3940 AS06

Main

Question 1

(i): True

(ii): True

(iii): False

(iv): True

(v): True

Question 2

If $ax^2 + bx + c = 0$ is divided by $x + 3$, $x - 5$, and $x - 1$, the remainders are 21, 61, and 9 respectively. Use Gaussian elimination method to evaluate the value of a, b, and c.

Solution 2

Given $f(x) = ax^2 + bx + c$, when $f(x)$ is divided by $x + 3$, $x - 5$ and $x - 1$, the remainders are 21, 61, and 9.

Therefore, $f(x) = ax^2 + bx + c = 0$ could be rewrite like $k_1(x + i)(x + k_2) + j = 0$, when $i = 3, -5, -1$, the corresponding j is $21, 61, 9$.

So, let $x = -i$, we have $f(-i) = j$

Hence,

$$\begin{aligned} f(-3) &= 9a - 3b + c = 21 \\ f(5) &= 25a + 5b + c = 61 \\ f(1) &= a + b + c = 9 \end{aligned}$$

So, the augmented matrix is

$$\widetilde{A} = \left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & 0 & -32 & -192 \end{array}\right]$$

Therefore, $a = 2,\ b = 1,\ c = 6$

Question 3

Let A = $\begin{bmatrix} \alpha & 1 & 0 \\ \beta & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, find all the value of α and β for which

(a) A is singular (b) A is strictly diagonal dominant (c) A is symmetric (d) A is positive definite*

*A matrix A is positive definite if it is symmetric and if $x^t Ax > 0$ for every n-dimensional vector $x \neq 0$

Solution 3

(a) A is singular

$$\begin{aligned} \det A &= 0 \\ 4\alpha - 2\beta - \alpha &= 0 \\ \alpha &= \frac{2}{3}\beta \end{aligned}$$

(b) A is strictly diagonal dominant

$$\begin{aligned} \alpha &> 1 \\ \beta &< 2 \end{aligned}$$

(c) A is symmetric

$$A^T = A$$

$$\therefore \alpha \in R, \beta = 1$$

(d) A is positive definite*

From Google, a matrix is said to be positive definite if it is symmetric and each of its leading principal sub matrices has a positive determinant.

The submatrices are

$$\begin{bmatrix} \alpha & 1 & 0 \\ \beta & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} \alpha & 1 \\ \beta & 2 \end{bmatrix}, [\alpha]$$

Since the matrix is symmetric, $\beta = 1$, so the submatrices are

$$\begin{bmatrix} \alpha & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} \alpha & 1 \\ 1 & 2 \end{bmatrix}, [\alpha]$$

So,

$$\det \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 3\alpha - 2 > 0,$$
$$\det \begin{bmatrix} \alpha & 1 \\ 1 & 2 \end{bmatrix} = 2\alpha - 1 > 0,$$
$$[\alpha] = \alpha > 0$$

Therefore, $\alpha > 1.5, \beta = 1$

Question 4

Find the permutation matrix P so that PA can be factored into the product LU, where L is lower triangular with ones on its diagonal and U is upper triangular for these matrices. Consider the following matrix, $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

Solution 4

During the LU, we need to swap $R_3 \leftrightarrow R_1, R_4 \leftrightarrow R_3$

And if we perform the same operation for I_4, we will get the permutation matrix P

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Question 5

Find $||x||_\infty$ and $||x||_2$$ for the following vectors:

(a) $\mathbf{x} = (3, -4, 0, 3/2)^T$ (b) $\mathbf{x} = (\sin k, \cos k, 2^k)^T$ for a fixed positive integer k.

Solution 5

(a) $\mathbf{x} = (3, -4, 0, 3/2)^T$

calculated on matlab

$$||x||_\infty = 4, ||x||_2 = 5.2202$$

(b) $\mathbf{x} = (\sin k, \cos k, 2^k)^T$ **for a fixed positive integer k**

$$||x||_\infty = \max_i |x_i| = 2^k$$

$$||x||_2 = \sqrt{\sum_{i=1}^{N} |x_i|^2} = \sqrt{1 + 2^{k+1}}$$

Question 6

(a) Verify that the function $||.||_1$ defined on R^n by $||x||_1 = \sum_{i=1}^{n} |x_i|$ is a norm on R^n.

(b) Show by example that $||.||_*$, defined by $||A||_* = \max_{1 \leq i,j \leq n} |a_{ij}|$, does not defined a matrix norm

Solution 6

Question 7

Compute the eigenvalues, associated eigenvectors and spectral radius of the following matrix

$$\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$$

Solution 7

$$\det[\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix} - \lambda] = \lambda^3 + 9\lambda^2 - 11\lambda - 21 = 0$$

Therefore, the eigenvalues are $\lambda = -1, 3, 7$

By the eigenvalues, we have the eigenvectors:

$$(1, 0, 0)^T, (1, 2, 0)^T, (1, 4, 4)^T$$

The spectral radius ρ of the matrix is 7

Question 8

The linear system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 7, \\ x_1 + x_2 + x_3 &= 2, \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

has solution $(1, 2, -1)^t$,

(a) Find the value of $\rho(T_j)$ and $\rho(T_g)$

(b) Use the Jacobi method with $x(0) = 0$ to approximate the solution to the linear system within 10^{-5} in the l_∞ norm

Solution 8

(a) Find the value of $\rho(T_j)$ and $\rho(T_g)$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The inverse of D is

$$D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For T_j, we obtain

$$\begin{aligned} T_j &= D^{-1}(L+U) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \end{aligned}$$

so that

$$T_j - \lambda I = \begin{bmatrix} -\lambda & -2 & 1 \\ -1 & -\lambda & -1 \\ -2 & -2 & -\lambda \end{bmatrix}$$

Therefore,

$$\det(T_j - \lambda I) = \begin{bmatrix} -\lambda & -2 & 1 \\ -1 & -\lambda & -1 \\ -2 & -2 & -\lambda \end{bmatrix} = -\lambda^3 + 2\lambda - 2 = 0$$

Thus,

$$\rho(T_j) = -1.76929$$

For T_g, we obtain

$$\begin{aligned} T_g &= (D-L)^{-1}U \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

Therefore,

$$\det\left(T_g - \lambda I\right) = \begin{bmatrix} -\lambda & -2 & 1 \\ 0 & 2-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{bmatrix} = -\lambda(-\lambda+2)^2$$

Thus,

$$\rho(T_g) = 2$$

(b) Use the Jacobi method with $x(0) = 0$ to approximate the solution to the linear system within 10^{-5} in the l_∞ norm

$$\begin{aligned} x_1 &= -2x_2 + 2x_3 + 7, \\ x_2 &= 2 - x_1 - x_3, \\ x_3 &= 5 - 2x_1 - 2x_2 \end{aligned}$$

From the initial approximation $x(0) = 0$, we have $x(1)$ given by

$$\begin{aligned} x_1^{(1)} &= -2x_2^{(0)} + 2x_3^{(0)} + 7 = 7 \\ x_2^{(1)} &= 2 - x_1^{(0)} - x_3^{(0)} = 2 \\ x_3^{(1)} &= 5 - 2x_1^{(0)} - 2x_2^{(0)} = 5 \end{aligned}$$

Additional iterates, $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})^t$ are generated in a similar manner and are summarized follows

k	**0**	**1**	**2**	**3**	**4**
$x_1^{(k)}$	0	7	13	1	1
$x_2^{(k)}$	0	2	-10	2	2
$x_3^{(k)}$	0	5	-13	-1	-1

The process was stopped after 4 iterations because

$$\frac{||x^{(10)} - x^{(9)}||_\infty}{||x^{(10)}||_\infty} < 10^{-5}$$

And the approximate solution is $(1, 2, -1)$

Question 9

The linear system of equation is defined as

$$\begin{aligned} 10x_1 - x_2 &= 9 \\ -x_1 + 10x_2 - 2x_3 &= 7 \\ -2x_2 + 10x_3 &= 6 \end{aligned}$$

(a) Find the first two iterations of the SOR method with $\omega = 1.1$, using $\mathbf{x}^{(0)} = \mathbf{0}$

(b) If the above matrix is tridiagonal and positive definite, then Repeat (a) using the optimal choice of ω

Solution 9

(a) Find the first two iterations of the SOR method with $\omega = 1.1$, using $\mathbf{x}^{(0)} = \mathbf{0}$

Let

$$A = \begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

So the linear system $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = \begin{bmatrix} 0.1095 \\ 0.0947 \\ 0.4189 \end{bmatrix}$

So, the SOR method for the linear system could be written as below:

$$\begin{aligned} x_1^{(k)} &= (1-\omega)x_1^{(k-1)} + \frac{\omega}{a_{11}}[b_1 - \sum_{j=2}^{3} a_{1j}x_j^{(k-1)}] \\ &= (1-\omega)x_1^{(k-1)} + \frac{\omega}{a_{11}}[b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)}] \\ x_2^{(k)} &= (1-\omega)x_2^{(k-1)} + \frac{\omega}{a_{22}}[b_2 - \sum_{j=1}^{1} a_{2j}x_j^{(k)} - \sum_{j=3}^{3} a_{2j}x_j^{(k-1)}] \\ &= (1-\omega)x_2^{(k-1)} + \frac{\omega}{a_{22}}[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k-1)}] \\ x_3^{(k)} &= (1-\omega)x_3^{(k-1)} + \frac{\omega}{a_{33}}[b_3 - \sum_{j=1}^{2} a_{3j}x_j^{(k)}] \\ &= (1-\omega)x_3^{(k-1)} + \frac{\omega}{a_{33}}[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}] \end{aligned}$$

Putting the values into these equation, we have

$$\begin{aligned}
x_1^{(k)} &= -0.1x_1^{(k-1)} + \frac{1.1}{10}[1 + x_2^{(k-1)}] \\
x_2^{(k)} &= -0.1x_2^{(k-1)} + \frac{1.1}{10}[x_1^{(k)} + 2x_3^{(k-1)}] \\
x_3^{(k)} &= -0.1x_3^{(k-1)} + \frac{1.1}{10}[4 + 2x_2^{(k)}]
\end{aligned}$$

Using the initial condition $\mathbf{x^{(0)}} = \mathbf{0}$, the first iteration gives:

$$\begin{aligned}
x_1^{(1)} &= -0.1x_1^{(0)} + \frac{1.1}{10}[1 + x_2^{(0)}] = 0.11 \\
x_2^{(1)} &= -0.1x_2^{(0)} + \frac{1.1}{10}[x_1^{(1)} + 2x_3^{(0)}] = 0.0121 \\
x_3^{(1)} &= -0.1x_3^{(0)} + \frac{1.1}{10}[4 + 2x_2^{(1)}] = 0.442662
\end{aligned}$$

The second iteration gives:

$$\begin{aligned}
x_1^{(2)} &= -0.1x_1^{(1)} + \frac{1.1}{10}[1 + x_2^{(1)}] = 0.100331 \\
x_2^{(2)} &= -0.1x_2^{(1)} + \frac{1.1}{10}[x_1^{(2)} + 2x_3^{(1)}] = 0.10721205 \\
x_3^{(2)} &= -0.1x_3^{(1)} + \frac{1.1}{10}[4 + 2x_2^{(2)}] = 0.419320451
\end{aligned}$$

(b) If the above matrix is tridiagonal and positive definite, then Repeat (a) using the optimal choice of ω

If the above matrix is tridiagonal and positive definite, then $\rho(T_g) = [\rho(T_j)]^2 < 1$, and the optimal choice of ω for the SOR method is

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_J)]^2}}$$

Given,

$$A = \begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{bmatrix}, D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The inverse of D is

$$D^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

and

$$T_j = D^{-1}(L+U)$$
$$= \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & 0.2 \\ 0 & 0.2 & 0 \end{bmatrix}$$

so that

$$T_j - \lambda I = \begin{bmatrix} -\lambda & 0.1 & 0 \\ 0.1 & -\lambda & 0.2 \\ 0 & 0.2 & -\lambda \end{bmatrix}$$

Therefore,

$$\det\left(T_j - \lambda I\right) = \begin{bmatrix} -\lambda & 0.1 & 0 \\ 0.1 & -\lambda & 0.2 \\ 0 & 0.2 & -\lambda \end{bmatrix} = -\lambda(\lambda^2 - 0.05)$$

Thus,

$$\rho(T_j) = \sqrt{0.05}$$

and

$$\omega = \frac{2}{1+\sqrt{1-[\rho(T_j)]^2}} \approx 1.01$$

So, **the optimal choice of ω for the SOR method is** 1.01.

Then, we then Repeat (a) using the optimal choice of ω:

$$x_1^{(k)} = -0.01x_1^{(k-1)} + \frac{1.01}{10}[1 + x_2^{(k-1)}]$$
$$x_2^{(k)} = -0.01x_2^{(k-1)} + \frac{1.01}{10}[x_1^{(k)} + 2x_3^{(k-1)}]$$
$$x_3^{(k)} = -0.01x_3^{(k-1)} + \frac{1.01}{10}[4 + 2x_2^{(k)}]$$

Using the initial condition $\mathbf{x}^{(0)} = \mathbf{0}$, the first iteration gives:

$$x_1^{(1)} = -0.01x_1^{(0)} + \frac{1.01}{10}[1 + x_2^{(0)}] = 0.101$$
$$x_2^{(1)} = -0.01x_2^{(0)} + \frac{1.01}{10}[x_1^{(1)} + 2x_3^{(0)}] = 0.010201$$
$$x_3^{(1)} = -0.01x_3^{(0)} + \frac{1.01}{10}[4 + 2x_2^{(1)}] = 0.405030301$$

The second iteration gives:

$$
\begin{aligned}
x_1^{(2)} &= -0.01x_1^{(1)} + \frac{1.01}{10}[1 + x_2^{(1)}] = 0.101020301 \\
x_2^{(2)} &= -0.01x_2^{(1)} + \frac{1.01}{10}[x_1^{(2)} + 2x_3^{(1)}] = 0.0919171612 \\
x_3^{(2)} &= -0.01x_3^{(1)} + \frac{1.01}{10}[4 + 2x_2^{(2)}] = 0.4185169636
\end{aligned}
$$

Question 10

Compute the condition number of the following matrix relative to $||x||_\infty$

$$
\begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}
$$

Solution 10

The matrix

$$
A = \begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}
$$

has inverse

$$
A^{-1} = \begin{bmatrix} 27.58620\ldots & -0.68965\ldots & 0.03448\ldots \\ -11.49425\ldots & 1.95402\ldots & 0.06896\ldots \\ -1.14942\ldots & -0.80459\ldots & 0.20689\ldots \end{bmatrix}
$$

and the condition number could be calculated as below

$$
k_\infty(A) = ||A||_\infty ||A^{-1}||_\infty = 7 \times 28.31 = 198.17
$$