Ambit – A FEniCS-based cardiovascular physics solver

Dr.-Ing. Marc Hirschvogel

April 25, 2023

Contents

1 Solid mechanics

- solid mechanics are formulated in a Total Lagrangian frame

- displacement-based strong form: primary variable $\boldsymbol{u}$

$$\begin{aligned}
\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{u},\boldsymbol{v}) + \hat{\boldsymbol{b}}_0 &= \rho_0 \boldsymbol{a} && \text{in } \Omega_0 \times [0,T], && (1)\\
\boldsymbol{u} &= \hat{\boldsymbol{u}} && \text{on } \Gamma_0^{\mathrm{D}} \times [0,T], && (2)\\
\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 &= \hat{\boldsymbol{t}}_0 && \text{on } \Gamma_0^{\mathrm{N}} \times [0,T], && (3)\\
\boldsymbol{u}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{u}}_0(\boldsymbol{x}_0) && \text{in } \Omega_0, && (4)\\
\boldsymbol{v}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}_0) && \text{in } \Omega_0, && (5)
\end{aligned}$$

- strong form for incompressible solid mechanics: primary variables $\boldsymbol{u}$ and p

$$
\begin{aligned}
\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{u},p,\boldsymbol{v}) + \hat{\boldsymbol{b}}_0 &= \rho_0 \boldsymbol{a} && \text{in } \mathit{\Omega}_0 \times [0,T], && (6)\\
J - 1 &= 0 && \text{in } \mathit{\Omega}_0 \times [0,T], && (7)\\
\boldsymbol{u} &= \hat{\boldsymbol{u}} && \text{on } \mathit{\Gamma}_0^{\mathrm{D}} \times [0,T], && (8)\\
\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 &= \hat{\boldsymbol{t}}_0 && \text{on } \mathit{\Gamma}_0^{\mathrm{N}} \times [0,T], && (9)\\
\boldsymbol{u}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{u}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (10)\\
\boldsymbol{v}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (11)
\end{aligned}
$$

with velocity and acceleration $\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$ and $\boldsymbol{a} = \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}$, respectively

2 Fluid mechanics

2.1 Eulerian reference frame

- incompressible Navier-Stokes equations in Eulerian reference frame

- strong form with primary variables velocity $\boldsymbol{v}$ and pressure p

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{v},p) + \hat{\boldsymbol{b}} &= \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{v}\right) && \text{in } \mathit{\Omega} \times [0,T], && (12)\\
\boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 && \text{in } \mathit{\Omega} \times [0,T], && (13)\\
\boldsymbol{v} &= \hat{\boldsymbol{v}} && \text{on } \mathit{\Gamma}^{\mathrm{D}} \times [0,T], && (14)\\
\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} &= \hat{\boldsymbol{t}} && \text{on } \mathit{\Gamma}^{\mathrm{N}} \times [0,T], && (15)\\
\boldsymbol{v}(\boldsymbol{x},0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}) && \text{in } \mathit{\Omega}, && (16)
\end{aligned}
$$

2.2 ALE reference frame

- to be written...

3 Fluid-reduced-solid-interaction (FrSI)

3.1 Physics-reduced solid

$$
\mathit{\Omega}_0^{\mathrm{s}} \longmapsto \mathit{\Gamma}_0^{\text{f-s}}, \qquad \rightsquigarrow \int\limits_{\mathit{\Omega}_0^{\mathrm{s}}} (\cdot)\,\mathrm{d}V_0 = \int\limits_{\mathit{\Gamma}_0^{\text{f-s}}} h_0\,(\cdot)\,\mathrm{d}A_0, \tag{17}
$$

h_0: reduced solid's wall thickness parameter

3.1.1 Kinematics

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}} \boldsymbol{x} = \boldsymbol{I} + \nabla_{\boldsymbol{X}} \boldsymbol{u}_{\mathrm{f}}, \quad \dot{\boldsymbol{F}} = \nabla_{\boldsymbol{X}} \boldsymbol{v}_{\mathrm{f}}, \quad \boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}} \boldsymbol{F}, \tag{18}$$

- fluid domain displacement:

$$\boldsymbol{u}_{\mathrm{f}} = \int_0^t \boldsymbol{v}_{\mathrm{f}}(\boldsymbol{X}, t) \,\mathrm{d}t \tag{19}$$

- in-plane deformation and rate of deformation gradient:

$$\boldsymbol{F}^0 = \boldsymbol{F} - \boldsymbol{F}\boldsymbol{n}_0 \otimes \boldsymbol{n}_0 \quad \text{and} \quad \dot{\boldsymbol{F}}^0 = \dot{\boldsymbol{F}} - \dot{\boldsymbol{F}}\boldsymbol{n}_0 \otimes \boldsymbol{n}_0, \tag{20}$$

- plane strain representation of the right Cauchy-Green tensor:

$$\boldsymbol{C}^{||} = \boldsymbol{F}^{0^{\mathrm{T}}} \boldsymbol{F}^0 + \boldsymbol{n}_0 \otimes \boldsymbol{n}_0 \tag{21}$$

- relation of in-plane and out-of-plane stretches

$$I\!I\!I_C^{||} = \det \boldsymbol{C}^{||} = \lambda_\xi^2 \lambda_\eta^2 = \frac{1}{\lambda_\zeta^2}, \qquad \rightsquigarrow \lambda_\zeta = \frac{1}{\sqrt{I\!I\!I_C^{||}}} \tag{22}$$

- membrane right Cauchy-Green deformation tensor

$$\tilde{\boldsymbol{C}} = \boldsymbol{F}^{0^{\mathrm{T}}} \boldsymbol{F}^0 + \frac{1}{I\!I\!I_C^{||}} \boldsymbol{n}_0 \otimes \boldsymbol{n}_0. \tag{23}$$

- rate:

$$\dot{\tilde{\boldsymbol{C}}} = \dot{\boldsymbol{F}}^{0^{\mathrm{T}}} \boldsymbol{F}^0 + \boldsymbol{F}^{0^{\mathrm{T}}} \dot{\boldsymbol{F}}^0 - \frac{\dot{I\!I\!I}_C^{||}}{I\!I\!I_C^{||^2}} \boldsymbol{n}_0 \otimes \boldsymbol{n}_0, \tag{24}$$

- time derivative of Eq. (22) with Jacobi's formula:

$$\dot{I\!I\!I}_C^{||} = \dot{\overline{\det \boldsymbol{C}^{||}}} = \det \boldsymbol{C}^{||} \operatorname{tr}\left(\boldsymbol{C}^{||^{-1}} \dot{\boldsymbol{C}}^{||} \right) \tag{25}$$

$$= I\!I\!I_C^{||} \operatorname{tr}\left(\boldsymbol{C}^{||^{-1}} \left(\dot{\boldsymbol{F}}^{0^{\mathrm{T}}} \boldsymbol{F}^0 + \boldsymbol{F}^{0^{\mathrm{T}}} \dot{\boldsymbol{F}}^0 \right) \right). \tag{26}$$

3.1.2 Constitutive equations

General isotropic hyperelasticity - exemplified for general isotropic hyperelasticity

$$\Psi = \Psi(I_{\tilde{C}}, I\!I_{\tilde{C}}) - \frac{1}{2} p_{\mathrm{s}} (I\!I\!I_{\tilde{C}} - 1), \tag{27}$$

where $I_{\tilde{C}}$, $I\!I_{\tilde{C}}$ and $I\!I\!I_{\tilde{C}}$ are the principal invariants of Eq. (23)

- 2nd Piola-Kirchhoff stress [3]:

$$\tilde{\boldsymbol{S}} = -p_{\mathrm{s}}\tilde{\boldsymbol{C}}^{-1} + 2\left(\frac{\partial \Psi}{\partial I_{\tilde{C}}} + I_{\tilde{C}}\frac{\partial \Psi}{\partial I\!I_{\tilde{C}}}\right)\boldsymbol{I} - 2\frac{\partial \Psi}{\partial I\!I_{\tilde{C}}}\tilde{\boldsymbol{C}} \tag{28}$$

- hydrostatic pressure p_{s} recovered by plane stress assumption for 2-dimensional continua [3, 4]:

$$p_{\mathrm{s}} = 2\left(\frac{1}{\lambda_\xi^2\lambda_\eta^2}\frac{\partial \Psi}{\partial I_{\tilde{C}}} - \lambda_\xi^2\lambda_\eta^2\frac{\partial \Psi}{\partial I\!I_{\tilde{C}}}\right) = 2\left(\frac{1}{I\!I\!I_C^{||}}\frac{\partial \Psi}{\partial I_{\tilde{C}}} - I\!I\!I_C^{||}\frac{\partial \Psi}{\partial I\!I_{\tilde{C}}}\right), \tag{29}$$

- 1st Piola-Kirchhoff stress then is computed by the push-forward operation

$$\tilde{\boldsymbol{P}}_{\mathrm{s}} = \boldsymbol{F}^0\tilde{\boldsymbol{S}}, \tag{30}$$

Cardiac mechanics - isotropic exponential strain energy [2]

$$\Psi(\boldsymbol{C}, p_{\mathrm{s}}) = \frac{a_0}{2b_0}\left(e^{b_0(\mathrm{tr}\boldsymbol{C}-3)} - 1\right) - \frac{1}{2}p_{\mathrm{s}}(\det\boldsymbol{C} - 1) \tag{31}$$

- viscous pseudo-potential [1]

$$\Psi_{\mathrm{v}}(\dot{\boldsymbol{C}}) = \frac{\eta}{8}\dot{\boldsymbol{C}} : \dot{\boldsymbol{C}} \tag{32}$$

- for 3D incompressible hyper-viscoelastic active cardiac mechanics:

$$\boldsymbol{S} = \boldsymbol{S}_{\mathrm{e}} + \boldsymbol{S}_{\mathrm{v}} + \boldsymbol{S}_{\mathrm{a}} = 2\frac{\partial \Psi(\boldsymbol{C}, p_{\mathrm{s}})}{\partial \boldsymbol{C}} + 2\frac{\partial \Psi_{\mathrm{v}}(\dot{\boldsymbol{C}})}{\partial \dot{\boldsymbol{C}}} + \tau_{\mathrm{a}}(t)\boldsymbol{f}_0 \otimes \boldsymbol{f}_0 \tag{33}$$

- for reduced solid:

$$\tilde{\boldsymbol{S}} = \tilde{\boldsymbol{S}}_{\mathrm{e}} + \tilde{\boldsymbol{S}}_{\mathrm{v}} + \tilde{\boldsymbol{S}}_{\mathrm{a}} = 2\frac{\partial \Psi(\tilde{\boldsymbol{C}})}{\partial \tilde{\boldsymbol{C}}} + 2\frac{\partial \Psi_{\mathrm{v}}(\dot{\tilde{\boldsymbol{C}}})}{\partial \dot{\tilde{\boldsymbol{C}}}} + \tau_{\mathrm{a}}(t)\tilde{\boldsymbol{M}}_0 \tag{34}$$

- note that in presence of viscosity according to Eq. (32), hydrostatic pressure for hyperelastic solid, Eq. (29), needs to be updated to balance the viscous normal stresses:

$$p_{\mathrm{s}} \leftarrow p_{\mathrm{s}} - \frac{\eta}{2}\frac{\dot{I\!I\!I}_C^{||}}{{I\!I\!I_C^{||}}^3} \tag{35}$$

- reduced (wall-averaged) structural tensor in circumferential and longitudinal directions $\boldsymbol{c}_0$ and $\boldsymbol{l}_0$, respectively:

$$\tilde{\boldsymbol{M}}_0 = \bar{\omega}\,\boldsymbol{c}_0 \otimes \boldsymbol{c}_0 + \bar{\iota}\,\boldsymbol{l}_0 \otimes \boldsymbol{l}_0 + 2\bar{\gamma}\,\mathrm{sym}(\boldsymbol{c}_0 \otimes \boldsymbol{l}_0). \tag{36}$$

Balance equation

$$\tilde{r}_{\mathrm{s}}(\boldsymbol{v}_{\mathrm{f}};\delta\boldsymbol{v}_{\mathrm{f}}) := \int\limits_{\Gamma_0^{\text{f-s}}} h_0\,\rho_{0,\mathrm{s}}\,D_t\boldsymbol{v}_{\mathrm{f}}\cdot\delta\boldsymbol{v}_{\mathrm{f}}\,\mathrm{d}A_0 + \int\limits_{\Gamma_0^{\text{f-s}}} h_0\,\tilde{\boldsymbol{P}}_{\mathrm{s}}(\boldsymbol{u}_{\mathrm{f}}(\boldsymbol{v}_{\mathrm{f}}),\boldsymbol{v}_{\mathrm{f}}) : \nabla_{\tilde{\boldsymbol{X}}}\delta\boldsymbol{v}_{\mathrm{f}}\,\mathrm{d}A_0 \tag{37}$$

3.2 Projection-based reduction

$$\hat{\mathbf{S}}_v^{\Gamma} = \mathbb{I}_{\Gamma}\begin{bmatrix}\mathbf{v}_1^{\mathrm{f}} & \dots & \mathbf{v}_{m_v}^{\mathrm{f}}\end{bmatrix} = \begin{bmatrix}\mathbb{I}_{\Gamma}^1 & & & \\ & \mathbb{I}_{\Gamma}^2 & & \\ & & \ddots & \\ & & & \mathbb{I}_{\Gamma}^{n_v}\end{bmatrix}\begin{bmatrix} v_1^1 & \dots & v_{m_v}^1 \\ v_1^2 & \dots & v_{m_v}^2 \\ \vdots & \ddots & \vdots \\ v_1^{n_v} & \dots & v_{m_v}^{n_v}\end{bmatrix}, \tag{38}$$

with

$$\mathbb{I}_{\Gamma}^i = \begin{cases} 1, & \text{if } i \in \mathcal{N}_{\Gamma_0^{\text{f-s}}}, \\ 0, & \text{else,}\end{cases} \tag{39}$$

3.3 Discrete FSI equations

- original FSI-ALE-flow0d discrete residual problem: for staggered solve

$$\mathbf{r}_{n+1}^{\mathrm{FSI}} = \begin{bmatrix} \mathbf{r}_v^{\mathrm{s}}(\mathbf{v}^{\mathrm{s}},\mathbf{p}^{\mathrm{s}},\boldsymbol{\lambda}) \\ \mathbf{r}_p^{\mathrm{s}}(\mathbf{v}^{\mathrm{s}}) \\ \mathbf{r}_v^{\mathrm{f}}(\mathbf{v}^{\mathrm{f}},\mathbf{p}^{\mathrm{f}},\boldsymbol{\lambda},\boldsymbol{\Lambda}) \\ \mathbf{r}_p^{\mathrm{f}}(\mathbf{v}^{\mathrm{f}}) \\ \mathbf{r}_{\lambda}^{\text{f-s}}(\mathbf{v}^{\mathrm{s}},\mathbf{v}^{\mathrm{f}}) \\ \mathbf{r}_{\Lambda}^{\text{f-0d}}(\boldsymbol{\Lambda},\mathbf{v}^{\mathrm{f}})\end{bmatrix}_{n+1} = \mathbf{0}, \qquad \mathbf{r}_{n+1}^{\mathrm{ALE}}(\mathbf{w}) = \mathbf{0}, \qquad \mathbf{r}_{n+1}^{\mathrm{0d}}(\mathbf{y}) = \mathbf{0}, \tag{40}$$

- FSI linear system

$$\begin{bmatrix} \mathbf{K}_{vv}^{\mathrm{s}} & \mathbf{K}_{vp}^{\mathrm{s}} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{v\lambda}^{\mathrm{s}} & \mathbf{0} \\ \mathbf{K}_{pv}^{\mathrm{s}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{vv}^{\mathrm{f}} & \mathbf{K}_{vp}^{\mathrm{f}} & \mathbf{K}_{v\lambda}^{\mathrm{f}} & \mathbf{K}_{v\Lambda}^{\mathrm{f}} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{pv}^{\mathrm{f}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{\lambda v}^{\mathrm{s}} & \mathbf{0} & \mathbf{K}_{\lambda v}^{\mathrm{f}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{\Lambda v}^{\mathrm{f}} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\Lambda\Lambda}^{\text{f-0d}} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\mathbf{v}^{\mathrm{s}} \\ \Delta\mathbf{p}^{\mathrm{s}} \\ \Delta\mathbf{v}^{\mathrm{f}} \\ \Delta\mathbf{p}^{\mathrm{f}} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\Lambda} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{r}_v^{\mathrm{s}} \\ \mathbf{r}_p^{\mathrm{s}} \\ \mathbf{r}_v^{\mathrm{f}} \\ \mathbf{r}_p^{\mathrm{f}} \\ \mathbf{r}_{\lambda}^{\text{f-s}} \\ \mathbf{r}_{\Lambda}^{\text{f-0d}} \end{bmatrix}_{n+1}^{k} \tag{41}$$

- ALE and 0D linear systems:

$$\mathbf{K}_{n+1}^{\mathrm{ALE},k}\Delta\mathbf{w}_{n+1}^{k+1} = -\mathbf{r}_{n+1}^{\mathrm{ALE},k}, \qquad \mathbf{K}_{n+1}^{\mathrm{0d},k}\Delta\mathbf{y}_{n+1}^{k+1} = -\mathbf{r}_{n+1}^{\mathrm{0d},k} \tag{42}$$

3.4 Discrete F3DrSI equations

$$\mathbf{r}_{n+1}^{\mathrm{F3DrSI}} = \begin{bmatrix} \mathbf{V}_v^{\mathrm{T}}\mathbf{r}_v^{\mathrm{s}}(\mathbf{V}_v\tilde{\mathbf{v}}^{\mathrm{s}}, \mathbf{V}_p\tilde{\mathbf{p}}^{\mathrm{s}}, \boldsymbol{\lambda}) \\ \mathbf{V}_p^{\mathrm{T}}\mathbf{r}_p^{\mathrm{s}}(\mathbf{V}_v\tilde{\mathbf{v}}^{\mathrm{s}}) \\ \mathbf{r}_v^{\mathrm{f}}(\mathbf{v}^{\mathrm{f}}, \mathbf{p}^{\mathrm{f}}, \boldsymbol{\lambda}, \boldsymbol{\Lambda}) \\ \mathbf{r}_p^{\mathrm{f}}(\mathbf{v}^{\mathrm{f}}) \\ \mathbf{r}_\lambda^{\mathrm{f\text{-}s}}(\mathbf{V}_v\tilde{\mathbf{v}}^{\mathrm{s}}, \mathbf{v}^{\mathrm{f}}) \\ \mathbf{r}_\Lambda^{\mathrm{f\text{-}0d}}(\boldsymbol{\Lambda}, \mathbf{v}^{\mathrm{f}}) \end{bmatrix}_{n+1} = \mathbf{0}, \qquad \mathbf{r}_{n+1}^{\mathrm{ALE}}(\mathbf{w}) = \mathbf{0}, \qquad \mathbf{r}_{n+1}^{\mathrm{0d}}(\mathbf{y}) = \mathbf{0}. \tag{43}$$

$$\frac{\partial(\mathbf{V}_v^{\mathrm{T}}\mathbf{r}_v^{\mathrm{s}})}{\partial\tilde{\mathbf{v}}^{\mathrm{s}}} = \mathbf{V}_v^{\mathrm{T}}\frac{\partial\mathbf{r}_v^{\mathrm{s}}}{\partial\mathbf{v}^{\mathrm{s}}}\frac{\partial\mathbf{v}^{\mathrm{s}}}{\partial\tilde{\mathbf{v}}^{\mathrm{s}}} = \mathbf{V}_v^{\mathrm{T}}\mathbf{K}_{vv}^{\mathrm{s}}\mathbf{V}_v \tag{44}$$

$$\begin{bmatrix} \mathbf{V}_v^{\mathrm{T}}\mathbf{K}_{vv}^{\mathrm{s}}\mathbf{V}_v & \mathbf{V}_v^{\mathrm{T}}\mathbf{K}_{vp}^{\mathrm{s}}\mathbf{V}_p & \mathbf{0} & \mathbf{0} & \mathbf{V}_v^{\mathrm{T}}\mathbf{K}_{v\lambda}^{\mathrm{s}} & \mathbf{0} \\ \mathbf{V}_p^{\mathrm{T}}\mathbf{K}_{pv}^{\mathrm{s}}\mathbf{V}_v & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{vv}^{\mathrm{f}} & \mathbf{K}_{vp}^{\mathrm{f}} & \mathbf{K}_{v\lambda}^{\mathrm{f}} & \mathbf{K}_{v\Lambda}^{\mathrm{f}} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{pv}^{\mathrm{f}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{\lambda v}^{\mathrm{s}}\mathbf{V}_v & \mathbf{0} & \mathbf{K}_{\lambda v}^{\mathrm{f}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{\Lambda v}^{\mathrm{f}} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\Lambda\Lambda}^{\mathrm{f\text{-}0d}} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\tilde{\mathbf{v}}^{\mathrm{s}} \\ \Delta\tilde{\mathbf{p}}^{\mathrm{s}} \\ \Delta\mathbf{v}^{\mathrm{f}} \\ \Delta\mathbf{p}^{\mathrm{f}} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\Lambda} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{V}_v^{\mathrm{T}}\mathbf{r}_v^{\mathrm{s}} \\ \mathbf{V}_p^{\mathrm{T}}\mathbf{r}_p^{\mathrm{s}} \\ \mathbf{r}_v^{\mathrm{f}} \\ \mathbf{r}_p^{\mathrm{f}} \\ \mathbf{r}_\lambda^{\mathrm{f\text{-}s}} \\ \mathbf{r}_\Lambda^{\mathrm{f\text{-}0d}} \end{bmatrix}_{n+1}^{k} \tag{45}$$

- ALE and 0D linear systems: Eq. (41)

3.5 Discrete FrSI equations

- discrete fluid domain displacement:

$$\mathbf{u}_{n+1}^{\mathrm{f}} = \theta\Delta t\,\mathbf{v}_{n+1}^{\mathrm{f}} + (1-\theta)\Delta t\,\mathbf{v}_n^{\mathrm{f}} + \mathbf{u}_n^{\mathrm{f}} \tag{46}$$

$$\mathbf{r}_{n+1}^{\mathrm{FrSI}} = \begin{bmatrix} \mathbf{V}_v^{\Gamma\,\mathrm{T}}\mathbf{r}_v^{\mathrm{f}}(\mathbf{V}_v^{\Gamma}\tilde{\mathbf{v}}^{\mathrm{f}}, \mathbf{p}^{\mathrm{f}}, \boldsymbol{\Lambda}) \\ \mathbf{r}_p^{\mathrm{f}}(\mathbf{V}_v^{\Gamma}\tilde{\mathbf{v}}^{\mathrm{f}}) \\ \mathbf{r}_\Lambda^{\mathrm{f\text{-}0d}}(\boldsymbol{\Lambda}, \mathbf{V}_v^{\Gamma}\tilde{\mathbf{v}}^{\mathrm{f}}) \end{bmatrix}_{n+1} = \mathbf{0}, \qquad \mathbf{r}_{n+1}^{\mathrm{ALE}}(\mathbf{w}) = \mathbf{0}, \qquad \mathbf{r}_{n+1}^{\mathrm{0d}}(\mathbf{y}) = \mathbf{0} \tag{47}$$

$$
\begin{bmatrix} \mathbf{V}_v^{\Gamma\mathrm{T}} \mathbf{K}_{vv}^{\mathrm{f}} \mathbf{V}_v^{\Gamma} & \mathbf{V}_v^{\Gamma\mathrm{T}} \mathbf{K}_{vp}^{\mathrm{f}} & \mathbf{V}_v^{\Gamma\mathrm{T}} \mathbf{K}_{v\Lambda}^{\mathrm{f}} \\ \mathbf{K}_{pv}^{\mathrm{f}} \mathbf{V}_v^{\Gamma} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{\Lambda v}^{\mathrm{f}} \mathbf{V}_v^{\Gamma} & \mathbf{0} & \mathbf{K}_{\Lambda\Lambda}^{\mathrm{f\text{-}0d}} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\tilde{\mathbf{v}}^{\mathrm{f}} \\ \Delta\mathbf{p}^{\mathrm{f}} \\ \Delta\mathbf{\Lambda} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{V}_v^{\Gamma\mathrm{T}} \mathbf{r}_v^{\mathrm{f}} \\ \mathbf{r}_p^{\mathrm{f}} \\ \mathbf{r}_{\Lambda}^{\mathrm{f\text{-}0d}} \end{bmatrix}_{n+1}^{k} \qquad (48)
$$

- ALE and 0D linear systems: Eq. (41)

3.6 Reduced physics solid

4 Lumped parameter (0D) models

4.1 "Syspul" circulation model

left heart and systemic circulation

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{\ell} = \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{pul}}} q_{\mathrm{ven},i}^{\mathrm{pul}} - q_{\mathrm{v,in}}^{\ell} && \text{left atrium flow balance} \\
&q_{\mathrm{v,in}}^{\ell} = q_{\mathrm{mv}}(p_{\mathrm{at}}^{\ell} - p_{\mathrm{v}}^{\ell}) && \text{mitral valve momentum} \qquad (49) \\
&-Q_{\mathrm{v}}^{\ell} = q_{\mathrm{v,in}}^{\ell} - q_{\mathrm{v,out}}^{\ell} && \text{left ventricle flow balance} \\
&q_{\mathrm{v,out}}^{\ell} = q_{\mathrm{av}}(p_{\mathrm{v}}^{\ell} - p_{\mathrm{ar}}^{\mathrm{sys}}) && \text{aortic valve momentum} \qquad (50) \\
&-Q_{\mathrm{aort}}^{\mathrm{sys}} = q_{\mathrm{v,out}}^{\ell} - q_{\mathrm{ar,p}}^{\mathrm{sys}} - \mathbb{I}^{\mathrm{cor}} \sum_{i=1}^{2} q_{\mathrm{ar,cor,in},i}^{\mathrm{sys}} && \text{aortic root flow balance} \\
&I_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t} + Z_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar,p}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ar,d}}^{\mathrm{sys}} && \text{aortic root inertia} \\
&C_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar}}^{\mathrm{sys}} && \text{systemic arterial flow balance} \\
&L_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar}}^{\mathrm{sys}} = p_{\mathrm{ar,d}}^{\mathrm{sys}} - p_{\mathrm{ven}}^{\mathrm{sys}} && \text{systemic arterial momentum} \\
&C_{\mathrm{ven}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{sys}} - \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{sys}}} q_{\mathrm{ven},i}^{\mathrm{sys}} && \text{systemic venous flow balance} \\
&L_{\mathrm{ven},i}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ven},i}^{\mathrm{sys}}\, q_{\mathrm{ven},i}^{\mathrm{sys}} = p_{\mathrm{ven}}^{\mathrm{sys}} - p_{\mathrm{at},i}^{r} && \text{systemic venous momentum} \\
&\qquad i \in \{1, ..., n_{\mathrm{ven}}^{\mathrm{sys}}\}
\end{aligned}
$$

right heart and pulmonary circulation

$$\begin{aligned}
&-Q_{\mathrm{at}}^{r} = \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{sys}}} q_{\mathrm{ven},i}^{\mathrm{sys}} - \mathbb{I}^{\mathrm{cor}} q_{\mathrm{ven,cor,out}}^{\mathrm{sys}} - q_{\mathrm{v,in}}^{r} && \text{right atrium flow balance} \\
&q_{\mathrm{v,in}}^{r} = q_{\mathrm{tv}}(p_{\mathrm{at}}^{r} - p_{\mathrm{v}}^{r}) && \text{tricuspid valve momentum} && (51) \\
&-Q_{\mathrm{v}}^{r} = q_{\mathrm{v,in}}^{r} - q_{\mathrm{v,out}}^{r} && \text{right ventricle flow balance} \\
&q_{\mathrm{v,out}}^{r} = q_{\mathrm{pv}}(p_{\mathrm{v}}^{r} - p_{\mathrm{ar}}^{\mathrm{pul}}) && \text{pulmonary valve momentum} && (52) \\
&C_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{v,out}}^{r} - q_{\mathrm{ar}}^{\mathrm{pul}} && \text{pulmonary arterial flow balance} \\
&L_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{pul}}\, q_{\mathrm{ar}}^{\mathrm{pul}} = p_{\mathrm{ar}}^{\mathrm{pul}} - p_{\mathrm{ven}}^{\mathrm{pul}} && \text{pulmonary arterial momentum} \\
&C_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{pul}} - \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{pul}}} q_{\mathrm{ven},i}^{\mathrm{pul}} && \text{pulmonary venous flow balance} \\
&L_{\mathrm{ven},i}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ven},i}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ven},i}^{\mathrm{pul}}\, q_{\mathrm{ven},i}^{\mathrm{pul}} = p_{\mathrm{ven}}^{\mathrm{pul}} - p_{\mathrm{at},i}^{\ell} && \text{pulmonary venous momentum} \\
&\qquad i \in \{1, ..., n_{\mathrm{ven}}^{\mathrm{pul}}\}
\end{aligned}$$

with:

$$Q_{\mathrm{at}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{at}}^{\ell}}{\mathrm{d}t}, \quad Q_{\mathrm{v}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{v}}^{\ell}}{\mathrm{d}t}, \quad Q_{\mathrm{at}}^{r} := -\frac{\mathrm{d}V_{\mathrm{at}}^{r}}{\mathrm{d}t}, \quad Q_{\mathrm{v}}^{r} := -\frac{\mathrm{d}V_{\mathrm{v}}^{r}}{\mathrm{d}t}, \quad Q_{\mathrm{aort}}^{\mathrm{sys}} := -\frac{\mathrm{d}V_{\mathrm{aort}}^{\mathrm{sys}}}{\mathrm{d}t}$$

and:

$$\mathbb{I}^{\mathrm{cor}} = \begin{cases} 1, & \text{if} \quad \text{CORONARY\_MODEL}, \\ 0, & \text{else} \end{cases}$$

The volume V of the heart chambers (0D) is modeled by the volume-pressure relationship

$$V(t) = \frac{p}{E(t)} + V_{\mathrm{u}}, \tag{53}$$

with the unstressed volume V_{u} and the time-varying elastance

$$E(t) = (E_{\max} - E_{\min}) \cdot \hat{y}(t) + E_{\min}, \tag{54}$$

where $E_{\max}$ and $E_{\min}$ denote the maximum and minimum elastance, respectively. The normalized activation function $\hat{y}(t)$ is input by the user.

Flow-pressure relations for the four valves, eq. (48), (49), (50), (51), are functions of the pressure difference $p - p_{\mathrm{open}}$ across the valve. The following valve models can be defined:

Valve model `pwlin_pres`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} R_{\max}, & p < p_{\text{open}} \\ R_{\min}, & p \geq p_{\text{open}} \end{cases}$$

Remark: Non-smooth flow-pressure relationship

Valve model `pwlin_time`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} \begin{cases} R_{\max}, & t < t_{\text{open}} \text{ and } t \geq t_{\text{close}} \\ R_{\min}, & t \geq t_{\text{open}} \text{ or } t < t_{\text{close}} \end{cases}, & t_{\text{open}} > t_{\text{close}} \\ \begin{cases} R_{\max}, & t < t_{\text{open}} \text{ or } t \geq t_{\text{close}} \\ R_{\min}, & t \geq t_{\text{open}} \text{ and } t < t_{\text{close}} \end{cases}, & \text{else} \end{cases}$$

Remark: Non-smooth flow-pressure relationship with resistance only dependent on timings, not the pressure difference!

Valve model `smooth_pres_resistance`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = 0.5\,(R_{\max} - R_{\min})\left(\tanh\frac{p - p_{\text{open}}}{\epsilon} + 1\right) + R_{\min}$$

Remark: Smooth but potentially non-convex flow-pressure relationship!

Valve model `smooth_pres_momentum`:

$$q(p - p_{\text{open}}) = \begin{cases} \frac{p - p_{\text{open}}}{R_{\max}}, & p < p_{\text{open}} - 0.5\epsilon \\ h_{00}p_0 + h_{10}m_0\epsilon + h_{01}p_1 + h_{11}m_1\epsilon, & p \geq p_{\text{open}} - 0.5\epsilon \text{ and } p < p_{\text{open}} + 0.5\epsilon \\ \frac{p - p_{\text{open}}}{R_{\min}}, & p \geq p_{\text{open}} + 0.5\epsilon \end{cases}$$

with

$$p_0 = \frac{p_{\text{open}} - 0.5\epsilon - p}{R_{\max}}, \qquad m_0 = \frac{1}{R_{\max}}, \qquad p_1 = \frac{p_{\text{open}} + 0.5\epsilon - p}{R_{\min}}, \qquad m_1 = \frac{1}{R_{\min}}$$

and

$$\begin{aligned} h_{00} &= 2s^3 - 3s^2 + 1, & h_{01} &= -2s^3 + 3s^2, \\ h_{10} &= s^3 - 2s^2 + s, & h_{11} &= s^3 - s^2 \end{aligned}$$

with

$$s = \frac{p - p_{\text{open}} + 0.5\epsilon}{\epsilon}$$

Remarks:
– Collapses to valve model `pwlin_pres` for $\epsilon = 0$
– Smooth and convex flow-pressure relationship

Valve model `pw_pres_regurg`:

$$q(p - p_{\mathrm{open}}) = \begin{cases} cA_{\mathrm{o}}\sqrt{p - p_{\mathrm{open}}}, & p < p_{\mathrm{open}} \\ \frac{p - p_{\mathrm{open}}}{R_{\min}}, & p \geq p_{\mathrm{open}} \end{cases}$$

Remark: Model to allow a regurgitant valve in the closed state, degree of regurgitation can be varied by specifying the valve regurgitant area A_{o}

Coronary circulation model:

$$\begin{aligned}
&C_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\left(\frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{sys},\ell}}{\mathrm{d}t} - Z_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\frac{\mathrm{d}q_{\mathrm{cor,p,in}}^{\mathrm{sys},\ell}}{\mathrm{d}t}\right) = q_{\mathrm{cor,p,in}}^{\mathrm{sys},\ell} - q_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\\
&R_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\, q_{\mathrm{cor,p}}^{\mathrm{sys},\ell} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{cor,d}}^{\mathrm{sys},\ell} - Z_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\, q_{\mathrm{cor,p,in}}^{\mathrm{sys},\ell}\\
&C_{\mathrm{cor,d}}^{\mathrm{sys},\ell}\frac{\mathrm{d}(p_{\mathrm{cor,d}}^{\mathrm{sys},\ell} - p_{\mathrm{v}}^{\ell})}{\mathrm{d}t} = q_{\mathrm{cor,p}}^{\mathrm{sys},\ell} - q_{\mathrm{cor,d}}^{\mathrm{sys},\ell}\\
&R_{\mathrm{cor,d}}^{\mathrm{sys},\ell}\, q_{\mathrm{cor,d}}^{\mathrm{sys},\ell} = p_{\mathrm{cor,d}}^{\mathrm{sys},\ell} - p_{\mathrm{at}}^{r}\\
&C_{\mathrm{cor,p}}^{\mathrm{sys},r}\left(\frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{sys},r}}{\mathrm{d}t} - Z_{\mathrm{cor,p}}^{\mathrm{sys},r}\frac{\mathrm{d}q_{\mathrm{cor,p,in}}^{\mathrm{sys},r}}{\mathrm{d}t}\right) = q_{\mathrm{cor,p,in}}^{\mathrm{sys},r} - q_{\mathrm{cor,p}}^{\mathrm{sys},r}\\
&R_{\mathrm{cor,p}}^{\mathrm{sys},r}\, q_{\mathrm{cor,p}}^{\mathrm{sys},r} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{cor,d}}^{\mathrm{sys},r} - Z_{\mathrm{cor,p}}^{\mathrm{sys},r}\, q_{\mathrm{cor,p,in}}^{\mathrm{sys},r}\\
&C_{\mathrm{cor,d}}^{\mathrm{sys},r}\frac{\mathrm{d}(p_{\mathrm{cor,d}}^{\mathrm{sys},r} - p_{\mathrm{v}}^{\ell})}{\mathrm{d}t} = q_{\mathrm{cor,p}}^{\mathrm{sys},r} - q_{\mathrm{cor,d}}^{\mathrm{sys},r}\\
&R_{\mathrm{cor,d}}^{\mathrm{sys},r}\, q_{\mathrm{cor,d}}^{\mathrm{sys},r} = p_{\mathrm{cor,d}}^{\mathrm{sys},r} - p_{\mathrm{at}}^{r}\\
&0 = q_{\mathrm{cor,d}}^{\mathrm{sys},\ell} + q_{\mathrm{cor,d}}^{\mathrm{sys},r} - q_{\mathrm{cor,d,out}}^{\mathrm{sys}}
\end{aligned}$$

4.2 "Syspulcap" circulation model

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{\ell} = q_{\mathrm{ven}}^{\mathrm{pul}} - q_{\mathrm{v,in}}^{\ell}\\
&\tilde{R}_{\mathrm{v,in}}^{\ell}\, q_{\mathrm{v,in}}^{\ell} = p_{\mathrm{at}}^{\ell} - p_{\mathrm{v}}^{\ell}\\
&-Q_{\mathrm{v}}^{\ell} = q_{\mathrm{v,in}}^{\ell} - q_{\mathrm{v,out}}^{\ell}\\
&\tilde{R}_{\mathrm{v,out}}^{\ell}\, q_{\mathrm{v,out}}^{\ell} = p_{\mathrm{v}}^{\ell} - p_{\mathrm{ar}}^{\mathrm{sys}}\\
&0 = q_{\mathrm{v,out}}^{\ell} - q_{\mathrm{ar,p}}^{\mathrm{sys}}\\
&I_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t} + Z_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar,p}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ar,d}}^{\mathrm{sys}}\\
&C_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar}}^{\mathrm{sys}}\\
&L_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar}}^{\mathrm{sys}} = p_{\mathrm{ar,d}}^{\mathrm{sys}} - p_{\mathrm{ar,peri}}^{\mathrm{sys}}\\
&\left(\sum_{j\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\right\}} C_{\mathrm{ar},j}^{\mathrm{sys}}\right) \frac{\mathrm{d}p_{\mathrm{ar,peri}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{sys}} - \sum_{j\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\right\}} q_{\mathrm{ar},j}^{\mathrm{sys}}\\
&R_{\mathrm{ar},i}^{\mathrm{sys}}\, q_{\mathrm{ar},i}^{\mathrm{sys}} = p_{\mathrm{ar,peri}}^{\mathrm{sys}} - p_{\mathrm{ven},i}^{\mathrm{sys}}, \quad i\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\right\}\\
&C_{\mathrm{ven},i}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar},i}^{\mathrm{sys}} - q_{\mathrm{ven},i}^{\mathrm{sys}}, \quad i\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\right\}\\
&R_{\mathrm{ven},i}^{\mathrm{sys}}\, q_{\mathrm{ven},i}^{\mathrm{sys}} = p_{\mathrm{ven},i}^{\mathrm{sys}} - p_{\mathrm{ven}}^{\mathrm{sys}}, \quad i\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\right\}\\
&C_{\mathrm{ven}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} = \sum_{j=\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}} q_{\mathrm{ven},j}^{\mathrm{sys}} - q_{\mathrm{ven}}^{\mathrm{sys}}\\
&L_{\mathrm{ven}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ven}}^{\mathrm{sys}}\, q_{\mathrm{ven}}^{\mathrm{sys}} = p_{\mathrm{ven}}^{\mathrm{sys}} - p_{\mathrm{at}}^{r}
\end{aligned}
$$

$$
\begin{aligned}
& -Q_{\mathrm{at}}^{r} = q_{\mathrm{ven}}^{\mathrm{sys}} - q_{\mathrm{v,in}}^{r} \\
& \tilde{R}_{\mathrm{v,in}}^{r}\, q_{\mathrm{v,in}}^{r} = p_{\mathrm{at}}^{r} - p_{\mathrm{v}}^{r} \\
& -Q_{\mathrm{v}}^{r} = q_{\mathrm{v,in}}^{r} - q_{\mathrm{v,out}}^{r} \\
& \tilde{R}_{\mathrm{v,out}}^{r}\, q_{\mathrm{v,out}}^{r} = p_{\mathrm{v}}^{r} - p_{\mathrm{ar}}^{\mathrm{pul}} \\
& C_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{v,out}}^{r} - q_{\mathrm{ar}}^{\mathrm{pul}} \\
& L_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{pul}}\, q_{\mathrm{ar}}^{\mathrm{pul}} = p_{\mathrm{ar}}^{\mathrm{pul}} - p_{\mathrm{cap}}^{\mathrm{pul}} \\
& C_{\mathrm{cap}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{cap}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{pul}} - q_{\mathrm{cap}}^{\mathrm{pul}} \\
& R_{\mathrm{cap}}^{\mathrm{pul}}\, q_{\mathrm{cap}}^{\mathrm{pul}} = p_{\mathrm{cap}}^{\mathrm{pul}} - p_{\mathrm{ven}}^{\mathrm{pul}} \\
& C_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{cap}}^{\mathrm{pul}} - q_{\mathrm{ven}}^{\mathrm{pul}} \\
& L_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ven}}^{\mathrm{pul}}\, q_{\mathrm{ven}}^{\mathrm{pul}} = p_{\mathrm{ven}}^{\mathrm{pul}} - p_{\mathrm{at}}^{\ell}
\end{aligned}
$$

with:

$$
Q_{\mathrm{at}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{at}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\mathrm{v}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{v}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\mathrm{at}}^{r} := -\frac{\mathrm{d}V_{\mathrm{at}}^{r}}{\mathrm{d}t}, \qquad Q_{\mathrm{v}}^{r} := -\frac{\mathrm{d}V_{\mathrm{v}}^{r}}{\mathrm{d}t}
$$

4.3 "Syspulcapcor" circulation model

$$
\begin{aligned}
&-Q_{\text{at}}^{\ell} = q_{\text{ven}}^{\text{pul}} - q_{\text{v,in}}^{\ell}\\
&\tilde{R}_{\text{v,in}}^{\ell}\, q_{\text{v,in}}^{\ell} = p_{\text{at}}^{\ell} - p_{\text{v}}^{\ell}\\
&-Q_{\text{v}}^{\ell} = q_{\text{v,in}}^{\ell} - q_{\text{v,out}}^{\ell}\\
&\tilde{R}_{\text{v,out}}^{\ell}\, q_{\text{v,out}}^{\ell} = p_{\text{v}}^{\ell} - p_{\text{ar}}^{\text{sys}}\\
&0 = q_{\text{v,out}}^{\ell} - q_{\text{ar,p}}^{\text{sys}} - q_{\text{ar,cor,in}}^{\text{sys}}\\
&I_{\text{ar}}^{\text{sys}} \frac{\mathrm{d}q_{\text{ar,p}}^{\text{sys}}}{\mathrm{d}t} + Z_{\text{ar}}^{\text{sys}}\, q_{\text{ar,p}}^{\text{sys}} = p_{\text{ar}}^{\text{sys}} - p_{\text{ar,d}}^{\text{sys}}\\
&C_{\text{ar,cor}}^{\text{sys}} \frac{\mathrm{d}p_{\text{ar}}^{\text{sys}}}{\mathrm{d}t} = q_{\text{ar,cor,in}}^{\text{sys}} - q_{\text{ar,cor}}^{\text{sys}}\\
&R_{\text{ar,cor}}^{\text{sys}}\, q_{\text{ar,cor}}^{\text{sys}} = p_{\text{ar}}^{\text{sys}} - p_{\text{ven,cor}}^{\text{sys}}\\
&C_{\text{ar}}^{\text{sys}} \frac{\mathrm{d}p_{\text{ar,d}}^{\text{sys}}}{\mathrm{d}t} = q_{\text{ar,p}}^{\text{sys}} - q_{\text{ar}}^{\text{sys}}\\
&L_{\text{ar}}^{\text{sys}} \frac{\mathrm{d}q_{\text{ar}}^{\text{sys}}}{\mathrm{d}t} + R_{\text{ar}}^{\text{sys}}\, q_{\text{ar}}^{\text{sys}} = p_{\text{ar,d}}^{\text{sys}} - p_{\text{ar,peri}}^{\text{sys}}\\
&\left(\sum_{j\in\left\{\substack{\text{spl,espl,}\\ \text{msc,cer}}\right\}} C_{\text{ar},j}^{\text{sys}}\right) \frac{\mathrm{d}p_{\text{ar,peri}}^{\text{sys}}}{\mathrm{d}t} = q_{\text{ar}}^{\text{sys}} - \sum_{j\in\left\{\substack{\text{spl,espl,}\\ \text{msc,cer}}\right\}} q_{\text{ar},j}^{\text{sys}}\\
&R_{\text{ar},i}^{\text{sys}}\, q_{\text{ar},i}^{\text{sys}} = p_{\text{ar,peri}}^{\text{sys}} - p_{\text{ven},i}^{\text{sys}}, \quad {\scriptstyle i\in\left\{\substack{\text{spl,espl,}\\ \text{msc,cer}}\right\}}\\
&C_{\text{ven},i}^{\text{sys}} \frac{\mathrm{d}p_{\text{ven},i}^{\text{sys}}}{\mathrm{d}t} = q_{\text{ar},i}^{\text{sys}} - q_{\text{ven},i}^{\text{sys}}, \quad {\scriptstyle i\in\left\{\substack{\text{spl,espl,}\\ \text{msc,cer}}\right\}}\\
&R_{\text{ven},i}^{\text{sys}}\, q_{\text{ven},i}^{\text{sys}} = p_{\text{ven},i}^{\text{sys}} - p_{\text{ven}}^{\text{sys}}, \quad {\scriptstyle i\in\left\{\substack{\text{spl,espl,}\\ \text{msc,cer}}\right\}}\\
&C_{\text{ven}}^{\text{sys}} \frac{\mathrm{d}p_{\text{ven}}^{\text{sys}}}{\mathrm{d}t} = \sum_{j=\substack{\text{spl,espl,}\\ \text{msc,cer}}} q_{\text{ven},j}^{\text{sys}} - q_{\text{ven}}^{\text{sys}}\\
&L_{\text{ven}}^{\text{sys}} \frac{\mathrm{d}q_{\text{ven}}^{\text{sys}}}{\mathrm{d}t} + R_{\text{ven}}^{\text{sys}}\, q_{\text{ven}}^{\text{sys}} = p_{\text{ven}}^{\text{sys}} - p_{\text{at}}^{r}\\
&C_{\text{ven,cor}}^{\text{sys}} \frac{\mathrm{d}p_{\text{ven,cor}}^{\text{sys}}}{\mathrm{d}t} = q_{\text{ar,cor}}^{\text{sys}} - q_{\text{ven,cor}}^{\text{sys}}\\
&R_{\text{ven,cor}}^{\text{sys}}\, q_{\text{ven,cor}}^{\text{sys}} = p_{\text{ven,cor}}^{\text{sys}} - p_{\text{at}}^{r}
\end{aligned}
$$

$$
\begin{aligned}
& -Q^r_{\mathrm{at}} = q^{\mathrm{sys}}_{\mathrm{ven}} + q^{\mathrm{sys}}_{\mathrm{ven,cor}} - q^r_{\mathrm{v,in}} \\
& \tilde{R}^r_{\mathrm{v,in}}\, q^r_{\mathrm{v,in}} = p^r_{\mathrm{at}} - p^r_{\mathrm{v}} \\
& -Q^r_{\mathrm{v}} = q^r_{\mathrm{v,in}} - q^r_{\mathrm{v,out}} \\
& \tilde{R}^r_{\mathrm{v,out}}\, q^r_{\mathrm{v,out}} = p^r_{\mathrm{v}} - p^{\mathrm{pul}}_{\mathrm{ar}} \\
& C^{\mathrm{pul}}_{\mathrm{ar}} \frac{\mathrm{d}p^{\mathrm{pul}}_{\mathrm{ar}}}{\mathrm{d}t} = q^r_{\mathrm{v,out}} - q^{\mathrm{pul}}_{\mathrm{ar}} \\
& L^{\mathrm{pul}}_{\mathrm{ar}} \frac{\mathrm{d}q^{\mathrm{pul}}_{\mathrm{ar}}}{\mathrm{d}t} + R^{\mathrm{pul}}_{\mathrm{ar}}\, q^{\mathrm{pul}}_{\mathrm{ar}} = p^{\mathrm{pul}}_{\mathrm{ar}} - p^{\mathrm{pul}}_{\mathrm{cap}} \\
& C^{\mathrm{pul}}_{\mathrm{cap}} \frac{\mathrm{d}p^{\mathrm{pul}}_{\mathrm{cap}}}{\mathrm{d}t} = q^{\mathrm{pul}}_{\mathrm{ar}} - q^{\mathrm{pul}}_{\mathrm{cap}} \\
& R^{\mathrm{pul}}_{\mathrm{cap}}\, q^{\mathrm{pul}}_{\mathrm{cap}} = p^{\mathrm{pul}}_{\mathrm{cap}} - p^{\mathrm{pul}}_{\mathrm{ven}} \\
& C^{\mathrm{pul}}_{\mathrm{ven}} \frac{\mathrm{d}p^{\mathrm{pul}}_{\mathrm{ven}}}{\mathrm{d}t} = q^{\mathrm{pul}}_{\mathrm{cap}} - q^{\mathrm{pul}}_{\mathrm{ven}} \\
& L^{\mathrm{pul}}_{\mathrm{ven}} \frac{\mathrm{d}q^{\mathrm{pul}}_{\mathrm{ven}}}{\mathrm{d}t} + R^{\mathrm{pul}}_{\mathrm{ven}}\, q^{\mathrm{pul}}_{\mathrm{ven}} = p^{\mathrm{pul}}_{\mathrm{ven}} - p^{\ell}_{\mathrm{at}}
\end{aligned}
$$

with:

$$
Q^{\ell}_{\mathrm{at}} := -\frac{\mathrm{d}V^{\ell}_{\mathrm{at}}}{\mathrm{d}t}, \qquad Q^{\ell}_{\mathrm{v}} := -\frac{\mathrm{d}V^{\ell}_{\mathrm{v}}}{\mathrm{d}t}, \qquad Q^{r}_{\mathrm{at}} := -\frac{\mathrm{d}V^{r}_{\mathrm{at}}}{\mathrm{d}t}, \qquad Q^{r}_{\mathrm{v}} := -\frac{\mathrm{d}V^{r}_{\mathrm{v}}}{\mathrm{d}t}
$$

Bibliography

[1] D. Chapelle, P. L. Tallec, P. Moireau, and M. Sorine. Energy-preserving muscle tissue model: formulation and compatible discretizations. *Journal for Multiscale Computational Engineering*, 10(2):189–211, 2012.

[2] H. Demiray. A note on the elasticity of soft biological tissues. *Journal of Biomechanics*, 5(3):309–311, 1972.

[3] G. A. Holzapfel. *Nonlinear Solid Mechanics – A Continuum Approach for Engineering.* Wiley Press Chichester, 2000.

[4] G. A. Holzapfel, R. Eberlein, P. Wriggers, and H. W. Weizsäcker. Large strain analysis of soft biological membranes: Formulation and finite element analysis. *Computer Methods in Applied Mechanics and Engineering*, 132(1–2):45–61, 1996.