Stochastic Finance (FIN 519)

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Course Overview

Goal

- Understand stochastic processes, i.e., probability distribution changing over time
- Model financial variables such as stock price, interest rate, foreign exchange rate with stochastic processes

Key Concepts

- Brownian motion (BM), Wiener process, normal process
- Random walk (RW) and diffusion
- Derivative pricing: Black-Scholes-Merton model, Geometric Brownian motion (GBM) or log-normal process

Quantitative finance courses in PHBS

- Y1-M3: Stochastic Finance by Jaehyuk Choi **[required for Qfin MA]**
- Y1-M4: Derivative Pricing by Lei (Jack) Sun
- Y2-M1: Applied Stochastic Processes by Jaehyuk CHOI
  Application, Programming, Course project
- Y2-M3: Topics in Quantitative Finance by Jaehyuk CHOI
  Machine Learning for Finance (Mon-Thurs 1:30 PM)
- Y2-M3: Numerical Methods and Analysis by Jake ZHAO (Mon-Thurs 3:30 PM)
- Y2-M3: Bayesian Statistics by Qian CHEN (Mon-Thurs 10:30 AM)

- Random Variable (RV): U, X, Y, Z
- Probability density function (PDF): $f_X(x)$
- Cumulative density function (CDF): $F_X(x) = \int f_X(x)dx$
- Standard deviation, variance:

$$\mathrm{Var}(X) = E((X-\bar{X})^2) = E(X^2) - E(X)^2, \quad \sigma_X = \sqrt{Var(X)}$$

- (Centralized) Moments: $M_k(X) = E((X-\bar{X})^k) = \int (x-\bar{X})^k f_X(x)dx$
- Moment generating function (MGF): $M_X(t) = E(e^{tX})$

$$M_X(t) = 1 + tM_1 + \frac{t^2}{2!}M_2 + \cdots + \frac{t^k}{k!}M_k$$

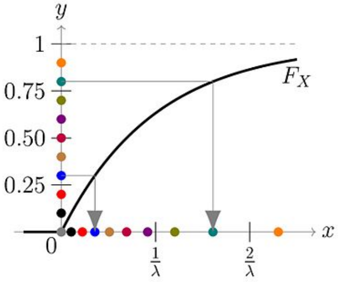
- Characteristic function (CF): $\phi_X(t) = E(e^{itX}) + \cdots$
- Covariance: $\mathrm{Cov}(X,Y) = E((X-\bar{X})(Y-\bar{Y})) = E(XY) - E(X)E(Y)$
- Correlation: $\rho(X,Y) = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = \mathrm{Cov}(X,Y)/(\sigma_X\sigma_Y)$

Properties

- Support: $[0, 1]$
- PDF: $f(x) = 1$
- CDF: $F(x) = x$
- Mean: $E(U) = 1/2$
- Var: $\mathrm{Var}(U) = 1/12$

Uniform distribution is a fundamental RV which can be generated by computer. Once U is generated, any RV X is generated by inverse transform sampling

$$X = F_X^{-1}(U)$$

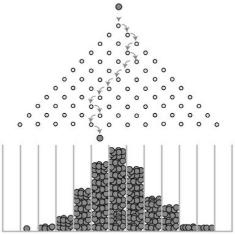


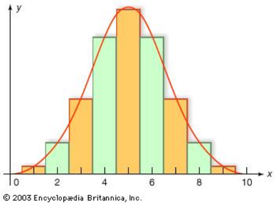
Bernoulli distribution

- $P(X=1)=p, \quad P(X=0)=q=(1-p)$
- $E(X)=p, \quad \mathrm{Var}(X)=pq$

Binomial distribution

- $Y=\sum_1^n X_k \sim N(n,p)$ for i.i.d. Bernoulli $\{X_k\}$ with p.
- $P(Y=k)=\binom{n}{k}p^k q^{(n-k)}$
- $E(Y)=np, \quad \mathrm{Var}(Y)=\sum_1^n \mathrm{Var}(X_k)=npq$.
- Approximated as normal dist. for large n: $B(n,p)\approx N(np,npq)$





Prob. Distribution: Event (default, arrival) at a constant rate λ

Exponential distribution

- Distribution for the survival time or the interval between the events, T
- PDF: $f(t) = \lambda e^{-\lambda t}$, CDF: $F(t) = 1 - e^{-\lambda t}$
- $E(T) = 1/\lambda$, $\mathrm{Var}(T) = 1/\lambda^2$.
- Memoryless: past events have no impact on the future!

Poisson distribution (discrete)

- The number of occurrences X of a Poisson-type event in a unit time interval $T = 1$
- PDF: $P(X = k) = \lambda^k e^{-\lambda}/k!$
- $E(X) = \mathrm{Var}(X) = \lambda$

Gamma distribution

- The distribution of time X before the next k Poisson-type events occur
- $X \sim \Gamma(\alpha, \beta)$ where $\alpha = k$, $\beta = \lambda$.
- PDF: $f(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$ for $x \geq 0$ and $\alpha, \beta > 0$.
- $E(X) = \alpha/\beta, \mathrm{Var}(X) = \alpha/\beta^2$.

Normal (Gaussian) Distribution

- $X \sim N(\mu, \sigma^2), \quad Z \sim N(0,1)$
- PDF: $f_X(x) = \frac{1}{\sqrt{2\sigma^2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \frac{1}{\sigma} n(\frac{x-\mu}{\sigma})$
- CDF: $F_X(x) = N(\frac{x-\mu}{\sigma})$
- MGF: $M_X(x) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right), \quad M_k = \sigma^k (k-1)!!$ for even k.
- Skewness: $s = M_3/\sigma^3 = 0$, Kurtosis $\kappa = M_4/\sigma^4 = 3$ (Ex-kurtosis: 0).

Variations

- Multivariate normal distribution: $(X_1, \cdots, X_n)$
- Log-normal distribution: $Y \sim e^{\mu + \sigma Z}$ for standard normal Z.

- Evaluation of the normal inverse CDF, $N^{-1}(U)$ is expensive and has been a topic of research.
- For two-dimensional Gaussian random numbers (z_1, z_2)

$$P\{z_1^2 + z_2^2 < R^2\} = \int_{z_1^2+z_2^2<R^2} e^{-\frac{1}{2}(z_1^2+z_2^2)} dz_1 dz_2 = \frac{2\pi}{\sqrt{2\pi}^2} \int_R Re^{-R^2/2} dR = 1 - e^{-R^2/2}$$

The CDF of the variable, $Y = R^2 = Z_1^2 + Z_2^2$, is $1 - e^{-Y/2}$.

- This means the inverse function,

$$Y = -2\log(1 - U_1) \quad \text{for a uniform RV } U_1$$

is the correct sampling of Y and

$$R = \sqrt{Y} = \sqrt{-2\log U_1}$$

is the correct sampling of R.

- Now, Z_1 and Z_2 can be split as

$$Z_1 = \sqrt{-2\log U_1}\cos(2\pi U_2), \quad Z_2 = \sqrt{-2\log U_1}\sin(2\pi U_2)$$

so we get two Gaussian RNs.

- The improvement of Box-Muller method
- $V_{1,2} = 2U_{1,2} - 1$ so that $V_{1,2}$ is uniform RVs between -1 and 1.
- Take if $0 < W = V_1^2 + V_2^2 < 1$ and reject otherwise so that (V_1, V_2) is uniform random point on the unit circle.
- W has a uniform distribution on $[0, 1]$, so can replace U_1,

$$P\{W < x\} = \pi(\sqrt{x})^2/\pi = x$$

- Using the trigonometric properties,

$$(\frac{V_1}{\sqrt{W}}, \frac{V_2}{\sqrt{W}}) = \Big(\cos(2\pi U_2), \sin(2\pi U_2)\Big)$$

- Finally we get

$$Z_1 = \sqrt{-2\log W}\, V_1/\sqrt{W} = V_1\sqrt{-2(\log W)/W}$$
$$Z_2 = \sqrt{-2\log W}\, V_2/\sqrt{W} = V_2\sqrt{-2(\log W)/W}$$

- Even though reject 21% of (V_1, V_2) pairs, we can avoid expensive sin and cos evaluations, so Marsaglia-Polar method is more efficient than Box-Muller.

Conditional Probability and Independence

Conditional Probability

A probability of an event A given that an event B has occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Independence

The two events A and B are (statistically) independent if $\boxed{P(A \cap B) = P(A)P(B)}$.
Equivalently,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) \quad \text{if} \quad P(B) \neq 0$$
$$\text{and} \quad P(B|A) = P(B) \quad \text{if} \quad P(A) \neq 0$$

- Joint CDF: $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ $\left(\mathrm{P}(X \leq x, Y < y) = \mathrm{P}(X \leq x)P(Y \leq y)\right)$
- Joint PDF: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$
- $E(XY) = E(X)E(Y)$, $\mathrm{Cov}(X, Y) = \rho(X, Y) = 0$. However, $\rho(X, Y) = 0$ does not imply independence.