3940 AS 02

Main

Question 1

(a) From Descartes's rule of signs, conclude that the equation $x^{2n}-1=0$ has $2n-2$ imaginary roots.

$$f(-x)=(-x)^{2n}-1=x^{2n}-1=0 \text{ Sign Has Changed Once}$$

So, $f(x)=0$ has one negative real root, that is -1.

Also, $f(x)=0$ has one positive real root $x=1$ ($f(x)$ is symmetric about the y-axis)

Since the equation has two real roots, the number of imaginary roots is $2n-2$

(ii) Without actually obtaining the roots, show that $x^6-5x^5-7x^2+8x+20=0$ possess imaginary roots.

From Descartes's rule of signs, since there are 2 sign changed for $f(x)=0$. So, the equation has 2 positive real roots.

Similarly, since there are 2 sign changed for $f(-x)=x^6+5x^5-7x^2-8x+20=0$, so the equation has 2 negative real roots.

Since the degree of $f(x)$ is 6, which means $f(x)$ has 6 roots.

Hence, the equation $f(x)=0$ has two imaginary roots.

Question 2

The equation $f(x)=x^2-2e^x=0$ has a solution in the interval [-1,1].

(a) With $p_0=-1$ and $p_1=1$, calculate p_2 using the Secant method.

From Secant method,

$$\begin{aligned} x_2 &= x_1-\frac{f(x_1)}{\frac{f(x_1)-f(x_0)}{x_1-x_0}} \\ &= \frac{x_0 f(x_1)-x_1 f(x_0)}{f(x_1)-f(x_0)} \end{aligned}$$

Since

$$f(-1) = (-1)^2 - 2e^{-1} = 1 - \frac{2}{e}$$
$$f(1) = 1^2 - 2e^1 = 1 - 2e$$

So,

$$x_2 = \frac{-2 + 2e + \frac{2}{e}}{\frac{2}{e} - 2e} \approx -0.8876$$

(b) With p_2 from part (a), calculate p_3 using the Newton's method.

From the Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Since,

$$f'(x) = 2x - 2e^x$$

The value of $f(p_2)$ and $f'(p_2)$ could be calculated as below,

$$f(-0.8876) = (-0.8876)^2 - 2e^{-0.8876} \approx -0.03543$$
$$f'(-0.8876) = 2(-0.8876) - 2e^{-0.8876} \approx -2.59856$$

Therefore,

$$x_3 = -0.8876 - \frac{-0.03543}{-2.59856} = -0.90123$$

Question 3

The equation $f(x) = 2 - x^2 \sin x = 0$ has a solution in the interval [-1,2].

(a) Verify that the Bisection method can be applied to the function $f(x)$ on [−1,2].

$$f(-1) = 2 - \sin(-1) = 2.814 > 0$$
$$f(2) = 2 - 2^2 \sin(2) = -1.6368 < 0$$

So $f(-1)f(2) > 0$

Therefore, the equation has a solution in the interval $[-1, 2]$, and the Bisection method can be applied to the function $f(x)$

(b) Using the error formula for the Bisection method, find the number of iterations needed for accuracy 10^{-7}

The error formula for the Bisection method is $n \geq \frac{\log(b-a)-\log\sigma}{\log 2}$ where $[a,b]=[-1,2], \sigma=10^{-7}$

So,

$$n \geq \frac{\log(2+1)-\log 10^{-6}}{\log 2} = \frac{\log(3)+7}{\log 2} \geq 24.8$$

Therefore, the number of iterations needed for accuracy 10^{-7} is 25

(c) Compute p_3 for the Bisection method

$$\begin{aligned}
p_0 &= f(0.5) = 2 - 0.5^2\sin(0.5) \approx 1.88015 > 0 \\
p_1 &= \frac{0.5+2}{2} = 1.25 \\
f(p_1) &= f(1.25) = 2 - 1.25^2\sin(1.25) > 0 \\
p_2 &= \frac{1.25+2}{2} = 1.625 \\
f(p_2) &= f(1.625) = 2 - 1.625^2\sin(1.625) \approx -0.64 < 0 \\
\therefore p_3 &= \frac{1.25+1.625}{2} = 1.4375
\end{aligned}$$

Question 4

The following refer to the fixed-point problem:

(a) State the theorem which gives conditions for a fixed-point sequence to converge to a unique fixed point.

If $g(x)=x_{n+1}$, theorem state that $|g'(x)|<1$ then sequence is converge to fixed point.

(b) Given $g(x)=\frac{2-x^3+2x}{3}$, use the theorem to show that the fixed-point sequence will converge to unique fixed-point of $g(x)$ for any p_0 in $[-1,1.1]$.

From the question, we have $g'(x)$,

$$g'(x) = \frac{-3x^2+2}{3} = -x^2+\frac{2}{3}$$

Since $|g'(0)|=\frac{2}{3}<1, |g'(1)|=\frac{1}{3}<1,$

Sequence is converge to fixed point.

(c) With $p_0 = 0.5$ generate p_3

$$x_{x+1} = \frac{2 - x_n^3 + 2x_n}{3},\ x_0 = 0.5$$

$$\begin{aligned} p_1 &= \frac{2 - 0.5^3 + 2 \times 0.5}{3} \approx 0.9583 \\ p_2 &= \frac{2 - 0.9583^3 + 2 \times 0.9583}{3} \approx 1.0122 \\ p_3 &= \frac{2 - 1.0122^3 + 2 \times 1.0122}{3} \approx 0.9957 \end{aligned}$$

Therefore, $p_3 = 0.9957$

Question 5

What is the main difference between secant and regula falsi method? Compute the root of the equation $x^2 e^{-x/2} = 1$ in the interval $[0, 2]$ correct to three decimal places using both secant method and regula falsi method.

Solution

The regula falsi is a bracketing algorithm. It iterates through intervals that always contain a root whereas the secant method is basically Newton's method without explicitly computing the derivative at each iteration. The secant is faster but may not converge at all.

Given equation $f(x) = x^2 e^{-x/2} - 1 = 0$ has a solution in the interval $[0, 2]$

using the Secant method

From Secant method,

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{\frac{f(x_1) - f(x_0)}{x_1 - x_0}} \\ &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \end{aligned}$$

Since

$$f(0) = -1$$

$$f(2) = 2^2 e^{-1} - 1 = \frac{4}{e} - 1$$

So,

$$x_2 = \frac{2}{\frac{4}{e}} = \frac{e}{2} \approx 1.359$$

using the Newton's method.

From the Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Since,

$$f'(x) = 2xe^{-x/2} - \frac{1}{2}x^2e^{-x/2}$$

The value of $f(x_1)$ and $f'(x_1)$ could be calculated as below,

$$f(2) = \frac{4}{e} - 1$$

$$f'(2) = \frac{4}{e} - \frac{2}{e} = \frac{2}{e}$$

Therefore,

$$x_2 = 2 - \frac{\frac{4}{e} - 1}{\frac{2}{e}} = 2 - \frac{4-e}{2} = \frac{e}{2} \approx 1.359$$

Question 6

(a) Suppose the function $f(x)$ has a unique zero p in the interval $[a, b]$. Further, suppose $f\prime\prime(x)$ exists and is continuous on the interval $[a, b]$. Under what conditions, Newton's method give a quadratically convergent sequence to p?

From the Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n)$$

Since r is a root of $f(x) = 0, r = g(r)$, so we have

$$\begin{aligned} x_{n+1} - r &= g(x_n) - g(r) \\ &= g(r) + g'(r)(x_n - r) + \frac{g''(\xi)}{2}(x_n - r)^2 - g(r) \\ &= g'(r)(x_n - r) + \frac{g''(\xi)}{2}(x_n - r)^2 \end{aligned}$$

where ξ lies in the interval $[x_n, r]$

Since

$$g'(r) = \frac{f(r)f'''(r)}{[f'(r)]^2} = 0,\ f(r) = 0$$

we have

$$(x_{n+1} - r) = \frac{g''(\xi)}{2}(x_n - r)^2, \text{ where r is a constant}$$

Therefore, Newton's method give a quadratically convergent sequence to p.

(b) Using Newton's method, construct an iterative formula to find $\sqrt[k]{N}^k\sqrt{N}\ k$ i.e. k^{th} root of any number N.

Given

$f(x) = \sqrt[k]{x} = x^{1/k}$

By the Newton's method,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^{\frac{1}{k}}}{\frac{1}{k}x_n^{\frac{1}{k}-1}} \\ &= x_n - \frac{kx_n^{\frac{1}{k}}}{x_n^{\frac{1}{k}-1}} \\ &= x_n - kx_n = (1-k)x_n \end{aligned}$$