

Ambit – A FEniCS-based cardiovascular physics solver

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1 Solid mechanics

- solid mechanics are formulated in a Total Lagrangian frame

1.1 Strong form

1.1.1 Displacement-based

- primary variable: displacement \mathbf{u}

$$\nabla_0 \cdot \mathbf{P}(\mathbf{u}, \mathbf{v}) + \hat{\mathbf{b}}_0 = \rho_0 \mathbf{a} \quad \text{in } \Omega_0 \times [0, T], \quad (1)$$

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Gamma_0^D \times [0, T], \quad (2)$$

$$\mathbf{t}_0 = \mathbf{P} \mathbf{n}_0 = \hat{\mathbf{t}}_0 \quad \text{on } \Gamma_0^N \times [0, T], \quad (3)$$

$$\mathbf{u}(\mathbf{x}_0, 0) = \hat{\mathbf{u}}_0(\mathbf{x}_0) \quad \text{in } \Omega_0, \quad (4)$$

$$\mathbf{v}(\mathbf{x}_0, 0) = \hat{\mathbf{v}}_0(\mathbf{x}_0) \quad \text{in } \Omega_0, \quad (5)$$

1.1.2 Incompressible mechanics

- primary variables: displacement \mathbf{u} and pressure p

$$\nabla_0 \cdot \mathbf{P}(\mathbf{u}, p, \mathbf{v}(\mathbf{u})) + \hat{\mathbf{b}}_0 = \rho_0 \mathbf{a}(\mathbf{u}) \quad \text{in } \Omega_0 \times [0, T], \quad (6)$$

$$J(\mathbf{u}) - 1 = 0 \quad \text{in } \Omega_0 \times [0, T], \quad (7)$$

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Gamma_0^D \times [0, T], \quad (8)$$

$$\mathbf{t}_0 = \mathbf{P}\mathbf{n}_0 = \hat{\mathbf{t}}_0 \quad \text{on } \Gamma_0^N \times [0, T], \quad (9)$$

$$\mathbf{u}(\mathbf{x}_0, 0) = \hat{\mathbf{u}}_0(\mathbf{x}_0) \quad \text{in } \Omega_0, \quad (10)$$

$$\mathbf{v}(\mathbf{x}_0, 0) = \hat{\mathbf{v}}_0(\mathbf{x}_0) \quad \text{in } \Omega_0, \quad (11)$$

with velocity and acceleration $\mathbf{v} = \frac{d\mathbf{u}}{dt}$ and $\mathbf{a} = \frac{d^2\mathbf{u}}{dt^2}$, respectively

1.2 Weak form

1.2.1 Displacement-based

- primary variable: displacement \mathbf{u}

- Principal of Virtual Work:

$$r(\mathbf{u}; \delta\mathbf{u}) := \delta\mathcal{W}_{\text{kin}}(\mathbf{u}; \delta\mathbf{u}) + \delta\mathcal{W}_{\text{int}}(\mathbf{u}; \delta\mathbf{u}) - \delta\mathcal{W}_{\text{ext}}(\mathbf{u}; \delta\mathbf{u}) = 0, \quad \forall \delta\mathbf{u} \quad (12)$$

- kinetic virtual work:

$$\delta\mathcal{W}_{\text{kin}}(\mathbf{u}; \delta\mathbf{u}) = \int_{\Omega_0} \rho_0 \mathbf{a}(\mathbf{u}) \cdot \delta\mathbf{u} \, dV \quad (13)$$

- internal virtual work:

$$\delta\mathcal{W}_{\text{int}}(\mathbf{u}; \delta\mathbf{u}) = \int_{\Omega_0} \mathbf{P}(\mathbf{u}, \mathbf{v}(\mathbf{u})) : \nabla_{\mathbf{X}} \delta\mathbf{u} \, dV \quad (14)$$

- external virtual work (conservative Neumann load):

$$\delta\mathcal{W}_{\text{ext}}(\delta\mathbf{u}) = \int_{\Gamma_0^N} \hat{\mathbf{t}}_0 \cdot \delta\mathbf{u} \, dA \quad (15)$$

1.2.2 Incompressible mechanics: 2-field displacement and pressure variables

- primary variables: displacement \mathbf{u} and pressure p

$$r_u(\mathbf{u}, p; \delta \mathbf{u}) := \delta \mathcal{W}_{\text{kin}}(\mathbf{u}; \delta \mathbf{u}) + \delta \mathcal{W}_{\text{int}}(\mathbf{u}, p; \delta \mathbf{u}) - \delta \mathcal{W}_{\text{ext}}(\mathbf{u}; \delta \mathbf{u}) = 0, \quad \forall \delta \mathbf{u} \quad (16)$$

$$r_p(\mathbf{u}; \delta p) := \delta \mathcal{W}_{\text{pres}}(\mathbf{u}; \delta p) = 0, \quad \forall \delta p \quad (17)$$

- kinetic virtual work: (13)
- internal virtual work:

$$\delta \mathcal{W}_{\text{int}}(\mathbf{u}, p; \delta \mathbf{u}) = \int_{\Omega_0} \mathbf{P}(\mathbf{u}, p, \mathbf{v}(\mathbf{u})) : \nabla_{\mathbf{X}} \delta \mathbf{u} \, dV \quad (18)$$

- pressure virtual work:

$$\delta \mathcal{W}_{\text{pres}}(\mathbf{u}; \delta p) = \int_{\Omega_0} (J(\mathbf{u}) - 1) \delta p \, dV \quad (19)$$

2 Fluid mechanics

2.1 Eulerian reference frame

- incompressible Navier-Stokes equations in Eulerian reference frame

2.1.1 Strong Form

- primary variables: velocity \mathbf{v} and pressure p

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{v}, p) + \hat{\mathbf{b}} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v}) \mathbf{v} \right) \quad \text{in } \Omega \times [0, T], \quad (20)$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_t \times [0, T], \quad (21)$$

$$\mathbf{v} = \hat{\mathbf{v}} \quad \text{on } \Gamma_t^{\text{D}} \times [0, T], \quad (22)$$

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} = \hat{\mathbf{t}} \quad \text{on } \Gamma_t^{\text{N}} \times [0, T], \quad (23)$$

$$\mathbf{v}(\mathbf{x}, 0) = \hat{\mathbf{v}}_0(\mathbf{x}) \quad \text{in } \Omega_t, \quad (24)$$

2.1.2 Weak Form

- primary variables: velocity \mathbf{v} and pressure p
- Principle of Virtual Power

$$r_v(\mathbf{v}, p; \delta \mathbf{v}) := \delta \mathcal{P}_{\text{kin}}(\mathbf{v}; \delta \mathbf{v}) + \delta \mathcal{P}_{\text{int}}(\mathbf{v}, p; \delta \mathbf{v}) - \delta \mathcal{P}_{\text{ext}}(\mathbf{v}; \delta \mathbf{v}) = 0, \quad \forall \delta \mathbf{v} \quad (25)$$

$$r_p(\mathbf{v}; \delta p) := \delta \mathcal{P}_{\text{pres}}(\mathbf{v}; \delta p), \quad \forall \delta p \quad (26)$$

- kinetic virtual power:

$$\delta\mathcal{P}_{\text{kin}}(\mathbf{v}; \delta\mathbf{v}) = \int_{\Omega_t} \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v}) \mathbf{v} \right) \cdot \delta\mathbf{v} \, dv \quad (27)$$

- internal virtual power:

$$\delta\mathcal{P}_{\text{int}}(\mathbf{v}, p; \delta\mathbf{v}) = \int_{\Omega_t} \boldsymbol{\sigma}(\mathbf{v}, p) : \nabla \delta\mathbf{v} \, dv \quad (28)$$

- pressure virtual power:

$$\delta\mathcal{P}_{\text{pres}}(\mathbf{v}; \delta p) = \int_{\Omega_t} (\nabla \cdot \mathbf{v}) \delta p \, dv \quad (29)$$

- external virtual power (Neumann load):

$$\delta\mathcal{P}_{\text{ext}}(\delta\mathbf{v}) = \int_{\Gamma_t^N} \hat{\mathbf{t}} \cdot \delta\mathbf{v} \, da \quad (30)$$

2.1.3 Stabilization

- Stabilization `supg_pspg2`:

- velocity residual operator (25) is augmented with the following terms:

$$r_v \leftarrow r_v + \int_{\Omega_t} d_1 ((\nabla \mathbf{v}) \mathbf{v}) \cdot (\nabla \delta\mathbf{v}) \mathbf{v} \, dv \quad (31)$$

$$+ \int_{\Omega_t} d_2 (\nabla \cdot \mathbf{v}) (\nabla \cdot \delta\mathbf{v}) \, dv \quad (32)$$

$$+ \int_{\Omega_t} d_3 (\nabla p) \cdot (\nabla \delta\mathbf{v}) \mathbf{v} \, dv \quad (33)$$

- pressure residual operator (26) is augmented with the following terms:

$$r_p \leftarrow r_p + \frac{1}{\rho} \int_{\Omega_t} d_1 ((\nabla \mathbf{v}) \mathbf{v}) \cdot (\nabla \delta p) \, dv \quad (34)$$

$$+ \frac{1}{\rho} \int_{\Omega_t} d_3 (\nabla p) \cdot (\nabla \delta p) \, dv \quad (35)$$

2.2 ALE reference frame

- to be written...

3 Fluid-reduced-solid-interaction (FrSI)

3.1 Physics-reduced solid

$$\Omega_0^s \mapsto \Gamma_0^{f-s}, \quad \rightsquigarrow \int_{\Omega_0^s} (\cdot) dV_0 = \int_{\Gamma_0^{f-s}} h_0(\cdot) dA_0, \quad (36)$$

h_0 : reduced solid's wall thickness parameter

3.1.1 Kinematics

$$\mathbf{F} = \nabla_{\mathbf{X}} \mathbf{x} = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{u}_f, \quad \dot{\mathbf{F}} = \nabla_{\mathbf{X}} \mathbf{v}_f, \quad \mathbf{C} = \mathbf{F}^T \mathbf{F}, \quad (37)$$

- fluid domain displacement:

$$\mathbf{u}_f = \int_0^t \mathbf{v}_f(\mathbf{X}, t) dt \quad (38)$$

- in-plane deformation and rate of deformation gradient:

$$\mathbf{F}^0 = \mathbf{F} - \mathbf{F} \mathbf{n}_0 \otimes \mathbf{n}_0 \quad \text{and} \quad \dot{\mathbf{F}}^0 = \dot{\mathbf{F}} - \dot{\mathbf{F}} \mathbf{n}_0 \otimes \mathbf{n}_0, \quad (39)$$

- plane strain representation of the right Cauchy-Green tensor:

$$\mathbf{C}^{\parallel} = \mathbf{F}^{0T} \mathbf{F}^0 + \mathbf{n}_0 \otimes \mathbf{n}_0 \quad (40)$$

- relation of in-plane and out-of-plane stretches

$$\mathbb{I}_C^{\parallel} = \det \mathbf{C}^{\parallel} = \lambda_{\xi}^2 \lambda_{\eta}^2 = \frac{1}{\lambda_{\zeta}^2}, \quad \rightsquigarrow \lambda_{\zeta} = \frac{1}{\sqrt{\mathbb{I}_C^{\parallel}}} \quad (41)$$

- membrane right Cauchy-Green deformation tensor

$$\tilde{\mathbf{C}} = \mathbf{F}^{0T} \mathbf{F}^0 + \frac{1}{\mathbb{I}_C^{\parallel}} \mathbf{n}_0 \otimes \mathbf{n}_0. \quad (42)$$

- rate:

$$\dot{\tilde{\mathbf{C}}} = \dot{\mathbf{F}}^{0T} \mathbf{F}^0 + \mathbf{F}^{0T} \dot{\mathbf{F}}^0 - \frac{\dot{\mathbb{I}}_C^{\parallel}}{\mathbb{I}_C^{\parallel 2}} \mathbf{n}_0 \otimes \mathbf{n}_0, \quad (43)$$

- time derivative of Eq. (41) with Jacobi's formula:

$$\dot{\mathbb{I}}_C^{\parallel} = \overline{\dot{\det \mathbf{C}^{\parallel}}} = \det \mathbf{C}^{\parallel} \operatorname{tr} \left(\mathbf{C}^{\parallel -1} \dot{\mathbf{C}}^{\parallel} \right) \quad (44)$$

$$= \mathbb{I}_C^{\parallel} \operatorname{tr} \left(\mathbf{C}^{\parallel -1} \left(\dot{\mathbf{F}}^{0T} \mathbf{F}^0 + \mathbf{F}^{0T} \dot{\mathbf{F}}^0 \right) \right). \quad (45)$$

3.1.2 Constitutive equations

General isotropic hyperelasticity - exemplified for general isotropic hyperelasticity

$$\Psi = \Psi(I_{\tilde{C}}, \mathbb{I}_{\tilde{C}}) - \frac{1}{2}p_s(\mathbb{III}_{\tilde{C}} - 1), \quad (46)$$

where $I_{\tilde{C}}$, $\mathbb{I}_{\tilde{C}}$ and $\mathbb{III}_{\tilde{C}}$ are the principal invariants of Eq. (42)

- 2nd Piola-Kirchhoff stress [3]:

$$\tilde{\mathbf{S}} = -p_s \tilde{\mathbf{C}}^{-1} + 2 \left(\frac{\partial \Psi}{\partial I_{\tilde{C}}} + I_{\tilde{C}} \frac{\partial \Psi}{\partial \mathbb{I}_{\tilde{C}}} \right) \mathbf{I} - 2 \frac{\partial \Psi}{\partial \mathbb{I}_{\tilde{C}}} \tilde{\mathbf{C}} \quad (47)$$

- hydrostatic pressure p_s recovered by plane stress assumption for 2-dimensional continua [3, 4]:

$$p_s = 2 \left(\frac{1}{\lambda_\xi^2 \lambda_\eta^2} \frac{\partial \Psi}{\partial I_{\tilde{C}}} - \lambda_\xi^2 \lambda_\eta^2 \frac{\partial \Psi}{\partial \mathbb{I}_{\tilde{C}}} \right) = 2 \left(\frac{1}{\mathbb{III}_{\tilde{C}}^{\parallel}} \frac{\partial \Psi}{\partial I_{\tilde{C}}} - \mathbb{III}_{\tilde{C}}^{\parallel} \frac{\partial \Psi}{\partial \mathbb{I}_{\tilde{C}}} \right), \quad (48)$$

- 1st Piola-Kirchhoff stress then is computed by the push-forward operation

$$\tilde{\mathbf{P}}_s = \mathbf{F}^0 \tilde{\mathbf{S}}, \quad (49)$$

Cardiac mechanics - isotropic exponential strain energy [2]

$$\Psi(\mathbf{C}, p_s) = \frac{a_0}{2b_0} (e^{b_0(\text{tr} \mathbf{C} - 3)} - 1) - \frac{1}{2}p_s(\det \mathbf{C} - 1) \quad (50)$$

- viscous pseudo-potential [1]

$$\Psi_v(\dot{\mathbf{C}}) = \frac{\eta}{8} \dot{\mathbf{C}} : \dot{\mathbf{C}} \quad (51)$$

- for 3D incompressible hyper-viscoelastic active cardiac mechanics:

$$\mathbf{S} = \mathbf{S}_e + \mathbf{S}_v + \mathbf{S}_a = 2 \frac{\partial \Psi(\mathbf{C}, p_s)}{\partial \mathbf{C}} + 2 \frac{\partial \Psi_v(\dot{\mathbf{C}})}{\partial \dot{\mathbf{C}}} + \tau_a(t) \mathbf{f}_0 \otimes \mathbf{f}_0 \quad (52)$$

- for reduced solid:

$$\tilde{\mathbf{S}} = \tilde{\mathbf{S}}_e + \tilde{\mathbf{S}}_v + \tilde{\mathbf{S}}_a = 2 \frac{\partial \Psi(\tilde{\mathbf{C}})}{\partial \tilde{\mathbf{C}}} + 2 \frac{\partial \Psi_v(\dot{\tilde{\mathbf{C}}})}{\partial \dot{\tilde{\mathbf{C}}}} + \tau_a(t) \tilde{\mathbf{M}}_0 \quad (53)$$

- note that in presence of viscosity according to Eq. (51), hydrostatic pressure for hyperelastic solid, Eq. (48), needs to be updated to balance the viscous normal stresses:

$$p_s \leftarrow p_s - \frac{\eta}{2} \frac{\dot{\mathbb{III}}_{\tilde{C}}^{\parallel}}{\mathbb{III}_{\tilde{C}}^{\parallel 3}} \quad (54)$$

- reduced (wall-averaged) structural tensor in circumferential and longitudinal directions \mathbf{c}_0 and \mathbf{l}_0 , respectively:

$$\tilde{\mathbf{M}}_0 = \bar{\omega} \mathbf{c}_0 \otimes \mathbf{c}_0 + \bar{\iota} \mathbf{l}_0 \otimes \mathbf{l}_0 + 2\bar{\gamma} \text{sym}(\mathbf{c}_0 \otimes \mathbf{l}_0). \quad (55)$$

Balance equation

$$\tilde{r}_s(\mathbf{v}_f; \delta \mathbf{v}_f) := \int_{\Gamma_0^{\text{f-s}}} h_0 \rho_{0,s} D_t \mathbf{v}_f \cdot \delta \mathbf{v}_f dA_0 + \int_{\Gamma_0^{\text{f-s}}} h_0 \tilde{\mathbf{P}}_s(\mathbf{u}_f(\mathbf{v}_f), \mathbf{v}_f) : \nabla_{\tilde{\mathbf{X}}} \delta \mathbf{v}_f dA_0 \quad (56)$$

3.2 Projection-based reduction

$$\hat{\mathbf{S}}_v^{\Gamma} = \mathbb{I}_{\Gamma} [\mathbf{v}_1^f \quad \dots \quad \mathbf{v}_{m_v}^f] = \begin{bmatrix} \mathbb{I}_{\Gamma}^1 & & \\ & \mathbb{I}_{\Gamma}^2 & \\ & & \ddots \\ & & & \mathbb{I}_{\Gamma}^{n_v} \end{bmatrix} \begin{bmatrix} v_1^1 & \dots & v_{m_v}^1 \\ v_1^2 & \dots & v_{m_v}^2 \\ \vdots & \ddots & \vdots \\ v_1^{n_v} & \dots & v_{m_v}^{n_v} \end{bmatrix}, \quad (57)$$

with

$$\mathbb{I}_{\Gamma}^i = \begin{cases} 1, & \text{if } i \in \mathcal{N}_{\Gamma_0^{\text{f-s}}}, \\ 0, & \text{else,} \end{cases} \quad (58)$$

3.3 Discrete FrSI equations

- discrete fluid domain displacement:

$$\mathbf{u}_{n+1}^f = \theta \Delta t \mathbf{v}_{n+1}^f + (1 - \theta) \Delta t \mathbf{v}_n^f + \mathbf{u}_n^f \quad (59)$$

$$\mathbf{r}_{n+1}^{\text{FrSI}} = \begin{bmatrix} \mathbf{V}_v^{\Gamma^T} \mathbf{r}_v^f(\mathbf{V}_v^{\Gamma} \tilde{\mathbf{v}}^f, \mathbf{p}^f, \boldsymbol{\Lambda}) \\ \mathbf{r}_p^f(\mathbf{V}_v^{\Gamma} \tilde{\mathbf{v}}^f) \\ \mathbf{r}_{\Lambda}^{\text{f-0d}}(\boldsymbol{\Lambda}, \mathbf{V}_v^{\Gamma} \tilde{\mathbf{v}}^f) \end{bmatrix}_{n+1} = \mathbf{0}, \quad \mathbf{r}_{n+1}^{\text{ALE}}(\mathbf{w}) = \mathbf{0}, \quad \mathbf{r}_{n+1}^{\text{0d}}(\mathbf{y}) = \mathbf{0} \quad (60)$$

$$\begin{bmatrix} \mathbf{V}_v^{\Gamma^T} \mathbf{K}_{vv}^f \mathbf{V}_v^{\Gamma} & \mathbf{V}_v^{\Gamma^T} \mathbf{K}_{vp}^f & \mathbf{V}_v^{\Gamma^T} \mathbf{K}_{v\Lambda}^f \\ \mathbf{K}_{pv}^f \mathbf{V}_v^{\Gamma} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{\Lambda v}^f \mathbf{V}_v^{\Gamma} & \mathbf{0} & \mathbf{K}_{\Lambda\Lambda}^{\text{f-0d}} \end{bmatrix}_{n+1}^k \begin{bmatrix} \Delta \tilde{\mathbf{v}}^f \\ \Delta \mathbf{p}^f \\ \Delta \boldsymbol{\Lambda} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{V}_v^{\Gamma^T} \mathbf{r}_v^f \\ \mathbf{r}_p^f \\ \mathbf{r}_{\Lambda}^{\text{f-0d}} \end{bmatrix}_{n+1}^k \quad (61)$$

- ALE and 0D linear systems: Eq. (??)

3.4 Reduced physics solid

4 Lumped parameter (0D) models

4.1 “Syspul” circulation model

left heart and systemic circulation

$$\begin{aligned}
 -Q_{\text{at}}^\ell &= \sum_{i=1}^{n_{\text{ven}}^{\text{pul}}} q_{\text{ven},i}^{\text{pul}} - q_{\text{v},\text{in}}^\ell && \text{left atrium flow balance} \\
 q_{\text{v},\text{in}}^\ell &= q_{\text{mv}}(p_{\text{at}}^\ell - p_{\text{v}}^\ell) && \text{mitral valve momentum} \\
 -Q_{\text{v}}^\ell &= q_{\text{v},\text{in}}^\ell - q_{\text{v},\text{out}}^\ell && \text{left ventricle flow balance}
 \end{aligned} \tag{62}$$

$$q_{\text{v},\text{out}}^\ell = q_{\text{av}}(p_{\text{v}}^\ell - p_{\text{ar}}^{\text{sys}}) \tag{63}$$

$$-Q_{\text{aort}}^{\text{sys}} = q_{\text{v},\text{out}}^\ell - q_{\text{ar},\text{p}}^{\text{sys}} - \mathbb{I}^{\text{cor}} \sum_{i=1}^2 q_{\text{ar},\text{cor},\text{in},i}^{\text{sys}} \tag{63}$$

$$I_{\text{ar}}^{\text{sys}} \frac{dq_{\text{ar},\text{p}}^{\text{sys}}}{dt} + Z_{\text{ar}}^{\text{sys}} q_{\text{ar},\text{p}}^{\text{sys}} = p_{\text{ar}}^{\text{sys}} - p_{\text{ar},\text{d}}^{\text{sys}} \tag{63}$$

$$C_{\text{ar}}^{\text{sys}} \frac{dp_{\text{ar},\text{d}}^{\text{sys}}}{dt} = q_{\text{ar},\text{p}}^{\text{sys}} - q_{\text{ar}}^{\text{sys}} \tag{63}$$

$$L_{\text{ar}}^{\text{sys}} \frac{dq_{\text{ar}}^{\text{sys}}}{dt} + R_{\text{ar}}^{\text{sys}} q_{\text{ar}}^{\text{sys}} = p_{\text{ar},\text{d}}^{\text{sys}} - p_{\text{ven}}^{\text{sys}} \tag{63}$$

$$C_{\text{ven}}^{\text{sys}} \frac{dp_{\text{ven}}^{\text{sys}}}{dt} = q_{\text{ar}}^{\text{sys}} - \sum_{i=1}^{n_{\text{ven}}^{\text{sys}}} q_{\text{ven},i}^{\text{sys}} \tag{63}$$

$$L_{\text{ven},i}^{\text{sys}} \frac{dq_{\text{ven},i}^{\text{sys}}}{dt} + R_{\text{ven},i}^{\text{sys}} q_{\text{ven},i}^{\text{sys}} = p_{\text{ven}}^{\text{sys}} - p_{\text{at},i}^r \tag{63}$$

$$i \in \{1, \dots, n_{\text{ven}}^{\text{sys}}\}$$

right heart and pulmonary circulation

$$-Q_{\text{at}}^r = \sum_{i=1}^{n_{\text{ven}}^{\text{sys}}} q_{\text{ven},i}^{\text{sys}} - \mathbb{I}^{\text{cor}} q_{\text{ven},\text{cor},\text{out}}^{\text{sys}} - q_{\text{v},\text{in}}^r \quad \text{right atrium flow balance}$$

$$q_{\text{v},\text{in}}^r = q_{\text{tv}}(p_{\text{at}}^r - p_{\text{v}}^r) \quad \text{tricuspid valve momentum} \quad (64)$$

$$-Q_{\text{v}}^r = q_{\text{v},\text{in}}^r - q_{\text{v},\text{out}}^r \quad \text{right ventricle flow balance}$$

$$q_{\text{v},\text{out}}^r = q_{\text{pv}}(p_{\text{v}}^r - p_{\text{ar}}^{\text{pul}}) \quad \text{pulmonary valve momentum} \quad (65)$$

$$C_{\text{ar}}^{\text{pul}} \frac{dp_{\text{ar}}^{\text{pul}}}{dt} = q_{\text{v},\text{out}}^r - q_{\text{ar}}^{\text{pul}} \quad \text{pulmonary arterial flow balance}$$

$$L_{\text{ar}}^{\text{pul}} \frac{dq_{\text{ar}}^{\text{pul}}}{dt} + R_{\text{ar}}^{\text{pul}} q_{\text{ar}}^{\text{pul}} = p_{\text{ar}}^{\text{pul}} - p_{\text{ven}}^{\text{pul}} \quad \text{pulmonary arterial momentum}$$

$$C_{\text{ven}}^{\text{pul}} \frac{dp_{\text{ven}}^{\text{pul}}}{dt} = q_{\text{ar}}^{\text{pul}} - \sum_{i=1}^{n_{\text{ven}}^{\text{pul}}} q_{\text{ven},i}^{\text{pul}} \quad \text{pulmonary venous flow balance}$$

$$L_{\text{ven},i}^{\text{pul}} \frac{dq_{\text{ven},i}^{\text{pul}}}{dt} + R_{\text{ven},i}^{\text{pul}} q_{\text{ven},i}^{\text{pul}} = p_{\text{ven}}^{\text{pul}} - p_{\text{at},i}^{\ell} \quad \text{pulmonary venous momentum}$$

$$i \in \{1, \dots, n_{\text{ven}}^{\text{pul}}\}$$

with:

$$Q_{\text{at}}^{\ell} := -\frac{dV_{\text{at}}^{\ell}}{dt}, \quad Q_{\text{v}}^{\ell} := -\frac{dV_{\text{v}}^{\ell}}{dt}, \quad Q_{\text{at}}^r := -\frac{dV_{\text{at}}^r}{dt}, \quad Q_{\text{v}}^r := -\frac{dV_{\text{v}}^r}{dt}, \quad Q_{\text{aort}}^{\text{sys}} := -\frac{dV_{\text{aort}}^{\text{sys}}}{dt}$$

and:

$$\mathbb{I}^{\text{cor}} = \begin{cases} 1, & \text{if CORONARY_MODEL,} \\ 0, & \text{else} \end{cases}$$

The volume V of the heart chambers (0D) is modeled by the volume-pressure relationship

$$V(t) = \frac{p}{E(t)} + V_{\text{u}}, \quad (66)$$

with the unstressed volume V_{u} and the time-varying elastance

$$E(t) = (E_{\text{max}} - E_{\text{min}}) \cdot \hat{y}(t) + E_{\text{min}}, \quad (67)$$

where E_{max} and E_{min} denote the maximum and minimum elastance, respectively. The normalized activation function $\hat{y}(t)$ is input by the user.

Flow-pressure relations for the four valves, eq. (62), (63), (64), (65), are functions of the pressure difference $p - p_{\text{open}}$ across the valve. The following valve models can be defined:

Valve model `pmlin_pres`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} R_{\text{max}}, & p < p_{\text{open}} \\ R_{\text{min}}, & p \geq p_{\text{open}} \end{cases}$$

Remark: Non-smooth flow-pressure relationship

Valve model `pmlin_time`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} \begin{cases} R_{\text{max}}, & t < t_{\text{open}} \text{ and } t \geq t_{\text{close}} \\ R_{\text{min}}, & t \geq t_{\text{open}} \text{ or } t < t_{\text{close}} \end{cases}, & t_{\text{open}} > t_{\text{close}} \\ \begin{cases} R_{\text{max}}, & t < t_{\text{open}} \text{ or } t \geq t_{\text{close}} \\ R_{\text{min}}, & t \geq t_{\text{open}} \text{ and } t < t_{\text{close}} \end{cases}, & \text{else} \end{cases}$$

Remark: Non-smooth flow-pressure relationship with resistance only dependent on timings, not the pressure difference!

Valve model `smooth_pres_resistance`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = 0.5 (R_{\text{max}} - R_{\text{min}}) \left(\tanh \frac{p - p_{\text{open}}}{\epsilon} + 1 \right) + R_{\text{min}}$$

Remark: Smooth but potentially non-convex flow-pressure relationship!

Valve model `smooth_pres_momentum`:

$$q(p - p_{\text{open}}) = \begin{cases} \frac{p - p_{\text{open}}}{R_{\text{max}}}, & p < p_{\text{open}} - 0.5\epsilon \\ h_{00}p_0 + h_{10}m_0\epsilon + h_{01}p_1 + h_{11}m_1\epsilon, & p \geq p_{\text{open}} - 0.5\epsilon \text{ and } p < p_{\text{open}} + 0.5\epsilon \\ \frac{p - p_{\text{open}}}{R_{\text{min}}}, & p \geq p_{\text{open}} + 0.5\epsilon \end{cases}$$

with

$$p_0 = \frac{-0.5\epsilon}{R_{\text{max}}}, \quad m_0 = \frac{1}{R_{\text{max}}}, \quad p_1 = \frac{0.5\epsilon}{R_{\text{min}}}, \quad m_1 = \frac{1}{R_{\text{min}}}$$

and

$$\begin{aligned} h_{00} &= 2s^3 - 3s^2 + 1, & h_{01} &= -2s^3 + 3s^2, \\ h_{10} &= s^3 - 2s^2 + s, & h_{11} &= s^3 - s^2 \end{aligned}$$

with

$$s = \frac{p - p_{\text{open}} + 0.5\epsilon}{\epsilon}$$

Remarks:

- Collapses to valve model `pwlin_pres` for $\epsilon = 0$
- Smooth and convex flow-pressure relationship

Valve model `pw_pres_regurg`:

$$q(p - p_{\text{open}}) = \begin{cases} cA_o \sqrt{p - p_{\text{open}}}, & p < p_{\text{open}} \\ \frac{p - p_{\text{open}}}{R_{\text{min}}}, & p \geq p_{\text{open}} \end{cases}$$

Remark: Model to allow a regurgitant valve in the closed state, degree of regurgitation can be varied by specifying the valve regurgitant area A_o

Coronary circulation model:

$$\begin{aligned} C_{\text{cor,p}}^{\text{sys},\ell} \left(\frac{dp_{\text{ar}}^{\text{sys},\ell}}{dt} - Z_{\text{cor,p}}^{\text{sys},\ell} \frac{dq_{\text{cor,p,in}}^{\text{sys},\ell}}{dt} \right) &= q_{\text{cor,p,in}}^{\text{sys},\ell} - q_{\text{cor,p}}^{\text{sys},\ell} \\ R_{\text{cor,p}}^{\text{sys},\ell} q_{\text{cor,p}}^{\text{sys},\ell} &= p_{\text{ar}}^{\text{sys}} - p_{\text{cor,d}}^{\text{sys},\ell} - Z_{\text{cor,p}}^{\text{sys},\ell} q_{\text{cor,p,in}}^{\text{sys},\ell} \\ C_{\text{cor,d}}^{\text{sys},\ell} \frac{d(p_{\text{cor,d}}^{\text{sys},\ell} - p_{\text{v}}^{\ell})}{dt} &= q_{\text{cor,p}}^{\text{sys},\ell} - q_{\text{cor,d}}^{\text{sys},\ell} \\ R_{\text{cor,d}}^{\text{sys},\ell} q_{\text{cor,d}}^{\text{sys},\ell} &= p_{\text{cor,d}}^{\text{sys},\ell} - p_{\text{at}}^r \\ C_{\text{cor,p}}^{\text{sys},r} \left(\frac{dp_{\text{ar}}^{\text{sys},r}}{dt} - Z_{\text{cor,p}}^{\text{sys},r} \frac{dq_{\text{cor,p,in}}^{\text{sys},r}}{dt} \right) &= q_{\text{cor,p,in}}^{\text{sys},r} - q_{\text{cor,p}}^{\text{sys},r} \\ R_{\text{cor,p}}^{\text{sys},r} q_{\text{cor,p}}^{\text{sys},r} &= p_{\text{ar}}^{\text{sys}} - p_{\text{cor,d}}^{\text{sys},r} - Z_{\text{cor,p}}^{\text{sys},r} q_{\text{cor,p,in}}^{\text{sys},r} \\ C_{\text{cor,d}}^{\text{sys},r} \frac{d(p_{\text{cor,d}}^{\text{sys},r} - p_{\text{v}}^{\ell})}{dt} &= q_{\text{cor,p}}^{\text{sys},r} - q_{\text{cor,d}}^{\text{sys},r} \\ R_{\text{cor,d}}^{\text{sys},r} q_{\text{cor,d}}^{\text{sys},r} &= p_{\text{cor,d}}^{\text{sys},r} - p_{\text{at}}^r \\ 0 &= q_{\text{cor,d}}^{\text{sys},\ell} + q_{\text{cor,d}}^{\text{sys},r} - q_{\text{cor,d,out}}^{\text{sys}} \end{aligned}$$

4.2 “Syspulcap” circulation model

$$\begin{aligned}
-Q_{\text{at}}^\ell &= q_{\text{ven}}^{\text{pul}} - q_{\text{v,in}}^\ell \\
\tilde{R}_{\text{v,in}}^\ell q_{\text{v,in}}^\ell &= p_{\text{at}}^\ell - p_{\text{v}}^\ell \\
-Q_{\text{v}}^\ell &= q_{\text{v,in}}^\ell - q_{\text{v,out}}^\ell \\
\tilde{R}_{\text{v,out}}^\ell q_{\text{v,out}}^\ell &= p_{\text{v}}^\ell - p_{\text{ar}}^{\text{sys}} \\
0 &= q_{\text{v,out}}^\ell - q_{\text{ar,p}}^{\text{sys}} \\
I_{\text{ar}}^{\text{sys}} \frac{dq_{\text{ar,p}}^{\text{sys}}}{dt} + Z_{\text{ar}}^{\text{sys}} q_{\text{ar,p}}^{\text{sys}} &= p_{\text{ar}}^{\text{sys}} - p_{\text{ar,d}}^{\text{sys}} \\
C_{\text{ar}}^{\text{sys}} \frac{dp_{\text{ar,d}}^{\text{sys}}}{dt} &= q_{\text{ar,p}}^{\text{sys}} - q_{\text{ar}}^{\text{sys}} \\
L_{\text{ar}}^{\text{sys}} \frac{dq_{\text{ar}}^{\text{sys}}}{dt} + R_{\text{ar}}^{\text{sys}} q_{\text{ar}}^{\text{sys}} &= p_{\text{ar,d}}^{\text{sys}} - p_{\text{ar,peri}}^{\text{sys}} \\
\left(\sum_{j \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor} \}} } C_{\text{ar},j}^{\text{sys}} \right) \frac{dp_{\text{ar,peri}}^{\text{sys}}}{dt} &= q_{\text{ar}}^{\text{sys}} - \sum_{j \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor} \}} } q_{\text{ar},j}^{\text{sys}} \\
R_{\text{ar},i}^{\text{sys}} q_{\text{ar},i}^{\text{sys}} &= p_{\text{ar,peri}}^{\text{sys}} - p_{\text{ven},i}^{\text{sys}}, \quad i \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor} \} \\
C_{\text{ven},i}^{\text{sys}} \frac{dp_{\text{ven},i}^{\text{sys}}}{dt} &= q_{\text{ar},i}^{\text{sys}} - q_{\text{ven},i}^{\text{sys}}, \quad i \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor} \} \\
R_{\text{ven},i}^{\text{sys}} q_{\text{ven},i}^{\text{sys}} &= p_{\text{ven},i}^{\text{sys}} - p_{\text{ven}}^{\text{sys}}, \quad i \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor} \} \\
C_{\text{ven}}^{\text{sys}} \frac{dp_{\text{ven}}^{\text{sys}}}{dt} &= \sum_{j = \substack{\text{spl,espl,} \\ \text{msc,cer,cor}}} q_{\text{ven},j}^{\text{sys}} - q_{\text{ven}}^{\text{sys}} \\
L_{\text{ven}}^{\text{sys}} \frac{dq_{\text{ven}}^{\text{sys}}}{dt} + R_{\text{ven}}^{\text{sys}} q_{\text{ven}}^{\text{sys}} &= p_{\text{ven}}^{\text{sys}} - p_{\text{at}}^r
\end{aligned}$$

$$\begin{aligned}
-Q_{\text{at}}^r &= q_{\text{ven}}^{\text{sys}} - q_{\text{v},\text{in}}^r \\
\tilde{R}_{\text{v},\text{in}}^r q_{\text{v},\text{in}}^r &= p_{\text{at}}^r - p_{\text{v}}^r \\
-Q_{\text{v}}^r &= q_{\text{v},\text{in}}^r - q_{\text{v},\text{out}}^r \\
\tilde{R}_{\text{v},\text{out}}^r q_{\text{v},\text{out}}^r &= p_{\text{v}}^r - p_{\text{ar}}^{\text{pul}} \\
C_{\text{ar}}^{\text{pul}} \frac{dp_{\text{ar}}^{\text{pul}}}{dt} &= q_{\text{v},\text{out}}^r - q_{\text{ar}}^{\text{pul}} \\
L_{\text{ar}}^{\text{pul}} \frac{dq_{\text{ar}}^{\text{pul}}}{dt} + R_{\text{ar}}^{\text{pul}} q_{\text{ar}}^{\text{pul}} &= p_{\text{ar}}^{\text{pul}} - p_{\text{cap}}^{\text{pul}} \\
C_{\text{cap}}^{\text{pul}} \frac{dp_{\text{cap}}^{\text{pul}}}{dt} &= q_{\text{ar}}^{\text{pul}} - q_{\text{cap}}^{\text{pul}} \\
R_{\text{cap}}^{\text{pul}} q_{\text{cap}}^{\text{pul}} &= p_{\text{cap}}^{\text{pul}} - p_{\text{ven}}^{\text{pul}} \\
C_{\text{ven}}^{\text{pul}} \frac{dp_{\text{ven}}^{\text{pul}}}{dt} &= q_{\text{cap}}^{\text{pul}} - q_{\text{ven}}^{\text{pul}} \\
L_{\text{ven}}^{\text{pul}} \frac{dq_{\text{ven}}^{\text{pul}}}{dt} + R_{\text{ven}}^{\text{pul}} q_{\text{ven}}^{\text{pul}} &= p_{\text{ven}}^{\text{pul}} - p_{\text{at}}^{\ell}
\end{aligned}$$

with:

$$Q_{\text{at}}^{\ell} := -\frac{dV_{\text{at}}^{\ell}}{dt}, \quad Q_{\text{v}}^{\ell} := -\frac{dV_{\text{v}}^{\ell}}{dt}, \quad Q_{\text{at}}^r := -\frac{dV_{\text{at}}^r}{dt}, \quad Q_{\text{v}}^r := -\frac{dV_{\text{v}}^r}{dt}$$

4.3 “Syspulcapcor” circulation model

$$\begin{aligned}
-Q_{\text{at}}^\ell &= q_{\text{ven}}^{\text{pul}} - q_{\text{v,in}}^\ell \\
\tilde{R}_{\text{v,in}}^\ell q_{\text{v,in}}^\ell &= p_{\text{at}}^\ell - p_{\text{v}}^\ell \\
-Q_{\text{v}}^\ell &= q_{\text{v,in}}^\ell - q_{\text{v,out}}^\ell \\
\tilde{R}_{\text{v,out}}^\ell q_{\text{v,out}}^\ell &= p_{\text{v}}^\ell - p_{\text{ar}}^{\text{sys}} \\
0 &= q_{\text{v,out}}^\ell - q_{\text{ar,p}}^{\text{sys}} - q_{\text{ar,cor,in}}^{\text{sys}} \\
I_{\text{ar}}^{\text{sys}} \frac{dq_{\text{ar,p}}^{\text{sys}}}{dt} + Z_{\text{ar}}^{\text{sys}} q_{\text{ar,p}}^{\text{sys}} &= p_{\text{ar}}^{\text{sys}} - p_{\text{ar,d}}^{\text{sys}} \\
C_{\text{ar,cor}}^{\text{sys}} \frac{dp_{\text{ar}}^{\text{sys}}}{dt} &= q_{\text{ar,cor,in}}^{\text{sys}} - q_{\text{ar,cor}}^{\text{sys}} \\
R_{\text{ar,cor}}^{\text{sys}} q_{\text{ar,cor}}^{\text{sys}} &= p_{\text{ar}}^{\text{sys}} - p_{\text{ven,cor}}^{\text{sys}} \\
C_{\text{ar}}^{\text{sys}} \frac{dp_{\text{ar,d}}^{\text{sys}}}{dt} &= q_{\text{ar,p}}^{\text{sys}} - q_{\text{ar}}^{\text{sys}} \\
L_{\text{ar}}^{\text{sys}} \frac{dq_{\text{ar}}^{\text{sys}}}{dt} + R_{\text{ar}}^{\text{sys}} q_{\text{ar}}^{\text{sys}} &= p_{\text{ar,d}}^{\text{sys}} - p_{\text{ar,peri}}^{\text{sys}} \\
\left(\sum_{j \in \left\{ \begin{smallmatrix} \text{spl,espl,} \\ \text{msc,cer} \end{smallmatrix} \right\}} C_{\text{ar},j}^{\text{sys}} \right) \frac{dp_{\text{ar,peri}}^{\text{sys}}}{dt} &= q_{\text{ar}}^{\text{sys}} - \sum_{j \in \left\{ \begin{smallmatrix} \text{spl,espl,} \\ \text{msc,cer} \end{smallmatrix} \right\}} q_{\text{ar},j}^{\text{sys}} \\
R_{\text{ar},i}^{\text{sys}} q_{\text{ar},i}^{\text{sys}} &= p_{\text{ar,peri}}^{\text{sys}} - p_{\text{ven},i}^{\text{sys}}, \quad i \in \left\{ \begin{smallmatrix} \text{spl,espl,} \\ \text{msc,cer} \end{smallmatrix} \right\} \\
C_{\text{ven},i}^{\text{sys}} \frac{dp_{\text{ven},i}^{\text{sys}}}{dt} &= q_{\text{ar},i}^{\text{sys}} - q_{\text{ven},i}^{\text{sys}}, \quad i \in \left\{ \begin{smallmatrix} \text{spl,espl,} \\ \text{msc,cer} \end{smallmatrix} \right\} \\
R_{\text{ven},i}^{\text{sys}} q_{\text{ven},i}^{\text{sys}} &= p_{\text{ven},i}^{\text{sys}} - p_{\text{ven}}^{\text{sys}}, \quad i \in \left\{ \begin{smallmatrix} \text{spl,espl,} \\ \text{msc,cer} \end{smallmatrix} \right\} \\
C_{\text{ven}}^{\text{sys}} \frac{dp_{\text{ven}}^{\text{sys}}}{dt} &= \sum_{j = \begin{smallmatrix} \text{spl,espl,} \\ \text{msc,cer} \end{smallmatrix}} q_{\text{ven},j}^{\text{sys}} - q_{\text{ven}}^{\text{sys}} \\
L_{\text{ven}}^{\text{sys}} \frac{dq_{\text{ven}}^{\text{sys}}}{dt} + R_{\text{ven}}^{\text{sys}} q_{\text{ven}}^{\text{sys}} &= p_{\text{ven}}^{\text{sys}} - p_{\text{at}}^r \\
C_{\text{ven,cor}}^{\text{sys}} \frac{dp_{\text{ven,cor}}^{\text{sys}}}{dt} &= q_{\text{ar,cor}}^{\text{sys}} - q_{\text{ven,cor}}^{\text{sys}} \\
R_{\text{ven,cor}}^{\text{sys}} q_{\text{ven,cor}}^{\text{sys}} &= p_{\text{ven,cor}}^{\text{sys}} - p_{\text{at}}^r
\end{aligned}$$

$$\begin{aligned}
-Q_{\text{at}}^r &= q_{\text{ven}}^{\text{sys}} + q_{\text{ven,cor}}^{\text{sys}} - q_{\text{v,in}}^r \\
\tilde{R}_{\text{v,in}}^r q_{\text{v,in}}^r &= p_{\text{at}}^r - p_{\text{v}}^r \\
-Q_{\text{v}}^r &= q_{\text{v,in}}^r - q_{\text{v,out}}^r \\
\tilde{R}_{\text{v,out}}^r q_{\text{v,out}}^r &= p_{\text{v}}^r - p_{\text{ar}}^{\text{pul}} \\
C_{\text{ar}}^{\text{pul}} \frac{dp_{\text{ar}}^{\text{pul}}}{dt} &= q_{\text{v,out}}^r - q_{\text{ar}}^{\text{pul}} \\
L_{\text{ar}}^{\text{pul}} \frac{dq_{\text{ar}}^{\text{pul}}}{dt} + R_{\text{ar}}^{\text{pul}} q_{\text{ar}}^{\text{pul}} &= p_{\text{ar}}^{\text{pul}} - p_{\text{cap}}^{\text{pul}} \\
C_{\text{cap}}^{\text{pul}} \frac{dp_{\text{cap}}^{\text{pul}}}{dt} &= q_{\text{ar}}^{\text{pul}} - q_{\text{cap}}^{\text{pul}} \\
R_{\text{cap}}^{\text{pul}} q_{\text{cap}}^{\text{pul}} &= p_{\text{cap}}^{\text{pul}} - p_{\text{ven}}^{\text{pul}} \\
C_{\text{ven}}^{\text{pul}} \frac{dp_{\text{ven}}^{\text{pul}}}{dt} &= q_{\text{cap}}^{\text{pul}} - q_{\text{ven}}^{\text{pul}} \\
L_{\text{ven}}^{\text{pul}} \frac{dq_{\text{ven}}^{\text{pul}}}{dt} + R_{\text{ven}}^{\text{pul}} q_{\text{ven}}^{\text{pul}} &= p_{\text{ven}}^{\text{pul}} - p_{\text{at}}^{\ell}
\end{aligned}$$

with:

$$Q_{\text{at}}^{\ell} := -\frac{dV_{\text{at}}^{\ell}}{dt}, \quad Q_{\text{v}}^{\ell} := -\frac{dV_{\text{v}}^{\ell}}{dt}, \quad Q_{\text{at}}^r := -\frac{dV_{\text{at}}^r}{dt}, \quad Q_{\text{v}}^r := -\frac{dV_{\text{v}}^r}{dt}$$

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