Does the explanation of melting ice cubes requires a special cosmological hypothesis?

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1 Introduction

Why do ice cubes always melt under room temperature? As it might feel easy to quote 'because thermal physics tells us that entropy is always increasing!' from most physics textbooks, the time asymmetry involved in this process has really to be introduced with more scrutiny. In this essay, I provide the explanation of melting ice cubes in two thermal physical paradigms: thermodynamics and statistical mechanics. I aim to show that an explanation free of any cosmological hypotheses can be given in thermodynamics. However one might be concerned about whether an explanation offered by a principle theory can really be an explanation. I then attempt to formulate an explanation of why ice melts in both frameworks of statistical mechanics set up by Albert (2000) and Wallace (2011). Albert (2000)'s explanation requires a special cosmological hypothesis that requires the initial state of the universe to have a low entropy (i.e. Low Entropy Past hypothesis), while Wallace (2011) only requires a restriction on the initial microstate (i.e. Simple Past Hypothesis). We might claim that in Wallace (2011) a cosmological hypothesis is not required as the restriction is not imposed on the macrostate of the universe. However I later argue that a hypothesis on the macrostate of the universe (i.e. a restriction on the 'coarse-grained' level which then corresponds to (almost) the Low Entropy Past hypothesis) is necessary for an explanation of the irreversible process of ice cubes melting in his framework.

2 The explanation in thermodynamics

It's clearer to separate the explanation to answers of two questions one can ask about the process of ice melting: why do ice cubes melt and why do ice cubes not freeze back? Thermodynamics can produce an answer of both of these questions without introducing an extra cosmological principle. The textbook style answer to this question is usually the second law of thermodynamics. However, Brown et al. (2009) argues that this tendency to melt really lies in the notion of equilibrium defined in the 'Minus First Law' (also called the Equilibrium principle).

Let's first check out why the Second Law is inadequate to provide a full-blown 'direction to melt' on its own. Consider Kelvin's version of the Second Law:

> No **cyclic** process can have the sole effect of extracting heat from a reservoir and producing a corresponding amount of work.

The Second Law only refers to 'cycles'. Suppose gas confined in a cylinder is in equilibrium state A at some initial time, and then we suddenly pull out the piston. Through a series of non-equilibrium states, it spontaneously expands and arrives at a new equilibrium state B. This expansion is not a cycle, although it could be part of a cycle.

Can we derive "B $\rightarrow$ A" is impossible" from the Second Law? No, but only "he process from B to A is impossible, provided other processes connecting them are available (to complete it as a cycle)". Hence the Second Law cannot say anything on its own about why B cannot be attained at an earlier time before A, so it can't be the most fundamental point of entry of temporal directionality into thermodynamics.

2.1 An explanation by the -1st law

The Minus first law states that:

> An isolated system in an arbitrary initial state within a finite fixed volume will spontaneously attain a unique state of equilibrium.

It can be broken into three components:

1. The existence of equilibrium states for isolated systems, which are defined as states that remain constant in time once attained, if no external conditions are changed. This claim rules out spontaneous fluctuation around the equilibrium state!
2. The uniqueness of the equilibrium state
3. The spontaneous approach to equilibrium from non-equilibrium.

Brown et al. (2009) claims that the asymmetry of evolution enters thermodynamics at the most fundamental level through the first component. The notion of equilibrium state is time-asymmetric in the sense that no departure from it is possible after the time it is attained. We can contrast it with equilibrium state defined in Boltzmann's statistical mechanics in the next section.

We can now piece together explanations to both questions under the established framework:

1 Why do ice cubes melt?

P0: isolated system The system containing the room and the ice cubes is (approximately) an isolated system.

P1: Equilibrium The state of the system containing the room and melted ice cubes is the state of equilibrium of this system or it is one of the intermediate states before the equilibrium state (as the water needs to heat up to the room temperature to reach equilibrium).

P2: -1 **law** An isolated system in an arbitrary initial state within a finite fixed volume will spontaneously attain a unique state of equilibrium.

Therefore Ice cubes melt as the system attains an equilibrium state in which ice cubes melt.

2 Why do the melted ice cubes (i.e. water) do not freeze back to ice cubes?

P0&P1&P2

A component of -1 **law** The equilibrium state is such that the isolated system does not depart from them once attained.

Therefore The melted ice cubes (i.e. water) not freeze back to ice cubes

No cosmological principle is used in the thermodynamical explanation offered. However one thing to note is that this explanation is only *approximate*. P2 only *approximately* holds in the scenarios that we are under in P0 as the system is only *approximately* isolated. Therefore we might raise doubts to this explanation as it does not *derive* the melting of ice cubes. I argue that we should not reject this explanation for it not being a derivation, as being a derivation is normally not considered as the demarcation criterion of an explanation. Another possible objection might come from whether principle theories hold can hold any explanatory power. As an example, objections towards the explanatory power of special relativity as a principle theory can be found in Brown and Pooley (2006). We might question if the laws themselves need any further explanation. In the end, these concerns motivate us to seek an alternative explanation in the paradigm of statistical mechanics.

2.2 An alternative explanation from the 2nd and 2.5th laws

Nevertheless we might question if some element of time asymmetry already exists in the 2nd law, but its power cannot be exercised because of the condition of cyclic process. A 2.5th law might be able to bring out the temporal asymmetry that lies in 2nd law.

The 2.5th law of thermodynamics All quasi-static processes are possible.

Notice that this 2.5th law is time-reversal invariantly. The definition of quasi-static does not embed any time asymmetry.

Suppose that we have two processes from A to B, where A is a state where the syringe is half-filled with gas, with the piston pushed to half of its length, B is a state where the syringe is completely filled by gas with the piston pushed to its full length. Process (i) is such that A is put in a infinite heat bath such that the gas quasi-statically expands to arrive at state B; while process (ii) is such that the gas in B spontaneously gather at the lower half of the syringe, and the lack of pressure brings the piston back, arriving at A. Now the combination of processes (i) and (ii) is forbidden by the 2nd law as it is a full conversion from heat into work. We can make the following argument:

P1 (i)+(ii) (i.e. quasi-static expansion followed by spontaneous contraction) is impossible as it is forbidden by the 2nd law.

P2 (ii) is possible **by the 2.5th law**.

P3 If (i) and (ii) are possible respectively, then (i)+(ii) is possible.

Conclusion (i) is impossible.

Therefore the impossibility of non-cyclic processes can be derived from constructing an impossible cycle with other known possible processes. We can imagine the process of the re-freezing of ice cubes to be put in one of such impossibly cycles. Below is an impossible cycle that I propose:

Process (i) The quasi-static melting of ice cubes after being taken out of fridge.

Process (ii) The water molecules spontaneously 'swim' to their original positions in the ice cubes.

Insert these two processes back to the above argument, we have an explanation of why (ii) is impossible.

This alternative explanation does not involve a special cosmological hypothesis as well. This alternative explanation also shows that, the irreversible unfreezing of ice cubes can be explained by the 2nd law.

3 From reversible microphysics to irreversible macrophysics

In Boltzmann's approach to statistical mechanics, once reaching equilibrium, the state is free to fluctuate around the equilibrium state. Hence we know that irreversibility cannot be so easily attained in Boltzmannian statistical mechanics, and the recurrence problem is bound to occur (unless the concept of probability is tweaked such that tiny probability implies impossibility). Most importantly, even before reaching equilibrium, the microphysics that 'governs' the dynamics is time-reversible. So why do the ice cubes never freeze back after they melt?

The Past Hypothesis aims to provide the direction of melting, in which the entropy of the early universe is assumed to be very low compared to the entropy of the current universe. Brown made an interesting analogy: A Dyson heater is symmetric, but if placed against the wall, the hot air only blows in one direction.

Here we will look into Albert (2000)'s and then Wallace (2011)'s arguments on how the Past Hypothesis (though defined differently) enables an irreversible macrophysics to be derived from reversible microphysics.

3.1 Albert (2000)'s framework

Wallace (2011) summarises Albert (2000) as two claims:

1. The tendency of system's entropy to increase with time is just a consequence of the geometry of phase space. The region of phase space corresponding to a system being at equilibrium (i.e. at maximum entropy) is much large compare to the rest of phase space. Unless the initial state is "ridiculously special", then the system will in short time ends up in the equilibrium state.
2. The asymmetry of time in statistical physics (e.g. the tendency of entropy to increase) can be derived from time-symmetric microphysics given the Low Entropy Past Hypothesis (LEPH).

Albert's structure of statistical mechanics contains three components: **Newtonain mechanics (as microphysics), LEPH, and the Statistical Postulate**.

> **Statistical Postulate**: the right probability-distribution to use for making inferences about the past and the future is the one that's uniform, on the standard measure, over those regions of phase space which are compatible with whatever other information—either in the form of laws or in the form of contingent empirical facts—we happen to have.

And this is how the three can be used to make inference about the future:

> Start with a probability-distribution which is uniform – on the standard measure – over the world's present macrocondition. Conditionalize that distribution on all we take ourselves to know of the world's entire macroscopic past history (and this will amount to precisely the same thing – if you think it over as conditionalizing it on the past hypothesis).
> Then evolve this conditionalized present-distribution, by means of the equations of motion, into the future.
> This will yield everything we take ourselves to know of the future. (Albert, 2000)

So (1) tells us that, if our ice cubes (assuming it has the behaviour of a not ridiculously special, but 'typical' system) are put under room temperature, then it is much more probable (under the statistics described in the Statistical

Postulate, defined later) for it be in a melting/melted state compared to the state in which they stay frozen. This is just a result of the chosen statistics. The asymmetric boundary condition set by LEPH is what gives a directional solution to time-symmetric microphysics (i.e. ice cubes do not freeze back).

The necessity of LEPH can be argued this way: if we explain the fully-melted ice cubes ($t = 5$) after 5 minutes of being half-melted at $t = 0$ as the fully-melted state is more possible according to the statistics, then as the microphysics is reversible, we should postulate that it is also possible that, half-full ice cubes are formed from a splash of water under the same external conditions, 5 minutes ago before they start to melt ($t = -5$). The postulation is empirically false as this has never been observed. Therefore one might try to fix it by positing that at $t = -5$, the ice cubes are perfectly frozen according to our memory. But this could not solve the empirical failure as then it leads to the possible history of perfectly frozen ice forming from half-melted ice at $t = -10$. Therefore it is necessary to impose a perfectly frozen condition of mactrostate at the very beginning of time such that any possibilities of a more frozen past in inhibited as there is no such past.

An explanation for the melting of ice cubes can be structured in Albert (2000)'s style in this way:

1 Why do ice cubes melt?

> **P1: Statistical postulate** offers the correct probability distribution used for later inference.
>
> **P2: Equilibrium** The state of the system containing the room and melted ice cubes is the state of equilibrium of this system with the overwhelmingly highest probability.
>
> **P3: Dynamics** Applying the time-reversible microphysics to distribution, the resulting evolution is such that a very large space is taken by the states in which ice cubes melt.
>
> **Therefore** Ice cubes melt after a later time as the state of melted ice cubes is overwhelmingly more possible.

2 Why do the melted ice cubes (i.e. water) not freeze back to ice cubes?

> **LEPH, P1, P2, P3**
>
> **P4** The system has not yet attained the equilibrium state.
>
> **Therefore** The melted ice cubes (i.e. water) will not freeze back to ice cubes

This explanation involves a special cosmological principle (i.e. LEPH). However two problems arise within Albert (2000)'s framework of statistical mechanics as one might discover.

1. Albert (2000)'s claim is empirically false on the time-scale of the system's evolution to equilibrium. As he claimed, because of the much larger phase

space taken up by the equilibrium state, the system will 'in short time end up in the equilibrium state'. As Wallace (2011) questions, how come our universe has not yet reached that state after billion of years? The general idea of system ending up in equilibrium due to its high probability also does not seem to be in line with the well-developed dynamical theories of the system's evolution in chemical industry where a precise quantification of reaction/mixing time is provided. This feature is not fatal, however, as billions of year might be short compared the lifetime of the universe, but it definitely leaves vagueness in the framework.

2. The vague time-scale leads to a bigger problem. Albert (2000) still fails to explain why the ice cubes do not freeze back. Suppose the room containing the ice cubes is an isolated system, then after reaching the equilibrium state 'in a short time', they will be able to freeze back again. The process of fluctuation is time symmetric, which means once reaching equilibrium, directionality is lost. The recurrence problem occurs such that entropy could be equally likely to be higher or lower than earlier. LEPH is only sufficient in explaining that *prior to* the time of reaching equilibrium, under room conditions, a splash of water does not spontaneously freeze into ice cubes. However if we are under Albert (2000)'s Boltzmannian framework of statistical mechanics, then it is possible for the splash of water to freeze back to ice cubes *after* the time of equilibrium. The Boltzmannian picture is bound to have the problem of recurrence, which justifies one's reason to object the overall framework of explanation.

Hence later we will use Wallace (2011)'s formulation of the explanation in statistical mechanics. We now leave these problems behind and move to Wallace (2011)'s paradigm of statistical mechanics.

3.2 Wallace (2011)'s framework

Wallace (2011) laid out a derivation from reversible microdynamics and his version of Past hypothesis to irreversible macrodynamics in both classical and quantum cases. As Wallace (2011) adopt the Gibbs but not the Boltzmannian idea of equilibrium, this can hopefully provide a more promising account of 'why ice cubes melt' in statistical mechanics. His original argument only requires a version of past hypothesis on the microstates of the universe. Nevertheless I attempt to argue that LEPH cannot be avoided if melting ice cubes are to be derived from his framework.

Here is a sketch of the derivation.

Both classical and quantum mechanics (under the Everettian interpretation) are structured in the following form:

1. A state space (phase space for classical mechanics or (projective) Hilbert space for QM);

2. A deterministic rule determining how all points on that state space evolves over time;

3. Time reversibility in the sense that given the state at time t, the dynamics is as well suited to determine the state for times before t as for times after t.

Wallace (2011)'s framework studies the evolution of a distribution (or ensemble) ρ. In classical mechanics, if the dynamical law is presented by function ψ_t, such that for a point x in the phase space, $\psi_t(x)$ is the t-second evolution of x, the the entire phase space is evolved to $\psi_t * \rho = \rho \cdot \psi_t$. In quantum mechanics, the distribution corresponds to a mixed state (represented by the density matrix), and if the t-second evolution takes $|\psi\rangle$ to $\hat{U}_t |\psi\rangle$, then the t-second distributional variant evolution takes ρ to $\hat{U}_t \rho \hat{U}_t^{-1}$.

Then Wallace (2011) gives a purely mathematical formalism to how macro-properties can be extracted from microstates, which I won't go into much detail. This extraction scheme is purely descriptive and it does not provide any shortcut to calculate macro-properties from microstates, but we need the description scheme to connect results of microphysics to results of macrophysics. In the next section, he introduces *coarse-graining maps*, which are believed to represent what mathematically is going on in the approximation scheme of microstates in common practice of physics. These coarse-graining maps $\mathcal{C}$ commute with forward evolution given by microphysics.

The key property of these coarse-graining maps is that, given a set of macroscopic properties interested, $\mathcal{C}$ projects the original distribution space onto a subset of the distributions, such that the maps would leave the macroproperties unchanged (precisely, the probability of any given macroproperty being possessed by the system is approximately unchanged). The coarse-graining exemplar rule is that: construct equivalence classes of distributions with the same probability function over macroproperties; the coarse-graining map takes all elements of the equivalence class onto one element in that class. Most notably, the small-scale correlations and entanglements between spatially distant subsystems are erased.

A *forward dynamics induced by* $\mathcal{C}$ (or the $\mathcal{C}+$ dynamics) is defined as taking the following steps on a distribution: 1. coarse-grain it; 2. time-evolve it forward under the microdynamics by Δt; 3, repeat step 1 and then 2.

As its name suggests, $\mathcal{C}+$ dynamics is not typically time reversible, and the existence of reverse coarse-grained dynamical trajectories (obtained from coarse-grained dynamics) is prevented. The Gibbs entropy function S_G can be defined such that S_G is preserved under micro-dynamical evolution and time reversal, but for any ρ, $S_G(C\rho) \geq S_G(\rho)$. Therefore under $C+$ dynamics which consists of alternating microdynamical evolution and coarse-graining, S_G is non-decreasing. In the probabilistic case, the branching structure brought by the $\mathcal{C}+$ dynamics also indicates irreversibility.

Now we have concluded that, from a purely mathematical standpoint, irreversible macrodynamics is derivable from $C+$ dynamics, which is an accurate approximation of reversible microphysics. What's left to be proved is that the macrodynamics defined by the $\mathcal{C}+$ dynamics is the same as the macrodynamics induced by the microdynamics.

> **Simple Dynamical Conjecture (SDC)** (for a given system with coarse-graining $\mathcal{C}$): Any distribution whose structure is Simple is forward predictable by $\mathcal{C}$; any distribution *not so* predictable is not specifiable in any way except by stipulating that it is generated via evolving some other distribution in time (for instance, by starting with a simple distribution, evolving it forwards in time, and then time reversing it).

A distribution is Simple if it can be represented by $d(\vec{x}, \vec{p})$ as a Simple distribution is 'any distribution specifiable in a closed form in a simple way, without specifying it via the time evolution of some other distribution'.

After 19 pages of heavy technicalities, we have arrived at the **Simple** Dynamical Conjecture. SDC is to ensure the *forward compatibility* of the fine-grained dynamics with any coarse-graining rule. A distribution is *forward compatible* with a given $\mathcal{C}$ if the result of evolve it under the accurate microphysics then coarse-grain the distribution is the same as the result of evolve under coarse-grain dynamics. This conjecture is in the hope to justify for the approximation we applied on microphysics ($C+$ dynamics) such that we get the correct macroproperties by the end. A distribution will be forward compatible with a smearing coarse-graining rule whenever the microscopic details of the distribution do not affect the evolution of its overall spread across phase space.

> **Simple Past Hypothesis (SPH)**:
> classical version: There is some Simple distribution ρ over the phase space of the universe such that for any point x, $\rho(x)\delta V$ is the probability of the initial state of the universe being in some small region δV around x.
> quantum version: The initial quantum state of the Universe is Simple.

Assuming SDC, SPH ensures that the initial state of the universe is forward predictable by $\mathcal{C}+$ dynamics, so the macrodynamics defined by the $\mathcal{C}+$ dynamics is the same as the macrodynamics induced by the microdynamics. Therefore, since $\mathcal{C}+$ is time irreversible, the macrodynamics reached by the end is time irreversible.

Two more statements about Microphysics and Macrophysics:

> **Predictive accuracy of Microphysics (PAμ)**: our current best theory of microphysics is predictively accurate.
> **Predictive accuracy of Macrophysics (PAM)**: our current best

theory of macrophysics dervied from coarse-graining microdynamics is predictively accurate.

Finally we have gathered enough elements to produce Wallace's derivation from a microphysics that is time reversible to a macrophysics that is time irreversible:

SPH + SDC + PAμ $\rightarrow$ PAM.

He first establishes that our current physics theories is not time symmetric because of the approximation schemes. Then he postulated that these approximation schemes will give rise to the same, correct macrophysics. Therefore it can be derived that the correct macrophysics is not time symmetric.

We can structure an explanation in the framework of Wallace (2011) as follows:

1 Why do ice cubes melt?

P1: SPH The probabilities we assign to possible initial states of the Universe are given by a Simple probability distribution.

P2: SDC A simple distribution is forwardly predictable by $C+$ dynamics such that the macrodynamics defined by the $\mathcal{C}+$ dynamics is the same as the macrodynamics induced by the microdynamics

P3: PAμ Classical mechanics (i.e. the microphysics) is predictively successful.

Therefore: P4 PAM.

P5: Prediction Ice cubes melt as it is the prediction of $C+$ dynamics. (see objection to this later)

Therefore Ice cubes melt (because of PAM).

2 Why do the melted ice cubes (i.e. water) not freeze back to ice cubes?

PAM already entails exactly in which way ice cubes melt (i.e. irreversibly) given the same external conditions. The irreversibility of $C+$ dynamics is what gives the irreversibility of the melting of ice cubes.

This explanation does not require a special cosmological principle because SPH is only a condition on the initial microstate. As Wallace clarified in a later work (Wallace, 2021):

> Unlike the commonly-discussed past hypothesis that the initial state of the system (most commonly, the entire Universe) is low in entropy, the forward-compatibility conditions are conditions on the microstate, invisible at the coarse-grained level.

3.3 *Interesting* Past Hypothesis

However we might raise objection to P5. Can the physical fact that ice cubes melt under room temperature be derived from $C+$ dynamics? The problem is that we need to assume that the initial distribution is not *Boring* as defined in Wallace (2011). A distribution is *Boring* over a given time period if evolving its coarse-graining forward under the microphysics within the time period leads to only other coarse-grained distributions. Otherwise, the distribution is *Interesting*. If a distribution is *Boring*, then we cannot guarantee an increasing Gibbs entropy for that period of time. Suppose the universe starts in the equilibrium state, there will be no evolution at all according to the $C+$ dynamics. The forward time evolution of a stationary distribution under the microdynamics is the the distribution itself. Yet this supposition does not violate any premises in Wallace (2011)'s argument. Hence I argue that an additional premise ensuring that there is at least one segment of the dynamical trajectory that is not *Boring* (i.e. *Interesting*). What should be the start point of this segment? I suggest that it has to be imposed at the very beginning of of the universe because *Boring* distributions cannot evolve into *Interesting* states at later times if $C+$ dynamics is obeyed. Therefore, I conclude that this additional cosmological hypothesis has to be added to an explanation of any physical observation that involves an irreversible process. What would Wallace (2011) possibly say about this addition? I believe that he will happily accept it as it does not interfere with his original derivation of PAM. We are not saying that his argument lacks a logical connection, but solely pointing out that, with PAM only and no contingent facts about the initial state of the universe (SPH isn't restrictive enough), it is impossible to derive any irreversible evolution in a non-specified universe.

Interesting past hypothesis (IPH) The initial distribution of the Universe is *Interesting*.

Nevertheless, P5 could still not be derived from IPH. For P5 to be derived, a specification of the initial distribution of our universe has to be given such that the existence of ice is possible. However an explanation of some properties of the macrostate does not have to be a derivation of the exact macrostate. An explanation of why ice cubes melt (and do not freeze back) does not need to include an explanation of why ice cubes exist. I argue that IPH is enough for an **explanation** of the irreversible melting of ice cubes.

Is IPH less restrictive than LEPH? I believe that IPH is almost as restrictive as LEPH. If the universe starts with an Interesting initial distribution, then it must not start with a distribution of highest entropy as it cannot reach the equilibrium state at the beginning. From we know about the our universe is that it hasn't reached equilibrium after billions of years, and it won't reach equilibrium even in the distant future. Therefore *empirically* the interesting initial distribution must be the one that corresponds to LEPH.

Is IPH then a cosmological hypothesis? Yes because the restriction set by IPH is no longer 'condition on the microstate, invisible on the coarse-grained level'. Any conditions visible on the coarse-graining level are eventually, a condition on the macrostates. This can be understood by recalling the definition of coarse-graining maps mentioned earlier, in which C maps are defined to leave the macroproperties unchanged. The restriction is on how the microstate level evolves after being coarse-grained, and it is (almost) equivalent to restricting the initial state of the universe to have a very low entropy.

4 Conclusion

So do we need a special cosmological hypothesis, or not? As I have argued so far, the answer varies with the thermal physics paradigm. It is shown that in the paradigm of thermodynamics, no cosmological hypothesis is required to explain the irreversible melting of ice cubes. As one might worry about whether an explanation from thermodynamics is a satisfactory explanation since laws of thermodynamics might be thought to be not fundamental enough, I also attempt to provide an explanation in the paradigm of statistical mechanics. Albert (2000)'s framework of statistical mechanics is briefly discussed but was concluded as non-satisfactory as any explanation in the Bolztmann paradigm cannot avoid the recurrence problem. I then focus on an explanation in Wallace (2011) Gibbs framework of statistical mechanics, and conclude that both a hypothesis on the microstate (SPH) and on the macrostate/coarse-grained level (IPH) are required for explaining why ice cubes melt irreversibly. If IPH can be properly classified as a cosmological hypothesis, then the explanation of melting ice cubes indeed requires a special cosmological hypothesis.

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