# Notes for General Topology by Tom Leinster

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April 12, 2021

# Chapter 1

# Topological Spaces

## 1.1 Review of Metric Spaces

**Definition 1.1.1.** Let $X$ be a set. A *metric* on $X$ is a function $\rho : X \times X \to \mathbb{R}_{\geq 0}$, such that $\forall x, y, z \in X$, the following (metric axioms) holds:

M1. $\rho(x, y) = 0 \iff x = y$ (identity of indiscernibles);

M2. $\rho(x, y) = \rho(y, x)$ (symmetry).

M3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (triangle inequality);

A *metric space* is a set together with a metric on it, or more formally, a pair $(X, \rho)$ where $X$ is a set and $\rho$ is a metric on $X$.

**Example 1.1.1.**

(i) The function $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $\forall p \in \mathbb{R}_{\geq 1}$, $\forall x, y \in \mathbb{R}^n$,

$$\rho_p(x, y) = \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}},$$

is a metric on $\mathbb{R}^n$. Clearly it satisfies identity of indiscernibles and symmetry. For triangle inequality, it is suggested by Minkowski inequality.

Given $x \in \mathbb{R}^3$, $r \in \mathbb{R}_{\geq 0}$, and

$$B_\rho = \left\{ y \in \mathbb{R}^3 \mid \rho(x, y) \leq r \right\}.$$

$\forall p, q \in \mathbb{R}_{\geq 1}$, it is true that, $\forall x, y \in \mathbb{R}^n$,

$$p \leq q \implies \rho_p(x, y) \geq \rho_q(x, y).$$

Thus, $B_p \subseteq B_q$.

Geometrically, as $p = 1$, $B$ is a octahedron in $\mathbb{R}^3$ with center $x$ and radius $r$; as $p = 2$, $B$ is a sphere in $\mathbb{R}^3$ with center $x$ and radius $r$. It is easy to observe that as $p \to \infty$, $B$ tends to the cube in $\mathbb{R}^3$ with center $x$ and edge length $2r$; i.e.,

$$\rho_\infty(x, y) = \lim_{p\to\infty} \rho_p(x, y) = \sup_{i\in\{1,\dots,n\}} |x_i - y_i|.$$

(ii) Let $f : (X, \rho) \to \mathbb{R}^n$ with $X \subseteq \mathbb{R}^m$ be a continuous map on $X$. Let $x, y \in X$, then $\rho' : f[X] \times f[X] \to \mathbb{R}_{\geq 0}$ defined by

$$\rho'_p(x, y) = \int_0^1 f(\ell(t))d_p s(t)$$

where

$$\ell(t) = x + t(y - a)$$

and

$$d_p s(t) = \left( \sum_{i=1}^{m} \left| \frac{dg_i}{dt}(t) \right|^p \right)^{\frac{1}{p}} dt.$$

with $p \geq \mathbb{R}_{\geq 1}$ is a metric on $f[X]$.

Fix $x$ and given $r \in \mathbb{R}_{\geq 0}$, the set

$$B_p = \left\{ y \in \mathbb{R}^m : \rho'_p(x, y) \leq r \right\}$$

describes a set "attached" on $f[X]$ with center $x$. If $p = 2$, $m = 2$ and $n = 3$, and $f : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$f(\lambda, \phi) = \begin{cases} r \sin\lambda \cos\phi, \\ r \sin\lambda \sin\phi, \\ r \cos\phi, \end{cases}$$

then $\rho'_2$ here is a *great circle metric* defined by

$$p'_2(x, y) = r \arccos(\sin x_\phi \sin y_\phi + \cos x_\phi \cos y_\phi \cos(x_\lambda - y_\lambda)).$$

(iii) Let $a, b \in \mathbb{R}$ with $a \leq b$, and $p \in \mathbb{R}_{\geq 1}$, and $C[a, b]$ denote the set of continuous function $[a, b] \to \mathbb{R}$.

Then $d_p$ defined by $\forall f, g \in C[a, b]$,

$$\rho_p(f, g) = \left( \int_a^b |f - g|^p \right)^{\frac{1}{p}}$$

is a metric on $C[a, b]$.

Similar to $\rho_p$ on $\mathbb{R}^n$,

$$B_p = \{g \mid \rho(f, g) \leq r\}$$

defines a set with "center" $f$ and "radius" $r \in \mathbb{R}_{\geq 0}$.

It also implies that, on $C[a,b]$, $\forall p, q \in \mathbb{R}_{\geq 1}$, $\forall x, y \in \mathbb{R}^n$

$$p \leq q \implies d_p(f,g) \geq d_q(f,g),$$

and, naturally, $B_p \subseteq B_q$. This is a straight corollary from the same case of $d_p$ on $\mathbb{R}^n$.

(iv) Let $A$ be a set. The *Hamming metric* $\rho$ on a set $A^n$ is given by $\forall x, y \in A^n$

$$\rho(x,y) = \#\left\{i \in \{1, \ldots, n\} : x_i \neq y_i\right\}.$$

An example from Wikipedia. The word "karolin" and "kathrin" can be considered as tuples

$$x = (\mathrm{k}, \mathrm{a}, \mathrm{r}, \mathrm{o}, \mathrm{l}, \mathrm{i}, \mathrm{n}),\ y = (\mathrm{k}, \mathrm{a}, \mathrm{t}, \mathrm{h}, \mathrm{r}, \mathrm{i}, \mathrm{n}).$$

For all $i \in \{0, \ldots, 6\} \setminus \{0, 1, 4, 6\}$, $x_i \neq y_i$, and $\#(\{0, \ldots, 6\} \setminus \{0, 1, 4, 6\}) = 3$, thus

$$\rho(x,y) = 3.$$

(v) Let $(M, \rho)$ be a metric space (for example, $\rho = \rho_2$ on $\mathbb{R}^n$), and $X, Y \in \mathcal{P}(M)$. The Hausdorff metric $\rho_{\mathrm{H}}$ on $\mathcal{P}(M)$ is defined by

$$\rho_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x \in X} \rho(x, Y), \sup_{y \in Y} \rho(X, y)\right\},$$

where $\rho(a, B) = \inf_{b \in B} \rho(a, b)$ for all $B \in \mathcal{P}(M)$ and $a \in M$.

This metric can be used to measure how close two figures (as sets of points) are.

**Definition 1.1.2.** Let $X$ be a metric space, let $x \in X$, and $\varepsilon > 0$. The *open ball with center $x$ and radius $\varepsilon$*, or more briefly the *open $\varepsilon$-ball about $x$* is the subset

$$B(x, \varepsilon) = \{y \in X : \rho(x,y) \leq \varepsilon\} \subseteq X.$$

Similarly, the *closed $\varepsilon$-ball around $x$* is

$$\overline{B}(x, \varepsilon) = \{y \in X : \rho(x,y) \leq \varepsilon\} \subseteq X.$$

**Note 1.1.1.** Clearly, the word "ball" does not mean it should look like a ball. Clearly, for all $x \in \mathbb{R}^3$, the ball $\{y \in \mathbb{R}^3 : \rho_\infty(x,y) < 1\}$ is a cube without its surface.

And it is interesting to think that on $C[a,b]$ with conditions above,

$$\{g \in C[a,b] : \rho_p(f,g) < 1\}$$

defines a open ball in $C[a,b]$.

**Note 1.1.2.** For hamming metric $\rho$ with conditions above, for $\varepsilon \in \mathbb{R}_{(0,1)}$, the ball

$$\{y \in A^n : \rho(x,y) < 1\} = \{x\}.$$

is a singleton.