Notes on General Topology

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Chapter 1.

Topological Spaces

§1.1 Metric Spaces

How do we measure the distance between two points in a space? Take $\mathbb{R}^3$ for example, for any points $x, y \in \mathbb{R}^3$, the distance between x and y is usually means the length of the segments with x and y as its endpoints, which is given by

$$\rho(x,y) = \left(\sum_{i=1}^{3} |x_i - y_i|^2\right)^{\frac{1}{2}},$$

where for any $p \in \mathbb{R}^3$, p_i denotes the i-th component of p. Here, we consider ρ as a function from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$, called *Euclidean metric function* on $\mathbb{R}^3$. Then, ρ satisfies the following conditions: For any x, y, and $z \in \mathbb{R}^3$,

1. $\rho(x,y) = 0$ if and only if $x = y$;
2. $\rho(x,y) = \rho(y,x)$;
3. $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$; this property is also called *triangle inequality.*

In this case, we call the ordered pair $(\mathbb{R}^3, \rho)$ the 3-*dimensional Eulidean metric space.*

Just like how the first scientist defined the unite of 1 kilogram, a metric function is not entirely naturally given, but is chosen depend on what distance we need to find. In the example above, the set $\mathbb{R}^3$ can be replaced by any set X, and the metric function ρ can be any operation from $X \times X \to \mathbb{R}$ satisfying the 3 conditions above. And this is how metric spaces are defined.

Definition 1.1.1. Let X be any set. A mapping $\rho : X \times X \to \mathbb{R}$ is a *metric* on X if and only if it satisfies the *metric axioms*. That is, for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$,

(M1) $\rho(x, y) = 0$ if and only if $x = y$;

(M2) $\rho(x, y) = 0$ if and only if $x = y$;

(M3) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

An ordered pair (X, ρ) is a *metric space* if and only if ρ is a metric on X.

Some author also consider $\rho(x, y) \geq 0$ as an axiom in the list above, but, rigorously, it is a property deduced by the 3 axioms. By metric axiom M3, we have

$$\rho(x, y) + \rho(y, x) \geq \rho(x, x).$$

By M2, we have

$$\rho(x, y) + \rho(x, y) \geq \rho(x, x).$$

By M1, we have

$$2\rho(x, y) \geq 0.$$

Thus,

$$\rho(x, y) \geq 0.$$

So, if we are going to prove if an operation is a metric, this is an unnecessary progress.

Definition 1.1.2. Let (X, ρ) be a metric space, let $x \in X$, and let $\delta \in \mathbb{R}_{>0}$.

The *open δ-ball*, or simply *δ-ball*, of x is defined as the set

$$B(x, \delta) = \{y \in X : \rho(x, y) < \delta\}.$$

The "shape" of an open ball is determined by the metric and the set. In the 3-dimensional Euclidean metric space $(\mathbb{R}^3, \rho)$, for example, an open δ-ball of x is a sphere with x as its center and δ as its radius. But if ρ is a *taxicap metric* on $\mathbb{R}^3$, i.e.,

$$\rho(x, y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|,$$

then, an open δ-ball of x is no longer a sphere, but a box with x as its center and 2δ as the length of it edges.

If $<$ is replaced by $\leq$ in the definition, then we have the definition blew.

Definition 1.1.3. Let (X, ρ) be a metric space, let $x \in X$, and let $\delta \in \mathbb{R}_{>0}$.

The *closed δ-ball* of x is defined as the set

$$\overline{B}(x, \delta) = \{y \in X : \rho(x, y) \leq \delta\}.$$

Note that, in the both definitions above, we have the condition $y \in X$. This means, open (closed) balls are always subsets of X. For example, let

$$X = [0, 1] \times [0, 1],$$

and let ρ be an Euclidean metric on X. In this case, $B(0, 1)$ is not a disk, but disk sector.

§1.2 Some Examples on Metric Spaces

Example 1.2.1. Some metrics do not care about the any geometrical length. For example, let $\rho : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$ be defined as

$$\rho(x, y) = \begin{cases} \dfrac{x}{y} - 1 & : x \geq y; \\ \dfrac{y}{x} - 1 & : x < y, \end{cases}$$

then ρ is a metric on $\mathbb{R}_{>0}$ only cares about the ratio between any two points in the space.

There is another metric on $\mathbb{R}_{>0}$ which is quite similar. Let $\rho : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ be defined as

$$\rho(x, y) = \left| \log \left(\frac{x}{y} \right) \right|,$$

then ρ is a metric on $\mathbb{R}_{>0}$, and it can be proved by the properties of logarithm functions.

Example 1.2.2. The discrete metric ρ on X only cares about if any two points x and y in X coincide or not. That is,

$$\rho(x, y) = \begin{cases} 1 & : x \neq y; \\ 0 & : \text{else}. \end{cases}$$

Example 1.2.3. The 3-dimensional Euclidean metric space is one of p-product metric spaces. Let

$$X = \prod_{i=1}^{n} X_i.$$

Then, for any $p \in \mathbb{R}_{\geq 1}$, the p-product metric $\rho_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\rho_p(x, y) = \left(\sum_{i=1}^{n} \rho_i^p(x_i, y_i) \right)^{\frac{1}{p}},$$

where $\rho_i(x_i, y_i)$ can be the Euclidean metric on X_i for any $i \in \{1, \ldots, n\}$, but it is not required. Indeed, p-product metric spaces are metric space. It is easy to show that ρ_p satisfies the metric axiom 1 and 2. Now, we prove that the ρ_p satisfies the metric axiom 3. That is to show that

$$\left(\sum_{i=1}^{n} \rho_i(x_i, z_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \rho_i(x_i, y_i)^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \rho_i(y_i, z_i)^p \right)^{\frac{1}{p}}$$

for any $x, y, z \in X$.

Proof. By Minkowski's inequality,

$$\left(\sum_{i=1}^{n} (\rho_i(x_i, y_i) + \rho_i(y_i, z_i))^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} \rho_i(x_i, y_i)^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \rho_i(y_i, z_i)^p \right)^{\frac{1}{p}}.$$

As for any i, ρ_i is a metric on X_i. So ρ_i satisfies the open axiom 3. Thus,

$$\left(\sum_{i=1}^{n} \rho_i(x_i, z_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} (\rho_i(x_i, y_i) + \rho_i(y_i, z_i))^p \right)^{\frac{1}{p}}.$$

That is,

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z),$$

which is precisely the metric axiom 3. ■

Example 1.2.4. Let M be a set of all bounded functions $S \to (T, \rho_T)$, where ρ_T is a metric on T. Here, we treat all of these functions as points in M. Let $\rho : M \times M \to \mathbb{R}$ be defined as

$$\rho(f, g) = \sup_{x \in S} \rho_T(f(x), g(x)),$$

Then ρ is a metric on M. It is actually easy to prove that ρ is indeed a metric on M. Take the metric axiom 3 for example.

Proof. Let $f, g, h \in M$, then we have

$$\begin{aligned}\rho(f,g) + \rho(g,h) &= \sup_{x\in S} \rho_T(f(x), g(x)) + \sup_{x\in S} \rho_T(g(x), h(x)) \\ &= \sup_{x\in S} \left(\rho_T(f(x), g(x)) + \rho_T(g(x), h(x))\right).\end{aligned}$$

As ρ_T satisfies the metric axiom 3, we have

$$\ldots \geq \sup_{x\in S} \rho(f(x), h(x)) = \rho(f,h).$$

■

Example 1.2.5. Let (M, ρ) be a metric space. Let $\rho_H : \mathcal{P}(M) \setminus \{\emptyset\} \to \mathbb{R}$ be defined as

$$\rho_H(X,Y) = \max\left\{\sup_{x\in X} \rho(x,Y), \sup_{y\in Y} \rho(y,X)\right\},$$

where

$$\rho_H(a,B) = \inf_{b\in B} \rho(a,b).$$

ρ_H is called *Hausdorff metric*. It measures how two subsets X and Y of M are similar.

§1.3 Bases of Sets

Definition 1.3.1. Let X be a set, and let $\mathcal{B} \subseteq \mathcal{P}(X)$.

$\mathcal{B}$ is a *basis* of X iff

1. $\mathcal{B}$ is a cover of X, i.e., $X \subseteq \bigcup \mathcal{B}$; and
2. For any $B_1, B_2 \in \mathcal{B}$, there exists a $\mathcal{A} \subseteq \mathcal{B}$, such that $B_1 \cap B_2 = \bigcup \mathcal{A}$.

Note 1.3.1. Some authors also call bases of sets *synthetics sets*.

§1.4 Topological Spaces

There are actually at least two ways to define topological spaces: by open set axioms and by bases of sets. The first one might be the more popular one.

Definition 1.4.1. Let X be any set.

A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ is a *topology* for X iff it satisfies the *open set axioms*:

(O1): $X \in \mathcal{T}$;

(O2): $\mathcal{T}$ is closed under arbitrary union; explicitly,

$$\forall \mathcal{U} \subseteq \mathcal{T} : \bigcup \mathcal{U} \in \mathcal{T};$$

(O3): $\mathcal{T}$ is closed under finite intersection; explicitly,

$$\forall \mathcal{F} \subseteq \mathcal{T} : |\mathcal{F}| \in \mathbb{N} : \bigcap \mathcal{F} \in \mathcal{T}.$$

The ordered pair $(X, \mathcal{T})$ is a *topological space* iff $\mathcal{T}$ is a topology for X.

A subset $U \subseteq X$ is an *open set* of $(X, \mathcal{T})$, or an *open subset* of X, iff $U \in \mathcal{T}$.

Another way to define topological spaces is to consider any topological space as a collection *generated* by a basis of the given set. Given any set X, a basis $\mathcal{B}$ of X is a cover of X, where for any $A, B \in \mathcal{B}$, $A \cap B$ can be considered as the union of an $\mathcal{S} \subseteq \mathcal{B}$; i.e.,

$$A \cap B = \bigcup \mathcal{S}.$$

For example, if (X, ρ) is a metric space, then the set of all open balls in (X, ρ) is a basis of X.

Lemma 1.4.1. Let X be any set, and let $\mathcal{T} \subseteq \mathcal{P}(X)$.

Then, $\mathcal{T}$ is a topology for X if and only if there exists a basis $\mathcal{B}$ of X such that

$$\mathcal{T} = \left\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B} \right\}.$$

Proof. Assume $\mathcal{T}$ is a topology for X, then $\mathcal{T}$ itself is a basis of X. (Why?) Then,

$$\mathcal{T} = \left\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{T} \right\}.$$

□

On the other hand, assume there is a basis $\mathcal{B}$ of X such that $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B}\}$.

As $\mathcal{B}$ is a cover of X and $\mathcal{B} \subseteq \mathcal{B}$, we have $X = \bigcup \mathcal{B} \in \mathcal{T}$. So $\mathcal{T}$ satisfies the open set axiom O1.

Let $\mathcal{U} \subseteq \mathcal{T}$. For any $U \in \mathcal{U}$, let $\mathcal{A}_U \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{A}_U$. Thus

$$\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} \left(\bigcup \mathcal{A}_U \right) = \bigcup \left(\bigcup_{U \in \mathcal{U}} \mathcal{A}_U \right).$$

The union in the bracket is a subset of $\mathcal{B}$, so $\bigcup \mathcal{U} \in \mathcal{T}$. Thus, open set axiom 2 is satisfied.

Let $U, V \in \mathcal{T}$. Then there exists $\mathcal{A}_U, \mathcal{A}_V \mathcal{B}$ ∎

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Even if $\mathcal{T}$ is an infinite topology on an infinite set X, $\mathcal{T}$ is not needed to be closed under infinite intersection. For example, let $\mathcal{T}$

$$\mathcal{T} = \{[0, r) : r \in \mathbb{R}\}.$$

then $\mathcal{T}$ is a topology for $\mathbb{R}_{\geq 0}$. The collection

$$\left\{\left[0, \frac{1}{i}\right)\right\}_{i \in \mathbb{Z}_{>0}}$$

is a subset of $\mathcal{T}$, but its intersection is $\{0\} \notin \mathcal{T}$.

Lemma 1.4.2. Let $(X, \mathcal{T})$ be a topological space.

Then, $\emptyset \in \mathcal{T}$.

Proof. As $\emptyset$ is a subset of any set, $\emptyset \subseteq \mathcal{T}$. By the open set axiom 2, we have

$$\emptyset = \bigcup \emptyset \in \mathcal{T}.$$

∎

Example 1.4.1. Let $X = \{1, 2, 3\}$, and let

$$\mathcal{B} = \{\{1, 2\}, \{2, 3\}, \{2\}\},$$

and let $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B}\}$, then $\mathcal{T}$ is a topology for X.

Definition 1.4.2. Let X be any set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X.

$\mathcal{T}_1$ is said to be *finer* than $\mathcal{T}_2$, or $\mathcal{T}_2$ is said to be *coarser* than $\mathcal{T}_1$ iff $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Example 1.4.2. For any set X, the power set $\mathcal{P}(X)$ can be considered as a topology for X, called *discrete topology*. It is the *finest topology* on X.

Example 1.4.3. For any set X, the collection $\{\emptyset, X\}$ is a topology for X. It is called *indiscrete topology*, or *trivial topology*, which is the coarsest topology on X.

§1.5 Interiors and Closures

Definition 1.5.1. Let $(X, \mathcal{T})$ be a topological space, and let $U \subseteq X$.

The *interior* of A, denoted A° or $\operatorname{int}(A)$, in $(X, \mathcal{T})$ is defined as the union of all open sets contained in A. Explicitly, int_X can be considered as a mapping from $\mathcal{P}(X)$ to $\mathcal{P}(X)$, defined as

$$\operatorname{int}_X(A) := \bigcup(\mathcal{P}(A) \cap \mathcal{T}).$$

Note 1.5.1. Finding the interior of a subset requires the definition of the topology for the set. I mean, even for the same set X and the same subset $A \subseteq X$, if there are two different topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for X, the interior of A in $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ might be different. For example, in $\mathbb{R}$, let $\mathcal{T}_1$ be indiscrete topology for $\mathbb{R}$, and let $\mathcal{T}_2$ be the Euclidean topology for $\mathbb{R}$, then,

$$\operatorname{int}_{\mathcal{T}_1}([0,1)) = \emptyset, \text{ and } \operatorname{int}_{\mathcal{T}_2}([0,1)) = (0,1),$$

where $\operatorname{int}_{\mathcal{T}_1}(\cdot)$ and $\operatorname{int}_{\mathcal{T}_2}(\cdot)$ denotes the interior mapping for $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ respectively.

Note 1.5.2. By the definition, it is clear that for any topology $(X, \mathcal{T})$ and for any $A \subseteq X$, $\operatorname{int}(A) \in \mathcal{T}$.

Lemma 1.5.1. Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let $U \in \mathcal{T}$.

Then, $U \subseteq A$ if and only if $U \subseteq \operatorname{int}(A)$.

Proof. As $U \in \mathcal{P}(A)$ and $U \in \mathcal{T}$, $U \in \mathcal{P}(A) \cap \mathcal{T}$. Thus,

$$U \subseteq \bigcup(\mathcal{P}(A) \cap \mathcal{T}) = \operatorname{int}(A).$$

□

Conversely, as $\operatorname{int}(A) \subseteq A$, as $U \subseteq \operatorname{int}(A)$, $U \subseteq A$. ■

Lemma 1.5.2. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

$\operatorname{int}(A) = A$ if and only if $A \in \mathcal{T}$.

Proof. If $\operatorname{int}(A) = A$, then $A = \bigcup(\mathcal{P}(A) \cap \mathcal{T})$. As $\mathcal{P}(A) \cap \mathcal{T} \subseteq \mathcal{T}$, this union is an element of $\mathcal{T}$. □

Conversely, as $A \in \mathcal{P}(A)$ and $A \in \mathcal{T}$, $A \in \mathcal{P}(A) \cap \mathcal{T}$. For any $U \in \mathcal{P}(A)$, $U \subseteq A$. Then, we have $A \supseteq \bigcup(\mathcal{P}(A) \cap \mathcal{T})$; and as $A \subseteq \bigcup(\mathcal{P}(A) \cap \mathcal{T})$, we have

$$A = \bigcup(\mathcal{P}(A) \cap \mathcal{T}) = \operatorname{int}(A).$$

■

Lemma 1.5.3. Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$.

Then,

$$\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B).$$

Proof. Let $U \subseteq \operatorname{int}(A \cap B)$. Then,

$$U \subseteq \operatorname{int}(A \cap B) = \bigcup(\mathcal{P}(A \cap \mathcal{B}) \cap \mathcal{T}).$$

Note that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$, so $U \in \mathcal{P}(A \cap B) \cap \mathcal{T}$ iff $U \in \mathcal{P}(A) \cap \mathcal{T}$ and $\mathcal{P}(B) \cap \mathcal{T}$. We have

$$\begin{aligned} U \subseteq \bigcup(\mathcal{P}(A \cap \mathcal{B}) \cap \mathcal{T}) &\iff U \subseteq \bigcup(\mathcal{P}(A) \cap \mathcal{T}) \wedge U \subseteq \bigcup(\mathcal{P}(B) \cap \mathcal{T}) \\ &\iff U \subseteq \operatorname{int}(A) \wedge U \subseteq \operatorname{int}(B) \\ &\iff U \subseteq \operatorname{int}(A) \cap \operatorname{int}(B). \end{aligned}$$

Thus, $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$. ■

§1.6 Bases for Topologies

Definition 1.6.1. Let $(X, \mathcal{T})$ be a topological space.

A collection $\mathcal{B} \subseteq \mathcal{T}$ is an *analytic basis* for $\mathcal{T}$ iff for any $U \in \mathcal{T}$, there is an $\mathcal{A} \subseteq \mathcal{B}$, such that

$$U = \bigcup \mathcal{A}.$$

Lemma 1.6.1. Let $(X, \mathcal{T})$ be a topological space.

A collection $\mathcal{B} \subseteq \mathcal{T}$ is an analytic basis for $\mathcal{T}$ iff for any $U \in \mathcal{T}$ and for any $x \in U$, there exists a $B \subseteq \mathcal{B}$, such that

$$x \in B \subseteq \mathcal{B}.$$

Proof. Let $\mathcal{B}$ be an analytic basis for $\mathcal{T}$.

As $\mathcal{B}$ is a basis for $\mathcal{T}$, for any $U \in \mathcal{T}$, there exists a $B' \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}'$, which implies that for any $B' \in \mathcal{B}$, $B' \subseteq U$.

□

Conversely, let $\mathcal{B} \subseteq \mathcal{T}$ satisfies the condition after "iff".

Let $U \in \mathcal{T}$. For any $x \in U$, let $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$.

As $\bigcup \{x\}_{x \in U} = U$, and $\{x\} \subseteq B_x$, we have

$$U \subseteq \bigcup_{x \in U} B_x.$$

As every $B_x \subseteq U$, we have

$$\bigcup_{x \in U} B_x \subseteq U.$$

Thus,

$$U = \bigcup_{x \in U} B_x.$$

■

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Note 1.6.1. Explicitly, 2 can be considered as: for any $B_1, B_2 \in \mathcal{B}$, and for any $x \in B_1 \cap B_2$, there exists a $B_x \in \mathcal{B}$, such that

$$x \in B_x \subseteq B_1 \cap B_2.$$

(Why?)

Note 1.6.2. $\emptyset$ is not necessary be an element of $\mathcal{B}$.

Lemma 1.6.2. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be an analytic basis for $\mathcal{T}$.

Then, $\mathcal{B}$ is a synthetic basis of X.

Proof. Let $B_1, B_2 \in \mathcal{B}$.

As $\mathcal{B}$ is an analytic basis for $\mathcal{T}$, $\mathcal{B} \subseteq \mathcal{T}$, thus $B_1 \cap B_2 \in \mathcal{T}$.

Thus, there exists an $\mathcal{A} \subseteq \mathcal{B}$, such that

$$B_1 \cap B_2 = \bigcup \mathcal{A}.$$

This precisely satisfies the definition of synthetic basis. ■

Lemma 1.6.3. Let X be any set, and let $\mathcal{B}$ be a synthetic basis of X.

Let

$$\mathcal{T} = \left\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B} \right\}.$$

Then, $\mathcal{T}$ is a topology for X.

Proof. As $\mathcal{B}$ is a synthetic basis of X, $X \subseteq \bigcup \mathcal{B}$. As $\mathcal{B} \subseteq \mathcal{P}(X)$, $\bigcup \mathcal{B} \subseteq X$. Thus, $X = \bigcup \mathcal{B} \in \mathcal{T}$. □

Let $\mathcal{U} \subseteq \mathcal{T}$. For any $U \in \mathcal{U}$, there exists an $\mathcal{A}_U \subseteq \mathcal{B}$, such that $U = \bigcup \mathcal{A}_U$.

We have

$$\begin{aligned} \bigcup \mathcal{U} &= \bigcup \left\{ \bigcup \mathcal{A}_U \right\}_{U \in \mathcal{U}} \\ &= \bigcup \left(\bigcup \{\mathcal{A}_U\}_{U \in \mathcal{U}} \right) \end{aligned}$$

As for any $U \in \mathcal{U}$, $\mathcal{A}_U \subseteq \mathcal{B}$, thus,

$$\mathcal{U} = \bigcup \{\mathcal{A}_U\}_{U \in \mathcal{U}} \subseteq \mathcal{B}.$$

Thus, $\bigcup \mathcal{U} \in \mathcal{T}$. Therefore, $\mathcal{T}$ is closed under arbitrary union. □

Let $\mathcal{V}$ be a finite subset of $\mathcal{T}$. For any $V \in \mathcal{U}$, there exists an $\mathcal{A}_V \subseteq \mathcal{B}$, such that $U = \mathcal{A}_V$.

We have

$$\begin{aligned} \bigcap \mathcal{V} &= \bigcap \left\{ \bigcup \mathcal{A}_V \right\}_{V \in \mathcal{U}} \\ &= \bigcap \left(\bigcup \{\mathcal{A}_V\}_{V \in \mathcal{U}} \right). \end{aligned}$$

Similar to what we have proved above,

$$\mathcal{V} = \bigcup \{\mathcal{A}_V\}_{V \in \mathcal{V}} \subseteq \mathcal{B}.$$

Thus, $\bigcap \mathcal{V} \in \mathcal{T}$. Therefore, $\mathcal{T}$ is closed under finite intersection. ■

Lemma 1.6.4. Let X be any set, and let $\mathcal{C}$ be a cover of X.

The collection

$$\mathcal{B} = \left\{ \bigcap \mathcal{A} : \mathcal{A} \subseteq \mathcal{C} \wedge |\mathcal{A}| \in \mathbb{N} \right\}$$

is a synthetic basis of X.

Proof. Let $B_1, B_2 \in \mathcal{B}$. There exist $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$, such that $B_1 = \bigcup \mathcal{U}$ and $B_2 = \bigcup \mathcal{V}$. Then, we have

$$\begin{aligned} B_1 \cap B_2 &= \bigcup_{U \in \mathcal{U}} U \cap \bigcup_{V \in \mathcal{V}} V \\ &= \bigcup \{U \cap V\}_{U \in \mathcal{U}, V \in \mathcal{V}}. \end{aligned}$$

$\{U, V\} \subseteq \mathcal{C}$, so $U \cap V \in \mathcal{B}$. As U and V are arbitrarily taken from $\mathcal{U}$ and $\mathcal{V}$ respectively, $\{U \cap V\}_{U \in \mathcal{U}, V \in \mathcal{V}} \subseteq \mathcal{B}$.

Therefore, for any $B_1, B_2 \in \mathcal{B}$, there exists a finite $\mathcal{A} \subseteq \mathcal{B}$, such that $B_1 \cap B_2 = \bigcap \mathcal{A}$. ■

Note 1.6.3. In this note, we say that $\mathcal{C}$ *generates* $\mathcal{B}$.

Note 1.6.4. If $\mathcal{C}$ generates the synthetic basis $\mathcal{B}$, then $\mathcal{B}$ is the smallest synthetic basis containing $\mathcal{C}$. (Why?)

Definition 1.6.2. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a synthetic basis of X.

$\mathcal{T}$ is *generated by* $\mathcal{B}$ iff

$$\mathcal{T} = \left\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B} \right\}.$$

Example 1.6.1. In $\mathbb{R}^n$, for any $\mathbf{x} \in \mathbb{R}^n$, define

$$B(\mathbf{x}, \delta) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < \delta \wedge \delta \in \mathbb{R}_{>0}\}.$$

Let $\mathcal{B}$ be the set of all such $B(\mathbf{x}, \delta)$, then, $\mathcal{B}$ is a synthetic basis of $\mathbb{R}^n$, and it generates the *Euclidean topology* for X.

Example 1.6.2. In $\mathbb{R}^n$, let $\mathcal{I}$ be the set of all open intervals. $\mathcal{I}$ is a synthetic basis of $\mathbb{R}^n$, and it also generates the Euclidean topology for $\mathbb{R}^n$.

Example 1.6.3. An *ordered set* $(X, \preceq)$ is a set X together with an *ordering* $\preceq$ defined on X. That is, for any $x, y, z \in X$,

(i) (reflexive) $x \preceq x$;

(ii) (transitive) $x \preceq y$ and $y \preceq z$ implies $x \prec z$;

(iii) (antisymmetric) $x \preceq y$ and $y \preceq x$ implies $x = y$.

$(X, \preceq)$ is an *totally ordered set* iff $\preceq$ is *connected*. That is, for any $x, y \in X$, $x \neq y$ implies $x \prec y$ or $y \prec x$.

Now, let $(X, \preceq)$ be a totally ordered set, and let

$$\mathcal{A} = \{X_{\prec x} : x \in X\} \cup \{X_{\succ x} : x \in X\}.$$

Let $\mathcal{B}$ be the synthetic basis generated by $\mathcal{A}$.

Then, $\mathcal{B}$ generates an *order topology* for X.

If $\preceq$ is $\leq$ on $\mathbb{R}$, then, the order topology for $\mathbb{R}$ is exactly the same as its Euclidean topology.

Let

$$\mathcal{X} = \left\{ \prod_{i=1}^{n} \mathbb{R}_{<x_i} \right\} \cup \left\{ \prod_{i=1}^{n} \mathbb{R}_{>x_i} \right\},$$

let $\mathcal{B}$ be the synthetic basis for $\mathbb{R}^n$ generated by $\mathcal{X}$. Then $\mathcal{B}$ also generates the Euclidean topology for $\mathbb{R}^n$

Note 1.6.5. In the example above, if $(X, \preceq)$ is an ordered set, but the connectedness of $\preceq$ is not required, then $\mathcal{A}$ is not a cover of X, and it generates no synthetic basis of X.

Example 1.6.4. For any totally ordered set $(X, \preceq)$, the discrete topology for X can be generated by either the collection of all closed intervals in X or the collection of all singletons in X.

Example 1.6.5. Let $(X, \preceq)$ be a totally ordered set, and let C be a countable subset of X. The set

$$\mathcal{A} = \{X_{\prec x} : x \in C\}$$

is a countable synthetic basis of X, and it generates a countable topology for X.

Example 1.6.6. Let X be an countably infinite set, and let $\mathcal{B}$ be the partition of X. As $\mathcal{B}$ is a synthetic basis for X, let $\mathcal{T}$ be the topology generated by $\mathcal{B}$.

Then, $|\mathcal{T}| = |\mathcal{P}(\mathcal{B})| = 2^{|\mathcal{B}|}$. Thus,

(i) $\mathcal{T}$ is finite iff $\mathcal{B}$ is finite;

(ii) $\mathcal{T}$ is uncountable iff $\mathcal{B}$ is infinite (even if $\mathcal{B}$ is just countably infinite).

(iii) $\mathcal{T}$ can not be countably infinite.