Notes for General Topology

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Chapter 1.

Topological Space

§1.1 Basic Ideas

Definition 1.1.1. Let X be any set.

A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ is a *topology* for X iff it satisfies the *open set axioms*:

1. $X \in \mathcal{T}$;
2. $\mathcal{T}$ is closed under arbitrary union; explicitly,
$$\forall \mathcal{A} \subseteq \mathcal{T} : \bigcup \mathcal{A} \in \mathcal{T};$$
3. $\mathcal{T}$ is closed under finite intersection; explicitly,
$$\forall \mathcal{B} \subseteq \mathcal{T} : |\mathcal{B}| \in \mathbb{N} : \bigcap \mathcal{B} \in \mathcal{T}.$$

The ordered pair $(X, \mathcal{T})$ is a *topological space* iff $\mathcal{T}$ is a topology for X.

A subset $U \subseteq X$ is an *open set* of $(X, \mathcal{T})$, or an *open subset* of X, iff $U \in \mathcal{T}$.

Note 1.1.1. Even if $\mathcal{T}$ is an infinite topology on an infinite set X, $\mathcal{T}$ is not needed to be closed under infinite intersection. For example, let $\mathcal{T}$

$$\mathcal{T} = \{[0, r) : r \in \mathbb{R}\}.$$

then $\mathcal{T}$ is a topology for $\mathbb{R}_{\geq 0}$. The collection

$$\left\{\left[0, \frac{1}{i}\right)\right\}_{i \in \mathbb{Z}_{>0}}$$

is a subset of $\mathcal{T}$, but its intersection is $\{0\} \notin \mathcal{T}$.

Lemma 1.1.1. Let $(X, \mathcal{T})$ be a topological space.

Then, $\emptyset \in \mathcal{T}$.

Proof. As $\emptyset$ is a subset of any set, $\emptyset \subseteq \mathcal{T}$. By the open set axiom 2, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

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Definition 1.1.2. Let X be any set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X.

$\mathcal{T}_1$ is said to be *finer* than $\mathcal{T}_2$, or $\mathcal{T}_2$ is said to be *coarser* than $\mathcal{T}_1$ iff $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Example 1.1.1. For any set X, the power set $\mathcal{P}(X)$ can be considered as a topology for X, called *discrete topology.* It is the *finest topology* on X.

Example 1.1.2. For any set X, the collection $\{\emptyset, X\}$ is a topology for X. It is called *indiscrete topology*, or *trivial topology*, which is the coarsest topology on X.

§1.2 Bases for Topologies

Definition 1.2.1. Let $(X, \mathcal{T})$ be a topological space.

A collection $\mathcal{B} \subseteq \mathcal{T}$ is an *analytic basis* for $\mathcal{T}$ iff for any $U \in \mathcal{T}$, there is a $\mathcal{S} \subseteq \mathcal{B}$, such that

$$U = \bigcup \mathcal{S}.$$