WENZHOU-KEAN UNIVERSITY

NUMERICAL ANALYSIS
MATH 3940

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UNIT 1: Solutions of Equations in One Variable

- ✓ Root Finding Problem
- ✓ Graphical Interpretation of Roots
- ✓ Descarte’s Rule of Sign
- ✓ Bisection Method
- ✓ Examples

Linear Equations

- Equations of the form $y = mx+c$
- Degree of linear equation is 1
- Solution in the form of a graph is always a straight line.

Non-Linear Equations

- Equations containing algebraic ($P_n(x)=0$) or transcendental functions *i.e.* functions which are expressed by infinite series (e^x, log x, sin x etc.)

$$\text{e.g.,}\quad x^2+3x+5=0,\quad \log x^2+7=0,\quad \sin x^2-x-2=0$$

- Degree of non-linear polynomial is at least 2.

Root-Finding Problem

- We now consider one of the most basic problems of numerical approximation, namely the root-finding problem.
- This process involves finding a root, or solution, of an equation of the form

$$f(x) = 0$$

  for a given function f.
- A root of this equation is also called a zero of the function f.

- The problem of finding an approximation to the root of an equation can be traced back at least to 1700 B.C.E.

Applications

Marketing Problem: The design of a box specifies that its length is 4 inches greater than its width. The height is 1 inch less than the width. The volume of the box is 12 cubic inches. What is the width of the box?

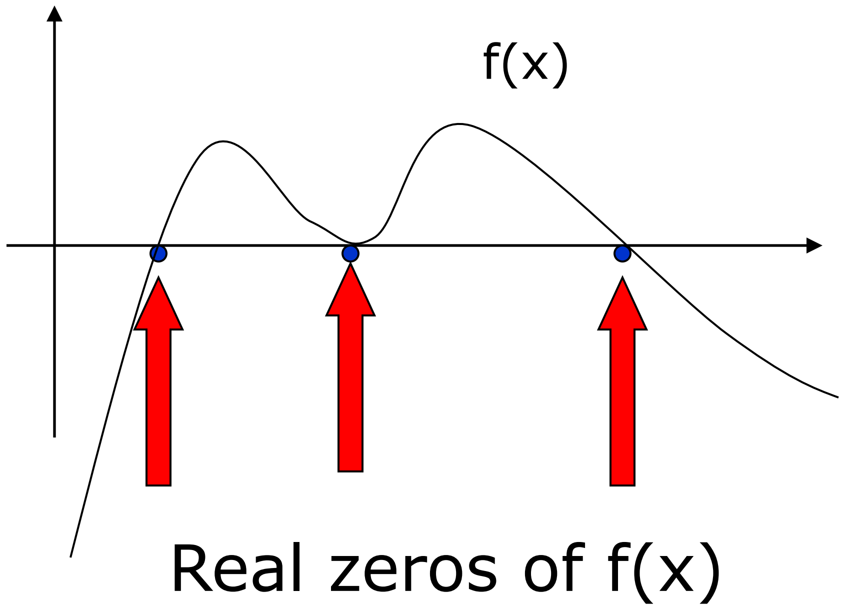
Population Growth Problem: Suppose that a certain population contains N(0)=1,000.000 individuals initially, that v=435,000 individuals immigrate into the community in the first year, and that N(l) = 1,564,000 individuals are present at the end of one year. Determine the birthrate λ of this population?

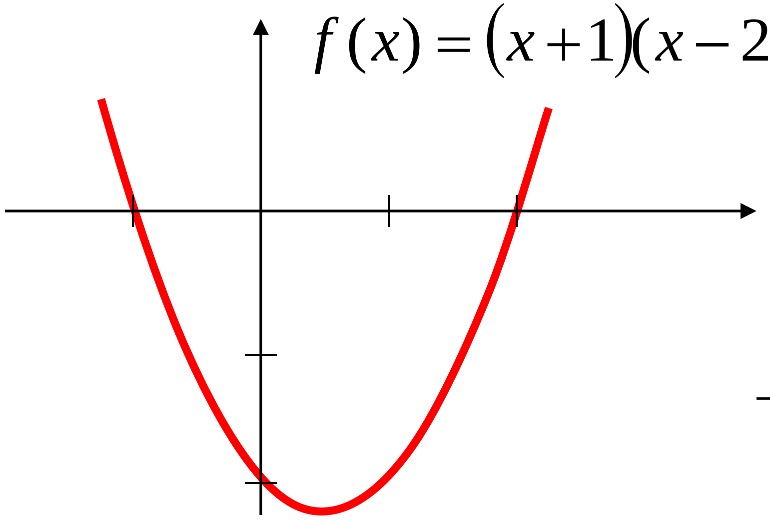
Note: The growth of a population can often be modeled over short periods of time by assuming that the population grows continuously with time at a rate proportional to the number present at that time

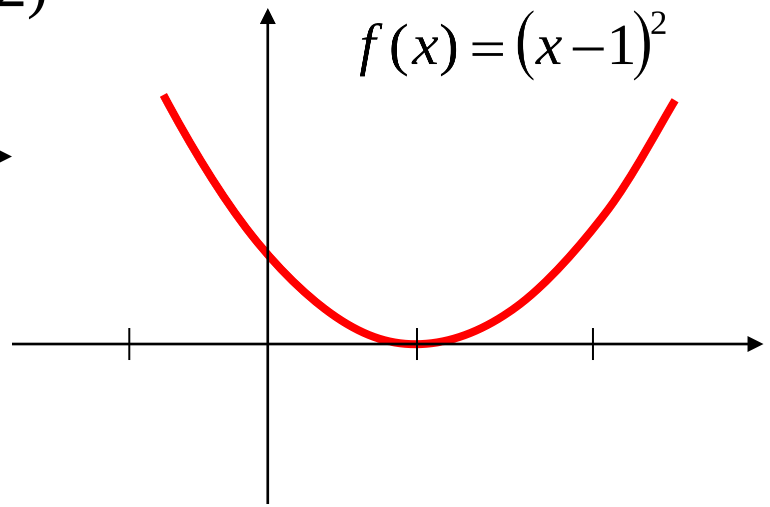
$$N(t) = N(0)e^{\lambda t} + \frac{v}{\lambda}(e^{\lambda t} - 1)$$

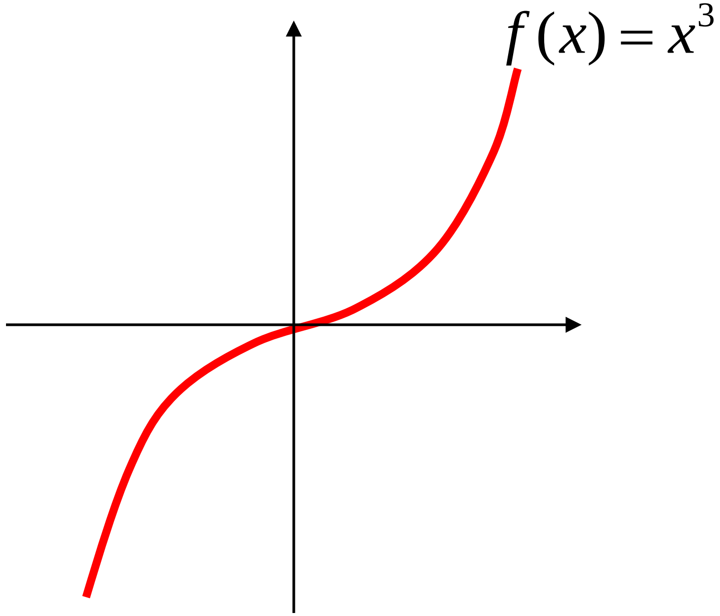
Graphical Interpretation of Zeros

- The real zeros of a function ***f(x)*** are the values of ***x*** at which the graph of the function crosses (or touches) the *x*-axis.
- Simple and Multiple Zeros









Facts

$$\boxed{\begin{array}{l} \qquad \mathrm{P_n}(x) = 0 \\ \Rightarrow \quad a_n x^n + a_{n\text{-}1} x^{n\text{-}1} + \ldots + a_1 x + a_0 = 0 \\ \text{coefficients of } x \text{ are real numbers.} \end{array}}$$

- Any n^{th} order polynomial has exactly n zeros (counting real and complex zeros with their multiplicities).
- Any polynomial with an odd order has at least one real zero.
- If (α+iβ) is a root then (α-iβ) will also be a root.
- If a function has a zero at ***x=r*** with multiplicity ***m*** then the function and its first ***(m-1)*** derivatives are zero at ***x=r*** *and the* **m**th *derivative at* **r** *is not zero*.

Descarte's Rule of Sign

- No. of positive roots can not exceed the no. of change in sign in f(x).
- No. of sign change in f($-x$) will give no. of negative roots.
- If f(a) and f(b) be the two values of f(x) *s.t.* $f(a)\times f(b) < 0$, then the equation $f(x)=0$ has at least one real root or an odd number of real roots in the interval (a,b).
- If $f(a)\times f(b) > 0$ then the no. of roots in (a, b) will be either 0 or even in numbers.

Ex: $f(x) = x^3 - 2x^2 + 3x + 1 = 0$

- There are two change in sign of f(x) so the equation f(x) = 0 can have at most two positive roots

$$f(-x) = -x^3 - 2x^2 - 3x + 1 = 0$$

- There is one change in sign of f(-x) so the equation f(x) = 0 can have at most one negative root.

- f(-1) = -5 = f(*a*)
- f(0) = 1 = f(*b*)

so, f(*a*) f(*b*) < 0. Hence, the negative root will lie in the interval (-1,0)

Descarte's Rule of Sign

$$f(x) = x^3 - 7x^2 - 10x - 8 = 0$$

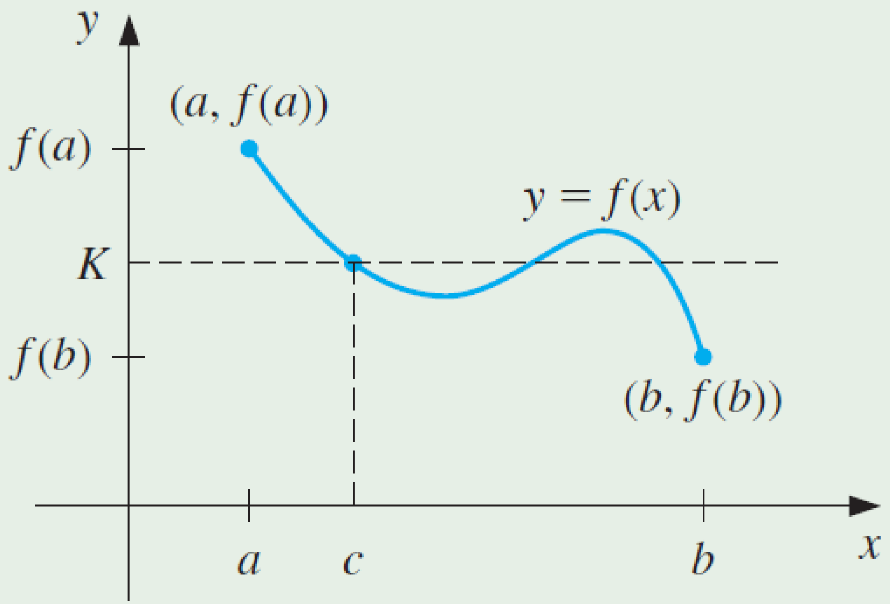
$$f(x) = 3x^4 + 2x^3 - 5x^2 - 8x + 1 = 0$$

The Bisection Method

- We first consider the Bisection (Binary search) Method which is based on the Intermediate Value Theorem (IVT). ▸ IVT Illustration
- Suppose a continuous function f, defined on $[a, b]$ is given with $f(a)$ and $f(b)$ of opposite sign.
- By the IVT, there exists a point $p \in (a, b)$ for which $f(p) = 0$. In what follows, it will be assumed that the root in this interval is unique.

Intermediate Value Theorem

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ for which $f(c) = K$.



(The diagram shows one of 3 possibilities for this function and interval.)

Bisection Technique

- Suppose f is a continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign.
- The Intermediate Value Theorem implies that a number p exists in (a, b) with $f(p) = 0$.
- Although the procedure will work when there is more than one root in the interval (a, b), we assume for simplicity that the root in this interval is unique.
- The method calls for a repeated halving (or bisecting) of subintervals of $[a, b]$ and, at each step, locating the half containing p.

Computation Steps

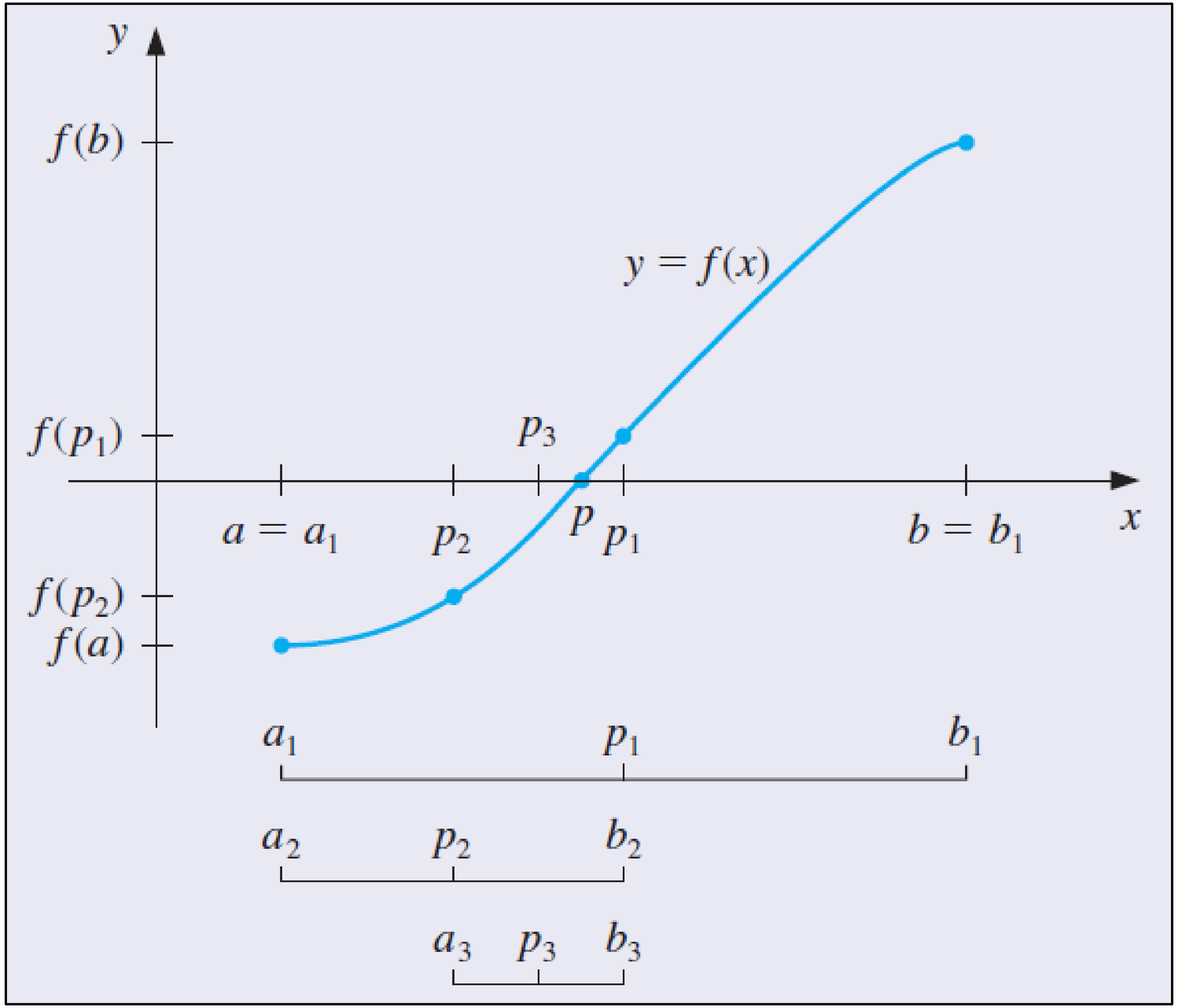
To begin, set $a_1 = a$ and $b_1 = b$, and let p_1 be the midpoint of $[a, b]$; that is,

$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}.$$

- If $f(p_1) = 0$, then $p = p_1$, and we are done.
- If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
  - If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.
  - If $f(p_1)$ and $f(a_1)$ have opposite signs, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Then re-apply the process to the interval $[a_2, b_2]$, etc.

Interval Halving to Bracket the Root



1. $a_1 = a$, $b_1 = b$, $p_0 = a$;
2. $i = 1$;
3. $p_i = \frac{1}{2}(a_i + b_i)$;
4. If $|p_i - p_{i-1}| < \epsilon$ or $|f(p_i)| < \epsilon$ then 10;
5. If $f(p_i)f(a_i) > 0$, then 6;
   If $f(p_i)f(a_i) < 0$, then 8;
6. $a_{i+1} = p_i$, $b_{i+1} = b_i$;
7. $i = i + 1$; go to 3;
8. $a_{i+1} = a_i$; $b_{i+1} = p_i$;
9. $i = i + 1$; go to 3;
10. End of Procedure.

Example

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$ and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

Example

Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$ and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}.

Note that, for this example, the iteration will be terminated when a bound for the relative error is less than 10^{-4}, implemented in the form:

$$\frac{|p_n - p_{n-1}|}{|p_n|} < 10^{-4}.$$

Example

Iter	a_n	b_n	p_n	$f(a_n)$	$f(p_n)$	RelErr
1	1.000000	2.000000	1.500000	-5.000	2.375	0.33333
2	1.000000	1.500000	1.250000	-5.000	-1.797	0.20000
3	1.250000	1.500000	1.375000	-1.797	0.162	0.09091

Example

Iter	a_n	b_n	p_n	$f(a_n)$	$f(p_n)$	RelErr
1	1.000000	2.000000	1.500000	-5.000	2.375	0.33333
2	1.000000	1.500000	1.250000	-5.000	-1.797	0.20000
3	1.250000	1.500000	1.375000	-1.797	0.162	0.09091
4	1.250000	1.375000	1.312500	-1.797	-0.848	0.04762
5	1.312500	1.375000	1.343750	-0.848	-0.351	0.02326
6	1.343750	1.375000	1.359375	-0.351	-0.096	0.01149
7	1.359375	1.375000	1.367188	-0.096	0.032	0.00571
8	1.359375	1.367188	1.363281	-0.096	-0.032	0.00287
9	1.363281	1.367188	1.365234	-0.032	0.000	0.00143
10	1.363281	1.365234	1.364258	-0.032	-0.016	0.00072
11	1.364258	1.365234	1.364746	-0.016	-0.008	0.00036
12	1.364746	1.365234	1.364990	-0.008	-0.004	0.00018
13	1.364990	1.365234	1.365112	-0.004	-0.002	0.00009

- After 13 iterations, $p_{13} = 1.365112305$ approximates the root p with an error

$$|p - p_{13}| < |b_{14} - a_{14}| = |1.3652344 - 1.3651123| = 0.0001221$$

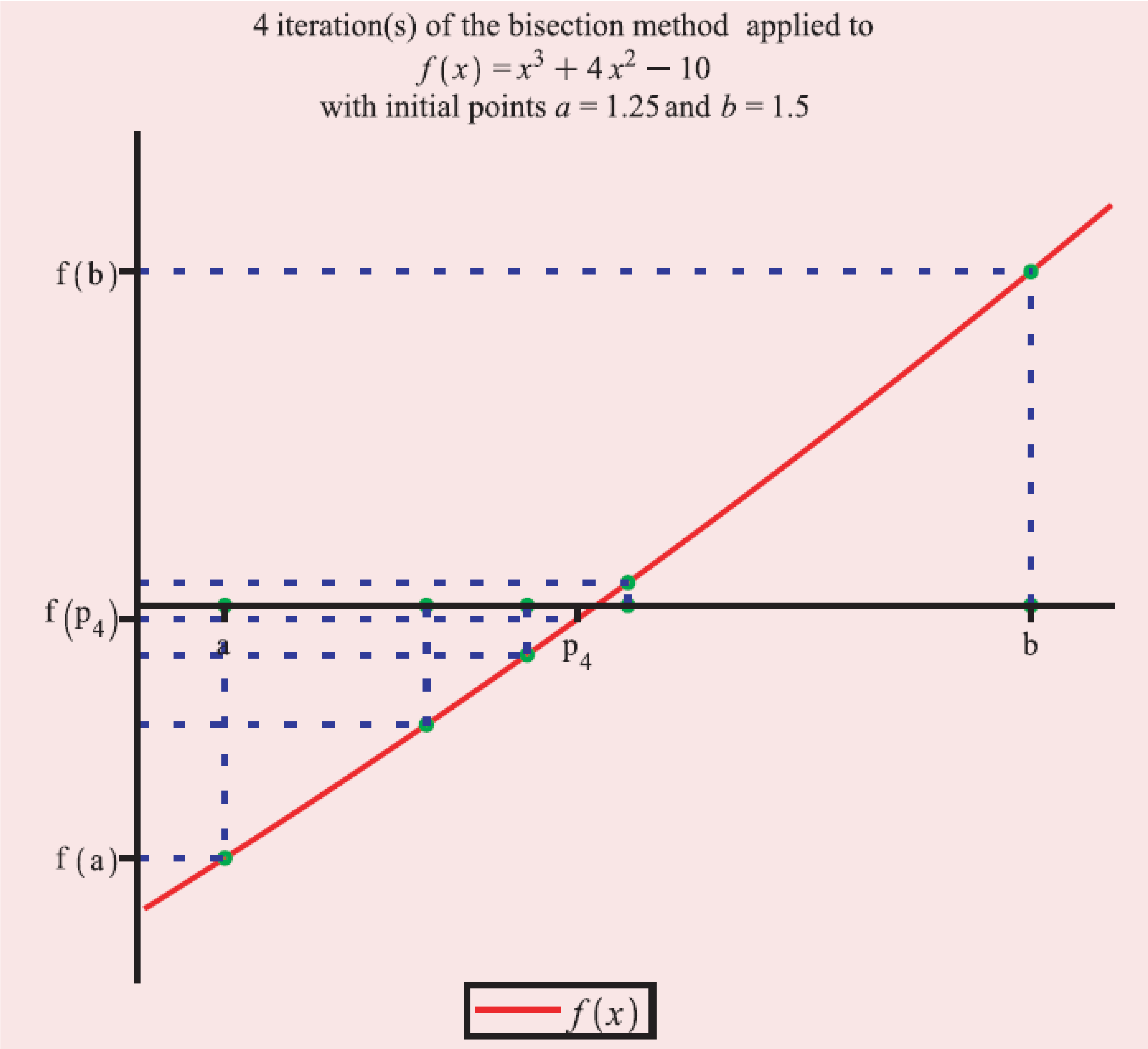
- Since $|a_{14}| < |p|$, we have

$$\frac{|p - p_{13}|}{|p|} < \frac{|b_{14} - a_{14}|}{|a_{14}|} \leq 9.0 \times 10^{-5},$$

so the approximation is correct to at least within 10^{-4}.

- The correct value of p to nine decimal places is $p = 1.365230013$

```
>> format long
>> p=[1 4 0 -10]

p =

     1     4     0   -10

>> roots(p)

ans =

 -2.682615006707049 + 0.358259359924043i
 -2.682615006707049 - 0.358259359924043i
  1.365230013414097 + 0.000000000000000i
```


4 iteration(s) of the bisection method applied to
$f(x) = x^3 + 4x^2 - 10$
with initial points $a = 1.25$ and $b = 1.5$
f(b)
f(p₄)
f(a)
a
p₄
b
$f(x)$

Theorem

Suppose that $f \in C[a,b]$ *and* $f(a) \cdot f(b) < 0$. *The Bisection method generates a sequence* $\{p_n\}_{n=1}^{\infty}$ *approximating a zero* p *of* f *with*

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \textit{when} \quad n \geq 1.$$

Proof.

For each $n \geq 1$, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \quad \text{and} \quad p \in (a_n, b_n).$$

Since $p_n = \frac{1}{2}(a_n + b_n)$ for all $n \geq 1$, it follows that

$$|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}.$$

$\square$

Conservative Error Bound

- It is important to realize that the theorem gives only a bound for approximation error and that this bound might be quite conservative.
- For example, this bound applied to the earlier problem, namely where

$$f(x) = x^3 + 4x^2 - 10$$

ensures only that

$$|p - p_9| \leq \frac{2 - 1}{2^9} \approx 2 \times 10^{-3},$$

but the actual error is much smaller:

$$|p - p_9| = |1.365230013 - 1.365234375| \approx 4.4 \times 10^{-6}.$$

Question!!

Determine the number of iterations necessary to solve
$f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.

Question!!

Find the positive root, between 0 and 1, of the equation $x = e^{-x}$ to a tolerance of 0.05%.

Final Remarks

1. **Convergence is guarenteed:** Bisection method is bracketing method and it is always convergent.
2. **Error can be controlled:** In Bisection method, increasing number of iteration always yields more accurate root.

1. **Slow Rate of Convergence:** Although convergence of Bisection method is guaranteed, it is generally slow.
2. **Choosing one guess close to root has no advantage:** Choosing one guess close to the root may result in requiring many iterations to converge.
3. Can not find root of some equations. For example: $f(x) = x^2$ as there are no bracketing values.

THANK YOU
ANY QUESTIONS???