Notes for General Topology by Tom Leinster

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April 13, 2021

Chapter 1

Topological Spaces

1.1 Review of Metric Spaces

Definition 1.1.1. Let X be a set. A *metric* on X is a function $\rho : X \times X \to \mathbb{R}_{\geq 0}$, such that $\forall x, y, z \in X$, the following (metric axioms) holds:

M1. $\rho(x, y) = 0 \iff x = y$ (identity of indiscernibles);

M2. $\rho(x, y) = \rho(y, x)$ (symmetry).

M3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (triangle inequality);

A *metric space* is a set together with a metric on it, or more formally, a pair (X, ρ) where X is a set and ρ is a metric on X.

Example 1.1.1.

1. The function $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $\forall p \in \overline{\mathbb{R}}_{\geq 1}$, $\forall x, y \in \mathbb{R}^n$,

   $$\rho_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}},$$

   is a metric on $\mathbb{R}^n$. Clearly it satisfies identity of indiscernibles and symmetry. For triangle inequality, it is suggested by Minkowski inequality.

   Given $x \in \mathbb{R}^3$, $r \in \mathbb{R}_{\geq 0}$, and

   $$B_\rho = \left\{ y \in \mathbb{R}^3 \mid \rho(x, y) \leq r \right\}.$$

   $\forall p, q \in \overline{\mathbb{R}}_{\geq 1}$, it is true that, $\forall x, y \in \mathbb{R}^n$,

   $$p \leq q \implies \rho_p(x, y) \geq \rho_q(x, y).$$

   Thus, $B_p \subseteq B_q$.

Geometrically, as $p = 1$, B is a octahedron in $\mathbb{R}^3$ with center x and radius r; as $p = 2$, B is a sphere in $\mathbb{R}^3$ with center x and radius r. It is easy to observe that as $p \to \infty$, B tends to a cube in $\mathbb{R}^3$ with center x and edge length $2r$; i.e.,

$$\rho_\infty(x, y) = \lim_{p\to\infty} \rho_p(x, y) = \sup_{i\in\{1,\ldots,n\}} |x_i - y_i|.$$

2. Let $f : (X, \rho) \to \mathbb{R}^n$ with $X \subseteq \mathbb{R}^m$ be a continuous map on X. Let $x, y \in X$, then $\rho' : f[X] \times f[X] \to \mathbb{R}_{\geq 0}$ defined by

$$\rho'_p(x, y) = \int_0^1 f(\ell(t)) d_p s(t)$$

where

$$\ell(t) = x + t(y - a)$$

and

$$d_p s(t) = \left(\sum_{i=1}^m \left| \frac{dg_i}{dt}(t) \right|^p \right)^{\frac{1}{p}} dt.$$

with $p \geq \overline{\mathbb{R}}_{\geq 1}$ is a metric on $f[X]$.

Fix x and given $r \in \mathbb{R}_{\geq 0}$, the set

$$B_p = \left\{ y \in \mathbb{R}^m : \rho'_p(x, y) \leq r \right\}$$

describes a set "attached" on $f[X]$ with center x. If $p = 2$, $m = 2$ and $n = 3$, and $f : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$ is defined by

$$f(\lambda, \phi) = \begin{cases} r \cos\lambda \sin\phi, \\ r \sin\lambda \sin\phi, \\ r \cos\phi, \end{cases}$$

then ρ'_2 here is a *great circle metric* defined by

$$\rho'_2(x, y) = r \arccos(\sin x_\phi \sin y_\phi + \cos x_\phi \cos y_\phi \cos(x_\lambda - y_\lambda)).$$

3. Let $a, b \in \mathbb{R}$ with $a \leq b$, and $p \in \overline{\mathbb{R}}_{\geq 1}$, and $C[a, b]$ denote the set of continuous function $[a, b] \to \mathbb{R}$.

Then d_p defined by $\forall f, g \in C[a, b]$,

$$\rho_p(f, g) = \left(\int_a^b |f - g|^p \right)^{\frac{1}{p}}$$

is a metric on $C[a, b]$.

Similar to ρ_p on $\mathbb{R}^n$,

$$B_p = \{g \mid \rho(f,g) \le r\}$$

defines a set with "center" f and "radius" $r \in \mathbb{R}_{\ge 0}$.

It also implies that, on $C[a,b]$, $\forall p, q \in \overline{\mathbb{R}}_{\ge 1}$, $\forall x, y \in \mathbb{R}^n$

$$p \le q \implies d_p(f,g) \ge d_q(f,g),$$

and, naturally, $B_p \subseteq B_q$. This is a straight corollary from the same case of d_p on $\mathbb{R}^n$.

4. Let A be a set. The *Hamming metric* ρ on a set A^n is given by $\forall x, y \in A^n$

$$\rho(x,y) = \#\left\{i \in \{1, \ldots, n\} : x_i \ne y_i\right\}.$$

An example from Wikipedia. The word "karolin" and "kathrin" can be considered as tuples

$$x = (\mathrm{k}, \mathrm{a}, \mathrm{r}, \mathrm{o}, \mathrm{l}, \mathrm{i}, \mathrm{n}), \; y = (\mathrm{k}, \mathrm{a}, \mathrm{t}, \mathrm{h}, \mathrm{r}, \mathrm{i}, \mathrm{n}).$$

For all $i \in \{0, \ldots, 6\} \setminus \{0, 1, 4, 6\}$, $x_i \ne y_i$, and $\#(\{0, \ldots, 6\} \setminus \{0, 1, 4, 6\}) = 3$, thus

$$\rho(x,y) = 3.$$

5. Let (M, ρ) be a metric space (for example, $\rho = \rho_2$ on $\mathbb{R}^n$), and $X, Y \in \mathcal{P}(M)$. The Hausdorff metric ρ_{H} on $\mathcal{P}(M)$ is defined by

$$\rho_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x \in X} \rho(x, Y), \sup_{y \in Y} \rho(X, y)\right\},$$

where $\rho(a, B) = \inf_{b \in B} \rho(a, b)$ for all $B \in \mathcal{P}(M)$ and $a \in M$.

This metric can be used to measure how close two figures (as sets of points) are.

Definition 1.1.2. Let X be a metric space, let $x \in X$, and $\varepsilon > 0$. The *open ball with center x and radius ε*, or more briefly the *open ε-ball about x* is the subset

$$B(x, \varepsilon) = \{y \in X : \rho(x,y) \le \varepsilon\} \subseteq X.$$

Similarly, the *closed ε-ball around x* is

$$\overline{B}(x, \varepsilon) = \{y \in X : \rho(x,y) \le \varepsilon\} \subseteq X.$$

Note 1.1.1. Clearly, the word "ball" does not mean it should look like a ball. Clearly, for all $x \in \mathbb{R}^3$, the ball $\{y \in \mathbb{R}^3 : \rho_\infty(x,y) < 1\}$ is a cube without its surface.

And it is interesting to think that on $C[a,b]$ with conditions above,

$$\{g \in C[a,b] : \rho_p(f,g) < 1\}$$

defines a open ball in $C[a,b]$.

Note 1.1.2. For hamming metric ρ with conditions above, for $\varepsilon \in \mathbb{R}_{(0,1)}$, the ball

$$\{y \in A^n : \rho(x,y) < 1\} = \{x\}.$$

is a singleton.

Definition 1.1.3. Let X be a metric space.

(i) A subset U of X is *open in* X (or an *open subset of* X) iff

$$\forall u \in U,\ \exists \varepsilon \in \mathbb{R}_{>0},\ \ B(u,\varepsilon) \subseteq U.$$

(ii) A subset V is *closed in* X iff $X \setminus V$ is open in X.

Note 1.1.3. Equivalently, U is open in X iff $\exists \varepsilon \in \mathbb{R}_{>0}$,

$$U = \bigcup_{x \in U} B(x,\varepsilon);$$

and V is closed in X iff

$$V = X \setminus \bigcup_{x \in U} B(x,\varepsilon) = \bigcap_{x \in U} (X \setminus B(x,\varepsilon)),$$

by De Morgan's Law.

Definition 1.1.4. Let X be a metric space, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and let $x \in X$. Then $\{x_n\}$ *converges* in X iff

$$\exists x \in X, \lim_{n\to\infty} d(x_n, x) = 0.$$

Explicitly, then, $\{x_n\}$ converges to x iff

$$\forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}_{\geq 1}, \forall n \in \mathbb{N}_{\geq N}, \quad d(x_n, x) < \varepsilon.$$

Note 1.1.4.

1. Equivalently, $\{x_n\}$ converges in X iff

   $$\exists x \in X,\ \forall \varepsilon \in \mathbb{R}_{>0}, \#(\{x_n\} \cap B(x,\varepsilon)) = \aleph_0 \wedge \#(\{x_n\} \setminus B(x,\varepsilon)) < \aleph_0.$$

   In other words, $B(x,\varepsilon)$ contains all but finitely many x_n.

2. Let $X \subseteq S$. $\{x_n\}$ converges to $x \in S$ does not means it need to converge in X. For example $\mathbb{Q} \subseteq \mathbb{R}$, the sequence

   $$\left\{x_n = \frac{1}{x} + r : r^2 = 2\right\}_{n \in \mathbb{N}}$$

   does converge to $\sqrt{2} \in \mathbb{R}$, but $\sqrt{2} \notin \mathbb{Q}$, so $\{x_n\}$ converges in $\mathbb{R}$, but does not converge in $\mathbb{Q}$.

Lemma 1.1.1. Let X be a metric space and $V \subseteq X$. Then V is closed in X iff

$$\forall \{x_n\}_{n=1}^{\infty} \subseteq V,\ \forall x \in X, \quad \lim_{n\to\infty} d(x_n, x) = 0 \implies x \in V.$$

Proof. Suppose V is closed in X, then $X \setminus V$ is open in X. Suppose $\exists x \in X \setminus V$, such that $\exists \{x_n\}_{n=1}^{\infty} \subseteq V$, $\{x_n\}$ converges to x, then $\forall \varepsilon \in \mathbb{R}_{>0}$, $B(x,\varepsilon) \cap \{x_n\} \neq \emptyset$. $\{x_n\} \subseteq V$, so $B(x,\varepsilon) \cap V \neq \emptyset$. This implies that $X \setminus V$ is not open, then V is not closed (for if V is closed, then $X \setminus V$ is open). It is contradicted to the assumption.

Now, suppose V is not closed in X, then $X \setminus V$ is not open. Then, $\exists p \in X \setminus V$, such that $\forall \varepsilon \in \mathbb{R}_{>0}$, $B(p,\varepsilon) \cap V \neq \emptyset$. This implies there are some $\{x_n\}_{n=1}^{\infty} \subseteq V$, such that $B(p,\varepsilon)$ contains all but finite elements in $\{x_n\}$. Thus, $\{x_n\}$ converges to $p \in X \setminus V$, contradicting to the conditions. □

Lemma 1.1.2. Let X be a metric space, and $\mathcal{T}$ be the family of open subsets of X. Then,

(i) $\mathcal{T}$ is closed under arbitrary union.

(ii) $\mathcal{T}$ is closed under finite intersection.

(iii) $\emptyset, X \in \mathcal{T}$.

Proof.

1. Let I be an index set. For all $i \in I$, let $U_i \in \mathcal{T}$. Then for some $\varepsilon \in \mathbb{R}_{>0}$,

   $$U_i = \bigcup_{x \in U_i} B(x,\varepsilon).$$

   Let $U = \bigcup_{i\in I} U_i$, then we have,

   $$U = \bigcup_{i\in I} \bigcup_{x\in U_i} B(x,\varepsilon) = \bigcup_{x \in U} B(x,\varepsilon)$$

   for some $\varepsilon \in \mathbb{R}_{>0}$.

2. Let $\mathcal{C}$ be the family of closed subsets of X, and let $U, V \in \mathcal{C}$. Then for all $\{u_n\}_{n=1}^{\infty} \subseteq U$, $\forall u \in X$, $\{u_n\}$ converges to u implies that $u \in U$. It also holds for $U \cup V \supseteq U$. Similarly, for all $\{v_m\}_{m=1}^{\infty}$, $\forall v \in X$, $\{v_m\}$ converges to v implies $v \in V$. It also holds for $U \cup V \supseteq V$. Thus $U \cup V$ is closed.

   Then, $X \setminus (U \cup V) = (X \setminus U) \cap (X \setminus V)$ is open, where $X \setminus U$ and $X \setminus V$ are open for U and V are closed.

3. $\emptyset = \bigcup_{i\in\emptyset} U_i$ for all $U_i \in \mathcal{T}$, so $\emptyset$ is open. $\emptyset = U \cap V$ for all mutually disjoint closed subsets $U, V \subseteq X$, so $\emptyset$ is closed, so $X = X \setminus \emptyset$ is open.

□

Lemma 1.1.3. Let X be a metric space, and $\mathcal{C}$ be the family of all closed subsets of X. Then,

(i) $\mathcal{C}$ is closed under arbitrary intersection.

(ii) $\mathcal{C}$ is closed under finite union.

(iii) $\emptyset, X \in \mathcal{C}$.

Proof. Let $\mathcal{T}$ be the family of all open subset of X, and let I be any index set.

1. It has been proved that $\mathcal{T}$ is closed under arbitrary union, so by De Morgan's law, for any $i \in I$, if $U_i \in \mathcal{T}$, then

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (X \setminus U_i) \in \mathcal{C}.$$

2. It has been proved in Lemma 1.1.2.

3. It has been proved that $\emptyset$ is open in X. So $X = X \setminus \emptyset$ is closed in X.

□

Definition 1.1.5. Let (X, ρ) and (Y, ρ') be metric spaces. A function $f : (X, \rho) \to (Y, \rho)$ is *continuous* on a point $p \in X$ iff

$$\forall \varepsilon \in \mathbb{R}_{>0},\ \exists \delta \in \mathbb{R}_{>0},\ \forall x \in B(p, \delta), \quad f(x) \in B(f(p), \varepsilon).$$

Note 1.1.5.

1. If ρ is a discrete metric on X, then $B(p, \delta) = \{p\}$ for all δ. Then, by definition, for all ε, $f(x) \in B(f(p), \varepsilon)$. So f is continuous everywhere.

2. On the contrary, if ρ' is a discrete metric on Y, but for all $p \in X$, ρ suffices for all $\delta \in \mathbb{R}_{>0}$, $\#B(p, \delta) \geq \aleph_0$, then for some $\varepsilon \in \mathbb{R}_{>0}$, for all $\delta \in \mathbb{R}_{>0}$, there exists $x \in B(p, \delta)$, such that $f(x) \notin B(f(p), \varepsilon)$. Thus f is not continuous on such p.

Lemma 1.1.4. Let (X, ρ) and (Y, ρ') be metric spaces and let $f : (X, \rho) \to (Y, \rho)$ be a function. The following are equivalent:

(i) f is continuous on X;

(ii) for all open $U \subseteq Y$, the preimage $f^{-1}[U] \subseteq X$ is open;

(iii) for all closed $V \subseteq Y$, the preimage $f^{-1}[V] \subseteq X$ is closed.

Proof. For (i) $\implies$ (ii), suppose f is continuous on X, then, by Definition 1.1.5, for all $p \in X$, there exists $B(p, \delta)$, such that $f(x) \in B(f(p), \varepsilon)$.

□