

Exercises from *Putnam Competition*

Exercise 2021.b4 Let F_0, F_1, \dots be the sequence of Fibonacci numbers, with $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $m > 2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m-1} k^k$ is divided by F_m . Prove that R_m is also a Fibonacci number.

Exercise 2020.b5 For $j \in \{1, 2, 3, 4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that $3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \neq 0$.

Exercise 2018.a5 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0, f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

Exercise 2018.b2 Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \dots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Exercise 2018.b4 Given a real number a , we define a sequence by $x_0 = 1, x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n , then the sequence is periodic.

Exercise 2018.b6 Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most $2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}$.

Exercise 2017.b3 Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

Exercise 2016.a6 Suppose that G is a finite group generated by the two elements g and h , where the order of g is odd. Show that every element of G can be written in the form $g^{m_1} h^{n_1} g^{m_2} h^{n_2} \dots g^{m_r} h^{n_r}$ with $1 \leq r \leq |G|$ and $m_1, n_1, m_2, n_2, \dots, m_r, n_r \in \{-1, 1\}$. (Here $|G|$ is the number of elements of G .)

Exercise 2015.b1 Let f be a three times differentiable function (defined on \mathbb{R} and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.