Exercises from *Complex analysis* by Elias M. Stein and Rami Shakarchi

Exercise 1.13a Suppose that f is holomorphic in an open set Ω. Prove that if $\mathrm{Re}(f)$ is constant, then f is constant.

Exercise 1.13b Suppose that f is holomorphic in an open set Ω. Prove that if $\mathrm{Im}(f)$ is constant, then f is constant.

Exercise 1.13c Suppose that f is holomorphic in an open set Ω. Prove that if $|f|$ is constant, then f is constant.

Exercise 1.18 Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

Exercise 1.19a Prove that the power series $\sum nz^n$ does not converge on any point of the unit circle.

Exercise 1.19b Prove that the power series $\sum zn/n^2$ converges at every point of the unit circle.

Exercise 1.19c Prove that the power series $\sum zn/n$ converges at every point of the unit circle except $z = 1$.

Exercise 1.22 Let $\mathbb{N} = 1, 2, 3, \ldots$ denote the set of positive integers. A subset $S \subset \mathbb{N}$ is said to be in arithmetic progression if $S = a, a + d, a + 2d, a + 3d, \ldots$ where $a, d \in \mathbb{N}$. Here d is called the step of S. Show that $\mathbb{N}$ cannot be partitioned into a finite number of subsets that are in arithmetic progression with distinct steps (except for the trivial case $a = d = 1$).

Exercise 1.26 Suppose f is continuous in a region Ω. Prove that any two primitives of f (if they exist) differ by a constant.

Exercise 2.2 Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Exercise 2.5 Suppose f is continuously complex differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that $\int_T f(z)dz = 0$.

Exercise 2.6 Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then $\int_T f(z)dz = 0$.

Exercise 2.9 Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0 \quad \text{and} \quad \varphi'(z_0) = 1$ then φ is linear.

Exercise 2.13 Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ is equal to 0. Prove that f is a polynomial.

Exercise 3.2 Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

Exercise 3.3 Show that $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = \pi \frac{e^{-a}}{a}$ for $a > 0$.

Exercise 3.4 Show that $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx = \pi e^{-a}$ for $a > 0$.

Exercise 3.9 Show that $\int_0^1 \log(\sin \pi x) dx = -\log 2$.

Exercise 3.14 Prove that all entire functions that are also injective take the form $f(z) = az + b$, $a, b \in \mathbb{C}$ and $a \neq 0$.

Exercise 3.22 Show that there is no holomorphic function f in the unit disc D that extends continuously to ∂D such that $f(z) = 1/z$ for $z \in \partial D$.

Exercise 4.4a Suppose Q is a polynomial of degree ≥ 2 with distinct roots, none lying on the real axis. Calculate $\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{Q(x)} dx$, $\xi \in \mathbb{R}$, in terms of the roots of Q.

Exercise 5.1 Prove that if f is holomorphic in the unit disc, bounded and not identically zero, and $z_1, z_2, \ldots, z_n, \ldots$ are its zeros ($|z_k| < 1$), then $\sum_n (1 - |z_n|) < \infty$.

Exercise 5.3 Show that $\sum \frac{z^n}{(n!)^\alpha}$ is an entire function of order $1/\alpha$.