Exercises from *Real Mathematical Analysis* by Charles Pugh

Exercise 2.12a Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$. The sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is called a rearrangement of (p_n). Show that if f is an injection, the limit of a sequence is unaffected by rearrangement.

Exercise 2.12b Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$. The sequence $(q_k)_{k\in\mathbb{N}}$ with $q_k = p_{f(k)}$ is called a rearrangement of (p_n). Show that if f is a surjection, the limit of a sequence is unaffected by rearrangement.

Exercise 2.26 Prove that a set $U \subset M$ is open if and only if none of its points are limits of its complement.

Exercise 2.29 Let $\mathcal{T}$ be the collection of open subsets of a metric space M, and $\mathcal{K}$ the collection of closed subsets. Show that there is a bijection from $\mathcal{T}$ onto $\mathcal{K}$.

Exercise 2.32a Show that every subset of $\mathbb{N}$ is clopen.

Exercise 2.41 Let $\|\cdot\|$ be any norm on $\mathbb{R}^m$ and let $B = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$. Prove that B is compact.

Exercise 2.46 Assume that A, B are compact, disjoint, nonempty subsets of M. Prove that there are $a_0 \in A$ and $b_0 \in B$ such that for all $a \in A$ and $b \in B$ we have $d(a_0, b_0) \leq d(a, b)$.

Exercise 2.48 Prove that there is an embedding of the line as a closed subset of the plane, and there is an embedding of the line as a bounded subset of the plane, but there is no embedding of the line as a closed and bounded subset of the plane.

Exercise 2.56 Prove that the 2-sphere is not homeomorphic to the plane.

Exercise 2.57 Show that if S is connected, it is not true in general that its interior is connected.

Exercise 2.79 Prove that if M is nonempty compact, locally path-connected and connected then it is path-connected.

Exercise 2.85 Suppose that M is compact and that $\mathcal{U}$ is an open covering of M which is redundant in the sense that each $p \in M$ is contained in at least two members of $\mathcal{U}$. Show that $\mathcal{U}$ reduces to a finite subcovering with the same property.

Exercise 2.92 Give a direct proof that the nested decreasing intersection of nonempty covering compact sets is nonempty.

Exercise 2.109 A metric on M is an ultrametric if for all $x, y, z \in M$, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. Show that a metric space with an ultrametric is totally disconnected.

Exercise 2.126 Suppose that E is an uncountable subset of $\mathbb{R}$. Prove that there exists a point $p \in \mathbb{R}$ at which E condenses.

Exercise 2.137 Let P be a closed perfect subset of a separable complete metric space M. Prove that each point of P is a condensation point of P.

Exercise 2.138 Given a Cantor space $M \subset R^2$, given a line segment $[p, q] \subset R^2$ with $p, q \notin M$, and given an $\epsilon > 0$, prove that there exists a path A in the ϵ-neighborhood of $[p, q]$ that joins p to q and is disjoint from M.

Exercise 3.1 Assume that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leq |t - x|^2$ for all t, x. Prove that f is constant.

Exercise 3.4 Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

Exercise 3.11a Let $f\colon (a, b) \to \mathbb{R}$ be given. If $f''(x)$ exists, prove that

$$\lim_{h \to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x).$$

Exercise 3.17c-i Show that the bump function $\beta(x) = e^2 e(1-x) \cdot e(x+1)$ is smooth.

Exercise 3.17c-ii Show that the bump function $\beta(x) = e^2 e(1-x) \cdot e(x+1)$ is identically 0 outside the interval $(-1, 1)$.

Exercise 3.18 Let L be any closed set in $\mathbb{R}$. Prove that there is a smooth function $f\colon \mathbb{R} \to [0,1]$ such that $f(x) = 0$ if and only if $x \in L$.

Exercise 3.43a Let $\psi(x) = x \sin 1/x$ for $0 < x \leq 1$ and $\psi(0) = 0$. If $f\colon [-1,1] \to \mathbb{R}$ is Riemann integrable, prove that $f \circ \psi$ is Riemann integrable.

Exercise 3.53 Given $f, g \in \mathcal{R}$, prove that $\max(f,g)$ and $\min(f,g)$ are Riemann integrable, where $\max(f,g)(x) = \max(f(x), g(x))$ and $\min(f,g)(x) = \min(f(x), g(x))$.

Exercise 3.63a Prove that $\sum 1/k(\log(k))^p$ converges when $p > 1$.

Exercise 3.63b Prove that $\sum 1/k(\log(k))^p$ diverges when $p \leq 1$.

Exercise 4.15a A continuous, strictly increasing function $\mu\colon (0,\infty) \to (0,\infty)$ is a modulus of continuity if $\mu(s) \to 0$ as $s \to 0$. A function $f\colon [a,b] \to \mathbb{R}$ has modulus of continuity μ if $|f(s) - f(t)| \leq \mu(|s-t|)$ for all $s, t \in [a,b]$. Prove that a function is uniformly continuous if and only if it has a modulus of continuity.

Exercise 4.15b A continuous, strictly increasing function $\mu\colon (0,\infty) \to (0,\infty)$ is a modulus of continuity if $\mu(s) \to 0$ as $s \to 0$. A function $f\colon [a,b] \to \mathbb{R}$ has modulus of continuity μ if $|f(s) - f(t)| \leq \mu(|s-t|)$ for all $s, t \in [a,b]$. Prove that a family of functions is equicontinuous if and only if its members have a common modulus of continuity.

Exercise 4.19 If M is compact and A is dense in M, prove that for each $\delta > 0$ there is a finite subset $\{a_1, \ldots, a_k\} \subset A$ which is δ-dense in M in the sense that each $x \in M$ lies within distance δ of at least one of the points $a_1, \ldots, a_k$.

Exercise 4.36a Suppose that the ODE $x' = f(x)$ on $\mathbb{R}$ is bounded, $|f(x)| \leq M$ for all x. Prove that no solution of the ODE escapes to infinity in finite time.

Exercise 4.42 Prove that $\mathbb{R}$ cannot be expressed as the countable union of Cantor sets.

Exercise 5.2 Let L be the vector space of continuous linear transformations from a normed space V to a normed space W. Show that the operator norm makes L a normed space.

Exercise 5.20 Assume that U is a connected open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$ is differentiable everywhere on U. If $(Df)_p = 0$ for all $p \in U$, show that f is constant.

Exercise 5.22 If Y is a metric space and $f\colon [a,b] \times Y \to \mathbb{R}$ is continuous, show that $F(y) = \int_a^b f(x,y)dx$ is continuous.

Exercise 5.43a Suppose that $T\colon R^n \to R^m$ has rank k. Show there exists a $\delta > 0$ such that if $S\colon R^n \to R^m$ and $||S - T|| < \delta$ then S has rank $\geq k$.

Exercise 6.38 If f and g are integrable prove that their maximum and minimum are integrable.

Exercise 6.39 Suppose that f and g are measurable and their squares are integrable. Prove that fg is measurable, integrable, and $\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}$.

Exercise 6.43 Prove that $g(y) = \int_0^\infty e^{-x}\sin(x+y)dx$ is differentiable and find $g'(y)$.

Exercise 6.49a Prove that $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if and only if the preimage of every Borel set is a Lebesgue measurable.