Course Notes

Stochastic Finance

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Contents

Chapter 1

Random Walks and First Step Analysis

Random walk is a probability process whose incremental change in unit time is up or down by random;

$$S_n = S_0 + X_1 + X_2 + \cdots X_n,$$

$$\text{where } X_k = 1 \text{ or } -1 \text{ with 50:50 chance.}$$

The process models the wealth of a gambler but it is easier to understand if S_n is the daily closing price of a stock and X_n is the profit and loss (P&L) of the n-th day.

For the rest of this chapter, except §1.5, we are interested in the event of S_n hitting A before hitting $-B$ (the gambler making \$$A$ first before losing \$$B$). Equivalently, the event is the stock price gaining A before losing B (assuming that you set a trading strategy of loss-cutting at $-B$ and profit-realizing at A).

For the purpose, the *stopping time* τ is introduced as the first time n when S_n hits either A or $-B$. So we know that $S_\tau = A$ or $-B$ although we don't know the value of τ (τ is a probability variable).

1.1 First Step Analysis

We first solve the probability of the event, $P(S_\tau = A \mid S_0 = 0)$. Generalizing the problem, let

$$f(k) = P(S_\tau = A \mid S_0 = k)$$

be the probability of the same event with the initial point being $S_0 = k$ rather than 0. The recurrence relation is given as

$$f(k) = \frac{1}{2} f(k-1) + \frac{1}{2} f(k+1) \quad \text{for} \quad -B < k < A \tag{1.1}$$

with the *boundary conditions* $f(A) = 1$ and $f(-B) = 0$. This basically means that $f(k)$ is a linear function.

After some algebra, we get

$$f(k) = \frac{k+B}{A+B}, \quad P(S_\tau = A \mid S_0 = 0) = f(0) = \frac{B}{A+B}.$$

The result is in line with the intuition that the probability goes to 1 when B gets bigger or goes to 0 when A gets bigger.

In relation to finance, almost all probability or expectation values can be thought of as the price of a security or a derivative. In this example, we can think of a derivative that pays \$1 when the underling stock price S_n hits A or expires worthless when S_n hits $-B$. This is a *derivative* security because the payoff is *derived* from the underlying stock S_n. Unlike the usual call or put options, the expiry of this derivative is infinite (sometimes such security is called *perpetual*). The probability we computed above, $P(S_\tau = A \mid S_0 = 0)$, can be understood as the current price of the derivative.

Quiz: (a hedging strategy) Imagine that you (as an investment bank) sold the derivative to investors. How would you *hedge* your position using the underlying stock?

1.2 Time and Infinity

In this section, we computes the expected number of bets, τ, until the gambler finishes the game, i.e., when he makes \$A or loses \$B. **SCFA** first proves that the expectation of τ (and any power) is finite. (See **SCFA** for detail.)

In a similar approach from the previous section, the generalized expectation, $g(k) = E(\tau \mid S_0 = k)$ satisfy the recurrence relation,

$$g(k) = \frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) + 1 \quad \text{for} \quad -B < k < A$$

with the boundary condition, $g(A) = g(-B) = 0$.

Notice that $\frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) - g(k)$ is the convexity (or curvature) operator. From the Taylor expansion, we know for small h,

$$\frac{1}{2}g(x+h) + \frac{1}{2}g(x-h) - g(x) \approx \frac{1}{2}g''(x)h^2.$$

So the recurrence relation above implies that $g(k)$ is a quadratic function on k with the second order coefficient is -1. Therefore we conclude that

$$g(k) = (A-k)(B+k) \quad \text{and} \quad \mathbb{E}(\tau \mid S_0 = 0) = AB$$

This quantity can be also thought of as the price of a financial contract, in which \$1 is accumulated each time unit and the money is paid to the investor when the event is triggered. This type of derivatives are generally called *accumulator*.

SCFA verifies the obtained result for the symmetric case of $A = B$. The standard deviation of S_n is $\sqrt{n}$. (The variance is n.) Since the stdev is the characteristic width (or scale) of the process, we can estimate that the time required for the scale to reach A is A^2, which is consistent with the result.

> Quiz (a popular interview question): Imagine that you keep tossing a fair coin (50% for head and 50% for tail) until you get two heads <u>in a row</u>. On average, how many times do you need to toss a coin?

1.3 Tossing an Unfair Coin

When the probability of X_1 is not fair and instead given as

$$X_n = 1 \text{ or } -1 \text{ with the chance of } p \text{ or } q \text{ respectively} \quad (p + q = 1),$$

we can still drive the equivalent results.

After some algebra,

$$f(k) = \frac{(q/p)^{k+B} - 1}{(q/p)^{A+B} - 1} \quad \text{and} \quad P(S_\tau = A | S_0 = 0) = f(0) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}.$$

$$\mathbb{E}(\tau \mid S_0 = 0) = \frac{B}{q-p} - \frac{A+B}{q-p}\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

One can recover the result of the fair bet case, if p and q are approaching to $\frac{1}{2}$, i.e., $p = \frac{1}{2} + \varepsilon$ and $q = \frac{1}{2} - \varepsilon$ for very small ε.

> Quiz (numerical implementation): If you want to implement the above results, i.e., $f(k)$ and $g(k)$ for a general value of $p = 1/2 + \varepsilon$, you will run into a small issue because you have to write a function for the two cases depending on $\varepsilon = 0$ or $\varepsilon \neq 0$. If ε is very small, then the formula may break. How would you resolve this issue?

1.4 Numerical Calculation and Intuition

I recommend that the students quickly verify the numbers in Table 1.1 using your favorite computer tool (R, Matlab, Python or even a calculator). It is quite noticeable that the probability for a gambler to win \$100 before losing \$100 is only 6×10^{-6} when $p = 0.47$.

1.5 First Steps with Generating Functions

The probability generating function is a powerful trick to obtain a series of values in one go, where the coefficients of the Taylor expansion is the values to seek. This chapter of **SCFA** considers the event of S_n hitting 1 for the first time (no longer the event of hitting A or $-B$) and wants to compute the probability of the event happening at time $\tau = 0, 1, 2, \cdots$ (the meaning of τ is also different from the previous sections!). The generating function is in the form of

$$\phi(z) = E(z^\tau \mid S_0 = 0) = \sum_{k=0}^{\infty} P(\tau = k \mid S_0 = 0)\, z^k,$$

i.e. the coefficient of z^k is the probability of S_n hitting 1 at time $\tau = k$ for the first time.

SCFA obtains the function $\phi(z)$ using the recurrence relation method. One important observation is that $\phi(z)^k$ is the generating function for the event of hitting k, which is from the property that the generating function for the sum of independent random variables is the product of the individual generating functions. For $k = 2$, let τ_1 is the first hitting time from 0 to 1 and τ_2 is the first hitting time from 1 to 2. Because τ_1 and τ_2 are independent (and identical) random variables,

$$E(z^{\tau_1+\tau_2}) = E(z^{\tau_1})E(z^{\tau_2}) = \phi(z)^2.$$

Thus, we end up the recurrent relation

$$\phi(z) = \frac{1}{2}\, z\, \phi(z)^2 + \frac{1}{2} z$$

and the $\phi(z)$ is finally given as

$$\phi(z) = \frac{1 - \sqrt{1 - z^2}}{z}.$$

The root with $+$ sign was excluded because the function has the term of $1/z$ and non-zero constant term (the probability for the negative or zero first hitting time should be zero).

1.6 Exercises