Stochastic Finance (M3, 2016-17)
Some Solutions for Exercise Problems

Exercise 3.1 (Brownian Bridge)

(a) This problem is based on a series representation of Brownian motion (also see p.286 of Steele),

$$B_t = tZ_0 + \sum_{k=0} + \sqrt{2}\sum_{k=1}^{\infty} Z_k \frac{\sin \pi k t}{\pi k}$$

for independent standard normal random variables, $\{Z_{k\geq 0}\}$. (But I think the author somehow dropped this in the current version of the textbook.) So, $\Delta_0(t) = t$ and $\lambda_0 = 1$. Because $B_1 = Z_0$ (from $\Delta_{k\geq 1}(1) = 0$), the first term can be expressed as $\lambda_0 Z_0 \Delta_0(t) = t\, B_1$. Therefore,

$$U_t = B_t - tB_1$$

(b)

$$\begin{aligned}\mathrm{Cov}(U_s, U_t) &= E\Big((B_s - sB_1)(B_t - tB_1)\Big) = E\Big(B_s B_t - sB_1 B_t - tB_s B_1 + stB_1^2\Big)\\ &= \min(s,t) - s\min(1,t) - t\min(s,t) + st = s(1-t)\end{aligned}$$

(c) We need to find any set of functions, $g(\cdot)$ and $h(\cdot)$, such that

$$\mathrm{Cov}(X_s, X_t) = g(s)g(t)\min(h(s), h(t)) = s(1-t) \quad \text{for} \quad s \leq t.$$

If we narrow down the search by assuming $h(\cdot)$ is monotonically increasing,

$$\mathrm{Cov}(X_s, X_t) = g(s)g(t)h(s) = s(1-t),$$

so we get

$$g(t) = 1-t, \quad h(s) = \frac{s}{1-s},$$

where $h(s)$ is indeed an increasing function. Therefore we obtained a representation of Brownian bridge,

$$X_t = (1-t)B_{\frac{t}{1-t}}$$

Since U_{1-t} is also a Brownian bridge due to the symmetry,

$$X_{1-t} = tB_{\frac{1-t}{t}}, \quad \left(g(t) = t,\ h(t) = \frac{1-t}{t}\right)$$

is also a valid solution.

(d) We use the inequality $s/(1+s) \leq t/(1+t)$ if $0 \leq s \leq t$.

$$\begin{aligned}\mathrm{Cov}(Y_s, Y_t) &= \mathrm{Cov}\Big((1+s)U_{\frac{s}{1+s}}(1+t)U_{\frac{t}{1+t}}\Big) = (1+s)(1+t)\mathrm{Cov}(U_{\frac{s}{1+s}}, U_{\frac{t}{1+t}})\\ &= (1+s)(1+t)\,\frac{s}{1+s}\Big(1 - \frac{t}{1+t}\Big) = s = \min(s,t).\end{aligned}$$

Exercise 3.2 (Cautionary Tale) Suppose X is a standard normal, consider an independent U such that $P(U = 1) = 1/2 = P(U = 1)$, and set $Y = UX$. Then, Y is also a standard normal as X and $-X$ are also standard normal.

In order to show X and Y are not independent, we need to show

$$\mathrm{Prob}(I_X \ \&\ J_Y) \neq \mathrm{Prob}(I_X)\mathrm{Prob}(J_Y)$$

for some event I_X and I_Y regarding X and Y respectively.

For any $h > 0$,

$$\begin{aligned}
\mathrm{Prob}(X > h \ \&\ Y > h) &= \frac{1}{2}\mathrm{Prob}(X > h \ \&\ X > h \mid U = 1) \\
&\quad + \frac{1}{2}\mathrm{Prob}(X > h \ \&\ -X > h \mid U = -1) \\
&= \frac{1}{2}(1 - \Phi(h)) + 0.
\end{aligned}$$

However,

$$\mathrm{Prob}(X > h)\mathrm{Prob}(Y > h) = (1 - \Phi(h))(1 - \Phi(h))$$

is not same as the previous value. Therefore

Exercise 3.3 (Multivariage Gaussians)

(a) Let us work on each components of the vectors and matrices; $V = (v_i)$, $\mu = (\mu_i)$, $A = (a_{ij})$ and $\Sigma = (\sigma_{ij})$.

$$E\Big((AV)_i\Big) = E\Big(\sum_j a_{ij}V_j\Big) = \sum_j a_{ij}E\Big(V_j\Big) = \sum_j a_{ij}\mu_j = (A\mu)_i$$

$$E\Big((AV)\Big) = A\mu$$

$$\begin{aligned}
\mathrm{Cov}\Big((AV)_i, (AV)_j\Big) &= \mathrm{Cov}\Big(\sum_l a_{il}V_l, \sum_m a_{jm}V_m\Big) = \sum_{l,m} a_{il}\mathrm{Cov}(V_l, V_m)a_{jm} \\
&= \sum_{l,m} a_{il}\sigma_{lm}a_{jm} = (A\Sigma A^T)_{ij}
\end{aligned}$$

Therefore,

$$\mathrm{Cov}\Big(AV, AV\Big) = A\Sigma A^T.$$

(b)

$$E(X \pm Y) = E(X) \pm E(Y) = 0 \pm 0 = 0$$

$$\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \pm 2\,\mathrm{Covar}(X, Y) = 1 + 1 + 0 = 2$$

$$\mathrm{Cov}(X + Y, X - Y) = \mathrm{Var}(X) - \mathrm{Var}(Y) = 1 - 1 = 0$$

(c) When $\mathrm{Cov}(X, Y) = 0$, the covariance matrix Σ is given as

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma 22 \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} 1/\sigma_{11} & 0 \\ 0 & 1/\sigma 22 \end{pmatrix}, \quad \det\Sigma = \sigma_{11}\,\sigma_{22}$$

The joint density function can be factored to the product of the single variable density function,

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sqrt{\sigma_{11}\,\sigma_{22}}}\exp\Big(-\frac{(x-\mu_X)^2}{2\sigma_{11}}-\frac{(y-\mu_Y)^2}{2\sigma_{22}}\Big) \\ &= \frac{1}{2\pi\sqrt{\sigma_{11}}}\exp\Big(-\frac{(x-\mu_X)^2}{2\sigma_{11}}\Big)\,\frac{1}{2\pi\sqrt{\sigma_{22}}}\exp\Big(-\frac{(y-\mu_Y)^2}{2\sigma_{22}}\Big) = f(x)f(y). \end{aligned}$$

Therefore X and Y are independent.

(d) We first find the matrix A such that, for the independent standard normal variables W and Z,

$$\begin{pmatrix} X-\mu_X \\ Y-\mu_Y \end{pmatrix} = A\begin{pmatrix} W \\ Z \end{pmatrix}$$

has the given covariance matrix

$$\begin{pmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{pmatrix} = A\,I\,A^T = A\,A^T.$$

One of the solution from Cholesky decomposition is

$$A = \begin{pmatrix} \sqrt{\sigma_{XX}} & 0 \\ \sigma_{XY}/\sqrt{\sigma_{XX}} & \sqrt{\sigma_{YY}-\sigma_{XY}^2/\sigma_{XX}} \end{pmatrix}.$$

Conditional on that $X = x$,

$$Y = \mu_Y + \frac{\sigma_{XY}}{\sqrt{\sigma_{XX}}}\frac{x-\mu_X}{\sqrt{\sigma_{XX}}} + Z\sqrt{\sigma_{YY}-\sigma_{XY}^2/\sigma_{XX}}.$$

Therefore

$$\begin{aligned} E(Y|X=x) &= \frac{\sigma_{XY}}{\sigma_{XX}}(x-\mu_X) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(x-\mu_X) \\ \mathrm{Var}(Y|X=x) &= \sigma_{YY} - \frac{\sigma_{XY}^2}{\sigma_{XX}} = \mathrm{Var}(Y) - \frac{\mathrm{Cov}^2(X,Y)}{\mathrm{Var}(X)} \end{aligned}$$

Exercise 3.4 (Auxilary Functions and Moments)

$$E(e^{tz}) = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \int_{-\infty}^{\infty} e^{t^2/2}\frac{1}{\sqrt{2\pi}}e^{-(z-t)^2/2}dz = e^{t^2/2}$$

If M_n is the $n-$th moment,

$$E(e^{tz}) = \sum_{k=0}^{\infty} M_k\frac{t^k}{k!} = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \cdots$$

By matching the coefficients, we get

$$\begin{aligned} M_0 &= 1 \\ M_1 &= M_3\ (= M_{2k-1}) = 0 \\ M_2 &= 1 \\ M_4 &= 4!/(2!\ 2^2) = 3 \\ M_6 &= 6!/(3!\ 2^3) = 15. \end{aligned}$$

For $t > 0$,

$$E(e^{tz^4}) = \int_{-\infty}^{\infty} e^{tz^4}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \to \infty,$$

so the MGF of Z^4 does not exist.

Exercise 4.6 The stopped process is a martingale. By the symmetry, $\mathrm{P}(B_\tau = A) = \mathrm{P}(B_\tau = -A) = 0.5$.

$$1 = E(X_\tau) = \frac{1}{2}e^{\alpha A}E(e^{-\alpha^2\tau/2}) + \frac{1}{2}e^{-\alpha A}E(e^{-\alpha^2\tau/2}).$$

Therefore, we have

$$E\left(e^{-\alpha^2\tau/2}\right) = \frac{1}{\cosh(\alpha A)}$$

or

$$\phi(\lambda) = E(e^{-\lambda\tau}) = \frac{1}{\cosh(A\sqrt{2\lambda})}.$$

In order to calculate $E(\tau^2)$, we need to obtain the x^4 term in the expansion of $1/\cosh(x)$ given that $\sqrt{\lambda}$ appears in the expression. From the expansion, $\cosh x \sim 1 + x^2/2! + x^4/4! + \cdots$,

$$\frac{1}{\cosh x} \sim \frac{1}{1 + (x^2/2! + x^4/4! + \cdots)} = 1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \cdots\right)^2 = 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + \cdots.$$

Finally we get

$$E(\tau^2) = 2\frac{5}{24}(A\sqrt{2\lambda})^4|_{\lambda=1} = \frac{5}{3}A^4$$

For the non-symmetric case ($A \neq B$), we can not use $\mathrm{P}(B_\tau = A) = \mathrm{P}(B_\tau = -B) = 0.5$ anymore.