Fall 2017 | Approximation Algorithms

Problem Set 1

Due September 14

Instructions: To solve these problems, you are allowed to consult your classmates, as well as the class textbook (*The Design of Approximation Algorithms* by Williamson and Shmoys) and notes, but no other sources. Everyone must write up their assignments separately. Please write clearly and concisely.

(1) [20 points] Given an undirected graph $G = (V, E)$, a legal coloring of this graph is an assignment of a color to every vertex $v \in V$ so that every neighboring pair of vertices is colored using a different color (i.e., for all $(u, v) \in E$, the colors of u and v are different). A k-coloring is a legal coloring using at most k colors, and a graph is called k-colorable if it has a k-coloring. (For example, one of the most amazing graph theory results of the last century is the 4-Color Theorem, which states that every planar graph is 4-colorable.) Determining if a graph is 3-colorable is NP-Complete (you do not have to prove this, it was covered in CS4020).

- **(a)** Prove that all graphs G are $(\Delta + 1)$-colorable, where Δ is the maximum degree of any vertex in G. Give a poly-time algorithm to find this coloring.
- **(b)** Prove that bipartite graphs are 2-colorable. Give a poly-time algorithm to find this coloring.
- **(c)** Using parts (a) and (b) above, give a poly-time algorithm for coloring a 3-colorable graph using at most $O(\sqrt{n})$ colors. Here n is the number of vertices.

  Hint: For any vertex v, consider the subgraph of its neighbors $N(v) = \{w|(v, w) \in E\}$. If the degree of v is large, color this subgraph, as well as node v, and remove them from the graph. Continue until every vertex has degree $\leq \sqrt{n}$. By the way, the best known algorithm for coloring 3-colorable graphs uses $O(n^{0.19996})$ colors.
- **(d)** Use the ideas from part (c) to give an algorithm for coloring 4-colorable graphs using $O(n^{2/3})$ colors.
- **(e)** The classic Vertex Coloring optimization problem is: Given a graph G, find the smallest number k such that G is k-colorable. Prove that Vertex Coloring is not approximable to a factor better than $4/3$ unless $P = NP$. You may use the fact that determining if a graph is 3-colorable is NP-Complete.

(2) [20 points] Solve problem **1.5** from the textbook. I suggest waiting until after our first lecture on Linear Programming to tackle this problem.