

# Notes for General Topology

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April 25, 2021

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# Chapter 1

## Topological Spaces

### 1.1 Topological Spaces

**Definition 1.1.1** (topology). Let  $X$  be a set, and let a family  $\mathcal{T} \subseteq \mathcal{P}(X)$ .  $\mathcal{T}$  is called a topology on  $X$  iff

- (i)  $\emptyset, X \in \mathcal{T}$ ;
- (ii)  $\mathcal{T}$  is closed under arbitrary union;
- (iii)  $\mathcal{T}$  is closed under finite intersection.

**Definition 1.1.2** (topological spaces). Let  $X$  be any set, and let  $\mathcal{T}$  be a topology on  $X$ , then the pair  $(X, \mathcal{T})$  is called a *topological space*. All subsets of  $X$  in  $\mathcal{T}$  are called *open sets* in  $(X, \mathcal{T})$ .

**Definition 1.1.3** (closed sets). Let  $(X, \mathcal{T})$  be a topological space. A subset  $V$  of  $X$  is said to be *closed* iff there is an open set  $U$  in  $X$  such that

$$V = X \setminus U.$$

**Proposition 1.1.1.** Let  $X$  be a set, and let  $\mathcal{C}$  be the family of all closed sets in  $X$ . Then

- (i)  $\emptyset, X \in \mathcal{C}$ ;
- (ii)  $\mathcal{C}$  is closed under arbitrary intersection;
- (iii)  $\mathcal{C}$  is closed under finite union.

**Definition 1.1.4** (finer and coarser topology). Let  $X$  be any set, and let  $\mathcal{T}, \mathcal{T}'$  be topologies on  $X$ .  $\mathcal{T}$  is said to be *finer* than  $\mathcal{T}'$  iff  $\mathcal{T} \supseteq \mathcal{T}'$ ; respectively,  $\mathcal{T}$  is said to be *coarser* than  $\mathcal{T}'$  iff  $\mathcal{T} \subseteq \mathcal{T}'$ .

**Definition 1.1.5** (neighbourhood). Given  $(X, \mathcal{T})$  as a topological space and a point  $x \in X$ , a subset  $N \subseteq X$  is called a *neighbourhood* iff it contains an open set  $U$  containing  $x$ .

**Proposition 1.1.2.** Given  $(X, \mathcal{T})$  as a topological space and  $U \subseteq X$ ,  $U$  is open iff for all  $x \in U$ , there is a neighbourhood  $N$  of  $x$  contained in  $U$ .

*Proof.* If  $U$  is open, then  $U$  itself is a neighbourhood of  $x$  contained in  $U$ .

Conversely, if for all  $x \in U$ , there is a neighbourhood  $N_x$  of  $x$  contained in  $U$ , then there is a open neighbourhood  $U_x \ni x$  contained in  $N_x$ . Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose  $U$  is not open, then  $U$  is a proper superset in the relation above. Then there exists  $y \in U$  which is not in any  $U_x$ . This implies that such a  $y$  does not have any neighbourhood  $N_y$  in  $U$ , for such an  $N_y$  must contains an open  $U_y \ni y$ . For if it does, then there must be a  $U_x$  contains  $y$ . This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open. □

## 1.2 Cover and Basis

**Definition 1.2.1** (cover). Let  $(X, \mathcal{T})$  be a topological space, and let  $U \subseteq X$ , then a family  $\mathcal{C} \subseteq \mathcal{P}(X)$  is called a *cover* of  $U$  iff the union of all sets in  $\mathcal{C}$  is a superset of  $U$ . That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If  $\mathcal{C} \subseteq \mathcal{T}$ , then we call  $\mathcal{C}$  an *open cover* of  $U$ .

Let  $\mathcal{S} \subseteq \mathcal{C}$ , iff the union of  $\mathcal{S}$  is still a superset of  $U$ , then we call  $\mathcal{S}$  a *subcover* of  $\mathcal{C}$ .

**Definition 1.2.2** (basis). Let  $(X, \mathcal{T})$  be a topological space, let  $U \subseteq X$ , and let  $\mathcal{B}$  be a open cover of  $X$ . We call  $\mathcal{B}$  a *base* of  $X$  iff the union of  $\mathcal{B}$  is precisely  $U$  itself, i.e.,

$$U = \bigcup \mathcal{B}.$$

**Definition 1.2.3** (synthetic basis). Let  $(X, \mathcal{T})$  be a topological space, and let  $\mathcal{B}$  be a base of  $X$ .  $\mathcal{B}$  is said to be *synthetic* iff for any  $A, B \in \mathcal{B}$ ,

$$A \cap B = \bigcup_{i=1}^n B_i, \quad B_i \in \mathcal{B}.$$

## 1.3 Continuity

**Definition 1.3.1** (continuous maps). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A map  $f : X \rightarrow Y$  is said to be *continuous* iff for any open set  $U$  in  $Y$ , its preimage in  $X$  under  $f$  is open.

**Proposition 1.3.1.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A map  $f : X \rightarrow Y$  is continuous at  $x \in X$  iff for any neighbourhood  $N_y$  of  $f(x)$ , there is a neighbourhood  $N_x$  of  $x$ , such that  $f[N_x] \subseteq N_y$ .

*Proof.* Let  $N_y$  be a neighbourhood of  $f(x)$ . Clearly, there exists an open set  $U_y$  contains  $y$ .

By Definition 1.3.1,  $f$  is continuous at  $x$  iff  $x \in f^{-1}[U_y] \in \mathcal{T}_X$ . Clearly,  $f^{-1}[U_y]$  is a neighbourhood of  $x$ . We have  $f[f^{-1}[U_y]] = U_y \subseteq N_y$ .

By Proposition 1.1.2, there  $U_x$  must contains at least one neighbourhood  $N_x$  of  $x$ , thus,  $f[N_x] \subseteq U_y$ .  $\square$

**Proposition 1.3.2.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be metrizable spaces. A map  $f : X \rightarrow Y$  is continuous at  $p \in X$  iff for any  $\varepsilon > 0$ , there is a  $\delta > 0$ , such that for all  $x \in B_X(p, \delta)$ ,  $f(x) \in B_Y(f(p), \varepsilon)$ , where  $B_X$  is defined by any metrics  $\rho_X$  induces  $\mathcal{T}_X$ , and  $B_Y$  is defined by any metrics  $\rho_Y$  induces  $\mathcal{T}_Y$ .

*Proof.* Clearly, for all  $\varepsilon > 0$ ,  $B_Y(f(x), \varepsilon)$  is an open neighbourhood of  $f(x)$ .

$f$  is not necessarily be injective, so  $f^{-1}[B_Y(f(x), \varepsilon)] = U \in x$ . By Definition 1.3.1,  $U$  is open, so for some  $\delta > 0$ ,  $B_X(x, \delta) \subseteq U$ . Thus, By Proposition 1.3.1,  $f$  is continuous iff  $f[B_X(x, \delta)] \subseteq B_Y(f(x), \varepsilon)$ . This satisfies the conditions we have.  $\square$

**Proposition 1.3.3.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A function  $f : X \rightarrow Y$  is said to be continuous iff for any closed set  $V$  in  $Y$ , its preimage in  $X$  under  $f$  is closed.

*Proof.* Let  $U_Y$  be any open set in  $Y$ , let  $U_X$  be the preimage of  $U_Y$  under  $f$ . By Definition 1.3.1,  $U_X$  is open in  $X$ . Let

$$V_X = f^{-1}[Y \setminus U_Y] = X \setminus U_X,$$

Then  $V_X$  is closed. □

## 1.4 Interiors and Closures

**Definition 1.4.1** (interiors). The *interior* of a set  $A$ , denoted  $A^\circ$ , is defined to be the union of all open subsets of  $A$ .

**Definition 1.4.2** (closure). The *closure* of a set  $A$ , denoted  $\overline{A}$ , is defined to be the intersection of all closed supersets of  $A$ .

**Definition 1.4.3** (dense sets). Let  $(X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .  $A$  is said to be *dense*, iff  $\overline{A} = X$ .

**Definition 1.4.4** (nowhere dense sets). A set  $A$  is said to be *nowhere dense* iff the interior of its closure is empty.

**Proposition 1.4.1** (properties of interiors). Let  $(X, \mathcal{T})$  be any topological space and  $A, B \subseteq X$ .

- (i) (Intensive)  $A^\circ \subseteq A$ .
- (ii)  $A$  is open iff  $A = A^\circ$ .
- (iii) (Idempotence)  $(A^\circ)^\circ = A^\circ$ .
- (iv)  $(A \cap B)^\circ = A^\circ \cap B^\circ$ .
- (v)  $A \subseteq B \implies A^\circ \subseteq B^\circ$ .
- (vi) If  $B$  is open, then  $B \subseteq A$  iff  $B \subseteq A^\circ$ .

*Proof.*

- (i) By Definition 1.4.1, naturally,  $A^\circ \subseteq A$ .
- (ii) By Definition 1.1.2,  $A^\circ$  is the union of open sets hence it is open.  $A$  is open iff it is the union of all open subsets of  $A$ . Thus  $A = A^\circ$ .
- (iii)  $A^\circ$  is open, thus  $(A^\circ)^\circ = A^\circ$ .

(iv) By Definition 1.4.1, we have

$$\begin{aligned}
(A \cap B)^\circ &= \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq A \cap B \right\} \\
&= \left\{ \bigcup U : (U \in \mathcal{T} \wedge U \subseteq A) \wedge (U \in \mathcal{T} \wedge U \subseteq B) \right\} \\
&= \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq A \right\} \cap \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq B \right\} \\
&= A^\circ \cap B^\circ.
\end{aligned}$$

(v) Clearly,  $A^\circ \subseteq A$ , thus,

$$A \subseteq B \implies A^\circ \subseteq B$$

Suppose  $A^\circ \not\subseteq B^\circ$ , then  $A^\circ \setminus B^\circ$  is not empty ( $\emptyset$  is the subset of any set, so  $A^\circ$  is not empty).

Then there exists  $x \in A^\circ$  with  $x \in \partial B$  ( $x \in B$  but  $x \notin B^\circ$ ). Then there exists neighbourhood  $N_x \ni x$ , and  $N_x \cap \partial B \neq \emptyset$ . But this is impossible, for  $A^\circ \subseteq B$  implies that  $A^\circ \cap \partial B = \emptyset$  (This is a straight consequence of  $A^\circ \cap \partial A = \emptyset$ . See Proposition 1.5.1), so such  $N_x$  does not exist. Thus,

$$A^\circ \subseteq B^\circ.$$

(vi) If  $B$  is open, then  $B = B^\circ$ . Then  $B \subseteq A$  iff  $B^\circ \subseteq A^\circ$ .

□

**Proposition 1.4.2** (properties of closures). Let  $(X, \mathcal{T})$  be a topological space, and let  $A, B \subseteq X$ .

- (i)  $\overline{A}$  is closed.
- (ii)  $A$  is closed iff  $A = \overline{A}$ .
- (iii)  $A \subseteq B$  implies  $\overline{A} \subseteq \overline{B}$ .
- (iv) If  $A$  is closed, then  $A \supseteq B$  iff  $A \supseteq \overline{B}$

*Proof.*

- (i) By Definition 1.4.2,  $\overline{A}$  is the intersection of closed sets. By Proposition 1.1.1,  $\overline{A}$  is closed.
- (ii) Proposition 1.1.1 implies that any closed set is the intersection of closed sets, this is precisely what Definition 1.4.2 says.

(iii)  $A \subseteq B$  iff  $X \setminus A \supseteq X \setminus B$ . Then we have

$$X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ$$

Clearly,  $(X \setminus A)^\circ$  is the union of all open set disjoint from  $A$ , then, by De Morgan's laws,  $X \setminus (X \setminus A)^\circ$  is the intersection of all closed sets containing  $A$ . By Definition 1.4.2, we have  $(X \setminus A)^\circ = \overline{A}$ . Thus

$$\overline{A} \subseteq \overline{B}.$$

(iv) If  $A$  is closed, then  $A = \overline{A}$ . Suppose  $B \subseteq A$ , then we have

$$\overline{B} \subseteq \overline{A} \iff \overline{B} \subseteq A.$$

□

## 1.5 Boundaries

**Definition 1.5.1** (boundaries). Let  $A$  be any set, the *boundary* of  $A$ , denoted  $\partial A$ , is defined to be the complement of the interior of  $A$  in the closure of  $A$ ; i.e.,

$$\partial A = \overline{A} \setminus A^\circ.$$

**Proposition 1.5.1** (properties of boundaries). Let  $(X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

- (i)  $\partial A$  is closed.
- (ii)  $A^\circ \cap \partial A = \emptyset$ .
- (iii)  $\overline{A} = A^\circ \cup \partial A$ .
- (iv)  $A$  is closed iff  $\partial A \subseteq A$ .
- (v)  $\partial A$  is nowhere dense.
- (vi)  $\partial \overline{A} \subseteq \partial A \subseteq \partial A^\circ$ .
- (vii)  $\partial A = \partial(X \setminus A)$ .
- (viii)  $A$  is dense iff  $\partial A = X \setminus A^\circ$ .

*Proof.*



(i)  $\overline{A}$  is closed, and  $X \setminus A^\circ$  is also closed. Thus

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (X \setminus A)$$

is closed.

(ii) By Definition 1.5.1, we have

$$\partial A = \overline{A} \setminus A^\circ \iff \partial A \cap A^\circ = \overline{A} \setminus A^\circ \cap A^\circ = \overline{A} \cap \emptyset = \emptyset.$$

(iii) We have

$$\begin{aligned} \partial A = \overline{A} \setminus A^\circ &\iff \partial A \cup A^\circ = \overline{A} \setminus A^\circ \cup A^\circ = \overline{A} \cap (X \setminus A^\circ \cup A^\circ) \\ &\iff \partial A \cup A^\circ = \overline{A} \cap X|_{\text{for } A^\circ} \subseteq X = \overline{A}. \end{aligned}$$

(iv) As  $A$  is closed,  $A = \overline{A}$  (this can be straightly proved by Definition 1.4.2).

By Definition 1.5.1, it is clear that  $\partial A \subseteq \overline{A}$ , thus  $\partial A \subseteq A$ .

(v) By Definition 1.4.4,  $\partial A$  is nowhere dense iff  $\overline{\partial A}^\circ$  is empty. We have

$$\begin{aligned} \overline{\partial A}^\circ &= \overline{\overline{A} \setminus A^\circ}^\circ \\ &= (\overline{A} \setminus A^\circ) \cup (\overline{A} \setminus A^\circ) \setminus (\overline{A} \setminus A^\circ) \\ &= \emptyset. \end{aligned}$$

(vi)  $\overline{A} \supseteq A^\circ$  implies  $\overline{A}^\circ \supseteq (A^\circ)^\circ = A^\circ$ , then we have,

$$\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^\circ \subseteq \overline{A} \setminus A^\circ = \partial A.$$

$A^\circ \subseteq A$  implies  $\overline{A}^\circ \subseteq \overline{A}$ , then we have,

$$\partial A^\circ = \overline{A^\circ} \setminus (A^\circ)^\circ \supseteq \overline{A} \setminus A^\circ.$$

(vii) We have

$$\begin{aligned} \partial(X \setminus A) &= \overline{X \setminus A} \setminus (X \setminus A)^\circ \\ &= X \setminus A^\circ \setminus (X \setminus \overline{A}) \\ &= X \setminus A^\circ \cap \overline{A} \\ &= \overline{A} \setminus A^\circ \\ &= \partial A. \end{aligned}$$

(viii) By Definition 1.4.3,  $A$  is dense in  $X$  iff  $\overline{A} = X$ . Then we have,

$$\begin{aligned} \overline{A} = X &\iff \overline{A} \setminus A^\circ = X \setminus A^\circ \\ &\iff \partial A = X \setminus A^\circ. \end{aligned}$$

□

## 1.6 Limit Points

**Definition 1.6.1** (limit points). Let  $(X, \mathcal{T}_X)$  be a topological space, and let  $A \subseteq X$ . A point  $x \in X$  is called a *limit point* of  $A$  iff for all neighbourhood  $N_x$  of  $x$ ,  $N_x \setminus \{x\}$  intersects  $A$ .

**Proposition 1.6.1.** Let  $A$  be any set, and let  $x$  be a limit point of  $A$ , then  $x$  is an element of the closure of  $A$ .

*Proof.* If  $A$  is empty, then this is vacuously true. So, suppose  $A$  is not empty.

By Definition 1.6.1, for all neighbourhood  $N_x$  of  $x$ ,  $N_x \setminus \{x\} \cap A$  is not empty. Naturally,  $N_x \cap A$  is not empty.

Assume that  $x \notin \overline{A}$ , then  $X \setminus \overline{A}$  is a neighbourhood of  $x$ , by Definition 1.1.5, and is disjoint from  $A$ . This is contradicted to the conditions.  $\square$

**Note 1.6.1.** In this proof, the proposition also holds for  $N_x \cap A^\circ = \emptyset$ . Because if it is true, then

$$N_x \cap \partial A \supseteq (N_x \cap A) \setminus (N_x \cap A^\circ) = N_x \cap A.$$

This implies that  $A \subseteq \partial A$ . In this case,  $\overline{A} = \partial A$ , for

Assume that  $x \notin \partial A$ , then we have the same conclusion.

Then  $A^\circ = A \setminus \partial A = \emptyset$ .

**Proposition 1.6.2.** A set is closed iff it contains all its limit point.

*Proof.* Let  $A$  be a set. By proposition 1.6.1, for every limit point of  $A$ , it is also an element of the closure  $\overline{A}$ . And  $A$  is closed iff  $A = \overline{A}$ .  $\square$

**Definition 1.6.2** (convergent sequences). Let  $(X, \mathcal{T}_X)$  be a topological space. A sequence  $\{x_n\}$  in  $X$  is said to be *convergence* in  $X$  iff there is an open set  $U$  contains all but finite terms of  $\{x_n\}$ .

## 1.7 Subspaces

**Definition 1.7.1** (subspace topology). Let  $(X, \mathcal{T})$  be a topological space and let  $A \subseteq X$ . The *subspace topology*  $\mathcal{T}_A$  on  $A$  is defined to be the family of the intersections of open sets in  $(X, \mathcal{T})$  and  $A$ . That is,

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}.$$

## Chapter 2

# Creating New Spaces

### 2.1 Quotient Spaces

**Definition 2.1.1** (quotient topology). Let  $(X, \mathcal{T})$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . The *quotient topology* is a topology on  $\mathcal{P}(X/\sim)$ ; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

### 2.2 Product Spaces

**Definition 2.2.1** (product topologies).

### 2.3 Homeomorphisms

**Definition 2.3.1** (homeomorphisms). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A bijection  $f : X \rightarrow Y$  is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

**Definition 2.3.2** (homeomorphic). Two topological spaces  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are said to be *homeomorphic* or *topologically equivalent*, denoted  $X \cong Y$ , iff there is an homeomorphism between them.

## Chapter 3

# Topological Properties

### 3.1 Cardinal Functions

### 3.2 Separation Axioms

**Definition 3.2.1** (topologically indistinguishable). Let  $(X, \mathcal{T})$  be a topological space. Two points  $x, y \in X$  are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let  $\mathcal{N}_x$  be the family of all neighbourhoods of  $x$  and let  $\mathcal{N}_y$  be the family of all neighbourhoods of  $y$ , we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively,  $x, y$  are said to be *topologically distinguishable* iff they are not topologically indistinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

**Definition 3.2.2** (separated sets). Let  $(X, \mathcal{T})$  be a topological space, and let  $A, B \in \mathcal{P}(X)$ .

- (i)  $A$  and  $B$  are said to be *separated* iff each is disjoint from other's closure.
- (ii)  $A$  and  $B$  are said to be *separated by neighbourhoods* iff there are neighbourhoods  $N_A$  of  $A$  and  $N_B$  of  $B$  such that  $N_A$  and  $N_B$  are disjoint.
- (iii)  $A$  and  $B$  are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods  $\overline{N}_A$  of  $A$  and  $\overline{N}_B$  of  $B$  such that  $\overline{N}_A$  and  $\overline{N}_B$  are disjoint.

- (iv)  $A$  and  $B$  are said to be *separated by a continuous function* iff there is a continuous function  $f : X \rightarrow \mathbb{R}$ , such that  $f[A] = \{0\}$  and  $f[B] = \{1\}$ .
- (v)  $A$  and  $B$  are said to be *precisely separated by a continuous function* iff there is a continuous function  $f : X \rightarrow \mathbb{R}$ , such that  $f^{-1}[\{0\}] = A$  and  $f^{-1}[\{1\}] = B$

**Definition 3.2.3** ( $T_0$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_0$  or *Kolmogorov*, iff all distinct points  $x, y \in X$  are *topologically distinguishable*.

**Example 3.2.1** (non- $T_0$  sets). The a set  $X$  with the discrete topology is  $T_0$  iff  $|X| \in \{0, 1\}$  (vacuously true).

**Definition 3.2.4** ( $R_0$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $R_0$  iff any two topologically distinguishable points in  $X$  are separated.

**Example 3.2.2** ( $R_0$  but not  $T_0$ ). Let  $X$  be any set, let  $U \subsetneq X$  with  $|X \setminus U| > 1$ , let  $V \subsetneq U$  with  $|V| = 1$ , and let

$$\mathcal{T} = \mathcal{P}(X) \setminus \mathcal{P}(X \setminus U) \setminus \mathcal{P}(V) \cup \{\emptyset, X\}.$$

For any two distinct points  $x, y \in U$ , the family  $\mathcal{N}_x$  of neighbourhoods of  $x$  and the family  $\mathcal{N}_y$  of neighbourhoods of  $y$  are different; and for all such  $x$  and  $y$ ,  $x \notin \overline{\{y\}} = \{y\}$  (i.e., they are separated; but, be caution, they are not necessarily be separated by neighbourhoods; for if  $y \in V$ , the smallest neighbourhood of  $y$  is  $X$ ). Thus  $(X, \mathcal{T})$  is  $R_0$ . But  $X$  is not  $T_0$ , because for two distinct points  $x, y \in X \setminus U \cup V$ , the families of their neighbourhoods are the same.

**Example 3.2.3** ( $T_0$  but not  $R_0$ ). Let  $(\mathbb{R}_{\geq 0}, \mathcal{T})$  be a topological space with

$$\mathcal{T} = \{U \subseteq \mathbb{R} : \forall i \in \mathbb{R}_{\geq 0}, U_i = [0, i)\},$$

Then for all  $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$ , if  $x \neq y$ , then there are  $|y - x|$  neighbourhoods  $N_x$  of  $x$  do not contain  $y$ . Thus, it is  $T_0$ .

On the other hand, it is not  $R_0$ , because for all  $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$  with  $x < y$ ,  $x \in [0, y]$

**Proposition 3.2.1** (alternative definitions of  $R_0$  spaces). Let  $(X, \mathcal{T})$  be  $R_0$ , then the following conditions are equivalent.

- (i) The closure of all singletons in  $X$  are not  $T_0$  subspace.
- (ii) For any two points  $x, y \in X$ ,  $x \in \overline{\{y\}}$  iff  $y \in \overline{\{x\}}$ .

**Definition 3.2.5** ( $T_1$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_1$  or *Fréchet* iff it is  $T_0$  and  $R_0$ .

**Proposition 3.2.2** (alternative definitions of  $T_1$  spaces). Let  $(X, \mathcal{T})$  be  $T_1$ , then the following conditions are equivalent.

- (i) All singletons in  $X$  are closed.
- (ii) Every subset of  $X$  is the intersection of all open sets containing it.
- (iii) Every cofinite subset of  $X$  is open.

**Proposition 3.2.3.** All singletons in a  $T_1$  space are closed, That is, if a topological space  $(X, \mathcal{T})$  is  $T_1$ , then

$$\forall x \in (X, \mathcal{T}) : \exists U \in \mathcal{T} : \{x\} = X \setminus U.$$

**Definition 3.2.6** ( $T_2$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_2$  or *Hausdorff* or *separated* iff any two distinct points in  $(X, \mathcal{T})$  are separated by neighbourhoods.

**Example 3.2.4** ( $T_2$  but not  $T_1$ ). Let  $X$  be an nonempty set, and let  $\mathcal{U} = \mathcal{P}(X \setminus \{x \in X\})$ . Then the topological space  $(X, \mathcal{T})$  with

$$\mathcal{T} = \mathcal{U} \cup \{X\}$$

is  $T_1$ . But it is  $T_2$  iff  $|X| = 1$  (This is vacuously true). As  $|X| > 1$ ,  $\{x\}$  is not open.

**Proposition 3.2.4.** All metric spaces are Hausdorff.

**Definition 3.2.7** ( $T_{2\frac{1}{2}}$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_{2\frac{1}{2}}$  or *Urysohn* iff two distinct points in  $X$  are separated by closed neighbourhoods.

**Definition 3.2.8** ( $T_3$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_3$  or *regular* iff it is  $T_0$  and given any point  $x \in (X, \mathcal{T})$  and closed set  $V \subseteq X$  with  $x \notin V$  are separated by neighbourhoods.

**Definition 3.2.9** ( $T_{3\frac{1}{2}}$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_{3\frac{1}{2}}$ , or *Tychonoff* or, *completely*  $T_3$ , or *completely regular*, iff it is  $T_0$  and given any point  $x$  and closed set  $V \subseteq X$  with  $x \notin V$ , they are separated by a continuous function.

**Definition 3.2.10** ( $T_4$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_4$  or *normal* iff it is Hausdorff and any two disjoint closed subsets of  $X$  are separated by neighbourhoods.

**Proposition 3.2.5** (Urysohn's lemma). A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

**Definition 3.2.11** ( $T_5$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_5$  or *completely  $T_4$*  iff it is  $T_1$  and any two separated sets are separated by neighbourhoods.

**Proposition 3.2.6.** Every subspace of a  $T_5$  space is normal.

**Definition 3.2.12** ( $T_6$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_6$ , or *perfectly  $T_4$*  or *perfectly normal* iff it is  $T_1$  and any two disjoint closed sets are precisely separated by a continuous function.

**Proposition 3.2.7** (Tietze extension theorem). Let  $(X, \mathcal{T})$  be normal topological space, and let  $f : A \rightarrow (\mathbb{R}, \mathcal{T}')$  be a continuous map where  $A$  is a closed subset of  $X$  and  $\mathcal{T}'$  is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \rightarrow (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$

### 3.3 Countability Axioms

### 3.4 Metrizable

### 3.5 Compactness

**Definition 3.5.1** (compactness). A topological space  $(X, \mathcal{T})$  is said to be *compact* iff every open cover of  $X$  has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

### 3.6 Connectedness

**Definition 3.6.1** (connectedness). Let  $(X, \mathcal{T})$  be a topological space.  $(X, \mathcal{T})$  is said to be *connected* iff  $X$  is not empty and it is not the union of any disjoint

open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

**Definition 3.6.2** (path-connectedness). Let  $(X, \mathcal{T})$  be a topological space.

- (i) A map  $\gamma : [0, 1] \rightarrow X$  is called a *path* in  $X$  iff it is continuous. If  $\gamma(0) = x$  and  $\gamma(1) = y$ , we say that  $\gamma$  is path from  $x$  to  $y$  in  $X$ .
- (ii)  $X$  is said to be *path-connected* iff for all  $x, y \in X$  there is a path from  $x$  to  $y$  in  $X$ .