3940 AS03

Main

Question 1

Let $f(x) = x^3 - e^{-x}$, $x_0 = 0$, $x_1 = 0.7$, $x_2 = 1.0$.

(a) Find the Lagrange polynomial, $P_2(x)$ of degree at most for $f(x)$ using x_0, x_1 and x_2

We first determine the coefficient polynomial $L_0(x)$, $L_1(x)$ and $L_2(x)$. In nested form they are

$$
\begin{aligned}
L_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\
&= \frac{(x - 0.7)(x - 1)}{(0 - 0.7)(0 - 1)} = \frac{10}{7}(x - 0.7)(x - 1) \\
L_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\
&= \frac{(x - 0)(x - 1)}{(0.7 - 0)(0.7 - 1)} = -\frac{100}{21}x(x - 1) \\
L_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{(x - 0)(x - 0.7)}{(1 - 0)(1 - 0.7)} = \frac{10}{3}x(x - 0.7)
\end{aligned}
$$

Also, $f(x_0) = f(0) = -1$, $f(x_1) = f(0.7) = 0.7^3 - e^{-0.7}$, $f(x_2) = f(1) = 1 - e^{-1}$, so

$$
\begin{aligned}
P_2(x) &= \sum_{k=0}^{2} f(x_k)L_k(x) \\
&= -\frac{10}{7}(x - 0.7)(x - 1) + (0.343 - e^{-0.7})\frac{100}{21}x(x - 1) + (1 - e^{-1})\frac{10}{3}x(x - 0.7)
\end{aligned}
$$

(b) Evaluate $P_2(0.8)$ and compute the actual error $|f(0.8) - P_2(0.8)|$.

From (a), we have

$$
\begin{aligned}
P_2(0.8) &= -\frac{10}{7}(0.8 - 0.7)(0.8 - 1) - (0.343 - e^{-0.7})\frac{100}{21}0.8(0.8 - 1) + (1 - e^{-1})\frac{10}{3}0.8(0.8 - 0.7) \\
&= 0.0801 \\
f(0.8) &= 0.0627
\end{aligned}
$$

So the actual error is $|f(0.8) - P_2(0.8)| = |0.0801 - 0.0627| = 0.0174$

Question 2

Let $f(x) = x^4 - 2x^3 + x^2 - 3$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$.

(a) Compute the interpolating polynomial, $P_3(x)$, of degree at most 3 for $f(x)$ using given nodes.

We first determine the coefficient polynomial $L_0(x)$, $L_1(x)$, $L_2(x)$ and $L_3(x)$. In nested form they are

$$
\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \\
&= \frac{(x-0.5)(x-1)(x-1.5)}{(0-0.5)(0-1)(0-1.5)} = -\frac{4}{3}(x-0.5)(x-1)(x-1.5) \\
L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\
&= \frac{(x-0)(x-1)(x-1.5)}{(0.5-0)(0.5-1)(0.5-1.5)} = -\frac{1}{4}x(x-1)(x-1.5) \\
L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \\
&= \frac{(x-0)(x-0.5)(x-1.5)}{(1-0)(1-0.5)(1-1.5)} = -\frac{1}{4}x(x-0.5)(x-1.5) \\
L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\
&= \frac{(x-0)(x-0.5)(x-1)}{(1.5-0)(1.5-0.5)(1.5-1)} = -\frac{4}{3}x(x-0.5)(x-1)
\end{aligned}
$$

Also, $f(x_0) = f(0) = -3$, $f(x_1) = f(0.5) = -2.9375$, $f(x_2) = f(1) = -3$, $f(x_3) = f(1.5) = -2.4375$, so

$$
\begin{aligned}
P_3(x) &= \sum_{k=0}^{3} f(x_k)L_k(x) \\
&= 4(x-0.5)(x-1)(x-1.5) + 2.9375 \times \frac{1}{4}x(x-1)(x-1.5) + \frac{3}{4}x(x-0.5)(x-1.5) + 2.4395 \times \frac{4}{3}x(x-0.5)(
\end{aligned}
$$

(b) Find the maximum error in using $P_3(x)$ to approximate $f(x)$ on the interval $[0, 2]$

Because $f(x) = x^4 - 2x^3 + x^2 - 3$, we have

$$
f'(x) = 4x^3 - 6x^2 + 2x,\ f''(x) = 12x^2 - 12x + 2,\ f'''(x) = 24x - 12,\ f''''(x) = 24
$$

As a consequence, the Lagrange polynomial has error form

$$
\frac{f''''(\xi(x))}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3) = x(x-0.5)(x-1)(x-1.5) \le 2(1.5)(1)(0.5) = 1.5
$$

Question 3

Find the missing term in the following table using Lagrange's interpolation:

x:	0	1	2	3	4
y:	1	3	9	-	81

$$L(x) = \frac{(x-1)\cdot(x-2)\cdot(x-4)}{(0-1)\cdot(0-2)\cdot(0-4)} + 3\frac{(x+0)\cdot(x-2)\cdot(x-4)}{(1+0)\cdot(1-2)\cdot(1-4)} + 9\frac{(x+0)\cdot(x-1)\cdot(x-4)}{(2+0)\cdot(2-1)\cdot(2-4)} + 81\frac{(x+0)\cdot(x-1)\cdot(x-2)}{(4+0)\cdot(4-1)\cdot(4-2)}$$

$$L(x) = \frac{(x-1)\cdot(x-2)\cdot(x-4)}{(-1)\cdot(-2)\cdot(-4)} + 3\frac{x\cdot(x-2)\cdot(x-4)}{1\cdot(-1)\cdot(-3)} + 9\frac{x\cdot(x-1)\cdot(x-4)}{2\cdot1\cdot(-2)} + 81\frac{x\cdot(x-1)\cdot(x-2)}{4\cdot3\cdot2}$$

$$L(x) = \frac{(x^2-3x+2)\cdot(x-4)}{2\cdot(-4)} + 3\frac{(x^2-2x)\cdot(x-4)}{(-1)\cdot(-3)} + 9\frac{(x^2-x)\cdot(x-4)}{2\cdot(-2)} + 81\frac{(x^2-x)\cdot(x-2)}{12\cdot2}$$

$$L(x) = \frac{(x^3-7x^2+14x-8)}{(-8)} + 3\frac{(x^3-6x^2+8x)}{3} + 9\frac{(x^3-5x^2+4x)}{(-4)} + 81\frac{(x^3-3x^2+2x)}{24}$$

$$L(x) = -\tfrac{1}{8}(x^3-7x^2+14x-8) + (x^3-6x^2+8x) - \tfrac{9}{4}(x^3-5x^2+4x) + \tfrac{27}{8}(x^3-3x^2+2x)$$

$$L(x) = (-\tfrac{1}{8}x^3 + \tfrac{7}{8}x^2 - \tfrac{7}{4}x + 1) + (x^3-6x^2+8x) + (-\tfrac{9}{4}x^3 + \tfrac{45}{4}x^2 - 9x) + (\tfrac{27}{8}x^3 - \tfrac{81}{8}x^2 + \tfrac{27}{4}x)$$

$$L(x) = 2x^3 - 4x^2 + 4x + 1$$

So, $f(3) \approx L(3) = 31$

Question 4

Let $f(x) = x \sin 2x - x^2$, $x_0 = 0$, $x_1 = 0.3$, $x_2 = 0.7$

(a) Find Newton's Divided-Difference form of the interpolating polynomial $P_2(x)$ for using the three given nodes.

From the question, we have

$$
\begin{aligned}
f(x_0) &= 0 \\
f(x_1) &= 0.3 \sin 0.6 - 0.09 \approx 0.079393 \\
f(x_2) &= 0.7 \sin 1.4 - 0.49 \approx 0.199815
\end{aligned}
$$

Therefore, the first divided difference is given by

$$
\begin{aligned}
f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
&= \frac{0.079393 - 0}{0.3 - 0} = 0.264643 \\
f[x_1, x_2] &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\
* &= 0.301055
\end{aligned}
$$

the second divided difference is given by

$$
f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = 0.052017
$$

So, the interpolating polynomial $P_2(x)$ of Newton's Divided-Difference form is

$$
P_2(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \qquad = 0.249038x + 0.052017x^2
$$

(b) Add a fourth node $x_3 = 0.9$ and compute the next interpolating polynomial $P_3(x)$

With the fourth node $x_3 = 0.9$, we have

$$
f(x_3) = f(0.9) \approx 0.066463
$$

So,

$$
\begin{aligned}
f[x_2, x_3] &= \frac{f(x_3) - f(x_2)}{x_3 - x_2} = -0.666761 \\
f[x_1, x_2, x_3] &= \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = -1.6130267 \\
f[x_0, x_1, x_2, x_3] &= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = -1.8500485
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
P_3(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] \\
= -0.139472x + 1.902065x^2 - 1.850048x^3
\end{aligned}
$$

Question 5

Let $f(x) = xe^{-x/2}$, $x_0 = 1$, $x_1 = 2$

(a) Construct the Hermite interpolating polynomial $H_3(x)$ for $f(x)$ using the given nodes

We first compute the Lagrange polynomials and their derivatives. This gives

$$
\begin{aligned}
L_{1,0}(x) &= \frac{x - x_1}{x_0 - x_1} = \frac{x - 2}{-1} = 2 - x, \qquad L'_{1,0}(x) = -1 \\
L_{1,1}(x) &= \frac{x - x_0}{x_1 - x_0} = \frac{x - 1}{1} = x - 1, \qquad L'_{1,1}(x) = 1
\end{aligned}
$$

The polynomials $H_{1,j}(x)$ and $\hat{H}_{1,j}(x)$ are then

$$
\begin{aligned}
H_{1,0}(x) &= [1 - 2(x - x_0) L'_{1,0}(x_0)] L^2_{1,0}(x) = (1 + 2(x - 1))(2 - x)^2 = (2x - 1)(x - 2)^2 \\
H_{1,1}(x) &= [1 - 2(x - x_1) L'_{1,1}(x_1)] L^2_{1,1}(x) = (1 - 2(x - 2))(2 - x)^2 = (5 - 2x)(x - 1)^2 \\
\hat{H}_{1,0}(x) &= [x - x_0] L^2_{1,0}(x) = (1 + 2(x - 1))(2 - x)^2 = (x - 1)(x - 2)^2 \\
\hat{H}_{1,1}(x) &= [x - x_1] L^2_{1,1}(x) = (1 - 2(x - 2))(2 - x)^2 = (x - 2)(x - 1)^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
H_3(x) &= f(x_0)H_{1,0}(x) + f(x_1)H_{1,1}(x) + f'(x_0)\hat{H}_{1,0}(x) + f'(x_1)\hat{H}_{1,1}(x) \\
&= \frac{e^{-0.5}}{2}(x-2)^2(5x-3) + 2e^{-1}(x-1)^2(5-2x)
\end{aligned}
$$

(b) Approximate $f(1.4)$ using $H_3(1.4)$

$$f(1.4) \approx H_3(1.4) = \frac{e^{-0.5}}{2}(1.4-2)^2(5\times 1.4-3) + 2e^{-1}(1.4-1)^2(5-2\times 1.4) \approx 0.6957$$

(c) Find the absolute error $|f(1.4) - H_3(1.4)|$

$$f(1.4) = 0.6952$$

Therefore, $|f(1.4) - H_3(1.4)| = 0.0005$

(d) Find a bound for the error using the error bound formula

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x)$$

So, the error bound formula is given by (simply from textbook Theorem 3.9)

$$f(x) - H_{2n+1}(x) = \frac{\prod_{i=0}^{n}(x-x_i)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$$

When $n = 1$,

$$f(x) - H_3(x) = \frac{\prod_{i=0}^{1}(x-x_i)^2}{4!} f^4(\xi(x)) = \frac{f^4(\xi(x))}{24}(x-x_0)(x-x_1)$$

Question 6

The following values of x and y are given:

x:	1	2	3	4
y:	1	2	5	11

Find the natural cubic splines and evaluate $y(1.5)$ and $y'(3)$.

The cubic spline formula is

$$f(x) = \frac{(x_i - x)^3}{6h}M_{i-1} + \frac{(x - x_{i-1})^3}{6h}M_i + \frac{x_i - x}{h}(y_{i-1} - \frac{h^2}{6}M_{i-1}) + \frac{x - x_{i-1}}{h}(y_i - \frac{h^2}{6}M_i)$$

We have $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$

Here $h = 1,\ n = 3,\ M_0 = 0,\ M_3 = 0$

When $i = 1$ for the second equation, we have $M_0 + 4M_1 + M_2 = \frac{6}{h^2}(y_0 - 2y_1 + y_2)$, or $4M_1 + M_2 = 12$

When $i = 2$ for the second equation, similarly, we have $M_1 + 4M_2 = 18$

So, $M_1 = 2, M_2 = 4$

When $i = 1$ for the first equation, we get cubic spline in first interval $[x_0, x_1] = [1, 2]$

$$f_1(x) = \frac{1}{3}(x^3 - 3x^2 + 5x)$$

When $i = 2$ for the first equation, we get cubic spline in second interval $[x_1, x_2] = [2, 3]$

$$f_2(x) = \frac{1}{3}(x^3 - 3x^2 + 5x)$$

When $i = 3$ for the first equation, we get cubic spline in third interval $[x_2, x_3] = [3, 4]$

$$f_3(x) = \frac{1}{3}(-2x^3 + 24x^2 - 76x + 81)$$

For $y(1.5)$, we get

$$f_1(1.5) = 1.375$$

For $y'(3)$, we have

$$f_2'(x)|_{x=3} = \frac{1}{3}(3x^2 - 6x + 5) = 4.6667$$