Exercises from *Topology* by James Munkres

Exercise 13.1 Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Exercise 13.3a Let X be a set, let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Show that $\mathcal{T}_c$ is a topology on the set X.

Exercise 13.3b Show that the collection

$$\mathcal{T}_\infty = \{U | X - U \text{ is infinite or empty or all of X}\}$$

is does not need to be a topology on the set X.

Exercise 13.4a1 If $\mathcal{T}_\alpha$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X.

Exercise 13.4a2 If $\mathcal{T}_\alpha$ is a family of topologies on X, show that $\bigcup \mathcal{T}_\alpha$ does not need to be a topology on X.

Exercise 13.4b1 Let $\mathcal{T}_\alpha$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$.

Exercise 13.4b2 Let $\mathcal{T}_\alpha$ be a family of topologies on X. Show that there is a unique largest topology on X contained in all the collections $\mathcal{T}_\alpha$.

Exercise 13.5a Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

Exercise 13.5b Show that if $\mathcal{A}$ is a subbasis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

Exercise 13.6 Show that the lower limit topology $\mathbb{R}_l$ and K-topology $\mathbb{R}_K$ are not comparable.

Exercise 13.8a Show that the collection $\{(a,b) \mid a < b, a \text{ and } b \text{ rational}\}$ is a basis that generates the standard topology on $\mathbb{R}$.

Exercise 13.8b Show that the collection $\{[a,b) \mid a < b, a \text{ and } b \text{ rational}\}$ is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Exercise 16.1 Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Exercise 16.4 A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Exercise 16.6 Show that the countable collection

$$\{(a,b) \times (c,d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

Exercise 16.9 Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology.

Exercise 17.2 Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Exercise 17.3 Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Exercise 17.4 Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

Exercise 18.8a Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

Exercise 18.8b Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous. Let $h : X \to Y$ be the function $h(x) = \min\{f(x), g(x)\}$. Show that h is continuous.

Exercise 18.13 Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

Exercise 19.4 Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.

Exercise 19.6a Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_i)$ converges to $\pi_\alpha(\mathbf{x})$ for each α.

Exercise 19.9 Show that the choice axiom is equivalent to the statement that for any indexed family of nonempty sets, $\{A_\alpha\}_{\alpha \in J}$ with $J \neq 0$, the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is not empty.

Exercise 20.2 Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Exercise 20.5 Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

Exercise 21.6a Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0,1]$.

Exercise 21.6b Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence (f_n) does not converge uniformly.

Exercise 21.8 Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

Exercise 22.2a Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

Exercise 22.2b If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

Exercise 22.5 Let $p\colon X \to Y$ be an open map. Show that if A is open in X, then the map $q\colon A \to p(A)$ obtained by restricting p is an open map.

Exercise 23.2 Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

Exercise 23.3 Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subset of X. Show that if $A \cap A_\alpha \neq \varnothing$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

Exercise 23.4 Show that if X is an infinite set, it is connected in the finite complement topology.

Exercise 23.6 Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects $\operatorname{Bd} A$.

Exercise 23.9 Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

Exercise 23.11 Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

Exercise 23.12 Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

Exercise 24.2 Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

Exercise 24.3a Let $f\colon X \to X$ be continuous. Show that if $X = [0,1]$, there is a point x such that $f(x) = x$. (The point x is called a fixed point of f.)

Exercise 24.4 Let X be an ordered set in the order topology. Show that if X is connected, then X is a linear continuum.

Exercise 24.6 Show that if X is a well-ordered set, then $X \times [0,1)$ in the dictionary order is a linear continuum.

Exercise 25.4 Let X be locally path connected. Show that every connected open set in X is path connected.

Exercise 25.9 Let G be a topological group; let C be the component of G containing the identity element e. Show that C is a normal subgroup of G.

Exercise 26.9 Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that $A \times B \subset U \times V \subset N$.

Exercise 26.11 Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then $Y = \bigcap_{A \in \mathcal{A}} A$ is connected.

Exercise 26.12 Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$. (Such a map is called a perfect map.) Show that if Y is compact, then X is compact.

Exercise 27.1 Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bound property.

Exercise 27.4 Show that a connected metric space having more than one point is uncountable.

Exercise 28.4 A space X is said to be countably compact if every countable open covering of X contains a finite subcollection that covers X. Show that for a T_1 space X, countable compactness is equivalent to limit point compactness.

Exercise 28.5 Show that X is countably compact if and only if every nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty sets of X has a nonempty intersection.

Exercise 28.6 Let (X, d) be a metric space. If $f : X \to X$ satisfies the condition $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$, then f is called an isometry of X. Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism.

Exercise 29.1 Show that the rationals $\mathbb{Q}$ are not locally compact.

Exercise 29.4 Show that $[0, 1]^\omega$ is not locally compact in the uniform topology.

Exercise 29.5 If $f\colon X_1 \to X_2$ is a homeomorphism of locally compact Hausdorff spaces, show that f extends to a homeomorphism of their one-point compactifications.

Exercise 29.6 Show that the one-point compactification of $\mathbb{R}$ is homeomorphic with the circle S^1.

Exercise 29.10 Show that if X is a Hausdorff space that is locally compact at the point x, then for each neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Exercise 30.10 Show that if X is a countable product of spaces having countable dense subsets, then X has a countable dense subset.

Exercise 30.13 Show that if X has a countable dense subset, every collection of disjoint open sets in X is countable.

Exercise 31.1 Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

Exercise 31.2 Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

Exercise 31.3 Show that every order topology is regular.

Exercise 32.1 Show that a closed subspace of a normal space is normal.

Exercise 32.2a Show that if $\prod X_\alpha$ is Hausdorff, then so is X_α. Assume that each X_α is nonempty.

Exercise 32.2b Show that if $\prod X_\alpha$ is regular, then so is X_α. Assume that each X_α is nonempty.

Exercise 32.2c Show that if $\prod X_\alpha$ is normal, then so is X_α. Assume that each X_α is nonempty.

Exercise 32.3 Show that every locally compact Hausdorff space is regular.

Exercise 33.7 Show that every locally compact Hausdorff space is completely regular.

Exercise 33.8 Let X be completely regular, let A and B be disjoint closed subsets of X. Show that if A is compact, there is a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Exercise 34.9 Let X be a compact Hausdorff space that is the union of the closed subspaces X_1 and X_2. If X_1 and X_2 are metrizable, show that X is metrizable.

Exercise 37.2 A collection $\mathcal{A}$ of subsets of X has the countable intersection property if every countable intersection of elements of $\mathcal{A}$ is nonempty. Show that X is a Lindelöf space if and only if for every collection $\mathcal{A}$ of subsets of X having the countable intersection property, $\bigcap_{A \in \mathcal{A}} \bar{A}$ is nonempty.

Exercise 38.4 Let Y be an arbitrary compactification of X; let $\beta(X)$ be the Stone-Čech compactification. Show there is a continuous surjective closed map $g\colon \beta(X) \to Y$ that equals the identity on X.

Exercise 38.6 Let X be completely regular. Show that X is connected if and only if the Stone-Čech compactification of X is connected.

Exercise 39.5 Show that if X has a countable basis, a collection $\mathcal{A}$ of subsets of X is countably locally finite if and only if it is countable.

Exercise 43.2 Let (X, d_X) and (Y, d_Y) be metric spaces; let Y be complete. Let $A \subset X$. Show that if $f\colon A \to Y$ is uniformly continuous, then f can be uniquely extended to a continuous function $g\colon \bar{A} \to Y$, and g is uniformly continuous.

Exercise 43.7 Show that the set of all sequences $(x_1, x_2, \ldots)$ such that $\sum x_i^2$ converges is complete in l^2-metric.