3940 AS04

Main

Question 1

If $f(x) = \frac{\cos x}{1+x^3}$, then find the approximate value of $f'(0.9)$ using the three-point midpoint formula with $h = 0.2$

Solution 1

The formula of three-point midpoint is

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

So, from the question, we have

$$f'(0.9) = \frac{f(0.9+0.2) - f(0.9-0.2)}{2 \times 0.2} = \frac{f(1.1) - f(0.7)}{0.4} = \frac{0.4289 - 0.7446}{0.4} = -0.78925$$

Question 2

Use the most accurate three-point formula to determine each missing entry in the following table

x	-0.3	-0.2	-0.1	0
f(x)	-0.27652	-0.25074	-0.16134	0
f'(x)	---	---	---	---

The above data is taken from the function $f(x) = e^{2x} - \cos 2x$. So, compute the actual errors.

Solution 2

$$\begin{aligned}
\overline{f'(-0.3)} &= \frac{-3f(-0.3) + 4f(-0.2) - f(-0.1)}{2 \times 0.1} = \frac{3 \times 0.27652 - 4 \times 0.25074 + 0.16134}{0.2} = -0.0603 \\
\overline{f'(-0.2)} &= \frac{f(-0.1) - f(-0.3)}{2 \times 0.1} = \frac{-0.16134 + 0.27652}{0.2} = 0.5759 \\
\overline{f'(-0.1)} &= \frac{f(0) - f(-0.2)}{0.2} = \frac{0.25074}{0.2} = 1.2537 \\
\overline{f'(0)} &= \frac{-3f(0) + 4f(-0.1) - f(-0.2)}{-2 \times 0.1} = \frac{-4 \times 0.16134 + 0.25074}{-0.2} = 1.9731
\end{aligned}$$

while the actual values of them are

$$\begin{aligned}
f'(x) &= 2e^{2x} + 2\sin 2x \\
f'(-0.3) &= -0.0316616746 \\
f'(-0.2) &= 0.56180340745 \\
f'(-0.1) &= 1.24012284457 \\
f'(0) &= 2
\end{aligned}$$

So, the actual errors are

$$\begin{aligned}
|f'(-0.3) - \overline{f'(-0.3)}| &= |-0.0316616746 + 0.0603| = 0.02864 \\
|f'(-0.2) - \overline{f'(-0.2)}| &= |0.56180340745 - 0.5759| = 0.01410 \\
|f'(-0.1) - \overline{f'(-0.1)}| &= |1.24012284457 - 1.2537| = 0.01358 \\
|f'(0) - \overline{f'(0)}| &= |-0.0316616746 + 0.0603| = 0.02864
\end{aligned}$$

Question 3

Let $f(x) = \cos \pi x$. Use the midpoint formula and the values of $f(x)$ at $x = 0.25, 0.5$ and 0.75 to approximate $f''(0.5)$. Compare this result to the exact value.

Solution 3

For the values of $f(x)$ at $x = 0.25, 0.5$ and 0.75, we have

$$f(0.25) = \frac{\sqrt{2}}{2}$$
$$f(0.5) = 0$$
$$f(0.75) = -\frac{\sqrt{2}}{2}$$

Since

$$f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12} f^{(4)}(\xi)$$

So

$$f''(0.5) = \frac{1}{h^2}[f(0.5 - h) - 2f(0.5) + f(0,5 + h)] - \frac{h^2}{12} f^{(4)}(\xi)$$

From the question, we could know that $h = 0.25$, therefore, the approximated value of $f''(0.5)$ is

$$f''(0.5) = \frac{1}{0.25^2}[f(0.25) - 2f(0.5) + f(0,75)] - \frac{0.25^2}{12} f^{(4)}(\xi) = -\frac{0.25^2}{12} f^{(4)}(\xi)$$

taken $\xi = 0.5$, so $f''(0.5) = -\frac{f^{(4)}(0.5)}{192}$

The exactly of $f''(0.5)$ could be calculated as below:

$$f''(0.5) = -\pi^2 \cos 0.5\pi = 0$$

Therefore, the absolute error is $|\frac{f^{(4)}(0.5)}{192}|$

Question 4

Apply the extrapolation process described to determine $N_3(h)$, an approximation to $f'(x_0)$, for the function $f(x) = x + e^x$, $x_0 = 0.0, h = 0.4$.

Solution 4

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(0.4) - f(0)}{0.4} \approx 2.22956174$$

So $N_1(0.4) = 2.22956174$

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(0.2) - f(0)}{0.2} \approx 2.10701379$$

So, $N_1(0.2) = 2.10701379$

Since

$$N_2(0.4) = N_1(0.2) + \frac{1}{3}[N_1(0.2) - N_1(0.4)]$$

Therefore

$$N_2(0.4) = 2.06616447$$

Since

$$N_1(0.1) = \frac{f(0.1) - f(0)}{0.1} = 2.05170918$$

$$N_2(0.2) = N_1(0.1) + \frac{1}{3}[N_1(0.1) - N_1(0.2)] = 2.03327431$$

Finally,

$$N_3(0.4) = \frac{1}{15}[16N_2(0.2) - N_2(0.4)] = 2.0310816$$

Question 6

The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ givens the values 4, and Simpson's rule gives the value 2. What is the value of $f(1)$.

Solution 6

$$\int_0^2 f(x)\,dx = f(2) + f(0) = 4$$

$$\int_0^2 f(x)\,dx = \frac{1}{3}(f(0) + 4f(1) + f(2)) = 2$$

Therefore, $f(1) = 0.5$

Question 7

Suppose that $f(0) = 1$, $f(0.5) = 2.5$, $f(1) = 2$ and $f(0.25) = f(0.75) = a$. Find the a, if the composite Trapezoidal rule with $n = 4$ gives the value 1.75 for $\int_0^1 f(x)\,dx$.

Solution 7

Since for Trapezoidal rule $n - 4$, $\Delta x = \frac{1}{4} \to x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1.0$

So,

$$\int_0^1 f(x)\,dx = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] = \frac{1}{8}(8 + 4a) = 1.75$$

Therefore, $a = 1.5$

Question 8

Determine the values of n and h required to approximate $\int_0^2 \frac{dx}{x+4}$ within 10^{-5} and compute the approximation using (a) composite Trapezoidal rule (b) composite Simpson's rule.

Solution 8

(a) Composite Trapezoidal Rule

$$f'(x) = -\frac{1}{(x+4)^2}$$
$$f''(x) = \frac{2}{(x+4)^3}$$
$$M = \text{Max}_{x \in (a,b)} |f''(x)|$$

$f''(x)$ will be maximum at 0.

$$M = \frac{2}{4^3} = \frac{1}{32}$$

So,

$$\text{error bound} \leq \frac{M(b-a)^2}{12n^2}$$
$$10^{-5} \leq \frac{M(b-a)^2}{12n^2}$$
$$n \approx 46$$

(b) Composite Simpson's Rule

$$f'''(x) = -\frac{6}{(x+4)^4},\ f^{(4)}(x) = \frac{24}{(x+4)^5}$$

$f'''(x)$ is max at $x = 0$ in $[0, 2]$

Therefore

$$\epsilon \leq \frac{(2-0)^5 \frac{6}{4^4}}{180n^4}$$
$$10^{-5} \leq \frac{(2-0)^5 \frac{6}{4^4}}{180n^4}$$
$$n \approx 4.5$$

Question 9

Use Romberg Integration to compute $R_{3,3}$ for the integral $\int_{-0.75}^{0.75} x \ln(x+1)\, dx$. Also, compare the results with exact value of the integral.

Solution 9

The exact value of the integral is $\int_{-0.75}^{0.75} x \ln(x+1)\, dx = -\dfrac{14\ln(4)+15}{64} - \dfrac{14\ln\left(\frac{7}{4}\right)-63}{64} = 0.32433$

$$
\begin{aligned}
R_{1,1} &= \frac{3}{4}[f(-0.75)+f(0.75)] = 1.09457\\
R_{2,1} &= \frac{3}{8}[f(-0.75)+2f(0)+f(0.75)] = 0.547287\\
R_{3,1} &= \frac{3}{16}[f(-0.75)+\ldots+f(0.75)] = 0.384520\\
R_{2,2} &= R_{2,1}+\frac{1}{3}(R_{2,1}-R_{1,1}) = 0.364858\\
R_{3,2} &= R_{3,1}+\frac{1}{3}(R_{3,1}-R_{2,1}) = 0.330265\\
R_{3,3} &= R_{3,2}+\frac{1}{15}(R_{3,2}-R_{2,2}) = 0.327957
\end{aligned}
$$

$$\epsilon = |0.327957 - 0.32433| = 0.003627$$

Question 10

Approximate the following integral $\int_0^{\frac{\pi}{4}} (\cos x)^2 dx$ using Gaussian quadrature with $n = 2, 3$ and 4. Also, compare your results to the exact values of the integral.

Solution 10

The exact value of the integral is $\int_0^{\frac{\pi}{4}} (\cos x)^2 dx = 0.64267$

$$x = \frac{b-a}{2}t + \frac{b+a}{2} = \frac{\pi}{8}t + \frac{\pi}{8} = \frac{\pi}{8}(t+1)$$

$$\therefore t = \frac{8x}{\pi} - 1 \to dt = \frac{8}{\pi}dx$$

$$\int_0^{\frac{\pi}{4}} (\cos x)^2 dx = \int_{-1}^{1} (\cos\frac{\pi}{8}(t+1))^2 \frac{\pi}{8} dt$$

When $n = 2$, the equation is:

$$\int_{-1}^{1} f(x)dx \approx f(-0.5773502692) + f(0.5773502692) = 0.64232$$

When $n = 3$, the equation is:

$$\int_{-1}^{1} f(x)dx \approx 0.\overline{5}f(-0.774596669) + 0.\overline{8}f(0) + 0.\overline{5}f((0.774596669) = 0.64270$$

When $n = 4$, the equation is:

$$\int_{-1}^{1} f(x)dx \approx 0.3478548451f(0.8611363116) + 0.6521451549f(0.3399810436) + 0.6521451549f(-0.3399810436) + 0.3$$
$$= 0.642670$$

$$
\begin{aligned}
\epsilon_{n=2} &= |0.64267 - 0.64232| = 0.00035\\
\epsilon_{n=3} &= |0.64267 - 0.64232| = 0.00003\\
\epsilon_{n=4} &= |0.64267 - 0.64232| = 0.00003
\end{aligned}
$$