Ambit – A FEniCS-based cardiovascular physics solver

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Contents

1 Solid mechanics

- solid mechanics are formulated in a Total Lagrangian frame

1.1 Strong form

1.1.1 Displacement-based

- primary variable: displacement $\boldsymbol{u}$

$$\begin{aligned}
\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{u},\boldsymbol{v}) + \hat{\boldsymbol{b}}_0 &= \rho_0 \boldsymbol{a} && \text{in } \mathit{\Omega}_0 \times [0,T], && (1)\\
\boldsymbol{u} &= \hat{\boldsymbol{u}} && \text{on } \mathit{\Gamma}_0^{\mathrm{D}} \times [0,T], && (2)\\
\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 &= \hat{\boldsymbol{t}}_0 && \text{on } \mathit{\Gamma}_0^{\mathrm{N}} \times [0,T], && (3)\\
\boldsymbol{u}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{u}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (4)\\
\boldsymbol{v}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (5)
\end{aligned}$$

1.1.2 Incompressible mechanics

- primary variables: displacement $\boldsymbol{u}$ and pressure p

$$\begin{aligned}
\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{u},p,\boldsymbol{v}(\boldsymbol{u})) + \hat{\boldsymbol{b}}_0 &= \rho_0 \boldsymbol{a}(\boldsymbol{u}) && \text{in } \mathit{\Omega}_0 \times [0,T], && (6)\\
J(\boldsymbol{u}) - 1 &= 0 && \text{in } \mathit{\Omega}_0 \times [0,T], && (7)\\
\boldsymbol{u} &= \hat{\boldsymbol{u}} && \text{on } \mathit{\Gamma}_0^{\mathrm{D}} \times [0,T], && (8)\\
\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 &= \hat{\boldsymbol{t}}_0 && \text{on } \mathit{\Gamma}_0^{\mathrm{N}} \times [0,T], && (9)\\
\boldsymbol{u}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{u}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (10)\\
\boldsymbol{v}(\boldsymbol{x}_0,0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}_0) && \text{in } \mathit{\Omega}_0, && (11)
\end{aligned}$$

with velocity and acceleration $\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$ and $\boldsymbol{a} = \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}$, respectively

1.2 Weak form

1.2.1 Displacement-based

- primary variable: displacement $\boldsymbol{u}$

- Principal of Virtual Work:

$$r(\boldsymbol{u};\delta\boldsymbol{u}) := \delta\mathcal{W}_{\mathrm{kin}}(\boldsymbol{u};\delta\boldsymbol{u}) + \delta\mathcal{W}_{\mathrm{int}}(\boldsymbol{u};\delta\boldsymbol{u}) - \delta\mathcal{W}_{\mathrm{ext}}(\boldsymbol{u};\delta\boldsymbol{u}) = 0, \quad \forall\, \delta\boldsymbol{u} \tag{12}$$

- kinetic virtual work:

$$\delta\mathcal{W}_{\mathrm{kin}}(\boldsymbol{u};\delta\boldsymbol{u}) = \int\limits_{\mathit{\Omega}_0} \rho_0\, \boldsymbol{a}(\boldsymbol{u}) \cdot \delta\boldsymbol{u}\,\mathrm{d}V \tag{13}$$

- internal virtual work:

$$\delta\mathcal{W}_{\text{int}}(\boldsymbol{u};\delta\boldsymbol{u}) = \int\limits_{\Omega_0} \boldsymbol{P}(\boldsymbol{u},\boldsymbol{v}(\boldsymbol{u})) : \boldsymbol{\nabla}_0\delta\boldsymbol{u}\,\mathrm{d}V \tag{14}$$

- external virtual work:

- conservative Neumann load:

$$\delta\mathcal{W}_{\text{ext}}(\delta\boldsymbol{u}) = \int\limits_{\Gamma_0^{\text{N}}} \hat{\boldsymbol{t}}_0(t)\cdot\delta\boldsymbol{u}\,\mathrm{d}A \tag{15}$$

- Neumann pressure load in current normal direction:

$$\delta\mathcal{W}_{\text{ext}}(\boldsymbol{u};\delta\boldsymbol{u}) = -\int\limits_{\Gamma_0^{\text{N}}} \hat{p}(t)\, J\boldsymbol{F}^{-\text{T}}\boldsymbol{n}_0\cdot\delta\boldsymbol{u}\,\mathrm{d}A \tag{16}$$

- general Neumann load in current normal direction:

$$\delta\mathcal{W}_{\text{ext}}(\boldsymbol{u};\delta\boldsymbol{u}) = \int\limits_{\Gamma_0} J\boldsymbol{F}^{-\text{T}}\,\hat{\boldsymbol{t}}_0(t)\cdot\delta\boldsymbol{u}\,\mathrm{d}A \tag{17}$$

- generalized Robin condition:

$$\delta\mathcal{W}_{\text{ext}}(\boldsymbol{u};\delta\boldsymbol{u}) = -\int\limits_{\Gamma_0^{\text{N}}} [k\,\boldsymbol{u} + c\,\boldsymbol{v}(\boldsymbol{u})]\cdot\delta\boldsymbol{u}\,\mathrm{d}A \tag{18}$$

- generalized Robin condition in reference surface normal direction:

$$\delta\mathcal{W}_{\text{ext}}(\boldsymbol{u};\delta\boldsymbol{u}) = -\int\limits_{\Gamma_0^{\text{N}}} (\boldsymbol{n}_0\otimes\boldsymbol{n}_0)\,[k\,\boldsymbol{u} + c\,\boldsymbol{v}(\boldsymbol{u})]\cdot\delta\boldsymbol{u}\,\mathrm{d}A \tag{19}$$

1.2.2 Incompressible mechanics: 2-field displacement and pressure variables

- primary variables: displacement $\boldsymbol{u}$ and pressure p

$$r_u(\boldsymbol{u},p;\delta\boldsymbol{u}) := \delta\mathcal{W}_{\text{kin}}(\boldsymbol{u};\delta\boldsymbol{u}) + \delta\mathcal{W}_{\text{int}}(\boldsymbol{u},p;\delta\boldsymbol{u}) - \delta\mathcal{W}_{\text{ext}}(\boldsymbol{u};\delta\boldsymbol{u}) = 0, \quad \forall\,\delta\boldsymbol{u} \tag{20}$$

$$r_p(\boldsymbol{u};\delta p) := \delta\mathcal{W}_{\text{pres}}(\boldsymbol{u};\delta p) = 0, \quad \forall\,\delta p \tag{21}$$

- kinetic virtual work: (13)
- internal virtual work:

$$\delta\mathcal{W}_{\text{int}}(\boldsymbol{u},p;\delta\boldsymbol{u}) = \int\limits_{\Omega_0} \boldsymbol{P}(\boldsymbol{u},p,\boldsymbol{v}(\boldsymbol{u})) : \nabla_{\boldsymbol{X}}\delta\boldsymbol{u}\,\mathrm{d}V \tag{22}$$

- pressure virtual work:

$$\delta\mathcal{W}_{\text{pres}}(\boldsymbol{u};\delta p) = \int\limits_{\Omega_0} (J(\boldsymbol{u}) - 1)\,\delta p\,\mathrm{d}V \tag{23}$$

2 Fluid mechanics

2.1 Eulerian reference frame

- incompressible Navier-Stokes equations in Eulerian reference frame

2.1.1 Strong Form

- primary variables: velocity $\boldsymbol{v}$ and pressure p

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{v},p) + \hat{\boldsymbol{b}} &= \rho\left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla}\boldsymbol{v})\,\boldsymbol{v}\right) && \text{in } \Omega_t \times [0,T], && (24)\\
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 && \text{in } \Omega_t \times [0,T], && (25)\\
\boldsymbol{v} &= \hat{\boldsymbol{v}} && \text{on } \Gamma_t^{\mathrm{D}} \times [0,T], && (26)\\
\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} &= \hat{\boldsymbol{t}} && \text{on } \Gamma_t^{\mathrm{N}} \times [0,T], && (27)\\
\boldsymbol{v}(\boldsymbol{x},0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}) && \text{in } \Omega_t, && (28)
\end{aligned}$$

with a Newtonian fluid constitutive law

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\,\boldsymbol{\gamma} = -p\boldsymbol{I} + \mu\left(\boldsymbol{\nabla}\boldsymbol{v} + (\boldsymbol{\nabla}\boldsymbol{v})^{\mathrm{T}}\right) \tag{29}$$

2.1.2 Weak Form

- primary variables: velocity $\boldsymbol{v}$ and pressure p

- Principle of Virtual Power

$$\begin{aligned}
r_v(\boldsymbol{v},p;\delta\boldsymbol{v}) &:= \delta\mathcal{P}_{\text{kin}}(\boldsymbol{v};\delta\boldsymbol{v}) + \delta\mathcal{P}_{\text{int}}(\boldsymbol{v},p;\delta\boldsymbol{v}) - \delta\mathcal{P}_{\text{ext}}(\boldsymbol{v};\delta\boldsymbol{v}) = 0, \quad \forall\,\delta\boldsymbol{v} && (30)\\
r_p(\boldsymbol{v};\delta p) &:= \delta\mathcal{P}_{\text{pres}}(\boldsymbol{v};\delta p), \quad \forall\,\delta p && (31)
\end{aligned}$$

- kinetic virtual power:

$$\delta\mathcal{P}_{\text{kin}}(\boldsymbol{v};\delta\boldsymbol{v}) = \int\limits_{\Omega_t} \rho\left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla}\boldsymbol{v})\,\boldsymbol{v}\right)\cdot\delta\boldsymbol{v}\,\mathrm{d}v \tag{32}$$

- internal virtual power:

$$\delta\mathcal{P}_{\text{int}}(\boldsymbol{v},p;\delta\boldsymbol{v}) = \int\limits_{\Omega_t} \boldsymbol{\sigma}(\boldsymbol{v},p) : \boldsymbol{\nabla}\delta\boldsymbol{v}\,\mathrm{d}v \tag{33}$$

- pressure virtual power:

$$\delta\mathcal{P}_{\mathrm{pres}}(\boldsymbol{v};\delta p) = \int\limits_{\Omega_t} (\boldsymbol{\nabla}\cdot\boldsymbol{v})\,\delta p\,\mathrm{d}v \tag{34}$$

- external virtual power:

- conservative Neumann load:

$$\delta\mathcal{P}_{\mathrm{ext}}(\delta\boldsymbol{v}) = \int\limits_{\Gamma_t^{\mathrm{N}}} \hat{\boldsymbol{t}}(t)\cdot\delta\boldsymbol{v}\,\mathrm{d}a \tag{35}$$

- pressure Neumann load:

$$\delta\mathcal{P}_{\mathrm{ext}}(\delta\boldsymbol{v}) = -\int\limits_{\Gamma_t^{\mathrm{N}}} \hat{p}(t)\,\boldsymbol{n}\cdot\delta\boldsymbol{v}\,\mathrm{d}a \tag{36}$$

2.1.3 Stabilization

- Stabilization `supg_pspg2`:
- velocity residual operator (30) is augmented with the following terms:

$$r_v \leftarrow r_v + \int\limits_{\Omega_t} d_1\,((\boldsymbol{\nabla}\boldsymbol{v})\,\boldsymbol{v})\cdot(\boldsymbol{\nabla}\delta\boldsymbol{v})\,\boldsymbol{v}\,\mathrm{d}v \tag{37}$$

$$+ \int\limits_{\Omega_t} d_2\,(\boldsymbol{\nabla}\cdot\boldsymbol{v})(\boldsymbol{\nabla}\cdot\delta\boldsymbol{v})\,\mathrm{d}v \tag{38}$$

$$+ \int\limits_{\Omega_t} d_3\,(\boldsymbol{\nabla}p)\cdot(\boldsymbol{\nabla}\delta\boldsymbol{v})\,\boldsymbol{v}\,\mathrm{d}v \tag{39}$$

- pressure residual operator (31) is augmented with the following terms:

$$r_p \leftarrow r_p + \frac{1}{\rho}\int\limits_{\Omega_t} d_1\,((\boldsymbol{\nabla}\boldsymbol{v})\,\boldsymbol{v})\cdot(\boldsymbol{\nabla}\delta p)\,\mathrm{d}v \tag{40}$$

$$+ \frac{1}{\rho}\int\limits_{\Omega_t} d_3\,(\boldsymbol{\nabla}p)\cdot(\boldsymbol{\nabla}\delta p)\,\mathrm{d}v \tag{41}$$

- discrete linear system

$$\begin{bmatrix} \mathbf{K}_{vv} & \mathbf{K}_{vp} \\ \mathbf{K}_{pv} & \mathbf{0} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\mathbf{v} \\ \Delta\mathbf{p} \end{bmatrix}_{n+1}^{k+1} = -\begin{bmatrix} \mathbf{r}_v \\ \mathbf{r}_p \end{bmatrix}_{n+1}^{k} \tag{42}$$

- note that we get a $\mathbf{K}_{pp}$ if we have stabilization

2.2 ALE reference frame

- incompressible Navier-Stokes equations in Arbitrary Lagrangian Eulerian (ALE) reference frame

- ALE domain problem deformation governed by linear-elastic or nonlinear hyperelastic solid, displacement field $\boldsymbol{d}$

- fluid mechanics formulated with respect to the reference frame, using ALE deformation gradient $\boldsymbol{F}(\boldsymbol{d}) = \boldsymbol{I} + \boldsymbol{\nabla}_0\boldsymbol{d}$ and its determinant, $J(\boldsymbol{d}) = \det \boldsymbol{F}(\boldsymbol{d})$

2.2.1 ALE problem

- displacement-based quasi-static solid

- primary variable: domain displacement $\boldsymbol{d}$

- strong form:

$$\begin{aligned} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^{\mathrm{G}}(\boldsymbol{d}) &= \boldsymbol{0} \quad \text{in } \mathit{\Omega}_0, && (43)\\ \boldsymbol{d} &= \hat{\boldsymbol{d}} \quad \text{on } \mathit{\Gamma}_0^{\mathrm{D}}, && (44) \end{aligned}$$

with

$$\boldsymbol{\sigma}^{\mathrm{G}}(\boldsymbol{d}) = E\,\frac{1}{2}\left(\boldsymbol{\nabla}_0\boldsymbol{d} + (\boldsymbol{\nabla}_0\boldsymbol{d})^{\mathrm{T}}\right) + \kappa\,(\boldsymbol{\nabla}_0 \cdot \boldsymbol{d})\ \boldsymbol{I} \tag{45}$$

- weak form:

$$r_d(\boldsymbol{d};\delta\boldsymbol{d}) := \int\limits_{\mathit{\Omega}_0} \boldsymbol{\sigma}^{\mathrm{G}}(\boldsymbol{d}) : \boldsymbol{\nabla}_0\delta\boldsymbol{d}\,\mathrm{d}V = 0, \quad \forall\,\delta\boldsymbol{d} \tag{46}$$

2.2.2 Strong form

- primary variables: velocity $\boldsymbol{v}$ and pressure p

$$\begin{aligned} \boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}(\boldsymbol{v},\boldsymbol{d},p) + \hat{\boldsymbol{b}} &= \rho\left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla}_0\boldsymbol{v}\,\boldsymbol{F}^{-1})\,\boldsymbol{v}\right) && \text{in } \mathit{\Omega}_0 \times [0,T], && (47)\\ \boldsymbol{\nabla}_0\boldsymbol{v} : \boldsymbol{F}^{-\mathrm{T}} &= 0 && \text{in } \mathit{\Omega}_0 \times [0,T], && (48)\\ \boldsymbol{v} &= \hat{\boldsymbol{v}} && \text{on } \mathit{\Gamma}_0^{\mathrm{D}} \times [0,T], && (49)\\ \boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} &= \hat{\boldsymbol{t}} && \text{on } \mathit{\Gamma}_0^{\mathrm{N}} \times [0,T], && (50)\\ \boldsymbol{v}(\boldsymbol{x},0) &= \hat{\boldsymbol{v}}_0(\boldsymbol{x}) && \text{in } \mathit{\Omega}_0, && (51) \end{aligned}$$

with a Newtonian fluid constitutive law

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{\gamma} = -p\boldsymbol{I} + \mu\left(\boldsymbol{\nabla}_0\boldsymbol{v}\,\boldsymbol{F}^{-1} + \boldsymbol{F}^{-\mathrm{T}}(\boldsymbol{\nabla}_0\boldsymbol{v})^{\mathrm{T}}\right) \tag{52}$$

2.2.3 Weak form

- primary variables: velocity $\boldsymbol{v}$, pressure p, and domain displacement $\boldsymbol{d}$

- Principle of Virtual Power

$$
\begin{aligned}
r_v(\boldsymbol{v}, p, \boldsymbol{d}; \delta\boldsymbol{v}) &:= \delta\mathcal{P}_{\mathrm{kin}}(\boldsymbol{v}, \boldsymbol{d}; \delta\boldsymbol{v}) + \delta\mathcal{P}_{\mathrm{int}}(\boldsymbol{v}, p, \boldsymbol{d}; \delta\boldsymbol{v}) - \delta\mathcal{P}_{\mathrm{ext}}(\boldsymbol{v}, \boldsymbol{d}; \delta\boldsymbol{v}) = 0, \quad \forall\, \delta\boldsymbol{v} && (53)\\
r_p(\boldsymbol{v}, \boldsymbol{d}; \delta p) &:= \delta\mathcal{P}_{\mathrm{pres}}(\boldsymbol{v}, \boldsymbol{d}; \delta p), \quad \forall\, \delta p && (54)
\end{aligned}
$$

- kinetic virtual power:

$$
\delta\mathcal{P}_{\mathrm{kin}}(\boldsymbol{v}, \boldsymbol{d}; \delta\boldsymbol{v}) = \int\limits_{\Omega_0} J\rho \left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla}_0\boldsymbol{v}\,\boldsymbol{F}^{-1})\,\boldsymbol{v}\right) \cdot \delta\boldsymbol{v}\,\mathrm{d}V \tag{55}
$$

- internal virtual power:

$$
\delta\mathcal{P}_{\mathrm{int}}(\boldsymbol{v}, p, \boldsymbol{d}; \delta\boldsymbol{v}) = \int\limits_{\Omega_0} J\boldsymbol{\sigma}(\boldsymbol{v}, p, \boldsymbol{d}) : \boldsymbol{\nabla}_0\delta\boldsymbol{v}\,\boldsymbol{F}^{-1}\,\mathrm{d}V \tag{56}
$$

- pressure virtual power:

$$
\delta\mathcal{P}_{\mathrm{pres}}(\boldsymbol{v}, \boldsymbol{d}; \delta p) = \int\limits_{\Omega_0} J\,\boldsymbol{\nabla}_0\boldsymbol{v} : \boldsymbol{F}^{-\mathrm{T}}\delta p\,\mathrm{d}V \tag{57}
$$

- external virtual power:

- conservative Neumann load:

$$
\delta\mathcal{P}_{\mathrm{ext}}(\delta\boldsymbol{v}) = \int\limits_{\Gamma_0^{\mathrm{N}}} \hat{\boldsymbol{t}}(t) \cdot \delta\boldsymbol{v}\,\mathrm{d}A \tag{58}
$$

- pressure Neumann load:

$$
\delta\mathcal{P}_{\mathrm{ext}}(\boldsymbol{d}; \delta\boldsymbol{v}) = -\int\limits_{\Gamma_0^{\mathrm{N}}} \hat{p}(t)\,J\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0 \cdot \delta\boldsymbol{v}\,\mathrm{d}A \tag{59}
$$

2.2.4 Stabilization

- Stabilization `supg_pspg2`:
- velocity residual operator (53) is augmented with the following terms:

$$
\begin{aligned}
r_v \leftarrow r_v &+ \int\limits_{\Omega_0} J\, d_1\, ((\boldsymbol{\nabla}_0 \boldsymbol{v}\, \boldsymbol{F}^{-1})\, \boldsymbol{v}) \cdot (\boldsymbol{\nabla}_0 \delta \boldsymbol{v}\, \boldsymbol{F}^{-1})\, \boldsymbol{v}\, \mathrm{d}V && (60)\\
&+ \int\limits_{\Omega_0} J\, d_2\, (\boldsymbol{\nabla}_0 \boldsymbol{v} : \boldsymbol{F}^{-\mathrm{T}})(\boldsymbol{\nabla}_0 \delta \boldsymbol{v} : \boldsymbol{F}^{-\mathrm{T}})\, \mathrm{d}V && (61)\\
&+ \int\limits_{\Omega_0} J\, d_3\, (\boldsymbol{F}^{-\mathrm{T}} \boldsymbol{\nabla}_0 p) \cdot (\boldsymbol{\nabla}_0 \delta \boldsymbol{v}\, \boldsymbol{F}^{-1})\, \boldsymbol{v}\, \mathrm{d}V && (62)
\end{aligned}
$$

- pressure residual operator (54) is augmented with the following terms:

$$
\begin{aligned}
r_p \leftarrow r_p &+ \frac{1}{\rho} \int\limits_{\Omega_0} J\, d_1\, ((\boldsymbol{\nabla}_0 \boldsymbol{v}\, \boldsymbol{F}^{-1})\, \boldsymbol{v}) \cdot (\boldsymbol{F}^{-\mathrm{T}} \boldsymbol{\nabla}_0 \delta p)\, \mathrm{d}V && (63)\\
&+ \frac{1}{\rho} \int\limits_{\Omega_0} J\, d_3\, (\boldsymbol{F}^{-\mathrm{T}} \boldsymbol{\nabla}_0 p) \cdot (\boldsymbol{F}^{-\mathrm{T}} \boldsymbol{\nabla}_0 \delta p)\, \mathrm{d}V && (64)
\end{aligned}
$$

- discrete linear system

$$
\begin{bmatrix} \mathbf{K}_{vv} & \mathbf{K}_{vp} & \mathbf{K}_{vd} \\ \mathbf{K}_{pv} & \mathbf{0} & \mathbf{K}_{pd} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{dd} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta \mathbf{v} \\ \Delta \mathbf{p} \\ \Delta \mathbf{d} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{r}_v \\ \mathbf{r}_p \\ \mathbf{r}_d \end{bmatrix}_{n+1}^{k} \tag{65}
$$

- note that we get a $\mathbf{K}_{pp}$ if we have stabilization

3 Coupling

3.1 Fluid–0D flow

- (30) augmented by following term:

$$
r_v \leftarrow r_v + \int\limits_{\Gamma_t^{\text{f-0d}}} \Lambda\, \boldsymbol{n} \cdot \delta \boldsymbol{v}\, \mathrm{d}a \tag{66}
$$

- multiplier constraint

$$r_\lambda(\varLambda, \boldsymbol{v}; \delta\varLambda) := \left(\int\limits_{\varGamma_t^{\text{f-0d}}} \boldsymbol{n} \cdot \boldsymbol{v} \,\mathrm{d}a - Q^{\text{0d}}(\varLambda) \right) \delta\varLambda, \quad \forall\, \delta\varLambda \tag{67}$$

- discrete linear system

$$\begin{bmatrix} \mathbf{K}_{vv} & \mathbf{K}_{vp} & \mathbf{K}_{v\varLambda} \\ \mathbf{K}_{pv} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{\varLambda v} & \mathbf{0} & \mathbf{K}_{\varLambda\varLambda} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\mathbf{v} \\ \Delta\mathbf{p} \\ \Delta\boldsymbol{\Lambda} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{r}_v \\ \mathbf{r}_p \\ \mathbf{r}_\varLambda \end{bmatrix}_{n+1}^{k} \tag{68}$$

3.2 ALE fluid–0D flow

- (53) augmented by following term:

$$r_v \leftarrow r_v + \int\limits_{\varGamma_0^{\text{f-0d}}} \varLambda\, J\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0 \cdot \delta\boldsymbol{v} \,\mathrm{d}A \tag{69}$$

- multiplier constraint

$$r_\lambda(\varLambda, \boldsymbol{v}, \boldsymbol{d}; \delta\varLambda) := \left(\int\limits_{\varGamma_0^{\text{f-0d}}} J\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0 \cdot (\boldsymbol{v} - \boldsymbol{w}(\boldsymbol{d})) \,\mathrm{d}A - Q^{\text{0d}}(\varLambda) \right) \delta\varLambda, \quad \forall\, \delta\varLambda \tag{70}$$

with $\boldsymbol{w}(\boldsymbol{d}) = \frac{\mathrm{d}\boldsymbol{d}}{\mathrm{d}t}$

- discrete linear system

$$\begin{bmatrix} \mathbf{K}_{vv} & \mathbf{K}_{vp} & \mathbf{K}_{v\varLambda} & \mathbf{K}_{vd} \\ \mathbf{K}_{pv} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{pd} \\ \mathbf{K}_{\varLambda v} & \mathbf{0} & \mathbf{K}_{\varLambda\varLambda} & \mathbf{K}_{\varLambda d} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{dd} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\mathbf{v} \\ \Delta\mathbf{p} \\ \Delta\boldsymbol{\Lambda} \\ \Delta\mathbf{d} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{r}_v \\ \mathbf{r}_p \\ \mathbf{r}_\varLambda \\ \mathbf{r}_d \end{bmatrix}_{n+1}^{k} \tag{71}$$

4 Fluid-reduced-solid-interaction (FrSI)

4.1 Physics-reduced solid

$$\mathit{\Omega}_0^{\mathrm{s}} \longmapsto \mathit{\Gamma}_0^{\text{f-s}}, \qquad \rightsquigarrow \int\limits_{\mathit{\Omega}_0^{\mathrm{s}}} (\cdot)\,\mathrm{d}V = \int\limits_{\mathit{\Gamma}_0^{\text{f-s}}} h_0\,(\cdot)\,\mathrm{d}A, \tag{72}$$

h_0: reduced solid's wall thickness parameter

4.1.1 Kinematics

$$\boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{x} = \boldsymbol{I} + \boldsymbol{\nabla}_0 \boldsymbol{u}_{\mathrm{f}}, \quad \dot{\boldsymbol{F}} = \boldsymbol{\nabla}_0 \boldsymbol{v}_{\mathrm{f}}, \quad \boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}, \tag{73}$$

- fluid domain displacement:

$$\boldsymbol{u}_{\mathrm{f}} = \int\limits_0^t \boldsymbol{v}(\boldsymbol{X},t)\,\mathrm{d}t \tag{74}$$

- in-plane deformation and rate of deformation gradient:

$$\boldsymbol{F}^0 = \boldsymbol{F} - \boldsymbol{F}\boldsymbol{n}_0 \otimes \boldsymbol{n}_0 \quad \text{and} \quad \dot{\boldsymbol{F}}^0 = \dot{\boldsymbol{F}} - \dot{\boldsymbol{F}}\boldsymbol{n}_0 \otimes \boldsymbol{n}_0, \tag{75}$$

- plane strain representation of the right Cauchy-Green tensor:

$$\boldsymbol{C}^{||} = \boldsymbol{F}^{0^{\mathrm{T}}}\boldsymbol{F}^0 + \boldsymbol{n}_0 \otimes \boldsymbol{n}_0 \tag{76}$$

- relation of in-plane and out-of-plane stretches

$$I\!I\!I_C^{||} = \det \boldsymbol{C}^{||} = \lambda_\xi^2 \lambda_\eta^2 = \frac{1}{\lambda_\zeta^2}, \qquad \rightsquigarrow \lambda_\zeta = \frac{1}{\sqrt{I\!I\!I_C^{||}}} \tag{77}$$

- membrane right Cauchy-Green deformation tensor

$$\tilde{\boldsymbol{C}} = \boldsymbol{F}^{0^{\mathrm{T}}}\boldsymbol{F}^0 + \frac{1}{I\!I\!I_C^{||}}\boldsymbol{n}_0 \otimes \boldsymbol{n}_0. \tag{78}$$

- rate:

$$\dot{\tilde{\boldsymbol{C}}} = \dot{\boldsymbol{F}}^{0^{\mathrm{T}}}\boldsymbol{F}^0 + \boldsymbol{F}^{0^{\mathrm{T}}}\dot{\boldsymbol{F}}^0 - \frac{\dot{I\!I\!I}_C^{||}}{I\!I\!I_C^{||^2}}\boldsymbol{n}_0 \otimes \boldsymbol{n}_0, \tag{79}$$

- time derivative of Eq. (77) with Jacobi's formula:

$$\dot{I\!I\!I}_C^{||} = \overline{\det \boldsymbol{C}^{||}}^{\,\cdot} = \det \boldsymbol{C}^{||}\,\mathrm{tr}\left(\boldsymbol{C}^{||^{-1}}\dot{\boldsymbol{C}}^{||}\right) \tag{80}$$

$$= I\!I\!I_C^{||}\,\mathrm{tr}\left(\boldsymbol{C}^{||^{-1}}\left(\dot{\boldsymbol{F}}^{0^{\mathrm{T}}}\boldsymbol{F}^0 + \boldsymbol{F}^{0^{\mathrm{T}}}\dot{\boldsymbol{F}}^0\right)\right). \tag{81}$$

4.1.2 Constitutive equations

General isotropic hyperelasticity - exemplified for general isotropic hyperelasticity

$$\varPsi = \varPsi(I_{\tilde{C}}, I\!I_{\tilde{C}}) - \frac{1}{2}p_{\mathrm{s}}(I\!I\!I_{\tilde{C}} - 1), \tag{82}$$

where $I_{\tilde{C}}$, $I\!I_{\tilde{C}}$ and $I\!I\!I_{\tilde{C}}$ are the principal invariants of Eq. (78)

- 2nd Piola-Kirchhoff stress [3]:

$$\tilde{\boldsymbol{S}} = -p_{\mathrm{s}}\tilde{\boldsymbol{C}}^{-1} + 2\left(\frac{\partial \varPsi}{\partial I_{\tilde{C}}} + I_{\tilde{C}}\frac{\partial \varPsi}{\partial I\!I_{\tilde{C}}}\right)\boldsymbol{I} - 2\frac{\partial \varPsi}{\partial I\!I_{\tilde{C}}}\tilde{\boldsymbol{C}} \tag{83}$$

- hydrostatic pressure p_{s} recovered by plane stress assumption for 2-dimensional continua [3, 4]:

$$p_{\mathrm{s}} = 2\left(\frac{1}{\lambda_\xi^2\lambda_\eta^2}\frac{\partial \varPsi}{\partial I_{\tilde{C}}} - \lambda_\xi^2\lambda_\eta^2\frac{\partial \varPsi}{\partial I\!I_{\tilde{C}}}\right) = 2\left(\frac{1}{I\!I\!I_C^{||}}\frac{\partial \varPsi}{\partial I_{\tilde{C}}} - I\!I\!I_C^{||}\frac{\partial \varPsi}{\partial I\!I_{\tilde{C}}}\right), \tag{84}$$

- 1st Piola-Kirchhoff stress then is computed by the push-forward operation

$$\tilde{\boldsymbol{P}}_{\mathrm{s}} = \boldsymbol{F}^0\tilde{\boldsymbol{S}}, \tag{85}$$

Cardiac mechanics - isotropic exponential strain energy [2]

$$\varPsi(\boldsymbol{C}, p_{\mathrm{s}}) = \frac{a_0}{2b_0}\left(e^{b_0(\mathrm{tr}\boldsymbol{C}-3)} - 1\right) - \frac{1}{2}p_{\mathrm{s}}(\det\boldsymbol{C} - 1) \tag{86}$$

- viscous pseudo-potential [1]

$$\varPsi_{\mathrm{v}}(\dot{\boldsymbol{C}}) = \frac{\eta}{8}\dot{\boldsymbol{C}} : \dot{\boldsymbol{C}} \tag{87}$$

- for 3D incompressible hyper-viscoelastic active cardiac mechanics:

$$\boldsymbol{S} = \boldsymbol{S}_{\mathrm{e}} + \boldsymbol{S}_{\mathrm{v}} + \boldsymbol{S}_{\mathrm{a}} = 2\frac{\partial \varPsi(\boldsymbol{C}, p_{\mathrm{s}})}{\partial \boldsymbol{C}} + 2\frac{\partial \varPsi_{\mathrm{v}}(\dot{\boldsymbol{C}})}{\partial \dot{\boldsymbol{C}}} + \tau_{\mathrm{a}}(t)\boldsymbol{f}_0 \otimes \boldsymbol{f}_0 \tag{88}$$

- for reduced solid:

$$\tilde{\boldsymbol{S}} = \tilde{\boldsymbol{S}}_{\mathrm{e}} + \tilde{\boldsymbol{S}}_{\mathrm{v}} + \tilde{\boldsymbol{S}}_{\mathrm{a}} = 2\frac{\partial \varPsi(\tilde{\boldsymbol{C}})}{\partial \tilde{\boldsymbol{C}}} + 2\frac{\partial \varPsi_{\mathrm{v}}(\dot{\tilde{\boldsymbol{C}}})}{\partial \dot{\tilde{\boldsymbol{C}}}} + \tau_{\mathrm{a}}(t)\tilde{\boldsymbol{M}}_0 \tag{89}$$

- note that in presence of viscosity according to Eq. (87), hydrostatic pressure for hyperelastic solid, Eq. (84), needs to be updated to balance the viscous normal stresses:

$$p_{\mathrm{s}} \leftarrow p_{\mathrm{s}} - \frac{\eta}{2}\frac{\dot{I\!I\!I}_C^{||}}{{I\!I\!I_C^{||}}^3} \tag{90}$$

- reduced (wall-averaged) structural tensor in circumferential and longitudinal directions $\boldsymbol{c}_0$ and $\boldsymbol{l}_0$, respectively:

$$\tilde{\boldsymbol{M}}_0 = \bar{\omega}\,\boldsymbol{c}_0 \otimes \boldsymbol{c}_0 + \bar{\iota}\,\boldsymbol{l}_0 \otimes \boldsymbol{l}_0 + 2\bar{\gamma}\,\mathrm{sym}(\boldsymbol{c}_0 \otimes \boldsymbol{l}_0). \tag{91}$$

Balance equation

$$\tilde{r}_v^{\mathrm{s}}(\boldsymbol{u}_{\mathrm{f}}(\boldsymbol{v});\delta\boldsymbol{v}) := \int\limits_{\Gamma_0^{\text{f-s}}} h_0\,\rho_0^{\mathrm{s}}\,\dot{\boldsymbol{v}}\cdot\delta\boldsymbol{v}\,\mathrm{d}A_0 + \int\limits_{\Gamma_0^{\text{f-s}}} h_0\,\tilde{\boldsymbol{P}}_{\mathrm{s}}(\boldsymbol{u}_{\mathrm{f}}(\boldsymbol{v}),\boldsymbol{v}) : \tilde{\boldsymbol{\nabla}}_0\delta\boldsymbol{v}\,\mathrm{d}A \tag{92}$$

4.2 Projection-based reduction

$$\hat{\mathbf{S}}_v^{\Gamma} = \mathbb{I}_{\Gamma}\begin{bmatrix}\mathbf{v}_1^{\mathrm{f}} & \dots & \mathbf{v}_{m_v}^{\mathrm{f}}\end{bmatrix} = \begin{bmatrix}\mathbb{I}_{\Gamma}^1 & & & \\ & \mathbb{I}_{\Gamma}^2 & & \\ & & \ddots & \\ & & & \mathbb{I}_{\Gamma}^{n_v}\end{bmatrix}\begin{bmatrix}v_1^1 & \dots & v_{m_v}^1\\ v_1^2 & \dots & v_{m_v}^2\\ \vdots & \ddots & \vdots\\ v_1^{n_v} & \dots & v_{m_v}^{n_v}\end{bmatrix}, \tag{93}$$

with

$$\mathbb{I}_{\Gamma}^i = \begin{cases}1, & \text{if } i\in\mathcal{N}_{\Gamma_0^{\text{f-s}}},\\ 0, & \text{else,}\end{cases} \tag{94}$$

4.3 Discrete FrSI equations

- discrete fluid domain displacement:

$$\mathbf{u}_{n+1}^{\mathrm{f}} = \theta\Delta t\,\mathbf{v}_{n+1} + (1-\theta)\Delta t\,\mathbf{v}_n + \mathbf{u}_n^{\mathrm{f}} \tag{95}$$

- trial projection:

$$\mathbf{v} = \mathbf{V}_v^{\Gamma}\tilde{\mathbf{v}} \tag{96}$$

- test projection

$$\mathbf{r}_{n+1}^{\mathrm{FrSI}} = \begin{bmatrix}\mathbf{V}_v^{\Gamma\,\mathrm{T}}\mathbf{r}_v(\mathbf{V}_v^{\Gamma}\tilde{\mathbf{v}},\mathbf{p},\boldsymbol{\Lambda},\mathbf{d})\\ \mathbf{r}_p(\mathbf{V}_v^{\Gamma}\tilde{\mathbf{v}},\mathbf{d})\\ \mathbf{r}_{\Lambda}(\boldsymbol{\Lambda},\mathbf{V}_v^{\Gamma}\tilde{\mathbf{v}},\mathbf{d})\\ \mathbf{r}_d(\mathbf{d})\end{bmatrix}_{n+1} = \mathbf{0} \tag{97}$$

$$\begin{bmatrix}\mathbf{V}_v^{\Gamma\,\mathrm{T}}\mathbf{K}_{vv}\mathbf{V}_v^{\Gamma} & \mathbf{V}_v^{\Gamma\,\mathrm{T}}\mathbf{K}_{vp} & \mathbf{V}_v^{\Gamma\,\mathrm{T}}\mathbf{K}_{v\Lambda} & \mathbf{V}_v^{\Gamma\,\mathrm{T}}\mathbf{K}_{vd}\\ \mathbf{K}_{pv}\mathbf{V}_v^{\Gamma} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{pd}\\ \mathbf{K}_{\Lambda v}\mathbf{V}_v^{\Gamma} & \mathbf{0} & \mathbf{K}_{\Lambda\Lambda} & \mathbf{K}_{\Lambda d}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{dd}\end{bmatrix}_{n+1}^{k}\begin{bmatrix}\Delta\tilde{\mathbf{v}}\\ \Delta\mathbf{p}\\ \Delta\boldsymbol{\Lambda}\\ \Delta\mathbf{d}\end{bmatrix}_{n+1}^{k+1} = -\begin{bmatrix}\mathbf{V}_v^{\Gamma\,\mathrm{T}}\mathbf{r}_v\\ \mathbf{r}_p\\ \mathbf{r}_{\Lambda}\\ \mathbf{r}_d\end{bmatrix}_{n+1}^{k} \tag{98}$$

4.4 Reduced physics solid

5 Lumped parameter (0D) models

5.1 "Syspul" circulation model

$$
\begin{aligned}
&\text{left heart and systemic circulation} && \\
&-Q_{\mathrm{at}}^{\ell}=\sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{pul}}} q_{\mathrm{ven},i}^{\mathrm{pul}}-q_{\mathrm{v,in}}^{\ell} && \text{left atrium flow balance} \\
&q_{\mathrm{v,in}}^{\ell}=q_{\mathrm{mv}}(p_{\mathrm{at}}^{\ell}-p_{\mathrm{v}}^{\ell}) && \text{mitral valve momentum} \qquad (99)\\
&-Q_{\mathrm{v}}^{\ell}=q_{\mathrm{v,in}}^{\ell}-q_{\mathrm{v,out}}^{\ell} && \text{left ventricle flow balance} \\
&q_{\mathrm{v,out}}^{\ell}=q_{\mathrm{av}}(p_{\mathrm{v}}^{\ell}-p_{\mathrm{ar}}^{\mathrm{sys}}) && \text{aortic valve momentum} \qquad (100)\\
&-Q_{\mathrm{aort}}^{\mathrm{sys}}=q_{\mathrm{v,out}}^{\ell}-q_{\mathrm{ar,p}}^{\mathrm{sys}}-\mathbb{I}^{\mathrm{cor}}\sum_{i=1}^{2} q_{\mathrm{ar,cor,in},i}^{\mathrm{sys}} && \text{aortic root flow balance} \\
&I_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t}+Z_{\mathrm{ar}}^{\mathrm{sys}}\,q_{\mathrm{ar,p}}^{\mathrm{sys}}=p_{\mathrm{ar}}^{\mathrm{sys}}-p_{\mathrm{ar,d}}^{\mathrm{sys}} && \text{aortic root inertia} \\
&C_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t}=q_{\mathrm{ar,p}}^{\mathrm{sys}}-q_{\mathrm{ar}}^{\mathrm{sys}} && \text{systemic arterial flow balance} \\
&L_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t}+R_{\mathrm{ar}}^{\mathrm{sys}}\,q_{\mathrm{ar}}^{\mathrm{sys}}=p_{\mathrm{ar,d}}^{\mathrm{sys}}-p_{\mathrm{ven}}^{\mathrm{sys}} && \text{systemic arterial momentum} \\
&C_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t}=q_{\mathrm{ar}}^{\mathrm{sys}}-\sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{sys}}} q_{\mathrm{ven},i}^{\mathrm{sys}} && \text{systemic venous flow balance} \\
&L_{\mathrm{ven},i}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t}+R_{\mathrm{ven},i}^{\mathrm{sys}}\,q_{\mathrm{ven},i}^{\mathrm{sys}}=p_{\mathrm{ven}}^{\mathrm{sys}}-p_{\mathrm{at},i}^{r} && \text{systemic venous momentum} \\
&\qquad\qquad i\in\{1,...,n_{\mathrm{ven}}^{\mathrm{sys}}\} &&
\end{aligned}
$$

right heart and pulmonary circulation

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{r} = \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{sys}}} q_{\mathrm{ven},i}^{\mathrm{sys}} - \mathbb{I}^{\mathrm{cor}} q_{\mathrm{ven,cor,out}}^{\mathrm{sys}} - q_{\mathrm{v,in}}^{r} && \text{right atrium flow balance} \\
&q_{\mathrm{v,in}}^{r} = q_{\mathrm{tv}}(p_{\mathrm{at}}^{r} - p_{\mathrm{v}}^{r}) && \text{tricuspid valve momentum} && (101) \\
&-Q_{\mathrm{v}}^{r} = q_{\mathrm{v,in}}^{r} - q_{\mathrm{v,out}}^{r} && \text{right ventricle flow balance} \\
&q_{\mathrm{v,out}}^{r} = q_{\mathrm{pv}}(p_{\mathrm{v}}^{r} - p_{\mathrm{ar}}^{\mathrm{pul}}) && \text{pulmonary valve momentum} && (102) \\
&C_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{v,out}}^{r} - q_{\mathrm{ar}}^{\mathrm{pul}} && \text{pulmonary arterial flow balance} \\
&L_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{pul}}\, q_{\mathrm{ar}}^{\mathrm{pul}} = p_{\mathrm{ar}}^{\mathrm{pul}} - p_{\mathrm{ven}}^{\mathrm{pul}} && \text{pulmonary arterial momentum} \\
&C_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{pul}} - \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{pul}}} q_{\mathrm{ven},i}^{\mathrm{pul}} && \text{pulmonary venous flow balance} \\
&L_{\mathrm{ven},i}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ven},i}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ven},i}^{\mathrm{pul}}\, q_{\mathrm{ven},i}^{\mathrm{pul}} = p_{\mathrm{ven}}^{\mathrm{pul}} - p_{\mathrm{at},i}^{\ell} && \text{pulmonary venous momentum} \\
&\qquad i \in \{1, ..., n_{\mathrm{ven}}^{\mathrm{pul}}\}
\end{aligned}
$$

with:

$$
Q_{\mathrm{at}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{at}}^{\ell}}{\mathrm{d}t}, \quad Q_{\mathrm{v}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{v}}^{\ell}}{\mathrm{d}t}, \quad Q_{\mathrm{at}}^{r} := -\frac{\mathrm{d}V_{\mathrm{at}}^{r}}{\mathrm{d}t}, \quad Q_{\mathrm{v}}^{r} := -\frac{\mathrm{d}V_{\mathrm{v}}^{r}}{\mathrm{d}t}, \quad Q_{\mathrm{aort}}^{\mathrm{sys}} := -\frac{\mathrm{d}V_{\mathrm{aort}}^{\mathrm{sys}}}{\mathrm{d}t}
$$

and:

$$
\mathbb{I}^{\mathrm{cor}} = \begin{cases} 1, & \text{if} \quad \text{CORONARY\_MODEL}, \\ 0, & \text{else} \end{cases}
$$

The volume V of the heart chambers (0D) is modeled by the volume-pressure relationship

$$
V(t) = \frac{p}{E(t)} + V_{\mathrm{u}}, \tag{103}
$$

with the unstressed volume V_{u} and the time-varying elastance

$$
E(t) = (E_{\max} - E_{\min}) \cdot \hat{y}(t) + E_{\min}, \tag{104}
$$

where $E_{\max}$ and $E_{\min}$ denote the maximum and minimum elastance, respectively. The normalized activation function $\hat{y}(t)$ is input by the user.

Flow-pressure relations for the four valves, eq. (99), (100), (101), (102), are functions of the pressure difference $p-p_{\mathrm{open}}$ across the valve. The following valve models can be defined:

Valve model `pwlin_pres`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} R_{\text{max}}, & p < p_{\text{open}} \\ R_{\text{min}}, & p \geq p_{\text{open}} \end{cases}$$

Remark: Non-smooth flow-pressure relationship

Valve model `pwlin_time`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} \begin{cases} R_{\text{max}}, & t < t_{\text{open}} \text{ and } t \geq t_{\text{close}} \\ R_{\text{min}}, & t \geq t_{\text{open}} \text{ or } t < t_{\text{close}} \end{cases}, & t_{\text{open}} > t_{\text{close}} \\ \begin{cases} R_{\text{max}}, & t < t_{\text{open}} \text{ or } t \geq t_{\text{close}} \\ R_{\text{min}}, & t \geq t_{\text{open}} \text{ and } t < t_{\text{close}} \end{cases}, & \text{else} \end{cases}$$

Remark: Non-smooth flow-pressure relationship with resistance only dependent on timings, not the pressure difference!

Valve model `smooth_pres_resistance`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = 0.5\,(R_{\text{max}} - R_{\text{min}}) \left(\tanh \frac{p - p_{\text{open}}}{\epsilon} + 1 \right) + R_{\text{min}}$$

Remark: Smooth but potentially non-convex flow-pressure relationship!

Valve model `smooth_pres_momentum`:

$$q(p - p_{\text{open}}) = \begin{cases} \frac{p - p_{\text{open}}}{R_{\text{max}}}, & p < p_{\text{open}} - 0.5\epsilon \\ h_{00} p_0 + h_{10} m_0 \epsilon + h_{01} p_1 + h_{11} m_1 \epsilon, & p \geq p_{\text{open}} - 0.5\epsilon \text{ and } p < p_{\text{open}} + 0.5\epsilon \\ \frac{p - p_{\text{open}}}{R_{\text{min}}}, & p \geq p_{\text{open}} + 0.5\epsilon \end{cases}$$

with

$$p_0 = \frac{-0.5\epsilon}{R_{\text{max}}}, \qquad m_0 = \frac{1}{R_{\text{max}}}, \qquad p_1 = \frac{0.5\epsilon}{R_{\text{min}}}, \qquad m_1 = \frac{1}{R_{\text{min}}}$$

and

$$\begin{aligned} h_{00} &= 2s^3 - 3s^2 + 1, & h_{01} &= -2s^3 + 3s^2, \\ h_{10} &= s^3 - 2s^2 + s, & h_{11} &= s^3 - s^2 \end{aligned}$$

with

$$s = \frac{p - p_{\text{open}} + 0.5\epsilon}{\epsilon}$$

Remarks:
– Collapses to valve model `pwlin_pres` for $\epsilon = 0$
– Smooth and convex flow-pressure relationship

Valve model `pw_pres_regurg`:

$$q(p - p_{\mathrm{open}}) = \begin{cases} cA_{\mathrm{o}}\sqrt{p - p_{\mathrm{open}}}, & p < p_{\mathrm{open}} \\ \frac{p-p_{\mathrm{open}}}{R_{\mathrm{min}}}, & p \geq p_{\mathrm{open}} \end{cases}$$

Remark: Model to allow a regurgitant valve in the closed state, degree of regurgitation can be varied by specifying the valve regurgitant area A_{o}

Coronary circulation model:

$$\begin{aligned}
&C_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\left(\frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{sys},\ell}}{\mathrm{d}t} - Z_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\frac{\mathrm{d}q_{\mathrm{cor,p,in}}^{\mathrm{sys},\ell}}{\mathrm{d}t}\right) = q_{\mathrm{cor,p,in}}^{\mathrm{sys},\ell} - q_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\\
&R_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\,q_{\mathrm{cor,p}}^{\mathrm{sys},\ell} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{cor,d}}^{\mathrm{sys},\ell} - Z_{\mathrm{cor,p}}^{\mathrm{sys},\ell}\,q_{\mathrm{cor,p,in}}^{\mathrm{sys},\ell}\\
&C_{\mathrm{cor,d}}^{\mathrm{sys},\ell}\frac{\mathrm{d}(p_{\mathrm{cor,d}}^{\mathrm{sys},\ell} - p_{\mathrm{v}}^{\ell})}{\mathrm{d}t} = q_{\mathrm{cor,p}}^{\mathrm{sys},\ell} - q_{\mathrm{cor,d}}^{\mathrm{sys},\ell}\\
&R_{\mathrm{cor,d}}^{\mathrm{sys},\ell}\,q_{\mathrm{cor,d}}^{\mathrm{sys},\ell} = p_{\mathrm{cor,d}}^{\mathrm{sys},\ell} - p_{\mathrm{at}}^{r}\\
&C_{\mathrm{cor,p}}^{\mathrm{sys},r}\left(\frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{sys},r}}{\mathrm{d}t} - Z_{\mathrm{cor,p}}^{\mathrm{sys},r}\frac{\mathrm{d}q_{\mathrm{cor,p,in}}^{\mathrm{sys},r}}{\mathrm{d}t}\right) = q_{\mathrm{cor,p,in}}^{\mathrm{sys},r} - q_{\mathrm{cor,p}}^{\mathrm{sys},r}\\
&R_{\mathrm{cor,p}}^{\mathrm{sys},r}\,q_{\mathrm{cor,p}}^{\mathrm{sys},r} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{cor,d}}^{\mathrm{sys},r} - Z_{\mathrm{cor,p}}^{\mathrm{sys},r}\,q_{\mathrm{cor,p,in}}^{\mathrm{sys},r}\\
&C_{\mathrm{cor,d}}^{\mathrm{sys},r}\frac{\mathrm{d}(p_{\mathrm{cor,d}}^{\mathrm{sys},r} - p_{\mathrm{v}}^{\ell})}{\mathrm{d}t} = q_{\mathrm{cor,p}}^{\mathrm{sys},r} - q_{\mathrm{cor,d}}^{\mathrm{sys},r}\\
&R_{\mathrm{cor,d}}^{\mathrm{sys},r}\,q_{\mathrm{cor,d}}^{\mathrm{sys},r} = p_{\mathrm{cor,d}}^{\mathrm{sys},r} - p_{\mathrm{at}}^{r}\\
&0 = q_{\mathrm{cor,d}}^{\mathrm{sys},\ell} + q_{\mathrm{cor,d}}^{\mathrm{sys},r} - q_{\mathrm{cor,d,out}}^{\mathrm{sys}}
\end{aligned}$$

5.2 "Syspulcap" circulation model

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{\ell} = q_{\mathrm{ven}}^{\mathrm{pul}} - q_{\mathrm{v,in}}^{\ell}\\
&\tilde{R}_{\mathrm{v,in}}^{\ell}\, q_{\mathrm{v,in}}^{\ell} = p_{\mathrm{at}}^{\ell} - p_{\mathrm{v}}^{\ell}\\
&-Q_{\mathrm{v}}^{\ell} = q_{\mathrm{v,in}}^{\ell} - q_{\mathrm{v,out}}^{\ell}\\
&\tilde{R}_{\mathrm{v,out}}^{\ell}\, q_{\mathrm{v,out}}^{\ell} = p_{\mathrm{v}}^{\ell} - p_{\mathrm{ar}}^{\mathrm{sys}}\\
&0 = q_{\mathrm{v,out}}^{\ell} - q_{\mathrm{ar,p}}^{\mathrm{sys}}\\
&I_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t} + Z_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar,p}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ar,d}}^{\mathrm{sys}}\\
&C_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar}}^{\mathrm{sys}}\\
&L_{\mathrm{ar}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar}}^{\mathrm{sys}} = p_{\mathrm{ar,d}}^{\mathrm{sys}} - p_{\mathrm{ar,peri}}^{\mathrm{sys}}\\
&\left(\sum_{j\in\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\}} C_{\mathrm{ar},j}^{\mathrm{sys}}\right) \frac{\mathrm{d}p_{\mathrm{ar,peri}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{sys}} - \sum_{j\in\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\}} q_{\mathrm{ar},j}^{\mathrm{sys}}\\
&R_{\mathrm{ar},i}^{\mathrm{sys}}\, q_{\mathrm{ar},i}^{\mathrm{sys}} = p_{\mathrm{ar,peri}}^{\mathrm{sys}} - p_{\mathrm{ven},i}^{\mathrm{sys}}, \quad i\in\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\}\\
&C_{\mathrm{ven},i}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar},i}^{\mathrm{sys}} - q_{\mathrm{ven},i}^{\mathrm{sys}}, \quad i\in\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\}\\
&R_{\mathrm{ven},i}^{\mathrm{sys}}\, q_{\mathrm{ven},i}^{\mathrm{sys}} = p_{\mathrm{ven},i}^{\mathrm{sys}} - p_{\mathrm{ven}}^{\mathrm{sys}}, \quad i\in\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}\}\\
&C_{\mathrm{ven}}^{\mathrm{sys}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} = \sum_{j=\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer,cor}}} q_{\mathrm{ven},j}^{\mathrm{sys}} - q_{\mathrm{ven}}^{\mathrm{sys}}\\
&L_{\mathrm{ven}}^{\mathrm{sys}} \frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ven}}^{\mathrm{sys}}\, q_{\mathrm{ven}}^{\mathrm{sys}} = p_{\mathrm{ven}}^{\mathrm{sys}} - p_{\mathrm{at}}^{r}
\end{aligned}
$$

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{r} = q_{\mathrm{ven}}^{\mathrm{sys}} - q_{\mathrm{v,in}}^{r} \\
&\tilde{R}_{\mathrm{v,in}}^{r}\, q_{\mathrm{v,in}}^{r} = p_{\mathrm{at}}^{r} - p_{\mathrm{v}}^{r} \\
&-Q_{\mathrm{v}}^{r} = q_{\mathrm{v,in}}^{r} - q_{\mathrm{v,out}}^{r} \\
&\tilde{R}_{\mathrm{v,out}}^{r}\, q_{\mathrm{v,out}}^{r} = p_{\mathrm{v}}^{r} - p_{\mathrm{ar}}^{\mathrm{pul}} \\
&C_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{v,out}}^{r} - q_{\mathrm{ar}}^{\mathrm{pul}} \\
&L_{\mathrm{ar}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{pul}}\, q_{\mathrm{ar}}^{\mathrm{pul}} = p_{\mathrm{ar}}^{\mathrm{pul}} - p_{\mathrm{cap}}^{\mathrm{pul}} \\
&C_{\mathrm{cap}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{cap}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{pul}} - q_{\mathrm{cap}}^{\mathrm{pul}} \\
&R_{\mathrm{cap}}^{\mathrm{pul}}\, q_{\mathrm{cap}}^{\mathrm{pul}} = p_{\mathrm{cap}}^{\mathrm{pul}} - p_{\mathrm{ven}}^{\mathrm{pul}} \\
&C_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} = q_{\mathrm{cap}}^{\mathrm{pul}} - q_{\mathrm{ven}}^{\mathrm{pul}} \\
&L_{\mathrm{ven}}^{\mathrm{pul}} \frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{pul}}}{\mathrm{d}t} + R_{\mathrm{ven}}^{\mathrm{pul}}\, q_{\mathrm{ven}}^{\mathrm{pul}} = p_{\mathrm{ven}}^{\mathrm{pul}} - p_{\mathrm{at}}^{\ell}
\end{aligned}
$$

with:

$$
Q_{\mathrm{at}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{at}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\mathrm{v}}^{\ell} := -\frac{\mathrm{d}V_{\mathrm{v}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\mathrm{at}}^{r} := -\frac{\mathrm{d}V_{\mathrm{at}}^{r}}{\mathrm{d}t}, \qquad Q_{\mathrm{v}}^{r} := -\frac{\mathrm{d}V_{\mathrm{v}}^{r}}{\mathrm{d}t}
$$

5.3 “Syspulcapcor” circulation model

$$
\begin{aligned}
&-Q_{\mathrm{at}}^{\ell} = q_{\mathrm{ven}}^{\mathrm{pul}} - q_{\mathrm{v,in}}^{\ell}\\
&\tilde{R}_{\mathrm{v,in}}^{\ell}\,q_{\mathrm{v,in}}^{\ell} = p_{\mathrm{at}}^{\ell} - p_{\mathrm{v}}^{\ell}\\
&-Q_{\mathrm{v}}^{\ell} = q_{\mathrm{v,in}}^{\ell} - q_{\mathrm{v,out}}^{\ell}\\
&\tilde{R}_{\mathrm{v,out}}^{\ell}\,q_{\mathrm{v,out}}^{\ell} = p_{\mathrm{v}}^{\ell} - p_{\mathrm{ar}}^{\mathrm{sys}}\\
&0 = q_{\mathrm{v,out}}^{\ell} - q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar,cor,in}}^{\mathrm{sys}}\\
&I_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t} + Z_{\mathrm{ar}}^{\mathrm{sys}}\,q_{\mathrm{ar,p}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ar,d}}^{\mathrm{sys}}\\
&C_{\mathrm{ar,cor}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,cor,in}}^{\mathrm{sys}} - q_{\mathrm{ar,cor}}^{\mathrm{sys}}\\
&R_{\mathrm{ar,cor}}^{\mathrm{sys}}\,q_{\mathrm{ar,cor}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ven,cor}}^{\mathrm{sys}}\\
&C_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar}}^{\mathrm{sys}}\\
&L_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{sys}}\,q_{\mathrm{ar}}^{\mathrm{sys}} = p_{\mathrm{ar,d}}^{\mathrm{sys}} - p_{\mathrm{ar,peri}}^{\mathrm{sys}}\\
&\left(\sum_{j\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer}}\right\}} C_{\mathrm{ar},j}^{\mathrm{sys}}\right)\frac{\mathrm{d}p_{\mathrm{ar,peri}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{sys}} - \sum_{j\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer}}\right\}} q_{\mathrm{ar},j}^{\mathrm{sys}}\\
&R_{\mathrm{ar},i}^{\mathrm{sys}}\,q_{\mathrm{ar},i}^{\mathrm{sys}} = p_{\mathrm{ar,peri}}^{\mathrm{sys}} - p_{\mathrm{ven},i}^{\mathrm{sys}}, \quad {\scriptstyle i\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer}}\right\}}\\
&C_{\mathrm{ven},i}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar},i}^{\mathrm{sys}} - q_{\mathrm{ven},i}^{\mathrm{sys}}, \quad {\scriptstyle i\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer}}\right\}}\\
&R_{\mathrm{ven},i}^{\mathrm{sys}}\,q_{\mathrm{ven},i}^{\mathrm{sys}} = p_{\mathrm{ven},i}^{\mathrm{sys}} - p_{\mathrm{ven}}^{\mathrm{sys}}, \quad {\scriptstyle i\in\left\{\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer}}\right\}}\\
&C_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} = \sum_{j=\substack{\mathrm{spl,espl,}\\ \mathrm{msc,cer}}} q_{\mathrm{ven},j}^{\mathrm{sys}} - q_{\mathrm{ven}}^{\mathrm{sys}}\\
&L_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ven}}^{\mathrm{sys}}\,q_{\mathrm{ven}}^{\mathrm{sys}} = p_{\mathrm{ven}}^{\mathrm{sys}} - p_{\mathrm{at}}^{r}\\
&C_{\mathrm{ven,cor}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven,cor}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,cor}}^{\mathrm{sys}} - q_{\mathrm{ven,cor}}^{\mathrm{sys}}\\
&R_{\mathrm{ven,cor}}^{\mathrm{sys}}\,q_{\mathrm{ven,cor}}^{\mathrm{sys}} = p_{\mathrm{ven,cor}}^{\mathrm{sys}} - p_{\mathrm{at}}^{r}
\end{aligned}
$$

$$
\begin{aligned}
&-Q^r_{\text{at}} = q^{\text{sys}}_{\text{ven}} + q^{\text{sys}}_{\text{ven,cor}} - q^r_{\text{v,in}}\\
&\tilde{R}^r_{\text{v,in}}\, q^r_{\text{v,in}} = p^r_{\text{at}} - p^r_{\text{v}}\\
&-Q^r_{\text{v}} = q^r_{\text{v,in}} - q^r_{\text{v,out}}\\
&\tilde{R}^r_{\text{v,out}}\, q^r_{\text{v,out}} = p^r_{\text{v}} - p^{\text{pul}}_{\text{ar}}\\
&C^{\text{pul}}_{\text{ar}} \frac{\mathrm{d}p^{\text{pul}}_{\text{ar}}}{\mathrm{d}t} = q^r_{\text{v,out}} - q^{\text{pul}}_{\text{ar}}\\
&L^{\text{pul}}_{\text{ar}} \frac{\mathrm{d}q^{\text{pul}}_{\text{ar}}}{\mathrm{d}t} + R^{\text{pul}}_{\text{ar}}\, q^{\text{pul}}_{\text{ar}} = p^{\text{pul}}_{\text{ar}} - p^{\text{pul}}_{\text{cap}}\\
&C^{\text{pul}}_{\text{cap}} \frac{\mathrm{d}p^{\text{pul}}_{\text{cap}}}{\mathrm{d}t} = q^{\text{pul}}_{\text{ar}} - q^{\text{pul}}_{\text{cap}}\\
&R^{\text{pul}}_{\text{cap}}\, q^{\text{pul}}_{\text{cap}} = p^{\text{pul}}_{\text{cap}} - p^{\text{pul}}_{\text{ven}}\\
&C^{\text{pul}}_{\text{ven}} \frac{\mathrm{d}p^{\text{pul}}_{\text{ven}}}{\mathrm{d}t} = q^{\text{pul}}_{\text{cap}} - q^{\text{pul}}_{\text{ven}}\\
&L^{\text{pul}}_{\text{ven}} \frac{\mathrm{d}q^{\text{pul}}_{\text{ven}}}{\mathrm{d}t} + R^{\text{pul}}_{\text{ven}}\, q^{\text{pul}}_{\text{ven}} = p^{\text{pul}}_{\text{ven}} - p^{\ell}_{\text{at}}
\end{aligned}
$$

with:

$$
Q^{\ell}_{\text{at}} := -\frac{\mathrm{d}V^{\ell}_{\text{at}}}{\mathrm{d}t}, \qquad Q^{\ell}_{\text{v}} := -\frac{\mathrm{d}V^{\ell}_{\text{v}}}{\mathrm{d}t}, \qquad Q^{r}_{\text{at}} := -\frac{\mathrm{d}V^{r}_{\text{at}}}{\mathrm{d}t}, \qquad Q^{r}_{\text{v}} := -\frac{\mathrm{d}V^{r}_{\text{v}}}{\mathrm{d}t}
$$

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