Homework 1

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1.

a.

Given a $(\Delta + 1)$-colorable graph, where Δ is the maximum degree of any vertex in $G = (V, E)$,

1. Find the vertex v in G with the maximum degree Δ and its neighbor $N(v) = \{w|(v, w) \in E, i = 1, \dots, \Delta\}$;
2. First color v with one color, then color its neighbors $w_i \in N(v)$ one by one. If w_i is also connected to $w_j, i \neq j$, then color them with different colors. In this way, we need at most $\Delta + 1$ colors to color v and its neighbor $N(v)$;
3. Then find another v' not colored yet and its neighbor $N(v') = \{w|(v', w) \in E, i \leq \Delta\}$. Color them in the same way as step (2) using the $\Delta + 1$ colors;
4. Continue to color all other vertexes with less than $\Delta + 1$ colors until the graph G is completely colored.

As we can color G in poly-time with at most $(\Delta + 1)$ colors, then G is $(\Delta + 1)$ colorable.

b.

Bipartite graph is an undirected graph with nodes partitioned into group X and Y. For each line, its one ending node is in group X and the other ending node is in Y.

Then we can color all the nodes in group X with one color and the nodes in group Y with the other color. Therefore, bipartite graphs are 2-colorable.

c.

Given undirected 3-colorable graph $G = (V, E)$, $|V| = n$. For each $v \in V$ vertext, let $d(v)$ be the degree of v.

If $d(v) \geq \sqrt{(n)}$, then remove v and its neighbors $N(v) = \{w_i|(v, w_i) \in E, i \geq \sqrt{(n)}\}$, and color v with the first color and then color its neighbor w_1 with the second color, and then color all other neighbors one by one. If two of its neighbors w_i, w_j are connected, then color them with the second and the third color, otherwise color the neighbors with the second color. We can color the subgraph with a poly-time $O(n)$.

As graph G is a 3 colorable graph and thus its subgraph is also a 3-colorable graph, thus we need at most 3 colors to color each subgraph.

Each time we remove at most $\sqrt{n}+1$ vetex from G, which has n vertex in total, thus we need to remove at most $\sqrt{(n)}$. As we need color each subgraph with at most 3 colors, then in order to color all these vertex and its neighbors, we need at most $O(\sqrt{n})$ colors.

Until all the vertex in G have degree less than $\sqrt{(n)}$, we can apply the algorithm in (a) to color them with $O(\sqrt{(n)})$ colors with a poly-time.

In summary, we need $O(\sqrt{n})$ colors.

d.

Given undirected 3-colorable graph $G = (V, E)$, $|V| = n$. For each $v \in V$ vertext, let $d(v)$ be the degree of v.
If $d(v) \geq n^{\frac{2}{3}}$, then remove v and its neighbors $N(v) = \{w_i | (v, w_i) \in E, i \geq \sqrt{(n)}\}$, and color v with the first color and then color its neighbor w_1 with the second color, and then color all other neighbors one by one. If two of its neighbors w_i, w_j are connected, then color them with the second and the third color, otherwise color the neighbors with the second color. We can color the subgraph with a poly-time $O(n)$.

As graph G is a 4 colorable graph and thus its subgraph is also a 3-colorable graph, thus we need at most 4 colors to color each subgraph.

Each time we remove at most $(n^{\frac{2}{3}} + 1)$ vetex from G, which has n vertex in total, thus we need to remove at most $n^{\frac{1}{3}}$. As we need color each subgraph with at most 4 colors, then in order to color all these vertex and its neighbors, we need at most $O(n^{\frac{1}{3}})$ colors.

Until all the vertex in G have degree less than $O(n^{\frac{1}{3}}) + O(n^{\frac{2}{3}})$, we can apply the algorithm in (a) to color them with $O(n^{\frac{2}{3}})$ colors with a poly-time.

In summary, we need $O(n^{\frac{2}{3}})$ colors.

e.

Claim: Suppose VERTEX COLORING is approximable to a factor better than $\frac{4}{3}k$ for a k-colorable graph when $P \neq NP$, then determining if a graph is 3-colorable is solvable in poly-time.

Proof: $\Rightarrow$

If VERTEX COLORING is approximable to a factor better than $\frac{4}{3}k$ when $P \neq NP$, then we can find a poly-time algorithm that using less $\frac{4}{3}k$ colors for a k-colorable graph.

When $k = 3$ and $\frac{4}{3}k = 4$, as $3 < \frac{4}{3}k$, then we can find a poly-time algorithm to determine if a graph is 3-colorable.

This is contradict to the fact that determining if a graph is 3-colorable is NP-Complete. Therefore, VERTEX COLORING is not approximable to a factor better than $\frac{4}{3}$ unless $P = NP$.