Ambit – A FEniCS-based cardiovascular physics solver

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October 18, 2023

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1 Preface

Ambit is an open-source software tool written in Python for parallel multi-physics simulations focusing on – but not limited to – cardiac mechanics. Amongst others, it contains re-implementations and generalizations of methods developed by the author for his PhD thesis [8]. Ambit makes use of the open-source finite element library FEniCS/dolfinx (https://fenicsproject.org) [13] along with the linear algebra package PETSc (https://petsc.org) [2]. It is constantly updated to ensure compatibility with a recent dolfinx development version, hence guaranteeing a state-of-the-art finite element and linear algebra backend.

Ambit is designed such that the user only needs to provide input files that define parameters through Python dictionaries, hence no programming or in-depth knowledge of any library-specific syntax is required.

Ambit provides general nonlinear (compressible or incompressible) finite strain solid dynamics [11], implementing a range of hyperelastic, viscous, and active material models. Specifically, the well-known anisotropic Holzapfel-Ogden [12] and Guccione models [7] for structural description of the myocardium are provided, along with a bunch of other models. It further implements strain- and stress-mediated volumetric growth models [6] that allow to model (maladaptive) ventricular shape and size changes. Inverse mechanics approaches to imprint loads into a reference state are implemented using the so-called prestressing method [4] in displacement formulation [14].

Furthermore, fluid dynamics in terms of incompressible Navier-Stokes/Stokes equations – either in Eulerian or Arbitrary Lagrangian-Eulerian (ALE) reference frames – are implemented. Taylor-Hood elements or equal-order approximations with SUPG/PSPG stabilization [15] can be used.

A variety of reduced 0D lumped models targeted at blood circulation modeling are implemented, including 3- and 4-element Windkessel models [16] as well as closed-loop full circulation [10] and coronary flow models [1].

Monolithic multi-physics coupling of solid, fluid, and ALE-fluid with 0D lumped models is implemented such that cardiovascular simulations with realistic boundary conditions can be performed. Monolithic fluid-solid interaction (FSI) will be provided in the near future once mixed domain functionality is fully supported in the finite element backend dolfinx.

Implementations for a recently proposed novel physics- and projection-based model reduction for FSI, denoted as fluid-reduced-solid interaction (FrSI) [9], are provided, along with POD-based Galerkin model reduction techniques [3] using full or boundary subspaces.

The nonlinear (single- or multi-field) problems are solved with a customized Newton solver

with PTC [5] adaptibity in case of divergence, providing robustness for numerically challenging problems. Linear solvers and preconditioners can be chosen from the PETSc repertoire, and specific block preconditioners are made available for coupled problems.

Avenues for future functionality include cardiac electrophysiology, scalar transport, or finite strain plasticity.

In the following, a brief description of the supported problem types is given, including the strong and weak form of the underlying equations as well as the discrete assembled systems that are solved.

Examples of input files for the respective problem types can be found in the folder `ambit/testing`.

2 Solid mechanics

– problem type: `solid`

– solid mechanics are formulated in a Total Lagrangian frame

2.1 Strong form

2.1.1 Displacement-based

– primary variable: displacement $\boldsymbol{u}$

$$\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{u}, \boldsymbol{v}(\boldsymbol{u})) + \hat{\boldsymbol{b}}_0 = \rho_0 \boldsymbol{a}(\boldsymbol{u}) \quad \text{in } \Omega_0 \times [0, T], \tag{1}$$

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \Gamma_0^{\mathrm{D}} \times [0, T], \tag{2}$$

$$\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 = \hat{\boldsymbol{t}}_0 \quad \text{on } \Gamma_0^{\mathrm{N}} \times [0, T], \tag{3}$$

$$\boldsymbol{u}(\boldsymbol{x}_0, 0) = \hat{\boldsymbol{u}}_0(\boldsymbol{x}_0) \quad \text{in } \Omega_0, \tag{4}$$

$$\boldsymbol{v}(\boldsymbol{x}_0, 0) = \hat{\boldsymbol{v}}_0(\boldsymbol{x}_0) \quad \text{in } \Omega_0, \tag{5}$$

2.1.2 Incompressible mechanics

– primary variables: displacement $\boldsymbol{u}$ and pressure p

$$\boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{u}, p, \boldsymbol{v}(\boldsymbol{u})) + \hat{\boldsymbol{b}}_0 = \rho_0 \boldsymbol{a}(\boldsymbol{u}) \quad \text{in } \Omega_0 \times [0, T], \tag{6}$$

$$J(\boldsymbol{u}) - 1 = 0 \quad \text{in } \Omega_0 \times [0, T], \tag{7}$$

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \Gamma_0^{\mathrm{D}} \times [0, T], \tag{8}$$

$$\boldsymbol{t}_0 = \boldsymbol{P}\boldsymbol{n}_0 = \hat{\boldsymbol{t}}_0 \quad \text{on } \Gamma_0^{\mathrm{N}} \times [0, T], \tag{9}$$

$$\boldsymbol{u}(\boldsymbol{x}_0, 0) = \hat{\boldsymbol{u}}_0(\boldsymbol{x}_0) \quad \text{in } \Omega_0, \tag{10}$$

$$\boldsymbol{v}(\boldsymbol{x}_0, 0) = \hat{\boldsymbol{v}}_0(\boldsymbol{x}_0) \quad \text{in } \Omega_0, \tag{11}$$

with velocity and acceleration $\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$ and $\boldsymbol{a} = \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}$, respectively

2.2 Weak form

2.2.1 Displacement-based

– primary variable: displacement $\boldsymbol{u}$

– Principal of Virtual Work:

$$r(\boldsymbol{u}; \delta\boldsymbol{u}) := \delta\mathcal{W}_{\mathrm{kin}}(\boldsymbol{u}; \delta\boldsymbol{u}) + \delta\mathcal{W}_{\mathrm{int}}(\boldsymbol{u}; \delta\boldsymbol{u}) - \delta\mathcal{W}_{\mathrm{ext}}(\boldsymbol{u}; \delta\boldsymbol{u}) = 0, \quad \forall\, \delta\boldsymbol{u} \tag{12}$$

– kinetic virtual work:

$$\delta\mathcal{W}_{\mathrm{kin}}(\boldsymbol{u}; \delta\boldsymbol{u}) = \int_{\Omega_0} \rho_0\, \boldsymbol{a}(\boldsymbol{u}) \cdot \delta\boldsymbol{u}\, \mathrm{d}V \tag{13}$$

– internal virtual work:

$$\delta\mathcal{W}_{\mathrm{int}}(\boldsymbol{u}; \delta\boldsymbol{u}) = \int_{\Omega_0} \boldsymbol{P}(\boldsymbol{u}, \boldsymbol{v}(\boldsymbol{u})) : \boldsymbol{\nabla}_0\delta\boldsymbol{u}\, \mathrm{d}V \tag{14}$$

– external virtual work:

- conservative Neumann load:

$$\delta\mathcal{W}_{\mathrm{ext}}(\delta\boldsymbol{u}) = \int_{\Gamma_0^{\mathrm{N}}} \hat{\boldsymbol{t}}_0(t) \cdot \delta\boldsymbol{u}\, \mathrm{d}A \tag{15}$$

- Neumann pressure load in current normal direction:

$$\delta\mathcal{W}_{\mathrm{ext}}(\boldsymbol{u}; \delta\boldsymbol{u}) = -\int_{\Gamma_0^{\mathrm{N}}} \hat{p}(t)\, J\boldsymbol{F}^{-\mathrm{T}}\boldsymbol{n}_0 \cdot \delta\boldsymbol{u}\, \mathrm{d}A \tag{16}$$

- general Neumann load in current direction:

$$\delta\mathcal{W}_{\mathrm{ext}}(\boldsymbol{u}; \delta\boldsymbol{u}) = \int_{\Gamma_0} J\boldsymbol{F}^{-\mathrm{T}}\, \hat{\boldsymbol{t}}_0(t) \cdot \delta\boldsymbol{u}\, \mathrm{d}A \tag{17}$$

- body force:

$$\delta\mathcal{W}_{\mathrm{ext}}(\delta\boldsymbol{u}) = \int_{\Omega_0} \hat{\boldsymbol{b}}_0(t) \cdot \delta\boldsymbol{u}\, \mathrm{d}V \tag{18}$$

- generalized Robin condition:

$$\delta\mathcal{W}_{\text{ext}}(\boldsymbol{u}; \delta\boldsymbol{u}) = -\int_{\Gamma_0^{\text{N}}} [k\,\boldsymbol{u} + c\,\boldsymbol{v}(\boldsymbol{u})] \cdot \delta\boldsymbol{u}\,\mathrm{d}A \tag{19}$$

- generalized Robin condition in reference surface normal direction:

$$\delta\mathcal{W}_{\text{ext}}(\boldsymbol{u}; \delta\boldsymbol{u}) = -\int_{\Gamma_0^{\text{N}}} (\boldsymbol{n}_0 \otimes \boldsymbol{n}_0)\,[k\,\boldsymbol{u} + c\,\boldsymbol{v}(\boldsymbol{u})] \cdot \delta\boldsymbol{u}\,\mathrm{d}A \tag{20}$$

2.2.2 Incompressible mechanics: 2-field displacement and pressure variables

– primary variables: displacement $\boldsymbol{u}$ and pressure p

$$r_u(\boldsymbol{u}, p; \delta\boldsymbol{u}) := \delta\mathcal{W}_{\text{kin}}(\boldsymbol{u}; \delta\boldsymbol{u}) + \delta\mathcal{W}_{\text{int}}(\boldsymbol{u}, p; \delta\boldsymbol{u}) - \delta\mathcal{W}_{\text{ext}}(\boldsymbol{u}; \delta\boldsymbol{u}) = 0, \quad \forall\,\delta\boldsymbol{u} \tag{21}$$

$$r_p(\boldsymbol{u}; \delta p) := \delta\mathcal{W}_{\text{pres}}(\boldsymbol{u}; \delta p) = 0, \quad \forall\,\delta p \tag{22}$$

– kinetic virtual work: (13)
– internal virtual work:

$$\delta\mathcal{W}_{\text{int}}(\boldsymbol{u}, p; \delta\boldsymbol{u}) = \int_{\Omega_0} \boldsymbol{P}(\boldsymbol{u}, p, \boldsymbol{v}(\boldsymbol{u})) : \boldsymbol{\nabla}_0\delta\boldsymbol{u}\,\mathrm{d}V \tag{23}$$

– pressure virtual work:

$$\delta\mathcal{W}_{\text{pres}}(\boldsymbol{u}; \delta p) = \int_{\Omega_0} (J(\boldsymbol{u}) - 1)\,\delta p\,\mathrm{d}V \tag{24}$$

3 Fluid mechanics

3.1 Eulerian reference frame

– problem type: `fluid`

– incompressible Navier-Stokes equations in Eulerian reference frame

3.1.1 Strong Form

– primary variables: velocity $\boldsymbol{v}$ and pressure p

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{v}, p) + \hat{\boldsymbol{b}} = \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla} \boldsymbol{v}) \, \boldsymbol{v} \right) \quad \text{in } \Omega_t \times [0, T], \tag{25}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \qquad\qquad \text{in } \Omega_t \times [0, T], \tag{26}$$

$$\boldsymbol{v} = \hat{\boldsymbol{v}} \qquad\qquad \text{on } \Gamma_t^{\mathrm{D}} \times [0, T], \tag{27}$$

$$\boldsymbol{t} = \boldsymbol{\sigma} \boldsymbol{n} = \hat{\boldsymbol{t}} \qquad\qquad \text{on } \Gamma_t^{\mathrm{N}} \times [0, T], \tag{28}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \hat{\boldsymbol{v}}_0(\boldsymbol{x}) \qquad\qquad \text{in } \Omega_t, \tag{29}$$

with a Newtonian fluid constitutive law

$$\boldsymbol{\sigma} = -p \boldsymbol{I} + 2\mu \, \boldsymbol{\gamma} = -p \boldsymbol{I} + \mu \left(\boldsymbol{\nabla} \boldsymbol{v} + (\boldsymbol{\nabla} \boldsymbol{v})^{\mathrm{T}} \right) \tag{30}$$

3.1.2 Weak Form

– primary variables: velocity $\boldsymbol{v}$ and pressure p

– Principle of Virtual Power

$$r_v(\boldsymbol{v}, p; \delta \boldsymbol{v}) := \delta \mathcal{P}_{\mathrm{kin}}(\boldsymbol{v}; \delta \boldsymbol{v}) + \delta \mathcal{P}_{\mathrm{int}}(\boldsymbol{v}, p; \delta \boldsymbol{v}) - \delta \mathcal{P}_{\mathrm{ext}}(\boldsymbol{v}; \delta \boldsymbol{v}) = 0, \quad \forall \, \delta \boldsymbol{v} \tag{31}$$

$$r_p(\boldsymbol{v}; \delta p) := \delta \mathcal{P}_{\mathrm{pres}}(\boldsymbol{v}; \delta p), \quad \forall \, \delta p \tag{32}$$

– kinetic virtual power:

$$\delta \mathcal{P}_{\mathrm{kin}}(\boldsymbol{v}; \delta \boldsymbol{v}) = \int_{\Omega_t} \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla} \boldsymbol{v}) \, \boldsymbol{v} \right) \cdot \delta \boldsymbol{v} \, \mathrm{d}v \tag{33}$$

– internal virtual power:

$$\delta \mathcal{P}_{\mathrm{int}}(\boldsymbol{v}, p; \delta \boldsymbol{v}) = \int_{\Omega_t} \boldsymbol{\sigma}(\boldsymbol{v}, p) : \boldsymbol{\nabla} \delta \boldsymbol{v} \, \mathrm{d}v \tag{34}$$

– pressure virtual power:

$$\delta \mathcal{P}_{\mathrm{pres}}(\boldsymbol{v}; \delta p) = \int_{\Omega_t} (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \, \delta p \, \mathrm{d}v \tag{35}$$

– external virtual power:

- conservative Neumann load:

$$\delta \mathcal{P}_{\mathrm{ext}}(\delta \boldsymbol{v}) = \int_{\Gamma_t^{\mathrm{N}}} \hat{\boldsymbol{t}}(t) \cdot \delta \boldsymbol{v} \, \mathrm{d}a \tag{36}$$

- pressure Neumann load:

$$\delta \mathcal{P}_{\text{ext}}(\delta \boldsymbol{v}) = - \int\limits_{\varGamma_t^{\text{N}}} \hat{p}(t) \, \boldsymbol{n} \cdot \delta \boldsymbol{v} \, \mathrm{d}a \tag{37}$$

- body force:

$$\delta \mathcal{P}_{\text{ext}}(\delta \boldsymbol{v}) = \int\limits_{\varOmega_t} \hat{\boldsymbol{b}}(t) \cdot \delta \boldsymbol{v} \, \mathrm{d}V \tag{38}$$

3.1.3 Stabilization

Streamline-upwind Petrov-Galerkin/pressure-stabilizing Petrov-Galerkin (SUPG/PSPG) methods are implemented, either using the full or a reduced scheme

Full scheme according to [15]: `supg_pspg`:
– velocity residual operator (31) is augmented with the following terms:

$$r_v \leftarrow r_v + \frac{1}{\rho} \int\limits_{\varOmega_t} \tau_{\text{SUPG}} \left(\boldsymbol{\nabla} \delta \boldsymbol{v} \right) \boldsymbol{v} \cdot \left[\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla} \boldsymbol{v}) \, \boldsymbol{v} \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{v}, p) \right] \mathrm{d}v \tag{39}$$

$$+ \int\limits_{\varOmega_t} \tau_{\text{LSIC}} \, \rho \left(\boldsymbol{\nabla} \cdot \delta \boldsymbol{v} \right) (\boldsymbol{\nabla} \cdot \boldsymbol{v}) \, \mathrm{d}v \tag{40}$$

– pressure residual operator (32) is augmented with the following terms:

$$r_p \leftarrow r_p + \frac{1}{\rho} \int\limits_{\varOmega_t} \tau_{\text{PSPG}} \left(\boldsymbol{\nabla} \delta p \right) \cdot \left[\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla} \boldsymbol{v}) \, \boldsymbol{v} \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{v}, p) \right] \mathrm{d}v \tag{41}$$

Reduced scheme (optimized for first-order): `supg_pspg2`:
– velocity residual operator (31) is augmented with the following terms:

$$r_v \leftarrow r_v + \int\limits_{\varOmega_t} d_1 \left((\boldsymbol{\nabla} \boldsymbol{v}) \, \boldsymbol{v} \right) \cdot (\boldsymbol{\nabla} \delta \boldsymbol{v}) \, \boldsymbol{v} \, \mathrm{d}v \tag{42}$$

$$+ \int\limits_{\varOmega_t} d_2 \left(\boldsymbol{\nabla} \cdot \boldsymbol{v} \right) (\boldsymbol{\nabla} \cdot \delta \boldsymbol{v}) \, \mathrm{d}v \tag{43}$$

$$+ \int\limits_{\varOmega_t} d_3 \left(\boldsymbol{\nabla} p \right) \cdot (\boldsymbol{\nabla} \delta \boldsymbol{v}) \, \boldsymbol{v} \, \mathrm{d}v \tag{44}$$

– pressure residual operator (32) is augmented with the following terms:

$$r_p \leftarrow r_p + \frac{1}{\rho} \int_{\Omega_t} d_1 \left((\boldsymbol{\nabla} \boldsymbol{v}) \, \boldsymbol{v} \right) \cdot (\boldsymbol{\nabla} \delta p) \, \mathrm{d}v \tag{45}$$

$$+ \frac{1}{\rho} \int_{\Omega_t} d_3 \left(\boldsymbol{\nabla} p \right) \cdot (\boldsymbol{\nabla} \delta p) \, \mathrm{d}v \tag{46}$$

– discrete linear system

$$\begin{bmatrix} \mathbf{K}_{vv} & \mathbf{K}_{vp} \\ \mathbf{K}_{pv} & \mathbf{K}_{pp} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta \mathbf{v} \\ \Delta \mathbf{p} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{r}_v \\ \mathbf{r}_p \end{bmatrix}_{n+1}^{k} \tag{47}$$

– note that $\mathbf{K}_{pp}$ is zero for Taylor-Hood elements (without stabilization)

3.2 ALE reference frame

– problem type: `fluid_ale`

– incompressible Navier-Stokes equations in Arbitrary Lagrangian Eulerian (ALE) reference frame

– ALE domain problem deformation governed by linear-elastic or nonlinear hyperelastic solid, displacement field $\boldsymbol{d}$

– fluid mechanics formulated with respect to the reference frame, using ALE deformation gradient $\boldsymbol{F}(\boldsymbol{d}) = \boldsymbol{I} + \boldsymbol{\nabla}_0 \boldsymbol{d}$ and its determinant, $J(\boldsymbol{d}) = \det \boldsymbol{F}(\boldsymbol{d})$

3.2.1 ALE problem

– displacement-based quasi-static solid

– primary variable: domain displacement $\boldsymbol{d}$

– strong form:

$$\boldsymbol{\nabla}_0 \cdot \boldsymbol{\sigma}^{\mathrm{G}}(\boldsymbol{d}) = \boldsymbol{0} \quad \text{in } \Omega_0, \tag{48}$$

$$\boldsymbol{d} = \hat{\boldsymbol{d}} \quad \text{on } \Gamma_0^{\mathrm{D}}, \tag{49}$$

with

$$\boldsymbol{\sigma}^{\mathrm{G}}(\boldsymbol{d}) = E \frac{1}{2} \left(\boldsymbol{\nabla}_0 \boldsymbol{d} + (\boldsymbol{\nabla}_0 \boldsymbol{d})^{\mathrm{T}} \right) + \kappa \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{d} \right) \boldsymbol{I} \tag{50}$$

– weak form:

$$r_d(\boldsymbol{d}; \delta\boldsymbol{d}) := \int_{\Omega_0} \boldsymbol{\sigma}^{\mathrm{G}}(\boldsymbol{d}) : \boldsymbol{\nabla}_0 \delta\boldsymbol{d} \, \mathrm{d}V = 0, \quad \forall \, \delta\boldsymbol{d} \tag{51}$$

3.2.2 Strong form

– primary variables: velocity $\boldsymbol{v}$ and pressure p

$$\boldsymbol{\nabla}_0 \boldsymbol{\sigma}(\boldsymbol{v}, \boldsymbol{d}, p) : \boldsymbol{F}^{-\mathrm{T}} + \hat{\boldsymbol{b}} = \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla}_0 \boldsymbol{v} \, \boldsymbol{F}^{-1}) \, \boldsymbol{v} \right) \quad \text{in } \Omega_0 \times [0, T], \tag{52}$$

$$\boldsymbol{\nabla}_0 \boldsymbol{v} : \boldsymbol{F}^{-\mathrm{T}} = 0 \qquad\qquad\qquad \text{in } \Omega_0 \times [0, T], \tag{53}$$

$$\boldsymbol{v} = \hat{\boldsymbol{v}} \qquad\qquad\qquad \text{on } \Gamma_0^{\mathrm{D}} \times [0, T], \tag{54}$$

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} = \hat{\boldsymbol{t}} \qquad\qquad \text{on } \Gamma_0^{\mathrm{N}} \times [0, T], \tag{55}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \hat{\boldsymbol{v}}_0(\boldsymbol{x}) \qquad\qquad \text{in } \Omega_0, \tag{56}$$

with a Newtonian fluid constitutive law

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{\gamma} = -p\boldsymbol{I} + \mu \left(\boldsymbol{\nabla}_0 \boldsymbol{v} \, \boldsymbol{F}^{-1} + \boldsymbol{F}^{-\mathrm{T}} (\boldsymbol{\nabla}_0 \boldsymbol{v})^{\mathrm{T}} \right) \tag{57}$$

3.2.3 Weak form

– primary variables: velocity $\boldsymbol{v}$, pressure p, and domain displacement $\boldsymbol{d}$

– Principle of Virtual Power

$$r_v(\boldsymbol{v}, p, \boldsymbol{d}; \delta\boldsymbol{v}) := \delta\mathcal{P}_{\mathrm{kin}}(\boldsymbol{v}, \boldsymbol{d}; \delta\boldsymbol{v}) + \delta\mathcal{P}_{\mathrm{int}}(\boldsymbol{v}, p, \boldsymbol{d}; \delta\boldsymbol{v}) - \delta\mathcal{P}_{\mathrm{ext}}(\boldsymbol{v}, \boldsymbol{d}; \delta\boldsymbol{v}) = 0, \quad \forall \, \delta\boldsymbol{v} \tag{58}$$

$$r_p(\boldsymbol{v}, \boldsymbol{d}; \delta p) := \delta\mathcal{P}_{\mathrm{pres}}(\boldsymbol{v}, \boldsymbol{d}; \delta p), \quad \forall \, \delta p \tag{59}$$

– kinetic virtual power:

$$\delta\mathcal{P}_{\mathrm{kin}}(\boldsymbol{v}, \boldsymbol{d}; \delta\boldsymbol{v}) = \int_{\Omega_0} J\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{\nabla}_0 \boldsymbol{v} \, \boldsymbol{F}^{-1}) \, \boldsymbol{v} \right) \cdot \delta\boldsymbol{v} \, \mathrm{d}V \tag{60}$$

– internal virtual power:

$$\delta\mathcal{P}_{\mathrm{int}}(\boldsymbol{v}, p, \boldsymbol{d}; \delta\boldsymbol{v}) = \int_{\Omega_0} J\boldsymbol{\sigma}(\boldsymbol{v}, p, \boldsymbol{d}) : \boldsymbol{\nabla}_0 \delta\boldsymbol{v} \, \boldsymbol{F}^{-1} \, \mathrm{d}V \tag{61}$$

– pressure virtual power:

$$\delta\mathcal{P}_{\mathrm{pres}}(\boldsymbol{v}, \boldsymbol{d}; \delta p) = \int_{\Omega_0} J \, \boldsymbol{\nabla}_0 \boldsymbol{v} : \boldsymbol{F}^{-\mathrm{T}} \delta p \, \mathrm{d}V \tag{62}$$

– external virtual power:

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- conservative Neumann load:

$$\delta \mathcal{P}_{\text{ext}}(\delta \boldsymbol{v}) = \int\limits_{\Gamma_0^{\text{N}}} \hat{\boldsymbol{t}}(t) \cdot \delta \boldsymbol{v} \, \mathrm{d}A \tag{63}$$

- pressure Neumann load:

$$\delta \mathcal{P}_{\text{ext}}(\boldsymbol{d}; \delta \boldsymbol{v}) = -\int\limits_{\Gamma_0^{\text{N}}} \hat{p}(t) \, J \boldsymbol{F}^{-\text{T}} \boldsymbol{n}_0 \cdot \delta \boldsymbol{v} \, \mathrm{d}A \tag{64}$$

- body force:

$$\delta \mathcal{P}_{\text{ext}}(\boldsymbol{d}; \delta \boldsymbol{v}) = \int\limits_{\Omega_0} J \, \hat{\boldsymbol{b}}(t) \cdot \delta \boldsymbol{v} \, \mathrm{d}V \tag{65}$$

3.2.4 Stabilization

ALE forms of stabilization introduced in sec. 3.1.3
`supg_pspg`:
– velocity residual operator (58) is augmented with the following terms:

$$r_v \leftarrow r_v + \frac{1}{\rho} \int\limits_{\Omega_0} J \, \tau_{\text{SUPG}} \left(\boldsymbol{\nabla}_0 \delta \boldsymbol{v} \, \boldsymbol{F}^{-1} \right) \boldsymbol{v} \cdot \tag{66}$$

$$\cdot \left[\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \left(\boldsymbol{\nabla}_0 \boldsymbol{v} \, \boldsymbol{F}^{-1} \right) (\boldsymbol{v} - \boldsymbol{w}) \right) - \boldsymbol{\nabla}_0 \boldsymbol{\sigma}(\boldsymbol{v}, \boldsymbol{d}, p) : \boldsymbol{F}^{-\text{T}} \right] \mathrm{d}V \tag{67}$$

$$+ \int\limits_{\Omega_0} J \, \tau_{\text{LSIC}} \, \rho \, (\boldsymbol{\nabla}_0 \delta \boldsymbol{v} : \boldsymbol{F}^{-\text{T}})(\boldsymbol{\nabla}_0 \boldsymbol{v} : \boldsymbol{F}^{-\text{T}}) \, \mathrm{d}V \tag{68}$$

– pressure residual operator (59) is augmented with the following terms:

$$r_p \leftarrow r_p + \frac{1}{\rho} \int\limits_{\Omega_0} J \, \tau_{\text{PSPG}} \left(\boldsymbol{F}^{-\text{T}} \boldsymbol{\nabla}_0 \delta p \right) \cdot \tag{69}$$

$$\cdot \left[\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \left(\boldsymbol{\nabla}_0 \boldsymbol{v} \, \boldsymbol{F}^{-1} \right) (\boldsymbol{v} - \boldsymbol{w}) \right) - \boldsymbol{\nabla}_0 \boldsymbol{\sigma}(\boldsymbol{v}, \boldsymbol{d}, p) : \boldsymbol{F}^{-\text{T}} \right] \mathrm{d}V \tag{70}$$

`supg_pspg2`:

– velocity residual operator (58) is augmented with the following terms:

$$r_v \leftarrow r_v + \int_{\Omega_0} J\, d_1 \left(\left(\boldsymbol{\nabla}_0 \boldsymbol{v}\, \boldsymbol{F}^{-1}\right)(\boldsymbol{v} - \boldsymbol{w})\right) \cdot \left(\boldsymbol{\nabla}_0 \delta \boldsymbol{v}\, \boldsymbol{F}^{-1}\right) \boldsymbol{v}\, \mathrm{d}V \tag{71}$$

$$+ \int_{\Omega_0} J\, d_2 \left(\boldsymbol{\nabla}_0 \boldsymbol{v} : \boldsymbol{F}^{-\mathrm{T}}\right)\left(\boldsymbol{\nabla}_0 \delta \boldsymbol{v} : \boldsymbol{F}^{-\mathrm{T}}\right) \mathrm{d}V \tag{72}$$

$$+ \int_{\Omega_0} J\, d_3 \left(\boldsymbol{F}^{-\mathrm{T}} \boldsymbol{\nabla}_0 p\right) \cdot \left(\boldsymbol{\nabla}_0 \delta \boldsymbol{v}\, \boldsymbol{F}^{-1}\right) \boldsymbol{v}\, \mathrm{d}V \tag{73}$$

– pressure residual operator (59) is augmented with the following terms:

$$r_p \leftarrow r_p + \frac{1}{\rho} \int_{\Omega_0} J\, d_1 \left(\left(\boldsymbol{\nabla}_0 \boldsymbol{v}\, \boldsymbol{F}^{-1}\right)(\boldsymbol{v} - \boldsymbol{w})\right) \cdot \left(\boldsymbol{F}^{-\mathrm{T}} \boldsymbol{\nabla}_0 \delta p\right) \mathrm{d}V \tag{74}$$

$$+ \frac{1}{\rho} \int_{\Omega_0} J\, d_3 \left(\boldsymbol{F}^{-\mathrm{T}} \boldsymbol{\nabla}_0 p\right) \cdot \left(\boldsymbol{F}^{-\mathrm{T}} \boldsymbol{\nabla}_0 \delta p\right) \mathrm{d}V \tag{75}$$

– discrete linear system

$$\begin{bmatrix} \boldsymbol{\mathsf{K}}_{vv} & \boldsymbol{\mathsf{K}}_{vp} & \boldsymbol{\mathsf{K}}_{vd} \\ \boldsymbol{\mathsf{K}}_{pv} & \boldsymbol{\mathsf{K}}_{pp} & \boldsymbol{\mathsf{K}}_{pd} \\ \boldsymbol{\mathsf{K}}_{dv} & 0 & \boldsymbol{\mathsf{K}}_{dd} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta \boldsymbol{\mathsf{v}} \\ \Delta \boldsymbol{\mathsf{p}} \\ \Delta \boldsymbol{\mathsf{d}} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \boldsymbol{\mathsf{r}}_v \\ \boldsymbol{\mathsf{r}}_p \\ \boldsymbol{\mathsf{r}}_d \end{bmatrix}_{n+1}^{k} \tag{76}$$

– note that $\boldsymbol{\mathsf{K}}_{pp}$ is zero for Taylor-Hood elements (without stabilization)

4 Coupling

4.1 Solid + 0D flow

– problem type: `solid_flow0d`

– (12) or (21) augmented by following term:

$$r_u \leftarrow r_u + \int_{\Gamma_0^{\text{s-0d}}} \Lambda\, J \boldsymbol{F}^{-\mathrm{T}} \boldsymbol{n}_0 \cdot \delta \boldsymbol{u}\, \mathrm{d}A \tag{77}$$

– multiplier constraint

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$$r_\lambda(\Lambda, \boldsymbol{u}; \delta\Lambda) := \left(\int_{\Gamma_0^{\text{s-0d}}} J\boldsymbol{F}^{-\text{T}}\boldsymbol{n}_0 \cdot \boldsymbol{v}(\boldsymbol{u})\,\mathrm{d}A - Q^{0\text{d}}(\Lambda) \right) \delta\Lambda, \quad \forall\,\delta\Lambda \tag{78}$$

– discrete linear system for displacement-based solid

$$\begin{bmatrix} \mathsf{K}_{uu} & \mathsf{K}_{u\Lambda} \\ \mathsf{K}_{\Lambda u} & \mathsf{K}_{\Lambda\Lambda} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\mathsf{u} \\ \Delta\boldsymbol{\Lambda} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathsf{r}_u \\ \mathsf{r}_\Lambda \end{bmatrix}_{n+1}^{k} \tag{79}$$

– discrete linear system for incompressible solid

$$\begin{bmatrix} \mathsf{K}_{uu} & \mathsf{K}_{up} & \mathsf{K}_{u\Lambda} \\ \mathsf{K}_{pu} & 0 & 0 \\ \mathsf{K}_{\Lambda u} & 0 & \mathsf{K}_{\Lambda\Lambda} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\mathsf{u} \\ \Delta\mathsf{p} \\ \Delta\boldsymbol{\Lambda} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathsf{r}_u \\ \mathsf{r}_p \\ \mathsf{r}_\Lambda \end{bmatrix}_{n+1}^{k} \tag{80}$$

4.2 Fluid + 0D flow

– problem type: `fluid_flow0d`

– (31) augmented by following term:

$$r_v \leftarrow r_v + \int_{\Gamma_t^{\text{f-0d}}} \Lambda\,\boldsymbol{n} \cdot \delta\boldsymbol{v}\,\mathrm{d}a \tag{81}$$

– multiplier constraint

$$r_\lambda(\Lambda, \boldsymbol{v}; \delta\Lambda) := \left(\int_{\Gamma_t^{\text{f-0d}}} \boldsymbol{n} \cdot \boldsymbol{v}\,\mathrm{d}a - Q^{0\text{d}}(\Lambda) \right) \delta\Lambda, \quad \forall\,\delta\Lambda \tag{82}$$

– discrete linear system

$$\begin{bmatrix} \mathsf{K}_{vv} & \mathsf{K}_{vp} & \mathsf{K}_{v\Lambda} \\ \mathsf{K}_{pv} & \mathsf{K}_{pp} & 0 \\ \mathsf{K}_{\Lambda v} & 0 & \mathsf{K}_{\Lambda\Lambda} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\mathsf{v} \\ \Delta\mathsf{p} \\ \Delta\boldsymbol{\Lambda} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathsf{r}_v \\ \mathsf{r}_p \\ \mathsf{r}_\Lambda \end{bmatrix}_{n+1}^{k} \tag{83}$$

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4.3   ALE fluid + 0D flow

– problem type: `fluid_ale_flow0d`

– (58) augmented by following term:

$$r_v \leftarrow r_v + \int_{\Gamma_0^{\text{f-0d}}} \Lambda \, J \boldsymbol{F}^{-\mathrm{T}} \boldsymbol{n}_0 \cdot \delta \boldsymbol{v} \, \mathrm{d}A \tag{84}$$

– multiplier constraint

$$r_\lambda(\Lambda, \boldsymbol{v}, \boldsymbol{d}; \delta\Lambda) := \left(\int_{\Gamma_0^{\text{f-0d}}} J \boldsymbol{F}^{-\mathrm{T}} \boldsymbol{n}_0 \cdot (\boldsymbol{v} - \boldsymbol{w}(\boldsymbol{d})) \, \mathrm{d}A - Q^{\mathrm{0d}}(\Lambda) \right) \delta\Lambda, \quad \forall \, \delta\Lambda \tag{85}$$

with $\boldsymbol{w}(\boldsymbol{d}) = \frac{\mathrm{d}\boldsymbol{d}}{\mathrm{d}t}$

– discrete linear system

$$\begin{bmatrix} \mathbf{K}_{vv} & \mathbf{K}_{vp} & \mathbf{K}_{v\Lambda} & \mathbf{K}_{vd} \\ \mathbf{K}_{pv} & \mathbf{K}_{pp} & 0 & \mathbf{K}_{pd} \\ \mathbf{K}_{\Lambda v} & 0 & \mathbf{K}_{\Lambda\Lambda} & \mathbf{K}_{\Lambda d} \\ \mathbf{K}_{dv} & 0 & 0 & \mathbf{K}_{dd} \end{bmatrix}_{n+1}^{k} \begin{bmatrix} \Delta\mathbf{v} \\ \Delta\mathbf{p} \\ \Delta\boldsymbol{\Lambda} \\ \Delta\mathbf{d} \end{bmatrix}_{n+1}^{k+1} = - \begin{bmatrix} \mathbf{r}_v \\ \mathbf{r}_p \\ \mathbf{r}_\Lambda \\ \mathbf{r}_d \end{bmatrix}_{n+1}^{k} \tag{86}$$

4.4   Fluid-Solid Interaction (FSI)

– problem type: `fsi`

– not yet fully implemented!

4.5   Fluid-Solid Interaction (FSI) + 0D flow

– problem type: `fsi_flow0d`

– not yet fully implemented!

5   0D flow: Lumped parameter models

– problem type: `flow0d`

5.1 "Syspul" circulation model

left heart and systemic circulation

$$-Q_{\mathrm{at}}^{\ell} = \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{pul}}} q_{\mathrm{ven},i}^{\mathrm{pul}} - q_{\mathrm{v,in}}^{\ell} \qquad\qquad \text{left atrium flow balance}$$

$$q_{\mathrm{v,in}}^{\ell} = q_{\mathrm{mv}}(p_{\mathrm{at}}^{\ell} - p_{\mathrm{v}}^{\ell}) \qquad\qquad \text{mitral valve momentum} \qquad (87)$$

$$-Q_{\mathrm{v}}^{\ell} = q_{\mathrm{v,in}}^{\ell} - q_{\mathrm{v,out}}^{\ell} \qquad\qquad \text{left ventricle flow balance}$$

$$q_{\mathrm{v,out}}^{\ell} = q_{\mathrm{av}}(p_{\mathrm{v}}^{\ell} - p_{\mathrm{ar}}^{\mathrm{sys}}) \qquad\qquad \text{aortic valve momentum} \qquad (88)$$

$$-Q_{\mathrm{aort}}^{\mathrm{sys}} = q_{\mathrm{v,out}}^{\ell} - q_{\mathrm{ar,p}}^{\mathrm{sys}} - \mathbb{I}^{\mathrm{cor}}\sum_{i=1}^{2} q_{\mathrm{ar,cor,in},i}^{\mathrm{sys}} \quad \text{aortic root flow balance}$$

$$I_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t} + Z_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar,p}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ar,d}}^{\mathrm{sys}} \qquad \text{aortic root inertia}$$

$$C_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar}}^{\mathrm{sys}} \qquad\qquad \text{systemic arterial flow balance}$$

$$L_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar}}^{\mathrm{sys}} = p_{\mathrm{ar,d}}^{\mathrm{sys}} - p_{\mathrm{ven}}^{\mathrm{sys}} \qquad \text{systemic arterial momentum}$$

$$C_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{sys}} - \sum_{i=1}^{n_{\mathrm{ven}}^{\mathrm{sys}}} q_{\mathrm{ven},i}^{\mathrm{sys}} \qquad \text{systemic venous flow balance}$$

$$L_{\mathrm{ven},i}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ven},i}^{\mathrm{sys}}\, q_{\mathrm{ven},i}^{\mathrm{sys}} = p_{\mathrm{ven}}^{\mathrm{sys}} - p_{\mathrm{at},i}^{r} \quad \text{systemic venous momentum}$$

$$i \in \{1, ..., n_{\mathrm{ven}}^{\mathrm{sys}}\}$$

right heart and pulmonary circulation

$$-Q_{\text{at}}^r = \sum_{i=1}^{n_{\text{ven}}^{\text{sys}}} q_{\text{ven},i}^{\text{sys}} - \mathbb{I}^{\text{cor}} q_{\text{ven,cor,out}}^{\text{sys}} - q_{\text{v,in}}^r \quad \text{right atrium flow balance}$$

$$q_{\text{v,in}}^r = q_{\text{tv}}(p_{\text{at}}^r - p_{\text{v}}^r) \qquad\qquad\qquad \text{tricuspid valve momentum} \qquad (89)$$

$$-Q_{\text{v}}^r = q_{\text{v,in}}^r - q_{\text{v,out}}^r \qquad\qquad\qquad \text{right ventricle flow balance}$$

$$q_{\text{v,out}}^r = q_{\text{pv}}(p_{\text{v}}^r - p_{\text{ar}}^{\text{pul}}) \qquad\qquad\qquad \text{pulmonary valve momentum} \qquad (90)$$

$$C_{\text{ar}}^{\text{pul}}\frac{\mathrm{d}p_{\text{ar}}^{\text{pul}}}{\mathrm{d}t} = q_{\text{v,out}}^r - q_{\text{ar}}^{\text{pul}} \qquad\qquad\quad \text{pulmonary arterial flow balance}$$

$$L_{\text{ar}}^{\text{pul}}\frac{\mathrm{d}q_{\text{ar}}^{\text{pul}}}{\mathrm{d}t} + R_{\text{ar}}^{\text{pul}} q_{\text{ar}}^{\text{pul}} = p_{\text{ar}}^{\text{pul}} - p_{\text{ven}}^{\text{pul}} \qquad \text{pulmonary arterial momentum}$$

$$C_{\text{ven}}^{\text{pul}}\frac{\mathrm{d}p_{\text{ven}}^{\text{pul}}}{\mathrm{d}t} = q_{\text{ar}}^{\text{pul}} - \sum_{i=1}^{n_{\text{ven}}^{\text{pul}}} q_{\text{ven},i}^{\text{pul}} \qquad\quad \text{pulmonary venous flow balance}$$

$$L_{\text{ven},i}^{\text{pul}}\frac{\mathrm{d}q_{\text{ven},i}^{\text{pul}}}{\mathrm{d}t} + R_{\text{ven},i}^{\text{pul}} q_{\text{ven},i}^{\text{pul}} = p_{\text{ven}}^{\text{pul}} - p_{\text{at},i}^\ell \quad \text{pulmonary venous momentum}$$

$$i \in \{1, ..., n_{\text{ven}}^{\text{pul}}\}$$

with:

$$Q_{\text{at}}^\ell := -\frac{\mathrm{d}V_{\text{at}}^\ell}{\mathrm{d}t}, \quad Q_{\text{v}}^\ell := -\frac{\mathrm{d}V_{\text{v}}^\ell}{\mathrm{d}t}, \quad Q_{\text{at}}^r := -\frac{\mathrm{d}V_{\text{at}}^r}{\mathrm{d}t}, \quad Q_{\text{v}}^r := -\frac{\mathrm{d}V_{\text{v}}^r}{\mathrm{d}t}, \quad Q_{\text{aort}}^{\text{sys}} := -\frac{\mathrm{d}V_{\text{aort}}^{\text{sys}}}{\mathrm{d}t}$$

and:

$$\mathbb{I}^{\text{cor}} = \begin{cases} 1, & \text{if } \text{CORONARY\_MODEL}, \\ 0, & \text{else} \end{cases}$$

The volume V of the heart chambers (0D) is modeled by the volume-pressure relationship

$$V(t) = \frac{p}{E(t)} + V_{\text{u}}, \qquad\qquad\qquad (91)$$

with the unstressed volume V_{u} and the time-varying elastance

$$E(t) = (E_{\text{max}} - E_{\text{min}}) \cdot \hat{y}(t) + E_{\text{min}}, \qquad\qquad (92)$$

where E_{max} and E_{min} denote the maximum and minimum elastance, respectively. The normalized activation function $\hat{y}(t)$ is input by the user.

Flow-pressure relations for the four valves, eq. (87), (88), (89), (90), are functions of the pressure difference $p - p_{\text{open}}$ across the valve. The following valve models can be defined:

Valve model `pwlin_pres`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} R_{\text{max}}, & p < p_{\text{open}} \\ R_{\text{min}}, & p \geq p_{\text{open}} \end{cases}$$

Remark: Non-smooth flow-pressure relationship

Valve model `pwlin_time`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = \begin{cases} \begin{cases} R_{\text{max}}, & t < t_{\text{open}} \text{ and } t \geq t_{\text{close}} \\ R_{\text{min}}, & t \geq t_{\text{open}} \text{ or } t < t_{\text{close}} \end{cases}, & t_{\text{open}} > t_{\text{close}} \\ \begin{cases} R_{\text{max}}, & t < t_{\text{open}} \text{ or } t \geq t_{\text{close}} \\ R_{\text{min}}, & t \geq t_{\text{open}} \text{ and } t < t_{\text{close}} \end{cases}, & \text{else} \end{cases}$$

Remark: Non-smooth flow-pressure relationship with resistance only dependent on timings, not the pressure difference!

Valve model `smooth_pres_resistance`:

$$q(p - p_{\text{open}}) = \frac{p - p_{\text{open}}}{\tilde{R}}, \quad \text{with } \tilde{R} = 0.5 \left(R_{\text{max}} - R_{\text{min}} \right) \left(\tanh \frac{p - p_{\text{open}}}{\epsilon} + 1 \right) + R_{\text{min}}$$

Remark: Smooth but potentially non-convex flow-pressure relationship!

Valve model `smooth_pres_momentum`:

$$q(p - p_{\text{open}}) = \begin{cases} \frac{p - p_{\text{open}}}{R_{\text{max}}}, & p < p_{\text{open}} - 0.5\epsilon \\ h_{00}p_0 + h_{10}m_0\epsilon + h_{01}p_1 + h_{11}m_1\epsilon, & p \geq p_{\text{open}} - 0.5\epsilon \text{ and } p < p_{\text{open}} + 0.5\epsilon \\ \frac{p - p_{\text{open}}}{R_{\text{min}}}, & p \geq p_{\text{open}} + 0.5\epsilon \end{cases}$$

with

$$p_0 = \frac{-0.5\epsilon}{R_{\text{max}}}, \qquad m_0 = \frac{1}{R_{\text{max}}}, \qquad p_1 = \frac{0.5\epsilon}{R_{\text{min}}}, \qquad m_1 = \frac{1}{R_{\text{min}}}$$

and

$$h_{00} = 2s^3 - 3s^2 + 1, \qquad h_{01} = -2s^3 + 3s^2,$$
$$h_{10} = s^3 - 2s^2 + s, \qquad h_{11} = s^3 - s^2$$

with

$$s = \frac{p - p_{\text{open}} + 0.5\epsilon}{\epsilon}$$

Remarks:

− Collapses to valve model `pwlin_pres` for $\epsilon = 0$

− Smooth and convex flow-pressure relationship

Valve model `pw_pres_regurg`:

$$q(p - p_{\text{open}}) = \begin{cases} cA_{\text{o}}\sqrt{p - p_{\text{open}}}, & p < p_{\text{open}} \\ \frac{p - p_{\text{open}}}{R_{\min}}, & p \geq p_{\text{open}} \end{cases}$$

Remark: Model to allow a regurgitant valve in the closed state, degree of regurgitation can be varied by specifying the valve regurgitant area A_{o}

Coronary circulation model:

$$C_{\text{cor,p}}^{\text{sys},\ell}\left(\frac{\mathrm{d}p_{\text{ar}}^{\text{sys},\ell}}{\mathrm{d}t} - Z_{\text{cor,p}}^{\text{sys},\ell}\frac{\mathrm{d}q_{\text{cor,p,in}}^{\text{sys},\ell}}{\mathrm{d}t}\right) = q_{\text{cor,p,in}}^{\text{sys},\ell} - q_{\text{cor,p}}^{\text{sys},\ell}$$

$$R_{\text{cor,p}}^{\text{sys},\ell}\, q_{\text{cor,p}}^{\text{sys},\ell} = p_{\text{ar}}^{\text{sys}} - p_{\text{cor,d}}^{\text{sys},\ell} - Z_{\text{cor,p}}^{\text{sys},\ell}\, q_{\text{cor,p,in}}^{\text{sys},\ell}$$

$$C_{\text{cor,d}}^{\text{sys},\ell}\frac{\mathrm{d}(p_{\text{cor,d}}^{\text{sys},\ell} - p_{\text{v}}^{\ell})}{\mathrm{d}t} = q_{\text{cor,p}}^{\text{sys},\ell} - q_{\text{cor,d}}^{\text{sys},\ell}$$

$$R_{\text{cor,d}}^{\text{sys},\ell}\, q_{\text{cor,d}}^{\text{sys},\ell} = p_{\text{cor,d}}^{\text{sys},\ell} - p_{\text{at}}^{r}$$

$$C_{\text{cor,p}}^{\text{sys},r}\left(\frac{\mathrm{d}p_{\text{ar}}^{\text{sys},r}}{\mathrm{d}t} - Z_{\text{cor,p}}^{\text{sys},r}\frac{\mathrm{d}q_{\text{cor,p,in}}^{\text{sys},r}}{\mathrm{d}t}\right) = q_{\text{cor,p,in}}^{\text{sys},r} - q_{\text{cor,p}}^{\text{sys},r}$$

$$R_{\text{cor,p}}^{\text{sys},r}\, q_{\text{cor,p}}^{\text{sys},r} = p_{\text{ar}}^{\text{sys}} - p_{\text{cor,d}}^{\text{sys},r} - Z_{\text{cor,p}}^{\text{sys},r}\, q_{\text{cor,p,in}}^{\text{sys},r}$$

$$C_{\text{cor,d}}^{\text{sys},r}\frac{\mathrm{d}(p_{\text{cor,d}}^{\text{sys},r} - p_{\text{v}}^{\ell})}{\mathrm{d}t} = q_{\text{cor,p}}^{\text{sys},r} - q_{\text{cor,d}}^{\text{sys},r}$$

$$R_{\text{cor,d}}^{\text{sys},r}\, q_{\text{cor,d}}^{\text{sys},r} = p_{\text{cor,d}}^{\text{sys},r} - p_{\text{at}}^{r}$$

$$0 = q_{\text{cor,d}}^{\text{sys},\ell} + q_{\text{cor,d}}^{\text{sys},r} - q_{\text{cor,d,out}}^{\text{sys}}$$

5.2 "Syspulcap" circulation model

$$-Q_{\text{at}}^{\ell} = q_{\text{ven}}^{\text{pul}} - q_{\text{v,in}}^{\ell}$$

$$\tilde{R}_{\text{v,in}}^{\ell} \, q_{\text{v,in}}^{\ell} = p_{\text{at}}^{\ell} - p_{\text{v}}^{\ell}$$

$$-Q_{\text{v}}^{\ell} = q_{\text{v,in}}^{\ell} - q_{\text{v,out}}^{\ell}$$

$$\tilde{R}_{\text{v,out}}^{\ell} \, q_{\text{v,out}}^{\ell} = p_{\text{v}}^{\ell} - p_{\text{ar}}^{\text{sys}}$$

$$0 = q_{\text{v,out}}^{\ell} - q_{\text{ar,p}}^{\text{sys}}$$

$$I_{\text{ar}}^{\text{sys}} \frac{\mathrm{d}q_{\text{ar,p}}^{\text{sys}}}{\mathrm{d}t} + Z_{\text{ar}}^{\text{sys}} \, q_{\text{ar,p}}^{\text{sys}} = p_{\text{ar}}^{\text{sys}} - p_{\text{ar,d}}^{\text{sys}}$$

$$C_{\text{ar}}^{\text{sys}} \frac{\mathrm{d}p_{\text{ar,d}}^{\text{sys}}}{\mathrm{d}t} = q_{\text{ar,p}}^{\text{sys}} - q_{\text{ar}}^{\text{sys}}$$

$$L_{\text{ar}}^{\text{sys}} \frac{\mathrm{d}q_{\text{ar}}^{\text{sys}}}{\mathrm{d}t} + R_{\text{ar}}^{\text{sys}} \, q_{\text{ar}}^{\text{sys}} = p_{\text{ar,d}}^{\text{sys}} - p_{\text{ar,peri}}^{\text{sys}}$$

$$\left(\sum_{j \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor}} \}} C_{\text{ar},j}^{\text{sys}} \right) \frac{\mathrm{d}p_{\text{ar,peri}}^{\text{sys}}}{\mathrm{d}t} = q_{\text{ar}}^{\text{sys}} - \sum_{j \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor}} \}} q_{\text{ar},j}^{\text{sys}}$$

$$R_{\text{ar},i}^{\text{sys}} \, q_{\text{ar},i}^{\text{sys}} = p_{\text{ar,peri}}^{\text{sys}} - p_{\text{ven},i}^{\text{sys}}, \quad i \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor}} \}$$

$$C_{\text{ven},i}^{\text{sys}} \frac{\mathrm{d}p_{\text{ven},i}^{\text{sys}}}{\mathrm{d}t} = q_{\text{ar},i}^{\text{sys}} - q_{\text{ven},i}^{\text{sys}}, \quad i \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor}} \}$$

$$R_{\text{ven},i}^{\text{sys}} \, q_{\text{ven},i}^{\text{sys}} = p_{\text{ven},i}^{\text{sys}} - p_{\text{ven}}^{\text{sys}}, \quad i \in \{ \substack{\text{spl,espl,} \\ \text{msc,cer,cor}} \}$$

$$C_{\text{ven}}^{\text{sys}} \frac{\mathrm{d}p_{\text{ven}}^{\text{sys}}}{\mathrm{d}t} = \sum_{j = \substack{\text{spl,espl,} \\ \text{msc,cer,cor}}} q_{\text{ven},j}^{\text{sys}} - q_{\text{ven}}^{\text{sys}}$$

$$L_{\text{ven}}^{\text{sys}} \frac{\mathrm{d}q_{\text{ven}}^{\text{sys}}}{\mathrm{d}t} + R_{\text{ven}}^{\text{sys}} \, q_{\text{ven}}^{\text{sys}} = p_{\text{ven}}^{\text{sys}} - p_{\text{at}}^{r}$$

$$-Q_{\text{at}}^r = q_{\text{ven}}^{\text{sys}} - q_{\text{v,in}}^r$$

$$\tilde{R}_{\text{v,in}}^r \, q_{\text{v,in}}^r = p_{\text{at}}^r - p_{\text{v}}^r$$

$$-Q_{\text{v}}^r = q_{\text{v,in}}^r - q_{\text{v,out}}^r$$

$$\tilde{R}_{\text{v,out}}^r \, q_{\text{v,out}}^r = p_{\text{v}}^r - p_{\text{ar}}^{\text{pul}}$$

$$C_{\text{ar}}^{\text{pul}} \frac{\mathrm{d}p_{\text{ar}}^{\text{pul}}}{\mathrm{d}t} = q_{\text{v,out}}^r - q_{\text{ar}}^{\text{pul}}$$

$$L_{\text{ar}}^{\text{pul}} \frac{\mathrm{d}q_{\text{ar}}^{\text{pul}}}{\mathrm{d}t} + R_{\text{ar}}^{\text{pul}} \, q_{\text{ar}}^{\text{pul}} = p_{\text{ar}}^{\text{pul}} - p_{\text{cap}}^{\text{pul}}$$

$$C_{\text{cap}}^{\text{pul}} \frac{\mathrm{d}p_{\text{cap}}^{\text{pul}}}{\mathrm{d}t} = q_{\text{ar}}^{\text{pul}} - q_{\text{cap}}^{\text{pul}}$$

$$R_{\text{cap}}^{\text{pul}} \, q_{\text{cap}}^{\text{pul}} = p_{\text{cap}}^{\text{pul}} - p_{\text{ven}}^{\text{pul}}$$

$$C_{\text{ven}}^{\text{pul}} \frac{\mathrm{d}p_{\text{ven}}^{\text{pul}}}{\mathrm{d}t} = q_{\text{cap}}^{\text{pul}} - q_{\text{ven}}^{\text{pul}}$$

$$L_{\text{ven}}^{\text{pul}} \frac{\mathrm{d}q_{\text{ven}}^{\text{pul}}}{\mathrm{d}t} + R_{\text{ven}}^{\text{pul}} \, q_{\text{ven}}^{\text{pul}} = p_{\text{ven}}^{\text{pul}} - p_{\text{at}}^{\ell}$$

with:

$$Q_{\text{at}}^{\ell} := -\frac{\mathrm{d}V_{\text{at}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\text{v}}^{\ell} := -\frac{\mathrm{d}V_{\text{v}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\text{at}}^r := -\frac{\mathrm{d}V_{\text{at}}^r}{\mathrm{d}t}, \qquad Q_{\text{v}}^r := -\frac{\mathrm{d}V_{\text{v}}^r}{\mathrm{d}t}$$

5.3 "Syspulcapcor" circulation model

$$-Q_{\mathrm{at}}^{\ell} = q_{\mathrm{ven}}^{\mathrm{pul}} - q_{\mathrm{v,in}}^{\ell}$$

$$\tilde{R}_{\mathrm{v,in}}^{\ell}\, q_{\mathrm{v,in}}^{\ell} = p_{\mathrm{at}}^{\ell} - p_{\mathrm{v}}^{\ell}$$

$$-Q_{\mathrm{v}}^{\ell} = q_{\mathrm{v,in}}^{\ell} - q_{\mathrm{v,out}}^{\ell}$$

$$\tilde{R}_{\mathrm{v,out}}^{\ell}\, q_{\mathrm{v,out}}^{\ell} = p_{\mathrm{v}}^{\ell} - p_{\mathrm{ar}}^{\mathrm{sys}}$$

$$0 = q_{\mathrm{v,out}}^{\ell} - q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar,cor,in}}^{\mathrm{sys}}$$

$$I_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar,p}}^{\mathrm{sys}}}{\mathrm{d}t} + Z_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar,p}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ar,d}}^{\mathrm{sys}}$$

$$C_{\mathrm{ar,cor}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,cor,in}}^{\mathrm{sys}} - q_{\mathrm{ar,cor}}^{\mathrm{sys}}$$

$$R_{\mathrm{ar,cor}}^{\mathrm{sys}}\, q_{\mathrm{ar,cor}}^{\mathrm{sys}} = p_{\mathrm{ar}}^{\mathrm{sys}} - p_{\mathrm{ven,cor}}^{\mathrm{sys}}$$

$$C_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ar,d}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,p}}^{\mathrm{sys}} - q_{\mathrm{ar}}^{\mathrm{sys}}$$

$$L_{\mathrm{ar}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ar}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ar}}^{\mathrm{sys}}\, q_{\mathrm{ar}}^{\mathrm{sys}} = p_{\mathrm{ar,d}}^{\mathrm{sys}} - p_{\mathrm{ar,peri}}^{\mathrm{sys}}$$

$$\left(\sum_{j\in\{^{\mathrm{spl,espl,}}_{\mathrm{msc,cer}}\}} C_{\mathrm{ar},j}^{\mathrm{sys}}\right)\frac{\mathrm{d}p_{\mathrm{ar,peri}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar}}^{\mathrm{sys}} - \sum_{j\in\{^{\mathrm{spl,espl,}}_{\mathrm{msc,cer}}\}} q_{\mathrm{ar},j}^{\mathrm{sys}}$$

$$R_{\mathrm{ar},i}^{\mathrm{sys}}\, q_{\mathrm{ar},i}^{\mathrm{sys}} = p_{\mathrm{ar,peri}}^{\mathrm{sys}} - p_{\mathrm{ven},i}^{\mathrm{sys}}, \quad i\in\{^{\mathrm{spl,espl,}}_{\mathrm{msc,cer}}\}$$

$$C_{\mathrm{ven},i}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven},i}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar},i}^{\mathrm{sys}} - q_{\mathrm{ven},i}^{\mathrm{sys}}, \quad i\in\{^{\mathrm{spl,espl,}}_{\mathrm{msc,cer}}\}$$

$$R_{\mathrm{ven},i}^{\mathrm{sys}}\, q_{\mathrm{ven},i}^{\mathrm{sys}} = p_{\mathrm{ven},i}^{\mathrm{sys}} - p_{\mathrm{ven}}^{\mathrm{sys}}, \quad i\in\{^{\mathrm{spl,espl,}}_{\mathrm{msc,cer}}\}$$

$$C_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} = \sum_{j=^{\mathrm{spl,espl,}}_{\mathrm{msc,cer}}} q_{\mathrm{ven},j}^{\mathrm{sys}} - q_{\mathrm{ven}}^{\mathrm{sys}}$$

$$L_{\mathrm{ven}}^{\mathrm{sys}}\frac{\mathrm{d}q_{\mathrm{ven}}^{\mathrm{sys}}}{\mathrm{d}t} + R_{\mathrm{ven}}^{\mathrm{sys}}\, q_{\mathrm{ven}}^{\mathrm{sys}} = p_{\mathrm{ven}}^{\mathrm{sys}} - p_{\mathrm{at}}^{r}$$

$$C_{\mathrm{ven,cor}}^{\mathrm{sys}}\frac{\mathrm{d}p_{\mathrm{ven,cor}}^{\mathrm{sys}}}{\mathrm{d}t} = q_{\mathrm{ar,cor}}^{\mathrm{sys}} - q_{\mathrm{ven,cor}}^{\mathrm{sys}}$$

$$R_{\mathrm{ven,cor}}^{\mathrm{sys}}\, q_{\mathrm{ven,cor}}^{\mathrm{sys}} = p_{\mathrm{ven,cor}}^{\mathrm{sys}} - p_{\mathrm{at}}^{r}$$

$$- Q_{\text{at}}^r = q_{\text{ven}}^{\text{sys}} + q_{\text{ven,cor}}^{\text{sys}} - q_{\text{v,in}}^r$$

$$\tilde{R}_{\text{v,in}}^r \, q_{\text{v,in}}^r = p_{\text{at}}^r - p_{\text{v}}^r$$

$$- Q_{\text{v}}^r = q_{\text{v,in}}^r - q_{\text{v,out}}^r$$

$$\tilde{R}_{\text{v,out}}^r \, q_{\text{v,out}}^r = p_{\text{v}}^r - p_{\text{ar}}^{\text{pul}}$$

$$C_{\text{ar}}^{\text{pul}} \frac{\mathrm{d}p_{\text{ar}}^{\text{pul}}}{\mathrm{d}t} = q_{\text{v,out}}^r - q_{\text{ar}}^{\text{pul}}$$

$$L_{\text{ar}}^{\text{pul}} \frac{\mathrm{d}q_{\text{ar}}^{\text{pul}}}{\mathrm{d}t} + R_{\text{ar}}^{\text{pul}} \, q_{\text{ar}}^{\text{pul}} = p_{\text{ar}}^{\text{pul}} - p_{\text{cap}}^{\text{pul}}$$

$$C_{\text{cap}}^{\text{pul}} \frac{\mathrm{d}p_{\text{cap}}^{\text{pul}}}{\mathrm{d}t} = q_{\text{ar}}^{\text{pul}} - q_{\text{cap}}^{\text{pul}}$$

$$R_{\text{cap}}^{\text{pul}} \, q_{\text{cap}}^{\text{pul}} = p_{\text{cap}}^{\text{pul}} - p_{\text{ven}}^{\text{pul}}$$

$$C_{\text{ven}}^{\text{pul}} \frac{\mathrm{d}p_{\text{ven}}^{\text{pul}}}{\mathrm{d}t} = q_{\text{cap}}^{\text{pul}} - q_{\text{ven}}^{\text{pul}}$$

$$L_{\text{ven}}^{\text{pul}} \frac{\mathrm{d}q_{\text{ven}}^{\text{pul}}}{\mathrm{d}t} + R_{\text{ven}}^{\text{pul}} \, q_{\text{ven}}^{\text{pul}} = p_{\text{ven}}^{\text{pul}} - p_{\text{at}}^{\ell}$$

with:

$$Q_{\text{at}}^{\ell} := -\frac{\mathrm{d}V_{\text{at}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\text{v}}^{\ell} := -\frac{\mathrm{d}V_{\text{v}}^{\ell}}{\mathrm{d}t}, \qquad Q_{\text{at}}^r := -\frac{\mathrm{d}V_{\text{at}}^r}{\mathrm{d}t}, \qquad Q_{\text{v}}^r := -\frac{\mathrm{d}V_{\text{v}}^r}{\mathrm{d}t}$$

Bibliography

[1] C. J. Arthurs, K. D. Lau, K. N. Asrress, S. R. Redwood, and C. A. Figueroa. A mathematical model of coronary blood flow control: simulation of patient-specific three-dimensional hemodynamics during exercise. *Am J Physiol Heart Circ Physiol*, 310(9):H1242–H1258, 2016.

[2] S. Balay, S. Abhyankar, M. F. Adams, S. Benson, J. Brown, P. Brune, K. Buschelman, E. Constantinescu, L. Dalcin, A. Dener, V. Eijkhout, W. D. Gropp, V. Hapla, T. Isaac, P. Jolivet, D. Karpeev, D. Kaushik, M. G. Knepley, F. Kong, S. Kruger, D. A. May, L. C. McInnes, R. T. Mills, L. Mitchell, T. Munson, J. E. Roman, K. Rupp, P. Sanan, J. Sarich, B. F. Smith, S. Zampini, H. Zhang, H. Zhang, and J. Zhang. PETSc/TAO users manual. Technical Report ANL-21/39 - Revision 3.17, Argonne National Laboratory, 2022.

[3] C. Farhat, P. Avery, T. Chapman, and J. Cortial. Dimensional reduction of nonlinear finite element dynamic models with finite rotations and energy-based mesh sampling and weighting for computational efficiency. *International Journal for Numerical Methods in Engineering*, 98(9):625–662, 2014.

[4] M. W. Gee, C. Förster, and W. A. Wall. A computational strategy for prestressing patient-specific biomechanical problems under finite deformation. *International Journal for Numerical Methods in Biomedical Engineering*, 26(1):52–72, 2010.

[5] M. W. Gee, C. T. Kelley, and R. B. Lehoucq. Pseudo-transient continuation for nonlinear transient elasticity. *International Journal for Numerical Methods in Engineering*, 78(10):1209–1219, 2009.

[6] S. Göktepe, O. J. Abilez, K. K. Parker, and E. Kuhl. A multiscale model for eccentric and concentric cardiac growth through sarcomerogenesis. *J Theor Biol*, 265(3):433–442, 2010.

[7] J. M. Guccione, K. D. Costa, and A. D. McCulloch. Finite element stress analysis of left ventricular mechanics in the beating dog heart. *J Biomech*, 28(10):1167–1177, 1995.

[8] M. Hirschvogel. *Computational modeling of patient-specific cardiac mechanics with model reduction-based parameter estimation and applications to novel heart assist technologies.* PhD thesis, Technische Universität München, 2018.

[9] M. Hirschvogel, M. Balmus, M. Bonini, and D. Nordsletten. Fluid-reduced-solid interaction (FrSI): Physics- and projection-based model reduction for cardiovascular applications. *Preprint, submitted to Elsevier*, 2022.

[10] M. Hirschvogel, M. Bassilious, L. Jagschies, S. M. Wildhirt, and M. W. Gee. A monolithic 3D-0D coupled closed-loop model of the heart and the vascular system: Experiment-based parameter estimation for patient-specific cardiac mechanics. *Int J Numer Method Biomed Eng*, 33(8):e2842, 2017.

[11] G. A. Holzapfel. *Nonlinear Solid Mechanics – A Continuum Approach for Engineering.* Wiley Press Chichester, 2000.

[12] G. A. Holzapfel and R. W. Ogden. Constitutive modelling of passive myocardium: A structurally based framework for material characterization. *Phil Trans R Soc A*, 367(1902):3445–3475, 2009.

[13] A. Logg, K.-A. Mardal, and G. N. Wells, editors. *Automated Solution of Differential Equations by the Finite Element Method – The FEniCS Book.* Springer, 2012.

[14] A. Schein and M. W. Gee. Greedy maximin distance sampling based model order reduction of prestressed and parametrized abdominal aortic aneurysms. *Advanced Modeling and Simulation in Engineering Sciences*, 8(18), 2021.

[15] T. E. Tezduyar and Y. Osawa. Finite element stabilization parameters computed from element matrices and vectors. *Computer Methods in Applied Mechanics and Engineering*, 190(3–4):411–430, 2000.

[16] N. Westerhof, J.-W. Lankhaar, and B. E. Westerhof. The arterial Windkessel. *Med Biol Eng Comput*, 47(2):H81–H88, 2009.