Notes for General Topology

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Chapter 1

Definitions

1.1 Basic Definitions

Definition 1.1.1 (Topology and Topological Space). Let X be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is called a topology on X iff

  (i) $\emptyset, X \in \mathcal{T}$;

  (ii) $\mathcal{T}$ is closed under arbitrary union;

  (iii) $\mathcal{T}$ is closed under finite intersection.

The pair $(X, \mathcal{T})$ is called a *topological space*. The elements of $\mathcal{T}$ are called *open sets* in $(X, \mathcal{T})$.

Definition 1.1.2 (Metrizable Topology). Let $(X, \mathcal{T})$ be a topological space.

1.2 Untitled

Definition 1.2.1 (Cover). Let $(X, \mathcal{T})$ be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of U iff the union of $\mathcal{C}$ is a superset of U. That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call $\mathcal{C}$ an *open cover* of U.

Let $\mathcal{C}' \subseteq \mathcal{C}$, iff the union of $\mathcal{C}'$ is still a superset of U, then we call $\mathcal{C}'$ a subcover of $\mathcal{C}$.

Definition 1.2.2 (Basis)**.** Let $(X, \mathcal{T})$ be a topological space, let $U \subseteq X$, and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a cover of X. We call $\mathcal{B}$ a *base* of $(X, \mathcal{T})$ iff $\mathcal{B} \subseteq \mathcal{T}$ and the union of $\mathcal{B}$ is exactly U itself. That is,

$$\mathcal{B} \subseteq \mathcal{T}, \text{ and } U = \bigcup \mathcal{B}.$$

Definition 1.2.3 (Subspace Topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The *subspace topology* $\mathcal{T}_A$ on A is defined to be the family of the intersections of open sets in $(X, \mathcal{T})$ and A. That is,

$$\mathcal{T}_A = \{ U \cap A : \ U \in \mathcal{T} \} .$$

Definition 1.2.4 (Quotient Topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{ U \in \mathcal{P}(X/\sim) : \{ x \in X : [x] \in U \} \in \mathcal{T}_X \} .$$

Definition 1.2.5 (Continuous Functions)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is said to be *continuous* iff for all open subset U of Y, the preimage $f^{-1}[U]$ is open in X. That is,

$$\forall U \in \mathcal{T}_Y, \quad f^{-1}[U] \in \mathcal{T}_X.$$

Definition 1.2.6 (Homeomorphisms)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

Definition 1.2.7 (Homeomorphic)**.** Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

Definition 1.2.8 (Compactness)**.** A topological space $(X, \mathcal{T})$ is said to be *compact* iff every open cover of X has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X, \ \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X, \quad |\mathcal{S}| < \aleph_0.$$

Definition 1.2.9 (Connectedness)**.** Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is said to be *connected* iff X is not empty and it it not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V, \quad U \cap V \neq \emptyset.$$

Definition 1.2.10 (Path-connectedness). Let $(X, \mathcal{T})$ be a topological space.

(i) A map $\gamma : [0,1] \to X$ is called a *path* in X iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that γ is path from x to y in X.

(ii) X is said to be *path-connected* iff for all $x, y \in X$ there is a path from x to y in X.

Definition 1.2.11 (T_0 space). A topological space $(X, \mathcal{T})$ is said to be T_0 or *Kolmogorov*, iff all distinct points $x, y \in X$ are *topologically distinguishable*. That is

$$\forall x, y \in X : x \neq y, \ \exists N \in \mathcal{T}, \quad x \in N \wedge y \notin N.$$

Definition 1.2.12 (R_0 space).