

Exercises from *Linear Algebra Done Right* by Sheldon Axler

Exercise 1.2 Show that $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1 (meaning that its cube equals 1).

Proof.

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{-1-\sqrt{3}i}{2},$$

hence

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{-1-\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} = 1$$

This means $\frac{-1+\sqrt{3}i}{2}$ is a cube root of 1. □

Exercise 1.3 Prove that $-(-v) = v$ for every $v \in V$.

Proof. By definition, we have

$$(-v) + (-(-v)) = 0 \quad \text{and} \quad v + (-v) = 0.$$

This implies both v and $-(-v)$ are additive inverses of $-v$, by the uniqueness of additive inverse, it follows that $-(-v) = v$. □

Exercise 1.4 Prove that if $a \in \mathbf{F}$, $v \in V$, and $av = 0$, then $a = 0$ or $v = 0$.

Exercise 1.6 Give an example of a nonempty subset U of \mathbf{R}^2 such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of \mathbf{R}^2 .

Exercise 1.7 Give an example of a nonempty subset U of \mathbf{R}^2 such that U is closed under scalar multiplication, but U is not a subspace of \mathbf{R}^2 .

Exercise 1.8 Prove that the intersection of any collection of subspaces of V is a subspace of V .

Exercise 1.9 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Exercise 2.1 Prove that if (v_1, \dots, v_n) spans V , then so does the list $(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n)$ obtained by subtracting from each vector (except the last one) the following vector.

Exercise 2.2 Prove that if (v_1, \dots, v_n) is linearly independent in V , then so is the list $(v_1 - v_2, v_2 - v_3, \dots, v_{n-1} - v_n, v_n)$ obtained by subtracting from each vector (except the last one) the following vector.

Exercise 2.6 Prove that the real vector space consisting of all continuous realvalued functions on the interval $[0, 1]$ is infinite dimensional.

Exercise 3.1 Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $a \in \mathbf{F}$ such that $Tv = av$ for all $v \in V$.

Exercise 3.8 Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and $\text{range } T = \{Tu : u \in U\}$.

Exercise 3.9 Prove that if T is a linear map from \mathbf{F}^4 to \mathbf{F}^2 such that $\text{null } T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}$, then T is surjective.

Exercise 3.10 Prove that there does not exist a linear map from \mathbf{F}^5 to \mathbf{F}^2 whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

Exercise 3.11 Prove that if there exists a linear map on V whose null space and range are both finite dimensional, then V is finite dimensional.

Exercise 4.4 Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m . Prove that p has m distinct roots if and only if p and its derivative p' have no roots in common.

Exercise 5.1 Suppose $T \in \mathcal{L}(V)$. Prove that if U_1, \dots, U_m are subspaces of V invariant under T , then $U_1 + \dots + U_m$ is invariant under T .

Exercise 5.4 Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\text{null}(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbf{F}$.

Exercise 5.11 Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Exercise 5.12 Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T . Prove that T is a scalar multiple of the identity operator.

Exercise 5.13 Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T . Prove that T is a scalar multiple of the identity operator.

Exercise 5.20 Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Exercise 5.24 Suppose V is a real vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T has even dimension.

Exercise 6.2 Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if $\|u\| \leq \|u + av\|$ for all $a \in \mathbf{F}$.

Exercise 6.3 Prove that $\left(\sum_{j=1}^n a_j b_j\right)^2 \leq \left(\sum_{j=1}^n j a_j^2\right) \left(\sum_{j=1}^n \frac{b_j^2}{j}\right)$ for all real numbers a_1, \dots, a_n and b_1, \dots, b_n .

Exercise 6.7 Prove that if V is a complex inner-product space, then $\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 - \|u-iv\|^2}{4}i$ for all $u, v \in V$.

Exercise 6.13 Suppose (e_1, \dots, e_m) is an orthonormal list of vectors in V . Let $v \in V$. Prove that $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_m \rangle|^2$ if and only if $v \in \text{span}(e_1, \dots, e_m)$.

Exercise 6.16 Suppose U is a subspace of V . Prove that $U^\perp = \{0\}$ if and only if $U = V$.

Exercise 6.17 Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\text{null } P$ is orthogonal to every vector in $\text{range } P$, then P is an orthogonal projection.

Exercise 6.18 Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and $\|Pv\| \leq \|v\|$ for every $v \in V$, then P is an orthogonal projection.

Exercise 6.19 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V . Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

Exercise 6.20 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V . Prove that U and U^\perp are both invariant under T if and only if $P_U T = T P_U$.

Exercise 6.29 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V . Prove that U is invariant under T if and only if U^\perp is invariant under T^* .

Exercise 7.4 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

Exercise 7.5 Show that if $\dim V \geq 2$, then the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

Exercise 7.6 Prove that if $T \in \mathcal{L}(V)$ is normal, then $\text{range } T = \text{range } T^*$.

Exercise 7.8 Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}(\mathbf{R}^3)$ such that $T(1, 2, 3) = (0, 0, 0)$ and $T(2, 5, 7) = (2, 5, 7)$.

Exercise 7.9 Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.

Exercise 7.10 Suppose V is a complex inner-product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Exercise 7.11 Suppose V is a complex inner-product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

Exercise 7.14 Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbf{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \epsilon$, then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

Exercise 7.15 Suppose U is a finite-dimensional real vector space and $T \in \mathcal{L}(U)$. Prove that U has a basis consisting of eigenvectors of T if and only if there is an inner product on U that makes T into a self-adjoint operator.