Ambit – A FEniCS-based cardiovascular physics solver

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1 Solid mechanics

- solid mechanics are formulated in a Total Lagrangian frame

- displacement-based strong form: primary variable $\boldsymbol{u}$

$$\begin{aligned}
\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{u},\boldsymbol{v}) + \hat{\boldsymbol{b}}_0 &= \rho_0 \boldsymbol{a} && \text{in } \mathit{\Omega}_0 \times [0,T], && (1)\\
\boldsymbol{u} &= \hat{\boldsymbol{u}} && \text{on } \mathit{\Gamma}_0^{\mathrm{D}} \times [0,T], && (2)\\
\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 &= \hat{\boldsymbol{t}}_0 && \text{on } \mathit{\Gamma}_0^{\mathrm{N}} \times [0,T], && (3)\\
\boldsymbol{u}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{u}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (4)\\
\boldsymbol{v}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (5)
\end{aligned}$$

- strong form for incompressible solid mechanics: primary variables $\boldsymbol{u}$ and p

$$\begin{aligned}
\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{u},p,\boldsymbol{v}) + \hat{\boldsymbol{b}}_0 &= \rho_0 \boldsymbol{a} && \text{in } \mathit{\Omega}_0 \times [0,T], && (6)\\
J-1 &= 0 && \text{in } \mathit{\Omega}_0 \times [0,T], && (7)\\
\boldsymbol{u} &= \hat{\boldsymbol{u}} && \text{on } \mathit{\Gamma}_0^{\mathrm{D}} \times [0,T], && (8)\\
\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 &= \hat{\boldsymbol{t}}_0 && \text{on } \mathit{\Gamma}_0^{\mathrm{N}} \times [0,T], && (9)\\
\boldsymbol{u}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{u}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (10)\\
\boldsymbol{v}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (11)
\end{aligned}$$

with velocity and acceleration $\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$ and $\boldsymbol{a} = \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}$, respectively

2 Fluid mechanics

- incompressible Navier-Stokes equations in Eulerian reference frame

- strong form with primary variables velocity $\boldsymbol{v}$ and pressure p

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{v},p)+\hat{\boldsymbol{b}} &= \rho\boldsymbol{a} && \text{in } \Omega\times[0,T], && (12)\\
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 && \text{in } \Omega\times[0,T], && (13)\\
\boldsymbol{v} &= \hat{\boldsymbol{v}} && \text{on } \Gamma^{\mathrm{D}}\times[0,T], && (14)\\
\boldsymbol{t}=\boldsymbol{\sigma}\boldsymbol{n} &= \hat{\boldsymbol{t}} && \text{on } \Gamma^{\mathrm{N}}\times[0,T], && (15)\\
\boldsymbol{v}(\boldsymbol{x},0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}) && \text{in } \Omega, && (16)
\end{aligned}$$

with acceleration $\boldsymbol{a}=\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{v}$

3 Lumped parameter (0D) models

3.1 "Syspul" circulation model

left heart and systemic circulation

$$\begin{aligned}
&-Q_{\mathrm{at}}^{\ell}=\sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{pul}}}q_{\mathrm{ven},i}^{\mathrm{pul}}-q_{\mathrm{v,in}}^{\ell} && \text{left atrium flow balance}\\
&q_{\mathrm{v,in}}^{\ell}=q_{\mathrm{mv}}(p_{\mathrm{at}}^{\ell}-p_{\mathrm{v}}^{\ell}) && \text{mitral valve momentum} && (17)\\
&-Q_{\mathrm{v}}^{\ell}=q_{\mathrm{v,in}}^{\ell}-q_{\mathrm{v,out}}^{\ell} && \text{left ventricle flow balance}\\
&q_{\mathrm{v,out}}^{\ell}=q_{\mathrm{av}}(p_{\mathrm{v}}^{\ell}-p_{\mathrm{ar}}^{\mathrm{sys}}) && \text{aortic valve momentum} && (18)\\
&-Q_{\mathrm{aort}}^{\mathrm{sys}}=q_{\mathrm{v,out}}^{\ell}-q_{\mathrm{ar,p}}^{\mathrm{sys}}-\mathbb{I}^{\mathrm{cor}}\sum_{i=1}^{2}q_{\mathrm{ar,cor,in},i}^{\mathrm{sys}} && \text{aortic root flow balance}\\
&I_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t}+Z_{\mathrm{ar}}^{\mathrm{sys}}\,q_{\mathrm{ar,p}}^{\mathrm{sys}}=p_{\mathrm{ar}}^{\mathrm{sys}}-p_{\mathrm{ar,d}}^{\mathrm{sys}} && \text{aortic root inertia}\\
&C_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t}=q_{\mathrm{ar,p}}^{\mathrm{sys}}-q_{\mathrm{ar}}^{\mathrm{sys}} && \text{systemic arterial flow balance}\\
&L_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t}+R_{\mathrm{ar}}^{\mathrm{sys}}\,q_{\mathrm{ar}}^{\mathrm{sys}}=p_{\mathrm{ar,d}}^{\mathrm{sys}}-p_{\mathrm{ven}}^{\mathrm{sys}} && \text{systemic arterial momentum}\\
&C_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t}=q_{\mathrm{ar}}^{\mathrm{sys}}-\sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{sys}}}q_{\mathrm{ven},i}^{\mathrm{sys}} && \text{systemic venous flow balance}\\
&L_{\mathrm{ven},i}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t}+R_{\mathrm{ven},i}^{\mathrm{sys}}\,q_{\mathrm{ven},i}^{\mathrm{sys}}=p_{\mathrm{ven}}^{\mathrm{sys}}-p_{\mathrm{at},i}^{r} && \text{systemic venous momentum}\\
&\qquad i\in\{1,\ldots,n_{\mathrm{ven}}^{\mathrm{sys}}\}
\end{aligned}$$

right heart and pulmonary circulation

$$\begin{aligned}
&-Q_{\mathrm{at}}^{r} = \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{sys}}} q_{\mathrm{ven},i}^{\mathrm{sys}} - \mathbb{I}^{\mathrm{cor}} q_{\mathrm{ven,cor,out}}^{\mathrm{sys}} - q_{\mathrm{v,in}}^{r} && \text{right atrium flow balance} \\
&q_{\mathrm{v,in}}^{r} = q_{\mathrm{tv}}(p_{\mathrm{at}}^{r} - p_{\mathrm{v}}^{r}) && \text{tricuspid valve momentum} && (19) \\
&-Q_{\mathrm{v}}^{r} = q_{\mathrm{v,in}}^{r} - q_{\mathrm{v,out}}^{r} && \text{right ventricle flow balance} \\
&q_{\mathrm{v,out}}^{r} = q_{\mathrm{pv}}(p_{\mathrm{v}}^{r} - p_{\mathrm{ar}}^{\mathrm{pul}}) && \text{pulmonary valve momentum} && (20) \\
&C_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{v,out}}^{r} - q_{\mathrm{ar}}^{\mathrm{pul}} && \text{pulmonary arterial flow balance} \\
&L_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{pul}}\, q_{\mathrm{ar}}^{\mathrm{pul}} = p_{\mathrm{ar}}^{\mathrm{pul}} - p_{\mathrm{ven}}^{\mathrm{pul}} && \text{pulmonary arterial momentum} \\
&C_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{pul}} - \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{pul}}} q_{\mathrm{ven},i}^{\mathrm{pul}} && \text{pulmonary venous flow balance} \\
&L_{\mathrm{ven},i}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ven},i}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ven},i}^{\mathrm{pul}}\, q_{\mathrm{ven},i}^{\mathrm{pul}} = p_{\mathrm{ven}}^{\mathrm{pul}} - p_{\mathrm{at},i}^{\ell} && \text{pulmonary venous momentum} \\
&\qquad i \in \{1, ..., n_{\mathrm{ven}}^{\mathrm{pul}}\}
\end{aligned}$$

with:

$$Q_{\mathrm{at}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{at}}^{\ell}}{\mathrm{d}t}, \quad Q_{\mathrm{v}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{v}}^{\ell}}{\mathrm{d}t}, \quad Q_{\mathrm{at}}^{r} := -\frac{\mathrm{d}V_{\mathrm{at}}^{r}}{\mathrm{d}t}, \quad Q_{\mathrm{v}}^{r} := -\frac{\mathrm{d}V_{\mathrm{v}}^{r}}{\mathrm{d}t}, \quad Q_{\mathrm{aort}}^{\mathrm{sys}} := -\frac{\mathrm{d}V_{\mathrm{aort}}^{\mathrm{sys}}}{\mathrm{d}t}$$

and:

$$\mathbb{I}^{\mathrm{cor}} = \begin{cases} 1, & \text{if CORONARY\_MODEL}, \\ 0, & \text{else} \end{cases}$$

The volume V of the heart chambers (0D) is modeled by the volume-pressure relationship

$$V(t) = \frac{p}{E(t)} + V_{\mathrm{u}}, \tag{21}$$

with the unstressed volume V_{u} and the time-varying elastance

$$E(t) = (E_{\max} - E_{\min}) \cdot \hat{y}(t) + E_{\min}, \tag{22}$$

where $E_{\max}$ and $E_{\min}$ denote the maximum and minimum elastance, respectively. The normalized activation function $\hat{y}(t)$ is input by the user.

Flow-pressure relations for the four valves, eq. (17), (18), (19), (20), are functions of the pressure difference $p - p_{\mathrm{open}}$ across the valve. The following valve models can be defined:

Valve model `pwlin_pres`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} R_{\max}, & p < p_{\text{open}} \\ R_{\min}, & p \geq p_{\text{open}} \end{cases}$$

Remark: Non-smooth flow-pressure relationship

Valve model `pwlin_time`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} \begin{cases} R_{\max}, & t < t_{\text{open}} \text{ and } t \geq t_{\text{close}} \\ R_{\min}, & t \geq t_{\text{open}} \text{ or } t < t_{\text{close}} \end{cases}, & t_{\text{open}} > t_{\text{close}} \\ \begin{cases} R_{\max}, & t < t_{\text{open}} \text{ or } t \geq t_{\text{close}} \\ R_{\min}, & t \geq t_{\text{open}} \text{ and } t < t_{\text{close}} \end{cases}, & \text{else} \end{cases}$$

Remark: Non-smooth flow-pressure relationship with resistance only dependent on timings, not the pressure difference!

Valve model `smooth_pres_resistance`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = 0.5\,(R_{\max} - R_{\min})\left(\tanh\frac{p - p_{\text{open}}}{\epsilon} + 1\right) + R_{\min}$$

Remark: Smooth but potentially non-convex flow-pressure relationship!

Valve model `smooth_pres_momentum`:

$$q(p - p_{\text{open}}) = \begin{cases} \frac{p - p_{\text{open}}}{R_{\max}}, & p < p_{\text{open}} - 0.5\epsilon \\ h_{00}p_0 + h_{10}m_0\epsilon + h_{01}p_1 + h_{11}m_1\epsilon, & p \geq p_{\text{open}} - 0.5\epsilon \text{ and } p < p_{\text{open}} + 0.5\epsilon \\ \frac{p - p_{\text{open}}}{R_{\min}}, & p \geq p_{\text{open}} + 0.5\epsilon \end{cases}$$

with

$$p_0 = \frac{p_{\text{open}} - 0.5\epsilon - p}{R_{\max}}, \qquad m_0 = \frac{1}{R_{\max}}, \qquad p_1 = \frac{p_{\text{open}} + 0.5\epsilon - p}{R_{\min}}, \qquad m_1 = \frac{1}{R_{\min}}$$

and

$$\begin{aligned} h_{00} &= 2s^3 - 3s^2 + 1, & h_{01} &= -2s^3 + 3s^2, \\ h_{10} &= s^3 - 2s^2 + s, & h_{11} &= s^3 - s^2 \end{aligned}$$

with

$$s = \frac{p - p_{\text{open}} + 0.5\epsilon}{\epsilon}$$

Remarks:
− Collapses to valve model `pwlin_pres` for $\epsilon = 0$
− Smooth and convex flow-pressure relationship

Valve model `pw_pres_regurg`:

$$q(p - p_{\text{open}}) = \begin{cases} cA_{\text{o}}\sqrt{p - p_{\text{open}}}, & p < p_{\text{open}} \\ \frac{p - p_{\text{open}}}{R_{\text{min}}}, & p \geq p_{\text{open}} \end{cases}$$

Remark: Model to allow a regurgitant valve in the closed state, degree of regurgitation can be varied by specifying the valve regurgitant area A_{o}

Coronary circulation model:

$$\begin{aligned}
&C_{\text{cor,p}}^{\text{sys},\ell}\left(\frac{\mathrm{d}p_{\text{ar}}^{\text{sys},\ell}}{\mathrm{d}t} - Z_{\text{cor,p}}^{\text{sys},\ell}\frac{\mathrm{d}q_{\text{cor,p,in}}^{\text{sys},\ell}}{\mathrm{d}t}\right) = q_{\text{cor,p,in}}^{\text{sys},\ell} - q_{\text{cor,p}}^{\text{sys},\ell}\\
&R_{\text{cor,p}}^{\text{sys},\ell}\, q_{\text{cor,p}}^{\text{sys},\ell} = p_{\text{ar}}^{\text{sys}} - p_{\text{cor,d}}^{\text{sys},\ell} - Z_{\text{cor,p}}^{\text{sys},\ell}\, q_{\text{cor,p,in}}^{\text{sys},\ell}\\
&C_{\text{cor,d}}^{\text{sys},\ell}\frac{\mathrm{d}(p_{\text{cor,d}}^{\text{sys},\ell} - p_{\text{v}}^{\ell})}{\mathrm{d}t} = q_{\text{cor,p}}^{\text{sys},\ell} - q_{\text{cor,d}}^{\text{sys},\ell}\\
&R_{\text{cor,d}}^{\text{sys},\ell}\, q_{\text{cor,d}}^{\text{sys},\ell} = p_{\text{cor,d}}^{\text{sys},\ell} - p_{\text{at}}^{r}\\
&C_{\text{cor,p}}^{\text{sys},r}\left(\frac{\mathrm{d}p_{\text{ar}}^{\text{sys},r}}{\mathrm{d}t} - Z_{\text{cor,p}}^{\text{sys},r}\frac{\mathrm{d}q_{\text{cor,p,in}}^{\text{sys},r}}{\mathrm{d}t}\right) = q_{\text{cor,p,in}}^{\text{sys},r} - q_{\text{cor,p}}^{\text{sys},r}\\
&R_{\text{cor,p}}^{\text{sys},r}\, q_{\text{cor,p}}^{\text{sys},r} = p_{\text{ar}}^{\text{sys}} - p_{\text{cor,d}}^{\text{sys},r} - Z_{\text{cor,p}}^{\text{sys},r}\, q_{\text{cor,p,in}}^{\text{sys},r}\\
&C_{\text{cor,d}}^{\text{sys},r}\frac{\mathrm{d}(p_{\text{cor,d}}^{\text{sys},r} - p_{\text{v}}^{\ell})}{\mathrm{d}t} = q_{\text{cor,p}}^{\text{sys},r} - q_{\text{cor,d}}^{\text{sys},r}\\
&R_{\text{cor,d}}^{\text{sys},r}\, q_{\text{cor,d}}^{\text{sys},r} = p_{\text{cor,d}}^{\text{sys},r} - p_{\text{at}}^{r}\\
&0 = q_{\text{cor,d}}^{\text{sys},\ell} + q_{\text{cor,d}}^{\text{sys},r} - q_{\text{cor,d,out}}^{\text{sys}}
\end{aligned}$$

3.2 "Syspulcap" circulation model

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{\ell}=q_{\mathrm{ven}}^{\mathrm{pul}}-q_{\mathrm{v,in}}^{\ell}\\
&\tilde{R}_{\mathrm{v,in}}^{\ell}\,q_{\mathrm{v,in}}^{\ell}=p_{\mathrm{at}}^{\ell}-p_{\mathrm{v}}^{\ell}\\
&-Q_{\mathrm{v}}^{\ell}=q_{\mathrm{v,in}}^{\ell}-q_{\mathrm{v,out}}^{\ell}\\
&\tilde{R}_{\mathrm{v,out}}^{\ell}\,q_{\mathrm{v,out}}^{\ell}=p_{\mathrm{v}}^{\ell}-p_{\mathrm{ar}}^{\mathrm{sys}}\\
&0=q_{\mathrm{v,out}}^{\ell}-q_{\mathrm{ar,p}}^{\mathrm{sys}}\\
&I_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t}+Z_{\mathrm{ar}}^{\mathrm{sys}}\,q_{\mathrm{ar,p}}^{\mathrm{sys}}=p_{\mathrm{ar}}^{\mathrm{sys}}-p_{\mathrm{ar,d}}^{\mathrm{sys}}\\
&C_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t}=q_{\mathrm{ar,p}}^{\mathrm{sys}}-q_{\mathrm{ar}}^{\mathrm{sys}}\\
&L_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t}+R_{\mathrm{ar}}^{\mathrm{sys}}\,q_{\mathrm{ar}}^{\mathrm{sys}}=p_{\mathrm{ar,d}}^{\mathrm{sys}}-p_{\mathrm{ar,peri}}^{\mathrm{sys}}\\
&\left(\sum_{j\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer,cor}}\right\}}C_{\mathrm{ar},j}^{\mathrm{sys}}\right)\frac{\mathrm{d}p_{\mathrm{ar,peri}}^{\mathrm{sys}}}{\mathrm{d}t}=q_{\mathrm{ar}}^{\mathrm{sys}}-\sum_{j\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer,cor}}\right\}}q_{\mathrm{ar},j}^{\mathrm{sys}}\\
&R_{\mathrm{ar},i}^{\mathrm{sys}}\,q_{\mathrm{ar},i}^{\mathrm{sys}}=p_{\mathrm{ar,peri}}^{\mathrm{sys}}-p_{\mathrm{ven},i}^{\mathrm{sys}},\quad i\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer,cor}}\right\}\\
&C_{\mathrm{ven},i}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t}=q_{\mathrm{ar},i}^{\mathrm{sys}}-q_{\mathrm{ven},i}^{\mathrm{sys}},\quad i\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer,cor}}\right\}\\
&R_{\mathrm{ven},i}^{\mathrm{sys}}\,q_{\mathrm{ven},i}^{\mathrm{sys}}=p_{\mathrm{ven},i}^{\mathrm{sys}}-p_{\mathrm{ven}}^{\mathrm{sys}},\quad i\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer,cor}}\right\}\\
&C_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t}=\sum_{j=\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer,cor}}}q_{\mathrm{ven},j}^{\mathrm{sys}}-q_{\mathrm{ven}}^{\mathrm{sys}}\\
&L_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t}+R_{\mathrm{ven}}^{\mathrm{sys}}\,q_{\mathrm{ven}}^{\mathrm{sys}}=p_{\mathrm{ven}}^{\mathrm{sys}}-p_{\mathrm{at}}^{r}
\end{aligned}
$$

$$
\begin{aligned}
&-Q^r_{\mathrm{at}} = q^{\mathrm{sys}}_{\mathrm{ven}} - q^r_{\mathrm{v,in}} \\
&\tilde{R}^r_{\mathrm{v,in}}\, q^r_{\mathrm{v,in}} = p^r_{\mathrm{at}} - p^r_{\mathrm{v}} \\
&-Q^r_{\mathrm{v}} = q^r_{\mathrm{v,in}} - q^r_{\mathrm{v,out}} \\
&\tilde{R}^r_{\mathrm{v,out}}\, q^r_{\mathrm{v,out}} = p^r_{\mathrm{v}} - p^{\mathrm{pul}}_{\mathrm{ar}} \\
&C^{\mathrm{pul}}_{\mathrm{ar}} \frac{\mathrm{d}p^{\mathrm{pul}}_{\mathrm{ar}}}{\mathrm{d}t} = q^r_{\mathrm{v,out}} - q^{\mathrm{pul}}_{\mathrm{ar}} \\
&L^{\mathrm{pul}}_{\mathrm{ar}} \frac{\mathrm{d}q^{\mathrm{pul}}_{\mathrm{ar}}}{\mathrm{d}t} + R^{\mathrm{pul}}_{\mathrm{ar}}\, q^{\mathrm{pul}}_{\mathrm{ar}} = p^{\mathrm{pul}}_{\mathrm{ar}} - p^{\mathrm{pul}}_{\mathrm{cap}} \\
&C^{\mathrm{pul}}_{\mathrm{cap}} \frac{\mathrm{d}p^{\mathrm{pul}}_{\mathrm{cap}}}{\mathrm{d}t} = q^{\mathrm{pul}}_{\mathrm{ar}} - q^{\mathrm{pul}}_{\mathrm{cap}} \\
&R^{\mathrm{pul}}_{\mathrm{cap}}\, q^{\mathrm{pul}}_{\mathrm{cap}} = p^{\mathrm{pul}}_{\mathrm{cap}} - p^{\mathrm{pul}}_{\mathrm{ven}} \\
&C^{\mathrm{pul}}_{\mathrm{ven}} \frac{\mathrm{d}p^{\mathrm{pul}}_{\mathrm{ven}}}{\mathrm{d}t} = q^{\mathrm{pul}}_{\mathrm{cap}} - q^{\mathrm{pul}}_{\mathrm{ven}} \\
&L^{\mathrm{pul}}_{\mathrm{ven}} \frac{\mathrm{d}q^{\mathrm{pul}}_{\mathrm{ven}}}{\mathrm{d}t} + R^{\mathrm{pul}}_{\mathrm{ven}}\, q^{\mathrm{pul}}_{\mathrm{ven}} = p^{\mathrm{pul}}_{\mathrm{ven}} - p^{\ell}_{\mathrm{at}}
\end{aligned}
$$

with:

$$
Q^{\ell}_{\mathrm{at}} := -\frac{\mathrm{d}V^{\ell}_{\mathrm{at}}}{\mathrm{d}t}, \qquad Q^{\ell}_{\mathrm{v}} := -\frac{\mathrm{d}V^{\ell}_{\mathrm{v}}}{\mathrm{d}t}, \qquad Q^{r}_{\mathrm{at}} := -\frac{\mathrm{d}V^{r}_{\mathrm{at}}}{\mathrm{d}t}, \qquad Q^{r}_{\mathrm{v}} := -\frac{\mathrm{d}V^{r}_{\mathrm{v}}}{\mathrm{d}t}
$$

3.3 "Syspulcapcor" circulation model

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{\ell} = q_{\mathrm{ven}}^{\mathrm{pul}} - q_{\mathrm{v,in}}^{\ell}\\
&\tilde{R}_{\mathrm{v,in}}^{\ell}\, q_{\mathrm{v,in}}^{\ell} = p_{\mathrm{at}}^{\ell} - p_{\mathrm{v}}^{\ell}\\
&-Q_{\mathrm{v}}^{\ell} = q_{\mathrm{v,in}}^{\ell} - q_{\mathrm{v,out}}^{\ell}\\
&\tilde{R}_{\mathrm{v,out}}^{\ell}\, q_{\mathrm{v,out}}^{\ell} = p_{\mathrm{v}}^{\ell} - p_{\mathrm{ar}}^{\mathrm{sys}}\\
&0 = q_{\mathrm{v,out}}^{\ell} - q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar,cor,in}}^{\mathrm{sys}}\\
&I_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t} + Z_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar,p}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ar,d}}^{\mathrm{sys}}\\
&C_{\mathrm{ar,cor}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,cor,in}}^{\mathrm{sys}} - q_{\mathrm{ar,cor}}^{\mathrm{sys}}\\
&R_{\mathrm{ar,cor}}^{\mathrm{sys}}\, q_{\mathrm{ar,cor}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ven,cor}}^{\mathrm{sys}}\\
&C_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar}}^{\mathrm{sys}}\\
&L_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar}}^{\mathrm{sys}} = p_{\mathrm{ar,d}}^{\mathrm{sys}} - p_{\mathrm{ar,peri}}^{\mathrm{sys}}\\
&\left(\sum_{j\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer}}\right\}} C_{\mathrm{ar},j}^{\mathrm{sys}}\right) \frac{\mathrm{d}p_{\mathrm{ar,peri}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{sys}} - \sum_{j\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer}}\right\}} q_{\mathrm{ar},j}^{\mathrm{sys}}\\
&R_{\mathrm{ar},i}^{\mathrm{sys}}\, q_{\mathrm{ar},i}^{\mathrm{sys}} = p_{\mathrm{ar,peri}}^{\mathrm{sys}} - p_{\mathrm{ven},i}^{\mathrm{sys}}, \quad {\scriptstyle i\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer}}\right\}}\\
&C_{\mathrm{ven},i}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar},i}^{\mathrm{sys}} - q_{\mathrm{ven},i}^{\mathrm{sys}}, \quad {\scriptstyle i\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer}}\right\}}\\
&R_{\mathrm{ven},i}^{\mathrm{sys}}\, q_{\mathrm{ven},i}^{\mathrm{sys}} = p_{\mathrm{ven},i}^{\mathrm{sys}} - p_{\mathrm{ven}}^{\mathrm{sys}}, \quad {\scriptstyle i\in\left\{\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer}}\right\}}\\
&C_{\mathrm{ven}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} = \sum_{j=\substack{\mathrm{spl,espl,}\\\mathrm{msc,cer}}} q_{\mathrm{ven},j}^{\mathrm{sys}} - q_{\mathrm{ven}}^{\mathrm{sys}}\\
&L_{\mathrm{ven}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ven}}^{\mathrm{sys}}\, q_{\mathrm{ven}}^{\mathrm{sys}} = p_{\mathrm{ven}}^{\mathrm{sys}} - p_{\mathrm{at}}^{r}\\
&C_{\mathrm{ven,cor}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ven,cor}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,cor}}^{\mathrm{sys}} - q_{\mathrm{ven,cor}}^{\mathrm{sys}}\\
&R_{\mathrm{ven,cor}}^{\mathrm{sys}}\, q_{\mathrm{ven,cor}}^{\mathrm{sys}} = p_{\mathrm{ven,cor}}^{\mathrm{sys}} - p_{\mathrm{at}}^{r}
\end{aligned}
$$

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{r} = q_{\mathrm{ven}}^{\mathrm{sys}} + q_{\mathrm{ven,cor}}^{\mathrm{sys}} - q_{\mathrm{v,in}}^{r}\\
&\tilde{R}_{\mathrm{v,in}}^{r}\, q_{\mathrm{v,in}}^{r} = p_{\mathrm{at}}^{r} - p_{\mathrm{v}}^{r}\\
&-Q_{\mathrm{v}}^{r} = q_{\mathrm{v,in}}^{r} - q_{\mathrm{v,out}}^{r}\\
&\tilde{R}_{\mathrm{v,out}}^{r}\, q_{\mathrm{v,out}}^{r} = p_{\mathrm{v}}^{r} - p_{\mathrm{ar}}^{\mathrm{pul}}\\
&C_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{v,out}}^{r} - q_{\mathrm{ar}}^{\mathrm{pul}}\\
&L_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{pul}}\, q_{\mathrm{ar}}^{\mathrm{pul}} = p_{\mathrm{ar}}^{\mathrm{pul}} - p_{\mathrm{cap}}^{\mathrm{pul}}\\
&C_{\mathrm{cap}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{cap}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{pul}} - q_{\mathrm{cap}}^{\mathrm{pul}}\\
&R_{\mathrm{cap}}^{\mathrm{pul}}\, q_{\mathrm{cap}}^{\mathrm{pul}} = p_{\mathrm{cap}}^{\mathrm{pul}} - p_{\mathrm{ven}}^{\mathrm{pul}}\\
&C_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{cap}}^{\mathrm{pul}} - q_{\mathrm{ven}}^{\mathrm{pul}}\\
&L_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ven}}^{\mathrm{pul}}\, q_{\mathrm{ven}}^{\mathrm{pul}} = p_{\mathrm{ven}}^{\mathrm{pul}} - p_{\mathrm{at}}^{\ell}
\end{aligned}
$$

with:

$$
Q_{\mathrm{at}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{at}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\mathrm{v}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{v}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\mathrm{at}}^{r} := -\frac{\mathrm{d}V_{\mathrm{at}}^{r}}{\mathrm{d}t}, \qquad Q_{\mathrm{v}}^{r} := -\frac{\mathrm{d}V_{\mathrm{v}}^{r}}{\mathrm{d}t}
$$