Notes for General Topology

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Chapter 1

Metric Spaces

Chapter 2

Topological Spaces

2.1 Topological Spaces

Definition 2.1.1 (topology)**.** Let X be a set, and let a family $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is called a topology on X iff

(i) $\emptyset, X \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under arbitrary union;

(iii) $\mathcal{T}$ is closed under finite intersection.

The pair $(X, \mathcal{T})$ is called a *topological space*. The elements of $\mathcal{T}$ are called *open sets* in $(X, \mathcal{T})$.

Definition 2.1.2 (topological spaces)**.** Let X be any set, and let $\mathcal{T}$ be a topology on X, then the pair $(X, \mathcal{T})$ is called a *topological space*. All subsets of X in $\mathcal{T}$ are called *open sets* of $(X, \mathcal{T})$.

Definition 2.1.3 (finer and coarser topology)**.** Let X be any set, and let $\mathcal{T}, \mathcal{T}'$ be topologies on X. $\mathcal{T}$ is said to be *finer* than $\mathcal{T}'$ iff $\mathcal{T} \supseteq \mathcal{T}'$; respectively, $\mathcal{T}$ is said to be *coarser* than $\mathcal{T}'$ iff $\mathcal{T} \subseteq \mathcal{T}'$.

Definition 2.1.4 (neighbourhood)**.** Given $(X, \mathcal{T})$ as a topological space and a point $x \in X$, a subset $N \subseteq X$ is called a *neighbourhood* iff it contains an open set U containing x.

Proposition 2.1.1. Given $(X, \mathcal{T})$ as a topological space and $U \subseteq X$, U is open iff for all $x \in U$, there is a neighbourhood N of x contained in U.

Proof. If U is open, then U itself is a neighbourhood of x contained in U.

Conversely, if for all $x \in U$, there is a neighbourhood N_x of x contained in U, then there is a open neighbourhood $U_x \ni x$ contained in N_x. Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose U is not open, then U is a proper superset in the relation above. Then there exists $y \in U$ which is not in any U_x. This implies that such a y does not have any neighbourhood N_y in U, for such an N_y must contains an open $U_y \ni y$. For if it does, then there must be a U_x contains y. This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open. $\square$

2.2 Untitled

Definition 2.2.1 (cover)**.** Let $(X, \mathcal{T})$ be a topological space, and let $U \subseteq X$, then a family $\mathcal{C} \subseteq \mathcal{P}(X)$ is called a *cover* of U iff the union of $\mathcal{C}$ is a superset of U. That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If $\mathcal{C} \subseteq \mathcal{T}$, then we call $\mathcal{C}$ an *open cover* of U.

Let $\mathcal{C}' \subseteq \mathcal{C}$, iff the union of $\mathcal{C}'$ is still a superset of U, then we call $\mathcal{C}'$ a subcover of $\mathcal{C}$.

Definition 2.2.2 (basis)**.** Let $(X, \mathcal{T})$ be a topological space, let $U \subseteq X$, and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a cover of X. We call $\mathcal{B}$ a *base* of $(X, \mathcal{T})$ iff $\mathcal{B} \subseteq \mathcal{T}$ and the union of $\mathcal{B}$ is exactly U itself. That is,

$$\mathcal{B} \subseteq \mathcal{T}, \text{ and } U = \bigcup \mathcal{B}.$$

Definition 2.2.3 (subspace topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The *subspace topology* $\mathcal{T}_A$ on A is defined to be the family of the intersections of open sets in $(X, \mathcal{T})$ and A. That is,

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}.$$

Definition 2.2.4 (quotient topology)**.** Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* is a topology on $\mathcal{P}(X/\sim)$; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

Definition 2.2.5 (continuous functions)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is said to be *continuous* iff for all open subset U of Y, the preimage $f^{-1}[U]$ is open in X. That is,

$$\forall U \in \mathcal{T}_Y : f^{-1}[U] \in \mathcal{T}_X.$$

Definition 2.2.6 (homeomorphisms)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A bijection $f : X \to Y$ is called a *homeomorphism* iff it is continuous and its inverse is also continuous.

Definition 2.2.7 (homeomorphic)**.** Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are said to be *homeomorphic* or *topologically equivalent*, denoted $X \cong Y$, iff there is an homeomorphism between them.

Definition 2.2.8 (compactness)**.** A topological space $(X, \mathcal{T})$ is said to be *compact* iff every open cover of X has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

Definition 2.2.9 (connectedness)**.** Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is said to be *connected* iff X is not empty and it it not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

Definition 2.2.10 (path-connectedness)**.** Let $(X, \mathcal{T})$ be a topological space.

(i) A map $\gamma : [0, 1] \to X$ is called a *path* in X iff it is continuous. If $\gamma(0) = x$ and $\gamma(1) = y$, we say that γ is path from x to y in X.

(ii) X is said to be *path-connected* iff for all $x, y \in X$ there is a path from x to y in X.

Definition 2.2.11 (topologically indistinguishable)**.** Let $(X, \mathcal{T})$ be a topological space. Two points $x, y \in X$ are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let $\mathcal{N}_x$ be the family of all neighbourhoods of x and let $\mathcal{N}_y$ be the family of all neibourhoods of y, we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively, x, y are said to be *topologically distinguishable* iff they are not topologically distinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

Definition 2.2.12 (saperated sets)**.** Let $(X, \mathcal{T})$ be a topological space, and let $A, B \in \mathcal{P}(X)$.

(i) A and B are said to be *separated* iff each is disjoint from other's closure.

(ii) A and B are said to be *separated by neighbourhoods* iff there are neighbourhoods N_A of A and N_B of B such that N_A and N_B are disjoint.

(iii) A and B are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods $\overline{N}_A$ of A and $\overline{N}_B$ of B such that $\overline{N}_A$ and $\overline{N}_B$ are disjoint.

(iv) A and B are said to be *separated by a continuous function* iff there is a continuous function $f : X \to \mathbb{R}$, such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

(v) A and B are said to be *precisely separated by a continuous function* iff there is a continuous function $f : X \to \mathbb{R}$, such that $f^{-1}[\{0\}] = A$ and $f^{-1}[\{1\}] = B$

See Wikipedia.org

Definition 2.2.13 (T_0 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_0 or *Kolmogorov*, iff all distinct points $x, y \in X$ are *topologically distinguishable.*

Definition 2.2.14 (R_0 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be R_0 iff any two topologically distinguishable points in X are separated.

Definition 2.2.15 (T_1 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_1 or *Fréchet* iff any two distinct points in X are separated.

Proposition 2.2.1. All singletons in a T_1 space are closed, That is, if a topological space $(X, \mathcal{T})$ is T_1, then

$$\forall x \in (X, \mathcal{T}) : \exists U \in \mathcal{T} : \{x\} = X \setminus U.$$

Definition 2.2.16 (T_2 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_2 or *Hausdorff* or *separated* iff any two distinct points in $(X, \mathcal{T})$ are separated by neighbourhoods.

Definition 2.2.17 ($T_{2^1/_2}$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_{2^1/_2}$ or *Urysohn* iff two distinct points in X are separated by closed neighbourhoods.

Definition 2.2.18 (T_3 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_3 or *regular* iff it is T_0 and given any point $x \in (X, \mathcal{T})$ and closed set $V \subseteq X$ with $x \notin V$ are separated by neighbourhoods.

Definition 2.2.19 ($T_{3^1/_2}$ spaces)**.** A topological space $(X, \mathcal{T})$ is said to be $T_{3^1/_2}$, or *Tychonoff* or, *completely* T_3, or *completely regular*, iff it is T_0 and given any point x and closed set $V \subseteq X$ with $x \notin V$, they are separated by a continuous function.

Definition 2.2.20 (T_4 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_4 or *normal* iff it is Hausdorff and any tow disjoint closed subsets of X are separated by neighbourhoods.

Proposition 2.2.2 (Urysohn's lemma)**.** A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

Definition 2.2.21 (T_5 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_5 or *completely* T_4 iff it is T_1 any two separated sets are separated by neighbourhoods.

Proposition 2.2.3. Every subspace of a T_5 space is normal.

Definition 2.2.22 (T_6 spaces)**.** A topological space $(X, \mathcal{T})$ is said to be T_6, or *perfectly* T_4 or *perfectly normal* iff it is T_1 and any two disjoint closed sets are precisely separated by a continuous function.

Proposition 2.2.4 (Tietze extension theorem)**.** Let $(X, \mathcal{T})$ be normal topological space, and let $f : A \to (\mathbb{R}, \mathcal{T}')$ be a continuous map where A is a closed subset of X and $\mathcal{T}'$ is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$