Course Notes

Stochastic Finance

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Chapter 1

Random Walks and First Step Analysis

Random walk is a probability process whose incremental change in unit time is up or down by random;

$$S_n = S_0 + X_1 + X_2 + \cdots X_n,$$

$$\text{where } X_k = 1 \text{ or } -1 \text{ with 50:50 chance.}$$

The process models the wealth of a gambler but it is easier to understand if S_n is the daily closing price of a stock and X_n is the profit and loss (P&L) of the n-th day.

For the rest of this chapter, except §1.5, we are interested in the event of S_n hitting A before hitting $-B$ (the gambler making \$$A$ first before losing \$$B$). Equivalently, the event is the stock price gaining A before losing B (assuming that you set a trading strategy of loss-cutting at $-B$ and profit-realizing at A).

For the purpose, the *stopping time* τ is introduced as the first time n when S_n hits either A or $-B$. So we know that $S_\tau = A$ or $-B$ although we don't know the value of τ (τ is a probability variable).

1.1 First Step Analysis

We first solve the probability of the event, $P(S_\tau = A \mid S_0 = 0)$. Generalizing the problem, let

$$f(k) = P(S_\tau = A \mid S_0 = k)$$

be the probability of the same event with the initial point being $S_0 = k$ rather than 0. The recurrence relation is given as

$$f(k) = \frac{1}{2} f(k-1) + \frac{1}{2} f(k+1) \quad \text{for} \quad -B < k < A \tag{1.1}$$

with the *boundary conditions* $f(A) = 1$ and $f(-B) = 0$. This basically means that $f(k)$ is a linear function.

After some algebra, we get

$$f(k) = \frac{k+B}{A+B}, \quad P(S_\tau = A \mid S_0 = 0) = f(0) = \frac{B}{A+B}.$$

The result is in line with the intuition that the probability goes to 1 when B gets bigger or goes to 0 when A gets bigger.

In relation to finance, almost all probability or expectation values can be thought of as the price of a security or a derivative. In this example, we can think of a derivative that pays \$1 when the underling stock price S_n hits A or expires worthless when S_n hits $-B$. This is a *derivative* security because the payoff is *derived* from the underlying stock S_n. Unlike the usual call or put options, the expiry of this derivative is infinite (sometimes such security is called *perpetual*). The probability we computed above, $P(S_\tau = A \mid S_0 = 0)$, can be understood as the current price of the derivative.

Quiz: (a hedging strategy) Imagine that you (as an investment bank) sold the derivative to investors. How would you *hedge* your position using the underlying stock?

1.2 Time and Infinity

In this section, we computes the expected number of bets, τ, until the gambler finishes the game, i.e., when he makes \$A or loses \$B. **SCFA** first proves that the expectation of τ (and any power) is finite. (See **SCFA** for detail.)

In a similar approach from the previous section, the generalized expectation, $g(k) = E(\tau \mid S_0 = k)$ satisfy the recurrence relation,

$$g(k) = \frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) + 1 \quad \text{for} \quad -B < k < A$$

with the boundary condition, $g(A) = g(-B) = 0$.

Notice that $\frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) - g(k)$ is the convexity (or curvature) operator. From the Taylor expansion, we know for small h,

$$\frac{1}{2}g(x+h) + \frac{1}{2}g(x-h) - g(x) \approx \frac{1}{2}g''(x)h^2.$$

So the recurrence relation above implies that $g(k)$ is a quadratic function on k with the second order coefficient is -1. Therefore we conclude that

$$g(k) = (A-k)(B+k) \quad \text{and} \quad \mathbb{E}(\tau \mid S_0 = 0) = AB$$

This quantity can be also thought of as the price of a financial contract, in which \$1 is accumulated each time unit and the money is paid to the investor when the event is triggered. This type of derivatives are generally called *accumulator*.

SCFA verifies the obtained result for the symmetric case of $A = B$. The standard deviation of S_n is $\sqrt{n}$. (The variance is n.) Since the stdev is the characteristic width (or scale) of the process, we can estimate that the time required for the scale to reach A is A^2, which is consistent with the result.

Quiz (a popular interview question): Imagine that you keep tossing a fair coin (50% for head and 50% for tail) until you get two heads <u>in a row</u>. On average, how many times do you need to toss a coin?

1.3 Tossing an Unfair Coin

When the probability of X_1 is not fair and instead given as

$$X_n = 1 \text{ or } -1 \text{ with the chance of } p \text{ or } q \text{ respectively} \quad (p+q=1),$$

we can still drive the equivalent results.

After some algebra,

$$f(k) = \frac{(q/p)^{k+B}-1}{(q/p)^{A+B}-1} \quad \text{and} \quad P(S_\tau = A|S_0 = 0) = f(0) = \frac{(q/p)^B - 1}{(q/p)^{A+B}-1}.$$

$$\mathbb{E}(\tau \mid S_0 = 0) = \frac{B}{q-p} - \frac{A+B}{q-p}\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

One can recover the result of the fair bet case, if p and q are approaching to $\frac{1}{2}$, i.e., $p = \frac{1}{2} + \varepsilon$ and $q = \frac{1}{2} - \varepsilon$ for very small ε.

Quiz (numerical implementation): If you want to implement the above results, i.e., $f(k)$ and $g(k)$ for a general value of $p = 1/2+\varepsilon$, you will run into a small issue because you have to write a function for the two cases depending on $\varepsilon = 0$ or $\varepsilon \neq 0$. If ε is very small, then the formula may break. How would you resolve this issue?

1.4 Numerical Calculation and Intuition

I recommend that the students quickly verify the numbers in Table 1.1 using your favorite computer tool (R, Matlab, Python or even a calculator). It is quite noticeable that the probability for a gambler to win \$100 before losing \$100 is only 6×10^{-6} when $p = 0.47$.

1.5 First Steps with Generating Functions

The probability generating function is a powerful trick to obtain a series of values in one go, where the coefficients of the Taylor expansion is the values to seek. This chapter of **SCFA** considers the event of S_n hitting 1 for the first time (no longer the event of hitting A or $-B$) and wants to compute the probability of the event happening at time $\tau = 0, 1, 2, \cdots$ (the meaning of τ is also different from the previous sections!). The generating function is in the form of

$$\phi(z) = E(z^\tau \mid S_0 = 0) = \sum_{k=0}^{\infty} P(\tau = k \mid S_0 = 0)\, z^k,$$

i.e. the coefficient of z^k is the probability of S_n hitting 1 at time $\tau = k$ for the first time.

SCFA obtains the function $\phi(z)$ using the recurrence relation method. One important observation is that $\phi(z)^k$ is the generating function for the event of hitting k, which is from the property that the generating function for the sum of independent random variables is the product of the individual generating functions. For $k = 2$, let τ_1 is the first hitting time from 0 to 1 and τ_2 is the first hitting time from 1 to 2. Because τ_1 and τ_2 are independent (and identical) random variables,

$$E(z^{\tau_1+\tau_2}) = E(z^{\tau_1})E(z^{\tau_2}) = \phi(z)^2.$$

Thus, we end up the recurrent relation

$$\phi(z) = \frac{1}{2}\, z\, \phi(z)^2 + \frac{1}{2} z$$

and the $\phi(z)$ is finally given as

$$\phi(z) = \frac{1 - \sqrt{1 - z^2}}{z}.$$

The root with + sign was excluded because the function has the term of $1/z$ and non-zero constant term (the probability for the negative or zero first hitting time should be zero).

1.6 Exercises

Chapter 2

First Martingale Steps

Martingale is one of the key concepts in stochastic process. Although it is a very formal mathematical concept, it will turn out that many practical results will be derived out of it. For the definition of the martingale, we refer to Wikipedia.

> In probability theory, a martingale is a model of a fair game where knowledge of past events never helps predict the mean of the future winnings. In particular, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values.

In **SCFA**, a stochastic process $\{M_n : 0 \leq n\}$ is a *martingale* with respect to another stochastic process $\{X_n : 0 \leq n\}$ if (i) the sequence M_n is determined from the past knowledge of $\{X_k : 0 \leq k \leq n\}$ and (ii) the next expectation value is equal to the present value of M_n (*fundamental martingale identity*),

$$E(M_{n+1} \mid X_1, X_2, \cdots, X_n) = M_n \text{ for all } n \geq 0.$$

In general, however, $\{M_n\}$ is simply a martingale if the next expectation value, conditional on the history of itself, is equal to the present value,

$$E(M_{n+1} \mid M_1, M_2, \cdots, M_n) = M_n \text{ for all } n \geq 0.$$

2.1 Classic Examples

SCFA gives 3 examples of martingales

Example 1 If the X_n are independent random variables with zero mean, the running sum, $S_n = \sum_0^n X_k$ is a martingale. The process S_n was the subject of Chapter 1. So the wealth of a gambler or a stock price are all martingale as long as the game is fair ($E(X_n) = 0$ and the no one can look into the future. In the case of the stock, this assumption is closely related to the efficient market hypothesis, where the stock prices reflect the market information immediately and fully. Since all the news are *priced in* the stock, the expectation for tomorrow's stock is same as the current value (no one know that tomorrow's news will be good or bad).

This observation gives us a good example of what is **not** a martingale. Imagine that a stock price has a momentum (or a positive auto-correlation) in that the stock price tends to be up (or down) in a day when the price was up (or down) in the previous day, i.e., X_n and X_{n+1} are positively correlated rather than independent. The stock price in that circumstance is not a martingale because one can look into the future (based on the past). Many technical analyses are indeed based on that stock markets have momentum. For a well-known strategy, see the turtle trading rule.

Example 2 On top of the assumptions of **Example 1**, let us assume that $\mathrm{Var}(X_n) = \sigma$. Then $M_n = S_n^2 - n\sigma^2$ is also a martingale. See the textbook for the detailed proof. Basically what it tells us is that the squared process S_n^2 increases by the σ^2 on average on each time step, so we need to add the correction term, $-n\sigma^2$ for the process M_n to be a martingale. This is an important precursor to the famous Itô's lemma which we will cover later!

Example 3 If $\{X_n\}$ are non-negative independent random variables with $E(X_n) = 1$, the running product $M_n = X_1 \cdot X_2 \cdots X_n$ is a martingale. See the textbook for the detailed proof. Out of any identical and independent random variables $\{Y_n\}$, we can construct such $\{X_n\}$ by

$$X_n = e^{\lambda Y_n}/\phi(\lambda) \quad \text{where} \quad \phi(\lambda) = E(e^{\lambda Y n})$$

and the resulting martingale is

$$M_n = \exp(\lambda \sum_{k=1}^{n} Y_k)/\phi(z)^n$$

Shortened Notation

This paragraph is about a rather formal mathematical background called *filtration*. While it is an important subject providing a mathematical background for the stochastic process, it is enough to understand what the notation mean in common sense. A filtration, $\{\mathcal{F}_n\}$, can be understood as the set of information available (or events that happened) up to time n. The set $\mathcal{F}_n$ not only contains the event at time n but also all the past events before n. Therefore the contents of $\mathcal{F}_n$ increases as n increases (time passes), i.e., $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. If $\{\mathcal{F}_n\}$ is the filtration that contains information with respect to a stochastic process $\{X_n\}$, i.e., $X_n \in \mathcal{F}_n$, we can shorten many of our previous statements. For example, we can now say a stochastic process $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$ and it satisfy

$$E(M_{n+1} \mid \mathcal{F}_n) = M_n \text{ for all } n \geq 0.$$

For a practical purpose, it can not go wrong even if you simply think that $\{\mathcal{F}_n\}$ represents *all* information known to time n, not just the information about $\{X_n\}$.

2.2 New Martingales from Old

The main idea of this section is the Martingale Transform Theorem (Theorem 2.1). Assume that $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$ representing the price of a stock (or a gambler's wealth). What if you change the unit of stock every day or the gambler changes the size of bet every time? Let A_n be such multiplier before the outcome of the n-th step. Then, the amount of the wealth will be

$$\widetilde{M}_n = M_0 + A_0(M_1 - M_0) + A_1(M_2 - M_1) + A_2(M_3 - M_2) + \cdots .$$

(<u>Note that the indexing of A_n here is slightly different from that of the textbook.</u>) This process $\{\widetilde{M}_n\}$ is called the martingale transform of $\{M_n\}$ by $\{A_n\}$. What the theorem is stating is a common-sense that if the <u>bounded</u> random variable A_n is determined from the information up to the time n (*non-anticipating* to $\{\mathcal{F}_n\}$ or $A_n \in \mathcal{F}_n$), the new process $\{\widetilde{M}_n\}$ is also a martingale. Again, the *no-fortune-telling* condition on $\{A_n\}$ is critical here.

Stopping times provide martingale transforms

In terms of the new martingale $\{\widetilde{M}_n\}$, we can think of a special type of trading (or betting) strategy where $A_k = 1$ if $k \leq \tau$ or 0 otherwise for a random variable τ. It means you have some kind of betting strategy (or trading strategy) such that you stop betting (or investing in stock) after the outcome at the τ-th step is just known. The random variable τ is a *stopping time* only when the stopping decision is made only from the information at each time step, not in the future (no-fortune-telling again!). Using the filtration notation above, we can say

$$\{\tau \leq n\} \in \mathcal{F}_n.$$

It seems quite confusing but what it means in simple words is that, when you are at n-th time step (so you know all information up to time n), you have to know for sure that either you want to stop $\tau = n$ or you already stopped before $\tau < n$ (so $\{\tau \leq n\}$ is already a known event time n). An example of a stopping strategy which is not a stopping time would be something like you stop you stop your bet at time n when you know you'll lose in the next bet, e.g., $M_{n+1} - M_n < 0$, which is obviously when you have a fortune-telling power. So the bottom line is that any τ associated with any stopping strategy you can imagine with common sense is a proper stopping time, so you don't need to worry too much about the stopping time.

In this regard, Theorem 2.2 is trivial from Theorem 2.1. Restating the theorem,

Theorem 2.2 (Stopping Time Theorem) The stopped process $\{M_{n\wedge\tau}\}$ $(n \wedge \tau = \min(n, \tau))$ derived from the original martingale $\{M_n\}$ is also a martingale.

2.3 Revisiting the Old Ruins

Given that we are armed with the knowledge of martingales and stopping times, the author derives the results of Chapter 1 in a much easier and more elegant way. First note that the first hitting time τ (of hitting A or $-B$) is a stopping time indeed. Please read the book for the detailed re-derivation.

2.4 Submartingales

We skip this section.

2.5 Doob's Inequalities

We skip this section.

2.6 Martingale Convergence

We skip this section.

2.7 Exercises