

Notes for General Topology

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Chapter 1

Metric Spaces

Chapter 2

Topological Spaces

2.1 Topological Spaces

Definition 2.1.1 (topology). Let X be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. \mathcal{T} is called a topology on X iff

- (i) $\emptyset, X \in \mathcal{T}$;
- (ii) \mathcal{T} is closed under arbitrary union;
- (iii) \mathcal{T} is closed under finite intersection.

Definition 2.1.2 (topological spaces). Let X be any set, and let \mathcal{T} be a topology on X , then the pair (X, \mathcal{T}) is called a *topological space*. All subsets of X in \mathcal{T} are called *open sets* in (X, \mathcal{T}) .

Definition 2.1.3 (closed sets). Let (X, \mathcal{T}) be a topological space. A subset V of X is said to be *closed* iff there is an open set U in X such that

$$V = X \setminus U.$$

Definition 2.1.4 (finer and coarser topology). Let X be any set, and let $\mathcal{T}, \mathcal{T}'$ be topologies on X . \mathcal{T} is said to be *finer* than \mathcal{T}' iff $\mathcal{T} \supseteq \mathcal{T}'$; respectively, \mathcal{T} is said to be *coarser* than \mathcal{T}' iff $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 2.1.5 (neighbourhood). Given (X, \mathcal{T}) as a topological space and a point $x \in X$, a subset $N \subseteq X$ is called a *neighbourhood* iff it contains an open set U containing x .

Proposition 2.1.1. Given (X, \mathcal{T}) as a topological space and $U \subseteq X$, U is open iff for all $x \in U$, there is a neighbourhood N of x contained in U .

Proof. If U is open, then U itself is a neighbourhood of x contained in U .

Conversely, if for all $x \in U$, there is a neighbourhood N_x of x contained in U , then there is a open neighbourhood $U_x \ni x$ contained in N_x . Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose U is not open, then U is a proper superset in the relation above. Then there exists $y \in U$ which is not in any U_x . This implies that such a y does not have any neighbourhood N_y in U , for such an N_y must contains an open $U_y \ni y$. For if it does, then there must be a U_x contains y . This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open. □

2.2 Metrizable Spaces

2.3 Continuity

Definition 2.3.1 (continuous maps). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f : X \rightarrow Y$ is said to be *continuous* iff for any open set U in Y , its preimage in X under f is open.

Proposition 2.3.1. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f : X \rightarrow Y$ is continuous at $x \in X$ iff for any neighbourhood N_y of $f(x)$, there is a neighbourhood N_x of x , such that $f[N_x] \subseteq N_y$.

Proof. Let N_y be a neighbourhood of $f(x)$. Clearly, there exists an open set U_y contains y .

By Definition 2.3.1, f is continuous at x iff $x \in f^{-1}[U_y] \in \mathcal{T}_X$. Clearly, $f^{-1}[U_y]$ is a neighbourhood of x . We have $f[f^{-1}[U_y]] = U_y \subseteq N_y$.

By Proposition 2.1.1, there U_x must contains at least one neighbourhood N_x of x , thus, $f[N_x] \subseteq U_y$. □

Proposition 2.3.2. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be metrizable spaces. A map $f : X \rightarrow Y$ is continuous at $p \in X$ iff for any $\varepsilon > 0$, there is a $\delta > 0$, such that

for all $x \in B_X(p, \delta)$, $f(x) \in B_Y(f(p), \varepsilon)$, where B_X is defined by any metrics ρ_X induces \mathcal{T}_X , and B_Y is defined by any metrics ρ_Y induces \mathcal{T}_Y .

Proof. Clearly, for all $\varepsilon > 0$, $B_Y(f(x), \varepsilon)$ is an open neighbourhood of $f(x)$.

f is not necessarily be injective, so $f^{-1}[B_Y(f(x), \varepsilon)] = U \in x$. By Definition 2.3.1, U is open, so for some $\delta > 0$, $B_X(x, \delta) \subseteq U$. Thus, By Proposition 2.3.1, f is continuous iff $f[B_X(x, \delta)] \subseteq B_Y(f(x), \varepsilon)$. This satisfies the conditions we have. \square

Proposition 2.3.3. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A function $f : X \rightarrow Y$ is said to be continuous iff for any closed set V in Y , its preimage in X under f is closed.

Proof. Let U_Y be any open set in Y , let U_X be the preimage of U_Y under f . By Definition 2.3.1, U_X is open in X . Let

$$V_X = f^{-1}[Y \setminus U_Y] = X \setminus U_X,$$

Then V_X is closed. \square

2.4 Cover

Definition 2.4.1 (cover). Let (X, \mathcal{T}) be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of U iff the union of all sets in \mathcal{C} is a superset of U . That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call \mathcal{C} an *open cover* of U .

Let $\mathcal{S} \subseteq \mathcal{C}$, iff the union of \mathcal{S} is still a superset of U , then we call \mathcal{S} a *subcover* of \mathcal{C} .

Definition 2.4.2 (basis). Let (X, \mathcal{T}) be a topological space, let $U \subseteq X$, and let \mathcal{B} be a open cover of X . We call \mathcal{B} a *base* of X iff the union of \mathcal{B} is precisely U itself, i.e.,

$$U = \bigcup \mathcal{B}.$$

Definition 2.4.3 (synthetic basis). Let (X, \mathcal{T}) be a topological space, and let \mathcal{B} be a base of X . \mathcal{B} is said to be *synthetic* iff for any $A, B \in \mathcal{B}$,

$$A \cap B = \bigcup_{i=1}^n B_i, \quad B_i \in \mathcal{B}.$$

2.5 Untitled

Definition 2.5.1 (subspace topology). Let (X, \mathcal{T}) be a topological space and let $A \subseteq X$. The *subspace topology* \mathcal{T}_A on A is defined to be the family of the intersections of open sets in (X, \mathcal{T}) and A . That is,

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}.$$

Definition 2.5.2 (quotient topology). Let (X, \mathcal{T}) be a topological space and let \sim be an equivalence relation on X . The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

Definition 2.5.3 (homeomorphisms). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A bijection $f : X \rightarrow Y$ is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

Definition 2.5.4 (homeomorphic). Two topological spaces (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

Definition 2.5.5 (compactness). A topological space (X, \mathcal{T}) is said to be *compact* iff every open cover of X has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

Definition 2.5.6 (connectedness). Let (X, \mathcal{T}) be a topological space. (X, \mathcal{T}) is said to be *connected* iff X is not empty and it is not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

Definition 2.5.7 (path-connectedness). Let (X, \mathcal{T}) be a topological space.

- (i) A map $\gamma : [0, 1] \rightarrow X$ is called a *path* in X iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that γ is path from x to y in X .
- (ii) X is said to be *path-connected* iff for all $x, y \in X$ there is a path from x to y in X .

Definition 2.5.8 (topologically indistinguishable). Let (X, \mathcal{T}) be a topological space. Two points $x, y \in X$ are said to be *topologically indistinguishable*

iff they share all their neighbourhoods. That is, let \mathcal{N}_x be the family of all neighbourhoods of x and let \mathcal{N}_y be the family of all neighbourhoods of y , we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively, x, y are said to be *topologically distinguishable* iff they are not topologically distinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

Definition 2.5.9 (separated sets). Let (X, \mathcal{T}) be a topological space, and let $A, B \in \mathcal{P}(X)$.

- (i) A and B are said to be *separated* iff each is disjoint from other's closure.
- (ii) A and B are said to be *separated by neighbourhoods* iff there are neighbourhoods N_A of A and N_B of B such that N_A and N_B are disjoint.
- (iii) A and B are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods \overline{N}_A of A and \overline{N}_B of B such that \overline{N}_A and \overline{N}_B are disjoint.
- (iv) A and B are said to be *separated by a continuous function* iff there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $f[A] = \{0\}$ and $f[B] = \{1\}$.
- (v) A and B are said to be *precisely separated by a continuous function* iff there is a continuous function $f : X \rightarrow \mathbb{R}$, such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$

[See Wikipedia.org](https://en.wikipedia.org/wiki/Topological_spaces)

Definition 2.5.10 (T_0 spaces). A topological space (X, \mathcal{T}) is said to be T_0 or *Kolmogorov*, iff all distinct points $x, y \in X$ are *topologically distinguishable*.

Definition 2.5.11 (R_0 spaces). A topological space (X, \mathcal{T}) is said to be R_0 iff any two topologically distinguishable points in X are separated.

Definition 2.5.12 (T_1 spaces). A topological space (X, \mathcal{T}) is said to be T_1 or *Fréchet* iff any two distinct points in X are separated.

Proposition 2.5.1. All singletons in a T_1 space are closed, That is, if a topological space (X, \mathcal{T}) is T_1 , then

$$\forall x \in (X, \mathcal{T}) : \exists U \in \mathcal{T} : \{x\} = X \setminus U.$$

Definition 2.5.13 (T_2 spaces). A topological space (X, \mathcal{T}) is said to be T_2 or *Hausdorff* or *separated* iff any two distinct points in (X, \mathcal{T}) are separated by neighbourhoods.

Definition 2.5.14 ($T_{21/2}$ spaces). A topological space (X, \mathcal{T}) is said to be $T_{21/2}$ or *Urysohn* iff two distinct points in X are separated by closed neighbourhoods.

Definition 2.5.15 (T_3 spaces). A topological space (X, \mathcal{T}) is said to be T_3 or *regular* iff it is T_0 and given any point $x \in (X, \mathcal{T})$ and closed set $V \subseteq X$ with $x \notin V$ are separated by neighbourhoods.

Definition 2.5.16 ($T_{31/2}$ spaces). A topological space (X, \mathcal{T}) is said to be $T_{31/2}$, or *Tychonoff* or, *completely* T_3 , or *completely regular*, iff it is T_0 and given any point x and closed set $V \subseteq X$ with $x \notin V$, they are separated by a continuous function.

Definition 2.5.17 (T_4 spaces). A topological space (X, \mathcal{T}) is said to be T_4 or *normal* iff it is Hausdorff and any two disjoint closed subsets of X are separated by neighbourhoods.

Proposition 2.5.2 (Urysohn's lemma). A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

Definition 2.5.18 (T_5 spaces). A topological space (X, \mathcal{T}) is said to be T_5 or *completely* T_4 iff it is T_1 any two separated sets are separated by neighbourhoods.

Proposition 2.5.3. Every subspace of a T_5 space is normal.

Definition 2.5.19 (T_6 spaces). A topological space (X, \mathcal{T}) is said to be T_6 , or *perfectly* T_4 or *perfectly normal* iff it is T_1 and any two disjoint closed sets are precisely separated by a continuous function.

Proposition 2.5.4 (Tietze extension theorem). Let (X, \mathcal{T}) be normal topological space, and let $f : A \rightarrow (\mathbb{R}, \mathcal{T}')$ be a continuous map where A is a closed subset of X and \mathcal{T}' is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \rightarrow (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$

2.6 Limit Points

Definition 2.6.1 (limit points). Let (X, \mathcal{T}_X) be a topological space, and let $A \subseteq X$. A point $x \in X$ is called a *limit point* of A iff for all neighbourhood N_x of x , $N_x \setminus \{x\}$ intersects A .

Proposition 2.6.1. Let A be any set, and let x be a limit point of A , then x is an element of the closure of A .

Proof. If A is empty, then this is vacuously true. So, suppose A is not empty.

By Definition 2.6.1, for all neighbourhood N_x of x , $N_x \setminus \{x\} \cap A$ is not empty. Naturally, $N_x \cap A$ is not empty. As A is not empty, $N_x \cap A^\circ$ is not empty, either, for if it is empty, then

$$N_x \cap \partial A = (N_x \cap A) \setminus (N_x \cap A^\circ) = N_x \cap A.$$

This is impossible, for if $\partial A = A$, then A° is empty.

Assume that $x \notin \overline{A}$, then $X \setminus \overline{A}$ is a neighbourhood of x , by Definition 2.1.5, and is disjoint from A . This is contradicted to the conditions. \square

Definition 2.6.2 (convergent sequences). Let (X, \mathcal{T}_X) be a topological space. A sequence $\{x_n\}$ in X is said to be *convergence* in X iff there is an open set U contains all but finite terms of $\{x_n\}$.

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Proposition 2.6.1. Let A be any set, and let x be a limit point of A , then x is an element of the closure of A .

Proof. By Definition 2.6.1, for all open neighbourhood U_x of x , $U_x \setminus \{x\} \cap A$ is not empty, naturally, $U_x \cap A^0$ is not empty and open, thus $U_x^c \cup A^{0c}$ is closed. If $x \notin \text{Int}A$. \square

Definition 2.6.2 (convergent sequences). Let (X, \mathcal{T}_X) be a topological space. A sequence $\{x_n\}$ in X is said to be *convergence* in X iff there is an open set U contains all but finite terms of $\{x_n\}$.