

# Notes for General Topology

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# Chapter 1

## Topological Spaces

### 1.1 Topological Spaces

**Definition 1.1.1** (topology). Let  $X$  be a set, and let a family  $\mathcal{T} \subseteq \mathcal{P}(X)$ .  $\mathcal{T}$  is called a *topology* on  $X$  iff it satisfies the *open set axioms*:

- O1.**  $X \in \mathcal{T}$ ;
- O2.**  $\mathcal{T}$  is closed under arbitrary union;
- O3.**  $\mathcal{T}$  is closed under finite intersection.

**Theorem 1.1.1.**  $\emptyset \in \mathcal{T}$ .

*Proof.* By O2 in Definition 1.1.1, for all  $\mathcal{A} \subseteq \mathcal{T}$ ,  $\bigcup \mathcal{A} \in \mathcal{T}$ . Clearly,  $\emptyset \in \mathcal{T}$ , then we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

□

**Definition 1.1.2** (topological spaces). With the condition in Definition 1.1.1, the pair  $(X, \mathcal{T})$  is called a *topological space*. All subsets of  $X$  in  $\mathcal{T}$  are said to be *open* in  $(X, \mathcal{T})$ .

**Theorem 1.1.2.** Let  $(X, \rho)$  be a metric space, and let

$$\mathcal{T}_\rho = \left\{ \bigcup_{x \in U} B_\rho(x, \varepsilon) : U \subseteq X \wedge \varepsilon \in \mathbb{R}_{>0} \right\}.$$

$(X, \mathcal{T}_\rho)$  is a topological space; i.e.,  $\mathcal{T}_\rho$  satisfies the open set axioms.

*Proof.* For O1. Apparently, there exists (for any)  $\varepsilon \in \mathbb{R}_{>0}$ ,

$$X = \bigcup_{x \in X} B(x, \varepsilon).$$

For O2. Let  $I$  be an index set, and let  $U_i \in \mathcal{T}$  for all  $i \in I$ . There exists  $\varepsilon \in \mathbb{R}_{>0}$  we can define

$$U = \bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{x \in U_i} B(x, \varepsilon) = \bigcup_{x \in U} B(x, \varepsilon).$$

Then, for all  $x \in U$ , there exists  $B(x, \varepsilon) \ni x$ . Thus  $U \in \mathcal{T}$ .

□

**Definition 1.1.3** (closed sets). Let  $(X, \mathcal{T})$  be a topological space. A subset  $V$  of  $X$  is said to be *closed* iff there is an open set  $U$  in  $X$  such that

$$V = X \setminus U.$$

**Proposition 1.1.1.** Let  $(X, \mathcal{T})$  be a topological space, and let  $\mathcal{C}$  be the family of all closed sets in  $X$ . Then

- (i)  $\emptyset, X \in \mathcal{C}$ ;
- (ii)  $\mathcal{C}$  is closed under arbitrary intersection;
- (iii)  $\mathcal{C}$  is closed under finite union.

*Proof.*

- (i)  $X \in \mathcal{T}$  implies  $X \setminus X = \emptyset \in \mathcal{C}$ ; and  $\emptyset \in \mathcal{T}$  implies  $X \setminus \emptyset = X \in \mathcal{C}$ ;
- (ii) As  $\mathcal{T}$  is closed under arbitrary union, then by Definition 1.1.3 and De Morgan's Law,  $\mathcal{C}$  is closed under arbitrary intersection.
- (iii) As  $\mathcal{T}$  is closed under finite intersection, then by Definition 1.1.3 and De Morgan's Law,  $\mathcal{C}$  is closed under finite union.

□

**Definition 1.1.4** (finer and coarser topology). Let  $X$  be any set, and let  $\mathcal{T}, \mathcal{T}'$  be topologies on  $X$ .  $\mathcal{T}$  is said to be *finer* than  $\mathcal{T}'$  iff  $\mathcal{T} \supseteq \mathcal{T}'$ ; respectively,  $\mathcal{T}$  is said to be *coarser* than  $\mathcal{T}'$  iff  $\mathcal{T} \subseteq \mathcal{T}'$ .

**Definition 1.1.5** (neighbourhood). Given  $(X, \mathcal{T})$  as a topological space and a point  $x \in X$ , a subset  $N \subseteq X$  is called a *neighbourhood* iff it contains an open set  $U$  containing  $x$ .

**Proposition 1.1.2.** Given  $(X, \mathcal{T})$  as a topological space and  $U \subseteq X$ ,  $U$  is open iff for all  $x \in U$ , there is a neighbourhood  $N$  of  $x$  contained in  $U$ .

*Proof.* If  $U$  is open, then  $U$  itself is a neighbourhood of  $x$  contained in  $U$ .

Conversely, if for all  $x \in U$ , there is a neighbourhood  $N_x$  of  $x$  contained in  $U$ , then there is a open neighbourhood  $U_x \ni x$  contained in  $N_x$ . Then we have

$$U \supseteq \bigcup_{x \in U} U_x.$$

Suppose  $U$  is not open, then  $U$  is a proper superset in the relation above. Then there exists  $y \in U$  which is not in any  $U_x$ . This implies that such a  $y$  does not have any neighbourhood  $N_y$  in  $U$ , for such an  $N_y$  must contains an open  $U_y \ni y$ . For if it does, then there must be a  $U_x$  contains  $y$ . This is a contradiction. Thus,

$$U = \bigcup_{x \in U} U_x$$

is open. □

## 1.2 Metrizable

**Definition 1.2.1** (metric spaces). Let  $X$  be any set. A *metric*  $\rho$  on  $X$  is a function  $\rho : X \times X \rightarrow \mathbb{R}$  satisfying the following conditions: for all  $x, y, z \in X$

- (i)  $\rho(x, y) \geq 0$ , and  $\rho(x, y) = 0$  iff  $x = y$ ;
- (ii)  $\rho(x, y) = \rho(y, z)$ ;
- (iii)  $\rho(x, z) + \rho(z, y) \geq \rho(x, y)$ .

**Definition 1.2.2** (balls). Let  $(X, \rho)$  be a metric space, let  $x \in X$ , and let  $\varepsilon \in \mathbb{R}_{>0}$ . The *open  $\varepsilon$ -ball about  $x$*  or just  *$\varepsilon$ -ball about  $x$*  is defined to be

$$B(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}.$$

The *closed  $\varepsilon$ -ball about  $x$*  is defined to be

$$\overline{B}(x, \varepsilon) = \{y \in X : \rho(x, y) \leq \varepsilon\}.$$

**Example 1.2.1.** Let  $X$  be any set, and let metric  $\rho_p$  on  $X^n$  ( $n \in \mathbb{Z}_{>0}$ ) defined by

$$\rho_p(x, y) = \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}},$$

where  $p \in \mathbb{R}_{\geq 1}$ .  $\rho_2$  is so called the *standard Euclidean metric*. If  $X = \mathbb{R}$ , then the metric space  $(\mathbb{R}^n, \rho_2)$  is so-called *Euclidean  $n$ -space*.

For all  $p, q \in \mathbb{R}_{\geq 1}$ , if  $p < q$ , then for all  $\varepsilon \in \mathbb{R}_{>0}$  and for all  $x, y \in X$ ,  $\rho_p(x, y) \geq \rho_q(x, y)$ ; in particular,  $\rho_p = \rho_q$  iff there is a unique  $k \in \{1, \dots, n\}$ , such that for all  $i \in \{1, \dots, n\} \setminus \{k\}$ ,  $x_i = 0$ . As  $\rho_p(x, y)$  is always “overestimated” than  $\rho_q(x, y)$ , we have  $B_{\rho_p}(x, \varepsilon) \supseteq B_{\rho_q}(x, \varepsilon)$ .

**Example 1.2.2.** Let  $X$  be any set. The *discrete metric*  $\rho$  on  $X$  is defined to be

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

**Example 1.2.3.** Let  $a, b \in \mathbb{R}$  with  $a < b$ , and let metric  $\rho_p$  on  $C[a, b]$  defined by

$$\rho_p(f, g) = \left( \int_a^b |f(t) - g(t)|^p dt \right)^{\frac{1}{p}},$$

where  $p \geq 1$ . In particular,

$$\rho_\infty(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|.$$

**Proposition 1.2.1.** Let  $(X, \rho)$  be a metric space, then for all  $x, y \in X$  ( $x \neq y$ ), there is an  $\varepsilon > 0$  such that  $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$ .

*Proof.* Suppose for all  $\varepsilon > 0$ ,  $B(x, \varepsilon) \cap B(y, \varepsilon) \neq \emptyset$ , then there must be a  $z \in X$  such that  $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$ .  $z \in B(x, \varepsilon)$  only if  $\rho(x, z) < \varepsilon$ , and  $z \in B(y, \varepsilon)$  only if  $\rho(y, z) < \varepsilon$ . Thus

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

As the assumption holds for all  $\varepsilon > 0$ , we may put

$$\varepsilon = \frac{\rho(x, y)}{2}.$$

Then, we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which is impossible. □

**Definition 1.2.3** (induced topologies). Let  $(X, \rho)$  be a metric space. A topology  $\mathcal{T}$  on  $X$  is said to be *induced* by  $\rho$  iff for all  $\varepsilon > 0$ , any  $U \in \mathcal{T}$  is the union of ball(s) in  $X$ ; i.e.,

$$\mathcal{T} = \left\{ U \subseteq X : U = \bigcup_{x \in X} B(x, \varepsilon) \right\}.$$

In this case,  $\mathcal{T}$  is called the *underlying topology* of  $\rho$ .

**Definition 1.2.4** (metrizable spaces). Let  $(X, \mathcal{T})$  be a topological space. If there is any  $\rho$  induce  $\mathcal{T}$ , then  $(X, \mathcal{T})$  is said to be *metrizable*.

**Definition 1.2.5** (Lipschitz equivalence). Let  $X$  be any set, and let  $\rho$  and  $\rho'$  be metrics on  $X$ .  $\rho$  and  $\rho'$  are said to be *Lipschitz equivalent* iff there exist  $c, C > 0$ , such that for all  $x, y \in X$ ,

$$c\rho(x, y) \leq \rho'(x, y) \leq C\rho(x, y).$$

**Proposition 1.2.2.** Lipschitz equivalence is an equivalence relation.

*Proof.* Clearly, Definition 1.2.5 also holds for  $\rho = \rho'$ . So, Lipschitz equivalence is reflexive. In Definition 1.2.5, the relation also holds for  $\frac{1}{C}\rho' \leq \rho \leq \frac{1}{c}\rho'$ . So Lipschitz equivalence is symmetric.

If there is another  $\rho''$  be Lipschitz equivalent to  $\rho'$ , then there is  $r, R > 0$ , such that for all  $x, y \in X$ ,

$$r\rho''(x, y) \leq \rho'(x, y) \leq R\rho''(x, y).$$

By the conditions in Definition 1.2.5, we have

$$\frac{c}{r}\rho(x, y) \leq \rho''(x, y) \leq \frac{C}{R}\rho(x, y),$$

i.e.,  $\rho$  and  $\rho''$  are also Lipschitz equivalent. So Lipschitz equivalence is transitive.

Above all, Lipschitz equivalence is an equivalence relation.  $\square$

**Proposition 1.2.3.** Let  $X$  be any set, and let  $\rho$  and  $\rho'$  be metrics on  $X$ . If  $\rho$  and  $\rho'$  are Lipschitz equivalent, then  $\rho$  and  $\rho'$  induce the same topology.

*Proof.* As  $\rho$  and  $\rho'$  are Lipschitz equivalent, by Definition 1.2.5, there is a  $c > 0$  such that for all  $x, y \in X$ ,

$$c\rho(x, y) \leq \rho'(x, y).$$

Given  $r \in \mathbb{R}_{>0}$  and for all  $x \in X$ , we have

$$B_{\rho'}(x, cr) \subseteq B_{c\rho}(x, r) = B_{\rho}\left(x, \frac{1}{c}r\right).$$

For all  $U \in \mathcal{T}_{\rho}$ , for all  $x \in U$ , there is an  $\varepsilon \in \mathbb{R}_{>0}$ , such that

$$B_{\rho'}(x, \varepsilon) \subseteq B_{\rho}(x, \varepsilon) \subseteq U.$$

So  $U \in \mathcal{T}_{\rho'}$ . Then we have  $\mathcal{T}_{\rho} \subseteq \mathcal{T}_{\rho'}$ .

Similarly,  $U \in \mathcal{T}_{\rho'}$  only if  $U \in \mathcal{T}_{\rho}$ . Then we have  $\mathcal{T}_{\rho'} \subseteq \mathcal{T}_{\rho}$ .

Above all,  $\mathcal{T}_{\rho} = \mathcal{T}_{\rho'}$ . □

**Note 1.2.1.** In this proposition,  $\mathcal{T}_{\rho}$  and  $\mathcal{T}_{\rho'}$  are said to be homeomorphic or topologically equivalent (see Definition 1.5.2). And  $\rho$  and  $\rho'$  are also said to be topologically equivalent.

**Example 1.2.4.** In Example 1.2.1, for all  $p, q \geq 1$ , all  $\rho_p$  and  $\rho_q$  induce the same topology. Let  $X$  be any subset of  $\mathbb{R}^n$ , then for all  $x, y \in X$ , if  $p < q$ , then

$$\rho_p(x, y) \geq \rho_q(x, y).$$

Thus, if  $\rho_1$  and  $\rho_{\infty}$  are Lipschitz equivalent, then any other  $\rho_p$  and  $\rho_q$  are Lipschitz equivalent. We have

$$\rho_1(x, y) = \sum_{i=1}^n |x_i - y_i| \geq \max_{i \in \{1, \dots, n\}} |x_i - y_i| = \rho_{\infty}(x, y).$$

Clearly,

$$\rho_{\infty}(x, y) \leq \rho_1(x, y) \leq n\rho_{\infty}(x, y).$$

By Definition 1.2.5,  $\rho_1$  and  $\rho_{\infty}$  are Lipschitz equivalent, hence for all  $p, q \geq 1$ ,  $\rho_p$  and  $\rho_q$  are Lipschitz equivalent. Thus, by Proposition 1.2.3, they induce the same topology.

### 1.3 Separation Axioms. From $T_0$ to Hausdorff

**Definition 1.3.1** (saperated). In a topological space, two sets are said to be *separated* iff each is disjoint from other's closure.

**Definition 1.3.2** (separated by neighbourhoods). In a topological space  $(X, \mathcal{T})$ , two sets  $A$  and  $B$  are said to be *separated by neighbourhood* iff there are neighbourhoods  $N_A$  of  $A$  and  $N_B$  of  $B$  such that  $N_A$  and  $N_B$  are disjoint.



**Definition 1.3.3** (topologically indistinguishable). Let  $(X, \mathcal{T})$  be a topological space. Two points  $x, y \in X$  are said to be *topologically indistinguishable* iff they share all their neighbourhoods. That is, let  $\mathcal{N}_x$  be the family of all neighbourhoods of  $x$  and let  $\mathcal{N}_y$  be the family of all neighbourhoods of  $y$ , we have

$$\mathcal{N}_x = \mathcal{N}_y.$$

Respectively,  $x, y$  are said to be *topologically distinguishable* iff they are not topologically indistinguishable; i.e.,

$$\mathcal{N}_x \neq \mathcal{N}_y.$$

**Example 1.3.1.** In an indiscrete topological space, all distinct points are topologically indistinguishable.

### $T_0$ Spaces

**Definition 1.3.4** ( $T_0$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_0$  or *Kolmogorov*, iff all distinct points  $x, y \in X$  are topologically distinguishable.

**Example 1.3.2.** Let  $X$  be any set and let  $\mathcal{T}$  be the indiscrete topology on  $X$ .  $(X, \mathcal{T})$  is  $T_0$  iff  $|X| \in \{0, 1\}$ .

### $T_1$ Spaces

**Definition 1.3.5** ( $R_0$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $R_0$  iff any two topologically distinguishable points in  $X$  are separated.

**Definition 1.3.6** ( $T_1$  Spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_1$  or *Fréchet* iff it is  $T_0$  and  $R_0$ ; i.e., all distinct points  $x, y \in X$  are separated.

**Example 1.3.3** ( $R_0$  but not  $T_0$ ). Let  $\mathcal{T}$  be a countable family of disjoint proper intervals on  $\mathbb{R}^n$ , and  $\bigcup \mathcal{T} = \mathbb{R}^n$ .  $(X, \mathcal{T})$  is  $R_0$ , but not  $T_0$ .

**Example 1.3.4** ( $T_0$  but not  $R_0$ ). Let  $(\mathbb{R}_{\geq 0}, \mathcal{T})$  be a topological space with

$$\mathcal{T} = \{U \subseteq \mathbb{R} : \forall i \in \mathbb{R}_{\geq 0}, U_i = [0, i)\},$$

Then for all  $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$ , if  $x \neq y$ , then there are  $|y - x|$  neighbourhoods  $N_x$  of  $x$  do not contain  $y$ . Thus, it is  $T_0$ .

On the other hand, it is not  $R_0$ , because for all  $x, y \in (\mathbb{R}_{\geq 0}, \mathcal{T})$  with  $x < y$ ,  $x \in \overline{\{y\}} = [0, y]$ .

**Example 1.3.5** ( $R_0$  but not  $T_1$ ). Let  $X$  be any set with  $|X| \geq 3$ , let  $U \subsetneq X$  with  $|U| \geq 2$ , let  $\mathcal{T}_{X \setminus U}$  be a  $T_1$  topology on  $X \setminus U$ , and let  $\mathcal{T}$

$$\mathcal{T} = \mathcal{T}_{X \setminus U} \cup \{X, U\}.$$

For all  $x, y \in X$ , if  $x \neq y$ , then they are separated. Thus, the space is  $R_0$ . But  $(X, \mathcal{T})$  is not  $T_1$ , because all  $\{u\} \in U$  share the same closure which is  $U$  itself.

**Proposition 1.3.1** (alternative definitions of  $R_0$  spaces). Let  $(X, \mathcal{T})$  be  $R_0$ , then the following conditions are equivalent.

- (i) The closure of all singletons in  $X$  are not  $T_0$  subspace.
- (ii) For any two points  $x, y \in X$ ,  $x \in \overline{\{y\}}$  iff  $y \in \overline{\{x\}}$ .
- (iii) Every open set is the union of closed sets.

*Proof.*

- (i) By Definition 1.3.5, if  $y$  and  $x$  are topologically distinguishable, by Definition 1.3.5,  $x$  and  $y$  are separated; i.e.,  $x \notin \overline{\{y\}}$  and  $y \notin \overline{\{x\}}$ .
- (ii) By Definition 1.3.5, for all  $x, y \in X$ ,  $x, y$  are not separated only if they are topologically indistinguishable. By Definition 1.3.3, they share all their neighbourhoods, thus they have the same closure; i.e.,  $\overline{\{x\}} = \overline{\{y\}}$ .
- (iii) For any  $U \in \mathcal{T}$ ,

$$U = \bigcup_{x \in U} \{x\}.$$

If  $(X, \mathcal{T})$  is  $T_1$ , then we are done. Suppose  $(X, \mathcal{T})$  is not  $T_1$ , then there exists  $A \in \mathcal{T}$  with  $|A| > 1$ , and for all  $B \subsetneq A$ ,  $B \notin \mathcal{T}$  (proof omitted). For such  $A$ ,  $X \setminus A$  is open, for  $X \setminus A = \bigcup (\mathcal{T} \setminus \{A\})$ , thus  $A$  is also closed.

Suppose for any such  $A$  with  $A \cap U \neq \emptyset$ ,  $A \subseteq U$ . Suppose it fails, i.e.,  $A \cap U \neq A$ , then we have  $A \cap U \subsetneq A$  and  $A \cap U \in \mathcal{T}$ , which is contradicted to the condition of  $A$ . Now we have

$$U = \bigcup \mathcal{A} \cup \bigcup_{x \in I} \{x\}$$

where  $\mathcal{A}$  is the family of such  $A$ , and  $I$  is the union of all closed singletons in  $U$ . Thus  $U$  is open.

□

**Proposition 1.3.2** (alternative definitions of  $T_1$  spaces). Let  $(X, \mathcal{T})$  be  $T_1$ , then the following conditions are equivalent.

- (i) All singletons in  $X$  are closed.
- (ii) Every subset of  $X$  is the intersection of all open sets containing it.
- (iii) Every cofinite subset of  $X$  is open.

*Proof.*

- (i) Suppose there exists  $\{x\} \subseteq X$  with  $\overline{\{x\}} \neq \{x\}$ , then there exists  $y \in \overline{\{x\}}$  with  $x \neq y$ . By Definition 1.3.6, this is impossible.

- (ii) For any  $A \subseteq X$ ,

$$A = \bigcup_{x \in A} \{x\}.$$

Let  $B = X \setminus A$ . By De Morgan's law,

$$B = \bigcap_{x \in A} X \setminus \{x\}.$$

$(X, \mathcal{T})$  is  $T_1$  iff all  $\{x\}$  are closed, in which case,  $B$  is the intersection of all open sets  $X \setminus \{x\} \supseteq B$ .

- (iii) Let  $A$  be a cofinite subset of  $X$ .  $X \setminus A$  is a finite union of singletons. As  $(X, \mathcal{T})$  is  $T_1$ , any singletons in  $X$  is closed. By Proposition 1.1.1,  $X \setminus A$  is closed. By Definition 1.1.3,  $A$  is open.

□

## Hausdorff Spaces

**Definition 1.3.7** ( $R_1$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $R_1$  iff any two topological distinguishable points in  $X$  are separated by neighbourhoods.

**Definition 1.3.8** (Hausdorff Spaces). A topological space  $(X, \mathcal{T})$  is said to be *Hausdorff* or  $T_2$  iff it is  $T_0$  and  $R_1$ ; i.e., all distinct points  $x, y \in X$  are separated by neighbourhoods.

**Proposition 1.3.3.** All metrizable spaces are Hausdorff

*Proof.* Let  $(X, \mathcal{T})$  be a metrizable space. There exists a metric  $\rho$  on  $X$  that induces  $\mathcal{T}$ . Given distinct points  $x, y \in X$ , suppose for all  $\varepsilon \in \mathbb{R}_{>0}$ , there exists  $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$ . Then  $\rho(x, z) < \varepsilon$  and  $\rho(y, z) < \varepsilon$ . Now we have

$$\rho(x, z) + \rho(y, z) < 2\varepsilon.$$

Put  $\rho(x, y) > 2\varepsilon$  as  $x$  and  $y$  are arbitrarily given. Then we have

$$\rho(x, z) + \rho(y, z) < \rho(x, y),$$

which implies that  $\rho$  is not a metric on  $X$ . Hence,  $(X, \mathcal{T})$  is not metrizable which is contradicted to the condition.  $\square$

**Proposition 1.3.4.** All singletons in a Hausdorff space are closed.

*Proof.* Let  $(X, \mathcal{T})$  be a Hausdorff space, and let  $x \in X$ . For all  $y \in X$  with  $x \neq y$ , there is a open neighbourhood  $U_y$  of  $y$  such that  $x \notin U_y$ . Then, for all such  $U_y$ , we have

$$\forall y \in X, x \in X \setminus U_y = \{x\} \iff x \in \bigcap_{y \in X \setminus \{x\}} X \setminus U_y = \{x\}.$$

As all  $X \setminus U_y$  are closed, their intersection  $\{x\}$  is closed.  $\square$

**Example 1.3.6** ( $T_1$  but not Hausdorff). Let  $X$  be a nonempty set, let  $p \in X$ , let  $\mathcal{T}'$  be a Hausdorff topology on  $X \setminus \{p\}$ , and let

$$\mathcal{T} = \{X\} \cup \mathcal{T}'.$$

Then, all  $x \in (X, \mathcal{T})$  are closed, thus  $(X, \mathcal{T})$  is Fréchet. But the only neighbourhood of  $p$  is  $X$ , so its closure is  $X$ . Then, for any  $x \in X \setminus \{p\}$ ,  $x$  and  $p$  are not separated, in which case  $(X, \mathcal{T})$  is not  $R_0$ . Thus,  $(X, \mathcal{T})$  is not Hausdorff.

## 1.4 Continuity

**Definition 1.4.1** (continuous maps). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A map  $f : X \rightarrow Y$  is said to be *continuous* iff for any open set  $U$  in  $Y$ , its preimage in  $X$  under  $f$  is open.

**Note 1.4.1.** In Definition 1.4.1, note that even if for any open set  $U$  in  $X$ ,  $f[X]$  is open in  $Y$ ,  $f$  is not necessarily continuous. For example, let  $X = (\mathbb{R}, \mathcal{T}_X)$

with  $\mathcal{T}_X$  induced by standard Euclidean metric, let  $Y = (\mathbb{R}, \mathcal{T}_Y)$  with  $\mathcal{T}_Y$  as a indiscrete topology, and define

$$f(x) = [x],$$

where  $[x]$  denotes the integer part of  $x$ . Then for all  $U \subseteq X$ ,  $f[U]$  is open in  $Y$ , but by Definition 1.4.1,  $f$  is not continuous.

**Note 1.4.2.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces, if  $\mathcal{T}_X$  is the discrete topology on  $X$ , then any function with domain  $X$  is continuous. If  $\mathcal{T}_Y$  is the indiscrete topology on  $Y$ , then any function with codomain  $Y$  is continuous.

**Note 1.4.3.** A function is continuous bijection does not implies that its inverse is continuous. For example, let  $X$  be any set and let  $\mathcal{T}$  and  $\mathcal{T}'$  be its topologies. If  $\mathcal{T}$  is finer than  $\mathcal{T}'$ , then any bijection  $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}')$  is continuous. In this case, however, if  $\mathcal{T} \neq \mathcal{T}'$ , then  $f^{-1}$  is not continuous.

**Proposition 1.4.1.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A map  $f : X \rightarrow Y$  is continuous at  $x \in X$  iff for any neighbourhood  $N_y$  of  $f(x)$ , there is a neighbourhood  $N_x$  of  $x$ , such that  $f[N_x] \subseteq N_y$ .

*Proof.* Let  $N_y$  be a neighbourhood of  $f(x)$ . Clearly, there exists an open set  $U_y$  contains  $y$ .

By Definition 1.4.1,  $f$  is continuous at  $x$  iff  $x \in f^{-1}[U_y] \in \mathcal{T}_X$ . Clearly,  $f^{-1}[U_y]$  is a neighbourhood of  $x$ . We have  $f[f^{-1}[U_y]] = U_y \subseteq N_y$ .

By Proposition 1.1.2, there  $U_x$  must contains at least one neighbourhood  $N_x$  of  $x$ , thus,  $f[N_x] \subseteq U_y$ .  $\square$

**Proposition 1.4.2.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be metrizable spaces. A map  $f : X \rightarrow Y$  is continuous at  $p \in X$  iff for any  $\varepsilon > 0$ , there is a  $\delta > 0$ , such that for all  $x \in B_X(p, \delta)$ ,  $f(x) \in B_Y(f(p), \varepsilon)$ , where  $B_X$  is defined by any metrics  $\rho_X$  induces  $\mathcal{T}_X$ , and  $B_Y$  is defined by any metrics  $\rho_Y$  induces  $\mathcal{T}_Y$ .

*Proof.* Clearly, for all  $\varepsilon > 0$ ,  $B_Y(f(x), \varepsilon)$  is an open neighbourhood of  $f(x)$ .  $f$  is not necessarily be injective, so  $f^{-1}[B_Y(f(x), \varepsilon)] = U \in x$ . By Definition 1.4.1,  $U$  is open, so for some  $\delta > 0$ ,  $B_X(x, \delta) \subseteq U$ . Thus, By Proposition 1.4.1,  $f$  is continuous iff  $f[B_X(x, \delta)] \subseteq B_Y(f(x), \varepsilon)$ . This satisfies the conditions we have.  $\square$

**Proposition 1.4.3.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A function  $f : X \rightarrow Y$  is said to be continuous iff for any closed set  $V$  in  $Y$ , its preimage in  $X$  under  $f$  is closed.

*Proof.* Let  $U_Y$  be any open set in  $Y$ , let  $U_X$  be the preimage of  $U_Y$  under  $f$ . By Definition 1.4.1,  $U_X$  is open in  $X$ . Let

$$V_X = f^{-1}[Y \setminus U_Y] = X \setminus U_X,$$

Then  $V_X$  is closed. □

**Definition 1.4.2** (convergence of sequences). Let  $(X, \mathcal{T})$  be a topological space, and let  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  is said to be *converges* in  $X$  iff there is an  $x \in X$ , such that for any open neighbourhood  $U_x$  of  $x$ , it contains a cofinite subset  $A \subseteq \{x_n\}$ . That is, there exists  $N$  in the domain of  $\{x_n\}$ , for any natural numbers  $n \geq N$ ,  $x_n \in U_x$ .

**Example 1.4.1.**

1. In a discrete topological space, a sequence  $\{x_n\}$  converges iff there is an  $N$  in the domain of  $\{x_n\}$ , for any natural numbers  $m > N$ ,  $x_m = x_N$ .
2. In an indiscrete topological space, any sequence  $\{x_n\}$  in  $X$  converges in  $X$ .  
And

$$\lim_{n \rightarrow \infty} \{x_n\} = X.$$

**Proposition 1.4.4.** In a Hausdorff space, any convergent sequence converges to a unique point in the space.

*Proof.* Let  $(X, \mathcal{T})$  be a Hausdorff space, and let  $\{x_n\}$  be a sequence in  $X$ . Suppose  $\{x_n\}$  converges to more than one point, say to  $x, y \in X$  with  $x \neq y$ , then, for all neighbourhoods  $N_x$  of  $x$  and  $N_y$  of  $y$ ,  $N_x$  contains a cofinite subset  $A \subseteq \{x_n\}$  and  $N_y$  contains a cofinite subset  $B \subseteq \{x_n\}$ . If this were true,  $N_x \cap N_y$  should be non-empty, otherwise  $N_x$  or  $N_y$  should be finite.

Then,  $x$  and  $y$  are not separated by neighbourhoods, thus  $(X, \mathcal{T})$  is not Hausdorff. This is a contradiction.

But, as  $(X, \mathcal{T})$  is Hausdorff, there must be mutually disjoint  $N_x$  and  $N_y$ . Thus, the assumption cause a contradiction. □

**Note 1.4.4.** As all metrizable spaces are Hausdorff, so any convergent sequence in a metrizable space converges to at most one point.

**Proposition 1.4.5.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological space, let  $f : X \rightarrow Y$  be a map, and let  $\{x_n\}$  be a convergent sequence in  $X$ . If  $f$  is continuous, then  $f[\{x_n\}]$  is a sequence convergent in  $Y$ .

*Proof.* Let  $U_y$  be any open neighbourhood of  $f(x)$ . By Definition 1.4.1,  $f^{-1}[U_y]$  is also an open neighbourhood of  $x$ . By Definition 1.4.2,  $f^{-1}[U_y]$  contains a cofinite subset  $A \subseteq \{x_n\}$ . Then  $f[A]$  is a cofinite subset of  $f[\{x_n\}]$ . As  $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$ ,  $f[\{x_n\}]$  converges in  $f[f^{-1}[U_y]] \supseteq f^{-1}[A]$ .  $\square$

**Note 1.4.5.** In this proposition, even if  $f[\{x_n\}]$  converges in  $Y$ ,  $f$  might be discontinuous. For example, let  $X$  any set, let  $\mathcal{T}$  be the indiscrete topology on  $X$ , let  $U$  be another cofinite subset of  $X$  with  $X \neq U$ , and let  $\mathcal{T}' = \{\emptyset, X, U\}$ . Let  $f : (X, \mathcal{T}) \rightarrow (X, \mathcal{T}')$  be defined by

$$f(x) = x.$$

By Definition 1.4.1,  $f$  is not continuous. But, for any convergent sequence  $\{x_n\}$  in  $(X, \mathcal{T})$ ,  $f[\{x_n\}]$  also convergent in  $(X, \mathcal{T})$ .

## 1.5 Homeomorphisms

**Definition 1.5.1** (homeomorphisms). Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. A bijection  $f : X \rightarrow Y$  is called a *homeomorphism* iff

- (i)  $f$  is a bijection;
- (ii)  $f$  is continuous;
- (iii)  $f^{-1}$  is continuous.

**Definition 1.5.2** (homeomorphic). Two topological spaces  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are said to be *homeomorphic* or *topologically equivalent*, denoted  $X \cong Y$ , iff there is an homeomorphism between them.

**Proposition 1.5.1.** Two topological spaces are homeomorphic only if they have the same cardinality.

*Proof.* Let  $X$  and  $Y$  be two sets with  $|X| < |Y|$ . There is no surjection from  $A$  to  $B$ .  $\square$

**Example 1.5.1.**  $|X| = |Y|$  does not imply  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are homeomorphic, even if they are finite. For example, let  $X = Y = \{1, \dots, n\}$ , and let  $\mathcal{T}_X$  be indiscrete topology on  $X$  and  $\mathcal{T}_Y = \mathcal{P}(X)$ . There is no homeomorphism between  $X$  and  $Y$ .

On the other hand, even if  $|X| = |Y| \geq \aleph_0$  and  $\mathcal{T}_X$  and  $\mathcal{T}_Y$  are induced by same metric,  $X$  and  $Y$  might not be homeomorphic. For example, if  $\mathcal{T}_X$  and  $\mathcal{T}_Y$  are both induced by standard Euclidean metric, and  $X = [a, b] \subseteq \mathbb{R}$  and  $Y = [c, d] \subseteq \mathbb{R}$  where  $a < b$  and  $c < d$ . No doubt,  $|X| = |Y| = \mathfrak{c}$ , but  $X$  and  $Y$  are not homeomorphic.

**Example 1.5.2.**  $\mathbb{R}^n$  and  $\mathbb{R}^m$  ( $n < m$ ) are not homeomorphic, although  $|\mathbb{R}^n| = |\mathbb{R}^m|$ .

**Example 1.5.3.** Let  $I$  be a proper interval in  $\mathbb{R}^n$ , let  $\mathcal{T}$  be standard Euclidean topology on  $\mathbb{R}^n$  and let  $\mathcal{T}_I$  be a subspace topology on  $I$ .  $I \cong \mathbb{R}^n$  iff  $I$  is an open interval.

But if  $\mathcal{T} = \mathcal{P}(\mathbb{R}^n)$ , then there exists bijection  $f : I \rightarrow \mathbb{R}^n$ , for  $|I| = |\mathbb{R}^n|$ , and such  $f$  can be bicontinuous, for any subset  $A \subseteq I$  is also open in  $\mathbb{R}^n$ , vise versa. In this case,  $I \cong \mathbb{R}^n$  whenever  $I$  is a closed, half-close, half-open, or open interval respect to standard Euclidean metric.

**Example 1.5.4.** Let  $S^n$  be an  $n$ -dimensional sphere with center  $o \in \mathbb{R}^{n+1}$  and radius  $r \in \mathbb{R}$ , i.e.,

$$S^n = \{x \in \mathbb{R}^{n+1} : \rho(o, x) = r\},$$

where  $\rho$  is the standard Euclidean metric on  $\mathbb{R}^{n+1}$ . For any  $x \in S^n$ , let  $U = B(x, \varepsilon) \cap S^n$  where  $0 \leq \varepsilon < \max_{x, y \in S^n} \rho(x, y)$  (here  $B(x, \varepsilon) = \{x\}$  if  $\varepsilon = 0$ ), then  $S^n \setminus U \cong \mathbb{R}^n$ .

**Example 1.5.5.** Indeed,  $S^1 \setminus \{x\} \cong \mathbb{R}$  where  $x \in S^1$ . But for any interval  $I \in \mathbb{R}$ ,  $S^1 \not\cong I$ .

## 1.6 Cover and Basis

**Definition 1.6.1** (cover). Let  $(X, \mathcal{T})$  be a topological space, and let  $U \subseteq X$ , then a family  $\mathcal{C} \subseteq \mathcal{P}(X)$  is called a *cover* of  $U$  iff the union of all sets in  $\mathcal{C}$  is a superset of  $U$ . That is,

$$U \subseteq \bigcup \mathcal{C}.$$

If  $\mathcal{C} \subseteq \mathcal{T}$ , then we call  $\mathcal{C}$  an *open cover* of  $U$ .

Let  $\mathcal{S} \subseteq \mathcal{C}$ , iff the union of  $\mathcal{S}$  is still a superset of  $U$ , then we call  $\mathcal{S}$  a *subcover* of  $\mathcal{C}$ .



**Definition 1.6.2** (basis). Let  $(X, \mathcal{T})$  be a topological space, let  $U \subseteq X$ , and let  $\mathcal{B}$  be a open cover of  $X$ . We call  $\mathcal{B}$  a *base* of  $X$  iff the union of  $\mathcal{B}$  is precisely  $U$  itself, i.e.,

$$U = \bigcup \mathcal{B}.$$

**Definition 1.6.3** (synthetic basis). Let  $(X, \mathcal{T})$  be a topological space, and let  $\mathcal{B}$  be a base of  $X$ .  $\mathcal{B}$  is said to be *synthetic* iff for any  $A, B \in \mathcal{B}$ ,

$$A \cap B = \bigcup_{i=1}^n B_i, \quad B_i \in \mathcal{B}.$$

**Definition 1.6.4** (generated by basis). Let  $X$  be any set and let  $\mathcal{B} \subseteq \mathcal{P}(X)$  be its cover. A topology  $\mathcal{T}$  on  $X$  is said to be *generated* by the base  $\mathcal{B}$  iff

- (i) for all  $U \in \mathcal{T}$ ,  $U$  is the union of  $\mathcal{B}$ -sets;
- (ii) for all  $U \in \mathcal{T}$ ,  $U$  is the finite intersection of  $\mathcal{B}$ -sets.

**Proposition 1.6.1.** Let  $(X, \mathcal{T})$  be a topological space be generated by a base  $\mathcal{B}$ . For all  $U \in \mathcal{T}$ , there is a  $B \in \mathcal{B}$  such that  $U \subseteq B$ .

*Proof.* By Definition 1.6.4, if  $\mathcal{T}$  is generated by  $\mathcal{B}$ , then for all  $U \in \mathcal{T}$ , there is an finite set  $I$ , such that

$$U = \bigcap_{i \in I} B_i, \quad B_i \in \mathcal{B}.$$

Thus, for at least one  $k \in I$ ,  $U \subseteq B_k$ . □

**Proposition 1.6.2.** Let  $X$  be any set, and let  $\mathcal{T}$  and  $\mathcal{T}'$  be its topologies generated by basis  $\mathcal{B}$  and  $\mathcal{B}'$  respectively. Then  $\mathcal{T}'$  is finer than  $\mathcal{T}$  iff for any  $B \in \mathcal{B}$ , there is a  $B' \in \mathcal{B}'$  such that  $B' \subseteq B$ .

*Proof.* If  $\mathcal{T}$  is generated by  $\mathcal{B}$ , then for all  $U' \in \mathcal{T}'$ ,

$$U' = \bigcup_{j \in J} B'_j,$$

where  $B_j \in \mathcal{B}$ .

As  $\mathcal{T}$  is generated by  $\mathcal{B}$ , then, certainly,  $\mathcal{B} \subseteq \mathcal{T}$ .

By the conditions we have,  $\mathcal{T} \subseteq \mathcal{T}'$  iff for all  $B \in \mathcal{B}$ , there is  $W' \in \mathcal{T}$  such that

$$B = W' = \bigcup_{i \in I} B'_i,$$

where  $B'_i \in \mathcal{B}'$ . Certainly, all such  $B'_i$  are contained in  $B$ . □

**Proposition 1.6.3.** Let  $X$  be any set, and let  $\mathcal{T} \subseteq \mathcal{P}(X)$ .  $\mathcal{T}$  is a topology on  $X$  iff it generates itself.

*Proof.* If  $\mathcal{T}$  is a topology on  $X$ , then, by Definition 1.6.4, any open set generated by  $\mathcal{T}$  is still a member of  $\mathcal{T}$ . On the other hand, if  $\mathcal{T}$  generates itself, then,  $\emptyset$  and  $X$  must be members of  $\mathcal{T}$ , and, by Definition 1.6.4,  $\mathcal{T}$  is a topology on  $X$ .  $\square$

## 1.7 Interiors and Closures

**Definition 1.7.1** (interiors). The *interior* of a set  $A$ , denoted  $A^\circ$ , is defined to be the union of all open subsets of  $A$ .

**Definition 1.7.2** (closure). The *closure* of a set  $A$ , denoted  $\overline{A}$ , is defined to be the intersection of all closed supersets of  $A$ .

**Definition 1.7.3** (dense sets). Let  $(X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .  $A$  is said to be *dense*, iff  $\overline{A} = X$ .

**Definition 1.7.4** (nowhere dense sets). A set  $A$  is said to be *nowhere dense* iff the interior of its closure is empty.

**Proposition 1.7.1** (properties of interiors). Let  $(X, \mathcal{T})$  be any topological space and  $A, B \subseteq X$ .

- (i) (Intensive)  $A^\circ \subseteq A$ .
- (ii)  $A$  is open iff  $A = A^\circ$ .
- (iii) (Idempotence)  $(A^\circ)^\circ = A^\circ$ .
- (iv)  $(A \cap B)^\circ = A^\circ \cap B^\circ$ .
- (v)  $A \subseteq B \implies A^\circ \subseteq B^\circ$ .
- (vi) If  $B$  is open, then  $B \subseteq A$  iff  $B \subseteq A^\circ$ .

*Proof.*

- (i) By Definition 1.7.1, naturally,  $A^\circ \subseteq A$ .
- (ii) By Definition 1.1.2,  $A^\circ$  is the union of open sets hence it is open.  $A$  is open iff it is the union of all open subsets of  $A$ . Thus  $A = A^\circ$ .

(iii)  $A^\circ$  is open, thus  $(A^\circ)^\circ = A^\circ$ .

(iv) By Definition 1.7.1, we have

$$\begin{aligned}(A \cap B)^\circ &= \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq A \cap B \right\} \\ &= \left\{ \bigcup U : (U \in \mathcal{T} \wedge U \subseteq A) \wedge (U \in \mathcal{T} \wedge U \subseteq B) \right\} \\ &= \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq A \right\} \cap \left\{ \bigcup U : U \in \mathcal{T} \wedge U \subseteq B \right\} \\ &= A^\circ \cap B^\circ.\end{aligned}$$

(v) Clearly,  $A^\circ \subseteq A$ , thus,

$$A \subseteq B \implies A^\circ \subseteq B$$

Suppose  $A^\circ \not\subseteq B^\circ$ , then  $A^\circ \setminus B^\circ$  is not empty ( $\emptyset$  is the subset of any set, so  $A^\circ$  is not empty).

Then there exists  $x \in A^\circ$  with  $x \in \partial B$  ( $x \in B$  but  $x \notin B^\circ$ ). Then there exists neighbourhood  $N_x \ni x$ , and  $N_x \cap \partial B \neq \emptyset$ . But this is impossible, for  $A^\circ \subseteq B$  implies that  $A^\circ \cap \partial B = \emptyset$  (This is a straight consequence of  $A^\circ \cap \partial A = \emptyset$ . See Proposition 1.8.1), so such  $N_x$  does not exist. Thus,

$$A^\circ \subseteq B^\circ.$$

(vi) If  $B$  is open, then  $B = B^\circ$ . Then  $B \subseteq A$  iff  $B^\circ \subseteq A^\circ$ .

□

**Proposition 1.7.2** (properties of closures). Let  $(X, \mathcal{T})$  be a topological space, and let  $A, B \subseteq X$ .

- (i)  $\overline{A}$  is closed.
- (ii)  $A$  is closed iff  $A = \overline{A}$ .
- (iii)  $A \subseteq B$  implies  $\overline{A} \subseteq \overline{B}$ .
- (iv) If  $A$  is closed, then  $A \supseteq B$  iff  $A \supseteq \overline{B}$

*Proof.*

- (i) By Definition 1.7.2,  $\overline{A}$  is the intersection of closed sets. By Proposition 1.1.1,  $\overline{A}$  is closed.

(ii) Proposition 1.1.1 implies that any closed set is the intersection of closed sets, this is precisely what Definition 1.7.2 says.

(iii)  $A \subseteq B$  iff  $X \setminus A \supseteq X \setminus B$ . Then we have

$$X \setminus (X \setminus A)^\circ \subseteq X \setminus (X \setminus B)^\circ$$

Clearly,  $(X \setminus A)^\circ$  is the union of all open set disjoint from  $A$ , then, by De Morgan's laws,  $X \setminus (X \setminus A)^\circ$  is the intersection of all closed sets containing  $A$ . By Definition 1.7.2, we have  $(X \setminus A)^\circ = \overline{A}$ . Thus

$$\overline{A} \subseteq \overline{B}.$$

(iv) If  $A$  is closed, then  $A = \overline{A}$ . Suppose  $B \subseteq A$ , then we have

$$\overline{B} \subseteq \overline{A} \iff \overline{B} \subseteq A.$$

□

## 1.8 Boundaries

**Definition 1.8.1** (boundaries). Let  $A$  be any set, the *boundary* of  $A$ , denoted  $\partial A$ , is defined to be the complement of the interior of  $A$  in the closure of  $A$ ; i.e.,

$$\partial A = \overline{A} \setminus A^\circ.$$

**Proposition 1.8.1** (properties of boundaries). Let  $(X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ .

- (i)  $\partial A$  is closed.
- (ii)  $A^\circ \cap \partial A = \emptyset$ .
- (iii)  $\overline{A} = A^\circ \cup \partial A$ .
- (iv)  $A$  is closed iff  $\partial A \subseteq A$ .
- (v)  $\partial A$  is nowhere dense.
- (vi)  $\partial \overline{A} \subseteq \partial A \subseteq \partial A^\circ$ .
- (vii)  $\partial A = \partial(X \setminus A)$ .

(viii)  $A$  is dense iff  $\partial A = X \setminus A^\circ$ .

*Proof.*

(i)  $\overline{A}$  is closed, and  $X \setminus A^\circ$  is also closed. Thus

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (X \setminus A)$$

is closed.

(ii) By Definition 1.8.1, we have

$$\partial A = \overline{A} \setminus A^\circ \iff \partial A \cap A^\circ = \overline{A} \setminus A^\circ \cap A^\circ = \overline{A} \cap \emptyset = \emptyset.$$

(iii) We have

$$\begin{aligned} \partial A = \overline{A} \setminus A^\circ &\iff \partial A \cup A^\circ = \overline{A} \setminus A^\circ \cup A^\circ = \overline{A} \cap (X \setminus A^\circ \cup A^\circ) \\ &\iff \partial A \cup A^\circ = \overline{A} \cap X|_{\text{for } A^\circ \subseteq X} = \overline{A}. \end{aligned}$$

(iv) As  $A$  is closed,  $A = \overline{A}$  (this can be straightly proved by Definition 1.7.2).

By Definition 1.8.1, it is clear that  $\partial A \subseteq \overline{A}$ , thus  $\partial A \subseteq A$ .

(v) By Definition 1.7.4,  $\partial A$  is nowhere dense iff  $\overline{\partial A}^\circ$  is empty. We have

$$\begin{aligned} \overline{\partial A}^\circ &= \overline{\overline{A} \setminus A^\circ}^\circ \\ &= (\overline{A} \setminus A^\circ) \cup (\overline{A} \setminus A^\circ) \setminus (\overline{A} \setminus A^\circ) \\ &= \emptyset. \end{aligned}$$

(vi)  $\overline{A} \supseteq A^\circ$  implies  $\overline{A}^\circ \supseteq (A^\circ)^\circ = A^\circ$ , then we have,

$$\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^\circ \subseteq \overline{A} \setminus A^\circ = \partial A.$$

$A^\circ \subseteq A$  implies  $\overline{A}^\circ \subseteq \overline{A}$ , then we have,

$$\partial A^\circ = \overline{A}^\circ \setminus (A^\circ)^\circ \supseteq \overline{A} \setminus A^\circ.$$

(vii) We have

$$\begin{aligned} \partial(X \setminus A) &= \overline{X \setminus A} \setminus (X \setminus A)^\circ \\ &= X \setminus A^\circ \setminus (X \setminus \overline{A}) \\ &= X \setminus A^\circ \cap \overline{A} \\ &= \overline{A} \setminus A^\circ \\ &= \partial A. \end{aligned}$$

(viii) By Definition 1.7.3,  $A$  is dense in  $X$  iff  $\overline{A} = X$ . Then we have,

$$\begin{aligned}\overline{A} = X &\iff \overline{A} \setminus A^\circ = X \setminus A^\circ \\ &\iff \partial A = X \setminus A^\circ.\end{aligned}$$

□

## 1.9 Limit Points

**Definition 1.9.1** (limit points). Let  $(X, \mathcal{T}_X)$  be a topological space, and let  $A \subseteq X$ . A point  $x \in X$  is called a *limit point* of  $A$  iff for all neighbourhood  $N_x$  of  $x$ ,  $N_x \setminus \{x\}$  intersects  $A$ .

**Proposition 1.9.1.** Let  $A$  be any set, and let  $x$  be a limit point of  $A$ , then  $x$  is an element of the closure of  $A$ .

*Proof.* If  $A$  is empty, then this is vacuously true. So, suppose  $A$  is not empty. By Definition 1.9.1, for all neighbourhood  $N_x$  of  $x$ ,  $N_x \setminus \{x\} \cap A$  is not empty. Naturally,  $N_x \cap A$  is not empty.

Assume that  $x \notin \overline{A}$ , then  $X \setminus \overline{A}$  is a neighbourhood of  $x$ , by Definition 1.1.5, and is disjoint from  $A$ . This is contradicted to the conditions. □

**Note 1.9.1.** In this proof, the proposition also holds for  $N_x \cap A^\circ = \emptyset$ . Because if it is true, then

$$N_x \cap \partial A \supseteq (N_x \cap A) \setminus (N_x \cap A^\circ) = N_x \cap A.$$

This implies that  $A \subseteq \partial A$ . In this case,  $\overline{A} = \partial A$ , for

Assume that  $x \notin \partial A$ , then we have the same conclusion.

Then  $A^\circ = A \setminus \partial A = \emptyset$ .

**Proposition 1.9.2.** A set is closed iff it contains all its limit point.

*Proof.* Let  $A$  be a set. By proposition 1.9.1, for every limit point of  $A$ , it is also an element of the closure  $\overline{A}$ . And  $A$  is closed iff  $A = \overline{A}$ . □

**Definition 1.9.2** (convergent sequences). Let  $(X, \mathcal{T}_X)$  be a topological space. A sequence  $\{x_n\}$  in  $X$  is said to be *convergence* in  $X$  iff there is an open set  $U$  contains all but finite terms of  $\{x_n\}$ .

## Chapter 2

# Creating New Spaces

### 2.1 Subspaces

**Definition 2.1.1** (subspace topology). Let  $(X, \mathcal{T})$  be a topological space and let  $A \subseteq X$ . The *subspace topology*  $\mathcal{T}_A$  on  $A$  is defined to be the family of the intersections of open sets in  $(X, \mathcal{T})$  and  $A$ . That is,

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}.$$

### 2.2 Quotient Spaces

**Definition 2.2.1** (quotient topology). Let  $(X, \mathcal{T})$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . The *quotient topology* is a topology on  $\mathcal{P}(X/\sim)$ ; it is defined as

$$\mathcal{T}_{X/\sim} = \{U \in \mathcal{P}(X/\sim) : \{x \in X : [x] \in U\} \in \mathcal{T}_X\}.$$

### 2.3 Product Spaces

**Definition 2.3.1** (product topologies).

## Chapter 3

# Topological Properties

### 3.1 Cardinal Functions

### 3.2 More on Separation Axioms

**Definition 3.2.1** (separated sets). Let  $(X, \mathcal{T})$  be a topological space, and let  $A, B \in \mathcal{P}(X)$ .

- (i)  $A$  and  $B$  are said to be *separated* iff each is disjoint from other's closure.
- (ii)  $A$  and  $B$  are said to be *separated by neighbourhoods* iff there are neighbourhoods  $N_A$  of  $A$  and  $N_B$  of  $B$  such that  $N_A$  and  $N_B$  are disjoint.
- (iii)  $A$  and  $B$  are said to be *separated by closed neighbourhoods* iff there are closed neighbourhoods  $\overline{N}_A$  of  $A$  and  $\overline{N}_B$  of  $B$  such that  $\overline{N}_A$  and  $\overline{N}_B$  are disjoint.
- (iv)  $A$  and  $B$  are said to be *separated by a continuous function* iff there is a continuous function  $f : X \rightarrow \mathbb{R}$ , such that  $f[A] = \{0\}$  and  $f[B] = \{1\}$ .
- (v)  $A$  and  $B$  are said to be *precisely separated by a continuous function* iff there is a continuous function  $f : X \rightarrow \mathbb{R}$ , such that  $f^{-1}[\{0\}] = A$  and  $f^{-1}[\{1\}] = B$

**Definition 3.2.2** ( $T_{2^{1/2}}$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_{2^{1/2}}$  or *Urysohn* iff two distinct points in  $X$  are separated by closed neighbourhoods.

**Example 3.2.1** ( $T_2$  but not  $T_{2^{1/2}}$ ).<sup>1</sup> (Remained as a problem)

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<sup>1</sup> See [MathPlanet](#).



**Definition 3.2.3** ( $T_3$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_3$  or *regular* iff it is  $T_0$  and given any point  $x \in (X, \mathcal{T})$  and closed set  $V \subseteq X$  with  $x \notin V$  are separated by neighbourhoods.

**Definition 3.2.4** ( $T_{3\frac{1}{2}}$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_{3\frac{1}{2}}$ , or *Tychonoff* or, *completely*  $T_3$ , or *completely regular*, iff it is  $T_0$  and given any point  $x$  and closed set  $V \subseteq X$  with  $x \notin V$ , they are separated by a continuous function.

**Definition 3.2.5** ( $T_4$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_4$  or *normal* iff it is Hausdorff and any two disjoint closed subsets of  $X$  are separated by neighbourhoods.

**Proposition 3.2.1** (Urysohn's lemma). A topological space is normal iff any two disjoint closed sets are separated by a continuous function.

**Definition 3.2.6** ( $T_5$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_5$  or *completely*  $T_4$  iff it is  $T_1$  any two separated sets are separated by neighbourhoods.

**Proposition 3.2.2.** Every subspace of a  $T_5$  space is normal.

**Definition 3.2.7** ( $T_6$  spaces). A topological space  $(X, \mathcal{T})$  is said to be  $T_6$ , or *perfectly*  $T_4$  or *perfectly normal* iff it is  $T_1$  and any two disjoint closed sets are precisely separated by a continuous function.

**Proposition 3.2.3** (Tietze extension theorem). Let  $(X, \mathcal{T})$  be normal topological space, and let  $f : A \rightarrow (\mathbb{R}, \mathcal{T}')$  be a continuous map where  $A$  is a closed subset of  $X$  and  $\mathcal{T}'$  is the standard topology (induced by Euclidean metric). Then there exists a continuous map

$$F : (X, \mathcal{T}) \rightarrow (\mathbb{R}, \mathcal{T}'),$$

such that

$$\forall x \in A : f(x) = g(x).$$

### 3.3 Countability Axioms

### 3.4 Compactness

**Definition 3.4.1** (compactness). A topological space  $(X, \mathcal{T})$  is said to be *compact* iff every open cover of  $X$  has a finite subcover. That is,

$$\forall \mathcal{C} \subseteq \mathcal{T} : \bigcup \mathcal{C} = X : \exists \mathcal{S} \subseteq \mathcal{C} : \bigcup \mathcal{S} = X : |\mathcal{S}| < \aleph_0.$$

### 3.5 Connectedness

**Definition 3.5.1** (connectedness). Let  $(X, \mathcal{T})$  be a topological space.  $(X, \mathcal{T})$  is said to be *connected* iff  $X$  is not empty and it is not the union of any disjoint open sets. That is,

$$\forall U, V \in \mathcal{T} : X = U \cup V : U \cap V \neq \emptyset.$$

**Definition 3.5.2** (path-connectedness). Let  $(X, \mathcal{T})$  be a topological space.

- (i) A map  $\gamma : [0, 1] \rightarrow X$  is called a *path* in  $X$  iff it is continuous. If  $\gamma(0) = x$  and  $\gamma(1) = y$ , we say that  $\gamma$  is path from  $x$  to  $y$  in  $X$ .
- (ii)  $X$  is said to be *path-connected* iff for all  $x, y \in X$  there is a path from  $x$  to  $y$  in  $X$ .