Exercises from *Cambridge Tripos*

Exercise 2022.IA.1-II-9D-a Let a_n be a sequence of real numbers. Show that if a_n converges, the sequence $\frac{1}{n}\sum_{k=1}^{n} a_k$ also converges and $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k = \lim_{n\to\infty} a_n$.

Exercise 2022.IA.1-II-10D-c Let a function $g:(0,\infty)\to\mathbb{R}$ be continuous and bounded. Show that for every $T>0$ there exists a sequence x_n such that $x_n\to\infty$ and $\lim_{n\to\infty}\left(g\left(x_n+T\right)-g\left(x_n\right)\right)=0$.

Exercise 2022.IA.4-I-1E-a By considering numbers of the form $3p_1\ldots p_k-1$, show that there are infinitely many primes of the form $3n+2$ with $n\in\mathbb{N}$.

Exercise 2022.IA.4-I-2D-a Prove that $\sqrt[3]{2}+\sqrt[3]{3}$ is irrational.

Exercise 2022.IB.3-II-13G-a-i Let $U\subset\mathbb{C}$ be a (non-empty) connected open set and let f_n be a sequence of holomorphic functions defined on U. Suppose that f_n converges uniformly to a function f on every compact subset of U. Show that f is holomorphic in U.

Exercise 2022.IB.3-II-11G-b Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be the map given by $f(x,y)=\left(\frac{\cos x+\cos y-1}{2},\cos x-\cos y\right)$. Prove that f has a fixed point.

Exercise 2022.IB.1-I-3G-i Show that $f(z)=\frac{z}{\sin z}$ has a removable singularity at $z=0$.

Exercise 2022.IB.3-I-1E-ii Let R be a subring of a ring S, and let J be an ideal in S. Show that $R+J$ is a subring of S and that $\frac{R}{R\cap J}\cong\frac{R+J}{J}$.

Exercise 2022.IIB.1-II-8F-a-i Let V be a finite dimensional complex inner product space, and let α be an endomorphism of V. Define its adjoint α^*. Assume that α is normal, i.e. α commutes with its adjoint: $\alpha\alpha^*=\alpha^*\alpha$. Show that α and α^* have a common eigenvector $\mathbf{v}$.

Exercise 2021.IIB.3-II-11F-ii Let X be an open subset of Euclidean space $\mathbb{R}^n$. Show that X is connected if and only if X is path-connected.

Exercise 2021.IIB.2-I-1G Let M be a module over a Principal Ideal Domain R and let N be a submodule of M. Show that M is finitely generated if and only if N and M/N are finitely generated.

Exercise 2021.IIB.3-I-1G-i Let G be a finite group, and let H be a proper subgroup of G of index n. Show that there is a normal subgroup K of G such that $|G/K|$ divides n ! and $|G/K| \geqslant n$

Exercise 2021.IIB.1-II-9G-v Let R be the ring of continuous functions $\mathbb{R} \rightarrow \mathbb{R}$. Show that R is not Noetherian.

Exercise 2018.IA.1-I-3E-b Let $f : \mathbb{R} \rightarrow (0, \infty)$ be a decreasing function. Let $x_1 = 1$ and $x_{n+1} = x_n + f(x_n)$. Prove that $x_n \rightarrow \infty$ as $n \rightarrow \infty$.