Course Notes

Stochastic Finance

Jaehyuk Choi

March 20, 2017

Contents

Chapter 1

Random Walks and First Step Analysis

Random walk is a probability process whose incremental change in unit time is up or down by random;

$$S_n = S_0 + X_1 + X_2 + \cdots X_n,$$
$$\text{where } X_k = 1 \text{ or } -1 \text{ with 50:50 chance.}$$

The process models the wealth of a gambler but it is easier to understand if S_n is the daily closing price of a stock and X_n is the profit and loss (P&L) of the n-th day.

For the rest of this chapter, except §1.5, we are interested in the event of S_n hitting A before hitting $-B$ (the gambler making \$$A$ first before losing \$$B$). Equivalently, the event is the stock price gaining A before losing B (assuming that you set a trading strategy of loss-cutting at $-B$ and profit-realizing at A).

For the purpose, the *stopping time* τ is introduced as the first time n when S_n hits either A or $-B$. So we know that $S_\tau = A$ or $-B$ although we don't know the value of τ (τ is a probability variable).

1.1 First Step Analysis

We first solve the probability of the event, $P(S_\tau = A \mid S_0 = 0)$. Generalizing the problem, let

$$f(k) = P(S_\tau = A \mid S_0 = k)$$

be the probability of the same event with the initial point being $S_0 = k$ rather than 0. The recurrence relation is given as

$$f(k) = \frac{1}{2} f(k-1) + \frac{1}{2} f(k+1) \quad \text{for} \quad -B < k < A \tag{1.1}$$

with the *boundary conditions* $f(A) = 1$ and $f(-B) = 0$. This basically means that $f(k)$ is a linear function.

After some algebra, we get

$$f(k) = \frac{k+B}{A+B}, \quad P(S_\tau = A \mid S_0 = 0) = f(0) = \frac{B}{A+B}.$$

The result is in line with the intuition that the probability goes to 1 when B gets bigger or goes to 0 when A gets bigger.

In relation to finance, almost all probability or expectation values can be thought of as the price of a security or a derivative. In this example, we can think of a derivative that pays \$1 when the underling stock price S_n hits A or expires worthless when S_n hits $-B$. This is a *derivative* security because the payoff is *derived* from the underlying stock S_n. Unlike the usual call or put options, the expiry of this derivative is infinite (sometimes such security is called *perpetual*). The probability we computed above, $P(S_\tau = A \mid S_0 = 0)$, can be understood as the current price of the derivative.

Quiz: (a hedging strategy) Imagine that you (as an investment bank) sold the derivative to investors. How would you *hedge* your position using the underlying stock?

1.2 Time and Infinity

In this section, we computes the expected number of bets, τ, until the gambler finishes the game, i.e., when he makes \$A or loses \$B. **SCFA** first proves that the expectation of τ (and any power) is finite. (See **SCFA** for detail.)

In a similar approach from the previous section, the generalized expectation, $g(k) = E(\tau \mid S_0 = k)$ satisfy the recurrence relation,

$$g(k) = \frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) + 1 \quad \text{for} \quad -B < k < A$$

with the boundary condition, $g(A) = g(-B) = 0$.

Notice that $\frac{1}{2}g(k-1) + \frac{1}{2}g(k+1) - g(k)$ is the convexity (or curvature) operator. From the Taylor expansion, we know for small h,

$$\frac{1}{2}g(x+h) + \frac{1}{2}g(x-h) - g(x) \approx \frac{1}{2}g''(x)h^2.$$

So the recurrence relation above implies that $g(k)$ is a quadratic function on k with the second order coefficient is -1. Therefore we conclude that

$$g(k) = (A-k)(B+k) \quad \text{and} \quad \mathbb{E}(\tau \mid S_0 = 0) = AB$$

This quantity can be also thought of as the price of a financial contract, in which \$1 is accumulated each time unit and the money is paid to the investor when the event is triggered. This type of derivatives are generally called *accumulator*.

SCFA verifies the obtained result for the symmetric case of $A = B$. The standard deviation of S_n is $\sqrt{n}$. (The variance is n.) Since the stdev is the characteristic width (or scale) of the process, we can estimate that the time required for the scale to reach A is A^2, which is consistent with the result.

Quiz (a popular interview question): Imagine that you keep tossing a fair coin (50% for head and 50% for tail) until you get two heads <u>in a row</u>. On average, how many times do you need to toss a coin?

1.3 Tossing an Unfair Coin

When the probability of X_1 is not fair and instead given as

$$X_n = 1 \text{ or } -1 \text{ with the chance of } p \text{ or } q \text{ respectively} \quad (p + q = 1),$$

we can still drive the equivalent results.

After some algebra,

$$f(k) = \frac{(q/p)^{k+B} - 1}{(q/p)^{A+B} - 1} \quad \text{and} \quad P(S_\tau = A | S_0 = 0) = f(0) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}.$$

$$\mathbb{E}(\tau \mid S_0 = 0) = \frac{B}{q-p} - \frac{A+B}{q-p} \frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

One can recover the result of the fair bet case, if p and q are approaching to $\frac{1}{2}$, i.e., $p = \frac{1}{2} + \varepsilon$ and $q = \frac{1}{2} - \varepsilon$ for very small ε.

Quiz (numerical implementation): If you want to implement the above results, i.e., $f(k)$ and $g(k)$ for a general value of $p = 1/2 + \varepsilon$, you will run into a small issue because you have to write a function for the two cases depending on $\varepsilon = 0$ or $\varepsilon \neq 0$. If ε is very small, then the formula may break. How would you resolve this issue?

1.4 Numerical Calculation and Intuition

I recommend that the students quickly verify the numbers in Table 1.1 using your favorite computer tool (R, Matlab, Python or even a calculator). It is quite noticeable that the probability for a gambler to win \$100 before losing \$100 is only 6×10^{-6} when $p = 0.47$.

1.5 First Steps with Generating Functions

The probability generating function is a powerful trick to obtain a series of values in one go, where the coefficients of the Taylor expansion is the values to seek. This chapter of **SCFA** considers the event of S_n hitting 1 for the first time (no longer the event of hitting A or $-B$) and wants to compute the probability of the event happening at time $\tau = 0, 1, 2, \cdots$ (the meaning of τ is also different from the previous sections!). The generating function is in the form of

$$\phi(z) = E(z^\tau \mid S_0 = 0) = \sum_{k=0}^{\infty} P(\tau = k \mid S_0 = 0)\, z^k,$$

i.e. the coefficient of z^k is the probability of S_n hitting 1 at time $\tau = k$ for the first time.

SCFA obtains the function $\phi(z)$ using the recurrence relation method. One important observation is that $\phi(z)^k$ is the generating function for the event of hitting k, which is from the property that the generating function for the sum of independent random variables is the product of the individual generating functions. For $k = 2$, let τ_1 is the first hitting time from 0 to 1 and τ_2 is the first hitting time from 1 to 2. Because τ_1 and τ_2 are independent (and identical) random variables,

$$E(z^{\tau_1+\tau_2}) = E(z^{\tau_1})E(z^{\tau_2}) = \phi(z)^2.$$

Thus, we end up the recurrent relation

$$\phi(z) = \frac{1}{2}\, z\, \phi(z)^2 + \frac{1}{2} z$$

and the $\phi(z)$ is finally given as

$$\phi(z) = \frac{1 - \sqrt{1 - z^2}}{z}.$$

The root with + sign was excluded because the function has the term of $1/z$ and non-zero constant term (the probability for the negative or zero first hitting time should be zero).

1.6 Exercises

Chapter 2

First Martingale Steps

Martingale is one of the key concepts in stochastic process. Although it is a very formal mathematical concept, it will turn out that many practical results will be derived out of it. For the definition of the martingale, we refer to Wikipedia.

> In probability theory, a martingale is a model of a fair game where knowledge of past events never helps predict the mean of the future winnings. In particular, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values.

In **SCFA**, a stochastic process $\{M_n : 0 \leq n\}$ is a *martingale* with respect to another stochastic process $\{X_n : 0 \leq n\}$ if (i) the sequence M_n is determined from the past knowledge of $\{X_k : 0 \leq k \leq n\}$ and (ii) the next expectation value is equal to the present value of M_n (*fundamental martingale identity*),

$$E(M_{n+1} \mid X_1, X_2, \cdots, X_n) = M_n \text{ for all } n \geq 0.$$

In general, however, $\{M_n\}$ is simply a martingale if the next expectation value, conditional on the history of itself, is equal to the present value,

$$E(M_{n+1} \mid M_1, M_2, \cdots, M_n) = M_n \text{ for all } n \geq 0.$$

2.1 Classic Examples

SCFA gives 3 examples of martingales

Example 1 If the X_n are independent random variables with zero mean, the running sum, $S_n = \sum_0^n X_k$ is a martingale. The process S_n was the subject of Chapter 1. So the wealth of a gambler or a stock price are all martingale as long as the game is fair ($E(X_n) = 0$ and the no one can look into the future. In the case of the stock, this assumption is closely related to the efficient market hypothesis, where the stock prices reflect the market information immediately and fully. Since all the news are *priced in* the stock, the expectation for tomorrow's stock is same as the current value (no one know that tomorrow's news will be good or bad).

This observation gives us a good example of what is **not** a martingale. Imagine that a stock price has a momentum (or a positive auto-correlation) in that the stock price tends to be up (or down) in a day when the price was up (or down) in the previous day, i.e., X_n and X_{n+1} are positively correlated rather than independent. The stock price in that circumstance is not a martingale because one can look into the future (based on the past). Many technical analyses are indeed based on that stock markets have momentum. For a well-known strategy, see the turtle trading rule.

Example 2 On top of the assumptions of **Example 1**, let us assume that $\text{Var}(X_n) = \sigma$. Then $M_n = S_n^2 - n\sigma^2$ is also a martingale. See the textbook for the detailed proof. Basically what it tells us is that the squared process S_n^2 increases by the σ^2 on average on each time step, so we need to add the correction term, $-n\sigma^2$ for the process M_n to be a martingale. This is an important precursor to the famous Itô's lemma which we will cover later!

Example 3 If $\{X_n\}$ are non-negative independent random variables with $E(X_n) = 1$, the running product $M_n = X_1 \cdot X_2 \cdots X_n$ is a martingale. See the textbook for the detailed proof. Out of any identical and independent random variables $\{Y_n\}$, we can construct such $\{X_n\}$ by

$$X_n = e^{\lambda Y_n}/\phi(\lambda) \quad \text{where} \quad \phi(\lambda) = E(e^{\lambda Y n})$$

and the resulting martingale is

$$M_n = \exp(\lambda \sum_{k=1}^{n} Y_k)/\phi(z)^n$$

Shortened Notation

This paragraph is about a rather formal mathematical background called *filtration.* While it is an important subject providing a mathematical background for the stochastic process, it is enough to understand what the notation mean in common sense. A filtration, $\{\mathcal{F}_n\}$, can be understood as the set of information available (or events that happened) up to time n. The set $\mathcal{F}_n$ not only contains the event at time n but also all the past events before n. Therefore the contents of $\mathcal{F}_n$ increases as n increases (time passes), i.e., $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. If $\{\mathcal{F}_n\}$ is the filtration that contains information with respect to a stochastic process $\{X_n\}$, i.e., $X_n \in \mathcal{F}_n$, we can shorten many of our previous statements. For example, we can now say a stochastic process $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$ and it satisfy

$$E(M_{n+1} \mid \mathcal{F}_n) = M_n \text{ for all } n \geq 0.$$

For a practical purpose, it can not go wrong even if you simply think that $\{\mathcal{F}_n\}$ represents *all* information known to time n, not just the information about $\{X_n\}$.

2.2 New Martingales from Old

The main idea of this section is the Martingale Transform Theorem (Theorem 2.1). Assume that $\{M_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$ representing the price of a stock (or a gambler's wealth). What if you change the unit of stock every day or the gambler changes the size of bet every time? Let A_n be such multiplier before the outcome of the n-th step. Then, the amount of the wealth will be

$$\widetilde{M}_n = M_0 + A_0(M_1 - M_0) + A_1(M_2 - M_1) + A_2(M_3 - M_2) + \cdots .$$

(<u>Note that the indexing of A_n here is slightly different from that of the textbook.</u>) This process $\{\widetilde{M}_n\}$ is called the martingale transform of $\{M_n\}$ by $\{A_n\}$. What the theorem is stating is a common-sense that if the <u>bounded</u> random variable A_n is determined from the information up to the time n (*non-anticipating* to $\{\mathcal{F}_n\}$ or $A_n \in \mathcal{F}_n$), the new process $\{\widetilde{M}_n\}$ is also a martingale. Again, the *no-fortune-telling* condition on $\{A_n\}$ is critical here.

Stopping times provide martingale transforms

In terms of the new martingale $\{\widetilde{M}_n\}$, we can think of a special type of trading (or betting) strategy where $A_k = 1$ if $k \leq \tau$ or 0 otherwise for a random variable τ. It means you have some kind of betting strategy (or trading strategy) such that you stop betting (or investing in stock) after the outcome at the τ-th step is just known. The random variable τ is a *stopping time* only when the stopping decision is made only from the information at each time step, not in the future (no-fortune-telling again!). Using the filtration notation above, we can say

$$\{\tau \leq n\} \in \mathcal{F}_n.$$

It seems quite confusing but what it means in simple words is that, when you are at n-th time step (so you know all information up to time n), you have to know for sure that either you want to stop $\tau = n$ or you already stopped before $\tau < n$ (so $\{\tau \leq n\}$ is already a known event time n). An example of a stopping strategy which is not a stopping time would be something like you stop you stop your bet at time n when you know you'll lose in the next bet, e.g., $M_{n+1} - M_n < 0$, which is obviously when you have a fortune-telling power. So the bottom line is that any τ associated with any stopping strategy you can imagine with common sense is a proper stopping time, so you don't need to worry too much about the stopping time.

In this regard, Theorem 2.2 is trivial from Theorem 2.1. Restating the theorem,

Theorem 2.2 (Stopping Time Theorem) The stopped process $\{M_{n\wedge\tau}\}$ $(n \wedge \tau = \min(n, \tau))$ derived from the original martingale $\{M_n\}$ is also a martingale.

2.3 Revisiting the Old Ruins

Given that we are armed with the knowledge of martingales and stopping times, the author derives the results of Chapter 1 in a much easier and more elegant way. First note that the first hitting time τ (of hitting A or $-B$) is a stopping time indeed. Please read the book for the detailed re-derivation.

2.4 Submartingales

We skip this section.

2.5 Doob's Inequalities

We skip this section.

2.6 Martingale Convergence

We skip this section.

2.7 Exercises

Problem 2.1 is a part of **HW** 2.

Chapter 3

Brownian Motion

Brownian Motion (BM) is the continuous version of the discrete random walk we covered in Chapter 1. Basically it is a stochastic process where normal distributions are repeated so that the stdev is increasing as $\sqrt{t}$. In other books, it is also called *Wiener process* after the Mathematician provided the Mathematical background of it.

Steele starts the chapter by stating that Brownian motion is the most important stochastic process, which I can not agree more. Brownian motion will be used a basic building block for about 99% of the stochastic processes that you'll see in financial modeling! So understanding BM is the single most important goal of this course.

BM has been closely linked to finance as well as physics. Although it is often overshadowed by the great success of Black-Scholes-Merton's option pricing theory (1973), a French mathematician, Bachelier made a first option pricing theory in his Ph.D. thesis *The Theory of Speculation* (1900) based on BM. And it was 5 years earlier than the Einstein's famous paper on BM (1905)!

BM is defined as below:

Definition 3.1 A *continuous-time stochastic process* $\{B_t : 0 \le t < T\}$ is called a Standard Brownian Motion on $[0,\ T)$ if (i) $B_0 = 0$, (ii) The increments of B_t, i.e., $B_{t_2} - B_{t_1},\ B_{t_3} - B_{t_2},\ \cdots$ for $0 \le t_1 < t_2 < t_3 < \cdots$, are independent, (iii) the increment $B_t - B_s$ for $s \le t$ has the Gaussian distribution with mean 0 and standard variation $\sqrt{t-s}$ and (iv) $B(t)$ is a continuous function.

The rest of this chapter is focused on how one can represent BM as a (infinite) sum of functions. Although it is an interesting topic (one of my research topic is related to this), we don't see an immediate practical use for our course, so we'll skip many of the following sections.

3.1 Covariances and Characteristic Functions

We skip the multivariate Gaussian distribution part for now. We will use some results of this section when we simulate multi-dimensional correlated BM's later. The covariance property of a single variable BM is important.

Covariance Functions and Gaussian Processes

A process, X_t is called a Gaussian process if the vectors $(X_{t_1}, X_{t_2}, \cdots, X_{t_n})$ form a multivariate Gaussian distribution for any finite set of $\{t_k\}$. On the other hand, the covariance for Brownian motion between time s and t $(s \le t)$ is given as

$$\mathrm{Cov}(B_s, B_t) = E(B_t B_s) = E(B_s B_s) + E((B_t - B_s)B_s) = s + 0 = s \wedge t$$

due to the property of the independent increments. The following lemma states that the opposite is also true, i.e, any process with covariance, $s \wedge t$, has independent increments.

Lemma 3.1 If a Gaussian process X_t has $E(X_t) = 0$ and $\mathrm{Cov}(X_s, X_t) = s \wedge t$, X_t has independent increments. Moreover if $X_0 = 0$ and X_t has continuous path, X_t is a standard Brownian motion.

3.2 Visions of a Series Approximation

We skip this section.

3.3 Two Wavelets

We skip this section.

3.4 Wavelet Representation of Brownian Motion

We skip this section.

3.5 Scaling and Inverting Brownian Motion

Proposition 3.2 For any $a > 0$, the following three processes defined by

$$X(t) = \frac{1}{\sqrt{a}} B(at) \text{ for } t \geq 0 \quad \text{(scaled process)},$$

$$Y(0) = 0 \text{ and } Y(t) = t\, B(1/t) \text{ for } t > 0 \quad \text{(inverted process)},$$

$$Z(t) = B(1) - B(1-t) \quad \text{(time-reversed process)}$$

are all standard BM's on $[\,0, \infty)$ for $X(t)$ and $Y(t)$ and on $[\,0, 1]$ for $Z(t)$.

3.6 Exercises

Exercise problems from 3.1 to 3.4 are recommended.

Chapter 4

Martingales: The next steps

This chapter introduces continuous-time martingales, thus parallels with Chapter 2. In a similar way we covered Chapter 2, we will focus on the intuition and skip the sections on the rigorous mathematical definition.

4.1 Foundation Stones

We skip this section.

4.2 Conditional Expectations

We skip this section.

4.3 Uniform Integrability

We skip this section.

4.4 Martingales in Continuous Time

We first introduce filtration under continuous time, $\{\mathcal{F}_t : \ 0 \leq t < \infty\}$. Again, set $\mathcal{F}_t$ represents all the (cumulative) information up to time t, thus $s \leq t$ implies that $\mathcal{F}_s \subset \mathcal{F}_t$. If a continuous filtration $\{\mathcal{F}_t\}$ contains all the information about a continuous stochastic process $\{X_t\}$, we say X_t is $\mathcal{F}_t-$*measurable* or X_t is *adapted* to the filtration $\mathcal{F}_t$. Similarly in Chapter 2, however, it is more convenient to assume that the filtration $\{\mathcal{F}_t\}$ is the filtration which contains *all* information, not just about a stochastic process. Now, the process $\{X_t\}$ is a martingale if

1. $E(|X_t|) < \infty$ for all $0 \leq t < \infty$ and

2. $E(X_t \mid \mathcal{F}_s) = X_s$ for all $0 \leq s \leq t < \infty$.

The Standard Brownian Filtration

This part discusses the minimal filtration Brownian motion is adapted to, i.e., the filtration having just enough information on Brownian motion. In the light of our convenient maximal assumption of the filtration $\mathcal{F}_t$, however, we skip this section.

Stopping Times

The stopping time under continuous time framework is almost same. Again, the stopping time is a stopping strategy under which one can determine stop or not based on the information up to now, not in the future.

A continuous random variable τ is a stopping time with respect to a filtration $\{\mathcal{F}_t\}$ if

$$\{w : \tau(w) \leq t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0.$$

Here w is a particular path or a realization of the process X_t. Also, the stopped variable X_τ (when τ is not ∞) is naturally $X_\tau(w) = X_t(w)$ for $\tau(w) = t$.

Doob's Stopping Time Theorem

Theorem 4.1 This theorem is the continuous-time version of Theorem 2.1, which states that the stopped process $M_{t\wedge\tau}$ derived from the original continuous process M_t and the stopping time τ is also a continuous martingale.

we will skip the rest of the section.

4.5 Classic Brownian Motion Martingles

Theorem 4.4 The following three processes associated with the standard Brownian motion are all continuous martingales

1. B_t,
2. $B_t^2 - t$,
3. $\exp(\alpha B_t - \alpha^2 t/2)$

Note that the three examples above are the continuous-time versions of the examples discussed in Section 2.1. Again, the second case gives the insight, $(dB_t)^2 = dt$, which leads to Itô's lemma. The third one is the geometric Brownian motion (log-normal process) which frequently appears in the derivation of the Black-Scholes-Merton formula if the parameter α is replaced with the volatility σ.

Ruin Probabilities for Brownian Motion

Theorem 4.5 Using the martingale property, we again obtain the same results,

$$P(B_\tau = A) = \frac{B}{A+B} \quad \text{and} \quad E(\tau) = AB,$$

where τ is the first hitting time of double-barrier, $\tau = \inf\{t : B_t = -B \text{ or } B_t = A\}$.

In the proof, we used the property from the second martingale example, $E(B_\tau^2) = E(\tau)$.

Hitting Time of a Level

Now we consider the first hitting time of a one-side barrier, $\tau_a = \min\{t : B_t = a\}$. We obtain the following important result:

Theorem 4.6 For any value real value a,

$$P(\tau_a < \infty) = 1 \quad \text{and} \quad E(e^{-\lambda\tau_a}) = e^{-|a|\sqrt{2\lambda}}.$$

In the proof of the second part, we used that the exponential Brownian motion is a (continuous-time) martingale,

$$1 = E(M_{\tau_a}) = \exp(\alpha a - \alpha^2 \tau_a/2).$$

By taking $\alpha = \text{sign}(a)\sqrt{2\lambda}$, we obtain the second part.

Note that, for a non-negative random variable X, $E(e^{-\lambda X})$ is the Laplace transform of the probability density function of X. The Laplace transform $E(e^{-\lambda X})$ is well-defined for the random variables with infinite moments, thus more useful than the moment generating function $E(e^{\lambda X})$ sometimes.

The second part also gives a pricing formula for a very plausible derivative, which pays \$1 when the underling stock following the standard BM B_t hits the level a. Then the interest rate is r, the second formula gives the present value of the derivative (perpetual digital option)

$$P = E(e^{-r\tau_a}) = e^{-|a|\sqrt{2r}}.$$

The fact that $P = 1$ when $r = 0$ is consistent with the first part, i.e., the probability of hitting the level is 1.

Quiz: how does the price formula modified if the underling stock follows a BM with volatility σ?

First Consequences

The Laplace transform (4.25) of the hitting-time density provides useful insights. For example, we can conclude $E(\tau_a) = \infty$ from that

$$E(\tau_a) = -\frac{d}{d\lambda} E(e^{-\lambda\tau_a})\Big|_{\lambda=0} = \frac{a}{\sqrt{2\lambda}} e^{-a\sqrt{2\lambda}}\Big|_{\lambda=0} = \infty$$

We can also show $E(1/\tau_a) = 1/a^2$ from

$$E\left(\frac{1}{\tau_a}\right) = \int_0^\infty E(e^{-\lambda\tau_a})d\lambda = \int_0^\infty e^{-a\sqrt{2\lambda}}d\lambda = \int_0^\infty ue^{-u}\frac{du}{a^2} = \frac{1}{a^2},$$

where we used the identity $1/t = \int_0^\infty e^{-\lambda t}d\lambda$ and the change of variable $u = a\sqrt{2\lambda}$.

The inverse Laplace transform is actually known, thus we have the analytic expression of the hitting-time density distribution as

$$f_{\tau_a}(t) = \frac{|a|}{\sqrt{2\pi t^3}} e^{-a^2/2t}$$

This is a special case (zero drift) of the Inverse Gaussian distribution family. We will elegantly derive this result using the reflection principle in Chapter 5.

Looking Back

The author argues that the functional form of the Laplace transform can be guessed from the symmetry argument.

4.6 Exercises

Exercise problem 4.6 is recommended.

Chapter 5

Richness of paths

5.1 Quantitative Smoothness

We skip this section.

5.2 Not Too Smooth

We skip this section.

5.3 Two Reflection Principles

We continue to explore quantitative properties of Brownian motion. In this section we elegantly derive the distributions related to the maximum process, $B_t^* = \max_{0\le s\le t} B_s$. We'll drive the joint distribution of $(B_t^*,\ B_t)$ and the distribution of B_t^* itself. Both results are important for pricing exotic options such as barrier and max options.

At the heart of the derivation is the reflection principle of Brownian motion. The author first states the principle for the (discrete) random walks, but there's no problem in understanding it directly for the continuous-time processes.

Proposition (5.1) The *reflected process*, $\tilde{B}_t$ of the standard BM, B_t defined as

$$\tilde{B}_t = \begin{cases} B_t & \text{if} \quad t < \tau \\ B_\tau - (B_t - B_\tau) & \text{if} \quad t \ge \tau \end{cases}$$

for any stopping time τ (usually the hitting-time at certain level) is also a Brownian motion.

The defined process B_t^* is flipping the original path of B_t for the portion after the stopping time $t > \tau$. The meaning of principal should be very intuitive even without the mathematical proof. Note that, conditional on $t = \tau$, the two subsequent paths, i.e., the original path B_t and the reflected path $\tilde{B}_t$ are equally probably due to the symmetry and the independence property of BM.

In the same way, we can also reflect the portion before the stopping time τ,

$$\tilde{B}_t = \begin{cases} B_\tau - (B_t - B_\tau) & \text{if} \quad t < \tau \\ B_t & \text{if} \quad t \geq \tau \end{cases}.$$

The definition makes sense only when we know the value of B_τ. In the case of the single barrier hitting-time, we are lucky to have $B_\tau = a$. So the reflected part of the path simply becomes $B_t^* = 2a - B_t$, which is a BM starting from $2a$.

From the reflection principle, we have the following equality of three probabilities, for $x, y \geq 0$,

$$P(B_t^* > x,\ B_t < x - y) = P(B_t^* > x,\ B_t > x + y) = P(B_t > x + y).$$

The first equality is directly from the reflection principle. The stopping time τ used here is the first hitting-time of the level x. The condition $S_t^* > x$ means that the path hit the level x before time t, so if $S_t > x + y$, the reflected path should satisfy $\tilde{S}_t < x - y$. The second equality is trivially due to the continuity of BM. Now we are ready to derive various probability densities.

Joint Distribution of B_t and B_t^*

The joint probability is given as

$$\begin{aligned} P(B_t^* < x,\ B_t < x - y) &= P(B_t < x - y) - P(B_t^* \geq x,\ B_t < x - y) \\ &= P(B_t < x - y) - P(B_t > x + y) \\ &= \Phi((x - y)/\sqrt{t}) + \Phi((x + y)/\sqrt{t}) - 1. \end{aligned}$$

Under the change of variables, $v = x,\ u = x - y$, we have the final result on the joint density,

$$\begin{aligned} P(B_t^* < v,\ B_t < u) &= \Phi(u/\sqrt{t}) + \Phi((2v - u)/\sqrt{t}) - 1 \\ &= \Phi(u/\sqrt{t}) - \Phi((u - 2v)/\sqrt{t}) \quad \text{(CDF)} \\ f_{(B_t^*, B_t)}(v, u) &= \frac{2(2v - u)}{t}\phi((2v - u)/\sqrt{t}) \quad \text{(PDF)} \end{aligned}$$

Density and Distribution of B_t^*

When $y = 0$, we have the cumulative distribution function,

$$\begin{aligned} P(B_t^* > x) = P(\tau_x < t) &= P(S_t^* > x,\ S_t > x) + P(S_t^* > x,\ S_t \le x) \\ &= P(S_t > x) + P(S_t^* > x,\ S_t \ge x) \\ &= 2P(S_t > x) = P(|S_t| > x) \\ &= 2 - 2\Phi(x/\sqrt{t}). \end{aligned}$$

Equivalently, we have the complementary value,

$$P(B_t^* < x) = P(\tau_x > t) = 2\Phi(x/\sqrt{t}) - 1.$$

The differentiation w.r.t. x gives the density on x,

$$f_{B_t^*}(x) = \frac{2}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) = \sqrt{\frac{2}{\pi t}}\, e^{-x^2/2t} \quad \text{for} \quad x \ge 0$$

Density of the Hitting Time τ_x

The differentiation w.r.t. t gives the density on τ_x,

$$f_{\tau_x}(t) = \frac{x}{t^{3/2}}\phi\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \ge 0,\ t \ge 0.$$

Note we saw the result in (4.27) from the inverse Laplace transform!

5.4 The Invariance Principle and Donsker's Theorem

It is worth to mention Donsker's Theorem, which connects (discrete) random walk and (continuous) Brownian motion.

If $\{X_n\}$ be a sequence of i.i.d. random variables with mean 0 and variance 1 (more general than $X_n = \pm 1$), we have the discrete-time random walk,

$$S_n = \sum_{k=1}^{n} X_n \quad (S_0 = 0).$$

From the central limit theorem (CLT), we know that $S_n/\sqrt{n}$ converges to $\Phi(0,1)$ as $n \to \infty$. Donsker's theorem is basically an extension of the CLT on the whole process of S_n.

Let us extend S_n into a continuous time process by interpolating the points at $t = n$ by

$$S_t^{(n)} = S_n + (t-n)X_{n+1} \quad \text{for} \quad n \le t < n+1$$

and define a scaled process,

$$B_t^{(n)} = S_{nt}^{(n)}/\sqrt{n}.$$

Donsker's theorem states that the process $B_t^{(n)}$ converges to B_t as $n \to \infty$. The CLT is obviously the special case of Donsker's theorem at $t = 1$,

$$B_1^{(n)} = S_n^{(n)}/\sqrt{n} = S_n/\sqrt{n} \to B_1 \quad \text{as} \quad n \to \infty.$$

It was not a coincidence that we saw same results between random walks and BM for many occasions such as the ruin probability and the expectation for the stopping time.

5.5 Random Walks Inside Brownian Motion

We skip this section.

5.6 Exercises