Homework 1

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1.

a.

Given a $(\Delta + 1)$-colorable graph, where Δ is the maximum degree of any vertex in $G = (V, E)$,

1. Find the vertex v in G with the maximum degree Δ and its neighbor $N(v) = \{w|(v, w) \in E, i = 1, \ldots, \Delta\}$;
2. First color v with one color, then color its neighbors $w_i \in N(v)$ one by one. If w_i is also connected to $w_j, i \neq j$, then color them with different colors. In this way, we need at most $\Delta + 1$ colors to color v and its neighbor $N(v)$;
3. Then find another v' not colored yet and its neighbor $N(v') = \{w|(v', w) \in E, i \leq \Delta\}$. Color them in the same way as step (2) using the $\Delta + 1$ colors;
4. Continue to color all other vertexes with less than $\Delta + 1$ colors until the graph G is completely colored.

As we can color G in poly-time with at most $(\Delta + 1)$ colors, then G is $(\Delta + 1)$ colorable.

b.

Bipartite graph is an undirected graph with nodes partitioned into group X and Y. For each line, its one ending node is in group X and the other ending node is in Y.

Then we can color all the nodes in group X with one color and the nodes in group Y with the other color. Therefore, bipartite graphs are 2-colorable.

c.

Given undirected 3-colorable graph $G = (V, E)$, $|V| = n$. For each $v \in V$ vertext, let $d(v)$ be the degree of v.

If $d(v) \geq \sqrt{(n)}$, then remove v and its neighbors $N(v) = \{w_i|(v, w_i) \in E, i \geq \sqrt{(n)}\}$, and then we can first assign v with the first color and then color those points that are only connected with v by the second color. Then remove all the points that are colored already together with the line between them.

If there are still points not colored, then these points can be colored with at most two colors, since G is a 3 colorable graph and all the remaining points are connected with v. Thus we can form a bipartite graph with them and color them with at most two colors. Specifically, for each of the remaining line, one ending point of the line is in one group while the other point is in the other group. Then color these two groups with two colors. As a result, we can color each v and its neighbor with at most 3 colors in this way in a poly-time.

We at most remove $\sqrt{n}$ times. The explanation is following: each time we remove at most $\sqrt{n}+1$ vetex from G, and there are n vertex in the graph, and thus we need to remove $O(\sqrt{n})$ times.

As we need at most 3 colors each time and totally $O(\sqrt{n})$ times, we need $O(\sqrt{n})$ colors to color those vertex having degree $\geq \sqrt{n}$ and their neighbors.

Until all the vertex in G have degree less than $\sqrt{n}$, we can apply the algorithm in (a) to color them with $O(\sqrt{n})$ colors with a poly-time.

In summary, we need $O(\sqrt{n})$ colors.

d.

Given undirected 4-colorable graph $G = (V, E)$, $|V| = n$. For each $v \in V$ vertext, let $d(v)$ be the degree of v.

If $d(v) \geq n^{\frac{2}{3}}$, then remove v and its neighbors $N(v) = \{w_i | (v, w_i) \in E, i \geq \sqrt{n}\}$, and we need remove $n^{\frac{1}{3}} = n/n^{\frac{2}{3}}$ times.

We first color v with one color. Then the subgraph of its neighbors is a 3 colorable graph since all its neighbors are connected with v and G is a 4 colorable graph.

According to the conclusion of (c) that the best known algorithm for coloring 3-colorable graphs uses $O(n^{0.19996})$ colors, we can color all the v and the subgraphs of their neighbors with $O(n^{0.19996}n^{\frac{1}{3}}) < O(\frac{2}{3})$ colors.

Until all the vertex in G have degree less than $n^{\frac{2}{3}}$, we can apply the algorithm in (a) to color them with $O(n^{\frac{2}{3}})$ colors with a poly-time.

In summary, we need $O(n^{\frac{2}{3}})$ colors.

e.

Claim: Suppose VERTEX COLORING is approximable to a factor better than $\frac{4}{3}k$ for a k-colorable graph when $P \neq NP$, then determining if a graph is 3-colorable is solvable in poly-time.

Proof: $\Rightarrow$

If VERTEX COLORING is approximable to a factor better than $\frac{4}{3}k$ when $P \neq NP$, then we can find an approximation algorithm that using less than $\frac{4}{3}k$ colors for a k-colorable graph.

For the 3-colorable graph ($k = 3$) and $\frac{4}{3}k = 4$, this approximation algorithm can determine whether this graph can be colored by less than 4 colors and thus it can determine whether this is a 3-colorable graph in poly-time.

This is contradict to the fact that determining if a graph is 3-colorable is NP-Complete problem and $P \neq NP$. Therefore, VERTEX COLORING is not approximable to a factor better than $\frac{4}{3}$ unless $P = NP$.

1.5

(a)

As an extreme point is a feasible solution of the linear program, we will show that if any extreme point is not in the set of $\{0, \frac{1}{2}, 1\}$, then it is not a feasible solution of the linear program by contradiction.

Proof: $\Rightarrow$

Suppose there is one extreme point $x_{i'}$ of the linear program that has $x_{i'} \notin \{0, \frac{1}{2}, 1\}$.

$\Rightarrow$ the optimal solution f^* of the linear program is

$$f^* = x_{i'}w_{i'} + \Sigma_{i \in V, i \neq i'} w_i x_i \tag{1}$$

where x_i are all the extreme points.

As $\forall (i,j) \in E, x_i + x_j \geq 1$ and there is at least one point x'_j that $(i', j') \in E$, we have $x_{i'} \geq 1 - x_{j'}$.
$\Rightarrow$ (1) becomes:

$$\begin{aligned} f^* &= x_{i'} w_{i'} + x_{j'} w_{j'} + \Sigma_{i \in V, i \neq i', i \neq j'} w_i x_i && (2) \\ &\geq (1 - x_{j'}) w_{i'} + x_{j'} w_{j'} + \Sigma_{i \in V, i \neq i', i \neq j'} w_i x_i && (3) \\ &= w_{i'} + x_{j'}(w_{j'} - w_{i'}) + \Sigma_{i \in V, i \neq i', i \neq j'} w_i x_i && (4) \end{aligned}$$

However, we can find other solutions when $x_{i'} \in \{0, \frac{1}{2}, 1\}$,

1. If $x_{i'} = 0$ and $x_{j'} = 1$, then the solution $f_1 = w_{j'} + \Sigma_{i \in V, i \neq i', i \neq j'} w_i x_i$
2. If $x_{i'} = \frac{1}{2}$ and $x_{j'} = \frac{1}{2}$, then the solution $f_2 = \frac{1}{2}(w_{i'} + w_{j'}) + \Sigma_{i \in V, i \neq i', i \neq j'} w_i x_i$
3. If $x_{i'} = 1$ and $x_{j'} = 0$, then the solution $f_3 = w_{i'} + \Sigma_{i \in V, i \neq i', i \neq j'} w_i x_i$

Then we will show the contradiction that f^* is not the minimum.

(1) . If $w_{i'} > w_{j'}$, then from (2), we have

$$\begin{aligned} f^* &\geq x_{i'} w_{j'} + x_{j'} w_{j'} + \Sigma_{i \in V, i \neq i', i \neq j'} w_i x_i && (5) \\ &\geq w_{j'} + \Sigma_{i \in V, i \neq i', i \neq j'} w_i x_i && (6) \\ &= f_1 && (7) \end{aligned}$$

(2) . If $w_{i'} = w_{j'}$, then from (4), we have $(w_{j'} - w_{i'}) = 0$ and $f^* \geq f_3$;

(3) . If $w_{i'} < w_{j'}$, then from (4), we have $(w_{j'} - w_{i'}) > 0$ and $f^* > f_3$.

Therefore, f^* is not the minimum and this is contradict to our assumption. Hence, we have demonstrated that any the extreme point of the linear program has the property that $x_i \in \{0, \frac{1}{2}, 1\}$ for all $i \in V$.

(b)

I first give an algorithm for the vertex cover problem when the input graph is planar, then I will show that it is a $\frac{3}{2}$-approximation algorithm.

My algorithm is as following:

1. Input the planar graph and run the 4-color for it (assign each vertex one of four colors such that for any edge $(i, j) \in E$, vertices i and j have been assigned different colors);
2. Find the color $*$ that is assigned to the maximum number of points;
3. Take all the points that are assigned by the other three colors rather than the color $*$. Then these points form the vertex cover of the graph.

Then I will demonstrate that this algorithm is $\frac{3}{2}$-approximation.

Proof: $\Rightarrow$

1. As running the 4-color algorithm and finding the color with the maximum number can be finished in poly-time, this algorithm is a poly-time method;
2. As each line is assigned with two distinct colors and at least one of them is not the color $*$, at least one of the points of each line is selected by this algorithm. Thus our solution is a vertex cover;

3. We employ the optimal solution of linear program LP-OPT as the lower bound of the vertex cover problem.

   When there are n points in the graph, then the LP-OPT$=\frac{n}{2}$ and the optimal solution of vertex cover LP-OPT $\leq$ OPT.

   As all the n points of the graph is assigned with one of the four colors, the maximal number of points $N(*)$ assigned by the color $*$ is larger than $\frac{n}{4}$, otherwise the total number of all these four colors is less than n. Thus the number of points selected by my algorithm S satisfies

$$\begin{aligned} S = n - N(*) &\leq n - \frac{n}{4} && (8)\\ &= \frac{3n}{4} && (9)\\ &= \frac{3}{2}\text{LP-OPT} && (10)\\ &\leq \frac{3}{2}\text{OPT} && (11) \end{aligned}$$

   Therefore, this algorithm is a $\frac{3}{2}$-approximation algorithm.

4. There is one example in Figure 1 that shows when the equal sign is reached. For different colors are assigned to these 4 points and our algorithm select 3 of them. The number of vertex cover is 2. Therefore, our algorithm is $\frac{3}{2}$ times of the OPT.

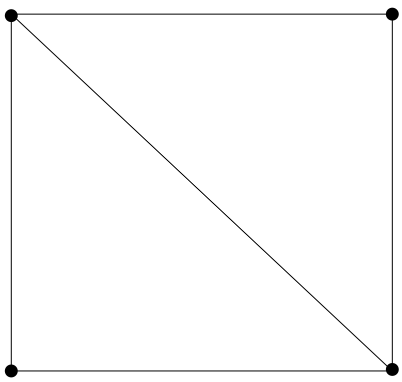

Figure 1: One example