + \mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \mathbf{0}_{\mathbb{R}^n}$$

Note 1.2.1. Rigorously, the uniqueness of ϕ is deduced by the reset of the definition.

There exists an $\alpha : \mathbb{R}^m \to \mathbb{R}^n$ with $\lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}} \alpha(\mathbf{t}) = \mathbf{0}_{\mathbb{R}^n}$, such that there exists an open subset $U \subseteq \mathbb{R}^m$ with $\mathbf{p} \in U$, such that for those $\mathbf{t} \in \mathbb{R}^n$ with $\mathbf{p} + \mathbf{t} \in U$,

$$\frac{f(\mathbf{p} + \mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \alpha(\mathbf{t}).$$

Assume Definition 1.2.1 holds for a linear mapping λ also, then, similarly, there exists a $\beta : \mathbb{R}^m \to \mathbb{R}^n$ with $\lim_{\mathbf{t} \to \mathbf{0}_{\mathbb{R}^m}} \beta(\mathbf{t}) = \mathbf{0}_{\mathbb{R}^n}$, such that there exists an open subset $U' \subseteq \mathbb{R}^m$ with $\mathbf{p} \in U'$, such that for those $\mathbf{t} \in \mathbb{R}^n$ with $\mathbf{p} + \mathbf{t} \in U'$,

$$\frac{f(\mathbf{p} + \mathbf{t}) - f(\mathbf{p}) - \lambda(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \beta(\mathbf{t}).$$

Let $\gamma = \phi - \lambda$. As ϕ and $-\lambda$ are both linear, then

$$\frac{\gamma(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^n}} = \alpha(\mathbf{t}) - \beta(\mathbf{t}) \iff \lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^m}} \gamma(\hat{\mathbf{t}}) = \lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^m}} (\alpha(\mathbf{t}) - \beta(\mathbf{t}))$$
$$\iff \gamma(\hat{\mathbf{t}}) = \mathbf{0}_{\mathbb{R}^n}.$$

As $\mathbf{t}$ is arbitrarily given in $U \cap U'$, and $U \cap U'$ is open in $\mathbb{R}^m$, the set

$$\left\{ \hat{\mathbf{t}} = \frac{\mathbf{t}}{\|\mathbf{t}\|} : \mathbf{t} \in U \cap U' - \mathbf{p} \right\}$$

gives the all possible directions in $\mathbb{R}^m$. And as $\gamma(s\hat{\mathbf{t}}) = \mathbf{0}_{\mathbb{R}^n}$, for all $s \in \mathbb{R}$, $\gamma(\mathbf{t}) = \mathbf{0}_{\mathbb{R}^n}$ constantly. This implies $\phi = \lambda$.

Lemma 1.2.1. With the condition in Definition 1.2.1, if f is differentiable at $\mathbf{p}$, then f is continuous at $\mathbf{p}$.

Proof. As f is differentiable at $\mathbf{p}$, by Definition 1.2.1,

$$\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^n}} \frac{f(\mathbf{p}+\mathbf{t}) - f(\mathbf{p}) - \phi(\mathbf{t})}{\|\mathbf{t}\|_{\mathbb{R}^m}} = \mathbf{0}_{\mathbb{R}^n}$$

for a unique $\phi : \mathbb{R}^m \to \mathbb{R}^n$ (Note 1.2.1). Then there exists an $\alpha : \mathbb{R}^m \to \mathbb{R}^n$ with $\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^m}} \alpha(\mathbf{t}) = \mathbf{0}_{\mathbb{R}^n}$ such that there exists an open subset $U \subseteq \mathbb{R}^m$ with $\mathbf{p} \in U$ such that for any $\mathbf{t} \in \mathbb{R}^m$ with $\mathbf{p} + \mathbf{t} \in U$,

$$f(\mathbf{p}+\mathbf{t}) - \phi(\mathbf{t}) = \alpha(\mathbf{t})\|\mathbf{t}\|_{\mathbb{R}^m} + f(\mathbf{p}).$$

Then, we have

$$\lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^m}} (f(\mathbf{p}+\mathbf{t}) - \phi(\mathbf{t})) = \lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^m}} (\alpha(\mathbf{t})\|\mathbf{t}\|_{\mathbb{R}^m} + f(\mathbf{p}))$$
$$\iff \lim_{\mathbf{t}\to\mathbf{0}_{\mathbb{R}^m}} f(\mathbf{p}+\mathbf{t}) = f(\mathbf{p}).$$

This, implies f is continuous at $\mathbf{p}$. ■

Lemma 1.2.2. With the condition in Definition 1.2.1, f is differentiable at $\mathbf{p}$, iff for any $g : \mathbb{R} \to \mathbb{R}^m$ with g differentiable at 0 and $g(0) = \mathbf{p}$, $f \circ g$ is differentiable at 0.

Proof. ■

§1.3 Directional Derivatives

Definition 1.3.1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$, let $\mathbf{u} \in \mathbb{R}^m \setminus \{\mathbf{0}_{\mathbb{R}^m}\}$, and let $\mathbf{p} \in \mathbb{R}^m$.

The *directional derivative* of f along $\mathbf{u}$ at $\mathbf{p}$ is defined as

$$\nabla_{\mathbf{u}} f(\mathbf{p}) := \lim_{t \to 0} \frac{f(\mathbf{p} + t\mathbf{u}) - f(\mathbf{p})}{t},$$

if the limit exists in $\mathbb{R}^n$.

Lemma 1.3.1. With the conditions in Definition 1.3.1, if $\nabla_{\mathbf{u}} f(\mathbf{p})$ exists at $\mathbf{p}$, then there exists open subset $U \subseteq \mathbb{R}^m$ with $\mathbf{p} \in U$ such that f is relative continuous on $U \cap \{\mathbf{p} + t\mathbf{u} : t \in \mathbb{R}\}$.

Proof. Let U be an open subset of $\mathbb{R}^m$.

Let $g : \mathbb{R} \to \mathbb{R}^m$ be defined as

$$g(t) = \mathbf{p} + t\mathbf{u}.$$

Then f is relative continuous on $U \cap \{\mathbf{p} + t\mathbf{u}\}$ iff $f \restriction_{g[\mathbb{R}]}$ is continuous on $U \cap g[\mathbb{R}]$.

Let $h = f \circ g$, then

$$\nabla_{\mathbf{u}} f(\mathbf{p}) = \lim_{t \to 0} \frac{h(t) - h(0)}{t} = \mathbf{v} \in \mathbb{R}^n.$$

Then, there exists an $\alpha : \mathbb{R} \to \mathbb{R}^n$ with $\lim_{t \to 0} \alpha(t) = \mathbf{0}_{\mathbb{R}^n}$, such that

$$h(t) = t\mathbf{v} + t\alpha(t) + h(0).$$

Then we have

$$\begin{aligned} & \lim_{t \to 0} h(t) = \lim_{t \to 0} \big(t\mathbf{v} + t\alpha(t) + h(0)\big) \\ \iff & \lim_{t \to 0} h(t) = h(0). \end{aligned}$$

Thus, h is continuous at 0.

As composition of mappings is associative, we have

$$\begin{aligned} h = f \circ g &\iff h \circ g^{-1} = f \circ g \circ g^{-1} \\ &\iff h \circ g^{-1} = f \circ (g \circ g^{-1}) \end{aligned}$$

As g is bijective, $g \circ g^{-1}$ is an identity map on $g[\mathbb{R}]$. Thus, we have

$$h \circ g^{-1} = f \restriction_{g[\mathbb{R}]} .$$

As h and g^{-1} are continuous, so is $f \restriction_{g[\mathbb{R}]}$. Thus f is relative continuous on $U \cap \{\mathbf{p} + t\mathbf{u} : t \in \mathbb{R}\}$. ■

Lemma 1.3.2. With the conditions in Definition 1.3.1, let $s \in \mathbb{R} \setminus \{0\}$, then

$$\nabla_{s\mathbf{u}} f(\mathbf{p}) = s \nabla_{\mathbf{u}} f(\mathbf{p})$$

if $\nabla_{\mathbf{u}} f(\mathbf{p})$ exists in $\mathbb{R}^n$.

Proof. By Definition 1.3.1,

$$\nabla_{s\mathbf{u}} f(\mathbf{p}) = s \lim_{ts \to 0} \frac{f(\mathbf{p} + ts\mathbf{u}) - f(\mathbf{p})}{ts} = s \nabla f(\mathbf{p}).$$

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§1.4 s