3940 AS06

Main

Question 1

(i): True

(ii): True

(iii): True

(iv): True

(v): False

Question 2

Use the Gersgorin Circle Theorem to determine bounds for the eigenvalues, and the spectral radius of the following matrix

$$\begin{bmatrix} 4.75 & 2.25 & -0.25 \\ 2.25 & 4.75 & 1.25 \\ -0.25 & 1.25 & 4.75 \end{bmatrix}$$

Solution 2

The circles in the Gersgorin Theorem are

$$R_1 = \{z \in C \mid |z - 4.75| \le 2.5\},\ R_2 = \{z \in C \mid |z - 4.75| \le 3.5\}, R_3 = \{z \in C \mid |z - 4.75| \le 5\}$$

So, there are three eigenvalues within R_1, R_2, R_3. Moreover, $\rho(A) = max_{1 \le i \le 3}$ so $0.25 \le \rho(A) \le 9.75$

Question 3

Consider the follow sets of vectors. (i) Show that the set is linearly independent; (ii) use the Gram-Schmidt process to find a set of orthogonal vectors; (iii) determine a set of orthonormal vectors from the vectors in (ii).

(a) $\mathbf{v_1} = (1, 1)^t,\ \mathbf{v_2} = (-2, 1)^t$

(b) $\mathbf{v_1} = (1, 1, 1, 1)^t,\ \mathbf{v_2} = (0, 2, 2, 2)^t,\ \mathbf{v_3} = (1, 0, 0, 1)^t$

Solution 3

(a) $\mathbf{v_1} = (1, 1)^t,\ \mathbf{v_2} = (-2, 1)^t$

$$(0, 0)^t = \alpha_1(1, 1)^t + \alpha_2(-2, 1)^t = (\alpha_1 - 2\alpha_2, \alpha_1 + \alpha_2)$$

so, the only solution to this system is $\alpha_1 = \alpha_2 = 0$, such that the set is linearly independent. (i)

We have the orthogonal vector a, b, given by

$$\begin{aligned} a &= v_1 = (1, 1)^t \\ b &= v_2 - \frac{a^t v_2}{v_1^t v_1} v_1 \\ &= (-2, 1)^t - \frac{(1, 1)(-2, 1)^t}{(1, 1)(1, 1)^t}(1, 1)^t \\ &= (-2, 1)^t + (1, 1)^t \\ &= (-1.5, 1.5)^t \end{aligned}$$

So, the orthogonal vectors are $a = (1, 1)^t, b = (-1.5, 1.5)^t$ (ii)

The vectors

$$\alpha = \frac{a}{||a||_2} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^t$$
$$\beta = \frac{b}{||b||_2} = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^t$$

from an orthonormal set, since they inherit orthogonality from a, b, and additionally,

$$||\alpha||_2 = ||\beta||_2 = 1$$

(b) $\mathbf{v_1} = (1,1,1,1)^t,\ \mathbf{v_2} = (0,2,2,2)^t,\ \mathbf{v_3} = (1,0,0,1)^t$

$(0,0,0,0)^t = \alpha_1(1,1,1,1)^t + \alpha_2(0,2,2,2)^t + \alpha_3(1,0,0,1)^t = (\alpha_1+\alpha_3, \alpha_1+2\alpha_2, \alpha_1+2\alpha_2, \alpha_1+2\alpha_2+\alpha_3)^t$

So, the only solution to this system is $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so this set is linear independent. (i)

We have the orthogonal vector a, b and c, given by (ii)

$$\begin{aligned}
a &= v_1 = (1,1,1,1)^t \\
b &= v_2 - \frac{a^t v_2}{v_1^t v_1} v_1 \\
&= (0,2,2,2)^t - \frac{(1,1,1,1)(0,2,2,2)^t}{(1,1,1,1)(1,1,1,1)^t}(1,1,1,1)^t \\
&= (-1.5, 0.5, 0.5, 0.5) \\
c &= v_3 - \frac{a^t v_3}{v_1^t v_1} v_1 - \frac{b^t v_3}{v_2^t v_2} v_2 \\
&= (1,0,0,1)^t - \frac{(1,1,1,1)(1,0,0,1)^t}{(1,1,1,1)(1,1,1,1)^t}(1,1,1,1)^t - \frac{(1,0,0,1)(-1.5,0.5,0.5,0.5)^t}{(-1.5,0.5,0.5,0.5)(-1.5,0.5,0.5,0.5)^t}(-1.5,0.5,0.5,0.5)^t \\
&= (1,0,0,1)^t - (0.5,0.5,0.5,0.5)^t + (-0.5, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})^t \\
&= (0, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})^t
\end{aligned}$$

The vectors

$$\alpha = \frac{a}{||a||_2} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t$$
$$\beta = \frac{b}{||b||_2} = (-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6})^t$$
$$\gamma = \frac{c}{||c||_2} = (0, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2})^t$$

from an orthonormal set, since they inherit orthogonality from a, b, and additionally,

$$||\alpha||_2 = ||\beta||_2 = ||\gamma||_2 = 1$$

Question 4

(i) For the following matrix determine if it diagonalizable and, if so, find P and D with $A = PDP^{-1}$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(ii) Determine if the above matrix is positive definite, and if so, (ii) construct an orthogonal matrix Q for which $Q^t AQ = D$, where D is a diagonal matrix.

Solution 4

(i) For the following matrix determine if it diagonalizable and, if so, find P and D with $A = PDP^{-1}$

The P matrix is the matrix of eigenvectors of A. So, by solving the equation

$$(\lambda I - A)x = 0$$

We have the eigenvectors, $(-1,1,0)^t, (-1,0,1)^t, (1,1,1)^t$

So,

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

And the diagonal matrix D is

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(ii) Determine if the above matrix is positive definite, and if so, (ii) construct an orthogonal matrix Q for which $Q^t AQ = D$, where D is a diagonal matrix.

$$\det\left(\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

So, the equation can be transformed into $-(\lambda-4)(\lambda-1)^2 = 0$ which has the solution $\lambda = 4, 1$

Since the solution (eigenvalues) is all positive and A is symmetric, the matrix is positive definite.

Given,

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The eigenvectors of the matrix A is

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
a &= (-1,1,0)^t \\
b &= (-1,0,1)^t - \frac{(-1,1,0)(-1,0,1)^t}{(-1,1,0)(-1,1,0)^t}(-1,1,0)^t \\
&= (-0.5,-0.5,0.5)^t \\
c &= (1,1,1)^t - \frac{(-1,1,0)(1,1,1)^t}{(-1,1,0)(-1,1,0)^t}(-1,1,0)^t - \frac{(-0.5,-0.5,0.5)(1,1,1)^t}{(-0.5,-0.5,0.5)(-0.5,-0.5,0.5)^t}(-0.5,-0.5,0.5)^t \\
&= (\frac{2}{3}, \frac{2}{3}, \frac{4}{3}) \\
Q &= \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{2\sqrt{6}}{6} \end{pmatrix}
\end{aligned}$$

Question 5

(a) Find the first three iterations obtained by the Power method applied to the following matrices

$$\begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

(b) Use the Power method to approximate the most dominant eigenvalue of the matrix. Iterate until a tolerance of 10^{-4} is achieved or until the number of iterations exceeds 25.

(c) Repeat above using Aitken's Δ^2 technique and the Power method for the most dominant eigenvalue.

Solution 5

(a) Find the first three iterations obtained by the Power method applied to the following matrices

Given matrix,

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

with vector $x^{(0)} = (1, 2, 1)^t$.

So, the three iterations are

$$\begin{aligned} x^{(1)} &= \begin{pmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{pmatrix} x^{(0)} = (9, 8, 7)^t \\ x^{(2)} &= \begin{pmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{pmatrix} x^{(1)} = (59, 38, 45)^t \\ x^{(3)} &= \begin{pmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{pmatrix} x^{(2)} = (357, 204, 277)^t \end{aligned}$$

(b) Use the Power method to approximate the most dominant eigenvalue of the matrix. Iterate until a tolerance of 10^{-4} is achieved or until the number of iterations exceeds 25.

1^{st} Iteration

$$Ax_0 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_1 = \frac{1}{9} \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.88888889 \\ 0.77777778 \end{bmatrix}$$

2^{nd} Iteration

$$Ax_1 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.88888889 \\ 0.77777778 \end{bmatrix} = \begin{bmatrix} 6.55555556 \\ 4.22222222 \\ 5 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_2 = \frac{1}{6.55555556} \begin{bmatrix} 6.55555556 \\ 4.22222222 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.6440678 \\ 0.76271186 \end{bmatrix}$$

3^{rd} Iteration

$$Ax_2 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6440678 \\ 0.76271186 \end{bmatrix} = \begin{bmatrix} 6.05084746 \\ 3.45762712 \\ 4.69491525 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_3 = \frac{1}{6.05084746} \begin{bmatrix} 6.05084746 \\ 3.45762712 \\ 4.69491525 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.57142857 \\ 0.77591036 \end{bmatrix}$$

4^{th} Iteration

$$Ax_3 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.57142857 \\ 0.77591036 \end{bmatrix} = \begin{bmatrix} 5.91876751 \\ 3.26610644 \\ 4.67507003 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_4 = \frac{1}{5.91876751} \begin{bmatrix} 5.91876751 \\ 3.26610644 \\ 4.67507003 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.55182205 \\ 0.78987222 \end{bmatrix}$$

5^{th} Iteration

$$Ax_4 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.55182205 \\ 0.78987222 \end{bmatrix} = \begin{bmatrix} 5.89351633 \\ 3.2352106 \\ 4.71131093 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_5 = \frac{1}{5.89351633} \begin{bmatrix} 5.89351633 \\ 3.2352106 \\ 4.71131093 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.54894403 \\ 0.79940577 \end{bmatrix}$$

6^{th} Iteration

$$Ax_5 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.54894403 \\ 0.79940577 \end{bmatrix} = \begin{bmatrix} 5.89729382 \\ 3.24564362 \\ 4.74656709 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_6 = \frac{1}{5.89729382} \begin{bmatrix} 5.89729382 \\ 3.24564362 \\ 4.74656709 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.55036152 \\ 0.80487207 \end{bmatrix}$$

7^{th} Iteration

$$Ax_6 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.55036152 \\ 0.80487207 \end{bmatrix} = \begin{bmatrix} 5.90559512 \\ 3.26082871 \\ 4.76984981 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_7 = \frac{1}{5.90559512} \begin{bmatrix} 5.90559512 \\ 3.26082871 \\ 4.76984981 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.55215921 \\ 0.80768317 \end{bmatrix}$$

8^{th} Iteration

$$Ax_7 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.55215921 \\ 0.80768317 \end{bmatrix} = \begin{bmatrix} 5.91200159 \\ 3.27184397 \\ 4.7828919 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_8 = \frac{1}{5.91200159} \begin{bmatrix} 5.91200159 \\ 3.27184397 \\ 4.7828919 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.55342407 \\ 0.80901397 \end{bmatrix}$$

9^{th} Iteration

$$Ax_8 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.55342407 \\ 0.80901397 \end{bmatrix} = \begin{bmatrix} 5.91586211 \\ 3.27830015 \\ 4.78947997 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_9 = \frac{1}{5.91586211} \begin{bmatrix} 5.91586211 \\ 3.27830015 \\ 4.78947997 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.55415425 \\ 0.80959966 \end{bmatrix}$$

10^{th} Iteration

$$Ax_9 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.55415425 \\ 0.80959966 \end{bmatrix} = \begin{bmatrix} 5.91790817 \\ 3.28166209 \\ 4.7925529 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_{10} = \frac{1}{5.91790817} \begin{bmatrix} 5.91790817 \\ 3.28166209 \\ 4.7925529 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.55453075 \\ 0.80983901 \end{bmatrix}$$

11^{th} Iteration

$$Ax_{10} = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.55453075 \\ 0.80983901 \end{bmatrix} = \begin{bmatrix} 5.91890052 \\ 3.28327029 \\ 4.7938868 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_{11} = \frac{1}{5.91890052} \begin{bmatrix} 5.91890052 \\ 3.28327029 \\ 4.7938868 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.55470949 \\ 0.8099286 \end{bmatrix}$$

12^{th} Iteration

$$Ax_{11} = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.55470949 \\ 0.8099286 \end{bmatrix} = \begin{bmatrix} 5.91934758 \\ 3.28398567 \\ 4.79442388 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_{12} = \frac{1}{5.91934758} \begin{bmatrix} 5.91934758 \\ 3.28398567 \\ 4.79442388 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.55478845 \\ 0.80995816 \end{bmatrix}$$

$\therefore$ The dominant eigenvalue $\lambda = 5.91934758 \cong 5.92$

(c) Repeat above using Aitken's Δ^2 technique and the Power method for the most dominant eigenvalue.

$$\hat{\lambda}_n = \lambda_n - \frac{(\lambda_{n+1} - \lambda_n)^2}{\lambda_{n+2} - 2\lambda_{n+1} + \lambda_n}$$

$$\ldots \hat{\lambda}_{10} = \lambda_{10} - \frac{(\lambda_{11} - \lambda_{10})^2}{\lambda_{12} - 2\lambda_{11} + \lambda_{10}}$$

$$= 5.91790817 - \frac{(5.91890052 - 5.91790817)^2}{5.91934758 - 2 * 5.91890052 + 5.91790817}$$

$$= 5.9197141054$$

$$\ldots$$

Question 6

Use Householder's method to place the following matrices in tridiagonal form

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

Solution 6

Given,

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

We calculate

$$\alpha = \sqrt{2} = 1.4142136, r = \sqrt{1 + \frac{\sqrt{2}}{2}} = 1.306563$$

This gives

$$w^{(1)} = (0, -0.9238795, -0.3826834, 0)$$

and

$$A^{(2)} = P^{(1)} A^{(1)} P^{(1)} = \begin{bmatrix} 4 & 1.4142136 & 0 & 0 \\ 1.4142136 & 4 & 0 & 1.4142136 \\ 0 & 0 & 4 & 0 \\ 0 & 1.4142136 & 0 & 4 \end{bmatrix}$$

So

$$\alpha = -1.4142136, r = 1$$

which means

$$w^{(1)} = (0, 0, 0.7071068, 0.7071068)$$

and

$$A^{(3)} = P^{(2)} A^{(2)} P^{(2)} = \begin{bmatrix} 4 & 1.4142136 & 0 & 0 \\ 1.4142136 & 4 & -1.4142136 & 0 \\ 0 & -1.4142136 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Question 7

Apply two iterations of the QR method without shifting to the following matrix.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution 7

Given,

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Let $A^{(1)} = A$,

When $n = 1,\ k = 1$,

$$A_1^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, b_2 = -1, x_1 = 2$$

$$s_2 = \frac{b_2}{\sqrt{b_2^2 + x_1^2}} = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

$$c_2 = \frac{x_1}{\sqrt{b_2^2 + x_1^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$P_2 = \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} & 0 \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2^{(1)} = P_2 A_1^{(1)} = \begin{bmatrix} \sqrt{5} & -\frac{4}{\sqrt{5}} & \frac{1}{5} \\ 0 & \frac{3}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & -1 & 2 \end{bmatrix}$$

When $n = 1,\ k = 2$,

$$A_2^{(1)} = \begin{bmatrix} \sqrt{5} & -\frac{4}{\sqrt{5}} & \frac{1}{5} \\ 0 & \frac{3}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & -1 & 2 \end{bmatrix}, b_3 = -1, x_2 = \frac{3}{\sqrt{5}}$$

$$s_3 = \frac{b_3}{\sqrt{b_3^2 + x_2^2}} = \frac{-1}{\sqrt{1 + \frac{9}{5}}} = -\frac{\sqrt{70}}{14}$$

$$c_3 = \frac{x_2}{\sqrt{b_3^2 + x_2^2}} = \frac{\frac{3}{\sqrt{5}}}{\sqrt{1 + \frac{9}{5}}} = -\frac{3\sqrt{14}}{14}$$

$$P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3\sqrt{14}}{14} & -\frac{\sqrt{70}}{14} \\ 0 & \frac{\sqrt{70}}{14} & \frac{3\sqrt{14}}{14} \end{pmatrix}$$

$$A_3^{(1)} = P_3 A_2^{(1)} = \begin{pmatrix} \sqrt{5} & \frac{\sqrt{35} - 12\sqrt{7}}{7\sqrt{10}} & \frac{3\sqrt{7} + 4\sqrt{35}}{7\sqrt{10}} \\ 0 & \frac{9 - 2\sqrt{5}}{\sqrt{70}} & \frac{-3\sqrt{35} - 6\sqrt{7}}{7\sqrt{10}} \\ 0 & \frac{2\sqrt{5} - 3}{\sqrt{14}} & \frac{3\sqrt{2} + \sqrt{\frac{5}{2}}}{\sqrt{7}} \end{pmatrix}$$

We factorize the matrix as $A^{(2)} = R^{(1)} Q^{(1)}$ with $R^{(1)} = A_3^{(1)}, Q^{(1)} = P_2^t P_3^t$ where

$$R^{(1)} = \begin{pmatrix} \sqrt{5} & \frac{\sqrt{35} - 12\sqrt{7}}{7\sqrt{10}} & \frac{3\sqrt{7} + 4\sqrt{35}}{7\sqrt{10}} \\ 0 & \frac{9 - 2\sqrt{5}}{\sqrt{70}} & \frac{-3\sqrt{35} - 6\sqrt{7}}{7\sqrt{10}} \\ 0 & \frac{2\sqrt{5} - 3}{\sqrt{14}} & \frac{3\sqrt{2} + \sqrt{\frac{5}{2}}}{\sqrt{7}} \end{pmatrix}$$

$$Q^{(1)} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} & \frac{1}{\sqrt{14}} \\ -\frac{1}{\sqrt{5}} & \frac{3\sqrt{2}}{\sqrt{35}} & \sqrt{\frac{2}{7}} \\ 0 & -\sqrt{\frac{5}{14}} & \frac{3}{\sqrt{14}} \end{pmatrix}$$

So we get

$$A^{(2)} = R^{(1)}Q^{(1)} = \begin{pmatrix} \frac{-\sqrt{35}+12\sqrt{7}+70\sqrt{2}}{35\sqrt{2}} & \frac{210\sqrt{5}-70\sqrt{14}-87\sqrt{70}}{70\sqrt{70}} & \frac{35\sqrt{2}+14\sqrt{35}-15\sqrt{7}}{14\sqrt{35}} \\ -\frac{9-2\sqrt{5}}{5\sqrt{14}} & \frac{84\sqrt{35}+15\sqrt{7}}{70\sqrt{35}} & -\frac{13}{14} \\ -\frac{2\sqrt{5}-3}{\sqrt{70}} & \frac{7\sqrt{5}-48}{14\sqrt{5}} & \frac{3\sqrt{\frac{5}{2}}+2\sqrt{10}+6\sqrt{2}}{7\sqrt{2}} \end{pmatrix}$$

Question 8

(a) Determine the singular values of the following matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

(b) Determine a singular value decomposition for the above matrix

Solution 8

(a) Determine the singular values of the following matrix

Given the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

So,

$$A^T A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

For the equation, $\det\left(A^T A - \lambda I\right) = 0$, we have

$$\begin{aligned} \det\left(A^T A - \lambda I\right) = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} &= 0 \\ (2-\lambda)^2 &= 0 \\ \lambda &= 2 \\ \sigma = \sqrt{\lambda} &= \sqrt{2} \end{aligned}$$

(b) Determine a singular value decomposition for the above matrix

From (a), we have the matrix $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

The columns of the matrix U as the normalized vectors $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, $v_i = \frac{1}{\sigma_i} \cdot A^T \cdot u_i$

Therefore,

$$V = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

The matrices U, Σ, V are such that the initial matrix $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = U\Sigma V^T$