Notes for General Topology

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Contents

Chapter 1.

Metric Spaces

§1.1 Metric Spaces

Definition 1.1.1. Let X be any set.

A function $d : X \times X \to \mathbb{R}_{\geq 0}$ is *metric function*, or, simply, *metric on* X iff it satisfies the *metric axioms*. That is, for any $x, y, z \in X$:

M1. $d(x, y) = 0$ iff $x = y$;

M2. $d(x, y) = d(y, x)$;

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 1.1.2. Let X be any set and let d be a structure on X.

The pair (X, d) is called a *metric space* iff d is a metric on X.

Definition 1.1.3. A $\mathbb{X} = (X, d)$ be a metric space, let $x \in X$ and let $\varepsilon \in \mathbb{R}_{>0}$.

An *open* ε*-ball*, or just ε-ball, about x is defined to be the set

$$B_\varepsilon(x; d) := \{y \in X : d(x, y) < \varepsilon\}.$$

A *closed ball* is defined to be the set

$$\overline{B}_\varepsilon(x; d) := \{y \in X : d(x, y) \leq \varepsilon\}.$$

Note 1.1.1. As

$$\mathbb{X}_0 = (X, d_0),\ \mathbb{X}_1 = (X, d_1),\ \mathbb{X}_2 = (X, d_2),\ \dots$$

are different although they share the same set X, for any $x \in X$ and any $\varepsilon \in \mathbb{R}_{>0}$,

$$B_\varepsilon(x; d_1),\ B_\varepsilon(x; d_2),\ B(x; d_3),\ \dots$$

are also different. However, if confusion is unlikely, we simply write "$B_\varepsilon(x)$" for "$B_\varepsilon(x; d)$".

Example 1.1.1. The *Euclidean metric space* $\mathbb{X} = (X, d)$ is an n-dimensional set X equipped with the *Euclidean metric* d defined as

$$d(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}.$$

This is also called *standard Euclidean metric*, in contrast to the *non-standard Euclidean metrics*

$$d_p(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

In particular,

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

Example 1.1.2. A *discrete metric space* $\mathbb{X} = (X, d)$ is a set X equiped with the *discrete metric* d_{dsic} defined as

$$d_{\text{disc}}(x, y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{else.} \end{cases}$$

This is an equivalent definition of the discrete metric:

$$d_{\text{disc}}(x, y) := (\operatorname{sgn}(d(x, y)))^2,$$

where $\operatorname{sgn}(\cdot)$ is a sign function, and d is any metric on X.

Example 1.1.3. [1] Let $\mathbb{I} = (C[a, b], d_p)$ be a metric space where $C[a, b]$ denotes the set of all continuous mapping $\mathbb{R}_{[a,b]} \to \mathbb{R}$, and $p > 0$, and the metric d_p is defined as

$$d_p(f, g) := \left(\int_a^b |f(t) - g(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

[1] See Minkowski inequality.

In particular,

$$d_\infty(f,g) := \sup_{t \in \mathbb{R}_{[a,b]}} |f(t) - g(t)|.$$

Example 1.1.4. [2] Let $\mathbb{X} = (X, d)$ be a metric space. The *Hausdorff metric* d_H on $2^X \setminus \{\emptyset\}$ is defined as

$$d_H := \max\left\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\right\},$$

where

$$d(x, Y) := \inf_{y \in Y}(x, y), \text{ and } d(y, X) := \inf_{x \in X}(y, x).$$

§1.2 Open Sets in Metric Spaces

Definition 1.2.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $U \subseteq X$.

U is said to be *open in* $\mathbb{X}$, iff for any $y \in U$, there exists $\varepsilon \in \mathbb{R}_{>0}$, such that $B_\varepsilon(y) \subseteq U$.

Lemma 1.2.1. Let $\mathbb{X} = (X, d)$ be a metric space, let $x \in A$ and let $\varepsilon \in \mathbb{R}_{>0}$.

For any $y \in B_\varepsilon(x)$, there is a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_\varepsilon(x)$.

Proof. For any $y \in B_\varepsilon(x)$, by the definition of open balls (Definition 1.1.3), we have $d(x, y) < \varepsilon$.

Let $\delta \in \mathbb{R}_{>0}$ such that $\delta + d(x, y) = \varepsilon$.

By M3 in metric axioms (Definition 1.1.1), for any $z \in A$ with $d(y, z) < \delta$, we have

$$d(x, z) \le d(y, z) + d(x, y) < \varepsilon.$$

Thus, again, by the definition of open balls, we have $B_\delta(y) \subseteq B_\varepsilon(x)$.

□

Theorem 1.2.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $U \subseteq X$.

U is open in $\mathbb{X}$ iff it is a union of open balls.

Proof. First, prove $\Rightarrow$.

As U is open, for any $y \in U$, there exists $\varepsilon_y \in \mathbb{R}_{>0}$ such that $B_{\varepsilon_y}(y) \subseteq U$.

[2] See Hausdorff distance.

Therefore,

$$U = \bigcup_{y \in U} B_{\varepsilon_y}(y).$$

$\square$

Now, prove $\Leftarrow$.

Aiming for a contradiction, suppose U is a union of open balls but not open.

As U is not open, there is a $y \in U$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \not\subseteq U$.

As U is a union of open balls, there is an $x \in U$ and $r \in \mathbb{R}_{>0}$ such that $y \in B_r(x)$.

By Lemma 1.2.1, there exists a $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(y) \subseteq B_r(x)$.

This is a contradiction by the assumption.

Thus, U has to be open.

$\blacksquare$

Theorem 1.2.2. Let $\mathbb{X} = (X, d)$ be any metric space.

$\mathbb{X}$ is *Hausdorff*. That is, For any distinct points $x, y \in X$, we can always find an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset.$$

Proof. Aiming for a contradiction, suppose there are $x, y \in X$ with $x \neq y$, such that for any $\varepsilon \in \mathbb{R}_{>0}$, we can always find a $z \in X$ such that

$$z \in B_\varepsilon(x) \cap B_\varepsilon(y).$$

Let $r = d(x, y)/2$, and let $z \in B_r(x) \cap B_r(y)$.

As $z \in B_r(x)$, by the definition of open balls (Definition 1.1.3), $d(x, z) < r$; as $z \in B_r(y)$, similarly, $d(y, z) < r$. Then we have

$$d(x, z) + d(y, z) < 2r = d(x, y).$$

This contradicts the metric axioms M3 (Definition 1.1.1).

Thus $\mathbb{X}$ is Hausdorff.

$\blacksquare$

Definition 1.2.2. Let $\mathbb{X} = (X, d)$ be any metric space, and let $V \subseteq X$.

V is said to be *closed* in $\mathbb{X}$, iff there is an open set U satisfies $X \setminus U = V$.

Lemma 1.2.2. In a metric space, any singleton is closed.

Proof. Let $\mathbb{X} = (X, d)$ be a metric space, let $x \in X$, and let $y \in X \setminus \{x\}$.

As M is Hausdorff (Theorem 1.2.2), there is an $\varepsilon \in \mathbb{R}_{>0}$ such that

$$0 < \varepsilon < d(x, y),$$

thus $X \setminus \{x\}$ is open, hence, by Definition 1.1.1, its complement $\{x\}$ is open.

$\square$

Theorem 1.2.3. Let $\mathbb{X} = (X, d)$ be a metric space, denote $\mathcal{T}$ for the family of open subsets of X.

Then $\mathcal{T}$ satisfies the following conditions:

O1. $X, \emptyset \in \mathcal{T}$;

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under arbitrary union;

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under finite intersection.

Proof.

O1. As $\emptyset$ is the subset of any set, $\emptyset \in \mathcal{T}$. $\bigcup \emptyset = \emptyset \in \mathcal{T}$.

By Definition 1.2.2, $X = X \setminus \emptyset$.

$\square$

O2. Let $\mathcal{U} \subseteq \mathcal{T}$, and denote $\mathcal{O}$ for the open balls in M.

For any $U \in \mathcal{U}$, there is an $\mathcal{O}_U \subseteq \mathcal{O}$ such that $U = \bigcup \mathcal{O}_U$.

Then we have

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \left(\bigcup \mathcal{O}_U \right) = \bigcup_{U \in \mathcal{U}} \mathcal{O}_U.$$

By Theorem 1.2.1, $\bigcup \mathcal{U}$ is open.

$\square$

O3. Let $\mathcal{V}$ be a finite subset of $\mathcal{T}$.

Aiming for a contradiction, suppose $\bigcap \mathcal{V}$ is not open.

By Definition 1.2.1, there exists a $y \in \bigcap \mathcal{V}$ such that for any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(y) \setminus \bigcap \mathcal{V} \neq \emptyset$.

By De Morgan's law, we have

$$\bigcup_{V \in \mathcal{V}} (B_\varepsilon(y) \setminus V) \neq \emptyset.$$

Thus, there exists $V \in \mathcal{V}$ such that $B_\varepsilon(y) \setminus V \neq \emptyset$.

As $V \in \mathcal{T}$ and ε is arbitrarily given, by Lemma 1.2.1, $y \notin V$. This is a contradiction.

Thus, $\bigcap \mathcal{V}$ is open.

□

Thus, the theorem is proved.

■

Theorem 1.2.4. Infinite intersections of open sets in some metric spaces are not necessarily open.

Proof. Consider $\mathbb{R}$ is a Euclidean metric space, and denote $\mathcal{T}$.

Clearly, for any $n \in \mathbb{N}_{>0}$ and for any $x \in X$, the open interval $B_{\frac{1}{n}}(x)$ is open, but

$$\bigcap \left\{ B_{\frac{1}{n}}(x) : n \in \mathbb{N}_{>0} \right\} = \{x\}.$$

For any $\varepsilon \in \mathbb{R}_{>0}$, $B_\varepsilon(x) \setminus \{x\}$ is not empty, thus $\{x\}$ is not open.

■

§1.3 Restrictions and Metric Subspaces

Restriction of metric function is a useful tool to describe the relation between metric spaces with different sets but "same" metric function on the sets.

As a restriction of a relation R on $X \times Y$ to a subset $A \times B \subseteq X \times Y$ is defined to be

$$R \restriction_{A \times B} := R \cap (X \times Y),$$

a restriction of a metric d on a set S to a subset $U \subseteq S$ is defined to be

$$d \restriction_{(U \times U) \times \mathbb{R}_{>0}} := d \cap ((U \times U) \times \mathbb{R}_{>0}).$$

If $B = Y$, customarily, we simply write $R \restriction_A$ for $R \restriction_{A \times B}$. Similarly, as the codomain of a metric function is alway $\mathbb{R}_{>0}$, so we simply write $d \restriction_{U \times U}$ instead of $d \restriction_{(U \times U) \times \mathbb{R}_{>0}}$.

Definition 1.3.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$.

The *metric on A induced by d*, or the *subspace metric of d with respect to A* is defined to be

$$d_A := d \restriction_{A \times A} .$$

Theorem 1.3.1. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$ and let $d_A := d \restriction_{A \times A}$.

Then $\mathbb{A} = (A, d_A)$ is a metric space.

Proof. As metric axioms (Definition 1.1.1) holds for any $x, y \in X$, and $A \subseteq X$, they also holds for any $a, b \in A$. As d_A is the subspace metric of d with respect to A, d_A is a metric on A.

Thus, $\mathbb{A}$ is a metric space.

Definition 1.3.2. Let $\mathbb{X} = (X, d)$ be a metric space, and let $A \subseteq X$.

$\mathbb{A} = (A, d_A)$ is a *metric subspace* of $\mathbb{X}$ iff d_A is a subspace metric of d with respect to A.

Chapter 2.

Topological Spaces

§2.1 Basic Definitions

Definition 2.1.1. Let X be any set, and let $\mathcal{T} \subseteq 2^X$.

$\mathcal{T}$ is a *topology on* X iff it satisfies the *open set axioms.* That is,

O1. $X \in \mathcal{T}$;

O2. For any $\mathcal{U} \subseteq \mathcal{T}$, $\bigcup \mathcal{U} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under arbitrary union.

O3. For any finite $\mathcal{V} \subseteq \mathcal{T}$, $\bigcap \mathcal{V} \in \mathcal{T}$; in words, $\mathcal{T}$ is closed under finite intersection.

A subset $U \subseteq X$ is said to be *open in* M iff it is an element of $\mathcal{T}$.

Definition 2.1.2. Let X be any set, and let $\mathcal{T}$ be a structure on X.

The pair $(X, \mathcal{T})$ is called a *topological space* iff $\mathcal{T}$ is a topology on X.

Theorem 2.1.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

Then $\emptyset \in \mathcal{T}$.

Proof. As empty set is an element of any set, it also an element of $\mathcal{T}$. Therefore, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

■

Definition 2.1.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space.

A subset $A \subseteq X$ is said to be *closed in* $\mathbb{X}$ iff there exists a $U \in \mathcal{T}$ such that $A = X \setminus U$.

Theorem 2.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and denote $\mathcal{C}$ for the family of all closed sets in M.

Then $\mathcal{C}$ satisfies the following conditions:

C1. $X, \emptyset \in \mathcal{C}$;

C2. For any $\mathcal{A} \subseteq \mathcal{C}$, $\bigcap \mathcal{A} \in \mathcal{C}$;

C3. For any finite $\mathcal{B} \subseteq \mathcal{C}$, $\bigcup \mathcal{B} \in \mathcal{C}$.

Proof.

C1. As $\emptyset \in \mathcal{T}$ and $X = X \setminus \emptyset$, by Definition 2.1.3, X is closed.

Similarly, as $X \in \mathcal{T}$ and $\emptyset = X \setminus X$, $\emptyset$ is closed.

$\square$

C2. For any $\mathcal{A} \subseteq \mathcal{C}$, there exists a $\mathcal{U} \subseteq \mathcal{T}$ such that

$$\forall A \in \mathcal{A} : \exists U \in \mathcal{U} : A = X \setminus U. \quad \text{(Definition 2.1.3.)}$$

Then we have

$$\begin{aligned} \mathcal{A} = \{X \setminus U : U \in \mathcal{U}\} &\iff \bigcap \mathcal{A} = \bigcap_{U \in \mathcal{U}} X \setminus U \\ &\iff \bigcap \mathcal{A} = X \setminus \bigcup \mathcal{U}. \end{aligned}$$

As $\bigcup \mathcal{U} \in \mathcal{T}$ by Definition 2.1.1 O2, its complement $\bigcap \mathcal{A} \in \mathcal{C}$ by Definition 2.1.3.

$\square$

C3. For any finite $\mathcal{B} \subseteq \mathcal{C}$, there exists a finite $\mathcal{U} \subseteq \mathcal{T}$ such that

$$\forall B \in \mathcal{B} : \exists U \in \mathcal{U} : A = X \setminus U. \quad \text{(Definition 2.1.3.)}$$

Then we have

$$\begin{aligned} \mathcal{B} = \{X \setminus U : U \in \mathcal{U}\} &\iff \bigcup \mathcal{B} = \bigcup_{U \in \mathcal{U}} X \setminus U \\ &\iff \bigcup \mathcal{B} = X \setminus \bigcap \mathcal{U}. \end{aligned}$$

As $\bigcap \mathcal{U} \in \mathcal{T}$ by Definition 2.1.1 O3, its complement $\bigcup \mathcal{A} \in \mathcal{C}$ by Definition 2.1.3.

□

Thus, the proof is done.

■

§2.2 Some Important Topologies

Definition 2.2.1. Let X be any set.

A family $\mathcal{T} \subseteq 2^X$ is a *discrete topology on* X iff $\mathcal{T} = 2^X$.

Definition 2.2.2. Let X be any set.

A family $\mathcal{T} \subseteq 2^X$ is an *indiscrete topology on* X iff $\mathcal{T} = \{X, \emptyset\}$.

Definition 2.2.3. Let $\mathbb{X} = (X, d)$ be a metric space.

A family $\mathcal{T} \subseteq 2^X$ is a *topology induced by* d iff $\mathcal{T}$ is the set of all open sets in $\mathbb{X}$.

§2.3 Comparison of Topologies

Definition 2.3.1. Let X be any set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X.

We say that $\mathcal{T}$ is *coarser* than $\mathcal{T}_1$, or $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$, iff $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Note 2.3.1. By the definition of cardinality and inclusion mapping, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$, it is certainly true that $|\mathcal{T}_1| \leq |\mathcal{T}_2|$. But, on the contrary, $|\mathcal{T}_1| \leq |\mathcal{T}_2|$ does not implies $\mathcal{T}_1 \subseteq \mathcal{T}_2$. It is easy to find counter-example about this.

Example 2.3.1. By Definition 2.3.1, for any set X, if a family $\mathcal{U}$ of open sets is given, then we can find the coarsest topology on X containing $\mathcal{U}$ by

$$\mathcal{T} = \left\{ \bigcup \mathcal{I}, \bigcap \mathcal{I}, X : \mathcal{I} \subseteq \mathcal{U} \right\}.$$

For example, let $X = \{1, 2, 3, 4, 5\}$, and let

$$\mathcal{U} = \{\{1, 2\}, \{2, 3\}, \{4\}\}.$$

Then a topology on X contains at least these sets:

$$\begin{gathered}\{1,2,3,4\},\{\},\\ \{1,2\},\{2,3\},\{4\},\\ \{1,2,3\},\{1,2,4\},\{2,3,4\},\\ \{2\}.\end{gathered}$$

Example 2.3.2. The discrete topology is the finest topology on any X, while the indiscrete topology is the coarsest.

§2.4 Interiors

Definition 2.4.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

The *interior* of A is defined as

$$\mathrm{Int}_{\mathcal{T}}(A) := \bigcup \left(\mathcal{T} \cap 2^A\right).$$

Note 2.4.1. Let $\mathbb{X}_1 = (X, \mathcal{T}_1)$, $\mathbb{X}_2 = (X, \mathcal{T}_2)$, and $A \subseteq X$. Then $\mathcal{T}_1 \neq \mathcal{T}_2$ iff $\mathrm{Int}_{\mathcal{T}_1}(A) \neq \mathrm{Int}_{\mathcal{T}_2}(A)$. In this case, the subscript for "Int" is necessary.

But, if the confusion is unlikely, we can also simply write $\mathrm{Int}(A)$ for $\mathrm{Int}_{\mathcal{T}} A$. In this case, it is also common to write A° for $\mathrm{Int}(A)$.

Theorem 2.4.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$A \in \mathcal{T}$ iff $A = A^\circ$.

Proof. First, prove $\Rightarrow$.

If $A \in \mathcal{T}$, then we have

$$\mathcal{T} \cap 2^A = \mathcal{T} \cap \{A\} \cap 2^A = \{A\} \cap 2^A = \{A\}.$$

By Definition 2.4.1,

$$A^\circ = \bigcup(\mathcal{T} \cap 2^A) = \bigcup\{A\} = A.$$

□

Now, prove $\Leftarrow$.

By Definition 2.4.1, we have

$$A = \bigcup(\mathcal{T} \cap 2^A).$$

As $\mathcal{T} \cap 2^A \subseteq \mathcal{T}$, thus, by open set axioms O2 (Definition 2.1.1 O2), $A \in \mathcal{T}$.

□

Thus, the proof is done.

■

Corollary 2.4.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \in \mathcal{T}$. For any $x \in A$, there is a $U \in \mathcal{T} \cap 2^A$ such that $x \in U$.

Proof.

$$\begin{aligned} x \in A &\iff x \in A^\circ && \text{(Theorem 2.4.1)} \\ &\iff x \in \bigcup (\mathcal{T} \cap 2^A) && \text{(Definition 2.4.1)} \\ &\iff \exists U \in \mathcal{T} \cap 2^A : x \in U. \end{aligned}$$

■

Lemma 2.4.1. Let X be any set, let I be an index set, and let $\mathcal{A}_i \subseteq 2^X$ for any $i \in I$.

Then we have

$$\bigcup \left(\bigcap_{i \in I} \mathcal{A}_i \right) \subseteq \bigcap_{i \in I} \left(\bigcup \mathcal{A}_i \right).$$

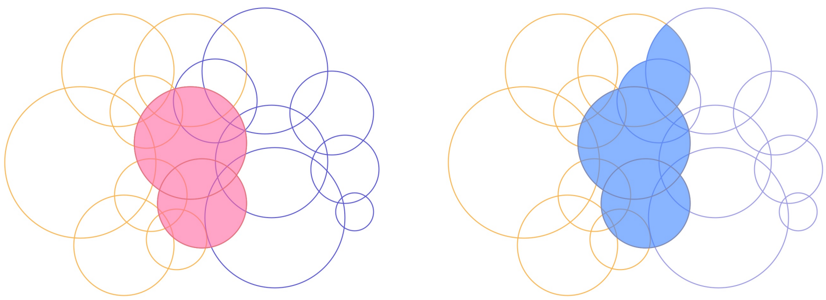

Figure 2.1: Diagram of the relation in Lemma 2.4.1.

Theorem 2.4.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $\mathcal{A} \subseteq 2^X$.

Then we have

$$\left(\bigcap \mathcal{A} \right)^\circ \subseteq \bigcap_{A \in \mathcal{A}} A^\circ.$$

Proof.

$$
\begin{aligned}
\left(\bigcap \mathcal{A}\right)^\circ &= \bigcup \left(\mathcal{T} \cap 2^{\bigcap \mathcal{A}}\right) && \text{(Definition 2.4.1)}\\
&= \bigcup \left(\mathcal{T} \cap \bigcap_{A \in \mathcal{A}} 2^A\right) && \text{(intersection of power sets)}\\
&= \bigcup \left(\bigcap_{A \in \mathcal{A}} (\mathcal{T} \cap 2^A)\right) && \text{(intersection is idempotent}\\
& && \text{and associative)}\\
&\subseteq \bigcap_{A \in \mathcal{A}} \left(\bigcup (\mathcal{T} \cap 2^A)\right) && \text{(Lemma 2.4.1)}\\
&= \bigcap_{A \in \mathcal{A}} A^\circ. && \text{(Definition 2.4.1)}
\end{aligned}
$$

■

Example 2.4.1. The equality in Theorem 2.4.2 may not hold.

Let $\mathbb{T} = (\mathbb{R}, \mathcal{T})$ be a topological space with

$$\mathcal{T} = \{X, (0,2), (1,3), \emptyset\}.$$

Then we have

$$((0,2) \cap (1,3))^\circ = \emptyset \quad \subsetneq \quad (0,2)^\circ \cap (1,3) = (1,2).$$

Theorem 2.4.3. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

If $A \subseteq B$, then $A^\circ \subseteq B^\circ$.

Proof.

$$
\begin{aligned}
A \subseteq B &\implies 2^A \subseteq 2^B && \text{(power set of subset)}\\
&\implies \mathcal{T} \cap 2^A \subseteq \mathcal{T} \cap 2^B\\
&\implies \bigcup (\mathcal{T} \cap 2^A) \subseteq \bigcup (\mathcal{T} \cap 2^B)\\
&\implies A^\circ \subseteq B^\circ && \text{(Definition 2.4.1)}
\end{aligned}
$$

■

Note 2.4.2. Note that, $A^\circ \subseteq B^\circ$ does not implies $A \subseteq B$. Consider $\mathbb{R}$ as a Euclidean metric space, and let

$$A = \{0\}, \quad B \subseteq \mathbb{R} \setminus \{0\}.$$

As $A^\circ = \emptyset$, $A^\circ \subseteq B^\circ$, but $A \setminus B = \{0\}$, so $A \not\subseteq B$.

§2.5 Limit Points and Isolated Points

Definition 2.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in X$ is a *limit point of* A iff for any neighbourhood N of x,

$$A \cap N \setminus \{x\} \neq \emptyset.$$

The *derived set of* A is the set of all limit points of X.

Definition 2.5.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A point $x \in A$ is said to be *isolated* iff there is a neighbourhood N of x,

$$A \cap N \setminus \{x\} = \emptyset.$$

Notations. The Derived set of A is usually denoted A'.[1] But sometime it is also necessary to know in which space (with its topology) the derived set of A is. For example, for topological spaces $\mathbb{X}_1 = (X, \mathcal{T}_1)$ and $\mathbb{X}_2 = (X, \mathcal{T}_2)$, if $\mathcal{T}_1 \neq \mathcal{T}_2$, the derived sets of a set A in $\mathbb{X}_1$ and $\mathbb{X}_2$ may be different. So, below, the notation A' is used only if the confusions are unlikely; else, we denote $\text{Ł}_{\mathcal{T}} A$ for A' with respect to the topology $\mathcal{T}$.

Sometime, the set of isolated points of A is denoted by A^i. For avoiding confusions, we denote $\mathrm{I}_{\mathcal{T}}(A)$ for A^i with respect to the topology $\mathcal{T}$.

Corollary 2.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then,

$$A \subseteq \text{Ł}(A) \sqcup \mathrm{I}(A).$$

Proof. By Definition 2.5.1, $x \notin \text{Ł}(A)$ iff there exists a neighbourhood N of x such that $A \cap N \setminus \{x\} = \emptyset$. This precisely satisfies Definition 2.5.2. Thus

$$A \subseteq \text{Ł}(A) \cup \mathrm{I}(A).$$

As Definition 2.5.1 and 2.5.2 are precisely logical complement for each other, $x \in \mathrm{I}(A) \cap \text{Ł}(A)$ always fails, i.e., $\mathrm{I}(A) \cap \text{Ł}(A) = \emptyset$. Thus

$$A \subseteq \text{Ł}(A) \sqcup \mathrm{I}(A).$$

[1] See ProofWiki and Wikipedia.

Theorem 2.5.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff $Ł(A) \subseteq A$.

Proof. First, prove $\Rightarrow$.

Aiming for a contradiction, suppose A is closed but there exists a $y \in Ł(A) \setminus A$.

By Definition 2.1.3, as A is closed, then $A^{\complement}$ is open.

As $y \in A^{\complement}$ and $A^{\complement}$ is open, then, by Corollary 2.4.1, there exists a $U \in \mathcal{T}$ with $y \in U$, such that $U \subseteq A^{\complement}$.

As U is a neighbourhood of y and $A \cap U \setminus \{y\} = \emptyset$, then $y \notin Ł(A)$. This contradicts the assumption.

Thus $Ł(A) \subseteq A$.

■

§2.6 Closures

Definition 2.6.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

The *closure of* A is defined as

$$\mathrm{Cl}_{\mathcal{T}}(A) := A \cup Ł(A).$$

When the confusions are unlikely, we simply write $\mathrm{Cl}(A)$, $\overline{A}$ or A^{-} for $\mathrm{Cl}_{\mathcal{T}}(A)$.

Corollary 2.6.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff $A = \mathrm{Cl}(A)$

Proof.

$$\begin{aligned} A \text{ is closed} &\iff A \supseteq Ł(A) && \text{(Corollary 2.5.1)} \\ &\iff A = A \cup Ł(A) \\ &\iff A = \mathrm{Cl}(A). && \text{(Definition 2.6.1)} \end{aligned}$$

■

Corollary 2.6.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

A is closed iff

$$A = \mathrm{I}(A) \sqcup Ł(A).$$

Proof. As A is closed, we have

$$\begin{aligned}
A &= \mathrm{Cl}(A) && \text{(Corollary 2.6.1)} \\
&= A \cup Ł(A) && \text{(Definition 2.6.1)} \\
&= A \setminus Ł(A) \sqcup Ł(A) \\
&= \mathrm{I}(A) \sqcup Ł(A). && \text{(Corollary 2.5.1)}
\end{aligned}$$

■

Theorem 2.6.1. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

Then we have

$$X \setminus A^\circ = (X \setminus A)^-.$$

Proof.

$$\begin{aligned}
X \setminus A^\circ &= X \setminus \bigcup \left(\mathcal{T} \cap 2^A\right) \\
&= \bigcap_{K \in \mathcal{T} \cap 2^A} (X \setminus K)
\end{aligned}$$

Theorem 2.6.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

If $A \subseteq B$, then $A^- \subseteq B^-$.

Proof.

$$\begin{aligned}
A \subseteq B &\iff A^\complement \supseteq B^\complement \\
&\implies \left(A^\complement\right)^\circ \supseteq \left(B^\complement\right)^\circ && \text{(Theorem 2.4.3)} \\
&\iff \left(\left(A^\complement\right)^\circ\right)^\complement \subseteq \left(\left(B^\complement\right)^\circ\right)^\complement \\
&\iff
\end{aligned}$$

Chapter 3.

Countable Axioms

§3.1 Covers and Basis

Definition 3.1.1. Let X be any set, and let $A \subseteq X$.
A family $\mathcal{C} \subseteq 2^X$ is a *cover for* A iff $A \subseteq \bigcup \mathcal{C}$.

Definition 3.1.2. Let $\mathbb{X} = (X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.
A family $\mathcal{C} \subseteq 2^X$ is an open cover for A iff $\mathcal{C}$ covers A and $\mathcal{C} \subseteq \mathcal{T}$.