Notes on General Topology

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Chapter 1.

Topological Spaces

§1.1 Metric Spaces

How do we measure the distance between two points in the space? Well, in the intuitive world, Pythagoras theorem might be the most popular way to do so.

Take the 2-dimensional Euclidean vector space $\mathbb{R}^2$ for example – Here we use the xy-plane. For any vector $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, their distance is defined as the length of segment with $\mathbf{a}$ and $\mathbf{b}$ as the end points. So, by Pythagoras theorem, it is by this formula:

$$\rho(\mathbf{a}, \mathbf{b}) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}.$$

Here, we consider ρ as an operation defined on $\mathbb{R}^2 \times \mathbb{R}^2$, called metric function, and its outputs are real numbers:

$$\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}.$$

In this case, ρ satisfies 4 conditions: (let me write down the precondition first) for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$

1. $\rho(\mathbf{a}, \mathbf{b}) \geq 0$. Obviously, there is no negative distance.
2. $\rho(\mathbf{a}, \mathbf{b}) = 0 \iff \mathbf{a} = \mathbf{b}$. That is, the distance between two points is zero if and only if they coincides.
3. $\rho(\mathbf{a}, \mathbf{b}) = \rho(\mathbf{b}, \mathbf{a})$. That is obvious, because a segment $\mathbf{ab}$ is actually the same as $\mathbf{ba}$.

4. $\rho(\mathbf{a}, \mathbf{b}) + \rho(\mathbf{b}, \mathbf{c}) \geq \rho(\mathbf{a}, \mathbf{c})$. We call this property the triangle inequality. It is a quite geometrical name, because if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form a triangle, then the sum of the length of any two sides of the triangle can't be less than the length of the third one.

So, in this case, we call the order pair $(\mathbb{R}^2, \rho)$ the 2-dimensional Euclidean metric space.

But, just like how we define the mass of 1 kg in Physics, the metric function ρ here is not so naturally given as what we might keep in mind.

In mathematics, the set here can be any set X. It can be $\mathbb{R}^n$, a collection of some sets, or a of some functions.

And the metric function ρ here can be any operation defined on $X \times X$, if it satisfies these conditions.

So, here we came up with the definition of metric spaces.

Definition 1.1.1.

§1.2 Bases of Sets

Definition 1.2.1. Let X be a set, and let $\mathcal{B} \subseteq \mathcal{P}(X)$.

$\mathcal{B}$ is a *basis* of X iff

1. $\mathcal{B}$ is a cover of X, i.e., $X \subseteq \bigcup \mathcal{B}$; and
2. For any $B_1, B_2 \in \mathcal{B}$, there exists a $\mathcal{A} \subseteq \mathcal{B}$, such that $B_1 \cap B_2 = \bigcup \mathcal{A}$.

Note 1.2.1. Some authors also call bases of sets *synthetics sets*.

§1.3 Topological Spaces

Definition 1.3.1. Let X be any set.

A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ is a *topology* for X iff it satisfies the *open set axioms*:

(O1): $X \in \mathcal{T}$;

(O2): $\mathcal{T}$ is closed under arbitrary union; explicitly,

$$\forall \mathcal{A} \subseteq \mathcal{T} : \bigcup \mathcal{A} \in \mathcal{T};$$

(O3): $\mathcal{T}$ is closed under finite intersection; explicitly,

$$\forall \mathcal{B} \subseteq \mathcal{T} : |\mathcal{B}| \in \mathbb{N} : \bigcap \mathcal{B} \in \mathcal{T}.$$

The ordered pair $(X, \mathcal{T})$ is a *topological space* iff $\mathcal{T}$ is a topology for X.

A subset $U \subseteq X$ is an *open set* of $(X, \mathcal{T})$, or an *open subset* of X, iff $U \in \mathcal{T}$.

Note 1.3.1. Even if $\mathcal{T}$ is an infinite topology on an infinite set X, $\mathcal{T}$ is not needed to be closed under infinite intersection. For example, let $\mathcal{T}$

$$\mathcal{T} = \{[0, r) : r \in \mathbb{R}\}.$$

then $\mathcal{T}$ is a topology for $\mathbb{R}_{\geq 0}$. The collection

$$\left\{ \left[0, \frac{1}{i}\right) \right\}_{i \in \mathbb{Z}_{>0}}$$

is a subset of $\mathcal{T}$, but its intersection is $\{0\} \notin \mathcal{T}$.

Lemma 1.3.1. Let $(X, \mathcal{T})$ be a topological space.

Then, $\emptyset \in \mathcal{T}$.

Proof. As $\emptyset$ is a subset of any set, $\emptyset \subseteq \mathcal{T}$. By the open set axiom 2, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

■

Example 1.3.1. Let $X = \{1, 2, 3\}$, and let

$$\mathcal{B} = \{\{1, 2\}, \{2, 3\}, \{2\}\},$$

and let $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B}\}$, then $\mathcal{T}$ is a topology for X.

Definition 1.3.2. Let X be any set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X.

$\mathcal{T}_1$ is said to be *finer* than $\mathcal{T}_2$, or $\mathcal{T}_2$ is said to be *coarser* than $\mathcal{T}_1$ iff $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Example 1.3.2. For any set X, the power set $\mathcal{P}(X)$ can be considered as a topology for X, called *discrete topology.* It is the *finest topology* on X.

Example 1.3.3. For any set X, the collection $\{\emptyset, X\}$ is a topology for X. It is called *indiscrete topology*, or *trivial topology*, which is the coarsest topology on X.

§1.4 Interiors and Closures

Definition 1.4.1. Let $(X, \mathcal{T})$ be a topological space, and let $U \subseteq X$.

The *interior* of A, denoted A° or $\mathrm{int}(A)$, in $(X, \mathcal{T})$ is defined as the union of all open sets contained in A. Explicitly, int_X can be considered as a mapping from $\mathcal{P}(X)$ to $\mathcal{P}(X)$, defined as

$$\mathrm{int}_X(A) := \bigcup(\mathcal{P}(A) \cap \mathcal{T}).$$

Note 1.4.1. Finding the interior of a subset requires the definition of the topology for the set. I mean, even for the same set X and the same subset $A \subseteq X$, if there are two different topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for X, the interior of A in $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ might be different. For example, in $\mathbb{R}$, let $\mathcal{T}_1$ be indiscrete topology for $\mathbb{R}$, and let $\mathcal{T}_2$ be the Euclidean topology for $\mathbb{R}$, then,

$$\mathrm{int}_{\mathcal{T}_1}([0,1)) = \emptyset, \text{ and } \mathrm{int}_{\mathcal{T}_2}([0,1)) = (0,1),$$

where $\mathrm{int}_{\mathcal{T}_1}(\cdot)$ and $\mathrm{int}_{\mathcal{T}_2}(\cdot)$ denotes the interior mapping for $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ respectively.

Note 1.4.2. By the definition, it is clear that for any topology $(X, \mathcal{T})$ and for any $A \subseteq X$, $\mathrm{int}(A) \in \mathcal{T}$.

Lemma 1.4.1. Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let $U \in \mathcal{T}$.

Then, $U \subseteq A$ if and only if $U \subseteq \mathrm{int}(A)$.

Proof. As $U \in \mathcal{P}(A)$ and $U \in \mathcal{T}$, $U \in \mathcal{P}(A) \cap \mathcal{T}$. Thus,

$$U \subseteq \bigcup(\mathcal{P}(A) \cap \mathcal{T}) = \mathrm{int}(A).$$

□

Conversely, as $\mathrm{int}(A) \subseteq A$, as $U \subseteq \mathrm{int}(A)$, $U \subseteq A$. ■

Lemma 1.4.2. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$\mathrm{int}(A) = A$ if and only if $A \in \mathcal{T}$.

Proof. If $\mathrm{int}(A) = A$, then $A = \bigcup(\mathcal{P}(A) \cap \mathcal{T})$. As $\mathcal{P}(A) \cap \mathcal{T} \subseteq \mathcal{T}$, this union is an element of $\mathcal{T}$. □

Conversely, as $A \in \mathcal{P}(A)$ and $A \in \mathcal{T}$, $A \in \mathcal{P}(A) \cap \mathcal{T}$. For any $U \in \mathcal{P}(A)$, $U \subseteq A$. Then, we have $A \supseteq \bigcup(\mathcal{P}(A) \cap \mathcal{T})$; and as $A \subseteq \bigcup(\mathcal{P}(A) \cap \mathcal{T})$, we have

$$A = \bigcup(\mathcal{P}(A) \cap \mathcal{T}) = \mathrm{int}(A).$$

■

Lemma 1.4.3. Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

Then,

$$\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B).$$

Proof. Let $U \subseteq \mathrm{int}(A \cap B)$. Then,

$$U \subseteq \mathrm{int}(A \cap B) = \bigcup(\mathcal{P}(A \cap \mathcal{B}) \cap \mathcal{T}).$$

Note that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$, so $U \in \mathcal{P}(A \cap B) \cap \mathcal{T}$ iff $U \in \mathcal{P}(A) \cap \mathcal{T}$ and $\mathcal{P}(B) \cap \mathcal{T}$. We have

$$\begin{aligned} U \subseteq \bigcup(\mathcal{P}(A \cap \mathcal{B}) \cap \mathcal{T}) &\iff U \subseteq \bigcup(\mathcal{P}(A) \cap \mathcal{T}) \wedge U \subseteq \bigcup(\mathcal{P}(B) \cap \mathcal{T}) \\ &\iff U \subseteq \mathrm{int}(A) \wedge U \subseteq \mathrm{int}(B) \\ &\iff U \subseteq \mathrm{int}(A) \cap \mathrm{int}(B). \end{aligned}$$

Thus, $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$. ■

§1.5 Bases for Topologies

Definition 1.5.1. Let $(X, \mathcal{T})$ be a topological space.

A collection $\mathcal{B} \subseteq \mathcal{T}$ is an *analytic basis* for $\mathcal{T}$ iff for any $U \in \mathcal{T}$, there is an $\mathcal{A} \subseteq \mathcal{B}$, such that

$$U = \bigcup \mathcal{A}.$$

Lemma 1.5.1. Let $(X, \mathcal{T})$ be a topological space.

A collection $\mathcal{B} \subseteq \mathcal{T}$ is an analytic basis for $\mathcal{T}$ iff for any $U \in \mathcal{T}$ and for any $x \in U$, there exists a $B \subseteq \mathcal{B}$, such that

$$x \in B \subseteq \mathcal{B}.$$

Proof. Let $\mathcal{B}$ be an analytic basis for $\mathcal{T}$.

As $\mathcal{B}$ is a basis for $\mathcal{T}$, for any $U \in \mathcal{T}$, there exists a $B' \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}'$, which implies that for any $B' \in \mathcal{B}$, $B' \subseteq U$.

□

Conversely, let $\mathcal{B} \subseteq \mathcal{T}$ satisfies the condition after "iff".

Let $U \in \mathcal{T}$. For any $x \in U$, let $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$.

As $\bigcup\{x\}_{x \in U} = U$, and $\{x\} \subseteq B_x$, we have

$$U \subseteq \bigcup_{x \in U} B_x.$$

As every $B_x \subseteq U$, we have

$$\bigcup_{x \in U} B_x \subseteq U.$$

Thus,

$$U = \bigcup_{x \in U} B_x.$$

■

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Note 1.5.1. Explicitly, 2 can be considered as: for any $B_1, B_2 \in \mathcal{B}$, and for any $x \in B_1 \cap B_2$, there exists a $B_x \in \mathcal{B}$, such that

$$x \in B_x \subseteq B_1 \cap B_2.$$

(Why?)

Note 1.5.2. $\emptyset$ is not necessary be an element of $\mathcal{B}$.

Lemma 1.5.2. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be an analytic basis for $\mathcal{T}$.

Then, $\mathcal{B}$ is a synthetic basis of X.

Proof. Let $B_1, B_2 \in \mathcal{B}$.

As $\mathcal{B}$ is an analytic basis for $\mathcal{T}$, $\mathcal{B} \subseteq \mathcal{T}$, thus $B_1 \cap B_2 \in \mathcal{T}$.

Thus, there exists an $\mathcal{A} \subseteq \mathcal{B}$, such that

$$B_1 \cap B_2 = \bigcup \mathcal{A}.$$

This precisely satisfies the definition of synthetic basis. ■

Lemma 1.5.3. Let X be any set, and let $\mathcal{B}$ be a synthetic basis of X.

Let

$$\mathcal{T} = \left\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B} \right\}.$$

Then, $\mathcal{T}$ is a topology for X.

Proof. As $\mathcal{B}$ is a synthetic basis of X, $X \subseteq \bigcup \mathcal{B}$. As $\mathcal{B} \subseteq \mathcal{P}(X)$, $\bigcup \mathcal{B} \subseteq X$. Thus, $X = \bigcup \mathcal{B} \in \mathcal{T}$. □

Let $\mathcal{U} \subseteq \mathcal{T}$. For any $U \in \mathcal{U}$, there exists an $\mathcal{A}_U \subseteq \mathcal{B}$, such that $U = \bigcup \mathcal{A}_U$.

We have

$$\begin{aligned} \bigcup \mathcal{U} &= \bigcup \left\{ \bigcup \mathcal{A}_U \right\}_{U \in \mathcal{U}} \\ &= \bigcup \left(\bigcup \{\mathcal{A}_U\}_{U \in \mathcal{U}} \right) \end{aligned}$$

As for any $U \in \mathcal{U}$, $\mathcal{A}_U \subseteq \mathcal{B}$, thus,

$$\mathcal{U} = \bigcup \{\mathcal{A}_U\}_{U \in \mathcal{U}} \subseteq \mathcal{B}.$$

Thus, $\bigcup \mathcal{U} \in \mathcal{T}$. Therefore, $\mathcal{T}$ is closed under arbitrary union. □

Let $\mathcal{V}$ be a finite subset of $\mathcal{T}$. For any $V \in \mathcal{U}$, there exists an $\mathcal{A}_V \subseteq \mathcal{B}$, such that $U = \mathcal{A}_V$.

We have

$$\begin{aligned} \bigcap \mathcal{V} &= \bigcap \left\{ \bigcup \mathcal{A}_V \right\}_{V \in \mathcal{U}} \\ &= \bigcap \left(\bigcup \{\mathcal{A}_V\}_{V \in \mathcal{U}} \right). \end{aligned}$$

Similar to what we have proved above,

$$\mathcal{V} = \bigcup \{\mathcal{A}_V\}_{V \in \mathcal{V}} \subseteq \mathcal{B}.$$

Thus, $\bigcap \mathcal{V} \in \mathcal{T}$. Therefore, $\mathcal{T}$ is closed under finite intersection. ■

Lemma 1.5.4. Let X be any set, and let $\mathcal{C}$ be a cover of X.

The collection

$$\mathcal{B} = \left\{ \bigcap \mathcal{A} : \mathcal{A} \subseteq \mathcal{C} \wedge |\mathcal{A}| \in \mathbb{N} \right\}$$

is a synthetic basis of X.

Proof. Let $B_1, B_2 \in \mathcal{B}$. There exist $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$, such that $B_1 = \bigcup \mathcal{U}$ and $B_2 = \bigcup \mathcal{V}$. Then, we have

$$\begin{aligned} B_1 \cap B_2 &= \bigcup_{U \in \mathcal{U}} U \cap \bigcup_{V \in \mathcal{V}} V \\ &= \bigcup \{U \cap V\}_{U \in \mathcal{U}, V \in \mathcal{V}}. \end{aligned}$$

$\{U, V\} \subseteq \mathcal{C}$, so $U \cap V \in \mathcal{B}$. As U and V are arbitrarily taken from $\mathcal{U}$ and $\mathcal{V}$ respectively, $\{U \cap V\}_{U \in \mathcal{U}, V \in \mathcal{V}} \subseteq \mathcal{B}$.

Therefore, for any $B_1, B_2 \in \mathcal{B}$, there exists a finite $\mathcal{A} \subseteq \mathcal{B}$, such that $B_1 \cap B_2 = \bigcap \mathcal{A}$. ■

Note 1.5.3. In this note, we say that $\mathcal{C}$ *generates* $\mathcal{B}$.

Note 1.5.4. If $\mathcal{C}$ generates the synthetic basis $\mathcal{B}$, then $\mathcal{B}$ is the smallest synthetic basis containing $\mathcal{C}$. (Why?)

Definition 1.5.2. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a synthetic basis of X.

$\mathcal{T}$ is *generated by* $\mathcal{B}$ iff

$$\mathcal{T} = \left\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B} \right\}.$$

Example 1.5.1. In $\mathbb{R}^n$, for any $\mathbf{x} \in \mathbb{R}^n$, define

$$B(\mathbf{x}, \delta) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < \delta \wedge \delta \in \mathbb{R}_{>0}\}.$$

Let $\mathcal{B}$ be the set of all such $B(\mathbf{x}, \delta)$, then, $\mathcal{B}$ is a synthetic basis of $\mathbb{R}^n$, and it generates the *Euclidean topology* for X.

Example 1.5.2. In $\mathbb{R}^n$, let $\mathcal{I}$ be the set of all open intervals. $\mathcal{I}$ is a synthetic basis of $\mathbb{R}^n$, and it also generates the Euclidean topology for $\mathbb{R}^n$.

Example 1.5.3. An *ordered set* $(X, \preceq)$ is a set X together with an *ordering* $\preceq$ defined on X. That is, for any $x, y, z \in X$,

(i) (reflexive) $x \preceq x$;

(ii) (transitive) $x \preceq y$ and $y \preceq z$ implies $x \prec z$;

(iii) (antisymmetric) $x \preceq y$ and $y \preceq x$ implies $x = y$.

$(X, \preceq)$ is an *totally ordered set* iff $\preceq$ is *connected.* That is, for any $x, y \in X$, $x \neq y$ implies $x \prec y$ or $y \prec x$.

Now, let $(X, \preceq)$ be a totally ordered set, and let

$$\mathcal{A} = \{X_{\prec x} : x \in X\} \cup \{X_{\succ x} : x \in X\} .$$

Let $\mathcal{B}$ be the synthetic basis generated by $\mathcal{A}$.

Then, $\mathcal{B}$ generates an *order topology* for X.

If $\preceq$ is $\leq$ on $\mathbb{R}$, then, the order topology for $\mathbb{R}$ is exactly the same as its Euclidean topology.

Let

$$\mathcal{X} = \left\{\prod_{i=1}^{n} \mathbb{R}_{<x_i}\right\} \cup \left\{\prod_{i=1}^{n} \mathbb{R}_{>x_i}\right\},$$

let $\mathcal{B}$ be the synthetic basis for $\mathbb{R}^n$ generated by $\mathcal{X}$. Then $\mathcal{B}$ also generates the Euclidean topology for $\mathbb{R}^n$

Note 1.5.5. In the example above, if $(X, \preceq)$ is an ordered set, but the connectedness of $\preceq$ is not required, then $\mathcal{A}$ is not a cover of X, and it generates no synthetic basis of X.

Example 1.5.4. For any totally ordered set $(X, \preceq)$, the discrete topology for X can be generated by either the collection of all closed intervals in X or the collection of all singletons in X.

Example 1.5.5. Let $(X, \preceq)$ be a totally ordered set, and let C be a countable subset of X. The set

$$\mathcal{A} = \{X_{\prec x} : x \in C\}$$

is a countable synthetic basis of X, and it generates a countable topology for X.

Example 1.5.6. Let X be an countably infinite set, and let $\mathcal{B}$ be the partition of X. As $\mathcal{B}$ is a synthetic basis for X, let $\mathcal{T}$ be the topology generated by $\mathcal{B}$.

Then, $|\mathcal{T}| = |\mathcal{P}(\mathcal{B})| = 2^{|\mathcal{B}|}$. Thus,

(i) $\mathcal{T}$ is finite iff $\mathcal{B}$ is finite;

(ii) $\mathcal{T}$ is uncountable iff $\mathcal{B}$ is infinite (even if $\mathcal{B}$ is just countably infinite).

(iii) $\mathcal{T}$ can not be countably infinite.